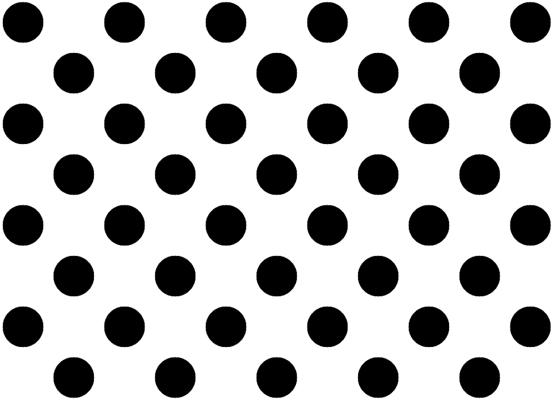
Basics for Enhanced Visualization: 3D/Data

Camera calibration



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Source: circles pattern from OpenCV.

Outline

1. Introduction — This class
2. Model and parameters
3. Calibration with a rig
   - General problem
   - Estimation of the camera matrix
   - Camera parameters from camera matrix
4. Conclusions

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5. Calibration with planes — Next class
   - Setting the calibration problem
   - Estimation of homographies
   - Camera parameters from homographies
6. Conclusions

Introduction

How to project a point $\begin{bmatrix} x_\mathrm{w} \\ y_\mathrm{w} \\ z_\mathrm{w} \end{bmatrix}$ in the image buffer?

Introduction

How to project a point $\begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix}$ in the image buffer?

- If you know the **camera matrix M** you do

$$\begin{bmatrix} x'_{im} \\ y'_{im} \\ w'_{im} \end{bmatrix} = \mathbf{M} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} x_{im} \\ y_{im} \end{bmatrix} = \begin{bmatrix} \frac{x'_{im}}{w'_{im}} \\ \frac{y'_{im}}{w'_{im}} \end{bmatrix}$$

where $\mathbf{M} = \mathbf{K}_s \, \mathbf{K}_f \, \mathbf{\Pi}_0 \, \mathbf{K}_{wc}$.

- But in general we do not know neither **M**, nor any of its parameters.

Camera calibration

- Retrieving **M** and/or its internal parameters is called **camera calibration**.
- When **M** and its internal parameters are known we say that the camera is **calibrated**.
- This is in general an estimation problem. We have to estimate the camera parameters from data.

Model and parameters

Overall model: the camera matrix

$$\mathbf{M} = \mathbf{K}_s \, \mathbf{K}_f \, \mathbf{\Pi}_0 \, \mathbf{K}_{\mathrm{wc}}$$

- The matrix **M** contains all the information from the camera.

Model and parameters

Overall model: the camera matrix

$$\mathbf{M} = \mathbf{K}_s\, \mathbf{K}_f\, \mathbf{\Pi}_0\, \mathbf{K}_{wc}$$

- The matrix **M** contains all the information from the camera.
- This **linear transformation** relates a 3D point in world space to a 2D point in the image buffer. **But both are in homogeneous coordinates!**
- To get the true relation we need to transform in Cartesian coordinates. **This is a non linear transformation!**
- If we know **M** we can retrieve its internal parameters: parameters within $\mathbf{K}_s$, $\mathbf{K}_f$ $\mathbf{\Pi}_0$, $\mathbf{K}_{wc}$.

Model and parameters

Overall model: the camera matrix

$$\mathbf{M} = \mathbf{K}_s \, \mathbf{K}_f \, \mathbf{\Pi}_0 \, \mathbf{K}_{\mathrm{wc}}$$

- Depending on the application we can
  - Estimate directly its internal parameters. **M** is a result.
    Hard but robust to noise.
  - Estimate **M** only, generally without constraining its structure.
    Easy but not robust to noise.
  - Estimate **M** and then the internal parameters.
    Medium.

Model and parameters

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- Depending on the application we can
  - Estimate directly its internal parameters. **M** is a result.
    Hard but robust to noise.
  - Estimate **M** only, generally without constraining its structure.
    Easy but not robust to noise.
  - Estimate **M** and then the internal parameters.
    Medium.
- We can also know in advance some of its internal parameters
  $\Longrightarrow$ the three options above need to be adapted to this case.
- To initialize the **Hard** approach we use the **Medium** approach.
- To simplify the **Medium** approach we use the **Easy** approach first to estimate **M**.

Intrinsic and extrinsic matrices

$$\mathbf{M} = \mathbf{M}_{\text{int}}\ \mathbf{M}_{\text{ext}}$$

- $\mathbf{M}_{\text{int}}$ is the **intrinsic matrix.** It contains the intrinsic parameters of the camera:

$$\mathbf{M}_{\text{int}} = \mathbf{K}_s\ \mathbf{K}_f = \begin{bmatrix} s_x f & s_\theta f & o_x \\ 0 & s_y f & o_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} f_x & f_\theta & o_x \\ 0 & f_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

Model and parameters

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- This matrix has 6 internal parameters. But we can retrieve uniquely only 5 (products *sf* have a scaling ambiguity).
- Sometimes these parameters are known if you have the camera specifications from the manufacturer.
- Note that this is an upper triangular matrix.
- In general we have $f_\theta = 0$ and for the assumption on the axis of the image buffer $f_y < 0$.

Model and parameters

Intrinsic and extrinsic matrices

$$\mathbf{M} = \mathbf{M}_{\text{int}}\, \mathbf{M}_{\text{ext}}$$

- If we assume $f_\theta = 0$ and that the center of the image buffer coincides with the center of the image plane, a fast and simple method to obtain the parameters in $\mathbf{M}_{\text{int}}$ is the following:

Model and parameters

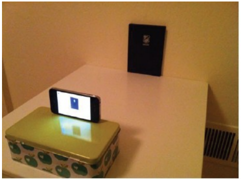
Intrinsic and extrinsic matrices

$$\mathbf{M} = \mathbf{M}_{\text{int}} \, \mathbf{M}_{\text{ext}}$$

- If we assume $f_\theta = 0$ and that the center of the image buffer coincides with the center of the image plane, a fast and simple method to obtain the parameters in $\mathbf{M}_{\text{int}}$ is the following:

  1. Retrieve the camera resolution in pixels p_x and p_y. This is normally available to you.
  2. Choose a planar rectangular object with known height H and width W, get an image from it by putting it at a distance d from the camera and in parallel with the image plane.
  3. Measure its height and width in pixels H_{im} and W_{im}.

