Lagrangian Data Assimilation

Flow model – two-layer QG with topography

$$\vec{v}(x,t) = \sum_k \hat{v}_k(t) e^{ik\cdot x} \vec{r}_k$$

$$\frac{\partial q_i}{\partial t} + J(\psi_i, q_i) + \beta \frac{\partial \psi_i}{\partial x} + U_i \frac{\partial}{\partial x} \Delta\psi_i + \frac{k_d^2}{2}\left(U_i \frac{\partial \psi_{3-i}}{\partial x} - U_{3-i}\frac{\partial \psi_i}{\partial x}\right)$$
$$= -\delta_{i2}\left(U_i \frac{\partial h}{\partial x} + \kappa \Delta\psi_i\right)$$

$$q_1 = \Delta\psi_1 + \frac{k_d^2}{2}(\psi_2 - \psi_1)$$
$$q_2 = \Delta\psi_2 + \frac{k_d^2}{2}(\psi_1 - \psi_2) + h$$

$$\begin{pmatrix} q_{1,k} \\ q_{2,k} \end{pmatrix} = \begin{pmatrix} -k^2 - \frac{k_d^2}{2} & \frac{k_d^2}{2} \\ \frac{k_d^2}{2} & -k^2 - \frac{k_d^2}{2} \end{pmatrix} \begin{pmatrix} \hat{\psi}_{1,k} \\ \hat{\psi}_{2,k} \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{h}_k \end{pmatrix}$$

$$\vec{v} = \nabla^\perp \psi_1 = i \sum_k (-k_y, k_x) \hat{\psi}_{1,k} e^{ik\cdot x}$$

$$\begin{bmatrix} \hat{q}_{1,k} \\ \hat{q}_{2,k} \end{bmatrix} = T \begin{bmatrix} \hat{\psi}_{1,k} \\ \hat{\psi}_{2,k} \end{bmatrix} + \begin{bmatrix} 0 \\ \hat{h}_k \end{bmatrix} \qquad \hat{\psi}_{1,k}(t) = T_1^{-1} \begin{bmatrix} \hat{q}_{1,k}(t) \\ \hat{q}_{2,k}(t) - \hat{h}_k \end{bmatrix}$$

$$q_\psi = (q_1 + q_2)\tfrac{1}{2} = \Delta\psi + \frac{h}{2}, \qquad \psi = \frac{\psi_1 + \psi_2}{2}$$
$$q_\tau = (q_1 - q_2)\tfrac{1}{2} = \Delta\tau - k_d^2 \tau - \frac{h}{2}, \qquad \tau = \frac{\psi_1 - \psi_2}{2}$$

$$\begin{pmatrix} \hat{q}_{\psi,k} - \frac{h_k}{2} \\ \hat{q}_{\tau,k} + \frac{h_k}{2} \end{pmatrix} = \begin{pmatrix} -k^2 & 0 \\ 0 & -k^2 - k_d^2 \end{pmatrix}$$

$$-\begin{bmatrix} k^{-2}\left(\hat{q}_{\psi,k} - \frac{h_k}{2}\right) \\ (k^2 + k_d^2)^{-1}\left(\hat{q}_{\tau,k} + \frac{h_k}{2}\right) \end{bmatrix} = \begin{pmatrix} \psi_k \\ \tau_k \end{pmatrix}$$

$$\hat{\psi}_{1,k} = \psi + \tau = \underbrace{-\frac{1}{k^2}\hat{q}_{\psi,k}(t) - \frac{1}{k^2 + k_d^2}\hat{q}_{\tau,k}(t)}_{= q_k(t)} + \underbrace{\frac{k_d^2}{2(k^2+k_d^2)k^2}\hat{h}_k}_{= h_k}$$

$$\vec{v}(x,t) = \sum_k \left(\hat{q}_k(t) + h_k\right) e^{ik\cdot x} \vec{r}_k \qquad \vec{r}_k = \begin{pmatrix} -ik_y \\ k_x \end{pmatrix} \qquad ?$$

$$H_k q_k = \left[-\frac{1}{k^2}, -\frac{1}{k^2 + k_d^2}\right], \qquad f_k = \frac{k_d^2}{2(k^2+k_d^2)}\hat{h}_k$$

Lagrangian tracer model

$$\frac{d\vec{x}_t^{\ell}}{dt} = \vec{u}(\vec{x}_t^{\ell}, t)\,dt + \Sigma\, dW_t^{\ell}$$

$$= \sum_k \left(q_k(t) + h_k\right) e^{i\vec{k}\cdot \vec{x}_t^{\ell}}\, \vec{r}_k\, dt + \Sigma\, dW_t^{\ell}$$

