RIGOROUS STATISTICAL UNCERTAINTY QUANTIFICATION FOR ONE-LAYER TURBULENT GEOPHYSICAL FLOWS

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9/27/2017

Contents

In this report, we discuss the stability theory of steady state jets and mean flow for the two-dimensional barotropic systems. Due to the internal and external instabilities generally existing in the turbulent systems, it is more reasonable to investigate the evolution of the probability distributions (specifically here, represented by *statistics of an ensemble of trajectories*). One interesting question is whether the mean steady state structures can be persistent due to perturbations from initial uncertainty and internal instabilities.

In the structure of the report, we begin with a general description of the statistical features of the topographic barotropic flow in various parameter regimes. First in the regime with parameter $\mu > 0$, equilibrium statistical mechanics predicts pseudo-energy with an invariant measure in the statistical steady state. While on the other hand with $\mu < 0$, negative coefficients in the pseudo-energy forces us to separate the system into a stable and unstable subspace. The following parts of the report will focus on these three situations:

- **i):** The saturation bounds for barotropic flow with mean flow interaction but without forcing and dissipation. The saturation statistical structure is related with the initial setup of the mean and variance in the ensemble since no energy source and sink;
- **ii):** The saturation bounds for barotropic flow with special forms of external forcing and damping. In this case the initial configuration becomes no longer important and the forcing and damping can introduce similar structure as in the initial conditions in the unforced-undamped case;
- **iii):** The saturation bounds for small scale barotropic flow without mean flow $U = 0$. The enstrophy is only conserved in this case with no beta-effect $\beta = 0$. And for the flow on the beta-plane, there could exist unbounded growth in the total statistical energy.

1. Statistical Properties of the Truncated Topographic Barotropic Equations and the Corresponding Fluctuation Equations

We consider the formulation of the barotropic system based on Galerkin projection of the state variables within wavenumbers in the range $1\leq|\mathbf{k}|\leq\Lambda$. Assuming periodic boundary condition, we have the spectral expansion of variables under Fourier modes

$$\psi_{\Lambda}=\sum_{1\leq|\mathbf{k}|\leq\Lambda}\hat{\psi}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}},\quad\omega_{\Lambda}=\sum_{1\leq|\mathbf{k}|\leq\Lambda}\hat{\omega}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}=-\sum_{1\leq|\mathbf{k}|\leq\Lambda}\frac{\hat{\omega}_{\mathbf{k}}}{|\mathbf{k}|^{2}}e^{i\mathbf{k}\cdot\mathbf{x}},\quad h_{\Lambda}=\sum_{1\leq|\mathbf{k}|\leq\Lambda}\hat{h}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}.$$

For the general topographic barotropic flow *with dissipation and forcing*, the system can be formulated in small scale vorticity ω and large scale background mean flow U as

$$\frac{\partial\omega_{\Lambda}}{\partial t}+\beta\frac{\partial\psi_{\Lambda}}{\partial x}+U\left(t\right)\frac{\partial q_{\Lambda}}{\partial x}+\mathcal{P}_{\Lambda}\left(\nabla^{\perp}\psi_{\Lambda}\cdot\nabla q_{\Lambda}\right)=-\mathcal{D}\left(\Delta\right)\omega_{\Lambda}+\mathcal{F}_{\Lambda},\tag{1.1a}$$

$$\frac{dU}{dt}+\fint\frac{\partial h_{\Lambda}}{\partial x}\psi_{\Lambda}\left(t\right)=-\mathcal{D}_{0}U+\mathcal{F}_{0},\tag{1.1b}$$

with $q_{\Lambda}=\omega_{\Lambda}+h_{\Lambda}$, $\omega_{\Lambda}=\Delta\psi_{\Lambda}$. The damping and forcing operators are chosen in the general way as

$$\mathcal{D}\left(\Delta\right)=\sum_{j=0}^{L}d_{j}\left(-1\right)^{j}\Delta^{j},\quad\mathcal{F}_{\Lambda}=\sum_{1\leq|\mathbf{k}|\leq\Lambda}\hat{F}_{\mathbf{k}}\left(t\right)e^{i\mathbf{k}\cdot\mathbf{x}}+\dot{W}_{\mathbf{k}}\hat{\sigma}_{\mathbf{k}}\left(t\right)e^{i\mathbf{k}\cdot\mathbf{x}},\quad\mathcal{F}_{0}=F_{0}+\dot{W}_{0}\sigma_{0}\left(t\right),$$

where $\mathcal{D}_0,\mathcal{F}_0$ are scalars for the damping and forcing on the mean flow field U. We also include stochastic components in the external forcing $\mathcal{F},\mathcal{F}_0$ to represent the unresolved small-scale effects. Note that the above equations (1.1) will reach the continuous limit as the truncation size approaches infinity $\Lambda\rightarrow\infty$.

1.1. Nonlinear stability and equilibrium statistical mechanics without forcing and damping.

In the first place, we focus on the internal interactions among the large and small scale modes in the system (1.1) without forcing and dissipation. Correspondingly, we can find the dynamical equations for mean flow and spectral vorticity modes $(U,\hat{\omega}_{\mathbf{k}})$ under Fourier decomposition. Nonlinearity in the system includes the mean-mode interaction in U, and the interactions between the small-scale modes with the triad relation $\mathbf{k}=\mathbf{m}+\mathbf{n}$. The topography plays the role that mediates the energy transfer between the small vortical modes, $\hat{\omega}_{\mathbf{k}}$, and large-scale mean flow, U.

Here we are interested in the *fluctuation* parts of the variables away from a presumed mean basic state, thus the statistical variability in the mean and variance can be investigated. The fluctuation equations together with the conserved pseudo-energy are discussed in the [*Majda-Wang book*]. In deriving the fluctuation equations, we decompose the quantities of interest into the *time-averaged steady mean state* (denoted by upper case letters) and the *statistical fluctuations about the mean* (denoted by lower case letters with tildes). That is, for state variables of the dynamical system such as the stream functions and the potential vorticity, we consider the following decomposition

$$\psi_{\Lambda}\left(\mathbf{x},t\right)=\Psi\left(\mathbf{x}\right)+\tilde{\psi}\left(\mathbf{x},t\right),\quad q_{\Lambda}\left(\mathbf{x},t\right)=Q\left(\mathbf{x}\right)+\tilde{\omega}\left(\mathbf{x},t\right),\quad U\left(t\right)=V+\tilde{U}\left(t\right).\tag{1.2}$$

In the rest parts we focus on the fluctuation components with tildes and abuse the notation a little and leave the 'tildes' in these components. We also decompose the large-scale zonal flow $U\left(t\right)$ into the time-averaged steady state mean V and the fluctuation $\tilde{U}\left(t\right)$ about this mean state. Note that in the general set-up with rotation and large-scale flow included, the full stream function Φ contains the large-scale mean flow $-Vy$ and the mean averaged stream functions, and the full potential vorticity P contains the large-scale part βy and small-scale relative vorticity, so that

$$P=Q+\beta y=\Delta\Psi+h+\beta y,\quad\Phi=\Psi-Vy.$$

In general, we assume P and Φ are functionally dependent, that is, $f\left(P\right)=\Phi$. In this way, the nonlinear interaction in (1.1a), $\nabla^{\perp}\Phi\cdot\nabla P$, is eliminated. Such assumption makes the potential vorticity equation above linear, it also makes the elliptic equation to solve the stream function through

$$f^{-1}\left(\Phi\right)=f^{-1}\left(\Psi-Vy\right)=\Delta\Psi+h+\beta y.$$

By solving the above nonlinear equation, we can discover a set of time-independent steady state exact solutions (Q,Ψ,V) of the barotropic equations without damping and forcing. Indeed, if we substitute the relations back to the original equations (1.1a) and (1.1b), it is easy to check (Q,V) is exactly the solution of the equations in (1.1).

We focus on *a special set of exact solutions* with linear dependence in the P-Φ relation. The special form of the exact mean steady state with linear dependence between the stream function and potential vorticity is then chosen as

$$Q_\mu = \mu\Psi_\mu = \Delta\Psi_\mu + h, \quad V_\mu = -\beta/\mu. \tag{1.3}$$

Now the exact solution is split into the large-scale mean flow and vorticity component. The parameter μ is taken to represent the linear dependence, that is, we take the functional $f = \mu^{-1} = \text{const.}$ from the general relation. μ thus can be viewed as the eigenvalue of the elliptic operator with associated eigenfunction given by Ψ. V represents the large-scale mean jet flow velocity. In the northern hemisphere with $\beta > 0$, a positive $\mu > 0$ represents *westward large-scale mean jet*, while a negative $\mu < 0$ represents a *eastward jet*. Especially for the spectral modes under Fourier basis, each steady state streamfunction mode is determined through the topographic structure in the flow system

$$\hat{\Psi}_{\mu,\mathbf{k}} = \frac{\hat{h}_\mathbf{k}}{\mu + |\mathbf{k}|^2} \quad \hat{Q}_{\mu,\mathbf{k}} = \frac{\mu\hat{h}_\mathbf{k}}{\mu + |\mathbf{k}|^2}, \quad h_\Lambda = \sum_{1\leq|\mathbf{k}|\leq\Lambda} \hat{h}_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

With the existence of topography, in general, solvable solution exists only if μ is not eigenvalue of the Laplacian Δ in the non-zero topographic mode wavenumber. Furthermore, it has been shown in [*Majda-Wang book*] through equilibrium statistical theory that there exists one *invariant Gibbs measure* for the truncated barotropic equation with no dissipation and forcing (1.1a)-(1.1b), which is a product of Gaussian measures with large and small scale mean, (V_μ, Q_μ), satisfying the linear relation in (1.3)

$$p_{\text{eq}}(U,\omega;\mu) = C^{-1}\exp\left(-\frac{\sigma_{\text{eq}}^{-2}}{2}\mu(U-V_\mu)^2 - \frac{\sigma_{\text{eq}}^{-2}}{2}\sum_\mathbf{k}\left(1+\mu|\mathbf{k}|^{-2}\right)\left|\hat{q}_\mathbf{k} - \hat{Q}_{\mu,\mathbf{k}}\right|^2\right), \tag{1.4}$$

with σ_{eq} some additional parameter for the equilibrium energy amplitude. One issue about the above invariance distribution (1.4) is that when $\mu < 0$, the equilibrium measure becomes ill-posed and is no longer valid as the invariant measure.

In the following part of this report, we will focus on the dynamical equations about above fluctuation parts in ω and ψ (the 'tildes' for fluctuation components and subscripts 'Λ' for truncated variables are neglected here and next), and investigate their statistical dynamics using the statistically conserved quantities. In the central question we concern about *the statistical stability in the evolution of an ensemble of trajectories in the system about the steady state*: i) whether the statistical mean state stay near the steady state basic solution in (1.3) with initial and external perturbations; and ii) how is the uncertainty in the mean flow and vorticity fluctuation modes characterized by the second-order variances.

1.1.1. *Fluctuation equations with no dissipation and forcing.* Here in the first step, we concentrate on the linear and nonlinear interaction parts in the fluctuation equations without the inclusion of dissipation and external forcing terms. Focusing on the nonlinear advection term, we get the equation for perturbations with the steady state mean satisfying the relation in (1.3) as

$$\begin{aligned}\frac{\partial\omega}{\partial t} + \mathcal{P}_\Lambda\left(\nabla^\perp\psi\cdot\nabla\omega\right) + \mathcal{P}_\Lambda\left(\nabla^\perp\Psi_\mu\cdot\nabla(\omega-\mu\psi)\right) \\ -\frac{\beta}{\mu}\frac{\partial}{\partial x}(\omega-\mu\psi) + \frac{\partial}{\partial x}(Q_\mu+\omega)U(t)\end{aligned} = 0, \tag{1.5a}$$

$$\frac{dU}{dt} + \fint \frac{\partial h}{\partial x}\psi(t) = 0. \tag{1.5b}$$

with $\omega = \Delta\psi$. (ω,ψ,U) represent the fluctuation components subtracting the steady state mean (Q_μ,Ψ_μ,V_μ) in (1.3) depending on the parameter μ. The first part $\mathcal{P}_\Lambda\left(\nabla^\perp\Psi_\mu\cdot\nabla(\omega-\mu\psi)\right)$ in the fluctuation equation is a linear operator reflecting the steady mean flow Ψ_μ (this part can be viewed as a skew-symmetric operator), and the second part $\mathcal{P}_\Lambda\left(\nabla^\perp\psi\cdot\nabla\omega\right)$ is the familiar nonlinear interactions between the fluctuation modes (the same as the original system, we can see this quadratic interaction conserves both energy and enstrophy and satisfies the detailed triad symmetry).

Besides the quadratic terms, two additional terms will enter the fluctuation system representing the effects due to rotation β and large-scale flow U. That is,

$$\beta\frac{\partial(\Psi_\mu+\psi)}{\partial x} + (V_\mu+U)\frac{\partial(Q_\mu+\omega)}{\partial x} = -\frac{\beta}{\mu}\frac{\partial}{\partial x}(\omega-\mu\psi) + \frac{\partial}{\partial x}(Q_\mu+\omega)U(t).$$

Above we simplify these terms using the linear relation (1.3) for steady state solutions. First note that the first term on the left hand side due to β-effect is the same as before in the original barotropic equation. It forms a skew-symmetric operator also and conserves both statistical energy and enstrophy. Thus the pseudo-energy is still conserved under the rotation effect and it will have no explicit effect for the statistical dynamics. Then the second term represents the effect from the large-scale mean flow $V_\mu + U$ (with V_μ the steady state mean flow and U the fluctuation about the steady state), which is balanced by the total topographic stress in the mean flow equation (1.5b). To summarize from the fluctuation equation (1.5), the *pseudo-energy* defined as a combination of the energy and enstrophy with the fluctuation part

$$E_\mu = \mathcal{E}_\Lambda + \mu E_\Lambda = \frac{\mu}{2}U^2 + \frac{1}{2}\fint \left(\omega^2 + \mu\left|\nabla\psi\right|^2\right), \quad \frac{dE_\mu}{dt} \equiv 0, \tag{1.6}$$

is conserved for this fluctuation dynamics (1.5a). Therefore the Theorem from [*Majda, PNAS* 2015] implies that we can derive the statistical energy conservation law according to this conserved pseudo-energy.

1.1.2. *Statistical nonlinear stability in the fluctuations.* Now we consider the statistical formulation for the pseudo-energy E^{stat} for a combination of the energy in the mean and second-order variances. We can decompose the fluctuation variables further into *the statistical mean state* and *the disturbance about the statistical mean* (here and after we will use overbar '$\bar{\cdot}$' to denote ensemble average)

$$U = \bar{U} + U',\ \omega = \bar{\omega} + \omega',\ \psi = \bar{\psi} + \psi', \quad \overline{U'} = \overline{\omega'} = \overline{\psi'} = 0, \quad \Delta\psi = \omega.$$

The mean state can describe the statistical bias in the fluctuation from the assumed steady state solution (V_μ, Q_μ), and the second order variance of the fluctuation variables is to describe the spread of the ensemble of particles to describe the statistical evolution of the system. Together the statistical mean and variance calibrate the total uncertainty (instability) in the fluctuation state variables about the steady state solution. Therefore as a combination of energy in the mean and energy in variance, we introduce the notion for ***statistical energy in each fluctuation mode*** in the form

$$E_{\mathbf{k}} \equiv \left\langle \left|\omega_{\mathbf{k}}\right|^2 \right\rangle \equiv \left|\bar{\omega}_{\mathbf{k}}\right|^2 + \overline{\left|\omega_{\mathbf{k}}'\right|^2}, \quad E_U \equiv \left\langle U^2 \right\rangle = \bar{U}^2 + \overline{U'^2}. \tag{1.7}$$

Further we can define the ***total statistical energy in fluctuation*** through the original pseudo-energy (1.6) as

$$E_\mu^{\text{stat}} \equiv \frac{\mu}{2}E_U + \frac{1}{2}\sum_{1\leq|\mathbf{k}|\leq\Lambda}\mu_{\mathbf{k}}E_{\mathbf{k}} = \frac{\mu}{2}\left\langle U^2\right\rangle + \frac{1}{2}\sum_{1\leq|\mathbf{k}|\leq\Lambda}\mu_{\mathbf{k}}\left\langle\left|\omega_{\mathbf{k}}\right|^2\right\rangle, \tag{1.8}$$

with the weighting coefficients $\mu_{\mathbf{k}} = 1 + \mu\left|\mathbf{k}\right|^{-2}$. On the right hand sides of the above definitions about the statistical energy from the fluctuation, the first part describes the perturbations in the statistical mean, $\left(\bar{U}^2, \left|\bar{\omega}_{\mathbf{k}}\right|^2\right)$, and the second part describes the variances in the corresponding mode, $\left(\overline{U'^2}, \overline{\left|\omega_{\mathbf{k}}'\right|^2}\right)$. It is useful to investigate the statistics in the first two moments rather than a single realization since they not only characterize the deviations from the steady state mean, but also illustrate the amplitude of uncertainty (variance) for this mean estimation. Thus this offers more reasonable and detailed characterization of the system especially when it becomes increasingly turbulent.

Now observe the fluctuation dynamics (1.5). The last quadratic part $\mathcal{P}_\Lambda\left(\nabla^\perp\psi\cdot\nabla\omega\right)$ is in exactly the same form as before in the barotropic flow. This term conserves both the energy and enstrophy, thus the pseudo-energy as a combination of these two energy is also conserved. Therefore through the conclusion in the main Theorem in [*Majda, PNAS* 2015], this part makes no explicit contribution to the statistical equation for E_μ^{stat}. Then we only need to focus on the second term $\mathcal{P}_\Lambda\left(\nabla^\perp\Psi\cdot\nabla\left(\omega - \mu\psi\right)\right)$, which is a linear operator including the steady mean effect from Ψ. It can be shown that this linear operator is actually skew-symmetric. Thus in the same way, from the conclusion in the major Theorem, this part also make no contribution to the statistical energy. Therefore we can conclude that the statistical energy E_μ^{stat} for fluctuation is invariant in time

$$\frac{d}{dt}E_\mu^{\text{stat}}\left(t\right) = 0. \tag{1.9}$$

in the case with no dissipation and external forcing.

It can be observed from the above statistical energy from fluctuation component (1.8) that the sign (and then instability properties) in the total statistical energy E_μ^{stat} depends on the choice for the parameter value μ:

- *Stable regime*: If $\mu > 0$, the statistical energy in fluctuation E_μ^{stat} is uniformly positive-definite for arbitrary truncation Λ in every small vortical mode and large scale mean flow U. Thus we can analyze the stability about the mean and variance perturbations all together for the total variability from the conservation properties of the statistical energy;
- *Unstable mean flow*: If $-1 < \mu < 0$, the statistical energy in the mean flow component U is negative while all the other small vortical modes stay positive. In this case, we need to separate the energy into the large-scale mean flow energy and small-scale energy to analyze them individually;
- *Unstable regime*: If $\mu < -1$, the positive-definite property of the statistical energy in all the small vortical modes may not be satisfied. The saturation of instability is reached at $\mu = -1$, and we may need to decompose the entire energy into the positive-definite and negative-definite part and analyze them separately. Especially in the case with non-positive-definite fluctuation energy $E_\mu^{\text{stat}} \leq 0$, the nonlinear interaction implies the control of instability in high wavenumber modes by a small number of low wavenumber modes.

In the following sections, we will begin with the simple stable regime $\mu > 0$ with positive-definite statistical energy; then we will turn to the non-positive-definite regime $\mu < 0$ for energy balance between the small and large scales. Especially it is an interesting case in regime $-1 < \mu < 0$ with an unstable mean flow U, since only topographic stress is included in the large-scale mean flow dynamics. Next we will consider the effect of external damping and forcing to the system. It can be shown that in a specific form, the Ekman damping and deterministic and stochastic forcing can play a similar role as the uncertainty in the initial values in the undamped and unforced case. Finally, we discuss a special case where there is no mean flow $U = 0$. In this case the enstrophy is no longer conserved and the previous conservation principle needs additional consideration.

