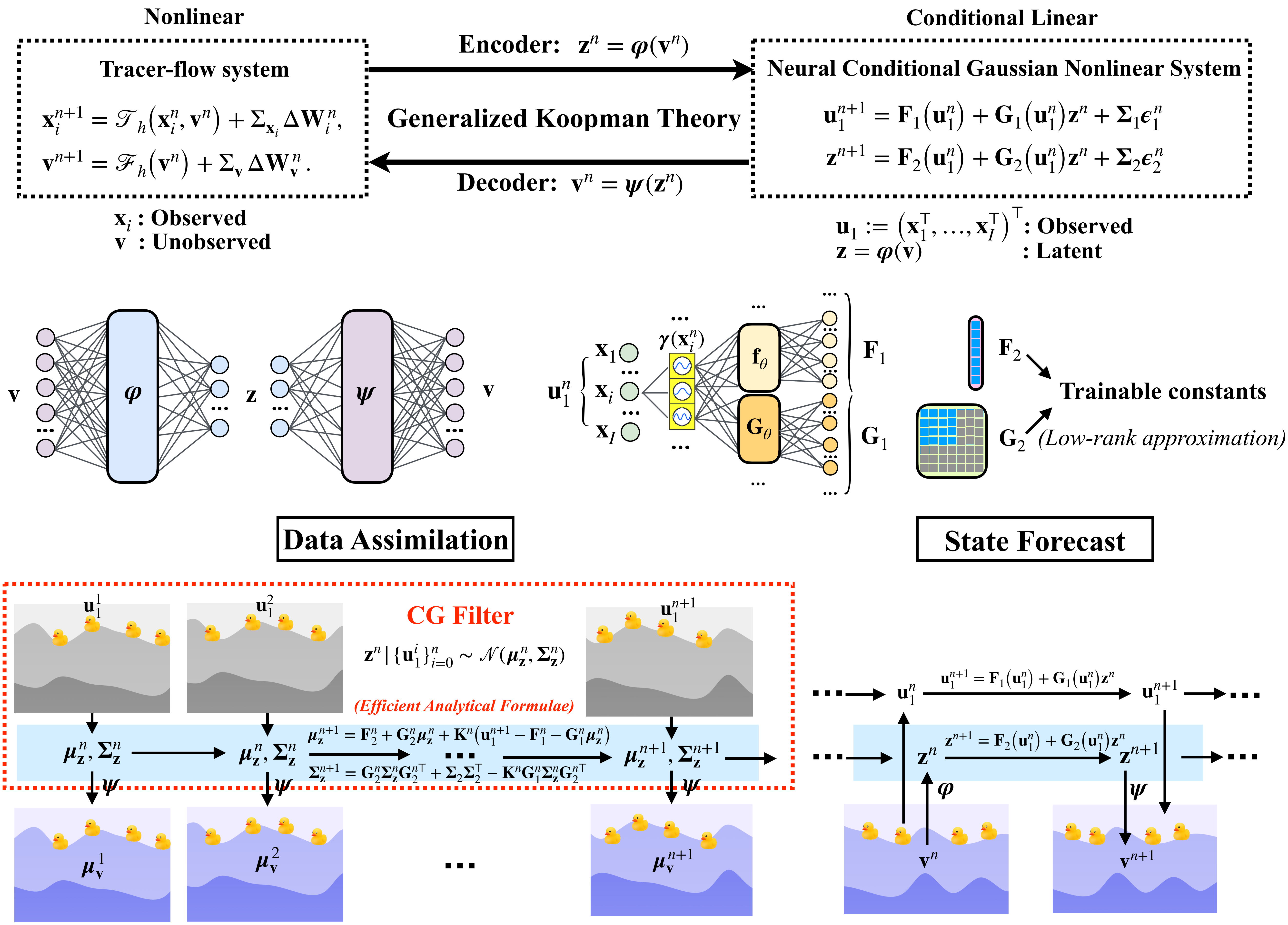

Nonlinear
Conditional Linear
Tracer-flow system
$\mathbf{x}_i^{n+1} = \mathscr{T}_h(\mathbf{x}_i^n, \mathbf{v}^n) + \Sigma_{\mathbf{x}_i} \Delta \mathbf{W}_i^n,$
$\mathbf{v}^{n+1} = \mathscr{F}_h(\mathbf{v}^n) + \Sigma_{\mathbf{v}} \Delta \mathbf{W}_{\mathbf{v}}^n.$
Encoder: $\mathbf{z}^n = \boldsymbol{\varphi}(\mathbf{v}^n)$
Generalized Koopman Theory
Decoder: $\mathbf{v}^n = \boldsymbol{\psi}(\mathbf{z}^n)$
Neural Conditional Gaussian Nonlinear System
$\mathbf{u}_1^{n+1} = \mathbf{F}_1(\mathbf{u}_1^n) + \mathbf{G}_1(\mathbf{u}_1^n)\mathbf{z}^n + \boldsymbol{\Sigma}_1 \boldsymbol{\epsilon}_1^n$
$\mathbf{z}^{n+1} = \mathbf{F}_2(\mathbf{u}_1^n) + \mathbf{G}_2(\mathbf{u}_1^n)\mathbf{z}^n + \boldsymbol{\Sigma}_2 \boldsymbol{\epsilon}_2^n$
$\mathbf{x}_i$: Observed
$\mathbf{v}$: Unobserved
$\mathbf{u}_1 := (\mathbf{x}_1^\top, \ldots, \mathbf{x}_I^\top)^\top$: Observed
$\mathbf{z} = \boldsymbol{\varphi}(\mathbf{v})$: Latent
$\mathbf{v}$
$\boldsymbol{\varphi}$
$\mathbf{z}$
$\boldsymbol{\psi}$
$\mathbf{v}$
$\mathbf{u}_1^n$
$\mathbf{x}_1$
$\mathbf{x}_i$
$\mathbf{x}_I$
$\gamma(\mathbf{x}_i^n)$
$\mathbf{f}_\theta$
$\mathbf{G}_\theta$
$\mathbf{F}_1$
$\mathbf{G}_1$
$\mathbf{F}_2$
Trainable constants
$\mathbf{G}_2$ (Low-rank approximation)
Data Assimilation
State Forecast
CG Filter
