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ORIGINAL RESEARCH

A derivative-free line search technique for Broyden-like method with applications to NCP, wLCP and SI

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Abstract
We propose a new derivative-free line search technique which contains the classical Li-Fukushima derivative-free line search [Optim. Methods Softw. 13 (3), 181–201, 2000] as a special case. The new line search can enable us to choose a larger step size at each iteration and reduce the number of function evaluations at each step. Based on the new line search, we prove that Broyden-like method for solving the nonlinear equation is globally and locally superlinearly convergent under appropriate assumptions. Moreover, we present some nonlinear equations arising from nonlinear complementarity problems (NCP), weighted linear complementarity problems (wLCP) and system of inequalities (SI). Numerical results show that Broyden-like method based on the new line search has better numerical performance than that based on Li-Fukushima line search.

Keywords Nonlinear equation · Broyden-like method · Derivative-free line search · Superlinear convergence

1 Introduction

Let $\mathbf{F}(x) := (\mathbf{F}_1(x), ..., \mathbf{F}_n(x))^T : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable mapping. Let $\mathbf{F}'(x)$ denote the Jacobian of $\mathbf{F}(x)$ at x. We consider the nonlinear equation

$$\mathbf{F}(x) = 0. \tag{1.1}$$

As it is well-known, quasi-Newton methods are regarded as one class of the most effective methods for solving (1.1). A quasi-Newton method for solving (1.1) generates an iteration sequence $\{x^k\}$ by following the iterative process

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$$x^{k+1} = x^k + d^k, k = 0, 1, ...,$$

starting from some initial point x^0. The direction d^k is called a quasi-Newton direction which is a solution of the following system of linear equations

$$\mathbf{B}_k d^k + \mathbf{F}(x^k) = 0,$$

where $\mathbf{B}_k$ is some matrix which is an approximation to $\mathbf{F}'(x^k)$. When $\mathbf{B}_k = \mathbf{F}'(x^k)$, the iterative process is the well-known Newton method. In the iterative process, the matrix $\mathbf{B}_{k+1}$ is obtained by updating $\mathbf{B}_k$ with some lower rank matrix. One of the well-known update formulae is the following Broyden's rank one formula (Broyden, 1965)

$$\mathbf{B}_{k+1} = \mathbf{B}_k + \frac{(y^k - \mathbf{B}_k s^k)(s^k)^T}{\|s^k\|^2},$$

where $s^k := x^{k+1} - x^k$ and $y^k := \mathbf{F}(x^{k+1}) - \mathbf{F}(x^k)$. Notice that the matrix $\mathbf{B}_{k+1}$ is generally not symmetric even if $\mathbf{B}_k$ is symmetric, and $\mathbf{B}_{k+1}$ may be singular even if $\mathbf{B}_k$ is nonsingular. So, the quasi-Newton direction d^k may not exist. As a remedy, Powell (Powell, 1970) proposed the so-called Broyden-like formula which takes the following update formulae

$$\mathbf{B}_{k+1} = \mathbf{B}_k + \theta_k \frac{(y^k - \mathbf{B}_k s^k)(s^k)^T}{\|s^k\|^2},$$

where the parameter $\theta_k \in (0, \theta)$ with some constant $\theta \in (0, 1)$ is chosen so that $\mathbf{B}_{k+1}$ is nonsingular. It has been proved that Broyden-like method for solving (1.1) is superlinearly convergent, see Dennis and Moré (1977)[Theorem 3.1]. Since for (1.1), the quasi-Newton direction d^k is generally not a descent direction of the merit function $\|\mathbf{F}(x)\|^2$ at x^k even if $\mathbf{B}_k$ is positive definite, line searches that require the computation of derivatives are not available for Broyden-like method. Thus, to globalize Broyden-like method, we need some derivative-free line search techniques. The earliest derivative-free line search was proposed by Griewank (1986) which makes it possible to design globally convergent Broyden-like method. However, (Griewank, 1986) pointed out that this line search may fail when $\mathbf{F}(x^k)^T \mathbf{F}'(x^k) d^k = 0$, also see Li and Fukushima (2000a)[Example]. Li and Fukushima proposed some new derivative-free line search techniques and developed globally convergent BFGS quasi-Newton method and DFP quasi-Newton method for solving symmetric nonlinear equations (Li & Fukushima, 1999a, b), mixed complementarity problems (Li & Fukushima, 2000b), nonconvex optimization problems (Li & Fukushima, 2001b, c), KKT systems in mathematical programming (Li et al., 2001). The first well-defined derivative-free line search for Broyden-like method for solving (1.1) was also proposed by Li and Fukushima (2000a). Their approach is the following.

Li-Fukushima Derivative-Free Line Search (LF-DFLS): Choose constants $\sigma, \delta \in (0, 1)$ and $\eta > 0$. Choose a positive sequence $\{\eta_k\}$ satisfying $\sum_{k=0}^{\infty} \eta_k \leq \eta < \infty$. Let the step size $\lambda_k := \delta^{l_k}$, where l_k is the smallest nonnegative integer l satisfying

$$\|\mathbf{F}(x^k + \delta^l d^k)\| \leq (1 + \eta_k)\|\mathbf{F}(x^k)\| - \sigma\|\delta^l d^k\|^2, \tag{1.2}$$

in which d^k is the generated quasi-Newton direction.

It is easy to see that the inequality (1.2) holds for all sufficiently large $l > 0$ as when $l \to \infty$, the left-hand side of (1.2) tends to $\|\mathbf{F}(x^k)\|$ but the right-hand side tends to $(1 + \eta_k)\|\mathbf{F}(x^k)\|$. Thus, LF-DFLS is well-defined. Based on LF-DFLS, (Li & Fukushima, 2000a) proved that

Broyden-like method for solving (1.1) is globally convergent and its convergence rate is superlinear under appropriate assumptions. After that, based on a derivative-free line search which is similar as LF-DFLS, (Li & Fukushima, 2001a) studied globally convergent Broyden-like method for solving semismooth equations. Recently, by applying LF-DFLS, many researchers have studied smoothing Broyden-like methods for solving nonlinear complementarity problems (Chen & Ma, 2011a, b; Fan, 2015; Ma et al., 2008), mixed complementarity problems (Tang & Liu, 2010), polyhedral cone constrained eigenvalue problems (Li et al., 2011) and second-order cone complementarity problems (Tang & Zhou, 2021). Very lately, based on LF-DFLS, (Zhou & Zhang, 2020) proposed a modified Broyden-like method for solving (1.1) and established its global and local superlinear convergence.

In this paper, we aim to introduce a new derivative-free line search technique for Broyden-like method which contains LF-DFLS as a special case. Our derivative-free line search has the same general form as the scheme of LF-DFLS, except that their "$\|\mathbf{F}(x^k)\|''$ is replaced by an convex combination of some values. More precisely, our approach is the following.

New Derivative-Free Line Search (N-DFLS): Choose constants σ, $\delta \in (0, 1)$, $\tau \in (0, 1]$ and $\eta > 0$. Choose a positive sequence $\{\eta_k\}$ satisfying $\sum_{k=0}^{\infty} \eta_k \leq \eta < \infty$. Let the step size $\lambda_k := \delta^{l_k}$, where l_k is the smallest nonnegative integer l satisfying

$$\|\mathbf{F}(x^k + \delta^l d^k)\| \leq (1 + \eta_k)\mathbf{\Phi}_k - \sigma\|\delta^l d^k\|^2, \tag{1.3}$$

in which $\mathbf{\Phi}_0 := \|\mathbf{F}(x^0)\|$ and for $k \geq 1$,

$$\mathbf{\Phi}_k := (1 - \tau)\frac{[(1 + \eta_{k-1})\mathbf{\Phi}_{k-1} + 1]\|\mathbf{F}(x^k)\|}{\|\mathbf{F}(x^k)\| + 1} + \tau\|\mathbf{F}(x^k)\|. \tag{1.4}$$

Obviously, if we choose $\tau = 1$ in N-DFLS, then $\mathbf{\Phi}_k = \|\mathbf{F}(x^k)\|$ for all k and in this case N-DFLS reduces to LF-DFLS. Hence, LF-DFLS can be viewed as a special case of N-DFLS. We show that N-DFLS is well-defined and it can enable us to choose a larger step size at each iteration and reduce the number of function evaluations at each step. Moreover, we prove that Broyden-like method based on N-DFLS is globally and locally superlinearly convergent under appropriate assumptions. In addition, we present some smooth nonlinear equations arising from nonlinear complementarity problems (NCP), weighted linear complementarity problems (wLCP) and system of inequalities (SI), and apply Broyden-like method to solve them. It is worth pointing out that many Broyden-like methods have been proposed for solving NCP (e.g., (Chen & Ma, 2011a, b; Fan, 2015; Ma et al., 2008)). However, to the best of our knowledge, there is no work on Broyden-like method for solving wLCP and SI. We also report some numerical results which show that Broyden-like method based on N-DFLS is much superior to that based on LF-DFLS.

The outline of this paper is as follows. In Section 2, we propose a Broyden-like method which is designed based on N-DFLS. In Section 3, we analyze its global and local superlinear convergence properties. In Section 4, we specialize our method to NCP, wLCP and SI. Numerical results are reported in Section 5. Some conclusions are given in Section 6.

2 Broyden-like method based on N-DFLS

In this section, we study Broyden-like method based on N-DFLS, denoted by N-BLM, which is described as follows.

Algorithm N-BLM

Step 0 Choose constants $\gamma, \rho, \delta, \sigma, \theta \in (0,1)$, $\tau \in (0,1]$ and $\eta > 0$. Choose a positive sequence $\{\eta_k\}$ satisfying $\sum_{k=0}^{\infty} \eta_k \leq \eta < \infty$. Choose $x^0 \in \mathbb{R}^n$ and let $\mathbf{\Phi}_0 := \|\mathbf{F}(x^0)\|$. Choose a nonsingular matrix $\mathbf{B}_0 \in \mathbb{R}^{n\times n}$. Set $k := 0$.
Step 1 If $\|\mathbf{F}(x^k)\| = 0$, then stop.
Step 2 Compute the search direction d^k by solving the linear equation

$$\mathbf{F}(x^k) + \mathbf{B}_k d^k = 0. \tag{2.1}$$

Step 3 If

$$\|\mathbf{F}(x^k + d^k)\| \leq \gamma\|\mathbf{F}(x^k)\| - \rho\|d^k\|^2, \tag{2.2}$$

then set $\lambda_k := 1$ and go to Step 5.
Step 4 Let l_k be the smallest nonnegative integer l satisfying

$$\|\mathbf{F}(x^k + \delta^l d^k)\| \leq (1+\eta_k)\mathbf{\Phi}_k - \sigma\|\delta^l d^k\|^2. \tag{2.3}$$

Set $\lambda_k := \delta^{l_k}$ and go to Step 5.
Step 5 Set $x^{k+1} := x^k + \lambda_k d^k$.
Step 6 Compute

$$\mathbf{T}_{k+1} := \frac{[(1+\eta_k)\mathbf{\Phi}_k + 1]\|\mathbf{F}(x^{k+1})\|}{\|\mathbf{F}(x^{k+1})\| + 1}. \tag{2.4}$$

Then update $\mathbf{\Phi}_k$ to get $\mathbf{\Phi}_{k+1}$ by

$$\mathbf{\Phi}_{k+1} := (1-\tau)\mathbf{T}_{k+1} + \tau\|\mathbf{F}(x^{k+1})\|. \tag{2.5}$$

Step 7 Update $\mathbf{B}_k$ to get $\mathbf{B}_{k+1}$ by Broyden-like formula

$$\mathbf{B}_{k+1} := \mathbf{B}_k + \theta_k \frac{(y^k - \mathbf{B}_k s^k)(s^k)^T}{\|s^k\|^2}, \tag{2.6}$$

where

$$s^k := \lambda_k d^k = x^{k+1} - x^k, \quad y^k := \mathbf{F}(x^{k+1}) - \mathbf{F}(x^k), \tag{2.7}$$

and the parameter θ_k is chosen so that $|\theta_k - 1| \leq \theta$ and $\mathbf{B}_{k+1}$ is nonsingular.
Step 8 Set $k := k+1$. Go to Step 1.

