Chapter 1

Problem 1.1. a) First note that

$$\begin{aligned}\mathbb{E}[(Y-c)^2] &= \mathbb{E}[Y^2 - 2Yc + c^2] = \mathbb{E}[Y^2] - 2c\mathbb{E}[Y] + c^2 \\ &= \mathbb{E}[Y^2] - 2c\mu + c^2.\end{aligned}$$

Find the extreme point by differentiating,

$$\frac{d}{dc}(\mathbb{E}[Y^2] - 2c\mu + c^2) = -2\mu + 2c = 0 \quad \Rightarrow c = \mu.$$

Since, $\frac{d^2}{dc^2}(\mathbb{E}[Y^2] - 2c\mu + c^2) = 2 > 0$ this is a min-point.
b) We have

$$\begin{aligned}\mathbb{E}[(Y-f(X))^2 \mid X] &= \mathbb{E}[Y^2 - 2Yf(X) + f^2(X) \mid X] \\ &= \mathbb{E}[Y^2 \mid X] - 2f(X)\mathbb{E}[Y \mid X] + f^2(X),\end{aligned}$$

which is minimized by $f(X) = \mathbb{E}[Y \mid X]$ (take $c = f(X)$ and $\mu = \mathbb{E}[Y \mid X]$ in a).
c) We have

$$\mathbb{E}[(Y-f(X))^2] = \mathbb{E}\big[\mathbb{E}[(Y-f(X))^2 \mid X]\big],$$

so the result follows from b).

Problem 1.4. a) For the mean we have

$$\mu_X(t) = \mathbb{E}[a + bZ_t + cZ_{t-2}] = a,$$

and for the autocovariance

$$\begin{aligned}\gamma_X(t+h,t) &= \mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}(a + bZ_{t+h} + cZ_{t+h-2}, a + bZ_t + cZ_{t-2}) \\ &= b^2\,\mathrm{Cov}(Z_{t+h}, Z_t) + bc\,\mathrm{Cov}(Z_{t+h}, Z_{t-2}) \\ &\quad + cb\,\mathrm{Cov}(Z_{t+h-2}, Z_t) + c^2\,\mathrm{Cov}(Z_{t+h-2}, Z_{t-2}) \\ &= \sigma^2 b^2 \mathbf{1}_{\{0\}}(h) + \sigma^2 bc\mathbf{1}_{\{-2\}}(h) + \sigma^2 cb\mathbf{1}_{\{2\}}(h) + \sigma^2 c^2 \mathbf{1}_{\{0\}}(h) \\ &= \begin{cases} (b^2+c^2)\sigma^2 & \text{if } h = 0, \\ bc\sigma^2 & \text{if } |h| = 2, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Since $\mu_X(t)$ and $\gamma_X(t+h,t)$ do not depend on t, $\{X_t : t \in \mathbb{Z}\}$ is (weakly) stationary.
b) For the mean we have

$$\mu_X(t) = \mathbb{E}[Z_1]\cos(ct) + \mathbb{E}[Z_2]\sin(ct) = 0,$$

and for the autocovariance

$$\begin{aligned}\gamma_X(t+h,t) &= \mathrm{Cov}(X_{t+h}, X_t) \\ &= \mathrm{Cov}(Z_1\cos(c(t+h)) + Z_2\sin(c(t+h)), Z_1\cos(ct) + Z_2\sin(ct)) \\ &= \cos(c(t+h))\cos(ct)\,\mathrm{Cov}(Z_1,Z_1) + \cos(c(t+h))\sin(ct)\,\mathrm{Cov}(Z_1,Z_2) \\ &\quad + \sin(c(t+h))\cos(ct)\,\mathrm{Cov}(Z_1,Z_2) + \sin(c(t+h))\sin(ct)\,\mathrm{Cov}(Z_2,Z_2) \\ &= \sigma^2(\cos(c(t+h))\cos(ct) + \sin(c(t+h))\sin(ct)) \\ &= \sigma^2\cos(ch)\end{aligned}$$

where the last equality follows since $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. Since $\mu_X(t)$ and $\gamma_X(t+h,t)$ do not depend on t, $\{X_t : t \in \mathbb{Z}\}$ is (weakly) stationary.
c) For the mean we have

