

# Heavy-ball-based optimal thresholding algorithms for sparse linear inverse problems

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**Abstract** Linear inverse problems arise in diverse engineering fields especially in signal and image reconstruction. The development of computational methods for linear inverse problems with sparsity tool is one of the recent trends in this area. The so-called optimal  $k$ -thresholding is a newly introduced method for sparse optimization and linear inverse problems. Compared to other sparsity-aware algorithms, the advantage of optimal  $k$ -thresholding method lies in that it performs thresholding and error metric reduction simultaneously and thus works stably and robustly for solving medium-sized linear inverse problems. However, the runtime of this method remains high when the problem size is relatively large. The purpose of this paper is to propose an acceleration strategy for this method. Specifically, we propose a heavy-ball-based optimal  $k$ -thresholding (HBOT) algorithm and its relaxed variants for sparse linear inverse problems. The convergence of these algorithms is shown under the restricted isometry property. In addition, the numerical performance of the heavy-ball-based relaxed optimal  $k$ -thresholding pursuit (HBROTP) has been studied, and simulations indicate that HBROTP admits robust capability for signal and image reconstruction even in noisy environments.

**Keywords** Sparse linear inverse problems · optimal  $k$ -thresholding · heavy-ball method · restricted isometry property · phase transition · peak signal-to-noise ratio

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## 1 Introduction

In recent years, the linear inverse problem has gained much attention in various fields such as wireless communication [9, 16], signal and image processing [38, 3, 41, 8, 46, 31]. The main task of the linear inverse problem is to recover the unknown data  $z \in \mathbb{R}^n$  from the linear measurements

$$y = \Phi z + \nu, \quad (1.1)$$

where  $y \in \mathbb{R}^m$  are the acquired measurements of  $z$ ,  $\nu \in \mathbb{R}^m$  is a noisy vector, and  $\Phi \in \mathbb{R}^{m \times n}$  is a given measurement matrix. In this paper, we only consider the underdetermined case  $m \ll n$ . In this case, it is generally impossible to reconstruct data  $z$  from the linear system (1.1) unless  $z$  possesses a certain special structure such as sparsity. When  $z$  in its current domain is not sparse enough, a practical approach to achieve sparsity is to consider a certain transformed domain in which  $z$  can be sparsely represented. For instance, many natural signal and image can be sparsely represented under the fourier transform or some wavelet transform. Suppose that  $z$  can be sparsely represented on an orthonormal basis  $\Psi \in \mathbb{R}^{n \times n}$ , i.e.,  $z = \Psi x$  where the coefficient vector  $x \in \mathbb{R}^n$  is  $k$ -sparse or  $k$ -compressible. That is,  $\|x\|_0 \leq k$  or  $x$  can be well approximated by a  $k$ -sparse vector, where  $\|\cdot\|_0$  denotes the number of nonzero entries of a vector. Via such a sparse representation, the non-sparse linear inverse model (1.1) can be transformed to the sparse linear inverse (SLI) problem

$$y = Ax + \nu, \quad (1.2)$$

where the product  $A = \Phi\Psi \in \mathbb{R}^{m \times n}$  is still referred to as the measurement (or sensing) matrix. The purpose of the SLI problem is to reconstruct the sparse signal  $x$  based on the observed (inaccurate) signal  $y$  and the adopted measurement matrix  $A$  and then reconstruct the original signal  $z = \Psi x$ . The SLI problem can be formulated as the sparse optimization problem [8, 16, 38, 43]

$$\min_u \{\|y - Au\|_2^2 : \|u\|_0 \leq k\} \quad (1.3)$$

or other related models such as the LASSO-type models [3, 9]. The problem (1.3) has been widely used in signal and image processing via compressed samplings.

The existing methods for (1.3) includes thresholding [22, 23, 25], greedy [17, 37, 42], convex optimization [11, 12, 15, 49], nonconvex optimization [14] and Bayesian methods [40, 45]. In this paper, we focus our attention on the thresholding method which was first proposed by Donoho and Johnstone [20]. It has experienced a significant development since 1994 and has evolved into a large family of algorithms including the hard thresholding [5, 7, 8, 24, 29, 33], soft thresholding [18, 19, 21] and optimal  $k$ -thresholding [34, 35, 48, 51]. It is worth stressing that the main advantage of thresholding methods is that the algorithms can always guarantee the points generated by the algorithms being feasible or nearly feasible to the problem (1.3), since the iterates are  $k$ -sparse or  $k$ -compressible. The simplest thresholding iterative method might be the so-called iterative hard thresholding [7] which is

a gradient-based thresholding method. The combination of IHT and orthogonal projection yields the well known hard thresholding pursuit (HTP) method [24]. Due to their low computational complexity, IHT, HTP and their modifications have been widely used in numerous areas such as compressive sampling and image processing [5, 6, 8, 29]. However, performing hard thresholding on an iterate does not necessarily leads to the reduction of the objective function value of (1.3) and might also cause numerical oscillation during the course of iterations [48, 51].

To alleviate the above-mentioned weakness of hard thresholding, the optimal  $k$ -thresholding (OT) operator was first introduced in [48] (see also [51]). Such an operator may reduce the objective value of (1.3) while performing thresholding of an iterate. Based on this operator, the basic framework of optimal  $k$ -thresholding (OT) and optimal  $k$ -thresholding pursuit (OTP) is proposed in [48]. The optimal  $k$ -thresholding of a vector  $v$  is defined as

$$\min\{\|y - A(v \otimes w)\|_2^2 : \mathbf{e}^T w = k, w \in \{0, 1\}^n\}, \quad (1.4)$$

where  $\mathbf{e} = (1, 1, \dots, 1)^T \in \mathbb{R}^n$ ,  $v \in \mathbb{R}^n$  is a given vector,  $\{0, 1\}^n$  is the set of  $n$ -dimensional binary vectors and  $v \otimes w := (v_1 w_1, \dots, v_n w_n)^T$  denotes the Hadamard product of two vectors. From a computational point of view, it is generally more convenient to solve the convex relaxation of (1.4) instead, which is the following quadratic convex optimization

$$\min\{\|y - A(v \otimes w)\|_2^2 : \mathbf{e}^T w = k, 0 \leq w \leq \mathbf{e}\}. \quad (1.5)$$

This problem is referred to as data compressing problem in [48, 51]. Based on (1.5), the relaxed optimal  $k$ -thresholding (ROT $\omega$ ) and relaxed optimal  $k$ -thresholding pursuit (ROTP $\omega$ ) algorithms were proposed in [48, 51], where  $\omega$  represents times of data compression that are performed in the algorithms. When  $\omega = 1$ , the algorithm is named as ROTP. Using partial gradient and Newton-type search direction, some modifications of ROTP were also studied in [34, 35] recently. However, solving the above-mentioned quadratic optimization problem (1.5) remains time-consuming for large-scale problems. It is important to further investigate how the computational cost of ROTP-type methods might be reduced or how these methods can be accelerated by incorporating such acceleration techniques as heavy-ball (HB) and Nesterov's method. By using linearization together with certain regularization method, the so-called nature thresholding algorithm is newly developed in [52] which has significantly lower the computational complexity than ROTP $\omega$ . In this paper, we investigate the ROTP-type algorithm from acceleration perspective by showing that the HB technique does be able to improve the efficiency of the ROTP-type algorithms.

The HB method introduced by Polyak [39] can be seen as a two-step method which combines the momentum term and gradient descent direction. In recent years, HB has gained significant attention and has found wide applications in image processing, data analysis, distributed optimization and undirected networks [2, 27, 28, 30, 32, 36, 44, 47]. The theoretical analysis (global convergence and local convergence rate) has been studied by several researchers. For example, the linear convergence rate of HB for unconstrained convex optimization problem was established by Aujol et. al [2]; Mohammadi et. al [36] analyzed the relation between the convergence rate of HB and its variance amplification when the objective functions of problems are strongly convex and quadratic; Xin and Khan [47] showed that

the distributed HB method with appropriate parameters attains a global  $R$ -linear rate, and it has potential acceleration compared with some first-order methods for the ill-conditioned problem. Besides the HB acceleration technique, the Nesterov's acceleration method are also studied in [28, 30, 32, 36].

In this paper, we study the acceleration issue of the optimal  $k$ -thresholding algorithms by incorporating the HB acceleration technique. Merging the optimal  $k$ -thresholding and heavy-ball techniques, we propose the following novel algorithms for the linear inverse problems reformulated as (1.3):

- Heavy-ball-based optimal  $k$ -thresholding (HBOT),
- Heavy-ball-based optimal  $k$ -thresholding pursuit (HBOTP),
- Heavy-ball-based relaxed optimal  $k$ -thresholding (HBROT $\omega$ ),
- Heavy-ball-based relaxed optimal  $k$ -thresholding pursuit (HBROTP $\omega$ ),

where the integer parameter  $\omega$  denotes the number of times for solving (1.5) at every iteration. The global convergence of these algorithms is established in this paper under the restricted isometry property (RIP) introduced by Candès and Tao [11], and the main results are summarized in Theorems 1 and 2. The performance of HBROTP (i.e., HBROTP $\omega$  with  $\omega = 1$ ) and several existing algorithms such as ROTP2 [48], PGROTP [35],  $\ell_1$ -minimization [15] and OMP [22] are compared through numerical experiments. The phase transition with Gaussian random data is adopted to illustrate the performance of the proposed algorithms for solving sparse signal reconstruction problems. Also, a few real images are used to demonstrate the reconstruction performance of the algorithms in noiseless situations.

This paper is organized as follows. In Section 2, we introduce some notations, definitions, basic inequalities and the algorithms. In Section 3, we discuss the error bounds and convergences of HBOT and HBOTP under the RIP. Similarly, the error bounds for HBROT $\omega$  and HBROTP $\omega$  are given in Section 4. Numerical results observed from synthetic signals and real images are estimated in Section 5. Conclusions are drawn in Section 6.

## 2 Preliminary and algorithms

### 2.1 Notations

Denote by  $N := \{1, 2, \dots, n\}$ . Given a subset  $\Omega \subseteq N$ ,  $\overline{\Omega} := N \setminus \Omega$  denotes the complement set of  $\Omega$  and  $|\Omega|$  denotes its cardinality. For a vector  $z \in \mathbb{R}^n$ , the support of  $z$  is represented as  $\text{supp}(z) := \{i \in N : z_i \neq 0\}$ , and the vector  $z_\Omega \in \mathbb{R}^n$  is obtained by zeroing out the elements of  $z$  supported on  $\overline{\Omega}$  and retaining other elements. Given the sparse level  $k$ ,  $\mathcal{L}_k(z)$  denotes the index set of the  $k$  largest absolute entries of  $z$ . As usual,  $\mathcal{H}_k(z) := z_{\mathcal{L}_k(z)}$  is called the hard thresholding of  $z$ . The symbols  $\|\cdot\|_1$  and  $\|\cdot\|_2$  represent  $\ell_1$ -norm and  $\ell_2$ -norm of a vector, respectively. Let  $\mathcal{W}^k := \{w : \mathbf{e}^T w = k, w \in \{0, 1\}^n\}$  and  $\mathcal{P}^k := \{w \in \mathbb{R}^n : \mathbf{e}^T w = k, 0 \leq w \leq \mathbf{e}\}$  be two subsets of  $\mathbb{R}^n$ .

## 2.2 Definitions and basic inequalities

Let us first recall the restricted isometry property (RIP) of a matrix  $A$  and the optimal  $k$ -thresholding operator  $Z_k^\#(\cdot)$  in the following two definitions respectively.

**Definition 1** [11] Given a matrix  $A \in \mathbb{R}^{m \times n}$  with  $m < n$ . The  $k$ th order restricted isometry constant (RIC) of  $A$ , denoted by  $\delta_k$ , is the smallest nonnegative number  $\delta$  such that

$$(1 - \delta)\|u\|_2^2 \leq \|Au\|_2^2 \leq (1 + \delta)\|u\|_2^2 \quad (2.1)$$

for all  $k$ -sparse vectors  $u \in \mathbb{R}^n$ . The matrix  $A$  is said to satisfy the RIP of order  $k$  if  $\delta_k < 1$ .

**Definition 2** [48, 51] Given a vector  $u \in \mathbb{R}^n$ , let  $w^*(u)$  be the solution of the binary optimization problem

$$\min_w \left\{ \|y - A(u \otimes w)\|_2^2 : w \in \mathcal{W}^k \right\}.$$

The optimal  $k$ -thresholding of  $u$  is denoted by  $Z_k^\#(u) := u \otimes w^*(u)$  which is a  $k$ -sparse vector.  $Z_k^\#(\cdot)$  is called the optimal  $k$ -thresholding operator.