Theorem 1 *Algorithm* N-BLM *is well-defined and its generated sequence* $\{x^k\}$ *satisfies* $\|\mathbf{F}(x^k)\| \leq \mathbf{\Phi}_k$ *for all* $k \geq 0$.

Proof Suppose that $\|\mathbf{F}(x^k)\| \leq \mathbf{\Phi}_k$ holds for some k. Since $\mathbf{B}_k$ is nonsingular, Step 2 is well-defined. Moreover, similarly as LF-DFLS, the inequality (2.3) holds for all sufficiently large $l > 0$. Thus, we can always find a step size λ_k and get $x^{k+1} = x^k + \lambda_k d^k$ in Step 5. Now we show $\|\mathbf{F}(x^{k+1})\| \leq \mathbf{\Phi}_{k+1}$. In fact, if λ_k is generated by Step 3, then from (2.2) we have

$$\|\mathbf{F}(x^{k+1})\| \leq \gamma\|\mathbf{F}(x^k)\| \leq \gamma\mathbf{\Phi}_k \leq (1+\eta_k)\mathbf{\Phi}_k. \tag{2.8}$$

Otherwise, λ_k is generated by Step 4 and from (2.3) we also have

$$\|\mathbf{F}(x^{k+1})\| \leq (1+\eta_k)\mathbf{\Phi}_k. \tag{2.9}$$

So, by (2.4) we have

$$\begin{aligned}\mathbf{T}_{k+1} &= \frac{[(1+\eta_k)\mathbf{\Phi}_k + 1]\|\mathbf{F}(x^{k+1})\|}{\|\mathbf{F}(x^{k+1})\| + 1} \\ &\geq \frac{[\|\mathbf{F}(x^{k+1})\| + 1]\|\mathbf{F}(x^{k+1})\|}{\|\mathbf{F}(x^{k+1})\| + 1} \\ &= \|\mathbf{F}(x^{k+1})\|,\end{aligned}$$

which together with (2.5) gives

$$\begin{aligned}\mathbf{\Phi}_{k+1} &= (1-\tau)\mathbf{T}_{k+1} + \tau\|\mathbf{F}(x^{k+1})\| \\ &\geq (1-\tau)\|\mathbf{F}(x^{k+1})\| + \tau\|\mathbf{F}(x^{k+1})\| \\ &= \|\mathbf{F}(x^{k+1})\|.\end{aligned}$$

Since we choose $\mathbf{\Phi}_0 = \|\mathbf{F}(x^0)\|$, $\|\mathbf{F}(x^k)\| \leq \mathbf{\Phi}_k$ is true for $k = 0$. Thus, by mathematical induction, we prove the theorem. □

Remark 1 Suppose that λ_k^{LF} and λ_k^{N} are step sizes defined by LF-DFLS and N-DFLS respectively. Since $\|\mathbf{F}(x^k)\| \leq \mathbf{\Phi}_k$, λ_k^{LF} must satisfy the inequality (2.3) and hence $\lambda_k^{\mathrm{LF}} \leq \lambda_k^{\mathrm{N}}$. This shows that we can choose a larger accepted step size by N-DFLS. In other words, λ_k^{N} is easier to find than λ_k^{LF} in practical computation.

Lemma 1 *Let $\{x^k\}$ and $\{\mathbf{\Phi}_k\}$ be generated by Algorithm* N-BLM. *Then for all $k \geq 0$, one has*:

(i) $\mathbf{\Phi}_{k+1} \leq (1+\eta_k)\mathbf{\Phi}_k$;
(ii) $\mathbf{\Phi}_k \leq e^{\eta}\|\mathbf{F}(x^0)\|$;
(iii) $\mathbf{\Phi}_k \leq \xi\|\mathbf{F}(x^k)\|$, *where* $\xi := (1-\tau)(1+\eta)e^{\eta}\|\mathbf{F}(x^0)\| + 1$.

Proof By (2.8) and (2.9), we have $\|\mathbf{F}(x^{k+1})\| \leq (1+\eta_k)\mathbf{\Phi}_k$ for all $k \geq 0$. Then it follows from (2.4) that for all $k \geq 0$,

$$\begin{aligned}\mathbf{T}_{k+1} &= \frac{[(1+\eta_k)\mathbf{\Phi}_k + 1]\|\mathbf{F}(x^{k+1})\|}{\|\mathbf{F}(x^{k+1})\| + 1} \\ &= \frac{(1+\eta_k)\mathbf{\Phi}_k\|\mathbf{F}(x^{k+1})\| + \|\mathbf{F}(x^{k+1})\|}{\|\mathbf{F}(x^{k+1})\| + 1} \\ &\leq \frac{(1+\eta_k)\mathbf{\Phi}_k\|\mathbf{F}(x^{k+1})\| + (1+\eta_k)\mathbf{\Phi}_k}{\|\mathbf{F}(x^{k+1})\| + 1} \\ &= (1+\eta_k)\mathbf{\Phi}_k,\end{aligned}$$

which together with (2.5) gives

$$\begin{aligned}\mathbf{\Phi}_{k+1} &= (1-\tau)\mathbf{T}_{k+1} + \tau\|\mathbf{F}(x^{k+1})\| \\ &\leq (1-\tau)(1+\eta_k)\mathbf{\Phi}_k + \tau(1+\eta_k)\mathbf{\Phi}_k \\ &= (1+\eta_k)\mathbf{\Phi}_k.\end{aligned}$$

This proves the result (i). According to the result (i), by following the proof of Li and Fukushima (2000a)[Lemma 2.1], we can obtain the result (ii). Furthermore, by (2.4), we have for all $k \geq 0$,

$$\begin{aligned}\mathbf{T}_{k+1} &= \frac{[(1+\eta_k)\mathbf{\Phi}_k + 1]\|\mathbf{F}(x^{k+1})\|}{\|\mathbf{F}(x^{k+1})\| + 1} \\ &\leq [(1+\eta_k)\mathbf{\Phi}_k + 1]\|\mathbf{F}(x^{k+1})\| \\ &\leq [(1+\eta)e^{\eta}\|\mathbf{F}(x^0)\| + 1]\|\mathbf{F}(x^{k+1})\|,\end{aligned}$$

and so by (2.5) it holds that

$$\begin{aligned}\mathbf{\Phi}_{k+1} &= (1-\tau)\mathbf{T}_{k+1} + \tau\|\mathbf{F}(x^{k+1})\| \\ &\leq [(1-\tau)(1+\eta)e^{\eta}\|\mathbf{F}(x^0)\| + 1]\|\mathbf{F}(x^{k+1})\| \\ &= \xi\|\mathbf{F}(x^{k+1})\|.\end{aligned}$$

By noticing that $\mathbf{\Phi}_0 = \|\mathbf{F}(x^0)\| \leq \xi\|\mathbf{F}(x^0)\|$, we prove the result (iii). □

Lemma 2 *Let* $\{x^k\}$ *and* $\{\mathbf{\Phi}_k\}$ *be generated by Algorithm* N-BLM. *Then there exists a constant* $\mathbf{\Phi}^* \geq 0$ *such that*

$$\lim_{k\to\infty}\|\mathbf{F}(x^k)\| = \lim_{k\to\infty}\mathbf{\Phi}_k = \mathbf{\Phi}^*. \tag{2.10}$$

Proof By Lemma 1 (i) and Li and Fukushima (2000a)[Lemma 2.2], the sequence $\{\mathbf{\Phi}_k\}$ is convergent and hence there exists a constant $\mathbf{\Phi}^* \geq 0$ such that $\lim_{k\to\infty}\mathbf{\Phi}_k = \mathbf{\Phi}^*$. By (2.4) and (2.5), we have for all $k \geq 1$,

$$\tau\|\mathbf{F}(x^k)\|^2 + \vartheta_k\|\mathbf{F}(x^k)\| - \mathbf{\Phi}_k = 0, \tag{2.11}$$

in which

$$\vartheta_k := (1-\tau)(1+\eta_{k-1})\mathbf{\Phi}_{k-1} - \mathbf{\Phi}_k + 1.$$

Since $\lim_{k\to\infty}\eta_k = 0$, it follows that $\lim_{k\to\infty}\vartheta_k = 1 - \tau\mathbf{\Phi}^*$. Thus, by (2.11) we have

$$\lim_{k\to\infty}\|\mathbf{F}(x^k)\| = \lim_{k\to\infty}\frac{-\vartheta_k + \sqrt{\vartheta_k^2 + 4\tau\mathbf{\Phi}_k}}{2\tau} = \mathbf{\Phi}^*.$$

The proof is completed. □

Lemma 3 *Let* $\{x^k\}$ *be the sequence generated by Algorithm* N-BLM. *Then*

$$\sum_{k=0}^{\infty}\|\lambda_k d^k\|^2 < \infty. \tag{2.12}$$

Proof If $\tau = 1$, then Algorithm N-BLM becomes Broyden-like method studied by Li and Fukushima (2000a), and we have (2.12) by Li and Fukushima (2000a)[Lemma 2.3]. Now we consider $0 < \tau < 1$. For any $k \geq 0$, if λ_k is generated by Step 3, then $\lambda_k = 1$ and

$$\begin{aligned}\|\mathbf{F}(x^{k+1})\| &\leq \gamma\|\mathbf{F}(x^k)\| - \rho\|\lambda_k d^k\|^2 \\ &\leq (1+\eta_k)\mathbf{\Phi}_k - \rho\|\lambda_k d^k\|^2.\end{aligned} \tag{2.13}$$