1.2. **Special forced-dissipative case with Ekman damping and forcing from equilibrium steady state.** In general the dissipation and forcing introduce the additional terms on the right hand sides of the original flow dynamics (1.1). With the inclusion of the dissipation and forcing effects, the pseudo-energy E_μ in (1.6) becomes no longer conserved, and so is the the statistical fluctuation due to the pseudo-energy. To get the dynamical equations for the statistical energy in fluctuation, we can still consider the full statistical energy in fluctuation defined in (1.8)

$$E_\mu = \frac{\mu}{2} E_U + \frac{1}{2} \sum_{1 \leq |\mathbf{k}| \leq \Lambda} \mu_{\mathbf{k}} E_{\mathbf{k}} = \frac{\mu}{2} \left\langle U^2 \right\rangle + \frac{1}{2} \sum_{1 \leq |\mathbf{k}| \leq \Lambda} \left(1 + \mu \left| \mathbf{k} \right|^{-2} \right) \left\langle \left| \omega_{\mathbf{k}} \right|^2 \right\rangle.$$

Correspondingly, we can define the general damping form at each spectral mode of statistical energy as

$$\mathcal{D}_{\mathbf{k}} = \mathcal{D}\left(-\left| \mathbf{k} \right|^2 \right) = \sum_{j=0}^{L} d_j \left| \mathbf{k} \right|^{2j}, \ 1 \leq \left| \mathbf{k} \right|^2 \leq \Lambda, \tag{1.10}$$

where L defines different orders of dissipation. Therefore, under the notations in (1.10), we can derive the general dynamical equation for the statistical energy in fluctuation including dissipation and external forcing effects as

$$\frac{dE_\mu}{dt} = -2\mathcal{D}E + \mu \bar{U} F_0 + \fint \left(\bar{\omega} - \mu \bar{\psi} \right) F + Q_\sigma. \tag{1.11}$$

Above the damping operator $\mathcal{D}E = \sum_{0 \leq |\mathbf{k}|^2 \leq \Lambda} \mathcal{D}_{\mathbf{k}} E_{\mathbf{k}}$ has inhomogeneous effect $\mathcal{D}_{\mathbf{k}}$ for each mode from (1.10) according to the general dissipation form. We use

$$Q_\sigma = \frac{1}{2} \mu \sigma_0^2 + \frac{1}{2} \sum_{1 \leq |\mathbf{k}|^2 \leq \Lambda} \left(1 + \mu \left| \mathbf{k} \right|^{-2} \right) \sigma_{\mathbf{k}}^2,$$

to represent the entire contribution from the stochastic (independent) white noises forcing in both small scale modes and large scales mean flow. Notice that especially in the unstable regime $\mu < 0$, the total stochastic forcing Q_σ may become negative in value with a large negative contribution to the total statistical energy from mean mode $\mu \sigma_0^2$. Besides the deterministic forcing effects on small scales can be written under spectral modes as

$$\fint F \left(\bar{\omega}_\Lambda - \mu \bar{\psi}_\Lambda \right) = \sum_{1 \leq |\mathbf{k}|^2 \leq \Lambda} \left(1 + \mu \left| \mathbf{k} \right|^{-2} \right) \hat{F}_{\mathbf{k}}^* \bar{\omega}_{\mathbf{k}}.$$

Thus only the statistical mean state in each mode is included in the contribution to the total statistical energy from the external forcing exerted. As a further comment, the dynamics of the total statistical energy E_μ combining mean and variance in (1.11) is determined through only the first order mean state only together with the external forcing and dissipation effects.

In the first place, for simplicity, we choose only uniform Ekman damping as the dissipation effect in the system. That is, let $D = dI$ in the general form of the dissipation (1.10). This Ekman damping is common to the large-scale pieces of the geophysical flow, arising from boundary layer effect. Besides we assume the first part of the deterministic forcing is introduced through the equilibrium steady state. Furthermore, we assume the damping rates in small vortical modes and large scale mean flow are of the same amplitude, d. Therefore, the forcing and dissipation terms from the above simplified set-up become

$$\begin{aligned} &\text{small scale} \quad -d\omega + d\bar{\omega}_{\mathrm{eq}} + F(\mathbf{x}) + \sigma_{\mathbf{k}}\dot{W}_{\mathbf{k}}, \\ &\text{large scale} \quad -dU + d\bar{U}_{\mathrm{eq}} + F_0 + \sigma_0\dot{W}_0. \end{aligned} \tag{1.12}$$

And the equilibrium mean states $\left(\bar{\omega}_{\mathrm{eq}}, \bar{U}_{\mathrm{eq}}\right)$ are determined from the steady state solutions in (1.3) depending on the parameter value of μ

$$\bar{\omega}_{\mathrm{eq},\mathbf{k}} = \frac{-\left|\mathbf{k}\right|^2}{\mu + \left|\mathbf{k}\right|^2}\hat{h}_{\mathbf{k}}, \quad \bar{U}_{\mathrm{eq}} = -\frac{\beta}{\mu}.$$

In addition we also assume additional deterministic forcing (F, F_0) and stochastic (σ, σ_0) forcing on both small and large scales. For simplicity, we further assume that (F, F_0) and (σ, σ_0) are both independent of time.

Therefore on the right hand sides of the fluctuation equations (1.5), only linear damping is applied on the fluctuation components in both small and large scale fluctuation variables $\tilde{\omega} = \omega - \bar{\omega}_{\mathrm{eq}}, \tilde{U} = U - \bar{U}_{\mathrm{eq}}$. Accordingly the **statistical fluctuation energy equation with Ekman damping** becomes

$$\frac{dE_\mu}{dt} = -2dE_\mu + \mu\bar{U}\cdot F_0 + \left\langle \bar{\omega}, F\right\rangle_\mu + Q_{\sigma,\mu}. \tag{1.13}$$

The inner product is defined as $\langle a, b\rangle = \sum\left(1 + \mu k^{-2}\right)a_k b_k^*$. And the random forcing has the total contribution

$$Q_{\sigma,\mu} = \frac{1}{2}\mu\sigma_0^2 + \frac{1}{2}\sum\left(1 + \mu\left|\mathbf{k}\right|^{-2}\right)\sigma_{\mathbf{k}}^2.$$

It is important to notice that in the unstable regimes, $\mu < 0$, both the deterministic and stochastic forcing can introduce negative effect in the total statistical energy on the right hand side in (1.13).

Remark. In fact, for the general dissipation form in (1.10), we can always find a constant independent of wavenumber $\mathbf{k}$ as

$$C_d \equiv \sum_{j=0}^{L} d_j \leq \sum_{j=0}^{L} d_j\left|\mathbf{k}\right|^{2j}, \quad \forall\ \left|\mathbf{k}\right| \geq 1.$$

Thus for the general form of dissipation, it is still useful to find the lower bounds of the entire damping effect as C_d, so that the above statistical energy conservation law (1.13) just becomes an inequality as

$$\frac{dE_\mu}{dt} \leq -2C_dE_\mu + \mu\bar{U}\cdot F_0 + \left\langle \bar{\omega}, F\right\rangle_\mu + Q_{\sigma,\mu}.$$

Thus the same strategy can apply.

1.3. **Illustration of flow structures and statistics with numerical simulations.** We first illustrate the flow structures in various parameter regimes through numerical simulations. A relatively small truncation size $|\mathbf{k}| \leq \Lambda = 12$ is used so that we can focus on the largest scales while the effect of nonlinear interactions are also not negligible. To achieve the statistics in the state variables, we run a Monte-Carlo simulation of the original topographic barotropic system (1.1) with an ensemble size $N = 1000$. More numerical simulations with larger ensemble size has confirmed that $N = 1000$ is large enough to capture the statistical mean and variance with admissible error. For the topographic structure, we assume a zonal structure on the largest scale with small perturbations in smaller scales such that

$$h = H\left(\sin x + \cos x\right) + H\sum_{2\leq|\mathbf{k}|\leq\Lambda}\left|\mathbf{k}\right|^{-2}e^{i(\mathbf{k}\cdot\mathbf{x}-\theta_{\mathbf{k}})}.$$

In the simulations we take the topographic strength $H = 3\sqrt{2}/4$ and uniform phase shift $\theta_{\mathbf{k}} = \frac{\pi}{4}$ (refs.). The beta-effect is introduced through the parameter $\beta = 1$. No additional dissipation and forcing are introduced in this test case in the system. The initial value of the ensemble is set with the mean from steady state solution

$$Q_{\mu,\mathbf{k}} = \frac{\mu\hat{h}_{\mathbf{k}}}{\mu + \left|\mathbf{k}\right|^2}, \quad V_\mu = -\frac{\beta}{\mu}.$$

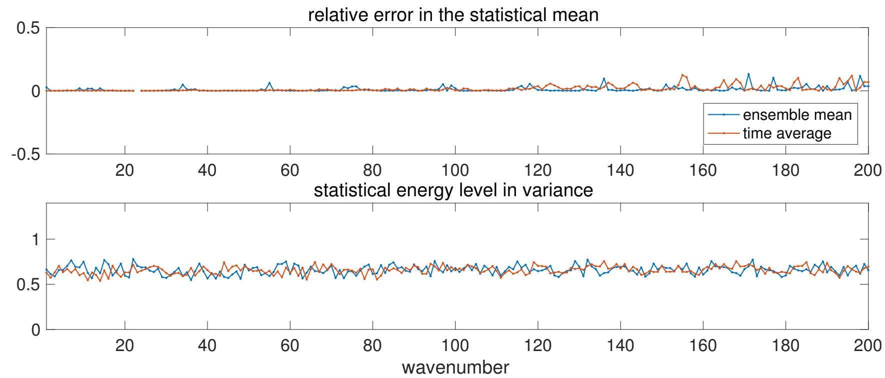


FIGURE 1.1. Statistical errors in statistical mean and variance in each mode with $\mu = 1$. The ensemble mean and variance are estimated from an ensemble of $N = 1000$ particles; and the time averages are achieved from time-averaging along a single realization of the solution in the equilibrium. The modes are ordered columnwisely. The dashed line marks the equilibrium statistical energy level $\sigma_{\text{eq}}^2 = 0.65$ from the invariance measure (1.4).

The parameter μ sets up the initial mean state of the flow field. And the initial variance of the ensemble is set among the mean flow U and only the unstable vortical modes $\hat{\omega}_{\mathbf{k}}, \mu + |\mathbf{k}|^2 < 0$ with uniform variance

$$\sigma_{U,0} = 1, \quad \sigma_{\text{un},0} = 1.$$

All the other modes are set with zero initial variances. Thus all the statistical energy in fluctuation is contained in the unstable large scales in the initial time. This enables us to monitor the energy cascades from unstable modes to stable ones in the model evolution. Throughout this report, we will always use the above model setups to test flow structures and statistics as the parameter values vary.

1.3.1. *Invariant measure and ergodicity in regime $\mu > 0$.* In the first place, we test the flow field in the regime with $\mu > 0$ with no forcing and dissipation on the right hand sides of (1.1). Here we use the parameter value $\mu = 1$ to illustrate the flow structure in statistical steady state. From the nonlinear stability and equilibrium statistical mechanics the flow statistics will converge to the Gaussian invariant measure in (1.4) with equilibrium mean and variance determined by the topography h, beta-effect β, and parameter μ

$$\bar{q}_{\mathbf{k}} = \frac{\mu \hat{h}_{\mathbf{k}}}{\mu + |\mathbf{k}|^2}, \quad \bar{V} = -\frac{\beta}{\mu}, \qquad \overline{|\hat{q}'_{\mathbf{k}}|^2} = \sigma_{\text{eq}}^2 \left(1 + \mu\, |\mathbf{k}|^{-2}\right)^{-1}, \quad \overline{V'^2} = \sigma_{\text{eq}}^2 \mu^{-1}.$$

In Figure 1.1 we plot the (relative) errors from the numerical simulation results compared with the theoretical steady state solutions (V_μ, Q_μ), that is,

$$\text{Err}_{\bar{u}} = \frac{|\bar{u} - u_\mu|^2}{|u_\mu|^2}, \quad \text{Err}_{u^2} = \frac{\overline{(u - \bar{u})^2}}{r_\mu}.$$

The initial statistics in the numerical simulations are set the same as the Gaussian invariant measure. Above u could be either the large scale flow U or vorticity mode $q_{\mathbf{k}}$. $\text{Err}_{\bar{u}}$ measures the relative error in the estimated mean state compared with the steady state solutions (V_μ, Q_μ); Err_{u^2} measures the ratio between the estimated variance and the invariance approximation with $r_\mu = \mu^{-1}, \left(1 + \mu\, |\mathbf{k}|^{-2}\right)^{-1}$. If the numerical solution converges to the invariant measure above, $\text{Err}_{\bar{u}}$ will be uniformly zero and Err_{u^2} will reach the uniform equilibrium energy level σ_{eq}^2 at each mode. From Figure 1.1, the numerical mean of all the modes goes to the theoretical prediction uniformly with small fluctuations; and the numerical variance is consistent with the invariant measure with the estimated equilibrium energy level $\sigma_{\text{eq}}^2 = 0.65$. Since here we have no damping and forcing in the system, the parameter σ_{eq} is determined by the initial total energy of the system. Furthermore to check the ergodicity of the system, we also compare the ensemble statistics (which are estimated at the final time with ensemble average) with the time averaged results (which are averaged along one single trajectory of the solution). Consistency can also be observed between the ensemble average and trajectory average. Note that the relatively larger errors at the small scales in the relative errors in the mean are due to the small mean steady state solution u_μ in high wavenumber modes.

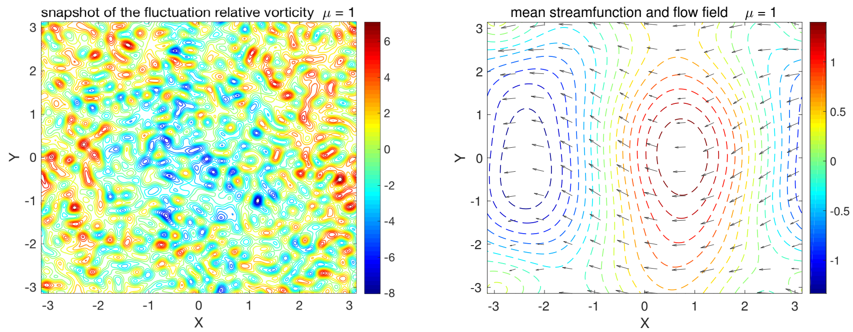


FIGURE 1.2. Snapshots of the relative vorticity field in fluctuation component $\tilde{\omega}$ and the corresponding mean stream function ψ with flow vector field (including mean flow U) with parameter $\mu = 1$ at final equilibrium state.

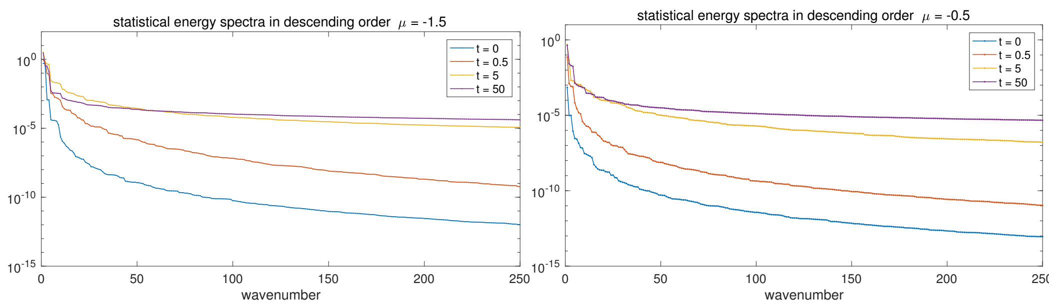


FIGURE 1.3. The fluctuation statistical energy spectra at different time instant with parameter $\mu = -1.5$ and $\mu = -0.5$. The $\mu = -1.5$ case gets instability in the mean flow U and the first ground mode $1 + \mu |\mathbf{k}|^{-2} < 0$; and the $\mu = -0.5$ case has instability from the mean flow U only. The modes are ordered in descending order.

In Figure 1.2 we show the flow vorticity and stream function at final equilibrium with parameter $\mu = 1$. The mean stream function and flow field is determined by the steady state solution with topography and beta-effect $\mu\Psi_\mu = \Delta\Psi_\mu + h, V_\mu = -\beta/\mu$ in (1.3). The vorticity fluctuation is isotropic in the spectral domain, and the mean flow field and stream function get a steady westward mean flow predicted from the steady state solution Ψ_μ, V_μ.

1.3.2. *Flow statistics in nonlinearly unstable regime* $\mu < 0$. Here we display the flow statistical structures in the two typical nonlinear unstable regimes $\mu < -1$ and $-1 < \mu < 0$. First to check the time evolution of the statistics in each mode, Figure 1.3 plots the statistical energy spectra at different time instants in the two parameter regimes. The modes are ordered in a descending order, which in this case is basically from the largest scale to the small scales. In the case $\mu = -1.5$ the unstable modes include the large-scale mean flow U and the leading vortical modes with $|\mathbf{k}| = 1$, while all the other smaller scale modes are stable. In the initial time $t = 0$, we only set initial variance in the largest scales with instability. As the system evolves in time, the smaller scales becomes more and more energetic due to the topographic forcing and energy cascades from the larger scale modes. Due to instability, both large and small scales get more energetic as the statistical mean and variance increase before the statistical steady state. Especially in the final statistical steady state, the leading modes become a little less energetic and more energy gets cascaded to the smaller scale modes. In a similar way, we also observe spectra for the case $\mu = -0.5$. In this case, only the mean flow U is unstable, and all the other vortical modes are stable with positive coefficient in the pseudo-energy. Still we can observe the energy cascades to the smaller scales as the system evolves in time, while in this case the instability is only from the large-scale flow U and the major source of energy is from the topographic stress to transfer the energy in the mean flow to smaller scale modes. Thus the drop in energy in the leading largest vortical mode in the steady state is not as obvious as before in the $\mu = -1.5$ case.

Next in Figure 1.4 we compare the statistical steady state spectra in the fluctuation component away from the steady state solution in each mode. First in the case $\mu < -1$, the statistical energy inside the small scale modes in

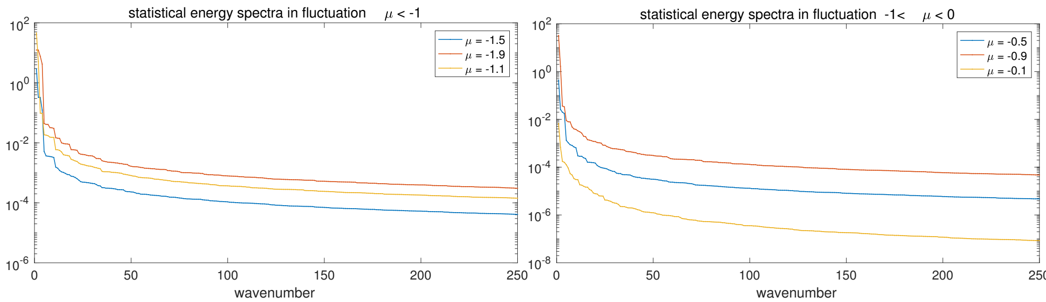


FIGURE 1.4. The fluctuation statistical energy spectra in regimes with $\mu < -1$ and $-1 < \mu < 0$. The modes are ordered in descending order.

μ	-1.9	-1.5	-1.1	-0.9	-0.5	-0.1	0.5	1
$\bar{U}$	-0.8222	-2.4816	-2.9737	0.3401	0.2282	0.6974	-2.0498	-1.1617
$-\beta/\mu$	0.5263	0.6667	0.9091	1	1	1	-2	-1

TABLE 1. Statistical mean large-scale flow $\bar{U}$ in steady state compared with the assumed steady state solution $\bar{U}_{\mathrm{eq}} = -\beta/\mu$.

the case $\mu = -1.5$ is relatively small; while in the other two cases, $\mu = -1.9$ and $\mu = -1.1$ larger statistical energy gets cascaded downwardly to the small modes in the tails. This infers larger instability as the parameter approaches the two limits, $\mu \to -1, -2$. On the other hand in the regime $-1 < \mu < 0$, the steady state statistical energy in small scales shown in the tails gets smaller monotonically as the parameter μ approaches 0. This illustrates that no stability exists any more near the limit $\mu \to 0-$ even though it is implies from the equilibrium statistical mechanics and the invariance measure (1.4).

We also compare the flow structure in steady state for the test regimes. Figure 1.5 compare the relative vorticity field $\tilde{\omega}$ when the parameter values change $\mu = -1.1$, $\mu = -1.5$, and $\mu = -1.9$. The color scales of the plots are normalized in the same range for comparison. Obviously in the vorticity field, with $\mu = -1.9$ and $\mu = -1.1$ larger small-scale vortical fluctuation is induced with stronger fluctuations compared with the $\mu = -1.5$ case. Notice the parameter μ only sets the initial structure in the largest scales $|\mathbf{k}| = 1$, thus the vortical fluctuations in much smaller scales are induced due to the instability. Also we compare the statistical mean field of the stream functions and the full flow vector field including the large scale flow U. The mean flow is always westward $\bar{U} < 0$ despite the defined steady state solution $\bar{U}_{\mathrm{eq}} = -\beta/\mu > 0$ in this regime. With $\mu = -1.5$, it can observe the dominant mode is $\mathbf{k} = (1, 0)$ consistent with the topography, while with $\mu = -1.9$ the flow develops stronger blocked structures and with $\mu = -1.1$ the stream lines are open with zonal westward jet flow. In contrast, Figure 1.6 compares the flow fields in the regimes $-1 < \mu < 0$ with $\mu = -0.9, -0.5, -0.1$. In this case, the vorticity fluctuation gets weaker and weaker as μ approaches 0. Also the flow stream functions are more aligned with the topography and the mean flow $\bar{U}$ becomes eastwards in this case consistent with the steady state solution $\bar{U}_{\mathrm{eq}}$.