The two lemmas below will be used for the analysis in Sections 3 and 4.

**Lemma 1** [24] Let  $u, v \in \mathbb{R}^n$ ,  $W \subseteq N$  be an index set and  $s, t \in N$  be two integers.

- (i) If  $|W \cup \text{supp}(v)| \leq t$ , then  $\|(I - A^T A)v\|_W \leq \delta_t \|v\|_2$ .
- (ii) If  $|W| \leq t$ , then  $\|(A^T v)_W\|_2 \leq \sqrt{1 + \delta_t} \|v\|_2$ .
- (iii) If  $\text{supp}(u) \cap \text{supp}(v) = \emptyset$ ,  $|\text{supp}(u)| \leq s$  and  $|\text{supp}(v)| \leq t$ , then  $|u^T A^T A v| \leq \delta_{s+t} \|u\|_2 \|v\|_2$ .

**Lemma 2** Let  $\{a^p\} \subseteq \mathbb{R}$  ( $p = 0, 1, \dots$ ) be a nonnegative sequence satisfying

$$a^{p+1} \leq b_1 a^p + b_2 a^{p-1} + b_3$$

for  $p \geq 1$ , where  $b_1, b_2$  and  $b_3 \geq 0$  are constants and  $b_1 + b_2 < 1$ . Then

$$a^p \leq \theta^{p-1} \left( a^1 + (\theta - b_1) a^0 \right) + \frac{b_3}{1 - \theta}$$

for  $p \geq 2$ , where  $0 \leq \theta < 1$  is a constant given by  $\theta = (b_1 + \sqrt{b_1^2 + 4b_2})/2 < 1$ .

The proof of Lemma 2 is straightforward and hence omitted.

### 2.3 Algorithms

Given iterates  $x^{p-1}$  and  $x^p$ , the heavy-ball search direction is defined as

$$d^p = \alpha A^T(y - Ax^p) + \beta(x^p - x^{p-1}),$$

where  $\alpha > 0$  and  $\beta \geq 0$  are two parameters. We use the optimal  $k$ -thresholding operator  $Z_k^\#(\cdot)$  to generate the new iterate  $x^{p+1}$ , i.e.,

$$x^{p+1} = Z_k^\#(x^p + d^p),$$

which is called the heavy-ball-based optimal  $k$ -thresholding (HBOT) of the iterate  $x^p + d^p$ . Combining HBOT with the orthogonal projection technique, we propose the heavy-ball-based optimal  $k$ -thresholding pursuit (HBOTP) algorithm. The two algorithms are described as follows. It is evident that HBOT and HBOTP are the multi-step extension of existing OT and OTP algorithms in [48, 51], respectively.

**HBOT and HBOTP algorithms.** Input the data  $(A, y, k)$  and two initial points  $x^0$  and  $x^1$ . Choose the parameters  $\alpha > 0$  and  $\beta \geq 0$ .

S1. At  $x^p$ , set

$$u^p = x^p - \alpha A^T(Ax^p - y) + \beta(x^p - x^{p-1}). \quad (2.2)$$

S2. Solve the optimization problem

$$w^* = \arg \min_w \{\|y - A(u^p \otimes w)\|_2^2 : \mathbf{e}^T w = k, w \in \{0, 1\}^n\}. \quad (2.3)$$

S3. Generate the next iterate  $x^{p+1}$  as follows:

For HBOT, let  $x^{p+1} = u^p \otimes w^*$ .

For HBOTP, let  $S^{p+1} = \text{supp}(u^p \otimes w^*)$  and  $x^{p+1}$  be the solution to the least squares problem

$$x^{p+1} = \arg \min_{x \in \mathbb{R}^n} \{\|y - Ax\|_2^2 : \text{supp}(x) \subseteq S^{p+1}\}. \quad (2.4)$$

Repeat S1-S3 until a certain stopping criterion is met.

In general, the computational cost for solving the binary optimization problem (2.3) is usually high [10, 13]. Replacing (2.3) by its convex relaxation

$$\arg \min_w \left\{ \|y - A(u \otimes w)\|_2^2 : w \in \mathcal{P}^k \right\}$$

yields the next heavy-ball-based relaxed optimal  $k$ -thresholding (HBROT $\omega$ ) and the heavy-ball-based relaxed optimal  $k$ -thresholding pursuit (HBROTP $\omega$ ) algorithms, where the parameter  $\omega$  represents the times of solving such a convex relaxation problem at each iteration (which, as pointed out in [48], can be interpreted as the times of data compression within each iteration). As  $\omega = 1$ , we simply use HBROT and HBROTP to denote the algorithms HBROT1 and HBROTP1, respectively. Clearly, as  $\alpha = 1$  and  $\beta = 0$ , HBROT $\omega$  and HBROTP $\omega$  reduce to ROT $\omega$  and ROTP $\omega$  in [51].

**HBROT $\omega$  and HBROTP $\omega$  algorithms.** Input the data  $(A, y, k)$ , two initial points  $x^0, x^1$  and  $\omega$ . Choose the parameters  $\alpha > 0$  and  $\beta \geq 0$ .

S1. At  $x^p$ , calculate  $u^p$  by (2.2).

S2. Set  $v \leftarrow u^p$ . Perform the following loops to produce the vectors  $w^{(j)}$  ( $j = 1, \dots, \omega$ ):  
**for**  $j = 1 : \omega$  **do**

$$w^{(j)} = \arg \min_w \{ \|y - A(v \otimes w)\|_2^2 : \mathbf{e}^T w = k, \ 0 \leq w \leq \mathbf{e} \}, \quad (2.5)$$

and set  $v \leftarrow v \otimes w^{(j)}$ .

**end**

S3. Let  $x^\# = \mathcal{H}_k(u^p \otimes w^{(1)} \otimes \dots \otimes w^{(\omega)})$ . Generate the next iterate  $x^{p+1}$  as follows:  
 For HBROT $\omega$ , let  $x^{p+1} = x^\#$ .  
 For HBOTP $\omega$ , let  $S^{p+1} = \text{supp}(x^\#)$ , and  $x^{p+1}$  be the solution to the least squares problem

$$x^{p+1} = \arg \min_{x \in \mathbb{R}^n} \{ \|y - Ax\|_2^2 : \text{supp}(x) \subseteq S^{p+1} \}. \quad (2.6)$$

Repeat S1-S3 until a certain stopping criterion is met.

The choice of stopping criterions depends on the application scenarios. For instance, one can simply prescribe the maximum number of iterations,  $p_{\max}$ , which allows the algorithm to perform a total number of  $p_{\max}$  iterations. One can also terminate the algorithm when  $\|y - Ax^p\|_2 \leq \varepsilon$ , where  $\varepsilon > 0$  is a prescribed tolerance depending on the noise level.

### 3 Analysis of HBOT and HBOTP

In this section, we establish the error bounds for HBOT and HBOTP under the RIP of order  $k$  or  $k+1$ . Taking into account of noise influence, the error bound provides an estimation for the distance between the problem solution and the iterates generated by the algorithms. Thus the error bound is an important measurement of the quality of iterates as the approximation to the sparse solution of the linear inverse problem. In noiseless situations, the error bound implies the global convergence of the algorithms under the RIP assumption. Let us first introduce the following property, which is a combination of Lemmas 3.3 and 3.6 in [51].

**Lemma 3** [51] *Let  $z$  be a  $(2k)$ -sparse vector. Then  $\|Az\|_2^2 \geq (1 - 2\delta_k - \delta_{k+s(k)})\|z\|_2^2$  with*

$$s(k) = \begin{cases} 1, & \text{if } k \text{ is an odd number,} \\ 0, & \text{if } k \text{ is an even number.} \end{cases} \quad (3.1)$$

Note that Lemma 3.4 in [51] was established for the sparsity level  $k$  being an even number. We now establish the similar result even when  $k$  is odd.

**Lemma 4** *Let  $h, z \in \mathbb{R}^n$  be two  $k$ -sparse vectors, and let  $\hat{w} \in \mathcal{W}^k$  be any  $k$ -sparse binary vector satisfied  $\text{supp}(h) \subseteq \text{supp}(\hat{w})$ , then*

$$\| \left[ (I - A^T A)(h - z) \right] \otimes \hat{w} \|_2 \leq \sqrt{5}\delta_{k+s(k)} \|h - z\|_2, \quad (3.2)$$

where  $s(k)$  is given by (3.1).

*Proof* For given vectors  $h, z, \hat{w}$  satisfying the conditions of the lemma, from [51, Lemma 3.4], we get

$$\|[(I - A^T A)(h - z)] \otimes \hat{w}\|_2 \leq \sqrt{5}\delta_k \|h - z\|_2 \quad (3.3)$$

for even number  $k$ . Therefore, we just need to show that (3.2) also holds when  $k$  is an odd number.

Indeed, assume that  $k$  is an odd number. Taking a  $(k+1)$ -sparse binary vector  $\bar{w} \in \mathcal{W}^{k+1}$  such that  $\text{supp}(\hat{w}) \subseteq \text{supp}(\bar{w})$ , we obtain

$$\begin{aligned} \|[ (I - A^T A)(h - z) ] \otimes \hat{w}\|_2 &= \|[ (I - A^T A)(h - z) ]_{\text{supp}(\hat{w})}\|_2 \\ &\leq \|[ (I - A^T A)(h - z) ]_{\text{supp}(\bar{w})}\|_2 \\ &= \|[ (I - A^T A)(h - z) ] \otimes \bar{w}\|_2. \end{aligned} \quad (3.4)$$

As  $h$  and  $z$  are two  $k$ -sparse vectors, they are also  $(k+1)$ -sparse vectors. Since  $\text{supp}(h) \subseteq \text{supp}(\hat{w}) \subseteq \text{supp}(\bar{w})$ , and since  $k+1$  is even (when  $k$  is odd), applying (3.3) to this case yields

$$\|[ (I - A^T A)(h - z) ] \otimes \bar{w}\|_2 \leq \sqrt{5}\delta_{k+1} \|h - z\|_2. \quad (3.5)$$

Combining (3.4) and (3.5), we obtain

$$\|[ (I - A^T A)(h - z) ] \otimes \hat{w}\|_2 \leq \sqrt{5}\delta_{k+1} \|h - z\|_2$$

for odd number  $k$ . Hence, we conclude that (3.2) holds for any positive integers  $k$ .  $\square$

The main results for HBOT and HBOTP are summarized as follows.

**Theorem 1** *Let  $x \in \mathbb{R}^n$  be a solution to the system  $y = Ax + \nu$  where  $\nu$  is a noise vector. Assume that the RIC,  $\delta_{k+s(k)}$ , of  $A$  and the parameters  $\alpha, \beta$  in HBOT and HBOTP satisfy that  $\delta_{k+s(k)} < \gamma^*$  and*

$$0 \leq \beta < \frac{1 + 1/\eta}{1 + \sqrt{5}\delta_{k+s(k)}} - 1, \quad \frac{1 + 2\beta - 1/\eta}{1 - \sqrt{5}\delta_{k+s(k)}} < \alpha < \frac{1 + 1/\eta}{1 + \sqrt{5}\delta_{k+s(k)}}, \quad (3.6)$$

where  $\gamma^* (\approx 0.2274)$  is the unique root of the equation  $5\gamma^3 + 5\gamma^2 + 3\gamma - 1 = 0$  in the interval  $(0, 1)$ ,  $s(k)$  is given by (3.1) and  $\eta := \sqrt{\frac{1+\delta_k}{1-2\delta_k-\delta_{k+s(k)}}}$ . Then the sequence  $\{x^p\}$  generated by HBOT or HBOTP obeys

$$\|x_S - x^p\|_2 \leq C_1 \theta^{p-1} + C_2 \|\nu'\|_2, \quad (3.7)$$

where  $S := \mathcal{L}_k(x)$ ,  $\nu' := \nu + Ax_{\bar{S}}$ , and the quantities  $C_1, C_2$  are defined as

$$C_1 = \|x_S - x^1\|_2 + (\theta - b)\|x_S - x^0\|_2, \quad C_2 = \frac{2 + (1 + \delta_k)\alpha}{(1 - \theta)\sqrt{1 - 2\delta_k - \delta_{k+s(k)}}}, \quad (3.8)$$

and  $\theta := (b + \sqrt{b^2 + 4\eta\beta})/2 < 1$  is ensured under the conditions (3.6) and the constant  $b$  is given by

$$b := \eta \left( |1 + \beta - \alpha| + \sqrt{5}\alpha\delta_{k+s(k)} \right). \quad (3.9)$$