Otherwise, λ_k is generated by Step 4 and it holds that

$$\|\mathbf{F}(x^{k+1})\| \leq (1+\eta_k)\mathbf{\Phi}_k - \sigma\|\lambda_k d^k\|^2. \tag{2.14}$$

Let $\chi := \min\{\rho, \sigma\}$. Then, from (2.13) and (2.14), we have for all $k \geq 0$,

$$\|\mathbf{F}(x^{k+1})\| \leq (1+\eta_k)\mathbf{\Phi}_k - \chi\|\lambda_k d^k\|^2. \tag{2.15}$$

By (2.4) and (2.15), we have for all $k \geq 0$,

$$\begin{aligned}
\mathbf{T}_{k+1} &= \frac{[(1+\eta_k)\mathbf{\Phi}_k + 1]\|\mathbf{F}(x^{k+1})\|}{\|\mathbf{F}(x^{k+1}) + 1\|} \\
&= \frac{(1+\eta_k)\mathbf{\Phi}_k\|\mathbf{F}(x^{k+1})\| + \|\mathbf{F}(x^{k+1})\|}{\|\mathbf{F}(x^{k+1})\| + 1} \\
&\leq \frac{(1+\eta_k)\mathbf{\Phi}_k\|\mathbf{F}(x^{k+1})\| + (1+\eta_k)\mathbf{\Phi}_k - \chi\|\lambda_k d^k\|^2}{\|\mathbf{F}(x^{k+1})\| + 1} \\
&= (1+\eta_k)\mathbf{\Phi}_k - \frac{\chi\|\lambda_k d^k\|^2}{\|\mathbf{F}(x^{k+1})\| + 1},
\end{aligned}$$

which together with (2.5) and the fact $\|\mathbf{F}(x^k)\| \leq \mathbf{\Phi}_k$ gives

$$\begin{aligned}
\mathbf{\Phi}_{k+1} &= (1-\tau)\mathbf{T}_{k+1} + \tau\|\mathbf{F}(x^{k+1})\| \\
&\leq (1-\tau)\left[(1+\eta_k)\mathbf{\Phi}_k - \frac{\chi\|\lambda_k d^k\|^2}{\|\mathbf{F}(x^{k+1})\| + 1}\right] + \tau\mathbf{\Phi}_{k+1}.
\end{aligned} \tag{2.16}$$

Since $\tau \in (0, 1)$, by (2.16) we have

$$\frac{\chi\|\lambda_k d^k\|^2}{\|\mathbf{F}(x^{k+1})\| + 1} \leq (1+\eta_k)\mathbf{\Phi}_k - \mathbf{\Phi}_{k+1}. \tag{2.17}$$

Since $\|\mathbf{F}(x^k)\| \leq \mathbf{\Phi}_k \leq e^{\eta}\|\mathbf{F}(x^0)\|$ for all $k \geq 0$, from (2.17) it follows that

$$\frac{\chi\|\lambda_k d^k\|^2}{e^{\eta}\|\mathbf{F}(x^0)\| + 1} \leq \mathbf{\Phi}_k - \mathbf{\Phi}_{k+1} + \eta_k\mathbf{\Phi}_k. \tag{2.18}$$

Since $\sum_{k=0}^{\infty} \eta_k \leq \eta$, we have (2.12) by summing both sides of (2.18). □

3 Global and local superlinear convergence

For a given $x^0 \in \mathbb{R}^n$, we define the level set of $\mathbf{F}(x)$ as

$$L(x^0) := \{x \in \mathbb{R}^n \,|\, \|\mathbf{F}(x)\| \leq e^{\eta}\|\mathbf{F}(x^0)\|\}, \tag{3.1}$$

where η is the constant given in Step 0. To establish the global and local superlinear convergence of Algorithm N-BLM, we make the following assumption introduced by Li and Fukushima (2000a).

Assumption 1 (i) The level set $L(x^0)$ is bounded.
(ii) $\mathbf{F}'(x)$ is Lipschitz continuous on $L(x^0)$.

Let $\{x^k\}$ be the sequence generated by Algorithm N-BLM. Let s^k and y^k be defined by (2.7). We define

$$\mathbf{A}_{k+1} := \int_0^1 \mathbf{F}'(x^k + ts^k)dt. \tag{3.2}$$

Then, $\mathbf{A}_{k+1}s^k = y^k$ and by the update formula (2.6) we have

$$\mathbf{B}_{k+1} = \mathbf{B}_k + \theta_k \frac{(\mathbf{A}_{k+1} - \mathbf{B}_k)s^k(s^k)^T}{\|s^k\|^2}. \tag{3.3}$$

Moreover, we define

$$\zeta_k := \frac{\|y^k - \mathbf{B}_k s^k\|}{\|s^k\|}. \tag{3.4}$$

Since $y^k = \mathbf{A}_{k+1}s^k$ and $s^k = x^{k+1} - x^k = \lambda_k d^k$, ζ_k can be rewritten as

$$\zeta_k = \frac{\|(\mathbf{A}_{k+1} - \mathbf{B}_k)s^k\|}{\|s^k\|} = \frac{\|(\mathbf{A}_{k+1} - \mathbf{B}_k)d^k\|}{\|d^k\|}. \tag{3.5}$$

3.1 Global convergence

Theorem 2 *If the condition* (i) *in Assumption 1 holds, then the sequence* $\{x^k\}$ *generated by Algorithm* N-BLM *is bounded.*

Proof The result holds by the fact

$$\|\mathbf{F}(x^k)\| \le \mathbf{\Phi}_k \le e^{\eta}\|\mathbf{F}(x^0)\| \tag{3.6}$$

for all $k \ge 0$. □

Under Assumption 1, we have the following lemma whose proof can be found in Li and Fukushima (2000a)[Lemma 2.6].

Lemma 4 *Assume that Assumption 1 holds and* $\{x^k\}$ *is the sequence generated by Algorithm* N-BLM. *Then the following results hold.*

(i) *If* $\sum_{k=0}^{\infty}\|s^k\|^2 < \infty$, *then we have* $\lim\limits_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\zeta_i^2 = 0$. *In particular, there is a subsequence* $\{\zeta_k\}_{k\in K}$ *of* $\{\zeta_k\}$ *such that* $\lim_{k(\in K)\to\infty}\zeta_k = 0$ *where* $K \subset \{0, 1, ...\}$.

(ii) *If* $\sum_{k=0}^{\infty}\|s^k\| < \infty$, *then we have* $\sum_{k=0}^{\infty}\zeta_k^2 < \infty$. *In particular,* $\lim\limits_{k\to\infty}\zeta_k = 0$.

Theorem 3 *Let Assumption 1 hold and* $\{x^k\}$ *be the sequence generated by Algorithm* N-BLM. *Assume that* $\mathbf{F}'(x^*)$ *is nonsingular for every accumulation point* x^* *of* $\{x^k\}$. *Then*

$$\lim_{k\to\infty}\|\mathbf{F}(x^k)\| = 0, \tag{3.7}$$

and every accumulation point x^* *of* $\{x^k\}$ *satisfies* $\mathbf{F}(x^*) = 0$.

Proof From Lemma 2, there exists a constant $\mathbf{\Phi}^* \ge 0$ such that

$$\lim_{k\to\infty}\|\mathbf{F}(x^k)\| = \lim_{k\to\infty}\mathbf{\Phi}_k = \mathbf{\Phi}^*. \tag{3.8}$$

Thus, if there are infinitely many k for which λ_k is determined by (2.2), then $\|\mathbf{F}(x^{k+1})\| \le \gamma\|\mathbf{F}(x^k)\|$ holds for infinitely many k. This implies that $\mathbf{\Phi}^* \le \gamma\mathbf{\Phi}^*$ which together with $\gamma \in (0, 1)$ yields $\mathbf{\Phi}^* = 0$, i.e., $\lim\limits_{k\to\infty}\|\mathbf{F}(x^k)\| = 0$.

Now we suppose that λ_k are determined by (2.3) for all sufficiently large k. By Lemma 3, we have

$$\sum_{k=0}^{\infty} \|s^k\|^2 = \sum_{k=0}^{\infty} \|\lambda_k d^k\|^2 < \infty. \tag{3.9}$$

This together with Lemma 4 (i) implies that there is a subsequence $\{\zeta_k\}_{k\in K}$ of $\{\zeta_k\}$ such that $\lim_{k(\in K)\to\infty} \zeta_k = 0$ where $K \subset \{0, 1, ...\}$. Since $\{x^k\}_{k\in K}$ is bounded, it has at least one accumulation point x^*. We can assume that $\lim_{k(\in K_1)\to\infty} x^k = x^*$ where $K_1 \subset K$. Then, by the continuity and (3.8), we have

$$\lim_{k(\in K_1)\to\infty} \|\mathbf{F}(x^k)\| = \|\mathbf{F}(x^*)\| = \mathbf{\Phi}^*, \tag{3.10}$$

$$\lim_{k(\in K_1)\to\infty} \mathbf{F}'(x^k) = \mathbf{F}'(x^*). \tag{3.11}$$

From (3.9), we have $\lim_{k\to\infty} s^k = 0$. Then, by (3.2) and (3.11), it holds that

$$\lim_{k(\in K_1)\to\infty} \mathbf{A}_{k+1} = \lim_{k(\in K_1)\to\infty} \int_0^1 \mathbf{F}'(x^k + ts^k)dt = \mathbf{F}'(x^*). \tag{3.12}$$

Since $\mathbf{F}'(x^*)$ is nonsingular, there exists a positive constant $M > 0$ such that $\|\mathbf{A}_{k+1}^{-1}\| \leq M$ for all sufficiently large $k \in K_1$. Thus, from (2.1) and (3.5), for all sufficiently large $k \in K_1$,

$$\begin{aligned}
\|d^k\| &= \|\mathbf{A}_{k+1}^{-1}[(\mathbf{A}_{k+1} - \mathbf{B}_k)d^k - \mathbf{F}(x^k)]\| \\
&\leq M(\|(\mathbf{A}_{k+1} - \mathbf{B}_k)d^k\| + \|\mathbf{F}(x^k)\|) \\
&= M(\zeta_k\|d^k\| + \|\mathbf{F}(x^k)\|),
\end{aligned}$$

which gives

$$\|d^k\| \leq \frac{M\|\mathbf{F}(x^k)\|}{1 - M\zeta_k} \leq 2M\|\mathbf{F}(x^k)\|, \tag{3.13}$$

where the second inequality holds because $\lim_{k(\in K_1)\to\infty} \zeta_k = 0$. This together with (3.6) implies that the sequence $\{d^k\}_{k\in K_1}$ is bounded and it has a convergent subsequence. We may assume that $\lim_{k(\in K_2)\to\infty} d^k = d^*$ where $K_2 \subset K_1$. By (3.5), we have

$$\|(\mathbf{A}_{k+1} - \mathbf{B}_k)d^k\| = \zeta_k\|d^k\|.$$

This together with (3.12) and $\lim_{k(\in K_2)\to\infty} \zeta_k = 0$ gives

$$\lim_{k(\in K_2)\to\infty} \mathbf{B}_k d^k = \lim_{k(\in K_2)\to\infty} \mathbf{A}_{k+1} d^k = \mathbf{F}'(x^*)d^*. \tag{3.14}$$