Simple intrinsic matrix calibration method



- Then we get

$$o_x = \frac{p_x}{2} \quad o_y = \frac{p_y}{2} \quad f_x = \frac{W_{\text{im}} d}{W} \quad f_y = -\frac{H_{\text{im}} d}{H}$$

- Note that if you have estimated $\mathbf{M}_{\text{int}}$ and $\mathbf{M}$, you can get

$$\mathbf{M}_{\text{ext}} = \mathbf{M}_{\text{int}}^{-1}\,\mathbf{M}$$

since $\mathbf{M}_{\text{int}}$ is always invertible (why?).

Source: Programming Computer Vision with Python, Jan Erik Solem, 2012.

Intrinsic and extrinsic matrices

$$\mathbf{M} = \mathbf{M}_{\text{int}}\,\mathbf{M}_{\text{ext}}$$

- $\mathbf{M}_{\text{ext}}$ is the **extrinsic matrix.** It contains the extrinsic parameters of the camera:

$$\mathbf{M}_{\text{ext}} = \mathbf{\Pi}_0\,\mathbf{K}_{\text{wc}} = \left[\ \mathbf{R}_{\text{wc}} \ \middle|\ -\mathbf{R}_{\text{wc}}\mathbf{t}_{\text{cw}}\ \right] = \left[\ \mathbf{R} \ \middle|\ \mathbf{t}\ \right]$$

Model and parameters

Intrinsic and extrinsic matrices

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- $\mathbf{M}_{\text{ext}}$ is the **extrinsic matrix**. It contains the extrinsic parameters of the camera:

$$\mathbf{M}_{\text{ext}} = \mathbf{\Pi}_0\ \mathbf{K}_{wc} = \left[\begin{array}{c|c} \mathbf{R}_{wc} & -\mathbf{R}_{wc}\mathbf{t}_{cw} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{R} & \mathbf{t} \end{array}\right]$$

- Matrix $\mathbf{R}$ is an orthogonal rotation matrix:

$$\mathbf{R} = \begin{bmatrix} \mathbf{r}_1^\top \\ \mathbf{r}_2^\top \\ \mathbf{r}_3^\top \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- Note that we have only 3 free parameters (why? Hint: $\mathbf{r}_1^\top \mathbf{r}_2 = 0$ and $\mathbf{r}_3 = \pm(\mathbf{r}_1 \times \mathbf{r}_2)$).
- $\mathbf{t} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$ is a translation vector.

Model and parameters

Intrinsic and extrinsic matrices

$$\mathbf{M} = \mathbf{M}_{\text{int}}\, \mathbf{M}_{\text{ext}}$$

- Camera model is commonly given in the following form

$$\mathbf{M} = \mathbf{M}_{\text{int}} \left[\begin{array}{c|c} \mathbf{R} & \mathbf{t} \end{array}\right]$$

Intrinsic and extrinsic matrices

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- Camera model is commonly given in the following form

$$\mathbf{M} = \mathbf{M}_{\text{int}} \left[\ \mathbf{R} \quad \middle| \quad \mathbf{t}\ \right]$$

- Note that **M** has its left 3 columns which factorize in an upper triangular matrix and an orthogonal matrix.
- Also remember that in homogeneous coordinates all matrices of the form

$$\mathbf{M} = \alpha \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix}$$

for $\alpha \neq 0$ are equivalent.

General camera calibration problem

Given a set of **N point correspondences** $(\mathbf{u}^1_{\mathbf{im}}, \mathbf{u}^1_{\mathbf{w}}), \cdots, (\mathbf{u}^N_{\mathbf{im}}, \mathbf{u}^N_{\mathbf{w}})$

where $\mathbf{u}^i_{\mathrm{im}}$ is the image buffer coordinates of the 3D point $\mathbf{u}^i_{\mathrm{w}}$ in Cartesian coordinates,

estimate the camera model M, and its parameters:

M_{int}	f_x, f_y, f_θ, o_x, o_y	
M_{ext}	$\mathbf{R}$, $\mathbf{t}$	$\mathbf{R}_{wc}$, $\mathbf{t}_{cw}$

Calibration with a rig

General camera calibration problem

Given a set of **N point correspondences** $(\mathbf{u}_{\mathbf{im}}^1, \mathbf{u}_{\mathbf{w}}^1), \cdots, (\mathbf{u}_{\mathbf{im}}^N, \mathbf{u}_{\mathbf{w}}^N)$

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- 3D coordinate points are normally explicitly given.
- Image points for given 3D points are either obtained by hand (mouse-clicking on the corresponding image points) or automatically from known features: corners, edges, lines, circles, ellipses, *etc*.

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estimate the camera model M, and its parameters:

M_{int}	$f_x, f_y, f_\theta, o_x, o_y$	
M_{ext}	$\mathbf{R}, \mathbf{t}$	$\mathbf{R}_{wc}, \mathbf{t}_{cw}$

- There are 11 internal parameters.

General camera calibration problem

- $\mathbf{u}_{\mathrm{im}}^{i}$ are noisy:
  - Circle of confusion from imperfect focusing.
  - Image is noisy.
  - Algorithm/user localizing image points is not ideal.

General camera calibration problem

- $\mathbf{u}^i_{\text{im}}$ are noisy:
  - Circle of confusion from imperfect focusing.
  - Image is noisy.
  - Algorithm/user localizing image points is not ideal.
- Consequences:
  - We cannot fit perfectly a camera model (with zero error).
  - We should take into account the perturbations: search for parameters minimizing image projection residuals ε^i_x and ε^i_y (to be defined later).
  - Minimization can be done in least squares sense.