*Proof* From (2.2), we have

$$u^p - x_S = (1 - \alpha + \beta)(x^p - x_S) + \alpha(I - A^T A)(x^p - x_S) - \beta(x^{p-1} - x_S) + \alpha A^T \nu', \quad (3.10)$$

where  $S = \mathcal{L}_k(x)$  and  $\nu' = \nu + Ax_{\bar{S}}$ . Let  $\hat{w} \in \mathcal{W}^k$  be a  $k$ -sparse binary vector such that  $\text{supp}(x_S) \subseteq \text{supp}(\hat{w})$ . Then  $x_S = x_S \otimes \hat{w}$ . Since  $(x_S - u^p) \otimes \hat{w}$  is a  $k$ -sparse vector and  $y = Ax_S + \nu'$ , we have

$$\begin{aligned} \|y - A(u^p \otimes \hat{w})\|_2 &= \|A[x_S - u^p \otimes \hat{w}] + \nu'\|_2 \\ &\leq \|A[(x_S - u^p) \otimes \hat{w}]\|_2 + \|\nu'\|_2 \\ &\leq \sqrt{1 + \delta_k} \|(x_S - u^p) \otimes \hat{w}\|_2 + \|\nu'\|_2, \end{aligned} \quad (3.11)$$

where the last inequality is obtained by using (2.1). From (3.10), one has

$$\begin{aligned} \|(x_S - u^p) \otimes \hat{w}\|_2 &\leq |1 - \alpha + \beta| \cdot \|(x^p - x_S) \otimes \hat{w}\|_2 + \alpha \|(I - A^T A)(x^p - x_S) \otimes \hat{w}\|_2 \\ &\quad + \beta \|(x^{p-1} - x_S) \otimes \hat{w}\|_2 + \alpha \|(A^T \nu') \otimes \hat{w}\|_2. \end{aligned} \quad (3.12)$$

Since  $x_S, x^p, \hat{w}$  are  $k$ -sparse vectors and  $\text{supp}(x_S) \subseteq \text{supp}(\hat{w})$ , by using Lemmas 4 and 1 (ii), we obtain

$$\|(I - A^T A)(x^p - x_S) \otimes \hat{w}\|_2 \leq \sqrt{5} \delta_{k+s(k)} \|x^p - x_S\|_2 \quad (3.13)$$

and

$$\|(A^T \nu') \otimes \hat{w}\|_2 = \|(A^T \nu')_{\text{supp}(\hat{w})}\|_2 \leq \sqrt{1 + \delta_k} \|\nu'\|_2. \quad (3.14)$$

Substituting (3.13) and (3.14) into (3.12) yields

$$\begin{aligned} \|(x_S - u^p) \otimes \hat{w}\|_2 &\leq |1 - \alpha + \beta| \cdot \|x^p - x_S\|_2 + \alpha \sqrt{5} \delta_{k+s(k)} \|x^p - x_S\|_2 \\ &\quad + \beta \|x^{p-1} - x_S\|_2 + \alpha \sqrt{1 + \delta_k} \|\nu'\|_2 \\ &= (|1 + \beta - \alpha| + \sqrt{5} \alpha \delta_{k+s(k)}) \|x^p - x_S\|_2 + \beta \|x^{p-1} - x_S\|_2 \\ &\quad + \alpha \sqrt{1 + \delta_k} \|\nu'\|_2. \end{aligned}$$

It follows from (3.11) that

$$\begin{aligned} \|y - A(u^p \otimes \hat{w})\|_2 &\leq \sqrt{1 + \delta_k} \left( |1 + \beta - \alpha| + \sqrt{5} \alpha \delta_{k+s(k)} \right) \|x^p - x_S\|_2 \\ &\quad + \beta \sqrt{1 + \delta_k} \|x^{p-1} - x_S\|_2 + [1 + (1 + \delta_k) \alpha] \|\nu'\|_2. \end{aligned} \quad (3.15)$$

Since  $x^{p+1} = u^p \otimes w^*$  in HBOT or  $x^{p+1}$  is the optimal solution of (2.4) in HBOTP, the iterates  $\{x^p\}$  generated by HBOT or HBOTP satisfies

$$\|y - Ax^{p+1}\|_2 \leq \|y - A(u^p \otimes w^*)\|_2 \leq \|y - A(u^p \otimes w)\|_2 \quad (3.16)$$

for all  $w \in \mathcal{W}^k$ , where the second inequality follows from (2.3). For  $\hat{w} \in \mathcal{W}^k$ , it follows from (3.16) that

$$\|y - Ax^{p+1}\|_2 \leq \|y - A(u^p \otimes \hat{w})\|_2. \quad (3.17)$$

As  $x_S - x^{p+1}$  is a  $(2k)$ -sparse vector, by using Lemma 3, one has

$$\begin{aligned} \|y - Ax^{p+1}\|_2 &= \|A(x_S - x^{p+1}) + \nu'\|_2 \\ &\geq \|A(x_S - x^{p+1})\|_2 - \|\nu'\|_2 \\ &\geq \sqrt{1 - 2\delta_k - \delta_{k+s(k)}} \|x_S - x^{p+1}\|_2 - \|\nu'\|_2. \end{aligned} \quad (3.18)$$

Combining (3.15), (3.17) and (3.18) yields

$$\begin{aligned} \|x^{p+1} - x_S\|_2 &\leq \eta(|1 + \beta - \alpha| + \sqrt{5}\alpha\delta_{k+s(k)}) \|x^p - x_S\|_2 + \eta\beta \|x^{p-1} - x_S\|_2 \\ &\quad + \frac{2 + (1 + \delta_k)\alpha}{\sqrt{1 - 2\delta_k - \delta_{k+s(k)}}} \|\nu'\|_2 \\ &= b \|x^p - x_S\|_2 + \eta\beta \|x^{p-1} - x_S\|_2 + (1 - \theta)C_2 \|\nu'\|_2, \end{aligned} \quad (3.19)$$

where  $\eta, b, \theta, C_2$  are given exactly as in Theorem 1. Since  $\delta_k \leq \delta_{k+s(k)} < \gamma^*$ , we have

$$\eta\sqrt{5}\delta_{k+s(k)} = \sqrt{5}\delta_{k+s(k)} \sqrt{\frac{1 + \delta_k}{1 - 2\delta_k - \delta_{k+s(k)}}} < \sqrt{5}\gamma^* \sqrt{\frac{1 + \gamma^*}{1 - 3\gamma^*}} = 1,$$

where the last equality follows from the fact that  $\gamma^*$  is the root of  $5\gamma^3 + 5\gamma^2 + 3\gamma = 1$  in  $(0, 1)$ . It implies that  $0 < \frac{1 + 1/\eta}{1 + \sqrt{5}\delta_{k+s(k)}} - 1$ , which shows that the range of  $\beta$  in (3.6) is well defined. Furthermore, the first inequality in (3.6) implies that

$$\frac{1 + 2\beta - 1/\eta}{1 - \sqrt{5}\delta_{k+s(k)}} < 1 + \beta < \frac{1 + 1/\eta}{1 + \sqrt{5}\delta_{k+s(k)}},$$

which indicates that the range for  $\alpha$  in (3.6) is also well defined. Combining (3.9) with (3.6), we deduce that

$$\begin{aligned} b &= \eta \left( |1 + \beta - \alpha| + \sqrt{5}\alpha\delta_{k+s(k)} \right) \\ &= \begin{cases} \eta [1 + \beta - \alpha(1 - \sqrt{5}\delta_{k+s(k)})], & \text{if } \frac{1 + 2\beta - 1/\eta}{1 - \sqrt{5}\delta_{k+s(k)}} < \alpha \leq 1 + \beta, \\ \eta [-1 - \beta + \alpha(1 + \sqrt{5}\delta_{k+s(k)})], & \text{if } 1 + \beta < \alpha < \frac{1 + 1/\eta}{1 + \sqrt{5}\delta_{k+s(k)}}, \end{cases} \\ &< 1 - \eta\beta, \end{aligned}$$

which means that the relation (3.19) obeys the conditions of Lemma 2. It follows from Lemma 2 that (3.7) holds with  $\theta = \frac{b + \sqrt{b^2 + 4\eta\beta}}{2} < 1$  and  $C_1, C_2$  are given by (3.8).  $\square$

The error bound (3.7) indicates that the iterates  $x^p$  generated by the algorithms can approximate the significant components  $x_S$  of the solution of the linear inverse problem. In particular, we immediately obtain the following convergence result for the algorithms.

**Corollary 1** *Let  $x \in \mathbb{R}^n$  be a  $k$ -sparse solution to the system  $y = Ax$ . Assume that the RIC,  $\delta_{k+s(k)}$ , of  $A$  and the parameters  $\alpha, \beta$  in HBOT and HBOTP satisfy  $\delta_{k+s(k)} < \gamma^*$ , where  $\gamma^*$  is the same constant given in Theorem 1. Then the sequence  $\{x^p\}$  generated by HBOT or HBOTP obeys that  $\|x - x^p\|_2 \leq C_1\theta^{p-1}$ , where the constant  $C_1$  is defined as in Theorem 1. Thus the iterates  $\{x^p\}$  generated by HBOT or HBOTP converges to  $x$ .*

#### 4 Analysis of HBROT $\omega$ and HBROTP $\omega$

In this section, we establish the error bounds for the two algorithms HBROT $\omega$  and HBROTP $\omega$ . The analysis is far from being trivial. We need a few useful technical results before we actually establish the error bounds. We first recall a helpful lemma concerning the polytope  $\mathcal{P}^k$ , which is the special case of Lemma 4.2 with  $\tau = k$  in [51].

**Lemma 5** [51] *Given an index set  $\Lambda \subseteq N$  and a vector  $w \in \mathcal{P}^k$ . Decompose  $w_\Lambda$  as the sum of  $k$ -sparse vectors:  $w_\Lambda = \sum_{j=1}^q w_{\Lambda_j}$ , where  $q := \lceil \frac{|\Lambda|}{k} \rceil$ ,  $\Lambda = \bigcup_{j=1}^q \Lambda_j$  and  $\Lambda_1 := \mathcal{L}_k(w_\Lambda)$ ,  $\Lambda_2 := \mathcal{L}_k(w_{\Lambda \setminus \Lambda_1})$ , and so on. Then*

$$\sum_{j=1}^q \|w_{\Lambda_j}\|_\infty < 2.$$

We now give an inequality concerning the norms  $\|\cdot\|_2$ ,  $\|\cdot\|_1$  and  $\|\cdot\|_\infty$ . This inequality is an modification of Lemma 6.14 in [25], but tailored to the need of the later analysis in this paper.

**Lemma 6** *Let  $h \in \mathbb{R}^r \setminus \{0\}$  be a vector with positive integer  $r \geq 2$ , and  $\zeta_1 > \zeta_2$  be two positive numbers such that  $\|h\|_1 \leq \zeta_1$  and  $\|h\|_\infty \leq \zeta_2$ . Then*

$$\|h\|_2 \leq \begin{cases} g(r), & \text{if } r \leq t_0, \\ \min\{g(t_0), g(t_0 + 1)\}, & \text{if } r \geq t_0 + 1, \end{cases} \quad (4.1)$$

where  $t_0 := \lfloor \frac{4\zeta_1}{\zeta_2} \rfloor$  and

$$g(j) := \frac{1}{\sqrt{j}}\zeta_1 + \frac{\sqrt{j}}{4}\zeta_2, \quad j \in (0, +\infty), \quad (4.2)$$

is strictly decreasing in the interval  $(0, \frac{4\zeta_1}{\zeta_2}]$  and strictly increasing in the interval  $[\frac{4\zeta_1}{\zeta_2}, +\infty)$ .

*Proof* Without loss of generality, we assume that  $h$  is a nonnegative vector since the norms of  $h$  are independent of the sign of every elements of it. Sort the components of  $h$  into descending order, and denote such ordered components by  $z_1 \geq z_2 \geq \dots \geq z_r \geq 0$  and  $z = (z_1, \dots, z_r)^T$ . Thus,  $\|z\|_q = \|h\|_q$  for  $q \geq 1$ . For a given positive integer  $s$  and  $a_1 \geq a_2 \geq \dots \geq a_s \geq 0$ , from [25, Lemma 6.14], one has

$$\sqrt{a_1^2 + \dots + a_s^2} \leq \frac{a_1 + \dots + a_s}{\sqrt{s}} + \frac{\sqrt{s}}{4}(a_1 - a_s). \quad (4.3)$$

The further discussion is divided into two cases according to the relation between  $r$  and  $t_0$ .