Furthermore, we list the statistical mean large scale flow $\bar{U}$ in the steady state as the parameter μ varies in Table 1. In the nonlinearly stable regime $\mu > 0$, the theoretical steady state solution $\bar{U}_{\mathrm{eq}} = -\beta/\mu$ is accurate with the numerical results for steady state mean flow $\bar{U}$. This implies little instability in the flow field in this regime. On the other hand, with $\mu < -1$ the steady state mean flow $\bar{U}$ gets opposite direction with the steady state solution $\bar{U}_{\mathrm{eq}}$. This is due to the interactions between small and large scale modes through topographic stress. In the regime $-1 < \mu < 0$, $\bar{U}$ and $\bar{U}_{\mathrm{eq}}$ also have difference in value but stay in the same direction. This corresponds to the weaker instability only in the large scale flow U.

1.3.3. *Flow field with forcing and dissipation.* In the final case we consider the situation with the additional linear damping and forcing in the form in (1.12) in the flow field. The linear Ekman damping parameter is taken as $d = 0.1, 0.5, 1$. The deterministic forcing is only from the equilibrium mean state of large and small variable, $F_0 = d\bar{U}_{\mathrm{eq}}$ and $\hat{F}_{\mathbf{k}} = d\bar{\omega}_{\mathrm{eq}}$. The stochastic forcing is only applied on the mean flow U and the first mode $|\mathbf{k}| = 1$, and the amplitude σ is taken so that $d^{-1}\sigma^2 = 1$. The steady state flow snapshots of the fluctuation relative vorticity

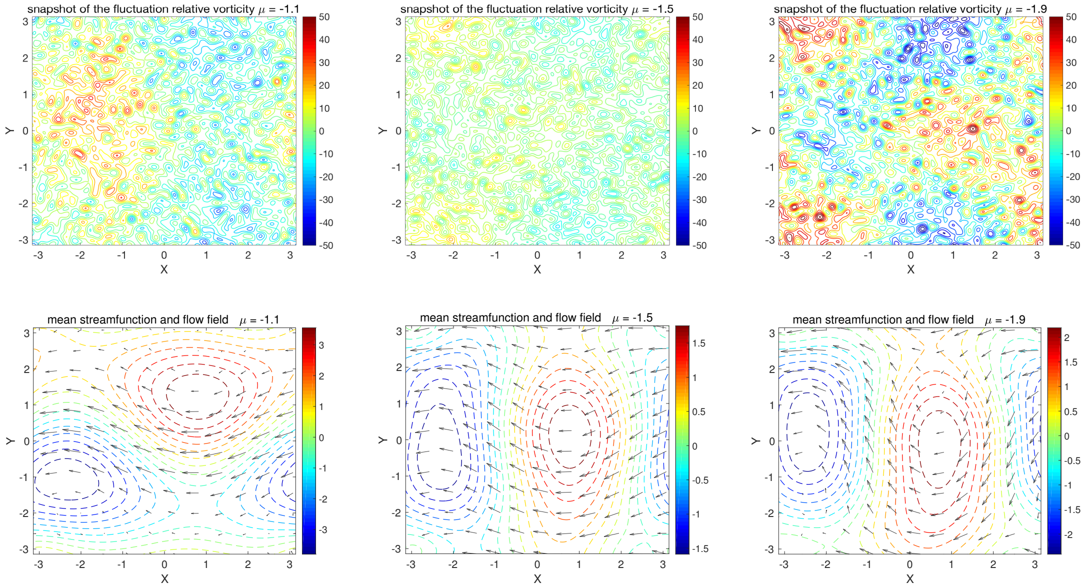


FIGURE 1.5. Snapshots of the relative vorticity field in fluctuation component $\tilde{\omega}$ with parameters $\mu = -1.1$, $\mu = -1.5$, and $\mu = -1.9$. The mean stream function (without mean flow U) and full flow filed (with mean flow U) for the three parameter values are shown next. The mean stream function changes from zonal flow to blocked circular streams as the parameter values change.

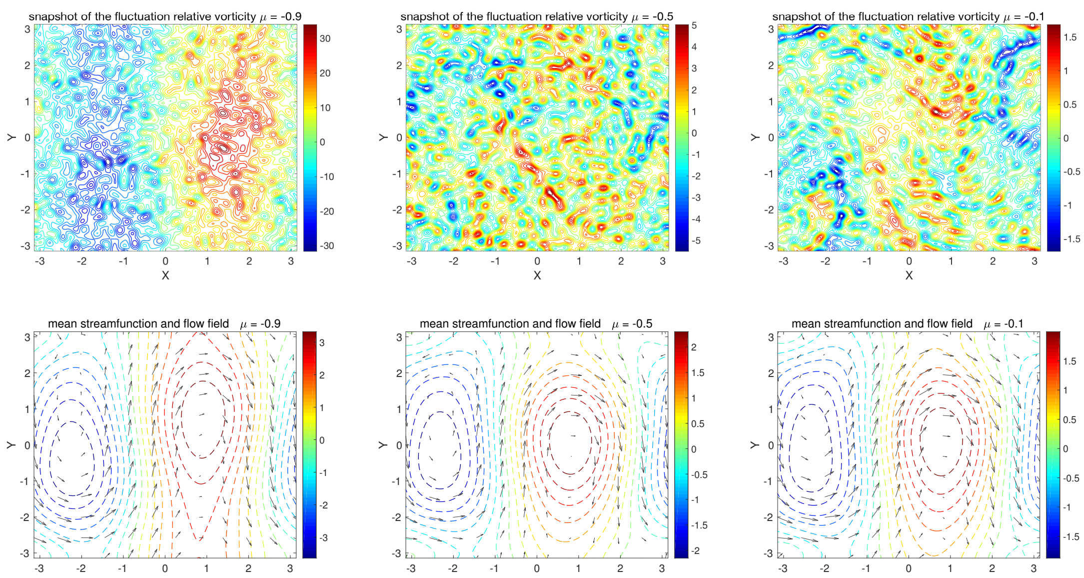


FIGURE 1.6. Snapshots of the relative vorticity field in fluctuation component $\tilde{\omega}$ with parameters $\mu = -0.1$, $\mu = -0.5$, and $\mu = -0.9$. The mean stream function (without mean flow U) and full flow filed (with mean flow U) for the three parameter values are shown next. The flow field reverses direction compared with the previous $\mu < -1$ case.

μ	-1.5				1			
d	0	0.1	0.5	1	0	0.1	0.5	1
$\bar{U}$	-2.4816	-0.1619	0.4908	0.6295	-1.1617	-1.3823	-1.1533	-1.0508
$\overline{U'^2}$	3.3292	0.6681	0.4876	0.4263	0.4719	0.4519	0.5131	0.5216

TABLE 2. Statistical mean and variance in large-scale flow U in steady state with different damping and forcing rate (d, F).

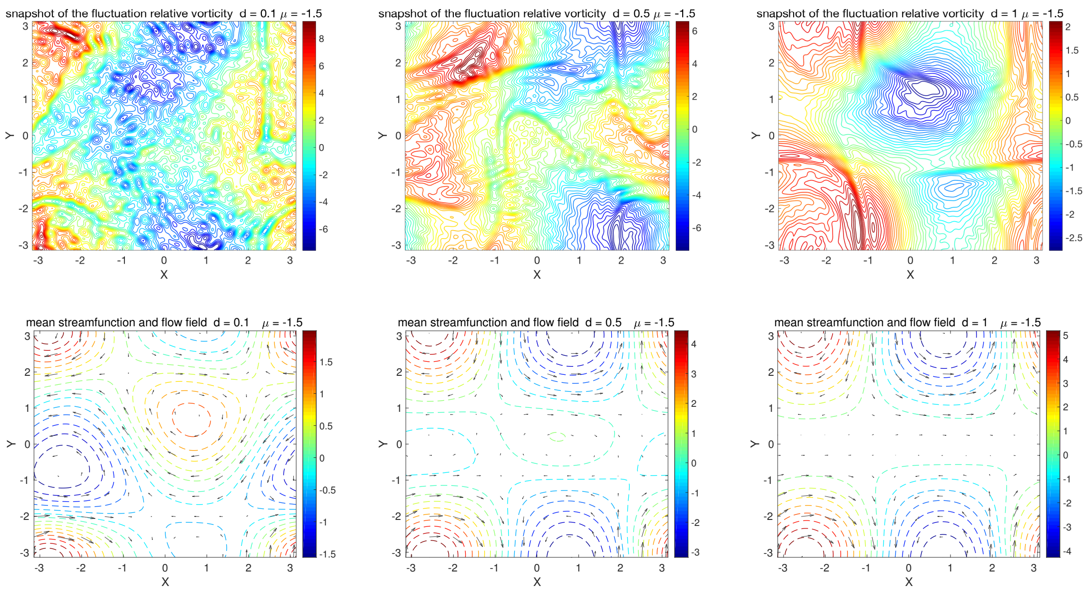


FIGURE 1.7. Snapshots of the relative vorticity field in fluctuation component $\tilde{\omega}$ with forcing and damping with parameters $\mu = -1.5$. Different damping rates $d = 0, 0.5, 1$ are compared. The mean stream functions (in small scale) and the flow vector field (including mean flow U) are shown in the following. The flow filed shifts from westward jet to eastward in background mean as d increases.

$\tilde{\omega}$ and the mean stream function as well as the entire flow vector fields are shown in Figure 1.7. The vorticity fields in fluctuation depict the deviation from the assumed steady state flow solution $\bar{\omega}_{\text{eq}}$. With weak damping and forcing $d = 0.1$, still there exist many small scale structures in the flow fluctuation. As the damping rate increases, the flow field becomes smoother with vortex filaments developed. Especially with strong damping $d = 1$, the small scale fluctuations are mostly dissipated in the steady state while only largest scale structures persist. In the steady state mean flow, numerical simulations get the steady mean flow $\bar{U} = -0.1619$ for $d = 0.1$, $\bar{U} = 0.4908$ for $d = 0.5$, and $\bar{U} = 0.6295$ for $d = 1$, compared with the steady state solution $\bar{U}_{\text{eq}} = -\beta/\mu = 0.6667$. The strong forcing and damping make sure the convergence to the exact steady state solution in the equilibrium, while the weaker forcing and damping cases introduce larger fluctuations. The background mean flow shifted from a major westward jet to eastward flow with blocked circulations as d increases in value.

2. Statistical Stability with Large-Scale Mean Flow in Regime $\mu > 0$

Now we consider the interaction with the large-scale zonal flow $U \neq 0$ in the statistical energy dynamics for fluctuations. With the inclusion of the large-scale flow, like the derivation before, we can find the conserved statistical pseudo-energy in fluctuation (1.8) becomes

$$E_{\mu}^{\text{stat}}(t) = \frac{\mu}{2}\left\langle U^2\right\rangle_t + \frac{1}{2}\sum_{1\leq|\mathbf{k}|\leq\Lambda}\left(1+\mu\left|\mathbf{k}\right|^{-2}\right)\left\langle\left|\omega_{\mathbf{k}}\right|^2\right\rangle_t.$$

The subscript 't' refers to ensemble average at time t. Without external forcing and damping effect, the statistical energy E_μ in fluctuation defined through the conservation of the pseudo-energy is conserved in time as shown in (1.9); while with the inclusion of forcing and dissipation effects, the change in total statistical energy fluctuation E_μ can purely determined from the first order mean state and forcing-damping structures shown in (1.13). Remember here that U represents the fluctuations from the steady state mean flow $V_\mu = -\beta/\mu$, and ω represents the fluctuations away from the steady state vorticity Q_μ. In this section, we first consider the simple case with $\mu > 0$, so that the coefficients $\mu_{\mathbf{k}}$ in the statistical energy E_μ are all positive.

2.1. **Statistical energy bound in fluctuation energy without forcing and dissipation.** We begin with the simple case in stable regime $\mu > 0$ and no damping and forcing terms on the right hand side of (1.5). Assume we have the initial perturbations $U(0) = \bar{U}_0 + U_0'$, $\omega(0) = \bar{\omega}_0 + \omega_0'$, where $\bar{U}_0, \bar{\omega}_0$ are the initial fluctuation mean states away from steady state V_μ, Q_μ, and U_0', ω_0' characterize the uncertainty (that is, ensemble variance) in the initial ensemble members. According to the steady state (V_μ, Q_μ) with initial statistical energy in perturbation, the initial statistical energy can be expressed as

$$E_\mu(0) = \frac{1}{2}\sum_{1\leq|\mathbf{k}|\leq\Lambda}\left(1+\mu\left|\mathbf{k}\right|^{-2}\right)E_{\mathbf{k},0} + \frac{\mu}{2}E_{U,0},$$

with $E_{U,0} = \bar{U}_0^2 + \overline{U_0'^2}$, and $E_{\mathbf{k},0} = \left|\bar{\omega}_{0,\mathbf{k}}\right|^2 + \overline{\left|\omega_{0,\mathbf{k}}'\right|^2}$. Especially we have the initial uncertainty from variance $\overline{\left|\omega_{0,\mathbf{k}}'\right|^2} = \overline{\left|q_{0,\mathbf{k}}'\right|^2}$, and the initial mean deviation for fluctuation component in the form $\left|\bar{\omega}_{0,\mathbf{k}}\right|^2 = \left|Q_{\mu,\mathbf{k}} - \bar{q}_{0,\mathbf{k}}\right|^2$ and $\bar{U}_0^2 = \left|V_\mu - \bar{V}_0\right|^2$. Therefore due to the conservation of total statistical energy we have the first statistical energy conservation relation

$$\sum_{1\leq|\mathbf{k}|\leq\Lambda}\mu_{\mathbf{k}}\left\langle\left|\omega_{\mathbf{k}}\right|^2\right\rangle_t + \mu\left\langle U^2\right\rangle_t \leq \sum_{1\leq|\mathbf{k}|\leq\Lambda}\mu_{\mathbf{k}}E_{\mathbf{k},0} + \mu E_{U,0}, \tag{2.1}$$

with $\mu_{\mathbf{k}} = 1 + \mu\left|\mathbf{k}\right|^{-2}$ the weighting coefficients due to the energy conserving inner-product metric. In fact in (2.1) equality can be reached in the case without forcing and dissipation, while the inequality is valid for cases with also damping terms included in the system. Notice that the above relation is valid for all the values of $\mu > 0$, whereas the statistics in $\left\langle\left|\omega_{\mathbf{k}}\right|^2(t)\right\rangle$ (and in fact only in the statistical mean fluctuation part $\left|\bar{\omega}_{\mathbf{k}}\right|^2$) will change accordingly with different values of μ due to the presumed mean state Q_μ. We can summarize the first statistical energy bound for the stable regime $\mu > 0$:

Proposition 1. *(Statistical energy conservation of fluctuation modes in regime $\mu > 0$) Consider the system of fluctuation equations away from the steady state solution (V_μ, Q_μ). For any parameter values $\mu > 0$ in the stable regime with $E_\mu > 0$, the total statistical variability in mean and variance, $\left\langle U^2\right\rangle \equiv \bar{U}^2 + \overline{U'^2}$, $\left\langle\left|\omega_{\mathbf{k}}\right|^2\right\rangle \equiv \left|\bar{\omega}_{\mathbf{k}}\right|^2 + \overline{\left|\omega_{\mathbf{k}}'\right|^2}$, can always be controlled by its initial statistical variability including initial mean and total variance as in the inequality (2.1). Especially, if there is no statistical mean perturbation in the initial time, $\bar{V}_0 = V_\mu$, $\bar{q}_0 = Q_\mu$, the total statistical energy of the system in the entire time can be controlled by the initial ensemble fluctuation variances $\sigma_{\mathbf{k},0}^2 = \overline{\left|\omega_{0,\mathbf{k}}'\right|^2}$ and $\sigma_{U,0}^2 = \overline{U'^2}$*

$$\sum_{1\leq|\mathbf{k}|\leq\Lambda}\left(1+\mu\left|\mathbf{k}\right|^{-2}\right)\left\langle\left|\omega_{\mathbf{k}}\right|^2\right\rangle_t + \mu\left\langle U^2\right\rangle_t \leq \sum_{1\leq|\mathbf{k}|\leq\Lambda}\left(1+\mu\left|\mathbf{k}\right|^{-2}\right)\sigma_{\mathbf{k},0}^2 + \mu\sigma_{U,0}^2.$$

Furthermore, we can see both the statistical mean and the statistical variance are bounded by their initial variability in this stable regime with the inclusion of dissipation $d > 0$.

As a further implication of the above inequality (2.1), we can have the estimation bound for the *total statistical enstrophy*, $\fint \left\langle \omega^2 \right\rangle$, and the *total statistical kinetic energy*, $U^2 + \fint \left\langle |\nabla\psi|^2 \right\rangle$. In the statistics in the vorticity (that is, the statistical enstrophy) in the stable regime $\mu > 0$, there exists the lower bound among all the positive coefficients for any wavenumber $\mathbf{k}$

$$\left(1+\mu\left|\mathbf{k}\right|^{-2}\right)\left\langle \left|\omega_{\mathbf{k}}\right|^2 \right\rangle \geq \left(1+\mu\Lambda^{-2}\right)\left\langle \left|\omega_{\mathbf{k}}\right|^2 \right\rangle;$$

and for the statistical kinetic energy for any wavenumber $\mathbf{k}$ the lower bound of the coefficients becomes

$$\left(\left|\mathbf{k}\right|^2+\mu\right)\left\langle \left|\mathbf{k}\right|^2\left|\psi_{\mathbf{k}}\right|^2 \right\rangle \geq \mu\left|\mathbf{k}\right|^2\left\langle \left|\psi_{\mathbf{k}}\left(t\right)\right|^2 \right\rangle.$$

Therefore we find the general bounds for *the total statistical enstrophy* $\fint \left\langle \omega^2 \right\rangle \equiv \sum \left\langle \left|\omega_{\mathbf{k}}\right|^2 \right\rangle$ and *the total statistical kinetic energy* $\left\langle U^2 \right\rangle + \fint \left\langle |\nabla\psi|^2 \right\rangle \equiv \left\langle U^2 \right\rangle + \sum \left|\mathbf{k}\right|^2 \left\langle \left|\psi_{\mathbf{k}}\right|^2 \right\rangle$ by the initial conditions

$$\begin{aligned} \sum_{1\leq|\mathbf{k}|\leq\Lambda}\left\langle \left|\omega_{\mathbf{k}}\right|^2 \right\rangle_t &\leq \sum_{1\leq|\mathbf{k}|\leq\Lambda}\frac{1+\mu\left|\mathbf{k}\right|^{-2}}{1+\mu\Lambda^{-2}}E^q_{\mathbf{k},0}+\frac{\mu}{1+\mu\Lambda^{-2}}E^U_0, \\ \left\langle U^2 \right\rangle_t + \sum_{1\leq|\mathbf{k}|\leq\Lambda}\left\langle \left|\mathbf{k}\right|^2\left|\psi_{\mathbf{k}}\right|^2 \right\rangle_t &\leq \sum_{1\leq|\mathbf{k}|\leq\Lambda}\mu^{-1}\left(\left|\mathbf{k}\right|^2+\mu\right)\left|\mathbf{k}\right|^2 E^v_{\mathbf{k},0}+E^U_0, \end{aligned} \tag{2.2}$$

where the right hand sides are from the initial fluctuation enstrophy/energy in mean and variance

$$E^q_{\mathbf{k},0}=\left|Q_{\mu,\mathbf{k}}-\bar{q}_{0,\mathbf{k}}\right|^2+\overline{\left|q'_{0,\mathbf{k}}\right|^2},\ E^v_{\mathbf{k},0}=\left|\Psi_{\mu,\mathbf{k}}-\bar{\psi}_{0,\mathbf{k}}\right|^2+\overline{\left|\psi'_{0,\mathbf{k}}\right|^2},\ E^U_0=\left|V_\mu-\bar{U}_0\right|^2+\overline{U_0'^2}.$$

The above bounds in (2.2) imply the stability in statistical mean and variance in fluctuation in both enstrophy and kinetic energy in the regime with $\mu > 0$. Especially the variance $\overline{U'^2}, \overline{\left|\omega'_{\mathbf{k}}\right|^2}$ in the fluctuation component is independent of the choices of the steady mean state Q_μ, and the variance is one component in the positive-definite statistical energy including mean and variance. In the above relations in (2.2), it illustrates that the uncertainty in the ensemble (or it can be described as the 'spread' of the ensemble of trajectories) can always be bounded by the 'initial noise' from the initial ensemble spread ($\overline{\left|q'_{0,\mathbf{k}}\right|^2}$ or $\overline{\left|\psi'_{0,\mathbf{k}}\right|^2}$) and the initial deviation in the statistical mean from the steady state solution $V_\mu, Q_{\mu,\mathbf{k}}$.