$\mathbf{z}^n \mid \{\mathbf{u}_1^i\}_{i=0}^n \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{z}}^n, \boldsymbol{\Sigma}_{\mathbf{z}}^n)$
(Efficient Analytical Formulae)
$\mathbf{u}_1^1$
$\mathbf{u}_1^2$
$\mathbf{u}_1^{n+1}$
$\boldsymbol{\mu}_{\mathbf{z}}^n, \boldsymbol{\Sigma}_{\mathbf{z}}^n$
$\boldsymbol{\mu}_{\mathbf{z}}^n, \boldsymbol{\Sigma}_{\mathbf{z}}^n$
$\boldsymbol{\mu}_{\mathbf{z}}^{n+1} = \mathbf{F}_2^n + \mathbf{G}_2^n \boldsymbol{\mu}_{\mathbf{z}}^n + \mathbf{K}^n(\mathbf{u}_1^{n+1} - \mathbf{F}_1^n - \mathbf{G}_1^n \boldsymbol{\mu}_{\mathbf{z}}^n)$
$\boldsymbol{\Sigma}_{\mathbf{z}}^{n+1} = \mathbf{G}_2^n \boldsymbol{\Sigma}_{\mathbf{z}}^n \mathbf{G}_2^{n\top} + \boldsymbol{\Sigma}_2 \boldsymbol{\Sigma}_2^\top - \mathbf{K}^n \mathbf{G}_1^n \boldsymbol{\Sigma}_{\mathbf{z}}^n \mathbf{G}_2^{n\top}$
$\boldsymbol{\mu}_{\mathbf{z}}^{n+1}, \boldsymbol{\Sigma}_{\mathbf{z}}^{n+1}$
$\boldsymbol{\psi}$
$\boldsymbol{\mu}_{\mathbf{v}}^1$
$\boldsymbol{\mu}_{\mathbf{v}}^2$
$\boldsymbol{\mu}_{\mathbf{v}}^{n+1}$
$\mathbf{u}_1^n$
$\mathbf{u}_1^{n+1} = \mathbf{F}_1(\mathbf{u}_1^n) + \mathbf{G}_1(\mathbf{u}_1^n)\mathbf{z}^n$
$\mathbf{u}_1^{n+1}$
$\mathbf{z}^n$
$\mathbf{z}^{n+1} = \mathbf{F}_2(\mathbf{u}_1^n) + \mathbf{G}_2(\mathbf{u}_1^n)\mathbf{z}^n$
$\mathbf{z}^{n+1}$
$\boldsymbol{\varphi}$
$\boldsymbol{\psi}$
$\mathbf{v}^n$
$\mathbf{v}^{n+1}$