7 Hypothesis Testing

Exercise 7.1

Assume that x_1, ..., x_n are n samples generated by a normal distribution with the unknown mean m and the known variance σ^2. The normal probability density function with mean m and variance σ^2 is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}.$$

We want to test $H_0 : \{m = m_0\}$ versus $H_1 : \{m = m_1\}$ with $m_1 > m_0$.

1. Calculate the decision function $d(x_1, \ldots, x_n)$ of the Neyman-Pearson test of size α. Simplify it as much as much possible by using the $\ln(\cdot)$ function.
2. What is the distribution of $d(x_1, \ldots, x_n)$?
3. Calculate the threshold h_α of the test in function of $\Phi^{-1}(\cdot)$ where $\Phi(t) = \Pr(U \le t)$ is the cumulative distribution function of the standard normal distribution, i.e., $U \sim \mathcal{N}(0, 1)$.
4. Describe carefully the critical region C_α of the test.
5. Calculate the power of the test, i.e., the probability to reject H_0 when H_1 is true.
6. Show that the Neyman-Pearson test is unchanged to test $H_0 : \{m = m_0\}$ versus $H_2 : \{m = m_2\}$ with $m_2 > m_0$ and $m_2 \neq m_1$. Is this test relevant to test $H_0 : \{m = m_0\}$ versus $H_3 : \{m : m > m_0\}$?

Exercise 7.2

This exercise is based on the previous one. It keeps the same notation. It is assumed that the sample $(x_1, \ldots, x_n)$ obeys to the hypothesis $H_0 : \{m = m_0\}$.

1. Show that $C_\alpha \subset C_{\alpha'}$ if $\alpha < \alpha'$.
2. Show that the p-value $\hat{p}$ of the sample $(x_1, \ldots, x_n)$ is
$$\hat{p} = \hat{p}(\bar{x}) = 1 - F\left(\frac{\bar{x} - m_0}{\frac{\sigma}{\sqrt{n}}}\right) \quad \text{with} \quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i.$$
3. Assume temporarily (just in this question) that $\bar{x}$ is a constant value. Show that $\hat{p} = \Pr(Z \ge \bar{x})$ where $Z \sim \mathcal{N}(m_0, \sigma^2/n)$.
4. Show that $\hat{p}$ is uniformly distributed over $(0, 1)$.

Exercise 7.3

Suppose that $X = (X_1, X_2, \ldots, X_n)$ is a random sample of size n from the Bernoulli distribution with success parameter p. The sample could represent the results of tossing a coin n times, where p is the probability of heads and $1-p$ is the probability of tails. We wish to test the simple hypotheses $H_0 : p = p_0$ versus $H_1 : p = p_1$, where $0 < p_0 < p_1 < 1$ are some specified values. In a nutshell, we know that the probability of heads is either p_0 or p_1, but we do not know which.

1. Recall the definition of the probability mass function of the Bernoulli distribution.
2. Show that $Y = Y(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} X_i$ is the decision function of the Neyman-Pearson test of size α.

3. Give the distribution of Y.
4. Let $F_{n,p}$ be the cumulative distribution function of Y. With R, plot $F_{n,p}$ for some different values of $n \in \{1, 5, 10\}$ and $p \in \{0.2, 0.6\}$.
5. What is the theoretical link between $F_{n,p}$ and the threshold h_α of the test ?
6. Is it possible to achieve any value of α ? Otherwise, how many values of α can you achieve ?
7. Compute all the type I and type II theoretical errors by using the R function of $F_{n,p}$.
8. Generate $M = 1000$ sequences of $n = 10$ random samples with R under H_0 and under H_1. You will choose $p_0 = 0.5$ and $p_1 = 0.8$.
9. Estimate numerically the type I and type II errors of the test when $\alpha = 0.1$.
10. Compare the estimated errors and the theoretical errors when $\alpha = 0.1$.