2.1.1. *Numerical verification of the statistical bound in regime* $\mu > 0$. Here we offer some simple numerical results to illustrate the statistical bounds in (2.1) and (2.2) in the stable regime $\mu > 0$. For simplicity, we assume there is no bias in the initial mean state, $\bar{U}_0 = V_\mu$, $\bar{q}_0 = Q_\mu$. And we propose two initial variances in the ensembles. The first only have non-zero initial variance in the large scale mean flow $\sigma_U = 1$; and the second case assign initial variance in the mean flow U and ground mode $|\mathbf{k}| = 1$, $\sigma_U = 1, \sigma_1 = 1$. The bounds in total statistical pseudo-energy (2.1) together with the statistical energy and enstrophy in (2.2) then can be simplified as

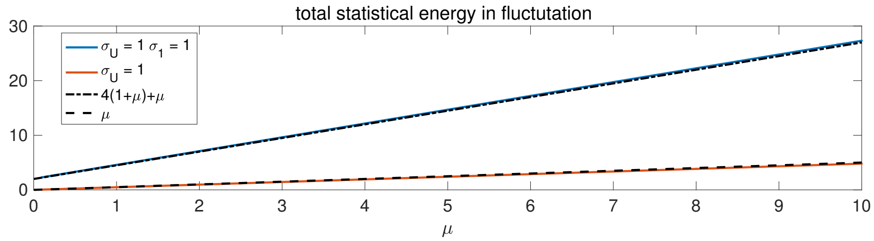
$$\sum_{1\leq|\mathbf{k}|\leq\Lambda}\left(1+\mu\left|\mathbf{k}\right|^{-2}\right)\left\langle \left|\omega_{\mathbf{k}}\right|^2 \right\rangle_t+\mu\left\langle U^2 \right\rangle_t=4\left(1+\mu\right)\sigma_1^2+\mu\sigma_U^2,\mu\sigma_U^2;$$

$$\left\langle U^2 \right\rangle_t+\sum_{1\leq|\mathbf{k}|\leq\Lambda}\left\langle \left|\mathbf{k}\right|^2\left|\psi_{\mathbf{k}}\right|^2 \right\rangle_t\leq 4\left(1+\mu\right)\sigma_1^2/\mu+\sigma_U^2,\sigma_U^2;$$

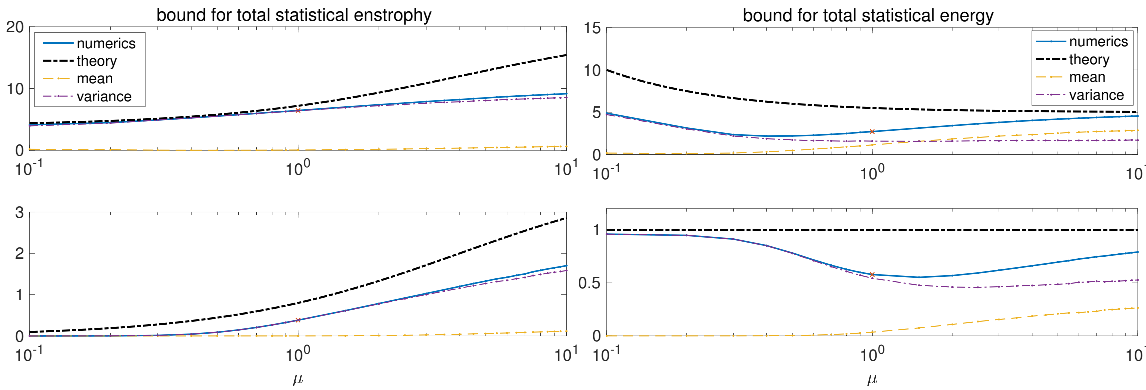
$$\sum_{1\leq|\mathbf{k}|\leq\Lambda}\left\langle \left|\omega_{\mathbf{k}}\right|^2 \right\rangle_t\leq\frac{4\left(1+\mu\right)\sigma_1^2+\mu\sigma_U^2}{1+\mu\Lambda^{-2}},\frac{\mu\sigma_U^2}{1+\mu\Lambda^{-2}};$$

Above on the right hand sides, the first term is for the bounds with initial variance in σ_U, σ_1 and the second term is for the bounds with only initial variance in the mean flow σ_U. Notice that in the first relation above equality is actually reached since the total statistical energy is conserved in this case with no damping and forcing. Besides according to the equilibrium statistical mechanics, if the invariant measure (1.4) is reached at the final equilibrium state the above statistical estimates $\left\langle \cdot \right\rangle_t$ at equilibrium get zero mean state in the fluctuation component and the variance in fluctuation component, $r_U \sim \mu^{-1}, r_{\mathbf{k}} \sim \left(1+\mu\left|\mathbf{k}\right|^{-2}\right)^{-1}$, according to the invariant measure.

Figure 2.1 shows the results in the total statistical pseudo-energy and statistical energy and enstrophy with changing values of μ. The statistical total pseudo-energy conservation from numerical calculations can be shown in the first panel exactly in agreement with the theoretical bounds from initial statistics. The statistical bounds for



(A) total statistical pseudo-energy



(B) statistical energy and enstrophy

FIGURE 2.1. Statistical energy bounds in statistical equilibrium with $0 < \mu < 10$. The solid lines are the numerical simulation results and the dashed lines are from the theoretical bounds. In the lower panel, the upper row is for the case with initial $\sigma_U = 1, \sigma_1 = 1$ and the lower row is the case with initial $\sigma_U = 1$, $\sigma_1 = 0$. Also the energy in the mean and variance are compared separately. The value for the flow fields shown in Figure 1.2 with $\mu = 1$ is marked with the red cross.

energy and enstrophy is also illustrated with different initial variances in next panel. The flow fields with $\mu = 1$ has been shown in Figure 1.2 in Section 2. The energy and enstrophy bounds from the pseudo-energy conservation in general can offer accurate estimation about the maximum amount of statistics as the steady state changes with parameter μ. We also compare the statistical mean and variance separately in the plots. With smaller values of μ, the invariant measure estimation in (1.4) is quite accurate. The fluctuation mean is near zero (thus the initial steady state solution (V_μ, Q_μ) is maintained) and the variance is in consistent with the equilibrium measure prediction. As μ becomes larger, there gradually developed a non-zero mean fluctuation. This implies that a new equilibrium steady state is reached in the mean, and correspondingly the variances in the system drop a little due to the transfer of energy to the mean state.

2.2. **The effects of additional deterministic and random external forcing in regime $\mu > 0$.** In the stable regime $\mu > 0$, we consider the effect with external forcing to the statistical stability in the mean and variance. In general, there could be deterministic component and a stochastic component represented by Gaussian white noise as the external forcing effects on both large mean flow and small vortical modes from (1.12)

$$\mathcal{F} = \sum_{\mathbf{k}\neq 0} \hat{F}_{\mathbf{k}}\left(t\right) e^{i\mathbf{k}\cdot\mathbf{x}} + \dot{W}_{\mathbf{k}}\hat{\sigma}_{\mathbf{k}}\left(t\right) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \mathcal{F}_0 = F_0 + \sigma_0\left(t\right)\dot{W}_0.$$

With minimal amount of stochastic forcing, it is shown that there is an ergodic invariant measure that attracts all the solutions (refs.). Thus in this forced-dissipated case the initial states become less important and we are more interested in the statistics on the long time limit as $t \to \infty$. Assuming there exists a statistical equilibrium state, thus we can first find the equilibrium relation in statistical mean and total statistical energy

$$2dE_{\mu,\infty} = \mu\bar{U}_\infty\bar{F}_0 + \sum\left(1 + \mu\left|\mathbf{k}\right|^{-2}\right)\bar{F}^*_{\mathbf{k}}\bar{\omega}_{\mathbf{k},\infty} + \bar{Q}_{\sigma,\mu,\infty}. \tag{2.3}$$

$E_{\mu,\infty}$ is the total statistical energy in the mean and variance, and it can be determined purely form the equilibrium mean state $\left(\bar{U}_\infty, \bar{\omega}_\infty\right)$ together with the forcing effects $\left(\bar{F}_0, \bar{F}_{\mathbf{k}}, \bar{Q}_\sigma\right)$.

Now to find the upper bound for the total statistical energy due to the effects of external forcing, we would like to determine the total energy in statistical steady state. First of all, we have the inequality in the interaction terms with the mean state

$$\begin{aligned}
\sum \left(1+\mu\left|\mathbf{k}\right|^{-2}\right) F_{\mathbf{k}}^{*}\cdot\bar{\omega}_{\mathbf{k}} &= \sum \left|1+\mu\left|\mathbf{k}\right|^{-2}\right|^{1/2} F_{\mathbf{k}}^{*}\cdot\left|1+\mu\left|\mathbf{k}\right|^{-2}\right|^{1/2}\bar{\omega}_{\mathbf{k}}\\
&\leq \frac{1}{4\lambda}\sum\left|1+\mu\left|\mathbf{k}\right|^{-2}\right|\left|\bar{\omega}_{\mathbf{k}}\right|^{2}+\lambda\sum\left|1+\mu\left|\mathbf{k}\right|^{-2}\right|\left|F_{\mathbf{k}}\right|^{2}\\
&< \frac{1}{4\lambda}\sum\left|1+\mu\left|\mathbf{k}\right|^{-2}\right| E_{\mathbf{k}}+\lambda\sum\left|1+\mu\left|\mathbf{k}\right|^{-2}\right|\left|F_{\mathbf{k}}\right|^{2},\quad \lambda>0.
\end{aligned}$$

$$\mu\bar{U}\cdot F_0\leq\frac{1}{4\lambda}\left|\mu\right|\bar{U}^{2}+\lambda\left|\mu\right|F_0^{2}<\frac{1}{4\lambda}\left|\mu\right|E_U+\lambda\left|\mu\right|F_0^{2}.$$

The above inequalities only holds in the stable regime $\mu>0$ since the coefficients on the right hand sides must always be positive. Notice that $E_{\mathbf{k}}=\left\langle\left|\omega_{\mathbf{k}}\right|^{2}\right\rangle$ is the total statistical mean and variance in small vortical modes and $E_U=\left\langle U^2\right\rangle$ is the statistical energy in the mean flow including both mean and variance. Thus the last inequality adds the variance to the original terms with purely the statistical mean. Thus the inflation at the last inequality could be large.

Therefore we can define the *effective statistical forcing* $Q_{F,\mu}$ combining the contributions in deterministic and stochastic forcing

$$Q_{F,\mu}\left(\lambda\right)=\mu\left(\lambda F_0^{2}+\frac{1}{2}\sigma_0^{2}\right)+\sum\left(1+\mu\left|\mathbf{k}\right|^{-2}\right)\left(\lambda\left|F_{\mathbf{k}}\right|^{2}+\frac{1}{2}\sigma_{\mathbf{k}}^{2}\right),\quad \mu>0;$$

and the *effective dissipation* in the statistical energy equation can be determined by changing the parameter value λ. The original system (1.13) contains the Ekman damping $-2dE_\mu$, thus the parameter $\lambda>0$ can be taken so that there still exist a negative damping effect in the total statistical energy dynamics

$$\bar{d}_F\left(\lambda\right)=2d-\left(2\lambda\right)^{-1}>0,\quad \lambda>\left(8d\right)^{-1}.$$

For simplicity we could just take $2\lambda=d^{-1}$ so that $\bar{d}_F=d$. With all these arrangements we have the differential inequality for the total statistical energy E_μ from (1.13) so that

$$\frac{dE_\mu}{dt}\leq-\bar{d}_F E_\mu+Q_{F,\mu}.$$

Using *Grönwall's inequality* to the above relation we get the upper bound for the total statistical energy E_μ due to the effect of damping and external forcing

$$\begin{aligned}
E_\mu\left(t\right) &\leq E_\mu\left(0\right)e^{-\bar{d}_F t}+\int_0^t e^{-\bar{d}_F\left(t-s\right)}Q_F\left(s\right)ds\\
&\leq \epsilon_T+\bar{d}_F^{-1}Q_{F,\mu}.
\end{aligned}\tag{2.4}$$

Above the first inequality is for the general time-dependent case in the forcing effect, and the second one is under the further assumption of constant forcing in time. If we just want to focus on the long time performance, the first term with initial statistics can be made arbitrarily small at the long time limit $t>T$, thus we need only focus on the second term above

$$Q_{F,\mu}=\mu\left(\frac{1}{2d}F_0^{2}+\frac{1}{2}\sigma_0^{2}\right)+\sum\left(1+\mu\left|\mathbf{k}\right|^{-2}\right)\left(\frac{1}{2d}\left|F_{\mathbf{k}}\right|^{2}+\frac{1}{2}\sigma_{\mathbf{k}}^{2}\right).$$

The stability can be developed in this forced-damped case in a similar way as before based on the relation (2.4). Then we can summarize the result in the following proposition:

Proposition 2. *(Statistical energy bounds with forcing and dissipation in regime $\mu>0$) Consider the forced-dissipated system of fluctuations (1.12) about the steady state solution (V_μ, Q_μ). For any parameter values $\mu>0$ the total statistical energy in mean and variance in the fluctuation component can be bounded by the inequality (2.4). Especially in the statistical steady state, the initial statistics get dissipated and the total statistical energy is purely determined by the external forcing and damping effects as*

$$E_\mu\left(t\right)\leq\frac{\mu}{2}\left(\left(d^{-1}F_0\right)^{2}+d^{-1}\sigma_0^{2}\right)+\frac{1}{2}\sum\left(1+\mu\left|\mathbf{k}\right|^{-2}\right)\left(\left|d^{-1}F_{\mathbf{k}}\right|^{2}+d^{-1}\sigma_{\mathbf{k}}^{2}\right).\tag{2.5}$$

Notice that (2.5) can be compared with the bound (2.1) from the non-forced non-damped case, where the deterministic forcing $d^{-1}F$ gets the similar role as the initial mean deviation and the stochastic forcing σ^2 gets the role of initial variance in the ensemble. The total energy bound in long time limit can be estimated based on the forcing and damping parameters. Above in (2.4), we simply assume that the deterministic forcing F and stochastic forcing σ are both independent in time. It should be easy to generalize the above bound to the time-dependent case.

2.2.1. *Statistical bounds for the mean and variance with time independent forcing.* In the statistical energy bound in (2.4) we can not only get the maximum total statistics in the mean and variance in final statistical equilibrium, but also an estimation of the transient energy decay rate from the original initial state by the effective damping rate $\bar{d}_F$. Here on the other hand if we only interested in the statistical in the final steady state, then the equilibrium relation (2.3) can be adopted directly. Then a tighter bound for the total statistical energy (especially for the total variance) could be achieved.

Directly from the equilibrium statistical relation (2.3), using the same estimation for the right hand side for the interaction with the mean state we have the equilibrium total statistical energy $E_{\mu,\infty}$ being controlled by the statistical mean states

$$2dE_{\mu,\infty} \leq \frac{1}{4\lambda}\left(|\mu|\,\bar{U}_\infty^2 + \sum\left|1+\mu\,|\mathbf{k}|^{-2}\right||\bar{\omega}_\mathbf{k}|_\infty^2\right) + \lambda\left(|\mu|\,F_0^2 + \sum\left|1+\mu\,|\mathbf{k}|^{-2}\right||F_\mathbf{k}|^2\right) + \bar{Q}_{\sigma,\mu,\infty}.$$

Unlike the previous strategy to inflate the energy in the mean to the total statistical energy, here we consider the mean and variant separately. Thus define the statistical energy in the mean E^m and in the variance E^v in each mode separately as

$$E_U^m = \bar{U}^2,\ E_\mathbf{k}^m = |\bar{\omega}_\mathbf{k}|^2, \quad E_U^v = \overline{U'^2},\ E_\mathbf{k}^v = \overline{|\omega_\mathbf{k}'|^2}.$$

Therefore we find the proper bound for the total statistical variance E^m and the total energy in the mean by simply separating the total statistical energy into the energy in the mean and variance, $E_\mu = E^m + E^v$. The above inequality can be rewritten accordingly for any parameter values $\lambda > 0$

$$\begin{aligned}&2d\left(|\mu|\,E_U^v + \sum\left|1+\mu\,|\mathbf{k}|^{-2}\right|E_\mathbf{k}^v\right) + \left(2d-(4\lambda)^{-1}\right)\left(|\mu|\,E_U^m + \sum\left|1+\mu\,|\mathbf{k}|^{-2}\right|E_\mathbf{k}^m\right)\\ \leq&\lambda\left(|\mu|\,F_0^2 + \sum\left|1+\mu\,|\mathbf{k}|^{-2}\right||F_\mathbf{k}|^2\right) + \bar{Q}_{\sigma,\mu,\infty}.\\ \Rightarrow\quad&\theta E^m + E^v \leq Q_{F,\mu}(\theta),\end{aligned} \tag{2.6}$$

where we define $\theta = 1-(8d\lambda)^{-1} < 1$ and the upper bound on the right hand side depending on the ratio θ becomes

$$Q_{F,\mu}(\lambda) = \mu\left((1-\theta)^{-1}\frac{F_0^2}{16d^2} + \frac{\sigma_0^2}{2d}\right) + \sum\left(1+\mu\,|\mathbf{k}|^{-2}\right)\left((1-\theta)^{-1}\frac{|F_\mathbf{k}|^2}{16d^2} + \frac{\sigma_\mathbf{k}^2}{2d}\right).$$

Especially when $\theta = 0$, we find the estimation for the total statistical variance

$$E^v = |\mu|\,E_U^v + \sum\left|1+\mu\,|\mathbf{k}|^{-2}\right|E_\mathbf{k}^v \leq \mu\left(\frac{F_0^2}{16d^2} + \frac{\sigma_0^2}{2d}\right) + \sum\left(1+\mu\,|\mathbf{k}|^{-2}\right)\left(\frac{|F_\mathbf{k}|^2}{16d^2} + \frac{\sigma_\mathbf{k}^2}{2d}\right). \tag{2.7}$$

Alternatively we can also take values $\theta < 0$, then we get the relation that the total first order mean E^m can be bounded by the total second order variance E^v. Hopefully the above relations can offer better bounds in the total variance and energy in the mean in the forced-damped case. This may need further investigations.

3. Statistical Saturation Bounds without Forcing and Dissipation with $\mu < 0$

It is clear that the steady state mean is stable with westward flow when $\mu > 0$; while the mean flow is typically unstable with eastward flow when $\mu < 0$. In the *statistically unstable regime with* $\mu < 0$, not all the weighting coefficients in the total statistical energy E_μ are positive. In this case, between two adjacent wave numbers $-\Lambda_{\mu+1}^2 < \mu < -\Lambda_\mu^2$ (in the notation, Λ_μ^2 and $\Lambda_{\mu+1}^2$ are two adjacent integer energy shells, while $\Lambda_\mu, \Lambda_{\mu+1}$ could be non-integer)

$$\begin{aligned}1+\mu\,|\mathbf{k}|^{-2} > 0,\quad |\mathbf{k}| \geq \Lambda_{\mu+1},\\ 1+\mu\,|\mathbf{k}|^{-2} < 0,\quad |\mathbf{k}| \leq \Lambda_\mu.\end{aligned} \tag{3.1}$$

Therefore, the total statistical energy E_μ in (1.8) can be decomposed into two components with a positive-definite part and a negative-definite part

$$\begin{aligned} E_\mu &= -E_\mu^L + E_\mu^S \\ E_\mu^L &= \frac{1}{2}\sum_{1\leq|\mathbf{k}|\leq\Lambda_\mu}\left|1+\mu\left|\mathbf{k}\right|^{-2}\right|\left\langle\left|\omega_{\mathbf{k}}\right|^2\right\rangle+\frac{|\mu|}{2}\left\langle U^2\right\rangle, \\ E_\mu^S &= \frac{1}{2}\sum_{|\mathbf{k}|\geq\Lambda_{\mu+1}}\left(1+\mu\left|\mathbf{k}\right|^{-2}\right)\left\langle\left|\omega_{\mathbf{k}}\right|^2\right\rangle. \end{aligned} \tag{3.2}$$

Above in (3.2) E_μ^L is the larger scale statistical energy with negative coefficients, and E_μ^S is the smaller scale statistical energy with positive coefficients. Especially in regime $-1<\mu<0$, only the large scale mean flow U is contained in E_μ^L. This is an interesting case where the interactions between the large mean flow U and small vortical modes ω become important through topographic stress.