So, by letting $k \to \infty$ with $k \in K_2$ in (2.1), we have

$$\mathbf{F}(x^*) + \mathbf{F}'(x^*)d^* = 0. \tag{3.15}$$

If $\lambda_k \geq c > 0$ for all $k \in K_2$ where c is a fixed constant, then by (2.12) we have $\lim_{k(\in K_2)\to\infty} d^k = d^* = 0$. It follows from (3.15) that $\mathbf{F}(x^*) = 0$ and by (3.8) and (3.10) we have $\lim_{k\to\infty} \|\mathbf{F}(x^k)\| = 0$. Otherwise, $\{\lambda_k\}$ has a subsequence converging to zero and we assume that $\lim_{k(\in K_3)\to\infty} \lambda_k = 0$ where $K_3 \subset K_2$. Then $\delta^{-1}\lambda_k$ does not satisfy the line search criterion (2.3) for all sufficiently large $k \in K_3$, i.e.,

$$\begin{aligned}\|\mathbf{F}(x^k + \delta^{-1}\lambda_k d^k)\| &> (1+\eta_k)\boldsymbol{\Phi}_k - \sigma\|\delta^{-1}\lambda_k d^k\|^2 \\ &> \|\mathbf{F}(x^k)\| - \sigma\|\delta^{-1}\lambda_k d^k\|^2.\end{aligned}$$

It follows that for all sufficiently large $k \in K_3$,

$$\frac{\|\mathbf{F}(x^k + \delta^{-1}\lambda_k d^k)\| - \|\mathbf{F}(x^k)\|}{\delta^{-1}\lambda_k} > -\sigma\delta^{-1}\lambda_k\|d^k\|^2. \tag{3.16}$$

Since $\mathbf{F}(x)$ is continuously differentiable at x^*, multiplying both sides of (3.16) by $\|\mathbf{F}(x^k + \delta^{-1}\lambda_k d^k)\| + \|\mathbf{F}(x^k)\|$ and letting $k \to \infty$ with $k \in K_3$, we have

$$\mathbf{F}(x^*)^T\mathbf{F}'(x^*)d^* \geq 0. \tag{3.17}$$

On the other hand, from (3.15) it holds that

$$\mathbf{F}(x^*)^T\mathbf{F}'(x^*)d^* = -\|\mathbf{F}(x^*)\|^2. \tag{3.18}$$

By (3.17) and (3.18), we have $\|\mathbf{F}(x^*)\| = 0$ which together with (3.8) and (3.10) gives $\lim_{k\to\infty} \|\mathbf{F}(x^k)\| = 0$. Therefore, we prove (3.7). Consequently, by the continuity of $\mathbf{F}$, every accumulation point x^* of $\{x^k\}$ satisfies $\mathbf{F}(x^*) = 0$. We complete the proof. □

Theorem 4 *If the conditions in Theorem 3 hold, then the whole sequence $\{x^k\}$ converges to a solution of $\mathbf{F}(x) = 0$.*

Proof From Theorem 2 and Theorem 3, it holds that $\{x^k\}$ has at least one accumulation point x^* and $\mathbf{F}(x^*) = 0$. Since $\mathbf{F}$ is continuously differentiable at x^*, when $x \to x^*$, we have

$$\|\mathbf{F}(x) - \mathbf{F}(x^*) - \mathbf{F}'(x^*)(x - x^*)\| = o(\|x - x^*\|).$$

Hence there exists a constant $\epsilon > 0$ such that for all $x \in N(x^*, \epsilon) := \{x \in \mathbb{R}^n \mid \|x - x^*\| \leq \epsilon\}$,

$$\|\mathbf{F}(x) - \mathbf{F}(x^*) - \mathbf{F}'(x^*)(x - x^*)\| \leq \frac{1}{2\|\mathbf{F}'(x^*)^{-1}\|}\|x - x^*\|.$$

It follows that for all $x \in N(x^*, \epsilon)$,

$$\begin{aligned}\|x - x^*\| &\leq \|x - x^*\| - \|\mathbf{F}'(x^*)^{-1}\mathbf{F}(x)\| + \|\mathbf{F}'(x^*)^{-1}\|\|\mathbf{F}(x)\| \\ &\leq \|\mathbf{F}'(x^*)^{-1}\mathbf{F}(x) - (x - x^*)\| + \|\mathbf{F}'(x^*)^{-1}\|\|\mathbf{F}(x)\| \\ &\leq \|\mathbf{F}'(x^*)^{-1}\|\|\mathbf{F}(x) - \mathbf{F}(x^*) - \mathbf{F}'(x^*)(x - x^*)\| + \|\mathbf{F}'(x^*)^{-1}\|\|\mathbf{F}(x)\| \\ &\leq \frac{1}{2}\|x - x^*\| + \|\mathbf{F}'(x^*)^{-1}\|\|\mathbf{F}(x)\|,\end{aligned}$$

that is,

$$\|x - x^*\| \leq 2\|\mathbf{F}'(x^*)^{-1}\|\|\mathbf{F}(x)\|. \tag{3.19}$$

So, for any $\bar{x} \in N(x^*, \epsilon)$, if $\bar{x}$ is an accumulation point of $\{x^k\}$, then by Theorem 3 we have $\mathbf{F}(\bar{x}) = 0$ which together with (3.19) gives $\|\bar{x} - x^*\| = 0$, i.e., $\bar{x} = x^*$. This indicates that x^* is an isolate accumulation point of $\{x^k\}$. Moreover, by Lemma 3, $\lim_{k\to\infty} \|x^{k+1} - x^k\| = \lim_{k\to\infty} \|\lambda_k d^k\| = 0$. Hence, by Facchinei and Pang (2003)[Proposition 8.3.10], we have $\lim_{k\to\infty} x^k = x^*$ and the proof is completed. □

3.2 Local superlinear convergence

In this subsection, we give the local superlinear convergence of N-BLM. First, by using completely same arguments as Li and Fukushima (2000a)[Lemma 2.7], we can obtain the following lemma.

Lemma 5 *Let the conditions in Theorem 3 hold. Then there exist an index $\bar{k}$ and a constant $\zeta > 0$ such that*

$$\|x^k + d^k - x^*\| = O(\zeta_k \|x^k - x^*\|) + o(\|x^k - x^*\|), \tag{3.20}$$
$$\|\mathbf{F}(x^k + d^k)\| \le \gamma \|\mathbf{F}(x^k)\| - \rho \|d^k\|^2 \le \gamma \|\mathbf{F}(x^k)\|, \tag{3.21}$$

hold for all $k \ge \bar{k}$ such that $\zeta_k \le \zeta$.

Theorem 5 *Assume that the conditions in Theorem 3 hold. If we choose $\gamma = \frac{1}{\xi^2}$ in Step 0 of Algorithm* N-BLM *where ξ is the constant given in Lemma 1 (iii), then the whole sequence $\{x^k\}$ converges to a solution of $\mathbf{F}(x) = 0$ and its convergence rate is Q-superlinear.*

Proof Let ζ and $\bar{k}$ be as specified in Lemma 5. By Lemma 4 (i) and (3.9), there is an index $\tilde{k}$ such that when $k \ge \tilde{k}$

$$\frac{1}{k}\sum_{i=0}^{k-1} \zeta_i^2 \le \frac{1}{2}\zeta^2.$$

This shows that for any $k \ge \tilde{k}$, there are at least $\lceil \frac{k}{2} \rceil$ many indices $i \le k$ such that $\zeta_i \le \zeta$. Let $\hat{k} := \max\{\bar{k}, \tilde{k}\}$. Then, by Step 3 of Algorithm N-BLM and Lemma 5, for any $k \ge 2\hat{k}$, there are at least $\lceil \frac{k}{2} \rceil - \hat{k}$ many indices $i \le k$ such that $\lambda_i = 1$ and

$$\|\mathbf{F}(x^{i+1})\| = \|\mathbf{F}(x^i + d^i)\| < \gamma \|\mathbf{F}(x^i)\|. \tag{3.22}$$

Let I_k be the set of indices for which (3.22) holds and let i_k be the number of elements in I_k. Then $i_k \ge \frac{k}{2} - \hat{k} - 1$. On the other hand, it is clear from Algorithm N-BLM that for each $i \notin I_k$, we have

$$\|\mathbf{F}(x^{i+1})\| \le (1 + \eta_i)\mathbf{\Phi}_i \le (1 + \eta_i)\xi \|\mathbf{F}(x^i)\|, \tag{3.23}$$

where the second inequality holds by Lemma 1 (iii). For any $k \ge 4\hat{k} + 8$, multiplying inequalities (3.22) for $i \in I_k$ and (3.23) for $i \notin I_k$ from $i = \hat{k}$ to k yields

$$\begin{aligned}
\|\mathbf{F}(x^{k+1})\| &\le \gamma^{i_k} \xi^{k-\hat{k}+1-i_k} \|\mathbf{F}(x^{\hat{k}})\| \prod_{i=\hat{k} \notin I_k}^{k} (1 + \eta_i) \\
&\le \gamma^{\frac{k}{2}-\hat{k}-1} \xi^{\frac{k}{2}+2} \|\mathbf{F}(x^{\hat{k}})\| \prod_{i=\hat{k}}^{k} (1 + \eta_i) \\
&\le \left(\frac{1}{\sqrt{\xi}}\right)^{k-4\hat{k}-8} \|\mathbf{F}(x^{\hat{k}})\| e^{\eta},
\end{aligned} \tag{3.24}$$

where the second and third inequalities hold because $i_k \geq \frac{k}{2} - \hat{k} - 1$, $\gamma = \frac{1}{\xi^2} < 1$ and $\sum_{k=0}^{\infty} \eta_k \leq \eta$. Since $\frac{1}{\sqrt{\xi}} < 1$, we have

$$\sum_{k=4\hat{k}+8}^{\infty} \left(\frac{1}{\sqrt{\xi}} \right)^{k-4\hat{k}-8} \|\mathbf{F}(x^{\hat{k}})\| e^{\eta} < \infty.$$

This together with (3.24) gives

$$\sum_{k=0}^{\infty} \|\mathbf{F}(x^k)\| < \infty.$$

So, by (3.19) we have

$$\sum_{k=0}^{\infty} \|x^k - x^*\| < \infty,$$

which gives

$$\sum_{k=0}^{\infty} \|s^k\| = \sum_{k=0}^{\infty} \|x^{k+1} - x^k\| \leq \sum_{k=0}^{\infty} \|x^{k+1} - x^*\| + \sum_{k=0}^{\infty} \|x^k - x^*\| < \infty.$$

Then, by Lemma 4 (ii), we deduce $\zeta_k \to 0$. Hence, from (3.21) it holds that for all sufficiently large k,

$$\|\mathbf{F}(x^k + d^k)\| \leq \gamma \|\mathbf{F}(x^k)\| - \rho \|d^k\|^2.$$

So, by Step 3 in Algorithm N-BLM, for all sufficiently large k,

$$x^{k+1} = x^k + d^k. \tag{3.25}$$

This together with (3.20) and $\zeta_k \to 0$ gives

$$\|x^{k+1} - x^*\| = o(\|x^k - x^*\|).$$

We complete the proof. □

Remark 2 By Theorem 1 and Lemma 1 (iii), we can see that N-DFLS can ensure the sequence $\{x^k\}$ generated by Algorithm N-BLM satisfy

$$\|\mathbf{F}(x^k)\| \leq \mathbf{\Phi}_k \leq \xi \|\mathbf{F}(x^k)\|, \tag{3.26}$$

where $\xi > 1$ is a constant. As it is shown in the proof of Theorem 5, the right inequality in (3.26) plays an important rule in establishing the local superlinear convergence of Algorithm N-BLM. Thus, although many derivative-free nonmonotone line search rules (e.g, (Cheng & Li, 2009; Diniz-Ehrhardta et al., 2008; Grippo & Rinaldi, 2015; Tang & Zhou, 2020)) can ensure the left inequality in (3.26) hold, they may be not suitable to establish the local superlinear convergence of Broyden-like method.