General camera calibration problem

Least squares minimization problem

$$\text{minimize} \quad \sum_{i=1}^{N} \left(\varepsilon_x^i\right)^2 + \left(\varepsilon_y^i\right)^2$$

$$\text{with respect to} \quad \text{all camera parameters}$$

$$\text{subject to} \quad \mathbf{R}^\mathsf{T}\mathbf{R} = \mathbf{I}$$

where

$$\varepsilon_x^i = x_{\text{im}}^i - \left[\frac{f_x(\mathbf{r}_1^\mathsf{T}\mathbf{u}_\text{w}^i + t_x) + f_\theta(\mathbf{r}_2^\mathsf{T}\mathbf{u}_\text{w}^i + t_y)}{\mathbf{r}_3^\mathsf{T}\mathbf{u}_\text{w}^i + t_z} + o_x \right]$$

$$\varepsilon_y^i = y_{\text{im}}^i - \left[\frac{f_y(\mathbf{r}_2^\mathsf{T}\mathbf{u}_\text{w}^i + t_y)}{\mathbf{r}_3^\mathsf{T}\mathbf{u}_\text{w}^i + t_z} + o_y \right]$$

Calibration with a rig

General camera calibration problem

Least squares minimization problem

minimize	$\sum_{i=1}^{N} \left(\varepsilon_x^i\right)^2 + \left(\varepsilon_y^i\right)^2$
with respect to	all camera parameters
subject to	$\mathbf{R}^\mathrm{T}\mathbf{R} = \mathbf{I}$

- This is the **hard but robust** approach.
- The parameters are obtained by minimizing this nonlinear, non-convex minimization problem.
- It is a small-dimensional optimization problem.

General camera calibration problem

Least squares minimization problem

minimize $\sum_{i=1}^{N} \left(\varepsilon_x^i\right)^2 + \left(\varepsilon_y^i\right)^2$

with respect to all camera parameters

subject to $\mathbf{R}^\mathsf{T}\mathbf{R} = \mathbf{I}$

- It is commonly solved without the constraint with a nonlinear solver: gradient, conjugate gradient or Levenberg-Marquardt algorithm.
- The constraint is then imposed from the resulting $\hat{\mathbf{R}}$. Final rotation matrix $\mathbf{R} = \mathbf{U}_R\mathbf{V}_R^\mathsf{T}$ where these matrices come from the SVD of the unconstrained matrix $\hat{\mathbf{R}} = \mathbf{U}_R\mathbf{S}\mathbf{V}_R^\mathsf{T}$.

General camera calibration problem

Least squares minimization problem

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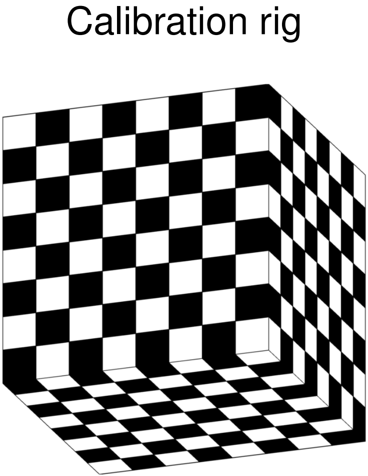
with respect to all camera parameters

subject to $\mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{I}$

- Since the problem is non-convex it requires a good initialization to get global minimum.
- It can be modified to include radial distortion parameters (how?).
- Parameters are identifiable (unique solution) if $N \geq 6$ and the points are not all co-planar $\Longrightarrow$ calibration with a rig.

Calibration with a rig



Source: Brian moore81 - Own work, CC BY-SA 4.0, `https://commons.wikimedia.org/w/index.php?curid=37119802`.

Direct camera model estimation

- We can also first estimate $\mathbf{M}$ from data.
- We know that in homogeneous coordinates $\mathbf{u}^i_{\mathrm{im}} = \alpha\,\mathbf{M}\,\mathbf{u}^i_{\mathrm{w}}$.
  In homogeneous coordinates these vectors are parallel.

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- This can be rewritten with cross-product:

$$\mathbf{u}^i_{\mathrm{im}} \times \mathbf{M}\,\mathbf{u}^i_{\mathrm{w}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

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- If we define $\mathbf{M} = \begin{bmatrix} \mathbf{m}_1^{\mathsf{T}} \\ \mathbf{m}_2^{\mathsf{T}} \\ \mathbf{m}_3^{\mathsf{T}} \end{bmatrix}$, the cross-product can be rewritten as:

$$\begin{bmatrix} 0 & -1 & y_{\text{im}}^{i} \\ 1 & 0 & -x_{\text{im}}^{i} \\ -y_{\text{im}}^{i} & x_{\text{im}}^{i} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{m}_1^{\mathsf{T}}\mathbf{u}_{\text{w}}^{i} \\ \mathbf{m}_2^{\mathsf{T}}\mathbf{u}_{\text{w}}^{i} \\ \mathbf{m}_3^{\mathsf{T}}\mathbf{u}_{\text{w}}^{i} \end{bmatrix} = \begin{bmatrix} -\mathbf{m}_2^{\mathsf{T}}\mathbf{u}_{\text{w}}^{i} + y_{\text{im}}^{i}\,\mathbf{m}_3^{\mathsf{T}}\mathbf{u}_{\text{w}}^{i} \\ \mathbf{m}_1^{\mathsf{T}}\mathbf{u}_{\text{w}}^{i} - x_{\text{im}}^{i}\,\mathbf{m}_3^{\mathsf{T}}\mathbf{u}_{\text{w}}^{i} \\ -y_{\text{im}}^{i}\,\mathbf{m}_1^{\mathsf{T}}\mathbf{u}_{\text{w}}^{i} + x_{\text{im}}^{i}\,\mathbf{m}_2^{\mathsf{T}}\mathbf{u}_{\text{w}}^{i} \end{bmatrix} = \mathbf{0}$$

Calibration with a rig

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- We can factor the unknown camera model vectors $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$:

$$\begin{bmatrix} \mathbf{0}^\top & -\mathbf{u}_{\text{w}}^{i\top} & y_{\text{im}}^i \, \mathbf{u}_{\text{w}}^{i\top} \\ \mathbf{u}_{\text{w}}^{i\top} & \mathbf{0}^\top & -x_{\text{im}}^i \, \mathbf{u}_{\text{w}}^{i\top} \\ -y_{\text{im}}^i \, \mathbf{u}_{\text{w}}^{i\top} & x_{\text{im}}^i \, \mathbf{u}_{\text{w}}^{i\top} & \mathbf{0}^\top \end{bmatrix} \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix} = \mathbf{0}$$

- Note that the third row is a linear combination of the first two. So we can delete it from the linear equations.