$$\mu_X(t) = \mathbb{E}[Z_t]\cos(ct) + \mathbb{E}[Z_{t-1}]\sin(ct) = 0,$$

and for the autocovariance

$$\begin{aligned}
\gamma_X(t+h,t) &= \operatorname{Cov}(X_{t+h}, X_t) \\
&= \operatorname{Cov}(Z_{t+h}\cos(c(t+h)) + Z_{t+h-1}\sin(c(t+h)), Z_t\cos(ct) + Z_{t-1}\sin(ct)) \\
&= \cos(c(t+h))\cos(ct)\operatorname{Cov}(Z_{t+h}, Z_t) + \cos(c(t+h))\sin(ct)\operatorname{Cov}(Z_{t+h}, Z_{t-1}) \\
&\quad + \sin(c(t+h))\cos(ct)\operatorname{Cov}(Z_{t+h-1}, Z_t) \\
&\quad + \sin(c(t+h))\sin(ct)\operatorname{Cov}(Z_{t+h-1}, Z_{t-1}) \\
&= \sigma^2\cos^2(ct)\mathbf{1}_{\{0\}}(h) + \sigma^2\cos(c(t-1))\sin(ct)\mathbf{1}_{\{-1\}}(h) \\
&\quad + \sigma^2\sin(c(t+1))\cos(ct)\mathbf{1}_{\{1\}}(h) + \sigma^2\sin^2(ct)\mathbf{1}_{\{0\}}(h) \\
&= \begin{cases} \sigma^2\cos^2(ct) + \sigma^2\sin^2(ct) = \sigma^2 & \text{if } h=0, \\ \sigma^2\cos(c(t-1))\sin(ct) & \text{if } h=-1, \\ \sigma^2\cos(ct)\sin(c(t+1)) & \text{if } h=1, \end{cases}
\end{aligned}$$

We have that $\{X_t : t \in \mathbb{Z}\}$ is (weakly) stationary for $c = \pm k\pi$, $k \in \mathbb{Z}$, since then $\gamma_X(t+h,t) = \sigma^2\mathbf{1}_{\{0\}}(h)$. For $c \neq \pm k\pi$, $k \in \mathbb{Z}$, $\{X_t : t \in \mathbb{Z}\}$ is not (weakly) stationary since $\gamma_X(t+h,t)$ depends on t.
d) For the mean we have

$$\mu_X(t) = \mathbb{E}[a + bZ_0] = a,$$

and for the autocovariance

$$\gamma_X(t+h,t) = \operatorname{Cov}(X_{t+h}, X_t) = \operatorname{Cov}(a + bZ_0, a + bZ_0) = b^2\operatorname{Cov}(Z_0, Z_0) = \sigma^2 b^2.$$

Since $\mu_X(t)$ and $\gamma_X(t+h,t)$ do not depend on t, $\{X_t : t \in \mathbb{Z}\}$ is (weakly) stationary.
e) If $c = k\pi$, $k \in \mathbb{Z}$ then $X_t = (-1)^{kt}Z_0$ which implies that X_t is weakly stationary when $c = k\pi$. For $c \neq k\pi$ we have

$$\mu_X(t) = \mathbb{E}[Z_0]\cos(ct) = 0,$$

and for the autocovariance

$$\begin{aligned}
\gamma_X(t+h,t) &= \operatorname{Cov}(X_{t+h}, X_t) = \operatorname{Cov}(Z_0\cos(c(t+h)), Z_0\cos(ct)) \\
&= \cos(c(t+h))\cos(ct)\operatorname{Cov}(Z_0, Z_0) = \cos(c(t+h))\cos(ct)\sigma^2.
\end{aligned}$$

The process $\{X_t : t \in \mathbb{Z}\}$ is (weakly) stationary when $c = \pm k\pi$, $k \in \mathbb{Z}$ and not (weakly) stationary when $c \neq \pm k\pi$, $k \in \mathbb{Z}$, see 1.4. c).
f) For the mean we have

$$\mu_X(t) = \mathbb{E}[Z_t Z_{t-1}] = 0,$$

and

$$\begin{aligned}
\gamma_X(t+h,t) &= \operatorname{Cov}(X_{t+h}, X_t) = \operatorname{Cov}(Z_{t+h}Z_{t+h-1}, Z_t Z_{t-1}) \\
&= \mathbb{E}[Z_{t+h}Z_{t+h-1}Z_t Z_{t-1}] = \begin{cases} \sigma^4 & \text{if } h=0, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Since $\mu_X(t)$ and $\gamma_X(t+h,t)$ do not depend on t, $\{X_t : t \in \mathbb{Z}\}$ is (weakly) stationary.