4 Applications to NCP, wLCP and SI

In this section, we pay particular attention to nonlinear equations arising from nonlinear complementarity problems (NCP), weighted linear complementarity problems (wLCP) and system of inequalities (SI). We show that the conditions in Assumption 1 hold for these nonlinear equations.

4.1 Nonlinear complementarity problem

The nonlinear complementarity problem (NCP) is to find $(x, s) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$\text{(NCP)} \quad x, s \geq 0, \quad s = f(x), \quad x^T s = 0,$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function. The NCP has received a lot of attention due to its various applications in operations research, economic equilibrium and engineering design, see the survey paper (Ferris & Pang, 1997).

To reformulate the NCP as a smooth nonlinear equation, we here consider the nonnegative NCP-function introduced by Chen and Pan (2008) which is defined by

$$\psi_p(a, b) := \frac{1}{2}|\phi_p(a, b)|^2, \quad \forall (a, b) \in \mathbb{R}^2, \tag{4.1}$$

where

$$\phi_p(a, b) := \sqrt[p]{|a|^p + |b|^p} - (a + b), \quad \forall (a, b) \in \mathbb{R}^2,$$

in which p is a fixed real number in the interval $(1, \infty)$. The following lemma shows that ψ_p is an NCP-function and it is smooth everywhere, whose proof can be found in Chen and Pan (2008)[Proposition 3.2].

Lemma 6 *The function ψ_p is an* NCP*-function, i.e., it satisfies*

$$\psi_p(a, b) = 0 \Longleftrightarrow a \geq 0, \ b \geq 0, \ ab = 0.$$

Moreover, ψ_p is continuously differentiable everywhere.

Let $X := (x, s) \in \mathbb{R}^{2n}$. By using ψ_p, we can reformulate the NCP as the smooth nonlinear equation

$$\mathbf{H}(X) := \begin{pmatrix} s - f(x) \\ \psi_p(x_1, s_1) \\ \vdots \\ \psi_p(x_n, s_n) \end{pmatrix} = 0 \tag{4.2}$$

and then apply Algorithm N-BLM to solve it.

The following propositions show that the conditions in Assumption 1 hold for $\mathbf{H}(X)$ given in (4.2) when f has some special properties.

Proposition 1 *If f is either strongly monotone or uniform P-function, then the level sets*

$$L(\zeta) := \{X \in \mathbb{R}^{2n} | \|\mathbf{H}(X)\| \leq \zeta\} \tag{4.3}$$

are bounded for any $\zeta > 0$.

Proof The result holds by Chen and Pan (2008)[Proposition 3.5]. □

Proposition 2 $\mathbf{H}'(X)$ *is Lipschitz continuous on* $\mathbb{R}^{2n}$ *if* $f'(x)$ *is Lipschitz continuous on* $\mathbb{R}^n$.

Proof By Chen (2006)[Lemma 3.1], the gradient $\nabla\psi_p(a,b)$ is Lipschitz continuous on $\mathbb{R}^2$ and so are $\nabla_a\psi_p(a,b)$ and $\nabla_b\psi_p(a,b)$. This together with the Lipschitz continuity of $f'(x)$ yields the desired result. □

4.2 Weighted linear complementarity problem

The weighted linear complementarity problem (wLCP) was introduced by Potra (2012) which is to find vectors $x \in \mathbb{R}^n$, $s \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ such that

$$\text{(wLCP)} \quad x, s \geq 0, \quad Px + Qs + Ry = d, \quad xs = w.$$

Here $P \in \mathbb{R}^{(n+m)\times n}$, $Q \in \mathbb{R}^{(n+m)\times n}$, $R \in \mathbb{R}^{(n+m)\times m}$ are given matrices, $d \in \mathbb{R}^{n+m}$ is a given vector, $w \geq 0$ is a given weight vector (the data of the problem) and xs is the componentwise product of the vectors x and s. The wLCP can be used for modeling a larger class of problems from science and engineering (Potra, 2012) and it has been extensively studied in recent years (see, (Asadi et al., 2020; Chi et al., 2019; Gowda, 2019; Tang & Zhang, 2021; Tang & Zhou, 2021a, b)).

For any fixed constant $c \geq 0$, we consider the following nonnegative function

$$\psi^c(a,b) := \frac{1}{2}\left[\sqrt{a^2+b^2+2c} - (a+b)\right]^2, \ \forall (a,b) \in \mathbb{R}^2. \tag{4.4}$$

Obviously, if $c = 0$, then ψ^c reduces to the function ψ_p in (4.1) with $p = 2$. The following lemma shows that ψ^c is a weighted complementarity function for wLCP and it is smooth everywhere.

Lemma 7 *The function* ψ^c *defined by (4.4) satisfies*

$$\psi^c(a,b) = 0 \Longleftrightarrow a \geq 0, \ b \geq 0, \ ab = c.$$

Moreover, ψ^c *is continuously differentiable at any* $(a,b) \in \mathbb{R}^2$ *and*

$$\nabla\psi^c(a,b) = \begin{bmatrix} \nabla_a\psi^c(a,b) \\ \nabla_b\psi^c(a,b) \end{bmatrix},$$

where $\nabla_a\psi^c(0,0) = \nabla_b\psi^c(0,0) = -\sqrt{2c}$ *and for any* $(a,b) \neq (0,0)$,

$$\nabla_a\psi^c(a,b) = \left(\frac{a}{\sqrt{a^2+b^2+2c}} - 1\right)\left[\sqrt{a^2+b^2+2c} - (a+b)\right],$$
$$\nabla_b\psi^c(a,b) = \left(\frac{b}{\sqrt{a^2+b^2+2c}} - 1\right)\left[\sqrt{a^2+b^2+2c} - (a+b)\right].$$

Proof The first result is easy to verify. From direct computation, we can obtain the second result. □

Let $X := (x, s, y) \in \mathbb{R}^{2n+m}$. By using ψ^c, we can reformulate the wLCP as the smooth nonlinear equation

$$\mathbf{G}(X) := \begin{pmatrix} Px + Qs + Ry - d \\ \psi^{w_1}(x_1, s_1) \\ \vdots \\ \psi^{w_n}(x_n, s_n) \end{pmatrix} = 0 \tag{4.5}$$

and then apply Algorithm N-BLM to solve it.

The following proposition shows that Assumption 1 (ii) holds for $\mathbf{G}(X)$ given in (4.5).

Proposition 3 *$\mathbf{G}'(X)$ is Lipschitz continuous on $\mathbb{R}^{2n+m}$.*

Proof Obviously, we only need to prove that the gradient $\nabla\psi^c(a, b)$ is Lipschitz continuous on $\mathbb{R}^2$. In fact, if $c = 0$, then $\nabla\psi^c$ is Lipschitz continuous on $\mathbb{R}^2$ by Chen (2006)[Lemma 3.1]. If $c > 0$, then ψ^c is twice continuously differentiable at any $(a, b) \in \mathbb{R}^2$. By using the same arguments as Chen (2006)[Lemma 3.1], we can also prove that $\nabla\psi^c$ is Lipschitz continuous on $\mathbb{R}^2$. □

4.3 System of inequalities

We consider the following system of inequalities:

$$\text{(SI)} \quad f(x) \leq 0,$$

where $f(x) := (f_1(x), ..., f_n(x))^T$ with $f_i : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. The SI has been extensively studied in recent years due to its wide range of applications in many fields such as data analysis, set separation problems, computer aided design problems and image reconstructions, see (Chen et al., 2016; Fan & Yan, 2019) and references therein.

To reformulate the SI as a smooth nonlinear equation, we introduce the following absolute value function:

$$\varphi(\alpha) := \alpha^3 + |\alpha|^3, \quad \forall\alpha \in \mathbb{R}. \tag{4.6}$$

Lemma 8 (a) *$\varphi(\alpha) = 0$ if and only if $\alpha \leq 0$.*
(b) *φ is continuously differentiable at any $\alpha \in \mathbb{R}$ with*

$$\varphi'(\alpha) = 3\alpha^2 + 3\alpha|\alpha|. \tag{4.7}$$

Proof The result (a) obviously holds. It is easy to see that φ is continuously differentiable at any $\alpha \neq 0$ with $\varphi'(\alpha)$ being given by (4.7). Moreover, we have

$$\varphi'(0) = \lim_{h\to 0} \frac{\varphi(0+h) - \varphi(0)}{h} = \lim_{h\to 0} \frac{h^3 + |h|^3}{h} = \lim_{h\to 0} [(h + |h|)(2h - |h|)] = 0.$$

We complete the proof. □

By using φ, we can reformulate the SI as the smooth nonlinear equation

$$\mathbf{E}(x) := \begin{pmatrix} \varphi(f_1(x)) \\ \vdots \\ \varphi(f_n(x)) \end{pmatrix} = 0 \tag{4.8}$$

and apply Algorithm N-BLM to solve it.

In the following, we give a condition which can ensure that Assumption 1 holds for $\mathbf{E}(x)$ given in (4.8).