Direct camera model estimation

- For N points we have $2N$ equations

$$\begin{bmatrix} \mathbf{0}^\top & -\mathbf{u}_{\mathrm{w}}^{1\,\top} & y_{\mathrm{im}}^{1}\,\mathbf{u}_{\mathrm{w}}^{1\,\top} \\ \mathbf{u}_{\mathrm{w}}^{1\,\top} & \mathbf{0}^\top & -x_{\mathrm{im}}^{1}\,\mathbf{u}_{\mathrm{w}}^{1\,\top} \\ & \vdots & \\ \mathbf{0}^\top & -\mathbf{u}_{\mathrm{w}}^{N\,\top} & y_{\mathrm{im}}^{N}\,\mathbf{u}_{\mathrm{w}}^{N\,\top} \\ \mathbf{u}_{\mathrm{w}}^{N\,\top} & \mathbf{0}^\top & -x_{\mathrm{im}}^{N}\,\mathbf{u}_{\mathrm{w}}^{N\,\top} \end{bmatrix} \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix} = \mathbf{A}\mathbf{m} = \mathbf{0}$$

Direct camera model estimation

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- $\mathbf{m}$ has 11 free parameters, it can be retrieved from the null-space of $\mathbf{A}$ and we need to correct its scaling (how to do it with SVD?).
- For uniqueness, we need $\dim\{\mathrm{null}(\mathbf{A})\} = 1$, thus we need $\mathrm{rank}(\mathbf{A}) = 11$.
- We can show that $\mathrm{rank}(\mathbf{A}) = 11$ if and only if we have at least $N \geq 6$ non co-planar calibration points.

Direct camera model estimation

- Problem I (Noise): in practice $\mathbf{u}^i_{\text{im}}$ are noisy, so we will never get equality.
- Solution: find the $\mathbf{m}$ that is closest to what we want:

$$\begin{aligned} &\text{minimize} && \|\mathbf{Am}\|_2^2 \\ &\text{with respect to} && \mathbf{m} \\ &\text{subject to} && \|\mathbf{m}\|_2^2 = 1 \end{aligned}$$

- The constraint is imposed to avoid trivial solution $\mathbf{m} = \mathbf{0}$.

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- The constraint is imposed to avoid trivial solution $\mathbf{m} = \mathbf{0}$.
- This problem has an analytic solution:

$$\mathbf{m} = \mathbf{v}_{\min}$$

  where $\mathbf{v}_{\min}$ is the singular vector of $\mathbf{A}$ corresponding to the smallest singular value (show it with KKT conditions!).

Direct camera model estimation

- Problem II (Ill-conditioning): in practice $\mathbf{u}_{\mathrm{w}}^{i}$ and $\mathbf{u}_{\mathrm{im}}^{i}$ are of different orders, ex.: $x_{\mathrm{im}} = 500$ pixels and $x_{\mathrm{w}} = 0.2$m, therefore $\mathbf{A}$ has entries in a wide range of values
  $\Longrightarrow$ **it can be very ill-conditioned**.
- Noise will produce large errors in $\mathbf{v}_{\min}$.

Direct camera model estimation

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- Noise will produce large errors in $\mathbf{v}_{\min}$.
- Solution: center and scale separately to unit standard deviation in each coordinate (like in PCA) $\mathbf{u}_{\mathrm{w}}^{i}$ and $\mathbf{u}_{\mathrm{im}}^{i}$.
- This can be done in homogeneous coordinates with linear transformations:

$$\tilde{\mathbf{u}}_{\mathrm{w}}^{i} = \mathbf{T}_{\mathrm{w}}\mathbf{u}_{\mathrm{w}}^{i} \quad \tilde{\mathbf{u}}_{\mathrm{im}}^{i} = \mathbf{T}_{\mathrm{im}}\mathbf{u}_{\mathrm{im}}^{i}$$

Direct camera model estimation

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- Previous minimization problem can be solved with $\tilde{\mathbf{A}}$ built with $\tilde{\mathbf{u}}_{\mathrm{w}}^{i}$ and $\tilde{\mathbf{u}}_{\mathrm{im}}^{i}$.
- Solution $\tilde{\mathbf{M}}$ is related to $\mathbf{M}$ as follows

$$\mathbf{M} = \mathbf{T}_{\mathrm{im}}^{-1}\tilde{\mathbf{M}}\mathbf{T}_{\mathrm{W}}$$

Direct camera model estimation

- Problem III (Scaling): we have estimated $\alpha\mathbf{M}$ with arbitrary α. How do we retrieve proper scaling?

Direct camera model estimation

- Problem III (Scaling): we have estimated $\alpha\mathbf{M}$ with arbitrary α. How do we retrieve proper scaling?
- Solution: to estimate the scaling α note that

$$\mathbf{m}_3^{1:3} = \left[m_{31}\ m_{32}\ m_{33}\right]^\mathsf{T} = \mathbf{r}_3$$

from the orthogonality constraint

$$\|\mathbf{r}_3\|_2 = 1$$

therefore the properly scaled matrix $\mathbf{M}_s$ is

$$\mathbf{M}_s = \frac{\mathbf{M}}{\|\mathbf{m}_3^{1:3}\|_2}$$

Direct camera model estimation

- Following all these steps to estimate $\mathbf{M}$ is called the **direct linear transformation (DLT) method**.
- It is the **easy but not robust** approach.