In general, E_μ^S will contain much more modes with high wavenumbers and E_μ^L usually only contains the modes in the largest scale (but often also of more interest). This implies the possible instability between the low wavenumber modes and high wavenumber modes in this regime. Still without the external damping and noise terms the total statistical energy conservation from (1.9) is valid,

$$E_\mu\left(t\right)=E_\mu\left(0\right).$$

Suppose negative initial statistical energy $E_0=E_{\mu,0}^S-E_{\mu,0}^L<0$, that is,

$$\sum_{|\mathbf{k}|\geq\Lambda_{\mu+1}}\left(1+\mu\left|\mathbf{k}\right|^{-2}\right)\left\langle\left|\omega_{\mathbf{k}}\right|^2\right\rangle_0\leq\sum_{|\mathbf{k}|\leq\Lambda_\mu}\left|1+\mu\left|\mathbf{k}\right|^{-2}\right|\left\langle\left|\omega_{\mathbf{k}}\right|^2\right\rangle_0+\left|\mu\right|\left\langle U^2\right\rangle_0. \tag{3.3}$$

This implies larger initial perturbations (both in mean and noise) in the unstable larger scales, and this should be a natural case that is easy to satisfy in many realistic scenarios [*Capter 2 of Majda & Wang book*]. As a result, the conservation law of the total statistical fluctuation energy predicts that the perturbed mean and variances in all the high wavenumber modes are 'slaved' by the low wavenumber perturbations in mean and variances by all the time, that is,

$$\sum_{|\mathbf{k}|\geq\Lambda_{\mu+1}}\left(1+\mu\left|\mathbf{k}\right|^{-2}\right)\left\langle\left|\omega_{\mathbf{k}}\right|^2\right\rangle_t\leq\sum_{|\mathbf{k}|\leq\Lambda_\mu}\left|1+\mu\left|\mathbf{k}\right|^{-2}\right|\left\langle\left|\omega_{\mathbf{k}}\right|^2\right\rangle_t+\left|\mu\right|\left\langle U^2\right\rangle_t. \tag{3.4}$$

Still this inequality cannot guarantee the general statistical stability in the total energy in mean and variance since both sides of (3.4) could grow (or decay) without bound at the same time. In the rest part of this section, we consider the saturation bounds in the total statistics in the system specially in the unstable regime $\mu<0$. No external forcing and dissipation is included in the first place, so that the problem is how the statistics in the system evolve in time according to the steady state solution (V_μ, Q_μ) as the initial state.

3.1. **Statistical energy saturation bound in unstable regime.** Here we make use of the conserved statistical functionals in Section 2 to derive the saturation bounds of the topographic barotropic flow without forcing and dissipation. In the first place, again we propose the stable 'reference state' with $\alpha>0$. Thus about the mean steady state in small scale stream functions and large scale mean flow

$$Q_{\alpha,\mathbf{k}}=\frac{\alpha\hat{h}_{\mathbf{k}}}{\alpha+\left|\mathbf{k}\right|^2},\quad\Psi_{\alpha,\mathbf{k}}=\frac{\hat{h}_{\mathbf{k}}}{\alpha+\left|\mathbf{k}\right|^2},\quad V_\alpha=-\frac{\beta}{\alpha},$$

the total statistical energy in fluctuation (1.8) is kept conserved depending on the initial state value, that is,

$$E_\alpha^{\mathrm{stat}}\left(t\right)=\frac{\alpha}{2}\left\langle\left(U-V_\alpha\right)^2\right\rangle_t+\frac{1}{2}\sum_{1\leq|\mathbf{k}|\leq\Lambda}\left(1+\alpha\left|\mathbf{k}\right|^{-2}\right)\left\langle\left|q_{\mathbf{k}}-Q_{\alpha,\mathbf{k}}\right|^2\right\rangle_t\equiv E_\alpha^{\mathrm{stat}}\left(0\right). \tag{3.5}$$

Now we consider the solutions in the unstable regime $\mu<0$ so that we can get two sets of decompositions with the real steady state μ and the reference state with α

$$U\left(t\right)=V_\mu+\tilde{U}\left(t\right)=V_\alpha+\hat{U}\left(t\right),\quad q\left(t\right)=Q_\mu+\tilde{\omega}\left(t\right)=Q_\alpha+\hat{\omega}\left(t\right),$$

thus we have the statistics in the fluctuation component $\left(\tilde{U},\tilde{\omega}\right)$ about the steady state solution (V_μ, Q_μ) (again we leave the tildes in the rest part) according to the previous stable reference state with parameter α as

$$\left\langle (U-V_\alpha)^2\right\rangle_t = \left(V_\mu - V_\alpha + \bar{U}\right)^2 + \overline{U'^2},$$
$$\left\langle |q_{\mathbf{k}} - Q_{\alpha,\mathbf{k}}|^2\right\rangle_t = |Q_{\mu,\mathbf{k}} - Q_{\alpha,\mathbf{k}} + \bar{\omega}_{\mathbf{k}}|^2 + \overline{|\omega'_{\mathbf{k}}|^2},$$

where we can define the constants between the steady state and the reference state as

$$V_{\mu,\alpha} \equiv V_\mu - V_\alpha = \frac{\mu-\alpha}{\alpha}\frac{\beta}{\mu}, \quad Q_{\mu,\alpha,\mathbf{k}} \equiv Q_{\mu,\mathbf{k}} - Q_{\alpha,\mathbf{k}} = \frac{\mu-\alpha}{\alpha+|\mathbf{k}|^2}\frac{|\mathbf{k}|^2\hat{h}_{\mathbf{k}}}{\mu+|\mathbf{k}|^2}. \tag{3.6}$$

Therefore in the same way, we get the first statistical energy bound for the fluctuation component $\left(\tilde{U},\tilde{\omega}\right)$ based on the conservation of (positive) total statistical fluctuation energy $E_\alpha(t) = E_\alpha(0)$ according to the reference state with parameter $\alpha > 0$. The initial statistical energy can be found from (3.5) with the initial mean $\left(\bar{U}_0, \bar{\omega}_0\right)$ and the initial variance $\left(\overline{U_0'^2}, \overline{|\omega_0'|^2}\right)$ in large scale mean flow and small vortical modes. The previous argument is based on the fact that the topographic barotropic system without forcing and dissipation always conserves the total statistical energy for any values of parameter $\alpha > 0$, thus we have the freedom to choose the optimal parameter value α in the conservation relation (3.5) for the saturation bound.

The goal here is to find the statistical stability of the steady state solution (V_μ, Q_μ) in the unstable regime $\mu < 0$. Without the inclusion of external forcing and dissipation, the problem is to track the evolution and amplification of the fluctuations in the ensemble of particles beginning with an unbiased initial steady state and proper amount of uncertainty among the particles. We propose the initial state with zero fluctuation about the steady state solution in mean state and prescribed variances in each mode

$$\bar{U}_0 = 0,\ \bar{\omega}_0 = 0, \quad \overline{U_0'^2} = \sigma_{U,0}^2,\ \overline{\left|\omega'_{\mathbf{k},0}\right|^2} = \sigma_{\mathbf{k},0}^2, \tag{3.7}$$

then the conservation of total statistical energy implies that the mean fluctuation and variance can be determined by the initial configuration of variance and the difference with the reference state

$$\begin{aligned}&\alpha\left[\left(V_{\mu,\alpha}+\bar{U}_t\right)^2 + \overline{U_t'^2}\right] + \sum_{1\leq|\mathbf{k}|\leq\Lambda}\left(1+\alpha|\mathbf{k}|^{-2}\right)\left[|Q_{\mu,\alpha,\mathbf{k}} + \bar{\omega}_{\mathbf{k},t}|^2 + \overline{\left|\omega'_{\mathbf{k},t}\right|^2}\right]\\ =&\alpha\left[V_{\mu,\alpha}^2 + \sigma_{U,0}^2\right] + \sum_{1\leq|\mathbf{k}|\leq\Lambda}\left(1+\alpha|\mathbf{k}|^{-2}\right)\left[Q_{\mu,\alpha,\mathbf{k}}^2 + \sigma_{\mathbf{k},0}^2\right].\end{aligned} \tag{3.8}$$

The above inequality is valid for all the values of $\alpha > 0$. Instead of the slaving relation (3.4) that separates the whole system into a stable and an unstable subspace with $\mu < 0$, the relation in (3.8) gets uniformly positive coefficients in every component of the statistical energy in mean and variance. Therefore immediately we get the statistical stability in each component of the fluctuation mean and variance that they will stay finite and stable as the system evolves in time since the right hand side in the initial value is finite with positive coefficients. That is, when we run an ensemble with initial steady state solution (V_μ, Q_μ) with uncertainties in particles the bias in the mean state and the spread of the particles will always keep finite in amplitude without unbounded growth. Still the bound in (3.8) is not desirable since it combines with the difference in reference state $V_{\mu,\alpha}$ and $Q_{\mu,\alpha}$. Next we try to find the optimal saturation bound for the statistics in fluctuation mean and variance by minimizing the right hand side among all the values of $\alpha > 0$. Especially we consider the saturation bounds for the total statistical kinetic energy in the mean, $\bar{U}^2 + \fint\left|\nabla\bar{\psi}\right|^2$, and in the variance, $\overline{U'^2} + \fint\overline{|\nabla\psi'|^2}$.

<u>Saturation bound for total variance based on the flow vorticity</u>. In the first place we can look at the saturation bound for the second order moments. In the conservation relation (3.8) to consider the statistical kinetic energy in second order variance only, just leave the first parts involving the mean states, then for all $\alpha > 0$ we have

$$\overline{U'^2} + \sum|\mathbf{k}|^2\overline{|\psi'_{\mathbf{k}}|^2} \leq \left[\left(V_{\mu,\alpha}+\bar{U}(t)\right)^2 + \overline{U'^2}(t)\right] + \sum\frac{|\mathbf{k}|^2+\alpha}{\alpha}|\mathbf{k}|^2\left[\left|\Psi_{\mu,\alpha,\mathbf{k}} + \bar{\psi}_{\mathbf{k}}\right|^2 + \overline{|\psi'_{\mathbf{k}}|^2}\right],$$

where the left hand side above defines the *total statistical kinetic energy* in the variance, and the right hand side is an reorganization of the total statistics E_μ. Using the relation in (3.8) to relate the right hand side above with the initial data and noting that the above inequality is valid for all values $\alpha > 0$, the saturation bound for the total

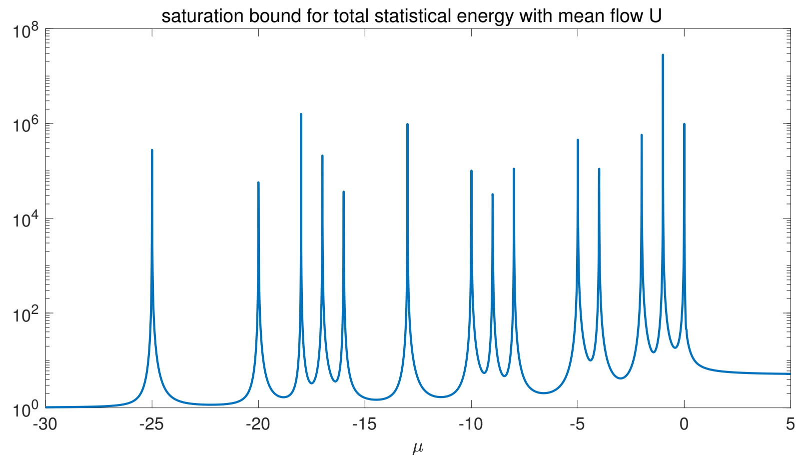


FIGURE 3.1. Saturation bound C_μ^v for the total variance in kinetic energy (3.9). Initial variance is only set to be non-zero among the mean flow $\sigma_{U,0}=1$ and the ground modes $|\mathbf{k}|=1$ with variance $\sigma_{1,0}=1$.

statistical kinetic energy variance can be calculated by minimizing the right hand side above with all the possible values of α so that

$$C_\mu^v=\min_{\alpha>0}\left[\frac{(\alpha-\mu)^2}{\alpha^2}V_\mu^2+\sigma_{U,0}^2\right]+\sum_{1\leq|\mathbf{k}|\leq\Lambda}\left[\frac{(\alpha-\mu)^2|\mathbf{k}|^2}{\alpha\left(\alpha+|\mathbf{k}|^2\right)}|\Psi_{\mu,\mathbf{k}}|^2+\left(|\mathbf{k}|^{-2}+\alpha^{-1}\right)\sigma_{\mathbf{k},0}^2\right],$$

with $V_\mu=-\beta/\mu$ and $\Psi_{\mu,\mathbf{k}}=\left(\mu+|\mathbf{k}|^2\right)^{-1}\hat{h}_\mathbf{k}$ the steady state solutions. The total variance of the flow fluctuation in both large scale mean flow and small vorticity can be reached at

$$\overline{U_t'^2}+\sum_{1\leq|\mathbf{k}|\leq\Lambda}|\mathbf{k}|^2\overline{\left|\psi_{\mathbf{k},t}'\right|^2}\leq C_\mu^v\left(h,\beta,\sigma_0,\Lambda\right),\tag{3.9}$$

where the bound C_μ is dependent on the truncation size Λ, topographic structure h, the beta-effect β and the initial noise in each mode σ_0. The saturation bound C_μ estimates the maximum amount of energy in variance the system could reach depending on the initial configuration.

In Figure 3.1, we show the saturation bounds C_μ with changing values of $\mu<0$ for statistical kinetic energy by minimization among values in the stable regime $\alpha>0$. The model parameters are the same with the previous setup in Section 2 with $\beta=1$, $\Lambda=12$, $h=\frac{3\sqrt{2}}{4}(\sin x+\cos x)$. Initial variance is only set to be non-zero among the mean flow $\sigma_{U,0}=1$ and the ground modes $|\mathbf{k}|=1$ with variance $\sigma_{1,0}=1$. The saturation bound C_μ blows up at the resonance points at $\mu=0$ due to the instability in the mean flow U and at $\mu=-|\mathbf{k}|^2$ with non-zero topographic mode $\hat{h}_\mathbf{k}\neq0$. The steady state solutions (V_μ,Q_μ) will blow up at these resonance points. For values of μ near these points, this large values of C_μ shows that the variances of the fluctuation component will also increase to much larger uncertainty due to the instability. On the other hand for values away from the resonance points the total statistical variance can be controlled within relatively small values, implying the small variability in the ensemble particles.

Non-optimal bound for statistical mean state. The above saturation bound estimation about the total fluctuation variance (3.9) could be accurate if the deviation from the mean $|Q_{\mu,\alpha,\mathbf{k}}+\bar{\omega}_\mathbf{k}|$ is small (for example, there is only topographic stress in small amplitude, $h\sim0$). Still the error due to the neglected mean from the term $Q_{\mu,\alpha}$ needs to be attended, and it is difficult to estimate the energy in the statistical mean fluctuation $\bar{\omega}$ from the previous inequalities. Especially when there are some values of $|\mathbf{k}|^2$ close to $-\mu$, the error from $Q_{\mu,\alpha}$ could be huge. There is possibility that large amount of energy cascades from the fluctuation variances back to the mean state due to the nonlinear interactions and drive the mean state $\bar{\omega}$ away to another distinct state as the system evolves.

Still it is possible to find a non-optimal estimation in the fluctuation statistical mean state. In general we have the inequality in the statistical mean difference

$$\begin{aligned}\left|Q_{\mu,\alpha,\mathbf{k}}+\bar{\omega}_{\mathbf{k}}\right|^{2} &= \left|Q_{\mu,\alpha,\mathbf{k}}\right|^{2}+\left|\bar{\omega}_{\mathbf{k}}\right|^{2}-2\mathfrak{Re}\left(Q_{\mu,\alpha,\mathbf{k}}\cdot\bar{\omega}_{\mathbf{k}}^{*}\right)\\ &\geq \left(1-\epsilon^{-1}\right)\left|Q_{\mu,\alpha,\mathbf{k}}\right|^{2}+\left(1-\epsilon\right)\left|\bar{\omega}_{\mathbf{k}}\right|^{2},\end{aligned}$$

where Cauchy's inequality is used with $\epsilon>0$ as a control parameter. Similarly for the large scale flow U we have the inequality to separate the mean fluctuation

$$\left(V_{\mu,\alpha}+\bar{U}\left(t\right)\right)^{2}\geq\left(1-\epsilon^{-1}\right)V_{\mu,\alpha}^{2}+\left(1-\epsilon\right)\bar{U}^{2}.$$

Still assuming initial statistical mean of the ensemble as the steady state mean (V_μ,Q_μ), and the initial ensemble variance with spectrum $\overline{U_0'^2}=\sigma_{U,0}^2$, $\overline{\left|\omega_{\mathbf{k},0}'\right|^2}=\sigma_{\mathbf{k},0}^2$, we have for the vortical modes

$$\left(1+\alpha\left|\mathbf{k}\right|^{-2}\right)\left(\left(1-\epsilon^{-1}\right)\left|Q_{\mu,\alpha,\mathbf{k}}\right|^{2}+\left(1-\epsilon\right)\left|\bar{\omega}_{\mathbf{k}}\right|^{2}+\overline{\left|\omega_{\mathbf{k}}'\left(t\right)\right|^{2}}\right)\leq\left(1+\alpha\left|\mathbf{k}\right|^{-2}\right)\left(\left|Q_{\mu,\alpha,\mathbf{k}}\right|^{2}+\sigma_{0,\mathbf{k}}^{2}\right),$$

since all the coefficients are positive for $\alpha>0$. The term with $\left|Q_{\mu,\alpha,\mathbf{k}}\right|^2$ cancels with each other on both sides of the above inequality. Substituting the above estimations back to the original stability inequality (3.8), we can derive the saturation bound combining the statistical mean and variance

$$C_{\mu}^{\theta}=\min_{\alpha>0}\frac{1}{1-\theta}\left[\frac{\left(\alpha-\mu\right)^{2}}{\alpha^{2}}V_{\mu}^{2}+\sum\frac{\left(\alpha-\mu\right)^{2}\left|\mathbf{k}\right|^{2}}{\alpha\left(\alpha+\left|\mathbf{k}\right|^{2}\right)}\left|\Psi_{\mu,\mathbf{k}}\right|^{2}\right]+\left[\sigma_{U,0}^{2}+\sum\left(\left|\mathbf{k}\right|^{-2}+\alpha^{-1}\right)\sigma_{\mathbf{k},0}^{2}\right].$$

Then the combination of statistical energy in the mean and statistical variance with a weighting parameter $\theta=1-\epsilon^{-1}<1$ becomes can be estimated by

$$\theta E^{m}\left(t\right)+E^{v}\left(t\right)\leq C_{\mu}^{\theta}\left(h,\beta,\sigma_{0},\Lambda\right),\tag{3.10}$$

where E^m is the statistical energy in the mean fluctuation and E^v is the statistical variance

$$\begin{aligned}E^{m}&=\bar{U}^{2}+\fint\left|\nabla\bar{\psi}\right|^{2}=\bar{U}^{2}+\sum\left|\mathbf{k}\right|^{2}\left|\bar{\psi}_{\mathbf{k}}\left(t\right)\right|^{2},\\ E^{v}&=\overline{U'^{2}}+\fint\overline{\left|\nabla\psi'\right|^{2}}=\overline{U'^{2}}+\sum\left|\mathbf{k}\right|^{2}\overline{\left|\psi_{\mathbf{k}}'\right|^{2}}.\end{aligned}$$

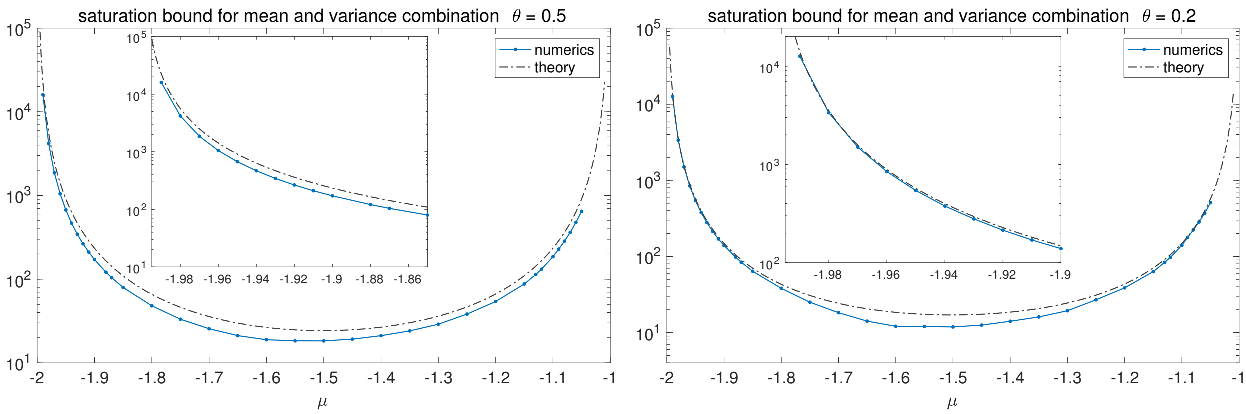
Comparing (3.10) with (3.9), C_μ^θ reduces to the variance bound C_μ^v when the parameter $\theta\rightarrow0$ with consistency. Unfortunately we cannot get the total statistical energy bound for $\theta=1$ in the above saturation bound C_μ^θ. Still we can find the *non-optimal bound* for the statistical mean state with a crude estimation $\theta E^m\leq\theta E^m+E^v\leq C_\mu^\theta$ from the statistical conservation (3.10) as

$$\bar{U}_{t}^{2}+\sum_{1\leq\left|\mathbf{k}\right|\leq\Lambda}\left|\mathbf{k}\right|^{2}\left|\bar{\psi}_{\mathbf{k},t}\right|^{2}\leq C_{\mu}^{m}=\min_{\theta<1}\theta^{-1}\left(1-\theta\right)^{-1}C_{\mu}^{v}=4C_{\mu}^{v}.\tag{3.11}$$

Notice that in (3.10) we can even have $\theta<0$, thus the inequality describes that the total variance in second order moments in the system can actually be controlled by the totally energy in the mean state in the first order moments.