Condition A For any sequence $\{x^k\}$ such that $\|x^k\| \to \infty$, one has

$$\max_{1\leq i\leq n} f_i(x^k) \to \infty.$$

Proposition 4 *Assume that $f(x)$ satisfies Condition* A. *Then the following results hold.*

Table 1 Numerical results of **N-BLM** and **LF-BLM** (I)

$\mathbf{F}(x)$	n	x^0	**N-BLM**			**LF-BLM**		
			N_{iter}	N_F	$\lVert\mathbf{F}(x^k)\rVert$	N_{iter}	N_F	$\lVert\mathbf{F}(x^k)\rVert$
F2	100	$-\mathbf{e}$	184	1472	9.8356e-11	208	1740	7.6136e-11
		$\mathbf{e}$	196	1605	8.1707e-11	181	1525	8.2528e-11
	500	$-\mathbf{e}$	625	5749	9.7525e-11	2000	30110	2.8272e-10
		$\mathbf{e}$	766	7280	6.9269e-11	2000	32058	1.9467e-10
F4	300	$3\mathbf{e}$	67	585	9.9279e-11	76	674	6.2811e-11
		$4\mathbf{e}$	83	670	9.5613e-11	80	657	7.0274e-11
	900	$3\mathbf{e}$	41	335	9.1363e-11	53	436	9.9044e-11
		$4\mathbf{e}$	52	380	9.0522e-11	2000	13430	NaN
F5	500	$-\mathbf{e}$	31	244	7.7831e-11	46	375	4.0760e-11
		$2\mathbf{e}$	40	322	7.3565e-11	2000	13327	NaN
	1000	$-\mathbf{e}$	33	268	5.0801e-11	42	351	3.5309e-11
		$2\mathbf{e}$	63	513	9.1119e-11	2000	13571	NaN
F6	500	$3\mathbf{e}$	16	118	1.2253e-11	36	281	5.1959e-11
		$5\mathbf{e}$	63	532	2.7558e-11	809	7882	6.4859e-12
	1000	$3\mathbf{e}$	37	290	9.6484e-11	77	655	4.4830e-11
		$5\mathbf{e}$	80	683	1.5861e-11	1509	15738	2.8180e-11
F8	200	$\mathbf{e}$	355	2982	8.7465e-11	1121	10345	9.7371e-11
		$2\mathbf{e}$	278	2088	3.8337e-11	265	2083	4.2551e-11
	500	$\mathbf{e}$	425	3297	8.3251e-11	2000	29701	298.3984
		$2\mathbf{e}$	515	4120	9.1489e-11	2000	12765	NaN
F10	500	$\mathbf{e}$	11	81	4.4596e-14	25	194	1.4287e-18
		$3\mathbf{e}$	32	250	1.8196e-18	131	1218	1.4630e-18
	1000	$\mathbf{e}$	23	178	9.8672e-19	28	218	3.8093e-11
		$3\mathbf{e}$	82	710	1.0155e-18	187	1776	3.6356e-11
F11	500	$-\mathbf{e}$	90	631	7.5325e-11	391	3128	9.4078e-11
		$-2\mathbf{e}$	92	645	8.4313e-11	392	3134	9.5392e-11
	1000	$-\mathbf{e}$	91	638	9.1462e-11	396	3168	9.7178e-11
		$-2\mathbf{e}$	93	652	8.9162e-11	391	3124	9.9599e-11
F12	500	$-\mathbf{e}$	119	786	9.4842e-11	87	601	9.3226e-11
		$2\mathbf{e}$	107	711	9.0832e-11	89	606	7.0647e-11
	1000	$-\mathbf{e}$	120	793	5.1575e-11	93	641	9.5601e-11
		$2\mathbf{e}$	147	1002	9.4013e-11	89	611	9.7089e-11
F15	500	$-\mathbf{e}$	271	1634	9.8361e-11	97	693	8.9357e-11
		$\mathbf{0}$	102	671	9.9940e-11	377	3030	9.6683e-11
	1000	$-\mathbf{e}$	1257	11211	7.4042e-11	1255	11235	9.1914e-11
		$\mathbf{0}$	105	692	6.2092e-11	1261	11301	8.8397e-11
F16	500	$2\mathbf{e}$	10	73	1.2900e-15	21	152	1.5263e-15
		$4\mathbf{e}$	12	84	4.7588e-13	24	172	2.6645e-15
	1000	$2\mathbf{e}$	10	71	1.7342e-15	22	159	1.6910e-15
		$4\mathbf{e}$	12	84	6.7284e-13	25	178	6.5305e-11

Table 1 continued

$\mathbf{F}(x)$	n	x^0	**N-BLM**			**LF-BLM**		
			N_{iter}	N_F	$\lVert\mathbf{F}(x^k)\rVert$	N_{iter}	N_F	$\lVert\mathbf{F}(x^k)\rVert$
F17	500	$-2\mathbf{e}$	640	6108	9.2865e-11	624	5932	6.1741e-11
		$2\mathbf{e}$	633	5915	5.6205e-11	633	5961	2.0954e-11
	1000	$-2\mathbf{e}$	1199	11632	4.3381e-11	1246	13698	8.1446e-11
		$2\mathbf{e}$	1175	10956	9.4648e-11	1236	11906	3.3829e-11

(i) *The level sets*

$$L(C) := \{x \in \mathbb{R}^n \,|\, \|\mathbf{E}(x)\| \le C\} \tag{4.9}$$

are bounded for any $C \ge 0$.

(ii) *For any* $C \ge 0$, $\mathbf{E}'(x)$ *is Lipschitz continuous on* $L(C)$ *if* $f(x)$ *and* $f'(x)$ *are Lipschitz continuous on* $L(C)$.

Proof For any sequence $\{x^k\}$, by (4.6) and (4.8), we have

$$\|\mathbf{E}(x^k)\|^2 = \sum_{i=1}^{n}[f_i(x^k)^3 + |f_i(x)|^3]^2 \ge \max_{1\le i\le n}[f_i(x^k)^3 + |f_i(x^k)|^3]^2.$$

Thus, by Condition A, if $\|x^k\| \to \infty$ as $k \to \infty$, then $\|\mathbf{E}(x^k)\| \to \infty$ as $k \to \infty$. This proves the result (i). By (4.7) and (4.8), we have

$$\mathbf{E}'(x) = \mathbf{diag}\big(a_i(x)\big) f'(x), \tag{4.10}$$

with

$$a_i(x) := 3f_i(x)^2 + 3f_i(x)|f_i(x)|, \; i = 1, ..., n. \tag{4.11}$$

By the result (i), the set $L(C)$ is closed and bounded. So, by the continuities of $f(x)$ and $f'(x)$, there exist constants $c_1, c_2 > 0$ such that for all $i = 1, ..., n$

$$|f_i(x)| \le \|f(x)\| \le c_1, \;\; \|f'(x)\| \le c_2, \;\; \forall\, x \in L(C). \tag{4.12}$$

Since $f(x)$ and $f'(x)$ are Lipschitz continuous on $L(C)$, there exist constants $l_1, l_2 > 0$ such that

$$\|f(x) - f(\tilde{x})\| \le l_1\|x - \tilde{x}\|, \;\; \|f'(x) - f'(\tilde{x})\| \le l_2\|x - \tilde{x}\|, \;\; \forall x, \tilde{x} \in L(C). \tag{4.13}$$

Then, for any $x, \tilde{x} \in L(C)$, by (4.11), (4.12) and (4.13), we have

$$\begin{aligned} |a_i(x) - a_i(\tilde{x})| &= 3|f_i(x)^2 - f_i(\tilde{x})^2 + f_i(x)|f_i(x)| - f_i(\tilde{x})|f_i(\tilde{x})|| \\ &\le 3|f_i(x) + f_i(\tilde{x})||f_i(x) - f_i(\tilde{x})| \\ &\quad +3||f_i(x)|(f_i(x) - f_i(\tilde{x}))| + 3|f_i(\tilde{x})(|f_i(x)| - |f_i(\tilde{x})|)| \\ &\le 12c_1|f_i(x) - f_i(\tilde{x})| \\ &\le 12c_1\|f(x) - f(\tilde{x})\| \\ &\le 12c_1 l_1\|x - \tilde{x}\|, \end{aligned}$$

Table 2 Numerical results of **N-BLM** and **LF-BLM** (II)

$\mathbf{F}(x)$	n	x^0	**N-BLM**			**LF-BLM**		
			N_{iter}	N_F	$\lVert\mathbf{F}(x^k)\rVert$	N_{iter}	N_F	$\lVert\mathbf{F}(x^k)\rVert$
F18	600	$-\mathbf{e}$	47	372	4.7391e-11	2000	12876	NaN
		$\mathbf{e}$	6	46	4.9791e-12	6	46	4.9791e-12
	1200	$-\mathbf{e}$	54	428	9.1734e-11	2000	12545	NaN
		$\mathbf{e}$	6	46	7.0495e-12	7	53	2.9071e-12
F21	300	$-\mathbf{e}$	39	320	7.5375e-11	195	1740	8.6086e-11
		$\mathbf{e}$	6	46	3.4916e-12	6	46	3.4916e-12
	900	$-\mathbf{e}$	59	480	2.8788e-11	2000	12513	NaN
		$\mathbf{e}$	6	46	6.0859e-12	7	53	1.5728e-12
F22	500	$2\mathbf{e}$	9	63	2.5340e-12	27	209	1.4222e-13
		$4\mathbf{e}$	34	266	7.8083e-13	71	601	1.8927e-12
	1000	$2\mathbf{e}$	9	64	4.3151e-12	30	234	1.3671e-12
		$4\mathbf{e}$	40	314	7.0607e-12	99	855	1.6741e-12
F23	100	$\mathbf{0}$	3	32	7.1233e-13	3	32	7.1233e-13
		$0.5\mathbf{e}$	4	40	2.2733e-11	4	40	2.2733e-11
	200	$\mathbf{0}$	3	33	9.3042e-13	3	33	9.3042e-13
		$0.5\mathbf{e}$	7	61	9.3173e-11	10	154	4.6586e-11
F25	500	$-\mathbf{e}$	30	220	3.9310e-11	23	169	9.9232e-11
		$\mathbf{0}$	20	148	3.2436e-11	21	151	1.8158e-11
	1000	$-\mathbf{e}$	22	158	9.2515e-11	22	158	9.2515e-11
		$\mathbf{0}$	22	158	5.4529e-11	22	158	5.4529e-11
F26	500	$2\mathbf{e}$	17	187	0	18	207	0
		$4\mathbf{e}$	19	218	0	20	239	0
	1000	$2\mathbf{e}$	19	221	0	20	241	0
		$4\mathbf{e}$	21	253	0	23	283	0
F30	600	$3\mathbf{e}$	16	125	1.0046e-13	17	127	9.5154e-11
		$5\mathbf{e}$	17	131	8.8005e-11	26	205	2.5350e-11
	1200	$3\mathbf{e}$	16	131	4.0068e-11	21	165	1.5529e-11
		$5\mathbf{e}$	19	149	4.2073e-11	33	261	2.2590e-11
F31	500	$\mathbf{e}$	94	798	8.7060e-11	434	4292	7.9016e-11
		$2\mathbf{e}$	494	4555	9.6150e-11	2000	12030	NaN
	1000	$\mathbf{e}$	108	877	9.9539e-11	2000	12429	NaN
		$2\mathbf{e}$	886	8401	9.9103e-11	2000	12030	NaN

which gives

$$\begin{aligned}\|\mathbf{diag}\big(a_i(x)\big) - \mathbf{diag}\big(a_i(\tilde{x})\big)\| &\le \|\mathbf{diag}\big(a_i(x)\big) - \mathbf{diag}\big(a_i(\tilde{x})\big)\|_F \\ &= \sqrt{\sum_{i=1}^{n}(a_i(x) - a_i(\tilde{x}))^2} \\ &\le 12\sqrt{n}c_1 l_1\|x - \tilde{x}\|, \end{aligned} \tag{4.14}$$