DLT method

1. Retrieve $N > 6$ not all co-planar point correspondences:
$$(\mathbf{u}_{\text{im}}^1, \mathbf{u}_{\text{w}}^1), \cdots, (\mathbf{u}_{\text{im}}^N, \mathbf{u}_{\text{w}}^N)$$
2. Center and scale the points with $\tilde{\mathbf{u}}_{\text{w}}^i = \mathbf{T}_{\text{w}}\mathbf{u}_{\text{w}}^i$ and $\tilde{\mathbf{u}}_{\text{im}}^i = \mathbf{T}_{\text{im}}\mathbf{u}_{\text{im}}^i$.
3. Construct matrix $\tilde{\mathbf{A}}$ (slide 22).
4. Retrieve $\tilde{\mathbf{m}} = \mathbf{v}_{\text{min}}$ with SVD$(\tilde{\mathbf{A}})$.
5. Build $\tilde{\mathbf{M}}$ from $\tilde{\mathbf{m}}$.
6. Retrieve $\mathbf{M}$ with $\mathbf{M} = \mathbf{T}_{\text{im}}^{-1}\tilde{\mathbf{M}}\mathbf{T}_{\text{W}}$.
7. Rescale $\mathbf{M}$ with $\mathbf{M} := \dfrac{\mathbf{M}}{\|\mathbf{m}_3^{1:3}\|_2}$

Camera parameters from camera matrix

- Now that we have **M**, how do we get the camera parameters?

Camera parameters from camera matrix

- Now that we have $\mathbf{M}$, how do we get the camera parameters?
- Remember that

$$\mathbf{M} = \mathbf{M}_{\text{int}} \left[\begin{array}{c|c} \mathbf{R} & \mathbf{t} \end{array} \right]$$

  Its first three columns are $\mathbf{M}^{(1:3)} = \mathbf{M}_{\text{int}} \mathbf{R}$.
- This is a factorization into an upper triangular matrix and an orthogonal matrix.
- Such a factorization is known in algebra: **RQ factorization**.

Camera parameters from camera matrix

- Now that we have $\mathbf{M}$, how do we get the camera parameters?
- Remember that

$$\mathbf{M} = \mathbf{M}_{\text{int}} \left[\; \mathbf{R} \quad | \quad \mathbf{t} \; \right]$$

  Its first three columns are $\mathbf{M}^{(1:3)} = \mathbf{M}_{\text{int}} \, \mathbf{R}$.

- This is a factorization into an upper triangular matrix and an orthogonal matrix.
- Such a factorization is known in algebra: **RQ factorization**.
- It is the "twin sister" of the QR factorization and many algorithms are available for this factorization (it can be retrieved with QR algorithms).

Camera parameters from camera matrix

- Retrieval of camera parameters - RQ factorization:

$$[\mathbf{M}_{\text{int}}, \mathbf{R}] = \text{RQ}(\mathbf{M}^{(1:3)})$$

- You can retrieve f_x, f_y, f_θ, o_x and o_y from $\mathbf{M}_{\text{int}}$.

Camera parameters from camera matrix

- Retrieval of camera parameters - RQ factorization:

$$[\mathbf{M}_{\text{int}}, \mathbf{R}] = \text{RQ}(\mathbf{M}^{(1:3)})$$

- You can retrieve f_x, f_y, f_θ, o_x and o_y from $\mathbf{M}_{\text{int}}$.
- RQ has sign ambiguities! You should correct it to have the right signs:
  1. f_x is normally positive
  2. f_y is normally negative
  3. You should have $\approx +1$ in the third diagonal element.
- Each time you multiply a column of $\mathbf{M}_{\text{int}}$ by -1 you should also multiply the corresponding row of $\mathbf{R}$.

Camera parameters from camera matrix

- Retrieval of camera parameters - $\mathbf{t}$, from the fourth column:

$$\mathbf{t} = \mathbf{M}_{\text{int}}{}^{-1}\mathbf{M}^{(4)}$$

- Trivially you have $\mathbf{R}_{\text{wc}} = \mathbf{R}$ and $\mathbf{t}_{\text{cw}} = -\mathbf{R}^{-1}\mathbf{t}$.

Camera parameters from camera matrix

- This is the **medium** approach. You can call it **camera calibration using DLT**.
- Its estimates are often used as an initialization for the **hard** approach.

Camera parameters using DLT method

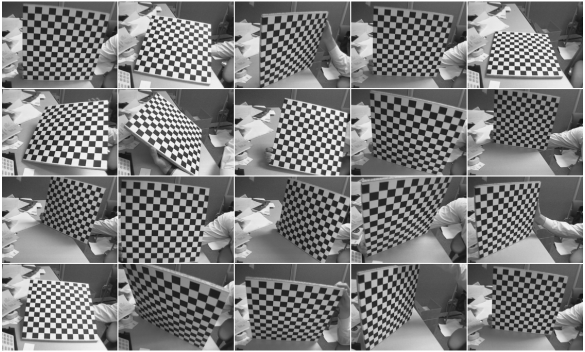
1. Retrieve $N > 6$ not all co-planar point correspondences:
$$(\mathbf{u}_{\mathrm{im}}^{1}, \mathbf{u}_{\mathrm{w}}^{1}), \cdots, (\mathbf{u}_{\mathrm{im}}^{N}, \mathbf{u}_{\mathrm{w}}^{N})$$
2. Estimate $\mathbf{M}$ using DLT method.
3. Retrieve $\mathbf{M}_{\mathrm{int}}$ and $\mathbf{R}$ with $[\mathbf{M}_{\mathrm{int}}, \mathbf{R}] = \mathrm{RQ}(\mathbf{M}^{(1:3)})$.
4. Correct RQ sign ambiguities by multiplying the columns of $\mathbf{M}_{\mathrm{int}}$ and corresponding rows of $\mathbf{R}$ by -1.
5. Retrieve f_x, f_y, f_θ, o_x and o_y from $\mathbf{M}_{\mathrm{int}}$.
6. Retrieve $\mathbf{t}$ with $\mathbf{t} = \mathbf{M}_{\mathrm{int}}{}^{-1}\mathbf{M}^{(4)}$.

