Correction TD 3

1) Le filtre est sans distorsion si

$$m_t = \sum_{j=-q}^{q} a_j m_{t-j} \quad m_t \in \{1, t, \dots, t^k\}$$

On remplace successivement :

$m_t = 1 \Rightarrow \sum_{j=-q}^{q} a_j \cdot 1 = \sum_{j=-q}^{q} a_j = 1$

$m_t = t$: $\sum_{j=-q}^{q} a_j (t-j) = t \Leftrightarrow t \underbrace{\sum_{j=-q}^{q} a_j}_{=1} - \sum_{j=-q}^{q} j a_j = t$

$\Rightarrow \sum_{j=-q}^{q} j\, a_j = 0$

Par récurrence si on supp que $\sum_{j=-q}^{q} j^r a_j = 0$, $r = 1, \dots, k-1$

si $m_t = t^k$: $\sum_{j=-q}^{q} a_j (t-j)^k = t^k$

$\Rightarrow t^k \underbrace{\sum_{j=-q}^{q} a_j}_{1} - \binom{k}{1} t^{k-1} \underbrace{\sum_{j=-q}^{q} a_j j}_{=0} + \dots + (-1)^k \sum_{j=-q}^{q} a_j j^k = t^k$

$\Rightarrow \sum_{j=-q}^{q} j^k a_j = 0$

2) Si $m_t = c_0 + c_1 t$ linéaire $\Rightarrow B^3 m_t = m_{t-3}$

$B^2 m_t = m_{t-2} \Rightarrow (1 + \alpha B + \beta B^2 + \gamma B^3) m_t = m_t + \alpha m_{t-1} + \beta m_{t-2} + \gamma m_{t-3}$

Si $m_t = m_t + \alpha m_{t-1} + \beta m_{t-2} + \gamma m_{t-3}$

$\Rightarrow \alpha + \beta + \gamma = 0$, $\alpha + 2\beta + 3\gamma = 0$

$(1 + \alpha B + \beta B^2 + \gamma B^3) S_t = S_t + \alpha S_{t-1} + \beta S_{t-2} + \gamma S_{t-3}$

~~$= (\gamma + 1) S_t + \alpha S_{t-1} + \beta S_{t-2}$~~

$= (\beta + 1) S_t + (\alpha + \gamma) S_{t-1}$

$$\begin{cases} \alpha+\beta+\gamma=0 \\ \alpha+2\beta+3\gamma=0 \\ (\beta+1)S_t+(\alpha+\gamma)(-S_t)=0 \end{cases} \qquad (S_t+S_{t+1}=0)$$

$$\Leftrightarrow \begin{cases} \alpha+\beta+\gamma=0 \\ \alpha+2\beta+3\gamma=0 \\ \beta+1-\alpha-\gamma=0 \end{cases} \quad \begin{aligned} &\Rightarrow \gamma=-\alpha-\beta = \alpha \\ &\Rightarrow \alpha+2\beta+3(-\alpha-\beta)=0 \\ &\quad \Downarrow \\ &\quad \beta=-2\alpha \end{aligned}$$

$$-2\alpha+1-\alpha-\alpha=0 \quad \Rightarrow \alpha=1/4$$

$$\beta=-1/2 \ , \ \gamma=1/4$$

④ a) $X_t = a+bt+S_t+Y_t$

$$\nabla_{12}X_t = X_t - X_{t-12} = -\cancel{a}+b(t-12)+\cancel{a}+bt+\cancel{S_t-S_{t-12}}$$
$$+Y_t - Y_{t-12} = 12b + Y_t - Y_{t-12}$$

$$\nabla\nabla_{12}X_t = \nabla(12b+Y_t-Y_{t-12}) = Y_t - Y_{t-12} - Y_{t-1} + Y_{t-13}$$

$\Rightarrow$ stationnaire

$$\gamma(\nabla\nabla_{12}X_t) = \mathrm{cov}(Y_{t+h}-Y_{t+h-12}-Y_{t+h-1}+Y_{t+h-13}, Y_t-Y_{t-12}-Y_{t-1}+\ldots$$
$$= \gamma_Y(h)-\gamma_Y(h+12)+\gamma_Y(h+1)+\gamma_Y(h+13)$$
$$-\gamma_Y(h-12)+\gamma_Y(h)+\gamma_Y(h-11)-\gamma_Y(h+1)$$
$$-\gamma_Y(h-1)+\gamma_Y(h+11)+\gamma_Y(h)-\gamma_Y(h+12)$$
$$+\gamma_Y(h-13)-\gamma_Y(h-1)-\gamma_Y(h-12)+\gamma_Y(h)$$

~~simplifier ensuite~~ ...

5) $(1-B^{12})X_t = (a+bt)s_t + Y_t -$

$- (a+b(t-12))s_{t-12} - Y_{t-12}$

$= 12b\,s_t + Y_t - Y_{t-12}$

$s_{t-12} = s_t$

$(1-B^{12})(1-B^{12})X_t = \cancel{12b}\,s_t + Y_t - Y_{t-12}$

$- \cancel{12b}\,s_{t-12} - Y_{t-12} + Y_{t-24}$

$= Y_t - 2Y_{t-12} + Y_{t-24} \Rightarrow$ stationnaire

$\gamma(\nabla_{12}^2 X_t) = Cov(Y_t - 2Y_{t-12} + Y_{t-24},\ Y_{t+h} - 2Y_{t+h-12} + Y_{t+h-24})$

on développe comme avant

$= \gamma_Y(h) - 2\gamma_Y(h-12) + \gamma_Y(h-24) +$