Strategic Complexity in Sequential Games

Ricky Zhao

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Overview

1. Motivating Example: What is strategic complexity, and what can we learn from applying it to this game?
2. Discussion: How does this concept generalize to other games?

Part I: Motivating Example

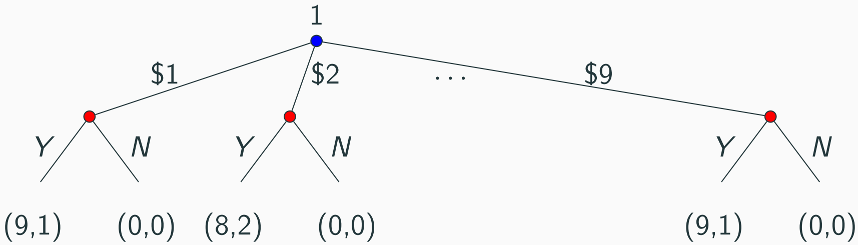
The Ultimatum Game

Two players are to split a fixed sum of money (\$10 in this example) in the following way:

- First, Player 1 proposes a division of the money (we'll denote this by the amount offered to Player 2).
- Player 2 can either accept or reject this proposal.
- If accepted, the money is split accordingly; if rejected, both players receive nothing.

Ultimatum Game

The Ultimatum Game (UG) has the following *extensive form*:



- In this structure, Player 2 faces many potential decisions
- What does a *strategy* for Player 2 look like?
- Standard definition: a complete action profile that assigns an action for every possible decision. Ex: "*RRARAAARA*"

Classical Analysis of UG

The previous definition of strategy is used in the *normal form* of the game:

	A^9	RA^8	ARA^7	...
1	(9,1)	(0,0)	(9,1)	
2	(8,2)	(8,2)	(0,0)	
...				

- This game has many Nash equilibria (a set for *each* P1 offer)
- There is a unique subgame-perfect equilibrium (P1 offers \$1, P2 accepts all offers)
- SPE drastically differs from behavior observed in experiments (P1 offers $40-60\%$ of the total, and P2 rejects low offers)
- **How can we explain these acts of altruism (for P1) and spite (for P2)?**

Strategic Complexity

We start with an observation about the Ultimatum game:

- Player 2's (normal-form) strategy is very complicated.
- This can mean two (slightly) different things:
  (a) Player 2's strategy set is very big (2^n strategies for a game with n offers)
  (b) Some of Player 2's strategies are more costly to implement than others (e.g. *ARRAAR* is more complex than *AAAAAA* or *RRAAAA*)

Cutoff Strategies

A natural simplification of normal-form strategies in the Ultimatum game is to only consider *cutoff strategies* — strategies that accept an offer iff it is greater than some cutoff value.

- This massively reduces the size of the strategy set (from 2^n to $n+1$)
- Cutoff strategies are also easier to implement cognitively
- Unfortunately, equilibrium analysis of this simplified game isn't all that different from the original.

Evolutionary Game Theory

We can think of behavior in games as a dynamic (learning) process, instead of a static (equilibrium) outcome. The process is generally defined (for each player) as follows:

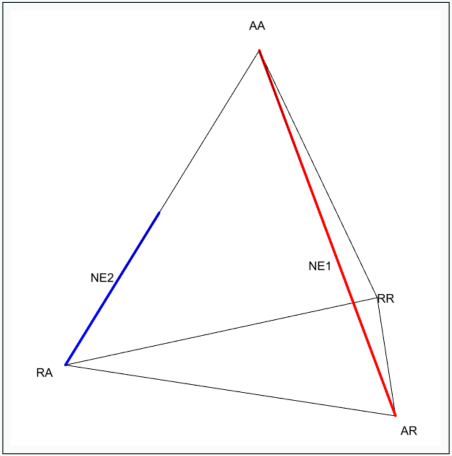
- Start with an arbitrary mixed strategy (no rationality involved)
- At each time step:
    - Calculate the expected payoff (or *fitness)* of each pure strategy e as well as your current mixed strategy s
    - If e does better (worse) than s, increase (decrease) the proportion of e in s for the next time step

One commonly studied evolutionary dynamic is the *replicator dynamic*:

$$\dot{p}_e = p(f_e - \phi)$$

Basins of Attraction (Not all NE are created equal!)