Above we offer two levels of estimations. The first separates the total statistical mean and total variance with a balance parameter θ. Note that larger value of θ (then more emphasis on the stability in statistical mean) leads larger value on the right hand side bound. This shows (3.10) is not so desirable a bound for estimating the variability in the statistical mean. In the third inequality, we separate the statistical mean state only. It shows the total statistical mean can not increase without bound with a largest value $2C_\mu$, while this bound is not optimal since C_μ could become huge. It is possible a tighter bound can be found for the statistical mean state.

Proposition 3. *(Statistical saturation bound for total fluctuation variance and statistical mean) For any general value of μ (and especially for unstable case $\mu<0$) in the topographic barotropic system without forcing and dissipation, assume zero initial statistical mean fluctuation and a general initial ensemble variance as (3.7). A saturation bound for the combination of statistical mean state and variance, θE^m+E^v, with a ratio parameter $\theta\in(0,1)$ can be reached from (3.10). Especially the total variance of the total fluctuation kinetic energy, E^v, in the flow field can be estimated with a saturation bound C_μ^v from (3.9); and for the statistical energy in the mean fluctuation only, a (non-optimal) estimation of the saturation bound $C_\mu^m=2C_\mu^v$ can be found as in (3.11).*



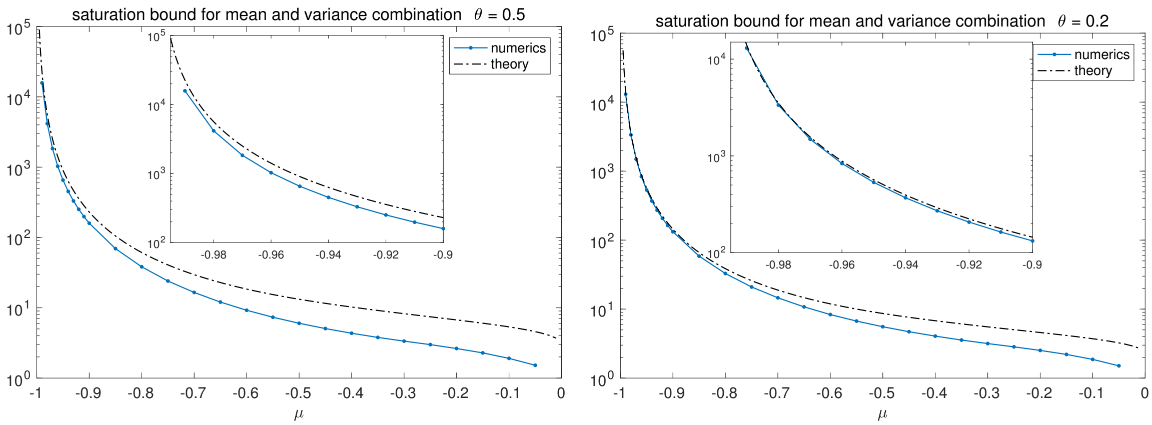
(A) $\theta = 0.5$ (B) $\theta = 0.2$

FIGURE 3.2. Saturation bound in unstable regime $-2 < \mu < -1$ for statistical mean and variance combined with parameter θ. It compares the combined statistical energy with $\theta E^m + E^v$ in statistical equilibrium with different weighting parameter $\theta = 0.5, 0.2$.

3.2. **Numerical test for the saturation bounds in unstable regimes.** In this final part, we verify these saturation bounds for both the variance and the mean state through numerical simulations. We test two regimes for the parameter $-2 < \mu < -1$ and $-1 < \mu < 0$. Instead of comparing the statistical energy in mean and variance separately, here we consider the combined bound C_μ^θ for the combination of mean fluctuation and variance $\theta E^m(t) + E^v(t)$.

3.2.1. *Saturation bound in unstable regime $\mu < -1$.* First we check the saturation bound for total variance and mean in the unstable regime with parameter values changing among $-2 < \mu < 1$. Figure 3.2 illustrates the bound for a combination of mean and variance with a parameter $\theta < 1$ from the inequality (3.10). θ sets the weight in the statistical mean state. We check two parameter values $\theta = 0.5$ and $\theta = 0.2$. With $\theta = 0.5$ the mean fluctuation becomes more important in the total statistical energy, while with $\theta = 0.2$ the statistical energy in the variance is dominant. First the dotted-dashed black lines illustrate the theoretical saturation bound C_μ^θ with changing values of μ. As expected from the theoretical results, instability with large will take place at the resonance points $\mu = -1, -2$. The saturation bound C_μ^θ for the total variance form a tight bound especially when μapproaches the two end points. For the intermediate values of $\mu$the statistical energy is relatively low and the saturation bound serves as a proper upper bound for the maximum of statistics in mean and variance the fluctuation component can reach. The instability with fluctuations in mean and variance increases as the parameter approaches the boundary. Especially in the case with $\theta = 0.2$ where the variance part is dominant, from the expanded plot in results near $\mu \to -2$ the saturation becomes extremely tight for the statistics in the system. This shows that the upper bound in C_μ^θ can offer an accurate estimation for the maximum of fluctuations the system can reach in the highly unstable regime. Furthermore, we can observe that the instability increases faster near the left side than that near the right side. The may be related with the linear instability in the left limit (see the Appendix).

3.2.2. *Saturation bound in regime with unstable mean flow $-1 < \mu < 0$.* In this second case, we check the saturation bound in regime with only an unstable mean flow U. We use smaller beta-effect $\beta = |\mu|$ as the parameter approaches zero $\mu \to 0$. This is due to the weaker variability for the flow system as μ decreases. Figure 3.3 shows the comparison between the numerical simulation results with the theoretical prediction as the parameter value changes. The theoretical saturation bound C_μ^θ gives overall good estimation for the total statistical fluctuation that the system can reach with instability. Similar with the previous case, the saturation bound for the variance is extremely tight near the unstable point on the left boundary. The instability near the other limit $\mu \to 0$ vanishes. This is consistent with the linear analysis (see Appendix) that no instability takes place as $\mu \to 0$. The system can be stabilized from the interactions between the large and small scales through topographic stress at this point $\mu = 0$.



(A) $\theta = 0.5$ (B) $\theta = 0.2$

FIGURE 3.3. Saturation bound in unstable regime $-1 < \mu < 0$ for statistical mean and variance combined with parameter θ. The beta-effect is taken as $\beta = |\mu|$. It compares the combined statistical energy with $\theta E^m + E^v$ with different partition value $\theta = 0.5, 0.2$.

4. Statistical Saturation Bounds with Forcing and Dissipation

In our previous discussions, we focus on the linear and nonlinear interactions for the barotropic flow in (1.1) without any external forcing and dissipation effects. Thus the total statistical energy is controlled by the initial state statistical mean and variance. On the other hand, for the performance of the energy in the mean and variance in the long time limit, geometric ergodicity for the truncated topographic barotropic model (1.1) is proved under dissipation, inhomogeneous deterministic forcing and minimal stochastic forcing [*Majda & Tong, JNLS 2015*]. Thus there exists an invariant measure that attracts all the solutions in the long time limit regardless of the initial values. In this section, we consider the statistical instability in this case with forcing and dissipation effects.

4.1. **Saturation bound with forcing and dissipation at long time limit in unstable regimes $\mu < 0$.** In the unstable regime with $\mu < 0$, just consider the special case of linear damping and forcing in the special form as in (4.1) without additional forcing $F_0 = 0, F = 0$

$$\begin{aligned} &\text{small scale} && -d\omega + d\bar{\omega}_{\text{eq}} + \sigma_{\mathbf{k}}\dot{W}_{\mathbf{k}}, \\ &\text{large scale} && -dU + d\bar{U}_{\text{eq}} + \sigma_0\dot{W}_0. \end{aligned} \tag{4.1}$$

Again we can consider the saturation bound using the 'reference state' with parameter $\alpha > 0$. Especially notice using the reference state with parameter α with the damping and forcing kept in the form (4.1) with parameter μ, we introduce the additional forcing terms

$$F_0 = d\left(V_\mu - V_\alpha\right) \equiv dV_{\mu,\alpha}, \quad F_{\mathbf{k}} = d\left(Q_{\mu,\mathbf{k}} - Q_{\alpha,\mathbf{k}}\right) \equiv dQ_{\mu,\alpha,\mathbf{k}}.$$

Assuming there is no additional forcing besides the above terms and using the inequality in (2.5), the statistical energy based on the reference state (3.5) can be also used in the forced-dissipated case

$$\begin{aligned} E_\alpha\left(t\right) &= \alpha\left[\left(V_{\mu,\alpha} + \bar{U}\left(t\right)\right)^2 + \overline{U'^2}\left(t\right)\right] + \sum\left(1 + \alpha\left|\mathbf{k}\right|^{-2}\right)\left[\left|Q_{\mu,\alpha,\mathbf{k}} + \bar{\omega}_{\mathbf{k}}\left(t\right)\right|^2 + \overline{\left|\omega'_{\mathbf{k}}\right|^2}\left(t\right)\right] \\ &\leq \bar{d}_F^{-1}Q_{F,\alpha} = \alpha\left(V_{\mu,\alpha}^2 + d^{-1}\sigma_0^2\right) + \sum\left(1 + \alpha\left|\mathbf{k}\right|^{-2}\right)\left(\left|Q_{\mu,\alpha,\mathbf{k}}\right|^2 + d^{-1}\sigma_{\mathbf{k}}^2\right). \end{aligned} \tag{4.2}$$

This becomes a similar case with the previous non-forced non-damped situation in (3.8) with dependence on initial values. It is useful to notice that the random forcing amplitudes $(\sigma_0, \sigma_{\mathbf{k}})$ play the same role as the initial ensemble variance in the unforced case; while the additional deterministic forcing including the equilibrium mean $\left(\bar{U}_{\text{eq}}, \bar{\omega}_{\text{eq}}\right)$ play the role of the initial mean deviation in the previous unforced case. Therefore we can again find the saturation bound in the forced-damped case following exactly as the previous procedure.

Saturation bound for total variance based on the flow vorticity. The saturation bound for the total statistical kinetic energy including the mean flow can be calculated by minimizing the right hand side above with all the possible values of α so that

$$C_{\mu}^{v}=\min_{\alpha>0}\left[\frac{(\alpha-\mu)^{2}}{\alpha^{2}}V_{\mu}^{2}+d^{-1}\sigma_{0}^{2}\right]+\sum_{1\leq|\mathbf{k}|\leq\Lambda}\left[\frac{(\alpha-\mu)^{2}|\mathbf{k}|^{2}}{\alpha\left(\alpha+|\mathbf{k}|^{2}\right)}\left|\Psi_{\mu,\mathbf{k}}\right|^{2}+\left(|\mathbf{k}|^{-2}+\alpha^{-1}\right)d^{-1}\sigma_{\mathbf{k}}^{2}\right],$$

with $V_{\mu}=-\beta/\mu$ and $\Psi_{\mu,\mathbf{k}}=\left(\mu+|\mathbf{k}|^{2}\right)^{-1}\hat{h}_{\mathbf{k}}$ the steady state solutions. The total variance of the flow fluctuation with forcing and dissipation can be reached at

$$\overline{U'^{2}}(t)+\sum_{1\leq|\mathbf{k}|\leq\Lambda}|\mathbf{k}|^{2}\,\overline{|\psi_{\mathbf{k}}'|^{2}}(t)\leq C_{\mu}^{v}(h,\beta,d,\sigma,\Lambda),\tag{4.3}$$

where the bound C_{μ}^{v} is dependent on the truncation size Λ, topographic structure h, the beta-effect β, Ekman friction rate d, and the stochastic forcing in each mode σ_0. Comparing this saturation bound C_{μ}^{v} with the previous case (3.9) in Section 3 with dependence on initial value, we find similar form can be reached in this forced-dissipated case. The deterministic forcing from the steady state solution can be compared with the initial mean state in the previous case, and the effective stochastic forcing amplitude $d^{-1}\sigma^{2}$ can be compared with the initial variance in the ensemble members.

Non-optimal bound for statistical mean state. Similarly for the mean state we have the estimation from Cauchy's inequality

$$\left(V_{\mu,\alpha}+\bar{U}(t)\right)^{2}\geq\left(1-\epsilon^{-1}\right)V_{\mu,\alpha}^{2}+(1-\epsilon)\,\bar{U}^{2},$$
$$\left|Q_{\mu,\alpha,\mathbf{k}}+\bar{\omega}_{\mathbf{k}}(t)\right|^{2}\geq\left(1-\epsilon^{-1}\right)\left|Q_{\mu,\alpha,\mathbf{k}}\right|^{2}+(1-\epsilon)\left|\bar{\omega}_{\mathbf{k}}\right|^{2}.$$

Substituting back to the original inequality (4.2), we can derive the saturation bound combining the statistical mean and variance

$$C_{\mu}^{\theta}=\min_{\alpha>0}\frac{1}{1-\theta}\left[\frac{(\alpha-\mu)^{2}}{\alpha^{2}}V_{\mu}^{2}+\sum\frac{(\alpha-\mu)^{2}|\mathbf{k}|^{2}}{\alpha\left(\alpha+|\mathbf{k}|^{2}\right)}\left|\Psi_{\mu,\mathbf{k}}\right|^{2}\right]+d^{-1}\left[\sigma_{U,0}^{2}+\sum\left(|\mathbf{k}|^{-2}+\alpha^{-1}\right)\sigma_{\mathbf{k},0}^{2}\right].$$

Then the combination of statistical energy in the damped and forced case becomes can be estimated by

$$\theta E^{m}(t)+E^{v}(t)\leq C_{\mu}^{\theta}(h,\beta,\sigma_{0},\Lambda),\tag{4.4}$$

with $\theta=1-\epsilon^{-1}<1$. Especially we can find the non-optimal bound for the statistical mean state as

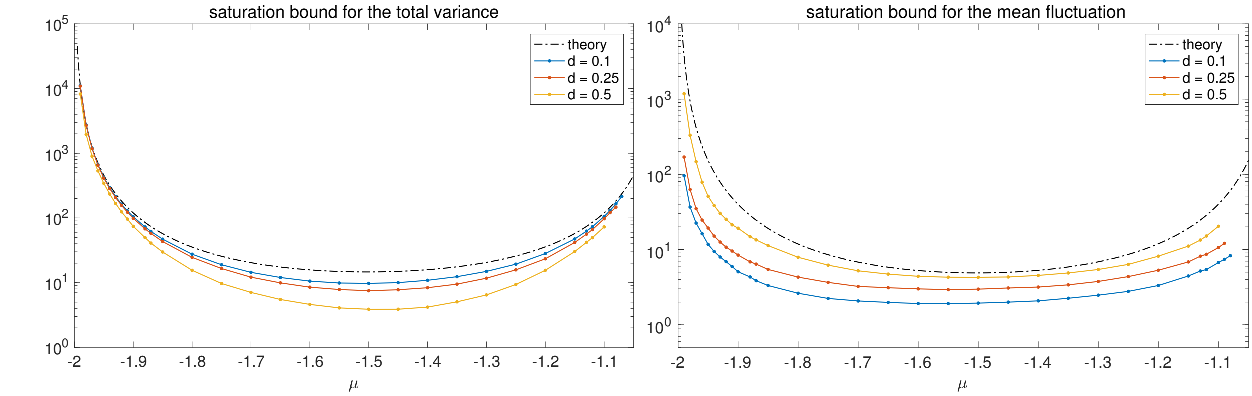
$$\bar{U}^{2}(t)+\sum_{1\leq|\mathbf{k}|\leq\Lambda}|\mathbf{k}|^{2}\left|\bar{\psi}_{\mathbf{k}}(t)\right|^{2}\leq C_{\mu}^{m}=\min_{\theta<1}\theta^{-1}(1-\theta)^{-1}C_{\mu}^{v}=4C_{\mu}^{v}.\tag{4.5}$$

Especially notice that in (4.4) we can even have $\theta<0$, thus the inequality describes that the total variance in second order moments in the system can actually be controlled by the totally energy in the mean state in the first order moments.

Proposition 4. *(Saturation bound for statistical mean and variance with damping and random forcing) With the special form of linear damping and forcing as in (4.1), the combined statistical mean and variance, $\theta E^{m}+E^{v}$, with the ratio parameter θ can be reached as in (4.4) with saturation bound C_{μ}^{θ}. Similarly the total variance of the flow fluctuation, E^{v}, is bounded by the saturation bound C_{μ}^{v} as in (4.3) depending on the topography h, linear damping d, and the random noise forcing σ; and the total statistical energy in mean fluctuation, E^{m}, is bounded by the (non-optimal) bound $C_{\mu}^{m}=4C_{\mu}^{v}$.*

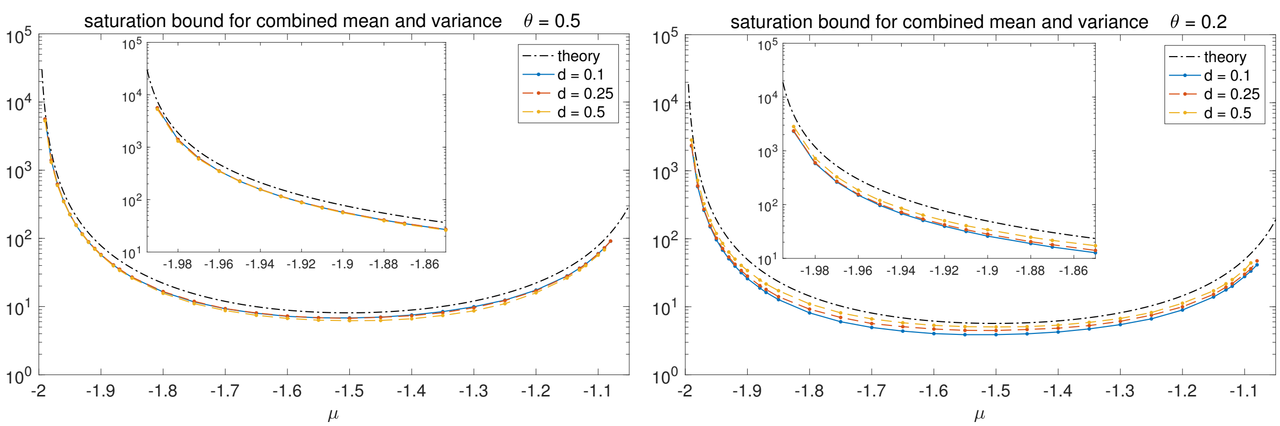
4.2. **Numerical verification of the saturation bound in the unstable regime with forcing and damping.** Here again we check the saturation bounds derived in (4.3), (4.4), and (4.5) using simple numerical simulations. The basic set-up is kept the same as before with the same set of parameters. Especially to make the bounds in the damped-forced case stay the same with the previous case, we choose the random forcing amplitude and damping in the relation $d_{k}^{-1}\sigma_{k}^{2}\equiv\sigma_{\mathrm{eq}}^{2}=1$ so exactly the same saturation bounds can be used in this case.

Similar like before, in Figure 4.1 we show compare the statistical energy in the mean and total variance with the saturation upper bound found in (4.3) and (4.5). Still the theoretical estimation sets a tight bound especially near the resonance point $\mu\to-2,-1$. We tested the mean and variance with different damping rates $d=0.1,0.25,0.5$.



(A) total variance (B) energy in the mean

FIGURE 4.1. Saturation bound with damping and forcing in unstable regime $-2 < \mu < -1$ for statistical mean and variance separately. Results with different damping rates $d = 0.1, 0.25, 0.5$ are shown. The left panel compares the total variance E^v and the right panel is the statistical energy in mean fluctuation E^m (in solid line) with the theoretical bound C^v_μ, C^m_μ (in dotted-dashed line).