Conclusions

- Camera calibration: information to project 3D points on images.
- Still not enough information to retrieve the 3D structure. Stereo vision next class.
- Two approaches that can be combined: optimization (non linear least squares) and algebra (DLT).
- It is suggested to have 5 times more equations than unknowns in practice, since noise is present. This gives you an idea on how many point correspondences you should use (how many?).

Setting the calibration problem

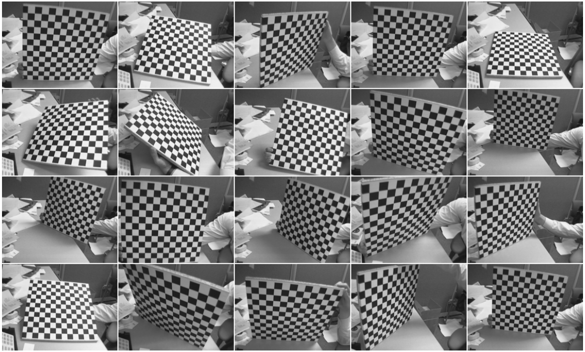
- Problem: it is often difficult to build a rig with good precision.



Source: https://commons.wikimedia.org/wiki/File:Multiple_chessboard_views.png.

Setting the calibration problem

- Problem: it is often difficult to build a rig with good precision.
- Solution: use a plane for calibration, a checkerboard for example, but in different positions and orientations.

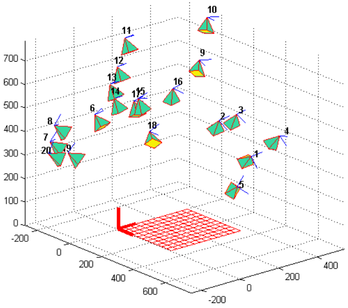


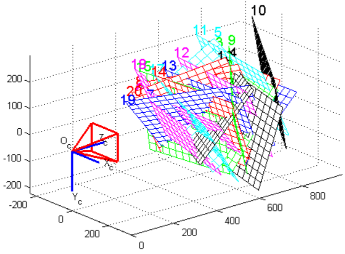
Source: https://commons.wikimedia.org/wiki/File:Multiple_chessboard_views.png.

Calibration with planes

Setting the calibration problem

- This is by far the most popular calibration method.
- The planes are all supposed to have $z_w = 0$ and the camera is supposed to change its pose (position and orientation). See left figure.
- Even if we actually change the plane pose. See right figure.





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Setting the calibration problem

- For K images (plane poses), we want to estimate the parameters of **1 intrinsic matrix** $\mathbf{M}_{\text{int}}$ and **K extrinsic matrices** $\mathbf{M}_{\text{ext}}^{1}, \cdots, \mathbf{M}_{\text{ext}}^{K}$.
- We can also use a non linear least squares approach, that is by appropriately modifying the previous **hard approach** (can you write the cost function?).

Setting the calibration problem

- For K images (plane poses), we want to estimate the parameters of **1 intrinsic matrix** $\mathbf{M}_{\text{int}}$ and **K extrinsic matrices** $\mathbf{M}_{\text{ext}}^{1}, \cdots, \mathbf{M}_{\text{ext}}^{K}$.
- We can also use a non linear least squares approach, that is by appropriately modifying the previous **hard approach** (can you write the cost function?).
- In practice, we first estimate all camera parameters using a modification of the medium approach with DLT.
- Then we use the results to initialize the non linear least squares optimization algorithm.

Estimation of homographies

- Can we use the DLT method in this case?

Estimation of homographies

- Can we use the DLT method in this case?
- Each plane has N^i point correspondences. From the assumption of $z_w = 0$ for all planes, for the j-th point correspondence, we have

$$
\begin{bmatrix} x^j_{\mathrm{im}} \\ y^j_{\mathrm{im}} \\ 1 \end{bmatrix} = \alpha \mathbf{M}_{\mathrm{int}} \begin{bmatrix} \mathbf{r}^1 & \mathbf{r}^2 & \mathbf{r}^3 & | & \mathbf{t} \end{bmatrix} \begin{bmatrix} x^j_{\mathrm{w}} \\ y^j_{\mathrm{w}} \\ \mathbf{0} \\ 1 \end{bmatrix}
$$

$$
= \alpha \mathbf{M}_{\mathrm{int}} \begin{bmatrix} \mathbf{r}^1 & \mathbf{r}^2 & | & \mathbf{t} \end{bmatrix} \begin{bmatrix} x^j_{\mathrm{w}} \\ y^j_{\mathrm{w}} \\ 1 \end{bmatrix}
$$

$$
= \alpha \mathbf{H}_i \begin{bmatrix} x^j_{\mathrm{w}} \\ y^j_{\mathrm{w}} \\ 1 \end{bmatrix}
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- $\mathbf{H}_i$ is a plane-to-plane homogeneous transformation, we call it an **homography**.

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$$\begin{bmatrix} x_{\mathrm{im}}^j \\ y_{\mathrm{im}}^j \\ 1 \end{bmatrix} = \alpha \mathbf{H}_i \begin{bmatrix} x_{\mathrm{w}}^j \\ y_{\mathrm{w}}^j \\ 1 \end{bmatrix}$$

- $\mathbf{H}_i$ is a plane-to-plane homogeneous transformation, we call it an **homography**.
- It is a matrix with 9 elements:

$$\mathbf{H}_i = \begin{bmatrix} \mathbf{h}_i^1 \ \mathbf{h}_i^2 \ \mathbf{h}_i^3 \end{bmatrix} = \begin{bmatrix} h_i^{11} & h_i^{12} & h_i^{13} \\ h_i^{21} & h_i^{22} & h_i^{23} \\ h_i^{31} & h_i^{32} & h_i^{33} \end{bmatrix}$$

  but only 8 are free due to free scaling α.