Data assimilation model $\quad \underline{K > L}$

$$\vec{u}_t = \begin{bmatrix} \vec{q}_1 \\ \vdots \\ \vec{q}_k \\ \vdots \\ \vec{q}_K \end{bmatrix} \in \mathbb{C}^{2K} \qquad \vec{q}_k = \begin{bmatrix} \hat{q}_{1,t,k} \\ \hat{q}_{2,t,k} \end{bmatrix}$$

$$\frac{d\vec{q}_k}{dt} + B_k(\vec{u}_t, \vec{u}_t) + L_k \vec{q}_k = -D_k \vec{q}_k$$

$$L = \begin{bmatrix} -L_1 - D_1 & & & \\ & L_2 - D_2 & & \\ & & \ddots & \\ & & & -L_K - D_K \end{bmatrix}_{2K\times 2K} \qquad \vec{B} = \begin{bmatrix} B_1 \\ \vdots \\ B_k \\ \vdots \\ B_K \end{bmatrix} \in \mathbb{C}^{2K}$$

① $$\frac{d\vec{u}_t}{dt} = \vec{B}(\vec{u}_t, \vec{u}_t) + L\vec{u}_t$$

$2\times 2K$-dim

$$\sim L\vec{u}_t \underbrace{- \Gamma^M \vec{u}_t + F^M + \Sigma^M \dot{B}_t}_{\text{Model parametrization} - 2\times 2 \text{ blocks}}$$

$$\vec{x}_t = \begin{bmatrix} \vec{x}_t^1 \\ \vdots \\ \vec{x}_t^{\ell} \\ \vdots \\ \vec{x}_t^L \end{bmatrix} \in \mathbb{R}^{2L}$$

$$\frac{d\vec{x}_t^{\ell}}{dt} = \left[H(\vec{x}_t^{\ell})\vec{u} + F(\vec{x}_t^{\ell})\right] + \Sigma W_t^{\ell}$$

$$= R\,P(\vec{x}_t^{\ell})\,\Gamma\,\vec{u}_t + R\,P(\vec{x}_t^{\ell})\,\vec{f} + \Sigma W_t$$

$$R^{2\times K} = [\vec{r}_1, \cdots, \vec{r}_k, \cdots \vec{r}_K]$$

$$P^{K\times K}(\vec{x}) = \mathrm{diag}\left(e^{i1\cdot x}, \cdots, e^{ik\cdot x}, \cdots, e^{iK\cdot x}\right)$$

$$\Gamma^{K\times 2K} = \begin{bmatrix} H_1 & & & & \\ & \ddots & & & \\ & & H_k & & \\ & & & \ddots & \\ & & & & H_K \end{bmatrix} \qquad H_k = \left[-\frac{1}{k^2}, -\frac{1}{k^2 + k_d^2}\right]$$

$$\vec{f}^{K\times 1} = \begin{bmatrix} f_1 \\ \vdots \\ f_K \end{bmatrix} \qquad f_k = \frac{k_d^2}{k^2 + k_d^2}\,\frac{\hat{h}_k}{2}$$

$$H = \left[\vec{r}_1 H_1 e^{i1\cdot x^{\ell}}, \cdots \vec{r}_k H_k e^{ik\cdot x^{\ell}}, \cdots \vec{r}_K H_K e^{iK\cdot x^{\ell}}\right]_{2\times 2K}$$

$$F = \left[\vec{r}_1 e^{i1\cdot x^{\ell}}, \cdots, \vec{r}_K e^{iK\cdot x^{\ell}}\right]_{2\times K} \begin{bmatrix} f_1 \\ \vdots \\ f_K \end{bmatrix}$$

$$\text{②} \quad \frac{d\vec{X}_t}{dt} = G(\vec{X}_t)\,\vec{u}_t + \mathbb{1}_L \otimes \underset{\underset{\text{"}}{}}{\overset{\overset{F_t}{\text{"}}}{F(\vec{X}_t)}} + \mathbb{1}_L \otimes \Sigma_x \vec{W}_t$$

2L-dim

$$G(\vec{X}_t) = \begin{bmatrix} RP(X_t^1)\Gamma \\ \vdots \\ RP(X_t^L)\Gamma \end{bmatrix}_{2L\times 2K} \qquad \begin{bmatrix} RP(X_t^1)f \\ \vdots \\ RP(X_t^L)f \end{bmatrix}_{2L\times 1}$$

Kalman-Bucy Filter (with observations $\tilde{X}_t$ and model parametrization Γ^M, F^M, Σ^M)

$$\begin{cases} \dfrac{d\mu_t}{dt} = [(L-\Gamma^M)\mu_t + F^M]\,dt \\ \qquad\qquad + \Sigma_x^{-2} R_t\, G^*(\tilde{X}_t)\left[\dfrac{d\tilde{X}_t}{dt} - G(\tilde{X}_t)\mu_t - F(\tilde{X}_t)\right] \\ \\ \dfrac{dR_t}{dt} = +(L-\Gamma^M)R_t + R_t(L-\Gamma^M)^* + \Sigma^M\Sigma^{M*} \\ \qquad\qquad - \Sigma_x^{-2} R_t\, G^*(\tilde{X}_t)\,G(\tilde{X}_t)\,R_t \end{cases}$$