**Case 1.** For  $r \leq t_0$ , by using (4.2) and (4.3), we have

$$\|z\|_2 \leq \frac{\|z\|_1}{\sqrt{r}} + \frac{\sqrt{r}}{4}(z_1 - z_r) \leq \frac{\|z\|_1}{\sqrt{r}} + \frac{\sqrt{r}}{4}\|z\|_\infty \leq \frac{1}{\sqrt{r}}\zeta_1 + \frac{\sqrt{r}}{4}\zeta_2 = g(r). \quad (4.4)$$

**Case 2.** For  $r \geq t_0 + 1$ , denote by  $t := \arg \min_j \{g(j) : j = t_0, t_0 + 1\}$  and let  $r_1, r_2$  be nonnegative integers such that  $r = r_1 t + r_2$  ( $0 \leq r_2 < t$ ). Decompose  $z$  as

the sum of  $t$ -sparse vectors:  $z = \sum_{j=1}^{r_1+1} z_{Q_j}$ , where  $Q_j := \{(j-1)t+1, \dots, jt\}$  ( $j = 1, \dots, r_1$ ) and  $Q_{r_1+1} := \{r_1t+1, \dots, r_1t+r_2\}$ .

Firstly, we consider the case when  $r_2 > 0$ . With the aid of (4.3), we see that

$$\|z_{Q_j}\|_2 \leq \frac{\|z_{Q_j}\|_1}{\sqrt{t}} + \frac{\sqrt{t}}{4} (z_{(j-1)t+1} - z_{jt}), \quad j = 1, \dots, r_1, \quad (4.5)$$

and

$$\|z_{Q_{r_1+1}}\|_2 \leq \frac{\|z_{Q_{r_1+1}}\|_1}{\sqrt{t}} + \frac{\sqrt{t}}{4} z_{r_1t+1}, \quad (4.6)$$

which is ensured under the condition  $a_1 = z_{r_1t+1}, \dots, a_{r_2} = z_{r_1t+r_2}, a_{r_2+1} = \dots = a_t = 0$ . Merging (4.5) with (4.6), one has

$$\|z\|_2 = \left\| \sum_{j=1}^{r_1+1} z_{Q_j} \right\|_2 \leq \sum_{j=1}^{r_1+1} \|z_{Q_j}\|_2 \leq \frac{1}{\sqrt{t}} \sum_{j=1}^{r_1+1} \|z_{Q_j}\|_1 + \frac{\sqrt{t}}{4} \mu$$

with

$$\mu := \sum_{j=1}^{r_1} (z_{(j-1)t+1} - z_{jt}) + z_{r_1t+1} = z_1 - \sum_{j=1}^{r_1} (z_{jt} - z_{jt+1}) \leq z_1,$$

where the inequality is resulted from  $z_1 \geq z_2 \geq \dots \geq z_r$ . It follows that

$$\|z\|_2 \leq \frac{1}{\sqrt{t}} \|z\|_1 + \frac{\sqrt{t}}{4} z_1 \leq \frac{1}{\sqrt{t}} \zeta_1 + \frac{\sqrt{t}}{4} \zeta_2 = g(t), \quad (4.7)$$

where the second inequality is ensured by  $\|z\|_1 = \|h\|_1 \leq \zeta_1$  and  $z_1 = \|h\|_\infty \leq \zeta_2$ .

Secondly, we now consider the case  $r_2 = 0$ , which means that  $Q_{r_1+1} = \emptyset$  and  $z = \sum_{j=1}^{r_1} z_{Q_j}$ . Hence, by using (4.3), we obtain

$$\|z\|_2 \leq \sum_{j=1}^{r_1} \|z_{Q_j}\|_2 \leq \frac{1}{\sqrt{t}} \sum_{j=1}^{r_1} \|z_{Q_j}\|_1 + \frac{\sqrt{t}}{4} \sum_{j=1}^{r_1} (z_{(j-1)t+1} - z_{jt}). \quad (4.8)$$

For  $z_1 \geq z_2 \geq \dots \geq z_r \geq 0$ , we have

$$\sum_{j=1}^{r_1} (z_{(j-1)t+1} - z_{jt}) = z_1 - \sum_{j=1}^{r_1-1} (z_{jt} - z_{jt+1}) - z_{r_1} \leq z_1 = \|z\|_\infty. \quad (4.9)$$

Merging (4.8) with (4.9) leads to

$$\|z\|_2 \leq \frac{1}{\sqrt{t}} \|z\|_1 + \frac{\sqrt{t}}{4} \|z\|_\infty \leq \frac{1}{\sqrt{t}} \zeta_1 + \frac{\sqrt{t}}{4} \zeta_2 = g(t). \quad (4.10)$$

Combining (4.4), (4.7), (4.10) with  $\|z\|_q = \|h\|_q$  ( $q \geq 1$ ), we obtain the relation (4.1) directly.  $\square$

Now, we use an example to show that the upper bound of  $\|h\|_2$  in Lemma 6 is tighter than that of Lemma 6.14 in [25] in some situations.

*Example 1* Let  $h = (1, \epsilon_1, \dots, \epsilon_{14}, \epsilon_0)^T \in \mathbb{R}^{16}$ , where  $\epsilon_j \geq \epsilon_0$  ( $j = 1, \dots, 14$ ),  $\sum_{j=1}^{14} \epsilon_j = 1 - \epsilon_0$  and  $\epsilon_0 \in (0, 1/15]$ . Hence,  $\|h\|_1 = 2$  and  $\|h\|_\infty = 1$ . Set  $\zeta_1 = 2$  and  $\zeta_2 = 1$ , then  $t_0 = \frac{4\zeta_1}{\zeta_2} = 8$ . The upper bound of  $\|h\|_2$  can be given by  $\|h\|_2 \leq 1.5 - \epsilon_0$  in (4.3) and  $\|h\|_2 \leq g(8) = \sqrt{2}$  in (4.1), respectively. Since  $1.5 - \epsilon_0 > \sqrt{2}$  for  $\epsilon_0 \in (0, 1/15]$ , we conclude that the upper bound of  $\|h\|_2$  given by (4.1) is tighter than that of (4.3) if  $r > t_0 + 1$  and  $\epsilon_0 = \min_{1 \leq i \leq r} |h_i|$  is small enough.

Taking  $h = (\|w_{\Lambda_1}\|_\infty, \dots, \|w_{\Lambda_q}\|_\infty)^T$  with  $q = \lceil \frac{|A|}{k} \rceil$ , by using Lemma 5, we have  $\|h\|_1 < 2$  and  $\|h\|_\infty = \|w_{\Lambda_1}\|_\infty \leq 1$ . Hence, by setting  $\zeta_1 = 2$  and  $\zeta_2 = 1$ , we get  $t_0 = 8$  in Lemma 6, which results in the following corollary.

**Corollary 2** *Under the conditions of Lemma 5, one has  $\left(\sum_{j=1}^q \|w_{\Lambda_j}\|_\infty^2\right)^{1/2} \leq \xi_q$ , where*

$$\xi_q = \begin{cases} 1, & \text{if } q = 1, \\ \frac{2}{\sqrt{q}} + \frac{\sqrt{q}}{4}, & \text{if } 2 \leq q < 8, \\ \sqrt{2}, & \text{if } q \geq 8, \end{cases} \quad (4.11)$$

which is strictly decreasing in the interval  $[2, 8]$  and  $\max_{q \geq 1} \xi_q = \xi_2 = \frac{5}{4}\sqrt{2}$ .

Using Lemmas 5 and 6, we now establish a helpful inequality in the following lemma.

**Lemma 7** *Let  $x \in \mathbb{R}^n$  be a vector satisfying  $y = Ax + \nu$  where  $\nu$  is a noise vector. Let  $S = \mathcal{L}_k(x)$  and  $V \subseteq N$  be any given index set obeys  $S \subseteq V$ . At the iterate  $x^p$ , the vectors  $w^{(j)}$  ( $j = 1, \dots, \omega$ ) are generated by HBROT $\omega$  or HBOT $P\omega$ . For every  $i \in \{1, \dots, \omega\}$ , we have*

$$\begin{aligned} \Theta^i &:= \|A[(u^p - x_S) \otimes (\bigotimes_{j=1}^i w^{(j)})]_{\overline{V}}\|_2 \\ &\leq \sqrt{1 + \delta_k} \left[ (\xi_q |1 - \alpha + \beta| + 2\alpha\delta_{3k}) \|x^p - x_S\|_2 + \beta \xi_q \|x^{p-1} - x_S\|_2 \right. \\ &\quad \left. + 2\alpha\sqrt{1 + \delta_k} \|\nu'\|_2 \right], \end{aligned} \quad (4.12)$$

where  $q = \lceil \frac{n-|V|}{k} \rceil$  and  $\xi_q$  is given by (4.11).

*Proof* Taking  $w = w^{(1)}$  and  $\Lambda = \overline{V}$  in Lemma 5 and Corollary 2, one has

$$\sum_{j=1}^q \|(w^{(1)})_{\Lambda_j}\|_\infty < 2, \quad \sqrt{\sum_{j=1}^q \|(w^{(1)})_{\Lambda_j}\|_\infty^2} \leq \xi_q, \quad (4.13)$$

where  $q = \lceil \frac{n-|V|}{k} \rceil$  and the definition of  $\Lambda_j$  ( $j = 1, \dots, q$ ) can be found in Lemma 5. Define the  $k$ -sparse vectors  $z^{(l)} := [(u^p - x_S) \otimes (\bigotimes_{j=1}^l w^{(j)})]_{\Lambda_l}$  ( $l = 1, \dots, q$ ), where  $u^p$  is given by (2.2). Since  $w^{(j)} \in \mathcal{P}^k$  ( $j = 1, \dots, \omega$ ), we have

$$\|z^{(l)}\|_2 \leq \|(\bigotimes_{j=1}^l w^{(j)})_{\Lambda_l}\|_\infty \cdot \|(u^p - x_S)_{\Lambda_l}\|_2 \leq \|(w^{(1)})_{\Lambda_l}\|_\infty \cdot \|(u^p - x_S)_{\Lambda_l}\|_2, \quad (4.14)$$

where the second inequality follows from  $0 \leq w^{(j)} \leq \mathbf{e}$  for  $j = 1, \dots, \omega$ . Since  $z^{(l)} (l = 1, \dots, q)$  are  $k$ -sparse vectors, from the definition of  $\Theta^i$  in (4.12) and (4.14), we obtain

$$\begin{aligned} \Theta^i &= \|A \sum_{l=1}^q z^{(l)}\|_2 \leq \sum_{l=1}^q \|Az^{(l)}\|_2 \leq \sqrt{1 + \delta_k} \sum_{l=1}^q \|z^{(l)}\|_2 \\ &\leq \sqrt{1 + \delta_k} \sum_{l=1}^q \|(w^{(1)})_{A_l}\|_\infty \cdot \|(u^p - x_S)_{A_l}\|_2, \end{aligned} \quad (4.15)$$

where the second inequality is given by (2.1). Since  $|A_l| \leq k$  and  $|\text{supp}(x^p - x_S) \cup A_l| \leq 3k (l = 1, \dots, q)$ , by using (3.10) and Lemma 1, we have

$$\begin{aligned} \|(u^p - x_S)_{A_l}\|_2 &\leq |1 - \alpha + \beta| \cdot \|(x^p - x_S)_{A_l}\|_2 + \alpha \|(I - A^T A)(x^p - x_S)_{A_l}\|_2 \\ &\quad + \beta \|(x^{p-1} - x_S)_{A_l}\|_2 + \alpha \|(A^T \nu')_{A_l}\|_2 \\ &\leq |1 - \alpha + \beta| \cdot \|(x^p - x_S)_{A_l}\|_2 + \alpha \delta_{3k} \|x^p - x_S\|_2 \\ &\quad + \beta \|(x^{p-1} - x_S)_{A_l}\|_2 + \alpha \sqrt{1 + \delta_k} \|\nu'\|_2. \end{aligned} \quad (4.16)$$

Substituting (4.16) into (4.15) yields

$$\begin{aligned} \frac{\Theta^i}{\sqrt{1 + \delta_k}} &\leq |1 - \alpha + \beta| \cdot \sum_{l=1}^q \|(w^{(1)})_{A_l}\|_\infty \cdot \|(x^p - x_S)_{A_l}\|_2 \\ &\quad + \alpha \delta_{3k} \sum_{l=1}^q \|(w^{(1)})_{A_l}\|_\infty \cdot \|x^p - x_S\|_2 \\ &\quad + \beta \sum_{l=1}^q \|(w^{(1)})_{A_l}\|_\infty \cdot \|(x^{p-1} - x_S)_{A_l}\|_2 \\ &\quad + \alpha \sqrt{1 + \delta_k} \sum_{l=1}^q \|(w^{(1)})_{A_l}\|_\infty \cdot \|\nu'\|_2. \end{aligned}$$