(A) $\theta = 0.5$ (B) $\theta = 0.2$

FIGURE 4.2. Saturation bound with forcing in unstable regime $-2 < \mu < -1$ for statistical mean and variance combined with parameter θ. Results with different damping rates $d = 0.1, 0.25, 0.5$ are shown. It compares the combined statistical energy with $\theta E^m + E^v$ with different partition value $\theta = 0.5, 0.2$.

With larger uniform damping rate along all the scales the total variance decreases due to the stronger dissipation on all the modes; while the statistics in the mean increases as the damping rate increases. Correspondingly larger mean and smaller variance will appear with smaller damping rate d. Then Figure 4.2 shows the combined mean and variance bounds with ratio parameter θ found in (4.4). In the combination of mean and variance together, the three cases with different damping d become near each other and are close to the theoretical saturation bound despite their differences in the mean fluctuation and variance.

5. Statistical Stability without Large-Scale Mean Flow $U = 0$

As a further discussion, we consider the small scale fluctuation potential vorticity $\tilde{\omega}$ only and assume there is no large scale mean flow $U \equiv 0$. Therefore the original vorticity equation (1.1) and the fluctuation equation (1.5) can be reduced to the form only with the small scale potential vorticity accordingly as

$$\frac{\partial \omega_\Lambda}{\partial t} + \beta \frac{\partial \psi_\Lambda}{\partial x} + \mathcal{P}_\Lambda \left(\nabla^\perp \psi_\Lambda \cdot \nabla q_\Lambda\right) = 0, \tag{5.1}$$

$$\frac{\partial \tilde{\omega}}{\partial t} + \beta \frac{\partial \tilde{\psi}}{\partial x} + \beta \frac{\partial \Psi_\mu}{\partial x} + \mathcal{P}_\Lambda \left(\nabla^\perp \Psi_\mu \cdot \nabla \left(\tilde{\omega} - \mu \tilde{\psi}\right)\right) + \mathcal{P}_\Lambda \left(\nabla^\perp \tilde{\psi} \cdot \nabla \tilde{\omega}\right) = 0, \tag{5.2}$$

where $\tilde{\omega}$ is the fluctuation component subtracting the steady mean Q_μ from the original potential vorticity ω_Λ

$$q_\Lambda (\mathbf{x}, t) = \omega_\Lambda (\mathbf{x}, t) + h_\Lambda (\mathbf{x}) = Q_\mu (\mathbf{x}) + \tilde{\omega} (\mathbf{x}, t) .$$

One additional term with the steady state stream function, $\beta \partial \Psi_\mu / \partial x$, enters the fluctuation equation as one additional forcing effect. One important feature in the barotropic flow (5.1) without large scale flow $U = 0$ is that only the total kinetic energy, $-\fint \psi \omega$, is conserved; while neither the potential enstrophy nor the relative enstrophy, $\fint q^2, \fint \omega^2$, stays conserved any more in general with $\beta \neq 0$. It is important to notice that the potential vorticity dynamics in fluctuation component (5.2) is subject to the parameter μ that due to the background steady state Ψ_μ. In the rest parts of this section, we will focus on the fluctuation equation (5.2) and again neglect the 'tildes' on the fluctuation variables.

In this case with $U = 0$, the pseudo-energy in (1.6) only contains the small scale part. We begin with the positive-definite case of the statistical energy E_μ containing statistical mean and variance in small-scale vortical modes with $\mu > -1$ so that

$$E_\mu = \frac{1}{2} \sum_{1 \leq |\mathbf{k}| \leq \Lambda} \left(1 + \mu |\mathbf{k}|^{-2}\right) \left\langle |\omega_\mathbf{k}|^2 \right\rangle \equiv \frac{1}{2} \sum_{1 \leq |\mathbf{k}| \leq \Lambda} \left(1 + \mu |\mathbf{k}|^{-2}\right) \left(|\bar{\omega}_\mathbf{k}|^2 + \overline{|\omega'_\mathbf{k}|^2}\right). \tag{5.3}$$

We use $\langle \cdot \rangle$ from (1.7) to represent the ensemble average among various realizations of the solution trajectories. For the fluctuation component in each wavenumber mode

$$q_\mathbf{k} = Q_{\mu,\mathbf{k}} + \tilde{\omega}_\mathbf{k}, \quad \tilde{\omega}_\mathbf{k} = \bar{\omega}_\mathbf{k} + \omega'_\mathbf{k}, \quad Q_{\mu,\mathbf{k}} = \Delta \Psi_{\mu,\mathbf{k}} + \hat{h}_\mathbf{k} = \mu \left(|\mathbf{k}|^2 + \mu\right)^{-1} \hat{h}_\mathbf{k}.$$

Notice that $\omega'_\mathbf{k}$ is the mean zero random variable with variance always the same as the original potential vorticity mode, independent of the choice of mean statistical state,

$$\bar{q}_\mathbf{k} = Q_{\mu,\mathbf{k}} + \bar{\omega}_\mathbf{k}, \quad \overline{\omega'_\mathbf{k}} \equiv 0, \quad \mathrm{var}\left\{q_\mathbf{k}\right\} = \mathrm{var}\left\{\omega'_\mathbf{k}\right\};$$

and $\bar{\omega}_\mathbf{k}$ measures the statistical mean deviation from the steady state solution $Q_\mathbf{k}$ (thus it depends on the parameter value of μ). In this case, the coefficients in E_μ are positive, $1 + \mu |\mathbf{k}|^{-2} > 0$, among all wavenumbers $|\mathbf{k}| \leq \Lambda$, and the component $E_\mathbf{k}$ in each scale is positive. Thus E_μ indeed define the positive-definite statistical energy.

It is important to notice that without mean flow $U = 0$ the statistical pseudo-energy $E_\mu (t)$ is no longer conserved due to the additional interaction term with steady state streamfunction $\beta \partial \Psi_\mu / \partial x$. Therefore we have the additional source term in the dynamical equation

$$\frac{dE_\mu}{dt} = -\beta \fint \frac{\partial \Psi_\mu}{\partial x} \left(\bar{\omega} - \mu \bar{\psi}^\mu\right) = \beta \sum \frac{i k_x \hat{h}^*_\mathbf{k}}{|\mathbf{k}|^2} \bar{\omega}_\mathbf{k}. \tag{5.4}$$

5.1. **Saturation bounds for total statistical energy on the f-plane with $\beta = 0$.** Here in the first place we consider the irrotational flow $\beta = 0$. In this case, both energy and enstrophy in the system are kept conserved. Thus the right hand side of (5.4) vanish, the total statistical energy E_μ becomes conserved again, that is,

$$E_\mu (t) = E_\mu (0) = \frac{1}{2} \sum_{1 \leq |\mathbf{k}| \leq \Lambda} \left(1 + \mu |\mathbf{k}|^{-2}\right) \left(|Q_{\mu,\mathbf{k}} - \bar{q}_{0,\mathbf{k}}|^2 + \overline{\left|q'_{0,\mathbf{k}}\right|^2}\right).$$

Thus it makes it possible for us to derive the saturation bound in this case among small scale vortical modes. Still we can suppose a series of *reference basic solutions*, Q_α with parameter $\alpha > -1$, compared with the *steady state solution*, $Q_\mu, \mu < -1$ so that we can define the stable steady solution Q_α and unstable steady solution Q_μ

$$Q_{\alpha,\mathbf{k}} = \frac{\alpha}{|\mathbf{k}|^2 + \alpha} \hat{h}_\mathbf{k}, \; Q_{\mu,\mathbf{k}} = \frac{\mu}{|\mathbf{k}|^2 + \mu} \hat{h}_\mathbf{k}, \quad q_\mathbf{k} = Q_{\mu,\mathbf{k}} + \omega_\mathbf{k}.$$

Notice that the deviation from statistical mean $\omega'_{\mathbf{k}}=\omega_{\mathbf{k}}-\bar{\omega}_{\mathbf{k}}$ does not change with the different choices of the steady state solution, and the statistical energy conservation relation is valid for any mean flow with parameter $\alpha>-1$. Thus the previous bound can be directly applied to the present flow with imposed artificial reference mean state Q_α, so that

$$\sum_{1\leq|\mathbf{k}|\leq\Lambda}\left(1+\alpha\left|\mathbf{k}\right|^{-2}\right)\left(\left|\tilde{Q}_{\alpha,\mu,\mathbf{k}}-\bar{\omega}_{\mathbf{k}}\left(t\right)\right|^{2}+\overline{\left|\omega'_{\mathbf{k}}\left(t\right)\right|^{2}}\right)\leq\sum_{1\leq|\mathbf{k}|\leq\Lambda}\left(1+\alpha\left|\mathbf{k}\right|^{-2}\right)\left(\left|\tilde{Q}_{\alpha,\mu,\mathbf{k}}-\bar{\omega}_{0,\mathbf{k}}\right|^{2}+\overline{\left|\omega'_{0,\mathbf{k}}\right|^{2}}\right).\tag{5.5}$$

Above $(\bar{\omega},\omega')$ are the fluctuation statistical mean and variance components in reference to the mean state Q_μ with $\mu<-1$ in the real steady state solution. The variance part $\overline{|\omega'_{\mathbf{k}}|^2}$ in the relation will not change with different choices of mean state, while we have the reference steady mean corresponding to the parameter α

$$\tilde{Q}_{\alpha,\mu,\mathbf{k}}=Q_{\alpha,\mathbf{k}}-Q_{\mu,\mathbf{k}}=\frac{\left(\alpha-\mu\right)\left|\mathbf{k}\right|^{2}}{\left(\alpha+\left|\mathbf{k}\right|^{2}\right)\left(\mu+\left|\mathbf{k}\right|^{2}\right)}\hat{h}_{\mathbf{k}}.$$

On the left hand side of the relation (5.5) the mean state of the fluctuation component $\bar{\omega}$ becomes related with parameters α,μ, thus it becomes difficult to estimate. On the other hand, the variances $\overline{\left|\omega'_{\mathbf{k}}\left(t\right)\right|^{2}}$ are always independent of the chosen reference steady state.

Saturation bound for total variance based on the flow vorticity. Again assume we assign initial statistical mean of the ensemble as the steady state mean Q_μ, and the initial uncertainty from ensemble variance C_0, such that

$$\bar{\omega}_{0,\mathbf{k}}=0,\quad\overline{\left|\omega'_{0,\mathbf{k}}\right|^{2}}=\sigma_{0,\mathbf{k}}^{2}.\tag{5.6}$$

The optimal bound from right hand side of (5.5) can be reached by minimizing among $\alpha>0$, using the explicit form of $\tilde{Q}_{\alpha,\mu,\mathbf{k}}$

$$C_{\mu}^{1}=\min_{\alpha>0}\sum_{1\leq|\mathbf{k}|\leq\Lambda}\left\{\frac{\left(\alpha-\mu\right)^{2}}{\left(\Lambda^{-1}\alpha+1\right)\left(\alpha+\left|\mathbf{k}\right|^{2}\right)}\left|\frac{\left|\mathbf{k}\right|\hat{h}_{\mathbf{k}}}{\mu+\left|\mathbf{k}\right|^{2}}\right|^{2}+\frac{1+\alpha\left|\mathbf{k}\right|^{-2}}{1+\alpha\Lambda^{-2}}\sigma_{0,\mathbf{k}}^{2}\right\}.$$

Above on the right hand side, the first part represents the error from initial statistical mean, and the second part is the initial random noise perturbation in the ensemble. Similarly for reference parameter regime $-1<\alpha<0$, the optimal bound from right hand side of (5.5) can be reached as

$$C_{\mu}^{2}=\min_{-1<\alpha<0}\sum_{1\leq|\mathbf{k}|\leq\Lambda}\left\{\frac{\left(\alpha-\mu\right)^{2}}{\left(\alpha+1\right)\left(\alpha+\left|\mathbf{k}\right|^{2}\right)}\left|\frac{\left|\mathbf{k}\right|\hat{h}_{\mathbf{k}}}{\mu+\left|\mathbf{k}\right|^{2}}\right|^{2}+\frac{1+\alpha\left|\mathbf{k}\right|^{-2}}{1+\alpha}\sigma_{0,\mathbf{k}}^{2}\right\}.$$

Combining the above two cases, the optimal saturation bound for the total variance of the flow fluctuation vorticity can be reached at

$$\sum_{1\leq|\mathbf{k}|\leq\Lambda}\overline{\left|\omega'_{\mathbf{k}}\left(t\right)\right|^{2}}\leq C_{\mu}\left(\Lambda,h,C_{0}\right)=\min\left\{C_{\mu}^{1},C_{\mu}^{2}\right\}.\tag{5.7}$$

The minimum value C_μ above depends on the truncation size Λ, the topography structure h, the initial ensemble variance C_0, as well as the steady state parameter μ. C_μ offers a rough upper bound according to any parameter value μ for the total variability in the ensemble of trajectories as they evolve in time.

Non-optimal bound for statistical mean state. The above bound estimation about the total fluctuation variance might be accurate if the deviation from the mean $\left|\tilde{Q}_{\alpha,\mu,\mathbf{k}}-\bar{\omega}_{\mathbf{k}}\right|$ is small (for example, there is only topographic stress in small amplitude, $h\sim0$). Still the error due to the neglected mean from the term $\tilde{Q}_{\alpha,\mu}$ needs to be attended, and it is difficult to estimate the error from the statistical mean fluctuation $\bar{\omega}$ from the previous inequalities. Especially when there are some values of $|\mathbf{k}|^2$ close to $-\mu$, the error from $\tilde{Q}_{\alpha,\mu}$ could be huge. There is possibility that large amount of energy cascades from the fluctuation variances back to the mean state due to the nonlinear interactions and drive the mean state $\bar{\omega}$ away to another distinct state as the system evolves.

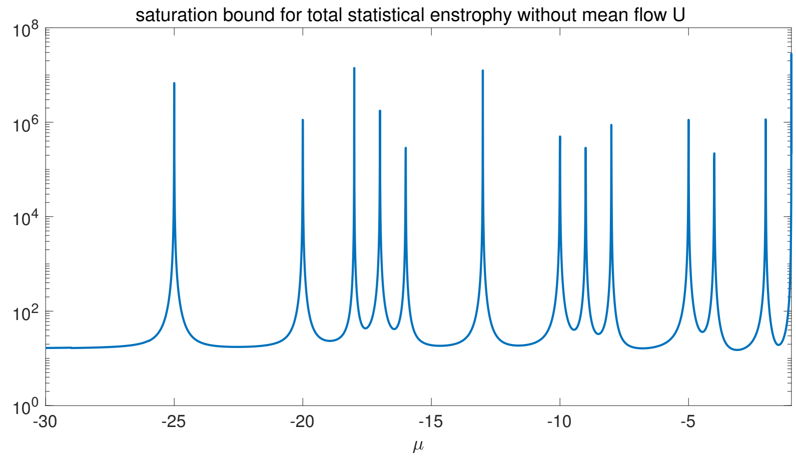


FIGURE 5.1. Saturation bound C_μ without mean flow $U=0$. Initial variance is only set to be non-zero among the ground modes $|\mathbf{k}|=1$ with variance $\sigma_{1,0}=1$.

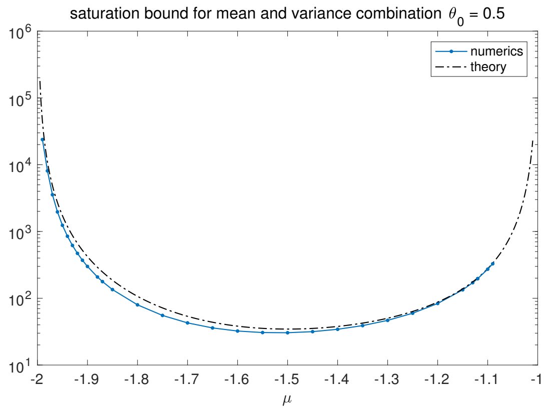
In a similar way, we get a general bound for a combination of the statistical mean and variance in all the time

$$\begin{aligned}\sum_{1\leq|\mathbf{k}|\leq\Lambda}\left(1+\alpha\left|\mathbf{k}\right|^{-2}\right)\left(\theta\left|\bar{\omega}_{\mathbf{k}}\right|^{2}+\overline{\left|\omega_{\mathbf{k}}'\left(t\right)\right|^{2}}\right) & \leq \sum_{1\leq|\mathbf{k}|\leq\Lambda}\left(1+\alpha\left|\mathbf{k}\right|^{-2}\right)\left(\left(1-\theta\right)^{-1}\left|\tilde{Q}_{\alpha,\mu,\mathbf{k}}\right|^{2}+C_{0,\mathbf{k}}\right)\\ \sum_{1\leq|\mathbf{k}|\leq\Lambda}\left(\theta\left|\bar{\omega}_{\mathbf{k}}\right|^{2}+\overline{\left|\omega_{\mathbf{k}}'\left(t\right)\right|^{2}}\right) & \leq \left(1-\theta\right)^{-1}C_{\mu},\qquad 0<\theta<1,\\ \frac{1}{2}\sum_{1\leq|\mathbf{k}|\leq\Lambda}\left|\bar{\omega}_{\mathbf{k}}\right|^{2} & \leq \frac{1}{2}\min_{\theta}\theta^{-1}\left(1-\theta\right)^{-1}C_{\mu}=2C_{\mu}.\end{aligned}\tag{5.8}$$

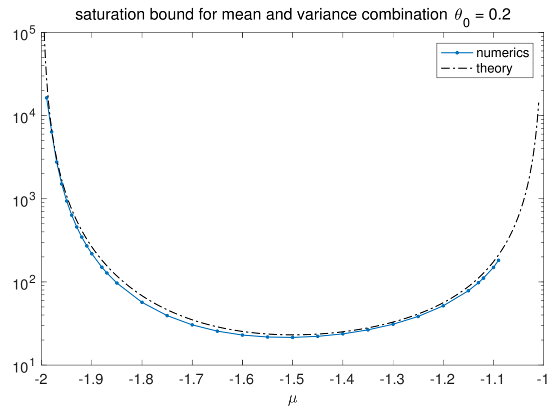
Above we offer three levels of estimations. The first row gives the most general inequality with changing coefficients in different wavenumbers. The second row separates the total statistical mean and total variance with a balance parameter θ. Note that larger value of θ (then more emphasis on the stability in statistical mean) leads larger value on the right hand side bound. This shows (5.8) is not so desirable a bound for estimating the variability in the statistical mean. In the third inequality, we separate the statistical mean state only. It shows the total statistical mean can not increase without bound with a largest value $2C_\mu$, while this bound is not optimal since C_μ could become huge. It is possible a tighter bound can be found for the statistical mean state. We summarize the above results as the following proposition.

Proposition 5. *(Statistical saturation bound for total fluctuation variance and statistical mean) For any general value of μ (and especially for unstable case $\mu<-1$), assume zero initial statistical mean fluctuation and a decaying spectrum in initial ensemble variance as (5.6). The total variance of the fluctuation vorticity, $\fint\overline{\omega'^2}$, in flow field without a mean flow $U\equiv0$ can be estimated with a saturation bound $C_\mu\left(\Lambda,h,C_0\right)$ from (5.7) for the entire time. Besides a non-optimal bound for the combination of statistical mean state and variance, $\theta\fint\bar{\omega}^2+\fint\overline{\omega'^2}$, with a parameter $\theta\in(0,1)$ can be reached from (5.8).*

5.2. **Numerical verification of the saturation bounds in small scales with $U=0$.** Here we also verify the saturation bounds in the statistical mean and variance in the barotropic equation without mean flow and beta-effect. First Figure 5.1 calculates the saturation upper bounds for the total statistical enstrophy. In Figure 5.2 numerical results from simulations are compared with the theoretical upper bounds.



(A) $\theta_0 = 0.5$



(B) $\theta_0 = 0.2$

FIGURE 5.2. Saturation bound with forcing in unstable regime $-2 < \mu < -1$ for statistical mean and variance combined with parameter θ_0 without mean flow. It compares the combined statistical enstrophy with $\theta \fint \bar{\omega}^2 + \fint \overline{\omega'^2}$ with different partition value $\theta_0 = 0.5, 0.2$.