Estimation of homographies

- For the point correspondences of each image, we can use the DLT method to estimate the $\mathbf{H}_i$ similarly to what we did for $\mathbf{M}$.
- We cannot rescale $\mathbf{H}_i$ at the end of the DLT procedure, since its third row is not unitary (we do not have the $\mathbf{r}^3$) column.
- We can show that the rank$(\mathbf{A}^i) = 8$ for $N^i \geq 4$ not all collinear points.

Camera parameters from homographies

- How to get $\mathbf{M}_{\text{int}}$ and $\mathbf{M}_{\text{ext}}^{1}, \cdots, \mathbf{M}_{\text{ext}}^{K}$ from $\mathbf{H}_1, \cdots, \mathbf{H}_k$?
- This is the tricky part, we cannot use the RQ decomposition.
- We are going to retrieve $\mathbf{M}_{\text{int}}$ first using all $\mathbf{H}_i$.

Calibration with planes

Camera parameters from homographies

- We will use 3 properties about our model:

1. $\left[\mathbf{h}_i^1\ \mathbf{h}_i^2\ \mathbf{h}_i^3\right] = \alpha \mathbf{M}_{\text{int}} \left[\mathbf{r}_i^1\ \mathbf{r}_i^2\ \mathbf{t}_i\right] \Longrightarrow \mathbf{M}_{\text{int}}^{-1} \left[\mathbf{h}_i^1\ \mathbf{h}_i^2\ \mathbf{h}_i^3\right] = \alpha \left[\mathbf{r}_i^1\ \mathbf{r}_i^2\ \mathbf{t}_i\right]$

   $\Longrightarrow \mathbf{M}_{\text{int}}^{-1} \mathbf{h}_i^1 = \alpha \mathbf{r}_i^1$ and $\mathbf{M}_{\text{int}}^{-1} \mathbf{h}_i^2 = \alpha \mathbf{r}_i^2$ Definition
2. $(\mathbf{r}^1)^{\mathsf{T}} \mathbf{r}^2 = 0$ Orthogonality
3. $\|\mathbf{r}^1\|_2 = \|\mathbf{r}^2\|_2$ Same norm

Camera parameters from homographies

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   $\Longrightarrow \mathbf{M}_{\text{int}}^{-1}\mathbf{h}_i^1 = \alpha \mathbf{r}_i^1$ and $\mathbf{M}_{\text{int}}^{-1}\mathbf{h}_i^2 = \alpha \mathbf{r}_i^2$ Definition
2. $(\mathbf{r}^1)^{\mathsf{T}}\mathbf{r}^2 = 0$ Orthogonality
3. $\|\mathbf{r}^1\|_2 = \|\mathbf{r}^2\|_2$ Same norm

- From properties 1 and 2 we have:

$$(\mathbf{h}_i^1)^{\mathsf{T}}(\mathbf{M}_{\text{int}}^{-1})^{\mathsf{T}}\mathbf{M}_{\text{int}}^{-1}\mathbf{h}_i^2 = 0$$

Camera parameters from homographies

- We will use 3 properties about our model:
  1. $\left[\mathbf{h}_i^1\ \mathbf{h}_i^2\ \mathbf{h}_i^3\right] = \alpha\mathbf{M}_{\text{int}}\left[\mathbf{r}_i^1\ \mathbf{r}_i^2\ \mathbf{t}_i\right] \implies \mathbf{M}_{\text{int}}^{-1}\left[\mathbf{h}_i^1\ \mathbf{h}_i^2\ \mathbf{h}_i^3\right] = \alpha\left[\mathbf{r}_i^1\ \mathbf{r}_i^2\ \mathbf{t}_i\right]$
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- From properties 1 and 2 we have:

$$(\mathbf{h}_i^1)^{\mathsf{T}}(\mathbf{M}_{\text{int}}^{-1})^{\mathsf{T}}\mathbf{M}_{\text{int}}^{-1}\mathbf{h}_i^2 = 0$$

- From properties 1 and 3 we have:

$$(\mathbf{h}_i^1)^{\mathsf{T}}(\mathbf{M}_{\text{int}}^{-1})^{\mathsf{T}}\mathbf{M}_{\text{int}}^{-1}\mathbf{h}_i^1 = (\mathbf{h}_i^2)^{\mathsf{T}}(\mathbf{M}_{\text{int}}^{-1})^{\mathsf{T}}\mathbf{M}_{\text{int}}^{-1}\mathbf{h}_i^2$$

Camera parameters from homographies

- We can define a symmetric matrix **B** with 6 parameters as follows:

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{bmatrix} = (\mathbf{M}_{\text{int}}^{-1})^{\mathsf{T}} \mathbf{M}_{\text{int}}^{-1}$$

- The 6 parameters can be stacked into a vector:

$$\mathbf{b} = [b_{11}\ b_{12}\ b_{22}\ b_{13}\ b_{23}\ b_{33}]^{\mathsf{T}}$$

Camera parameters from homographies

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- The two last equations in the previous slide can be rewritten as:

$$\begin{bmatrix} (\mathbf{g}_{12}^{i})^{\mathsf{T}} \\ (\mathbf{g}_{11}^{i} - \mathbf{g}_{22})^{\mathsf{T}} \end{bmatrix} \mathbf{b} = \mathbf{G}^{i} \mathbf{b} = \mathbf{0}$$

where $(\mathbf{g}_{kl}^{i})^{\mathsf{T}} = [h_{k1}^{i} h_{l1}^{i},\ h_{k1}^{i} h_{l2}^{i} + h_{k2}^{i} h_{l1}^{i},\ h_{k2}^{i} h_{l2}^{i},$
$h_{k3}^{i} h_{l1}^{i} + h_{k1}^{i} h_{l3}^{i},\ h_{k3}^{i} h_{l2}^{i} + h_{k2}^{i} h_{l3}^{i},\ h_{k3}^{i} h_{l3}^{i}]$