It follows from Cauchy-Schwarz inequality and (4.13) that

$$\begin{aligned} &\frac{\Theta^i}{\sqrt{1 + \delta_k}} \\ &\leq |1 - \alpha + \beta| \sqrt{\sum_{l=1}^q \|(w^{(1)})_{A_l}\|_\infty^2} \sqrt{\sum_{l=1}^q \|(x^p - x_S)_{A_l}\|_2^2} + 2\alpha \delta_{3k} \|x^p - x_S\|_2 \\ &\quad + \beta \sqrt{\sum_{l=1}^q \|(w^{(1)})_{A_l}\|_\infty^2} \sqrt{\sum_{l=1}^q \|(x^{p-1} - x_S)_{A_l}\|_2^2} + 2\alpha \sqrt{1 + \delta_k} \|\nu'\|_2 \\ &\leq |1 - \alpha + \beta| \xi_q \|x^p - x_S\|_2 + 2\alpha \delta_{3k} \|x^p - x_S\|_2 + \beta \xi_q \|x^{p-1} - x_S\|_2 \\ &\quad + 2\alpha \sqrt{1 + \delta_k} \|\nu'\|_2, \end{aligned}$$

where the last inequality can be obtained from the following relation  $\sum_{j=1}^q \|z_{A_l}\|_2^2 = \|z_{\overline{V}}\|_2^2 \leq \|z\|_2^2$  for any  $z \in \mathbb{R}^n$  due to  $\overline{V} = \bigcup_{j=1}^q A_j$  and  $A_j \cap A_l = \emptyset (j \neq l)$ . Thus (4.12) holds.  $\square$

We now establish the estimation of  $\|y - A[u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)})]\|_2$  by using Lemma 7.

**Lemma 8** *Let  $x \in \mathbb{R}^n$  be a vector satisfying  $y = Ax + \nu$  where  $\nu$  is a noise vector. At the iterate  $x^p$ , the vectors  $u^p$  and  $w^{(j)}$  ( $j = 1, \dots, \omega$ ) are generated by HBROT $\omega$  or HBROT $P\omega$ . Then*

$$\begin{aligned} & \|y - A[u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)})]\|_2 \\ & \leq c_{1,q} \sqrt{1 + \delta_k} \|x^p - x_S\|_2 + \sqrt{1 + \delta_k} \beta [\xi_q(\omega - 1) + 1] \|x^{p-1} - x_S\|_2 \\ & \quad + [\alpha(2\omega - 1)(1 + \delta_k) + 1] \|\nu'\|_2, \end{aligned} \quad (4.17)$$

where  $S := \mathcal{L}_k(x)$ ,  $q = \lceil \frac{n-k}{k} \rceil$ ,  $\xi_q$  is given by (4.11) and  $c_{1,q}$  is given as

$$c_{1,q} := (\xi_q(\omega - 1) + 1)|1 - \alpha + \beta| + \alpha(2(\omega - 1)\delta_{3k} + \delta_{2k}). \quad (4.18)$$

*Proof* Let  $\hat{w} \in \mathcal{W}^k$  be a binary vector satisfying  $\text{supp}(x_S) \subseteq \text{supp}(\hat{w})$  and  $V = \text{supp}(\hat{w})$ . From Lemma 4.3 in [51], we get

$$\begin{aligned} & \|y - A[u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)})]\|_2 \\ & \leq \|y - A(u^p \otimes \hat{w})\|_2 + \sum_{i=1}^{\omega-1} \|A[(u^p - x_S) \otimes (\bigotimes_{j=1}^i w^{(j)}) \otimes (\mathbf{e} - \hat{w})]\|_2. \end{aligned} \quad (4.19)$$

As  $V = \text{supp}(\hat{w})$  and  $|V| = k$ , it follows from (4.12) that

$$\begin{aligned} & \|A[(u^p - x_S) \otimes (\bigotimes_{j=1}^i w^{(j)}) \otimes (\mathbf{e} - \hat{w})]\|_2 = \|A[(u^p - x_S) \otimes (\bigotimes_{j=1}^i w^{(j)})]_{\overline{\text{supp}(\hat{w})}}\|_2 \\ & \leq \sqrt{1 + \delta_k} \left[ (\xi_q|1 - \alpha + \beta| + 2\alpha\delta_{3k}) \|x^p - x_S\|_2 + \beta\xi_q \|x^{p-1} - x_S\|_2 \right. \\ & \quad \left. + 2\alpha\sqrt{1 + \delta_k} \|\nu'\|_2 \right], \end{aligned} \quad (4.20)$$

where  $q = \lceil \frac{n-k}{k} \rceil$  and  $i = 1, \dots, \omega - 1$ .

We now estimate the term  $\|y - A(u^p \otimes \hat{w})\|_2$  in (4.19). Because  $|\text{supp}(x^p - x_S) \cup \text{supp}(\hat{w})| \leq 2k$ , by using (3.12) and Lemma 1, we obtain

$$\begin{aligned} & \|(x_S - u^p) \otimes \hat{w}\|_2 \\ & \leq |1 - \alpha + \beta| \cdot \|x^p - x_S\|_2 + \alpha \|(I - A^T A)(x^p - x_S)\|_{\text{supp}(\hat{w})} \\ & \quad + \beta \|x^{p-1} - x_S\|_2 + \alpha \|(A^T \nu')_{\text{supp}(\hat{w})}\|_2 \\ & \leq (|1 - \alpha + \beta| + \alpha\delta_{2k}) \|x^p - x_S\|_2 + \beta \|x^{p-1} - x_S\|_2 + \alpha\sqrt{1 + \delta_k} \|\nu'\|_2. \end{aligned}$$

It follows from (3.11) that

$$\begin{aligned} \|y - A(u^p \otimes \hat{w})\|_2 & \leq \sqrt{1 + \delta_k} \left[ (|1 - \alpha + \beta| + \alpha\delta_{2k}) \|x^p - x_S\|_2 \right. \\ & \quad \left. + \beta \|x^{p-1} - x_S\|_2 + \alpha\sqrt{1 + \delta_k} \|\nu'\|_2 \right] + \|\nu'\|_2. \end{aligned} \quad (4.21)$$

Combining (4.20), (4.21) with (4.19) yields (4.17).  $\square$

The following property of the hard thresholding operator  $\mathcal{H}_k(\cdot)$  is shown in [51, Lemma 4.1].

**Lemma 9** [51] *Let  $z, h \in \mathbb{R}^n$  be two vectors and  $\|h\|_0 \leq k$ . Then*

$$\|h - \mathcal{H}_k(z)\|_2 \leq \|(z - h)_{S \cup S^*}\|_2 + \|(z - h)_{S^* \setminus S}\|_2,$$

where  $S := \text{supp}(h)$  and  $S^* := \text{supp}(\mathcal{H}_k(z))$ .

Let us state a fundamental property of the orthogonal projection in the following lemma, which can be found in [24, Eq.(3.21)] and [48, p.49], and was extended to the general case in [50, Lemma 4.2].

**Lemma 10** [24, 48, 50] *Let  $x \in \mathbb{R}^n$  be a vector satisfying  $y = Ax + \nu$  where  $\nu$  is a noise vector. Let  $S^* \subseteq N$  be an index set satisfying  $|S^*| \leq k$  and*

$$z^* = \arg \min_{z \in \mathbb{R}^n} \{\|y - Az\|_2^2 : \text{supp}(z) \subseteq S^*\}.$$

Then

$$\|z^* - x_S\|_2 \leq \frac{1}{\sqrt{1 - (\delta_{2k})^2}} \|(z^* - x_S)_{\overline{S^*}}\|_2 + \frac{\sqrt{1 + \delta_k}}{1 - \delta_{2k}} \|\nu'\|_2,$$

where  $S := \mathcal{L}_k(x)$  and  $\nu' := \nu + Ax_{\overline{S}}$ .

We now establish the error bounds for HBROT $\omega$  and HBROTP $\omega$ .

**Theorem 2** *Suppose that  $n > 3k$  and denote  $\sigma := \lceil \frac{n-2k}{k} \rceil$ . Let  $x \in \mathbb{R}^n$  be a vector satisfying  $y = Ax + \nu$  where  $\nu$  is a noise vector. Denote*

$$t_k := \frac{\sqrt{1 + \delta_k}}{\sqrt{1 - \delta_{2k}}}, \quad z_k := \sqrt{1 - \delta_{2k}^2}. \quad (4.22)$$

- (i) *Assume that the  $(3k)$ -th order RIC,  $\delta_{3k}$ , of the matrix  $A$  and the pair  $(\alpha, \beta)$  satisfy  $\delta_{3k} < \gamma^*(\omega)$  and*

$$0 \leq \beta < \frac{1 - d_1}{1 + d_1 + d_2}, \quad \frac{(d_0 + d_2 + 2)\beta + d_0}{d_0 - d_1 + 1} < \alpha < \frac{d_0 + 2 - (d_2 - d_0)\beta}{d_0 + d_1 + 1}, \quad (4.23)$$

where  $\gamma^*(\omega)$  is the unique root of the equation  $G_\omega(\gamma) = 1$  in the interval  $(0, 1)$ , where

$$G_\omega(\gamma) := (2\omega + 1)\gamma \sqrt{\frac{1 + \gamma}{1 - \gamma}} + \gamma, \quad (4.24)$$

and the constants  $d_0, d_1, d_2$  are given as

$$\begin{cases} d_0 := t_k(\omega \xi_\sigma + 1), \\ d_1 := t_k(2\omega \delta_{3k} + \delta_{2k}) + \delta_{3k}, \\ d_2 := t_k[\xi_\sigma(\omega - 1) + 1] \frac{2\omega \delta_{3k} + \delta_{2k}}{2(\omega - 1)\delta_{3k} + \delta_{2k}}. \end{cases} \quad (4.25)$$

Then, the sequence  $\{x^p\}$  produced by HBROT $\omega$  obeys

$$\|x^p - x_S\|_2 \leq \theta_1^{p-1} \left[ \|x^1 - x_S\|_2 + (\theta_1 - b_1) \|x^0 - x_S\|_2 \right] + \frac{b_3}{1 - \theta_1} \|\nu'\|_2 \quad (4.26)$$



with  $\theta_1 := \frac{b_1 + \sqrt{b_1^2 + 4b_2}}{2}$ . The fact  $\theta_1 < 1$  is ensured under (4.23) and  $b_1, b_2, b_3$  are given as

$$\begin{aligned} b_1 &:= t_k c_\sigma + (|1 + \beta - \alpha| + \alpha \delta_{3k}), \quad b_2 := \beta t_k [\xi_\sigma(\omega - 1) + 1] \frac{c_\sigma}{c_{1,\sigma}} + \beta, \\ b_3 &:= \frac{\alpha(2\omega - 1)(1 + \delta_k) + 2}{\sqrt{1 - \delta_{2k}}} \cdot \frac{2\delta_{3k}}{2(\omega - 1)\delta_{3k} + \delta_{2k}} \cdot \frac{c_\sigma}{c_\sigma - c_{1,\sigma}} + \alpha\sqrt{1 + \delta_k}, \end{aligned} \quad (4.27)$$

where  $c_{1,\sigma}$  and  $\xi_\sigma$  are given by (4.18) and (4.11), respectively, and

$$c_\sigma := (\omega \xi_\sigma + 1)|1 - \alpha + \beta| + \alpha(2\omega \delta_{3k} + \delta_{2k}). \quad (4.28)$$

- (ii) Suppose that the  $(3k)$ -th order RIC,  $\delta_{3k}$ , of the matrix  $A$  and the pair  $(\alpha, \beta)$  satisfy  $\delta_{3k} < \gamma^\sharp(\omega)$  and

$$0 \leq \beta < \frac{z_k - d_1}{1 + d_1 + d_2}, \quad \frac{(d_0 + d_2 + 2)\beta + d_0 + 1 - z_k}{d_0 - d_1 + 1} < \alpha < \frac{d_0 + 1 + z_k - (d_2 - d_0)\beta}{d_0 + d_1 + 1}, \quad (4.29)$$

where the constants  $d_0, d_1, d_2$  are given by (4.25) and  $\gamma^\sharp(\omega)$  is the unique root of the equation  $\frac{1}{\sqrt{1 - \gamma^2}} G_\omega(\gamma) = 1$  in the interval  $(0, 1)$ , where  $G_\omega(\gamma)$  is given by (4.24). Then, the sequence  $\{x^p\}$  produced by HBROTP $\omega$  obeys

$$\begin{aligned} \|x^p - x_S\|_2 &\leq \theta_2^{p-1} \left[ \|x^1 - x_S\|_2 + \left(\theta_2 - \frac{b_1}{z_k}\right) \|x^0 - x_S\|_2 \right] \\ &\quad + \frac{1}{1 - \theta_2} \left( \frac{b_3}{z_k} + \frac{\sqrt{1 + \delta_k}}{1 - \delta_{2k}} \right) \|\nu'\|_2 \end{aligned} \quad (4.30)$$

with  $\theta_2 := \frac{b_1 + \sqrt{b_1^2 + 4b_2 z_k}}{2z_k}$ . The fact  $\theta_2 < 1$  is ensured under (4.29) and the constants  $b_i (i = 1, 2, 3)$  and  $z_k$  are given by (4.27) and (4.22), respectively.