Appendix A. Linear Stability Analysis of the Barotropic System with Topographic Stress

Here in the Appendix, we would like to illustrate the instability in the topographic barotropic system with linear stability analysis. It can be seen that despite the statistical stability bounds achieved in the main text, the system still contains strong linear internal growth rate among a wide parameter regimes. In the barotropic flow with topography, the instability is majorly form the interaction and energy transfer between the large-scale mean flow U and the small-scale vortical modes ω. Thus it is useful to consider the *layered topographic modes* (refs.) only along x-direction for simplicity

$$h=\sum_{k=-N}^{N}\hat{h}_{k}e^{ikx},\qquad\omega=\sum_{k=-N}^{N}\hat{\omega}_{k}e^{ikx}.$$

The above layered modes form a closed system. The quadratic nonlinear interactions between small-scale layered modes, $\nabla^{\perp}\psi\cdot\nabla\omega$ and $\nabla^{\perp}\Psi\cdot\nabla\left(\omega-\mu\psi\right)$, are eliminated since all the wavenumbers are along the same direction. This enable us to focus on the interactions between the large mean flow and small scale modes due to topographic stress and beta-effect. Therefore the original fluctuation equation (1.5) can be effectively simplified as in the spectral domain

$$\begin{aligned}\frac{d\hat{\omega}_{k}}{dt}-i\beta\frac{\mu+k^{2}}{\mu k}\hat{\omega}_{k}+i\frac{\mu k}{\mu+k^{2}}\hat{h}_{k}U\left(t\right)+ik\hat{\omega}_{k}U\left(t\right) &= 0,\\ \frac{dU}{dt}+\sum\frac{\hat{h}_{k}}{ik}\hat{\omega}_{k}^{*} &= 0.\end{aligned}\tag{A.1}$$

Notice that the state variables (U,ω) in (A.1) are already the fluctuation components about the steady state solution (V_{μ},Q_{μ}) defined in (1.3). Next we consider the linear growth rate in the beginning transient state in the statistics of the state variables due to the instability from topography using the layered model (A.1).

A.1. **Linear statistical stability in the layered model.** Here we investigate the linear statistical stability for the statistical formulation of the layered system. In the layered formulation (A.1), no nonlinear interactions between small scale modes are included. Thus the only source of instability (and positive linear growth rate) is induced by the interaction between large and small scales due to topographic stress h. For a better formulation of the linearized system, we decompose the complex spectral modes into real and imaginary part

$$\hat{\omega}_{k}=a_{k}+ib_{k},\quad\hat{h}_{k}=h_{k}^{r}+ih_{k}^{i},\qquad k=1,\cdots,N.$$

Thus the state variables of interest form the vector $\mathbf{u}=\left(a_{1},b_{1},\cdots,a_{N},b_{N},U\right)^{T}$ of length $2N+1$. From the layered equation (A.1) the dynamics of each wavenumber k can be written as

$$\begin{aligned}\frac{da_{k}}{dt} &= -\beta\frac{\mu+k^{2}}{\mu k}b_{k}+\frac{\mu k}{\mu+k^{2}}h_{k}^{i}U+kb_{k}U,\\ \frac{db_{k}}{dt} &= \beta\frac{\mu+k^{2}}{\mu k}a_{k}-\frac{\mu k}{\mu+k^{2}}h_{k}^{r}U-ka_{k}U,\\ \frac{dU}{dt} &= 2\sum_{k=1}^{N}k^{-1}\left(h_{k}^{r}b_{k}-h_{k}^{i}a_{k}\right).\end{aligned}\tag{A.2}$$

In this way, the small scale spectral modes (a_k,b_k) are decoupled with each other in (A.2), while the mean flow U combines all the feedbacks from small scale modes through the topographic stress.

We consider the statistical dynamics in the system (A.2). Especially here we want to check how the instability are induced due to the topographic stress. Therefore it is useful to consider the dynamical equation of the covariance matrix $R=\left\langle \mathbf{u}'\mathbf{u}'^{T}\right\rangle$ for fluctuations $\mathbf{u}'$ away from the statistical mean state $\bar{\mathbf{u}}_{k}=\left(\bar{a}_{k},\bar{b}_{k},\bar{U}\right)$. The linear growth rate of the covariance R illustrates how the uncertainty from the initial data grows due to the instability in the system; and the statistical mean state is the fixed point that steady state solution can be reached.

Notice that the only nonlinearity of the above system (A.2) comes from the mean flow and vortical modes interactions, $(a_k U, b_k U)$. The linearized part of the covariance dynamics can be written abstractly as

$$\frac{dR}{dt} = L_{\bar{\mathbf{u}}} R + R L_{\bar{\mathbf{u}}}^T + h.o.t., \quad R = R^T = \begin{bmatrix} \ddots & & & & \vdots \\ & \overline{a_k'^2} & \overline{a_k' b_k'} & \cdots & \overline{a_k' U'} \\ & \overline{b_k' a_k'} & \overline{b_k'^2} & & \overline{b_k' U'} \\ & \vdots & & \ddots & \vdots \\ \cdots & \overline{U' a_k'} & \overline{U' b_k'} & \cdots & \overline{U'^2} \end{bmatrix}_{(2N+1)\times(2N+1)}.$$

The linearized coefficient $L_{\bar{\mathbf{u}}}$ related with the statistical mean state $\bar{\mathbf{u}}$ can be calculated with a block-diagonal structure in the small scale modes

$$L_{\bar{\mathbf{u}}} = \begin{bmatrix} \ddots & & 0 & & \vdots \\ & 0 & -\beta\frac{\mu+k^2}{\mu k} + k\bar{U} & \cdots & \frac{\mu k}{\mu+k^2} h_k^i + k\bar{b}_k \\ & \beta\frac{\mu+k^2}{\mu k} - k\bar{U} & 0 & & -\frac{\mu k}{\mu+k^2} h_k^r - k\bar{a}_k \\ 0 & \vdots & & \ddots & \vdots \\ \cdots & -2k^{-1} h_k^i & 2k^{-1} h_k^r & \cdots & 0 \end{bmatrix}_{(2N+1)\times(2N+1)}. \tag{A.3}$$

Therefore the linear instability can be characterized by the positive eigenvalues of the linearized coefficient matrix $L_{\bar{\mathbf{u}}}$. The positive eigenvalues illustrate the linear growth rate of the uncertainty (in variance) from the initial ensemble of particles around the assumed steady state statistical mean $\bar{\mathbf{u}} = \left(\bar{a}_k, \bar{b}_k, \bar{U}\right)$. Larger growth rate implies that the variances in the modes may keep growing to diverge from the original statistical mean state $\bar{\mathbf{u}}$. Especially if we set zero mean state $\bar{a}_k = \bar{b}_k = \bar{U} = 0$, the eigenvalues of the above matrix $L_{\bar{\mathbf{u}}}$ give the *local Lyapunov exponents* of the original linearized system (A.2) that characterize the separation rate of two trajectories with close initial states.

A.1.1. *Relations in steady statistical mean state.* Now we consider the possible statistical mean $\left(\bar{a}_k, \bar{b}_k, \bar{U}\right)$ in steady state. The statistical mean dynamics can be derived by taking ensemble average about the original equations (A.2) so that

$$\begin{aligned} \frac{d\bar{a}_k}{dt} &= -\beta\frac{\mu+k^2}{\mu k}\bar{b}_k + \frac{\mu k}{\mu+k^2} h_k^i \bar{U} + k\bar{b}_k\bar{U} + k\overline{b_k' U'}, \\ \frac{d\bar{b}_k}{dt} &= \beta\frac{\mu+k^2}{\mu k}\bar{a}_k - \frac{\mu k}{\mu+k^2} h_k^r \bar{U} - k\bar{a}_k\bar{U} - k\overline{a_k' U'}, \\ \frac{d\bar{U}}{dt} &= 2\sum_{k=1}^{N} k^{-1}\left(h_k^r \bar{b}_k - h_k^i \bar{a}_k\right). \end{aligned}$$

In statistical steady state, the time derivatives on the left hand side vanish. Especially, we consider the statistical steady mean under the *homogeneous assumptions* that there is no cross-covariance in steady state and the mean dynamics vanish in each mode

$$h_k^i \bar{a}_k = h_k^r \bar{b}_k, \quad \overline{a_k' U'} = \overline{b_k' U'} = 0. \tag{A.4}$$

The above relations assume a homogeneous steady state without cross-covariances between modes in different scales. With the assumptions, then the statistical mean of each small-scale mode can be determined by the large scale flow mean $\bar{U}$, that is

$$\begin{aligned} \bar{a}_k &= \frac{\frac{k}{\mu+k^2}\mu\bar{U}}{\frac{\beta}{\mu}\frac{\mu+k^2}{k} - k\bar{U}} h_k^r, \\ \bar{b}_k &= \frac{\frac{k}{\mu+k^2}\mu\bar{U}}{\frac{\beta}{\mu}\frac{\mu+k^2}{k} - k\bar{U}} h_k^i. \end{aligned} \tag{A.5}$$

In the special case with $\frac{\beta}{\mu}\frac{\mu+k^2}{k} = k\bar{U}$, we have the steady mean state $\bar{a}_k = \bar{b}_k = 0$.

A.1.2. *Linear growth rate with interaction with single wavenumber.* We begin with the simple setup that there is one single small scale mode (a_k, b_k) interacting with the mean flow. Therefore the linearized coefficient matrix $L_{\bar{\mathbf{u}},k}$ is just a 3×3 matrix

$$L_{\bar{\mathbf{u}},k}=\begin{bmatrix} 0 & -\beta\left(\frac{k}{\mu}+\frac{1}{k}\right)+k\bar{U} & \frac{\mu k}{\mu+k^2}h_k^i+k\bar{b}_k\\ \beta\left(\frac{k}{\mu}+\frac{1}{k}\right)-k\bar{U} & 0 & -\frac{\mu k}{\mu+k^2}h_k^r-k\bar{b}_k\\ -2\frac{h_k^i}{k} & -2\frac{h_k^r}{k} & 0\end{bmatrix}.$$

Furthermore, we consider a special form of topography with only non-zero imaginary part

$$h_k^r\equiv 0,\ h_k^i=H\ \Rightarrow\ \bar{a}_k\equiv 0,\ \bar{b}_k=\frac{\frac{k}{\mu+k^2}\mu\bar{U}}{\frac{\beta}{\mu}\frac{\mu+k^2}{k}-k\bar{U}}H. \tag{A.6}$$

The coefficient matrix $L_{\bar{\mathbf{u}},k}$ first has one zero eigenvalue $\lambda=0$, and the other two eigenvalues can be solved by

$$\begin{aligned}\lambda^2&=-\left(\frac{\beta}{\mu}\frac{k^2+\mu}{k}-k\bar{U}\right)^2-2H\left(\frac{\mu}{k^2+\mu}H+\bar{b}_k\right)\\&=2H^2\left[k^2\bar{U}/\beta-\left(1+\mu^{-1}k^2\right)\right]^{-1}-\left(\frac{\beta}{\mu}\frac{k^2+\mu}{k}-k\bar{U}\right)^2.\end{aligned} \tag{A.7}$$

Linear instability takes place when the right hand side is positive. We can first have the following immediate result for the mean flow interaction with a single wavenumber vortical mode:

- One necessary condition for the existence of linear instability occurs when
$$k^2\bar{U}/\beta-\left(1+\mu^{-1}k^2\right)>0\ \Leftrightarrow\ \bar{U}+V_\mu>\beta k^{-2},$$
in the northern hemisphere $\beta>0$. This shows the lower bound for the mean flow $\bar{V}$ to induce instability.

As one specific example, we consider the case with zero steady mean state, $\bar{a}_k=\bar{b}_k=\bar{U}=0$. The eigenvalues (Lyapunov exponents) in (A.7) can be simplified as

$$\lambda^2=-\frac{2H^2}{1+\mu^{-1}k^2}-\beta^2\left(\frac{k}{\mu}+\frac{1}{k}\right)^2.$$

Explicitly we can calculate the regime of linear instability among the values of

$$-k^2<\mu<-\left[\left(\frac{2H^2}{\beta^2}\right)^{\frac{1}{3}}k^{-4/3}+k^{-2}\right]^{-1}\equiv\mu_c. \tag{A.8}$$

The linear growth rate $\lambda\to\infty$ as $\mu\to -k^2$; and the growth rate $\lambda\to 0$ as $\mu\to\mu_c$. Obviously the beta-effect works as a stabilizing effect so that larger value of β makes smaller regime of instability. On the other hand, the larger values of the topographic strength H will induce stronger instability into the system when the system becomes unstable, that is, $-k^2<\mu<0$.

A.2. **Numerical illustration of the linear growth rate.** In this final part, we illustrate the linear instability analyzed above using simple numerical results. We consider the linear growth rate from the full linearized coefficient matrix $L_{\bar{\mathbf{u}}}$ in (A.3) where mean flow interaction with multiple small scale spectral modes can be included.

A.2.1. *Local Lyapunov exponents.* Figure A.1 first shows the linear growth rates with single mode interaction with the mean state in each wavenumber $k=1,2,3,4,5$. The topographic is taken as $\hat{h}_k=Hk^{-2}e^{-i\theta_k}$ in each spectral mode with uniform phase shift $\theta_k=\frac{\pi}{4}$ in the same zonal structure as in the main text. We choose the other parameter values $\beta=1$ and $\bar{U}=0$ (thus the linear growth rate is the local Lyapunov exponents). Consistent with the analysis result in (A.8), large linear growth will be induced when the parameter μ reaches the resonance regime $-k^2$, and instability vanishes after the critical value μ_c. Also notice that there exists overlap between the unstable regimes of different wavenumbers.

We can also compute the eigenvalues directly from the full linearized coefficient matrix (A.3). In this way, the feedbacks to the mean flow from each small scale modes can be combined together. The dotted-dashed line in Figure A.1 compares the maximum eigenvalue from the linearized coefficient matrix as the parameter μ changes. The growth rate becomes large near the resonance regime as $\mu=-|k|^2$. And the growth rate gets reduced among

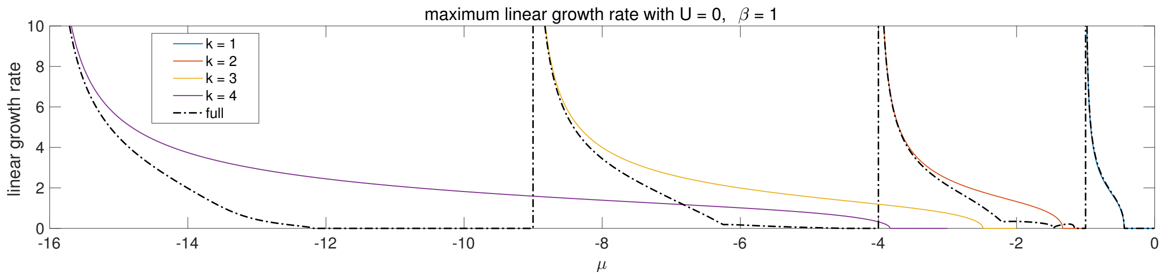


FIGURE A.1. Linear growth rate from the largest positive eigenvalue of the linearized coefficient matrix in the covariance equation with $\beta = 1, \bar{U} = 0$. The four solid lines are the growth rate from single mode interaction with the mean flow as in (A.7). The dotted-dashed line is from the combined interaction of the full matrix (A.3) of all first 5 modes.

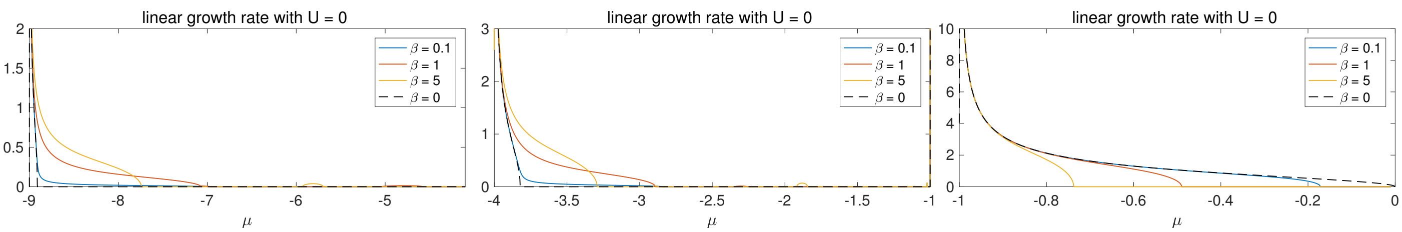


FIGURE A.2. Linear growth rates with multiple modes interaction including $k = 1, 2, 3, 4, 5$. The results with different values of $\beta = 0, 0.1, 1, 5$ are compared.

the overlapped regimes of different single mode instability. Especially noted that the unstable growth rate is one-sided. Positive growth rate only appear when μ approaches $-k^2$ from the right side, while no instability is generated from the left side. Similar phenomena can be observed from the model simulations for statistical instability in the main text.

We also compare the effect of beta effect in the linear stability in Figure A.2. Here we test four different values $\beta = 0, 0.1, 1, 5$. Consistent with the results before, the beta-effect can serve as a stabilizing factor. As the value of β increases, the size of the unstable regime with a positive growth rate reduces, while a larger growth rate will appear with larger β when the system becomes unstable. Especially in the regime $-1 < \mu < 0$, the entire regime is unstable when $\beta = 0$, and the area of the unstable regime gets reduced as the value of β increases.

A.2.2. *Linear growth rate with non-zero mean flow.* Further here we show the case with non-zero mean state $\bar{U} \neq 0$. In Figure A.3, as a further comparison, we show the linear growth rate of multiple modes interaction with dependence of steady state mean flow value $\bar{U}$. Compared with the previous case with zero mean state $\bar{U} = 0$, positive linear growth rates are also induce in the regime $\mu > 0$. The various regimes of positive growth rates show the large instability existing with the topographic barotropic flow in the general sense.

Further in Figure A.4, we show the regimes of unstable growth rates with different steady mean values $\bar{U}$ and parameter μ. As the wavenumber k increases, the unstable regime becomes narrower. As the steady mean state $\left|\bar{U}\right|$ increases, the instability reduces and finally vanishes. And especially in regime $\bar{U} > 0$, there exist two separated unstable regimes for $\mu > 0$ and $\mu < 0$ separately. Comparing with the single mode $k = 1$ case, with multiple small-scale modes interaction, the unstable regime with positive linear growth rate gets narrowed down. Still the two branches of linearly unstable regimes exist.

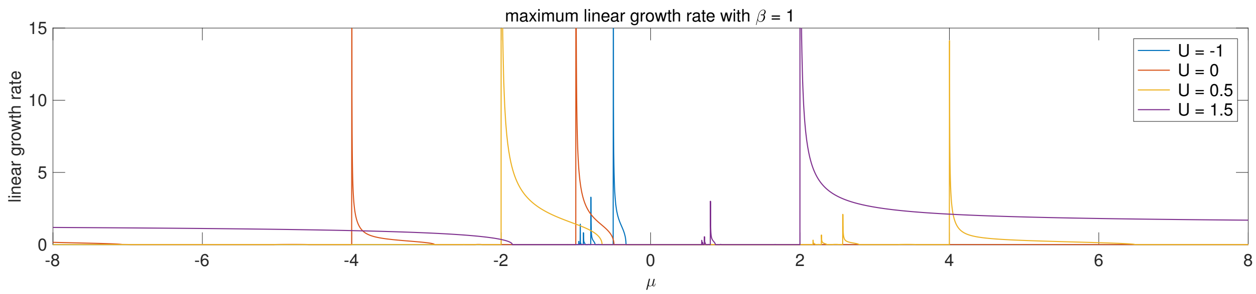
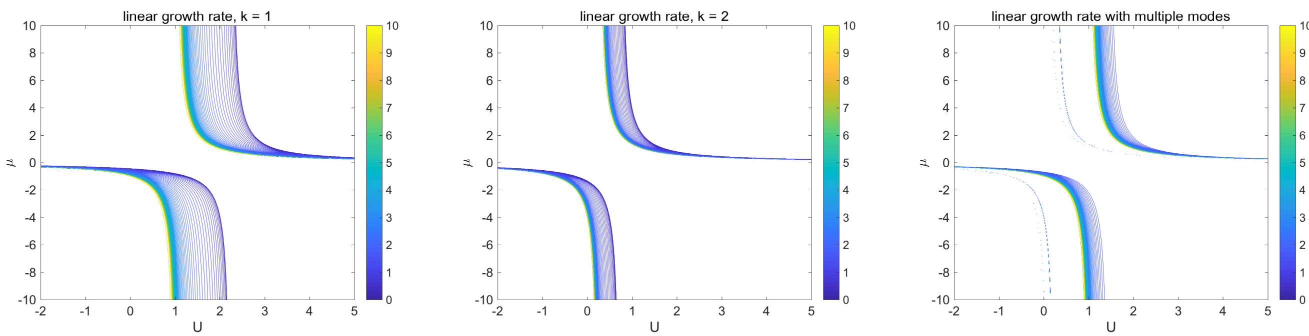


FIGURE A.3. Linear growth rate from the largest positive eigenvalue of the linearized coefficient matrix in the covariance equation with changing $\bar{U} = -1, 0, 0.5, 1.5$.



(A) $k = 1$

(B) $k = 2$

(C) multi-modes

FIGURE A.4. Linear growth rates with changing steady mean state $\bar{U}$ and parameter μ with $\beta = 1$. The first three wavenumbers $k = 1, 2, 3$ are shown.