Camera parameters from homographies

- What is the idea behind all this?
  1. We are going to stack all the K linear systems in a big linear system

$$\begin{bmatrix} \mathbf{G}^1 \\ \vdots \\ \mathbf{G}^K \end{bmatrix} \mathbf{b} = \mathbf{G}\mathbf{b} = \mathbf{0}$$

  2. Since there is noise in the homography estimation we are going to find a non trivial $\mathbf{b}$ that minimizes $\|\mathbf{G}\mathbf{b}\|_2$.
  3. After finding the best $\mathbf{b}$ up to a scaling factor α we can find the intrinsic parameters from the relation between $\mathbf{b}$ and $\mathbf{M}_{\text{int}}$:

$$b_1 = \frac{1}{f_x^2} \qquad b_2 = -\frac{f_\theta}{f_x^2 f_y} \qquad b_3 = \frac{f_\theta^2}{f_x^2 f_y} + \frac{1}{f_y^2}$$

$$b_4 = \frac{f_\theta o_y - f_y o_x}{f_x^2 f_y} \qquad b_5 = -\frac{o_x}{f_y^2} - \frac{f_\theta(f_\theta o_y - f_y o_x)}{f_x^2 f_y^2} \qquad b_6 = \frac{o_y^2}{f_y^2} - \frac{(f_\theta o_y - f_y o_x)^2}{f_x^2 f_y^2} + 1$$

Calibration with planes

Camera parameters from homographies

- We can solve the minimization problem

$$\begin{aligned} \text{minimize} \quad & \|\mathbf{Gb}\|_2^2 \\ \text{with respect to} \quad & \mathbf{b} \\ \text{subject to} \quad & \|\mathbf{b}\|_2^2 = 1 \end{aligned}$$

exactly as we solved it for DLT, using the SVD.

Camera parameters from homographies

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$$\begin{array}{cc} \text{minimize} & \|\mathbf{Gb}\|_2^2 \\ \text{with respect to} & \mathbf{b} \\ \text{subject to} & \|\mathbf{b}\|_2^2 = 1 \end{array}$$

  exactly as we solved it for DLT, using the SVD.
- It can be shown that this problem has a unique solution when $K > 3$ and the poses of the planes are different (2 equations per plane and 5 unknowns).

Calibration with planes

Camera parameters from homographies

- Intrinsic parameters and scaling factor can be retrieved from **b**:

$$o_y = \frac{b_2 b_4 - b_1 b_5}{b_1 b_3 - b_2^2} \qquad \lambda = \frac{1}{\alpha} = b_6 - \frac{b_4^2 + o_y(b_2 b_4 - b_1 b_5)}{b_1}$$

$$f_x = \sqrt{\frac{\lambda}{b_1}} \qquad f_y = -\sqrt{\frac{\lambda b_1}{b_1 b_3 - b_2^2}}$$

$$f_\theta = -b_2 \frac{f_x^2 f_y}{\lambda} \qquad o_x = \frac{f_\theta o_y}{f_y} - b_4 \frac{f_x^2}{\lambda}$$

Camera parameters from homographies

- After building $\mathbf{M}_{\text{int}}$ we can retrieve the extrinsic parameters from property 1 :

$$\left[\mathbf{r}_i^1 \ \mathbf{r}_i^2 \ \mathbf{t}_i\right] = \lambda \mathbf{M}_{\text{int}}^{-1} \left[\mathbf{h}_i^1 \ \mathbf{h}_i^2 \ \mathbf{h}_i^3\right]$$

- Vector $\mathbf{r}^3$ is given by $\mathbf{r}^3 = \mathbf{r}^1 \times \mathbf{r}^2$.

Calibration with planes

Camera parameters from homographies

- After building $\mathbf{M}_{\text{int}}$ we can retrieve the extrinsic parameters from property 1 :

$$\left[\mathbf{r}_i^1 \ \mathbf{r}_i^2 \ \mathbf{t}_i\right] = \lambda \mathbf{M}_{\text{int}}^{-1} \left[\mathbf{h}_i^1 \ \mathbf{h}_i^2 \ \mathbf{h}_i^3\right]$$

- Vector $\mathbf{r}^3$ is given by $\mathbf{r}^3 = \mathbf{r}^1 \times \mathbf{r}^2$.

- Due to noise estimated **R** may not be a rotation matrix. In this case we should search for the rotation matrix closest to **R** (solution given by SVD of **R** as previously presented).

Camera parameters from homographies

- You can call this method **camera calibration with homographies**.
- Its estimates are often used as an initialization for non linear least squares (**hard approach**).
- Non linear least squares can take into account radial distortion.

Camera calibration using homographies

---

1. Retrieve $N^i \geq 4$ point correspondences for $K \geq 3$ different planes.
2. Estimate all $\mathbf{H}^i$ using DLT method.
3. Build matrix $\mathbf{G}$ with the $\mathbf{H}^i$ (slides 39 and 40).
4. Find $\mathbf{b}$ using the SVD.
5. Retrieve intrinsic parameters and scaling from $\mathbf{b}$ (slide 42).
6. Build $\mathbf{M}_{\text{int}}$.
7. Retrieve the extrinsic parameters with $\left[\mathbf{r}_i^1\ \mathbf{r}_i^2\ \mathbf{t}_i\right] = \lambda \mathbf{M}_{\text{int}}^{-1}\left[\mathbf{h}_i^1\ \mathbf{h}_i^2\ \mathbf{h}_i^3\right]$ and $\mathbf{r}^3 = \mathbf{r}^1 \times \mathbf{r}^2$ (impose orthogonality on $\mathbf{R}$ if necessary).

Conclusions

- From all the presented methods calibration from planes is the most popular.
- If you already have intrinsic parameters, one plane is enough to estimate the extrinsic parameters (how do you do it?).
- It is suggested to have 5 times more equations than unknowns in practice, since noise is present. This gives you an idea on how many point correspondences and homographies you should use (how many?).