*Proof* For  $x^\sharp = \mathcal{H}_k(u^p \otimes (\bigotimes_{j=1}^\omega w^{(j)}))$  generated by the Algorithms, by using Lemma 9, we have

$$\|x_S - x^\sharp\|_2 \leq \|[u^p \otimes (\bigotimes_{j=1}^\omega w^{(j)}) - x_S]_{X \cup S}\|_2 + \|[u^p \otimes (\bigotimes_{j=1}^\omega w^{(j)}) - x_S]_{X \setminus S}\|_2, \quad (4.31)$$

where  $X = \text{supp}(x^\sharp)$  and  $w^{(j)} (j = 1, \dots, \omega)$  are given by (2.5). Using (3.10) and the triangle inequality, we have that

$$\begin{aligned} &\|[u^p \otimes (\bigotimes_{j=1}^\omega w^{(j)}) - x_S]_{X \setminus S}\|_2 \\ &= \|[u^p - x_S] \otimes (\bigotimes_{j=1}^\omega w^{(j)})\|_{X \setminus S} \leq \|[u^p - x_S]_{X \setminus S}\|_2 \\ &\leq |1 - \alpha + \beta| \cdot \|[x^p - x_S]_{X \setminus S}\|_2 + \alpha \|(I - A^T A)(x^p - x_S)\|_{X \setminus S} \\ &\quad + \beta \|[x^{p-1} - x_S]_{X \setminus S}\|_2 + \alpha \|(A^T \nu')_{X \setminus S}\|_2, \end{aligned}$$

where the first equality is ensured by  $(x_S)_{X \setminus S} = 0$  and the first inequality is due to  $0 \leq w^{(j)} \leq \mathbf{e} (j = 1, \dots, \omega)$ . Since  $|X \setminus S| \leq k$  and  $|\text{supp}(x^p - x_S) \cup (X \setminus S)| \leq 3k$ , by using Lemma 1, we see that

$$\begin{aligned} \| [u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)}) - x_S]_{X \setminus S} \|_2 &\leq (|1 + \beta - \alpha| + \alpha \delta_{3k}) \|x^p - x_S\|_2 \\ &\quad + \beta \|x^{p-1} - x_S\|_2 + \alpha \sqrt{1 + \delta_k} \|\nu'\|_2. \end{aligned} \quad (4.32)$$

Denote

$$\Theta_1 := \|A[u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)}) - x_S]_{X \cup S}\|_2, \quad \Theta_2 := \|A[u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)}) - x_S]_{\overline{X \cup S}}\|_2. \quad (4.33)$$

As  $|X \cup S| \leq 2k$ , by using (2.1), we obtain

$$\Theta_1 \geq \sqrt{1 - \delta_{2k}} \| (u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)}) - x_S)_{X \cup S} \|_2. \quad (4.34)$$

For any given  $\zeta \in (0, 1)$ , we consider the following two cases associated with  $\Theta_1$  and  $\Theta_2$ .

**Case 1.**  $\Theta_2 \leq \zeta \Theta_1$ . Since  $y = Ax_S + \nu'$ , by the triangle inequality and (4.33), we have

$$\begin{aligned} \|y - A[u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)})]\|_2 &= \|A[u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)}) - x_S] - \nu'\|_2 \\ &= \|A[u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)}) - x_S]_{X \cup S} + A[u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)}) - x_S]_{\overline{X \cup S}} - \nu'\|_2 \\ &\geq \Theta_1 - \Theta_2 - \|\nu'\|_2 \\ &\geq (1 - \zeta)\Theta_1 - \|\nu'\|_2. \end{aligned} \quad (4.35)$$

Merging (4.34), (4.35) with (4.17) yields

$$\begin{aligned} &\| [u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)}) - x_S]_{X \cup S} \|_2 \\ &\leq \frac{1}{(1 - \zeta)\sqrt{1 - \delta_{2k}}} (\|y - A[u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)})]\|_2 + \|\nu'\|_2) \\ &\leq \frac{t_k c_{1, q_1}}{1 - \zeta} \|x^p - x_S\|_2 + \frac{\beta t_k}{1 - \zeta} [\xi_{q_1}(\omega - 1) + 1] \|x^{p-1} - x_S\|_2 \\ &\quad + \frac{\alpha(2\omega - 1)(1 + \delta_k) + 2}{(1 - \zeta)\sqrt{1 - \delta_{2k}}} \|\nu'\|_2, \end{aligned} \quad (4.36)$$

where  $q_1 = \lceil \frac{n-k}{k} \rceil = \sigma + 1$  and  $t_k, c_{1, q_1}$  are given in (4.22) and (4.18), respectively.

**Case 2.**  $\Theta_2 > \zeta \Theta_1$ . From (4.33) and (4.34), we obtain

$$\| [u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)}) - x_S]_{X \cup S} \|_2 \leq \frac{1}{\zeta \sqrt{1 - \delta_{2k}}} \|A[(u^p - x_S) \otimes (\bigotimes_{j=1}^{\omega} w^{(j)})]_{\overline{X \cup S}}\|_2. \quad (4.37)$$

Taking  $V = X \cup S$  and  $i = \omega$  in (4.12), one has

$$\begin{aligned} & \|A[(u^p - x_S) \otimes (\bigotimes_{j=1}^{\omega} w^{(j)})]_{X \cup S}\|_2 \\ & \leq \sqrt{1 + \delta_k} \left[ c_{2,q_2} \|x^p - x_S\|_2 + \beta \xi_{q_2} \|x^{p-1} - x_S\|_2 + 2\alpha \sqrt{1 + \delta_k} \|\nu'\|_2 \right], \end{aligned} \quad (4.38)$$

with  $q_2 = \lceil \frac{n-|X \cup S|}{k} \rceil \geq \sigma$  which is due to  $|X \cup S| \leq 2k$ , and  $c_{2,q_2}$  is defined as

$$c_{2,q_2} := \xi_{q_2} |1 - \alpha + \beta| + 2\alpha \delta_{3k}. \quad (4.39)$$

Substituting (4.38) into (4.37), we get

$$\begin{aligned} \| [u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)}) - x_S]_{X \cup S} \|_2 & \leq \frac{t_k}{\zeta} [c_{2,q_2} \|x^p - x_S\|_2 + \beta \xi_{q_2} \|x^{p-1} - x_S\|_2 \\ & \quad + 2\alpha \sqrt{1 + \delta_k} \|\nu'\|_2]. \end{aligned} \quad (4.40)$$

From (4.11), we see that  $\xi_q$  is decreasing in  $[2, n]$ . For  $q_1 = \sigma + 1$  and  $q_2 \geq \sigma \geq 2$ , we have  $\xi_{q_1}, \xi_{q_2} \leq \xi_{\sigma}$ . It follows from (4.18) and (4.39) that  $c_{1,q_1} \leq c_{1,\sigma}$  and  $c_{2,q_2} \leq c_{2,\sigma}$ . Combining (4.36) and (4.40) leads to

$$\begin{aligned} & \| (u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)}) - x_S)_{X \cup S} \|_2 \\ & \leq t_k \max \left\{ \frac{c_{1,\sigma}}{1 - \zeta}, \frac{c_{2,\sigma}}{\zeta} \right\} \|x^p - x_S\|_2 \\ & \quad + \beta t_k \max \left\{ \frac{\xi_{\sigma}(\omega - 1) + 1}{1 - \zeta}, \frac{\xi_{\sigma}}{\zeta} \right\} \|x^{p-1} - x_S\|_2 \\ & \quad + \frac{1}{\sqrt{1 - \delta_{2k}}} \max \left\{ \frac{\alpha(2\omega - 1)(1 + \delta_k) + 2}{1 - \zeta}, \frac{2\alpha(1 + \delta_k)}{\zeta} \right\} \|\nu'\|_2 \end{aligned} \quad (4.41)$$

for any  $\zeta \in (0, 1)$ .

Next, we select a suitable parameter  $\zeta \in (0, 1)$  such that the right hand of (4.41) is as small as possible. For  $\delta_{2k} \leq \delta_{3k}$  and  $\xi_{\sigma} < 2$  in (4.11), we have

$$\frac{c_{2,\sigma}}{c_{1,\sigma}} = \frac{\xi_{\sigma} |1 - \alpha + \beta| + 2\alpha \delta_{3k}}{(\xi_{\sigma}(\omega - 1) + 1) |1 - \alpha + \beta| + \alpha[2(\omega - 1)\delta_{3k} + \delta_{2k}]} \leq \frac{2\delta_{3k}}{2(\omega - 1)\delta_{3k} + \delta_{2k}}. \quad (4.42)$$

It is easy to check that

$$\min_{\zeta \in (0,1)} \max \left\{ \frac{c_{1,\sigma}}{1 - \zeta}, \frac{c_{2,\sigma}}{\zeta} \right\} = c_{1,\sigma} + c_{2,\sigma} = c_{\sigma}, \quad (4.43)$$

where  $c_{\sigma}$  is given by (4.28) and its minimum attains at

$$\zeta^* = \frac{c_{2,\sigma}}{c_{1,\sigma} + c_{2,\sigma}} = \frac{\xi_{\sigma} |1 - \alpha + \beta| + 2\alpha \delta_{3k}}{(\omega \xi_{\sigma} + 1) |1 - \alpha + \beta| + \alpha(2\omega \delta_{3k} + \delta_{2k})}. \quad (4.44)$$

That is,

$$\max \left\{ \frac{c_{1,\sigma}}{1 - \zeta^*}, \frac{c_{2,\sigma}}{\zeta^*} \right\} = c_{\sigma}. \quad (4.45)$$

Moreover, noting that  $\xi_\sigma < 2$  and  $\delta_{2k} \leq \delta_{3k}$ , we have  $\zeta^* \geq \frac{\xi_\sigma}{\omega\xi_\sigma+1}$ . In particular, by taking  $\zeta = \zeta^*$  in (4.41), we deduce that

$$\max \left\{ \frac{\xi_\sigma(\omega-1)+1}{1-\zeta^*}, \frac{\xi_\sigma}{\zeta^*} \right\} = \frac{\xi_\sigma(\omega-1)+1}{1-\zeta^*} = \frac{\xi_\sigma(\omega-1)+1}{c_{1,\sigma}} c_\sigma,$$

and

$$\begin{aligned} & \max \left\{ \frac{\alpha(2\omega-1)(1+\delta_k)+2}{1-\zeta^*}, \frac{2\alpha(1+\delta_k)}{\zeta^*} \right\} \\ &= \frac{1}{\zeta^*} \max \left\{ [\alpha(2\omega-1)(1+\delta_k)+2] \frac{\zeta^*}{1-\zeta^*}, 2\alpha(1+\delta_k) \right\} \\ &= \frac{c_\sigma}{c_{2,\sigma}} \max \left\{ [\alpha(2\omega-1)(1+\delta_k)+2] \frac{c_{2,\sigma}}{c_{1,\sigma}}, 2\alpha(1+\delta_k) \right\} \\ &\leq \frac{c_\sigma}{c_{2,\sigma}} \max \left\{ [\alpha(2\omega-1)(1+\delta_k)+2] \frac{2\delta_{3k}}{2(\omega-1)\delta_{3k}+\delta_{2k}}, 2\alpha(1+\delta_k) \right\} \\ &= \left[ \alpha(2\omega-1)(1+\delta_k)+2 \right] \frac{2\delta_{3k}}{2(\omega-1)\delta_{3k}+\delta_{2k}} \cdot \frac{c_\sigma}{c_{2,\sigma}}, \end{aligned} \quad (4.46)$$

where the second equality is given by (4.44), the inequality above follows from (4.42), and the last equality holds owing to  $\delta_{2k} \leq \delta_{3k}$ . Merging (4.41) with (4.45)-(4.46), we obtain

$$\begin{aligned} & \| [u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)}) - x_S]_{X \cup S} \|_2 \\ &\leq t_k c_\sigma \|x^p - x_S\|_2 + \beta t_k \frac{\xi_\sigma(\omega-1)+1}{c_{1,\sigma}} c_\sigma \|x^{p-1} - x_S\|_2 \\ &\quad + \frac{\alpha(2\omega-1)(1+\delta_k)+2}{\sqrt{1-\delta_{2k}}} \cdot \frac{2\delta_{3k}}{2(\omega-1)\delta_{3k}+\delta_{2k}} \cdot \frac{c_\sigma}{c_{2,\sigma}} \|\nu'\|_2. \end{aligned} \quad (4.47)$$

Combining (4.32), (4.47) with (4.31), we have

$$\|x_S - x^\sharp\|_2 \leq b_1 \|x^p - x_S\|_2 + b_2 \|x^{p-1} - x_S\|_2 + b_3 \|\nu'\|_2, \quad (4.48)$$

where the constants  $b_1, b_2, b_3$  are given by (4.27).