Problem 1.5. a) We have

$$\begin{aligned}
\gamma_X(t+h,t) &= \operatorname{Cov}(X_{t+h}, X_t) = \operatorname{Cov}(Z_{t+h} + \theta Z_{t+h-2}, Z_t + \theta Z_{t-2}) \\
&= \operatorname{Cov}(Z_{t+h}, Z_t) + \theta\operatorname{Cov}(Z_{t+h}, Z_{t-2}) + \theta\operatorname{Cov}(Z_{t+h-2}, Z_t) \\
&\quad + \theta^2\operatorname{Cov}(Z_{t+h-2}, Z_{t-2}) \\
&= \mathbf{1}_{\{0\}}(h) + \theta\mathbf{1}_{\{-2\}}(h) + \theta\mathbf{1}_{\{2\}}(h) + \theta^2\mathbf{1}_{\{0\}}(h) \\
&= \begin{cases} 1+\theta^2 & \text{if } h=0, \\ \theta & \text{if } |h|=2. \end{cases} = \begin{cases} 1.64 & \text{if } h=0, \\ 0.8 & \text{if } |h|=2. \end{cases}
\end{aligned}$$

Hence the ACVF depends only on h and we write $\gamma_X(h) = \gamma_X(t+h,h)$. The ACF is then

$$\rho(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \begin{cases} 1 & \text{if } h = 0, \\ 0.8/1.64 \approx 0.49 & \text{if } |h| = 2. \end{cases}$$

b) We have

$$\begin{aligned} \operatorname{Var}\left(\frac{1}{4}(X_1+X_2+X_3+X_4)\right) &= \frac{1}{16}\operatorname{Var}(X_1+X_2+X_3+X_4) \\ = \frac{1}{16}\Big(&\operatorname{Var}(X_1)+\operatorname{Var}(X_2)+\operatorname{Var}(X_3)+\operatorname{Var}(X_4)+2\operatorname{Cov}(X_1,X_3) \\ &+2\operatorname{Cov}(X_2,X_4)\Big) \\ = \frac{1}{16}\big(4\gamma_X(0)+4\gamma_X(2)\big) &= \frac{1}{4}\big(\gamma_X(0)+\gamma_X(2)\big) = \frac{1.64+0.8}{4} = 0.61. \end{aligned}$$

c) $\theta = -0.8$ implies $\gamma_X(h) = -0.8$ for $|h| = 2$ so

$$\operatorname{Var}\left(\frac{1}{4}(X_1+X_2+X_3+X_4)\right) = \frac{1.64-0.8}{4} = 0.21.$$

Because of the negative covariance at lag 2 the variance in c) is considerably smaller.

Problem 1.8. a) First we show that $\{X_t : t \in \mathbb{Z}\}$ is $\mathrm{WN}(0,1)$. For t even we have $\mathbb{E}[X_t] = \mathbb{E}[Z_t] = 0$ and for t odd

$$\mathbb{E}[X_t] = \mathbb{E}\left[\frac{Z_{t-1}^2-1}{\sqrt{2}}\right] = \frac{1}{\sqrt{2}}\mathbb{E}[Z_{t-1}^2-1] = 0.$$

Next we compute the ACVF. If t is even we have $\gamma_X(t,t) = \mathbb{E}[Z_t^2] = 1$ and if t is odd

$$\gamma_X(t,t) = \mathbb{E}\left[\left(\frac{Z_{t-1}^2-1}{\sqrt{2}}\right)^2\right] = \frac{1}{2}\mathbb{E}[Z_{t-1}^4-2Z_{t-1}^2+1] = \frac{1}{2}(3-2+1) = 1.$$

If t is even we have

$$\gamma_X(t+1,t) = \mathbb{E}\left[\frac{Z_t^2-1}{\sqrt{2}}Z_t\right] = \frac{1}{\sqrt{2}}\mathbb{E}[Z_t^3-Z_t] = 0,$$

and if t is odd

$$\gamma_X(t+1,t) = \mathbb{E}\left[Z_{t+1}\frac{Z_{t-1}^2-1}{\sqrt{2}}\right] = \mathbb{E}[Z_{t+1}]\mathbb{E}\left[\frac{Z_{t-1}^2-1}{\sqrt{2}}\right] = 0.$$

Clearly $\gamma_X(t+h,t) = 0$ for $|h| \geq 2$. Hence

$$\gamma_X(t+h,h) = \begin{cases} 1 & \text{if } h = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\{X_t : t \in \mathbb{Z}\}$ is $\mathrm{WN}(0,1)$. If t is odd X_t and X_{t-1} is obviously dependent so $\{X_t : t \in \mathbb{Z}\}$ is *not* $\mathrm{IID}(0,1)$.
b) If n is odd

$$\mathbb{E}[X_{n+1} \mid X_1,\dots,X_n] = \mathbb{E}[Z_{n+1} \mid Z_0, Z_2, Z_4 \dots, Z_{n-1}] = \mathbb{E}[Z_{n+1}] = 0.$$

If n is even

$$\mathbb{E}[X_{n+1} \mid X_1,\dots,X_n] = \mathbb{E}\left[\frac{Z_n^2-1}{\sqrt{2}} \mid Z_0, Z_2, Z_4, \dots, Z_n\right] = \frac{Z_n^2-1}{\sqrt{2}} = \frac{X_n^2-1}{\sqrt{2}}.$$

This again shows that $\{X_t : t \in \mathbb{Z}\}$ is not $\mathrm{IID}(0,1)$.