Table 2 continued

$\mathbf{F}(x)$	n	x^0	N-BLM			LF-BLM		
			N_{iter}	N_F	$\|\mathbf{F}(x^k)\|$	N_{iter}	N_F	$\|\mathbf{F}(x^k)\|$
F32	500	2**e**	11	81	4.4686e-14	25	194	6.2172e-15
		4**e**	32	250	3.7682e-15	128	1185	7.9534e-15
	1000	2**e**	23	178	7.0217e-15	28	218	3.7931e-11
		4**e**	82	710	9.8203e-15	187	1776	3.6467e-11
F33	500	−**e**	16	113	6.5992e-11	9	74	2.3599e-11
		0	14	104	2.5065e-11	8	70	7.5819e-11
	1000	−**e**	19	136	9.0725e-11	2000	12034	NaN
		0	11	83	4.0581e-11	12	97	3.8789e-11
F34	100	−2**e**	623	6404	4.5255e-11	2000	32481	NaN
		−**e**	156	1290	4.4109e-11	150	1268	2.6022e-11
	200	−2**e**	609	6745	5.8466e-11	883	9786	8.0664e-11
		−**e**	306	2618	8.7010e-11	277	2370	7.1330e-11

Table 3 Numerical results of **N-BLM** and **LF-BLM** (III)

$\mathbf{F}(x)$	n	x^0	N-BLM			LF-BLM		
			N_{iter}	N_F	$\|\mathbf{F}(x^k)\|$	N_{iter}	N_F	$\|\mathbf{F}(x^k)\|$
F35	100	2**e**	155	1193	1.4286e-11	295	2967	9.1031e-11
		3**e**	236	1686	3.7581e-11	173	1178	6.5416e-11
	200	2**e**	300	2708	8.6177e-11	415	4246	8.8226e-11
		3**e**	365	2949	9.3263e-11	304	3109	6.4944e-11
F36	200	**e**	275	2351	5.2432e-11	280	2479	7.5684e-11
		2**e**	260	2009	7.9668e-11	454	5272	7.2841e-11
	500	**e**	583	5113	9.4973e-11	608	5401	6.8684e-11
		2**e**	709	6785	8.6035e-11	1054	14744	8.4366e-11
F37	500	2**e**	86	675	9.3383e-11	377	3671	4.7549e-11
		3**e**	23	172	4.4499e-12	25	194	3.6964e-11
	1000	2**e**	74	560	3.0650e-11	409	3959	3.8900e-11
		3**e**	23	170	1.8901e-11	27	210	6.5706e-11
F38	500	**e**	83	577	8.1062e-11	88	614	9.5127e-11
	1000	**e**	84	584	8.9338e-11	96	672	8.0689e-11
F39	500	**0**	144	1226	9.6886e-11	53	457	8.2553e-11
		0.5**e**	1577	16712	4.6402e-11	2000	24112	1.7195
	1000	**0**	122	1032	8.2415e-11	49	417	9.8215e-11
		0.5**e**	1737	19097	7.1623e-11	2000	26365	134.7387
F40	500	3**e**	4	30	4.0982e-14	6	43	6.3137e-11
		5**e**	13	95	3.6728e-14	36	282	5.2976e-11
	1000	3**e**	4	29	2.5882e-11	8	57	1.1233e-12
		5**e**	34	266	5.9684e-12	68	573	1.6294e-12

Table 3 continued

$\mathbf{F}(x)$	n	x^0	N-BLM			LF-BLM		
			N_{iter}	N_F	$\Vert\mathbf{F}(x^k)\Vert$	N_{iter}	N_F	$\Vert\mathbf{F}(x^k)\Vert$
F41	500	**3e**	88	539	8.0807e-11	109	720	8.5460e-11
		5e	90	553	7.9732e-11	133	932	8.1206e-11
	1000	**3e**	89	547	9.5654e-11	117	786	7.9562e-11
		5e	115	761	8.5851e-11	161	1187	8.5516e-11
F42	50	−2**e**	200	1698	8.0452e-11	203	1662	6.7527e-11
		0	203	1993	8.5479e-11	203	1993	8.5479e-11
	100	−2**e**	866	8163	9.7927e-11	891	8718	4.8258e-11
		0	194	1941	8.5250e-11	194	1941	8.5250e-11
F43	100	0.5**e**	304	2716	8.9707e-11	351	3097	9.6423e-11
		e	395	3605	9.8238e-11	406	3881	8.5077e-11
	200	0.5**e**	879	8204	9.7611e-11	1035	10005	9.3924e-11
		e	1239	11978	3.4833e-11	1247	12139	9.8196e-11
F44	50	−**e**	48	342	5.7018e-11	64	492	1.3596e-12
		e	73	560	9.8802e-11	69	534	7.8531e-11
	100	−**e**	155	1302	5.6106e-11	283	2268	9.9051e-11
		e	283	2268	9.9051e-11	387	3400	9.6762e-11

where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. Moreover, by (4.11) and (4.12), we have for any $x \in L(C)$

$$\|\mathbf{diag}\big(a_i(x)\big)\| \leq \|\mathbf{diag}\big(a_i(x)\big)\|_F = \sqrt{\sum_{i=1}^{n}(a_i(x))^2} \leq \sqrt{n}6c_1^2. \tag{4.15}$$

Therefore, by (4.10) and (4.12)–(4.15), for any $x, \tilde{x} \in L(C)$,

$$\begin{aligned}\|\mathbf{E}'(x) - \mathbf{E}'(\tilde{x})\| &= \|\mathbf{diag}\big(a_i(x)\big)f'(x) - \mathbf{diag}\big(a_i(\tilde{x})\big)f'(\tilde{x})\| \\ &\leq \|\mathbf{diag}\big(a_i(x)\big)\|\|f'(x) - f'(\tilde{x})\| \\ &\quad + \|f'(\tilde{x})\|\|\mathbf{diag}\big(a_i(x)\big) - \mathbf{diag}\big(a_i(\tilde{x})\big)\| \\ &\leq (\sqrt{n}6c_1^2 l_2 + 12\sqrt{n}c_1 l_1 c_2)\|x - \tilde{x}\|.\end{aligned}$$

The proof is completed. □

Remark 3 Note that the gradients of the functions ψ_p (4.1), ψ^c (4.4) and φ (4.6) are all zeros at the solution. However there does exist smooth functions possessing the property that the gradient is nonzero at the solution. For example, for the wLCP, we may choose the smooth complementarity function introduced in Tang and Zhou (2021a) whose gradient can take nonzeros at the solution. Hence the Jacobian of the function in nonlinear equations arising from NCP, wLCP and SI is greatly depending on the special smooth functions adopted to carry equivalent reformulation. In addition, semismooth equation reformulation is an alternative approach to solve the nonlinear equations. How to design and analyze Broyden-Like methods in this scenario deserves further research.

5 Numerical results

In this section, we give some numerical results of Algorithm N-BLM for solving the nonlinear equation (1.1). All experiments are performed on a PC with CPU of Inter(R) Core(TM)i7-7700 CPU @ 3.60 GHz and RAM of 8.00GB. The codes are written in MATLAB and run in MATLAB R2018a environment. In the experiments, we test the following two Broyden-like methods:

(i) Algorithm N-BLM with $\tau = 0.3$, denoted by **N-BLM**.
(ii) Algorithm N-BLM with $\tau = 1$, i.e., Broyden-like method studied by Li and Fukushima (2000a), denoted by **LF-BLM**.

First, we choose the test functions $\mathbf{F}(x)$ from La Cruz et al. (2006) which are described in La Cruz et al. (2004)[Appendix A: Test functions]. The parameters are chosen as $\delta = 0.25,\ \rho = 0.5,\ \gamma = 0.5,\ \sigma = 0.5,\ \theta_k = 1,\ \eta_k = \frac{1}{(k+1)^2},\ \mathbf{B}_0 = \mathbf{eye}(n)$. We take $x^0 = \alpha\mathbf{e}$ where $\alpha \in \mathbb{R}$ and $\mathbf{e} := (1, ..., 1)^T$, and let $-\mathbf{e} := (-1, ..., -1)^T$ and $\mathbf{0} := (0, ..., 0)^T$. Moreover, we use $\|\mathbf{F}(x^k)\| \leq 10^{-10}$ or $k \geq 2000$ as the stopping criterion. Numerical results are listed in Tables 1–3 in which Fn denotes the nth function listed in La Cruz et al. (2004)[Appendix A: Test functions], x^0 denotes the starting point, N_{iter} denotes the number of iterations, N_F denotes the total number of function evaluations, $\|\mathbf{F}(x^k)\|$ denotes the value of $\|\mathbf{F}(x)\|$ when the algorithm terminates.

In the experiments, we choose 32 functions from La Cruz et al. (2004)[Appendix A: Test functions] and every function is tested with two kinds of size. In all, we test **N-BLM** and **LF-BLM** by solving 64 problems. From numerical results in Tables 1–3, two observations can be made here.

- **N-BLM** can solve all tested problems for all starting points. While, **LF-BLM** fails to solve 17 tested problems for some starting points.
- As tested problems are solved, in most situations, **N-BLM** needs less number of iterations and less number of function evaluations than **FL-BLM**.

These observations indicate that our new derivative-free line search technique contributes a lot to the numerical performance and it is great helpful for the real application of Broyden-like method.