Next, we estimate  $\|x^{p+1} - x_S\|_2$  for HBROT $\omega$  and HBROTP $\omega$  based on the relation (4.48).

(i) Since  $x^{p+1} = x^\sharp$  in HBROT $\omega$ , (4.48) becomes

$$\|x^{p+1} - x_S\|_2 \leq b_1 \|x^p - x_S\|_2 + b_2 \|x^{p-1} - x_S\|_2 + b_3 \|\nu'\|_2. \quad (4.49)$$

Now, we consider the conditions of Lemma 2. Merging (4.42) with (4.43) produces

$$\frac{c_\sigma}{c_{1,\sigma}} \leq \frac{2\omega\delta_{3k}+\delta_{2k}}{2(\omega-1)\delta_{3k}+\delta_{2k}}.$$

It follows from (4.27) and (4.28) that

$$\begin{aligned} b_1 + b_2 &\leq t_k c_\sigma + (|1 + \beta - \alpha| + \alpha\delta_{3k}) \\ &\quad + \left\{ t_k \frac{2\omega\delta_{3k}+\delta_{2k}}{2(\omega-1)\delta_{3k}+\delta_{2k}} [\xi_\sigma(\omega-1)+1] + 1 \right\} \beta = F(\alpha, \beta), \end{aligned} \quad (4.50)$$

where

$$\begin{aligned} F(\alpha, \beta) &:= (d_0 + 1)|1 - \alpha + \beta| + d_1\alpha + (d_2 + 1)\beta, \\ &= \begin{cases} -(d_0 - d_1 + 1)\alpha + (d_0 + d_2 + 2)\beta + d_0 + 1, & \text{if } \alpha \leq 1 + \beta, \\ (d_0 + d_1 + 1)\alpha + (d_2 - d_0)\beta - (d_0 + 1), & \text{if } \alpha > 1 + \beta, \end{cases} \end{aligned} \quad (4.51)$$

with the constants  $d_0, d_1, d_2$  are given by (4.25).

Based on the fact  $\delta_k \leq \delta_{2k} \leq \delta_{3k} < \gamma^*(\omega)$  and the function  $G_\omega(\gamma)$  in (4.24) is strictly increasing in the interval  $(0, 1)$ , by using (4.25), we have

$$d_1 \leq [(2\omega + 1)t_k + 1]\delta_{3k} \leq G_\omega(\delta_{3k}) < G_\omega(\gamma^*(\omega)) = 1, \quad (4.52)$$

which shows that the range of  $\beta$  in (4.23) is well defined. From the first inequality in (4.23), we see that

$$\frac{(d_0 + d_2 + 2)\beta + d_0}{d_0 - d_1 + 1} < 1 + \beta < \frac{d_0 + 2 - (d_2 - d_0)\beta}{d_0 + d_1 + 1}, \quad (4.53)$$

which implies that the range of  $\alpha$  in (4.23) is also well defined. Merging (4.51)-(4.53) with the second inequality in (4.23), we see that if  $\frac{(d_0 + d_2 + 2)\beta + d_0}{d_0 - d_1 + 1} < \alpha \leq 1 + \beta$ , then

$$F(\alpha, \beta) < -(d_0 - d_1 + 1)\frac{(d_0 + d_2 + 2)\beta + d_0}{d_0 - d_1 + 1} + (d_0 + d_2 + 2)\beta + d_0 + 1 = 1,$$

and if  $1 + \beta < \alpha < \frac{d_0 + 2 - (d_2 - d_0)\beta}{d_0 + d_1 + 1}$ , then

$$F(\alpha, \beta) < (d_0 + d_1 + 1)\frac{d_0 + 2 - (d_2 - d_0)\beta}{d_0 + d_1 + 1} + (d_2 - d_0)\beta - (d_0 + 1) = 1.$$

It follows from (4.50) that  $b_1 + b_2 < 1$ . Hence, applying Lemma 2 to the relation (4.49), we conclude that (4.26) holds with  $\theta_1 = \frac{b_1 + \sqrt{b_1^2 + 4b_2}}{2} < 1$ .

(ii) Since  $x^{p+1}$  is given by (2.6) and  $S^{p+1} = \text{supp}(x^\sharp)$  in HBROTP $\omega$ , by setting  $S^* = S^{p+1}$  and  $z^* = x^{p+1}$  in Lemma 10, we have

$$\begin{aligned} \|x^{p+1} - x_S\|_2 &\leq \frac{1}{z_k} \|(x^\sharp - x_S)_{\overline{S^{p+1}}}\|_2 + \frac{\sqrt{1 + \delta_k}}{1 - \delta_{2k}} \|\nu'\|_2 \\ &\leq \frac{1}{z_k} \|x^\sharp - x_S\|_2 + \frac{\sqrt{1 + \delta_k}}{1 - \delta_{2k}} \|\nu'\|_2, \end{aligned} \quad (4.54)$$

where  $z_k$  is given in (4.22) and the first inequality follows from the fact  $(x^{p+1})_{\overline{S^{p+1}}} = (x^\sharp)_{\overline{S^{p+1}}} = 0$ . Combining (4.54) with (4.48), we have

$$\|x^{p+1} - x_S\|_2 \leq \frac{b_1}{z_k} \|x^p - x_S\|_2 + \frac{b_2}{z_k} \|x^{p-1} - x_S\|_2 + \left( \frac{b_3}{z_k} + \frac{\sqrt{1 + \delta_k}}{1 - \delta_{2k}} \right) \|\nu'\|_2. \quad (4.55)$$

Similar to the analysis in Part (i), we need to show  $\frac{b_1}{z_k} + \frac{b_2}{z_k} < 1$ .

From (4.29), we have  $\delta_{2k} \leq \delta_{3k} < \gamma^\sharp(\omega)$ . Since the function  $G_\omega(\gamma)$  in (4.24) is strictly increasing in  $(0, 1)$ , one has

$$d_1 \leq G_\omega(\delta_{3k}) < G_\omega(\gamma^\sharp(\omega)) = \sqrt{1 - (\gamma^\sharp(\omega))^2} < \sqrt{1 - (\delta_{2k})^2} = z_k,$$

where the first inequality is given by (4.52), the first equality follows from the fact that  $\gamma^\sharp(\omega)$  is the root of  $\frac{1}{\sqrt{1-\gamma^2}}G_\omega(\gamma) = 1$  in  $(0, 1)$  and the last equality is given by (4.22). It follows that the range of  $\beta$  in (4.29) is well defined. From the first inequality in (4.29), we derive

$$\frac{(d_0 + d_2 + 2)\beta + d_0 + 1 - z_k}{d_0 - d_1 + 1} < 1 + \beta < \frac{d_0 + 1 + z_k - (d_2 - d_0)\beta}{d_0 + d_1 + 1}, \quad (4.56)$$

which means that the range of  $\alpha$  in (4.29) is well defined. Combining (4.51), (4.56) with the second inequality in (4.29) leads to

$$F(\alpha, \beta) < \begin{cases} -(d_0 - d_1 + 1) \frac{(d_0 + d_2 + 2)\beta + d_0 + 1 - z_k}{d_0 - d_1 + 1} + (d_0 + d_2 + 2)\beta + d_0 + 1, \\ \quad \text{if } \frac{(d_0 + d_2 + 2)\beta + d_0 + 1 - z_k}{d_0 - d_1 + 1} < \alpha \leq 1 + \beta, \\ (d_0 + d_1 + 1) \frac{d_0 + 1 + z_k - (d_2 - d_0)\beta}{d_0 + d_1 + 1} + (d_2 - d_0)\beta - (d_0 + 1), \\ \quad \text{if } 1 + \beta < \alpha < \frac{d_0 + 1 + z_k - (d_2 - d_0)\beta}{d_0 + d_1 + 1}, \end{cases}$$

$= z_k.$

It follows from (4.50) that  $\frac{b_1}{z_k} + \frac{b_2}{z_k} < 1$ . Therefore, by Lemma 2, it follows from (4.55) that (4.30) holds with  $\theta_2 = \frac{b_1 + \sqrt{b_1^2 + 4b_2 z_k}}{2z_k} < 1$ .  $\square$

- Remark 1(i)* When  $\nu = 0$  and  $x$  is a  $k$ -sparse vector, from (4.26) and (4.30), we observe that the iterates  $\{x^p\}$  generated by HBROT $\omega$  or HBROTP $\omega$  converges to  $x$ .
- (ii) The condition  $n > 3k$  in Theorem 2 can be removed. If so, the constant  $\xi_\sigma$  will be replaced by  $\max_{q \geq 1} \xi_q = \frac{5}{4}\sqrt{2}$  (see Corollary 2). In addition, if  $n > 9k$ , then  $\sigma = \lceil \frac{n-2k}{k} \rceil \geq 8$ . In this case, we see from (4.11) that  $\xi_\sigma$  in Theorem 2 can be replaced by  $\min_{q \geq 2} \xi_q = \sqrt{2}$ .
- (iii) In the case  $\omega = 1$ , HBROT $\omega$  and HBROTP $\omega$  reduce to HBROT and HBROTP, respectively. In this case, the RIP bounds in Theorem 2 are reduced to  $\delta_{3k} < \gamma^*(1) \approx 0.2118$  for HBROT and  $\delta_{3k} < \gamma^\sharp(1) \approx 0.2079$  for HBROTP.
- (iv) It is not convenient to calculate the RIC of the matrix  $A$  and (4.29) is just a sufficient condition for the theoretical performance of HBROTP $\omega$ . In practical implementation, the parameters  $(\alpha, \beta)$  in HBROTP may be set simply as  $0 \leq \beta < 1/4$  and  $\alpha \geq 1 + \beta$  to roughly meet the conditions (4.29).

## 5 Numerical experiments

Sparse signal and image recovery/reconstruction through measurements  $y = Ax + \nu$ , where  $x$  denotes the original signal or image and  $\nu$  denotes the measurement errors, are typical linear inverse problems. In this section, we provide some experiment results for the HBROTP algorithm proposed in this paper and compare its performance with several existing methods. The experiments are performed on a server with the processor Intel(R) Xeon(R) CPU E5-2680 v3@ 2.50GHz and 256GB memory. All involved convex optimization problems are solved by CVX [26] with solver 'Mosek' [1]. The comparison of five algorithms including HBROTP,

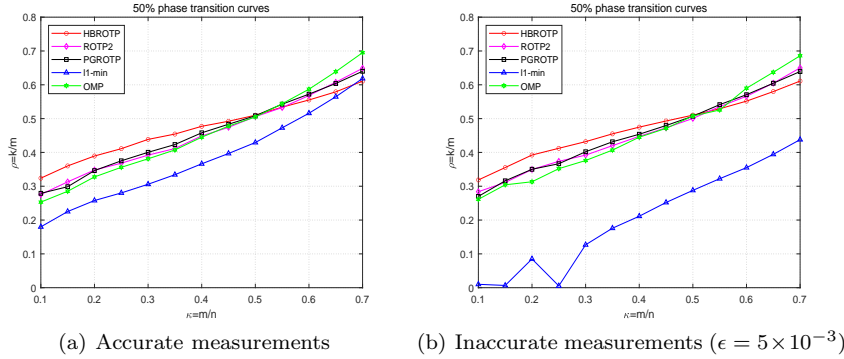
ROTP2, PGROTP,  $\ell_1$ -min and OMP are mainly made via the phase transition with synthetic data and the reconstruction of a few images are shown in sections 5.1 and 5.2, respectively.