Next, we test **N-BLM** and **LF-BLM** by solving the discretized two-point boundary value problem

$$\mathbf{F}(x) := Ax + \frac{1}{(n+1)^2} f(x) = 0, \tag{5.1}$$

where A is the $n \times n$ tridiagonal matrix given by

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix},$$

and $f(x) = (f_1(x), f_2(x), ..., f_n(x))^T$ with

$$f_i(x) = \arctan x_i - 1, \quad i = 1, 2, ..., n.$$

This problem has been tested by Li and Fukushima (2000a). In the experiments, we test **N-BLM** and **LF-BLM** by taking the parameters used in Li and Fukushima (2000a), i.e.,

Table 4 Numerical results of **N-BLM** and **LF-BLM** for solving (5.1)

x^0	n	N-BLM			LF-BLM		
		N_{iter}	N_F	$\|\mathbf{F}(x^k)\|$	N_{iter}	N_F	$\|\mathbf{F}(x^k)\|$
(1)	9	15	119	8.8919e-11	15	119	8.8919e-11
	29	86	642	7.3726e-11	85	623	1.6684e-11
	49	252	1873	9.2364e-11	171	1299	2.7101e-11
	69	414	3031	9.5796e-11	463	3405	9.2666e-11
	99	922	6630	8.7149e-11	1636	12337	7.3884e-11
(2)	9	15	117	7.2192e-11	15	117	7.2192e-11
	29	96	713	1.0897e-11	201	1470	6.1222e-11
	49	372	2706	8.9013e-11	410	2984	5.1249e-11
	69	561	4087	7.1750e-11	598	4335	9.0073e-11
	99	1265	8896	2.3071e-11	1159	8479	9.5945e-11
(3)	9	19	148	2.2713e-11	19	148	2.2713e-11
	29	198	1411	8.6793e-11	298	2115	7.3716e-11
	49	480	3386	9.0650e-11	552	3907	5.2319e-11
	69	733	5162	6.9744e-12	791	5608	8.0763e-11
	99	1081	7619	9.8391e-11	2000	15565	0.5598
(4)	9	23	175	6.1395e-11	28	213	5.2610e-11
	29	94	679	4.2654e-11	191	1371	4.3055e-11
	49	524	3714	9.8148e-11	508	3586	9.3517e-11
	69	878	6229	9.8863e-11	1033	7344	7.8315e-11
	99	1441	10302	6.9921e-12	1454	10404	6.7961e-11
(5)	9	23	175	6.1394e-11	28	213	5.2610e-11
	29	97	702	5.2435e-11	187	1337	8.3177e-12
	49	466	3286	9.9858e-11	508	3593	2.5160e-11
	69	906	6453	7.6762e-11	999	7144	8.2249e-11
	99	1600	11574	8.2356e-11	1563	11261	9.7190e-11

$\gamma = 0.9,\ \rho = \sigma = 0.001,\ \delta = 0.01,\ \theta_k = 1,\ \eta_k = \frac{1}{(k+1)^2},\ \mathbf{B}_0 = \mathbf{eye}(n)$. We also use $\|\mathbf{F}(x^k)\| \leq 10^{-10}$ or $k \geq 2000$ as the stopping criterion. The starting point x^0 is chosen as: (1) $(1, ..., 1)^T$; (2) $(10, ..., 10)^T$; (3) $(100, ..., 100)^T$; (4) $(1, 2, ..., n)^T$; (5) $(n, n-1, ..., 1)^T$. Numerical results are listed in Table 4 which also show that **N-BLM** has some advantages over **LF-BLM**.

6 Conclusions

In this paper we have proposed a new derivative-free line search technique (N-DFLS) which contains Li-Fukushima derivative-free line search (LF-DFLS) (Li & Fukushima, 2000a) as a special case. We have proved that Broyden-like method based N-DFLS has global and local superlinear convergence under appropriate assumptions. Moreover, we have shown that NCP, wLCP and SI can all be reformulated as the nonlinear equation (1.1) and be solved by our Broyden-like method. We have also reported some numerical results which indicate that

Broyden-like method based on N-DFLS is much superior to that based on LF-DFLS. The convergence properties of other quasi-Newton methods based on N-DFLS, such as BFGS method and DFP method, is an interesting issue deserved further research.

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References

Asadi, S., Darvay, Z., Lesaja, G., Mahdavi-Amiri, N., & Potra, F. A. (2020). A full-Newton step interior-point method for monotone weighted linear complementarity problems. *Journal of Optimization Theory and Applications, 186*, 864–878.

Broyden, C. G. (1965). A class of methods for solving nonlinear simultaneous equations. *Mathematics of Computation, 19*, 577–593.

Chen, B., & Ma, C. (2011). A new smoothing Broyden-like method for solving nonlinear complementarity problem with a P_0-function. *Journal of Global Optimization, 51*, 473–495.

Chen, B., & Ma, C. (2011). Superlinear/quadratic smoothing Broyden-like method for the generalized nonlinear complementarity problem. *Nonlinear Analysis: Real World Applications, 12*, 1250–1263.

Chen, J. S. (2006). The semismooth-related properties of a merit function and adescent method for the nonlinear complementarity problem. *Journal of Global Optimization, 36*, 565–580.

Chen, J. S., & Pan, S. H. (2008). A family of NCP functions and a descent method for the nonlinear complementarity problem. *Computational Optimization and Applications, 40*, 389–404.

Chen, J. S., Ko, C. H., Liu, Y. D., & Wang, S. P. (2016). New smoothing functions for solving a system of equalities and inequalities. *Pac. J. Optim., 12*(1), 185–206.

Cheng, W. Y., & Li, D. H. (2009). A derivative-free nonmonotone line search and its application to the spectral residual method. *IMA Journal of Numerical Analysis, 29*, 814–825.

Chi, X. N., Gowda, M. S., & Tao, J. (2019). The weighted horizontal linear complementarity problem on a Euclidean Jordan algebra. *Journal of Global Optimization, 73*, 153–169.

Dennis, J. E., Jr., & Moré, J. J. (1977). Quasi-Newton methods, motivation and theory. *SIAM Review, 19*, 46–89.

Diniz-Ehrhardta, M. A., Martíneza, J. M., & Raydan, M. (2008). A derivative-free nonmonotone line-search technique for unconstrained optimization. *Journal of Computational and Applied Mathematics, 219*, 383–397.

Facchinei, F., & Pang, J. S. (2003). *Finite-dimensional variational inequalities and complementarity problems*. New York: Springer.

Fan, B. (2015). A smoothing Broyden-like method with a nonmonotone derivative-free line search for nonlinear complementarity problems. *Journal of Computational and Applied Mathematics, 290*, 641–655.

Fan, X., & Yan, Q. (2019). Solving system of inequalities via a smoothing homotopy method. *Numerical Algorithms, 82*, 719–728.

Ferris, M. C., & Pang, J.-S. (1997). Engineering and economic applications of complementarity problems. *SIAM Review, 39*(4), 669–713.

Gowda, M. S. (2019). Weighted LCPs and interior point systems for copositive linear transformations on Euclidean Jordan algebras. *Journal of Global Optimization, 74*(4), 285–295.

Griewank, A. (1986). The “global” convergence of Broyden-like methods with suitable line search. *The ANZIAM Journal, 28,* 75–92.

Grippo, L., & Rinaldi, F. (2015). A class of derivative-free nonmonotone optimization algorithms employing coordinate rotations and gradient approximations. *Computational Optimization and Applications, 60*, 1–33.

La Cruz, W., Martinez, J. M., & Raydan, M. (2006). Spectral residual method without gradient information for solving large-scale nonlinear systems of equations. *Mathematics of Computation, 75*(255), 1429–1448.

La Cruz, W., Martinez, J.M., Raydan, M. (2004). Spectral residual method without gradient information for solving large-scale nonlinear systems: Theory and experiments, Technical Report RT-04-08, Dpto. de Computacion, UCV, Available at www.kuainasi.ciens.ucv.ve/ccct/mraydan_pub.html.

Li, D. H., & Fukushima, M. (1999). A globally and superlinearly convergent Gauss-Newton-based BFGS method for symmetric nonlinear equations. *SIAM Journal on Numerical Analysis, 37*(1), 152–172.

Li, D. H., & Fukushima, M. (1999). A derivative-free line search and DFP method for symmetric equations with global and superlinear convergence. *Numerical functional analysis and optimization, 20*, 59–77.

Li, D. H., & Fukushima, M. (2000). A derivative-free line search and global convergence of Broyden-like method for nonlinear equations. *Optimization methods and software, 13*(3), 181–201.

Li, D. H., & Fukushima, M. (2000). Smoothing Newton and quasi-Newton methods for mixed complementarity problems. *Computational Optimization and Applications, 17*, 203–230.

Li, D. H., & Fukushima, M. (2001). Globally convergent Broyden-like methods for semismooth equations and applications to VIP. *Annals of Operations Research Annals of Operations Research, 103*, 71–79.

Li, D. H., & Fukushima, M. (2001). A modified BFGS method and its global convergence in nonconvex minimization. *Journal of Computational and Applied Mathematics, 129*, 15–35.

Li, D. H., & Fukushima, M. (2001). On the global convergence of BFGS method for nonconvex unconstrained optimization problems. *SIAM Journal on Optimization, 11*, 1054–1064.

Li, D. H., Yamashita, N., & Fukushima, M. (2001). Nonsmooth equation based BFGS method for solving KKT systems in mathematical programming. *Journal of Optimization Theory and Applications, 109*(1), 123–167.

Li, Y. F., Sun, G., & Wang, Y. J. (2011). A smoothing Broyden-like method for polyhedral cone constrained eigenvalue problem. *Numerical Algebra, Control & Optimization, 1*(3), 529–537.

Ma, C. F., Chen, L., & Wang, D. (2008). A globally and superlinearly convergent smoothing Broyden-like method for solving nonlinear complementarity problem. *Applied Mathematics and Computation, 198*, 592–604.

Potra, F. A. (2012). Weighted complementarity problems-a new paradigm for computing equilibria. *SIAM Journal on Optimization, 22*(4), 1634–1654.

Powell, M.J.D. (1970). A fortran subroutine for solving systems of nonlinear algebraic equations. In: Rabinowitz, P. (Ed.), Numerical methods for nonlinear algebraic equations. Gordon and Breach, London (Chapter 7)

Tang, J. Y., & Zhang, H. C. (2021). A nonmonotone smoothing Newton algorithm for weighted complementarity problems. *Journal of Optimization Theory and Applications, 189*, 679–715.

Tang, J. Y., & Zhou, J. C. (2021). Quadratic convergence analysis of a nonmonotone Levenberg-Marquardt type method for the weighted nonlinear complementarity problem. *Computational Optimization and Applications, 80*, 213–244.

Tang, J. Y., & Zhou, J. C. (2021). A modified damped Gauss-Newton method for non-monotone weighted linear complementarity problems. *Optim. Methods Softw.* https://doi.org/10.1080/10556788.2021.1903007

Tang, J. Y., & Zhou, J. C. (2021). A smoothing quasi-Newton method for solving general second-order cone complementarity problems. *Journal of Global Optimization, 80*, 415–438.

Tang, J. Y., & Zhou, J. C. (2020). Smoothing inexact Newton method based on a new derivative-free nonmonotone line search for the NCP over circular cones. *Annals of Operations Research, 295*, 787–808.

Tang, J., & Liu, S. Y. (2010). A new smoothing Broyden-like method for solving the mixed complementarity problem with a P_0-function. *Nonlinear Analysis: Real World Applications, 11*, 2770–2786.

Zhou, W. J., & Zhang, L. (2020). A modified Broyden-like quasi-Newton method for nonlinear equations. *Journal of Computational and Applied Mathematics, 372*, 112744.

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