### 5.1 Phase transition

The first experiment is carried out to compare the performances of the algorithms through phase transition curves (PTC) [4, 5] and average recovery time. All sparse vectors  $x^* \in \mathbb{R}^n$  and matrices  $A \in \mathbb{R}^{m \times n}$  are randomly generated, and the position of nonzero elements of  $x^*$  follows the uniform distribution. In addition, all columns of  $A$  are normalized and the entries of  $A$  and the nonzeros of  $x^*$  are independent and identically distributed random variables following  $\mathcal{N}(0, 1)$ . In this experiment, we consider both accurate measurements  $y = Ax^*$  and inaccurate measurements  $y = Ax^* + \epsilon h$  with fixed  $n = 1000$ , where  $\epsilon = 5 \times 10^{-3}$  is the noise level and  $h \in \mathbb{R}^m$  is a normalized standard Gaussian noise. We let HBROTP start from  $x^1 = x^0 = 0$  with the fixed parameters  $\alpha = 5$  and  $\beta = 0.2$ , while other algorithms start from  $x^0 = 0$ . The maximum number of iterations of HBROTP, ROTP2 and PGROTP are set as 50, while OMP is performed exactly  $k$  iterations and  $\ell_1$ -min is performed by the solver 'Mosek' directly. Given the random data  $(A, x^*)$  or  $(A, x^*, h)$ , the recovery is counted as 'success' when the criterion

$$\|x^p - x^*\|_2 / \|x^*\|_2 \leq 10^{-3}$$

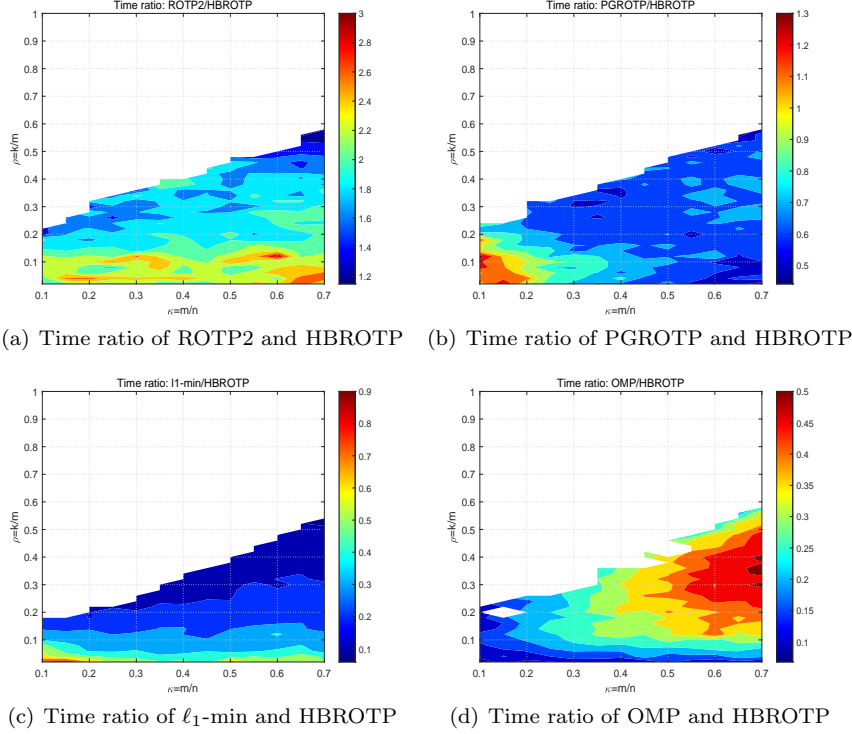
is satisfied, in which  $x^p$  is the approximated solution generated by algorithms.



**Fig. 1** The 50% success rate phase transition curves for algorithms.

Denote by  $\kappa = m/n$  and  $\rho = k/m$ , where  $\kappa$  is often called the sampling rate or the compression ratio. In the  $(\kappa, \rho)$ -space, the region below the PTC is called the 'success' recovery region, where the solution of the problem instances can be exactly or approximately recovered, while the region above the PTC corresponds to the 'failure' region. Thus if the region below the PTC is wider, the performance of an algorithm would be better. We now briefly describe the mechanism for plotting the PTC which is taken as the classical 50% logistic regression curves in this paper, and more detailed information can be found in [4, 5]. To generate the PTC,

13 groups of  $m = \lceil \kappa \cdot n \rceil$  are considered, where the sampling rate  $\kappa$  is ranged from 0.1 to 0.7 with stepsize 0.05. For any given  $m$ , by using the bisection method, the approximated recovery phase transition region  $[k_{\min}, k_{\max}]$  is produced for each algorithm, in which the success rate of recovery is at least 90% as  $k < k_{\min}$  and at most 10% as  $k > k_{\max}$ . The interval  $[k_{\min}, k_{\max}]$  will be equally divided into  $\min\{k_{\max} - k_{\min}, 50\}$  parts, and 10 problem instances are tested for each  $k$  to produce the recovery success rate for given algorithm. Thus PTC can be obtained from the logistic regression model in [4,5] directly.



**Fig. 2** The ratios of average CPU time of the algorithms.

The PTC for the experimented algorithms are shown in Fig. 1(a) and (b), which correspond to the accurate measurements and inaccurate measurements with the noise level  $\epsilon = 5 \times 10^{-3}$ , respectively. The results indicate that HBROTP has the highest PTC as  $\kappa \leq 0.5$ , which shows that the recovery capability of HBROTP is superior to the mentioned a few existing algorithms both in noiseless and the above-described noisy environments. However, the PTC of ROTP2, PGROTP and OMP may approach and might surpass that of HBROTP with a larger value of  $\kappa$ . The comparison between Fig. 1(a) and (b) demonstrates that all algorithms are robust for signal recovery when the measurements are slightly inaccurate except  $\ell_1$ -min. The comparison indicates that the overall performance of HBROTP is very comparable to that of mentioned existing methods.



In the intersection of the recovery regions of multiple algorithms, we compare the average CPU time for recovery of the algorithms. Specifically, for each above-mentioned  $\kappa$ , we test 10 problem instances for each algorithm with the mesh  $(\kappa, \rho)$ , wherein  $\rho$  is ranged from 0.02 to 1 with stepsize 0.02 until the success rate of recovery is less than 90%. The ratios of the average computational time of ROTP2, PGROTP,  $\ell_1$ -min and OMP against that of HBROTP are displayed in Fig. 2 (a)-(d), respectively. Fig. 2 (a) and (b) show that HBROTP is at least 1.6 times faster than ROTP2 in most areas and slower than PGROTP except in the region  $[0.1, 0.2] \times [0.02, 0.1]$ . On the other hand, from Fig. 2 (a)-(d), we observe that the ROT-type algorithms such as HBROTP, ROTP2 and PGROTP are always time-consuming than  $\ell_1$ -min and OMP, due to solving and resolving quadratic convex optimization problems.

## 5.2 Image reconstruction

In this section, we compare the performance of algorithms on the reconstruction of several images (Lena, Peppers and Baboon) of size  $512 \times 512$ . Only accurate measurements are used in the experiment, and the measurement matrices are  $m \times n$  normalized standard Gaussian matrices with  $n = 512$  and  $m = \lceil \kappa \cdot n \rceil$ , where  $\kappa$  is the sampling rate. The discrete wavelet transform with the ‘*sym8*’ wavelet is used to establish the sparse representation of the images. The input sparsity level is set as  $k = \lceil n/10 \rceil$  for HBROTP, ROTP2 and PGROTP. The peak signal-to-noise ratio (PSNR) is used to compare the reconstruction quality of images, which is defined by

$$PSNR := 10 \cdot \log_{10}(V^2/MSE),$$

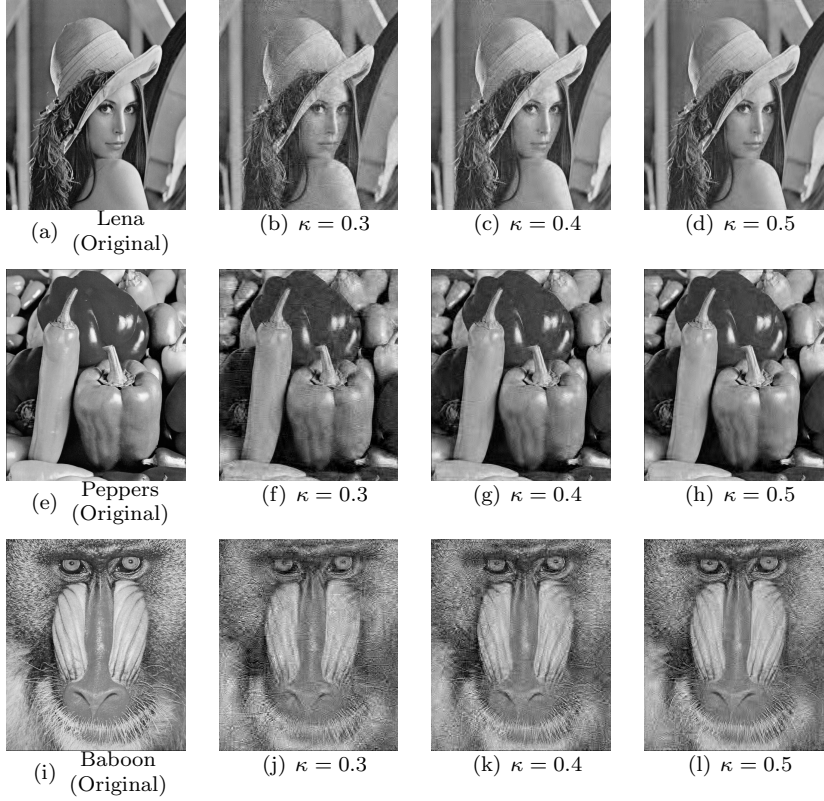
where  $MSE$  denotes the mean-squared error between the reconstructed and original image, and  $V$  represents the maximum fluctuation in the original image data type ( $V = 255$  is used in our experiments). Clearly, the larger the value of PSNR, the higher the reconstruction quality.

**Table 1** Comparison of PSNR (dB) for algorithms with different sampling rates

	$\kappa$	HBROTP	ROTP2	PGROTP	$\ell_1$ -min	OMP
Lena	0.3	32.60	32.34	33.12	33.63	31.37
	0.4	34.37	32.49	31.75	35.10	32.95
	0.5	35.63	33.11	31.93	37.04	34.34
Peppers	0.3	31.31	32.33	33.27	33.03	30.17
	0.4	33.10	31.78	31.60	34.08	31.66
	0.5	34.23	32.04	31.10	35.90	33.38
Baboon	0.3	28.70	31.35	32.33	29.90	28.35
	0.4	29.12	30.06	30.00	30.05	28.53
	0.5	29.37	30.06	30.07	30.20	28.78

The results in terms of PSNR with the sampling rate  $\kappa = 0.3, 0.4, 0.5$  are summarized in Tab. 1, from which we see that HBROTP is always superior to OMP and inferior to  $\ell_1$ -min in reconstruction quality. For ROTP-type algorithms with  $\kappa = 0.4, 0.5$ , the PSNR of HBROTP exceed that of ROTP2, PGROTP at least 1.88 dB for Lena and 1.32 dB for Peppers, respectively. In other cases, ROTP2

and PGROTP obtained better results than HBROTP in reconstruction quality. In particular, the performance of ROTP2 and PGROTP are always equivalent or superior to  $\ell_1$ -min for Baboon. In the meantime, the performance of HBROTP in the visual qualities with  $\kappa = 0.3, 0.4, 0.5$  are displayed in Fig. 3. It can be seen that the reconstruction quality has been significantly improved for three images as the sampling rate  $\kappa$  is ranged from 0.3 to 0.5, and the best visual results have been achieved around  $\kappa = 0.5$ .



**Fig. 3** Performance of HBROTP for three images with different sampling rates.

## 6 Conclusions

The new algorithms that combine the optimal  $k$ -thresholding and heavy-ball method are proposed in this paper. Such algorithms can be seen as the acceleration versions of the optimal  $k$ -thresholding method. The solution error bounds and convergence of the proposed algorithms have been shown mainly under the RIP of the matrices. The numerical performance of the proposed HBROTP algorithm has been evaluated through phase transition, average runtime and image reconstruction. The

experiment results indicate that HBROTP is robust signal recovery method, especially as the sampling rates  $\kappa \leq 0.5$ . The proposed algorithm is generally faster than the standard ROTP method due to integrating the heavy-ball acceleration technique.

## Declarations and statements

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