The replicator dynamic tells us not only the final outcomes (equilibria) of a game, but also the relative "likelihood" of each outcome given some initial distribution



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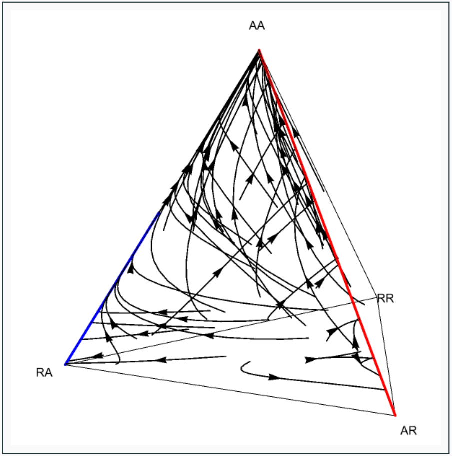

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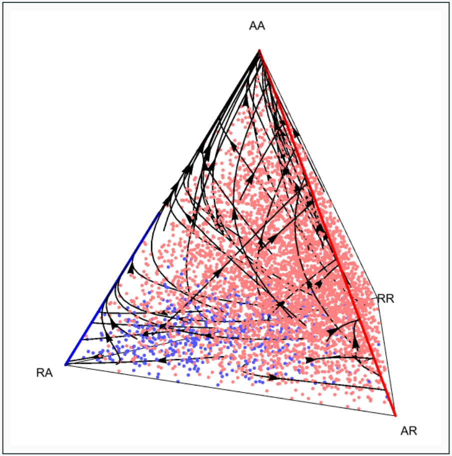
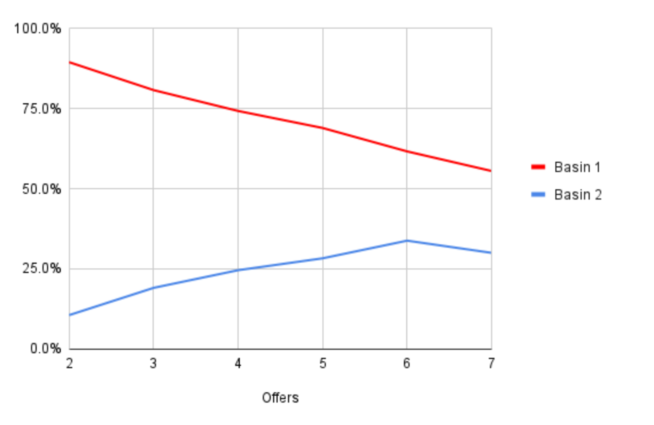


Replicator Dynamic on the Normal-form UG

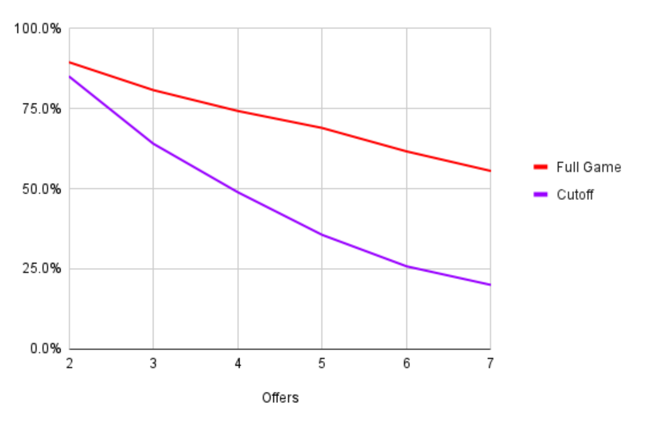
- In the normal form Ultimatum game, the NE set for the lowest offer has the largest basin.
- We can think of this as another form of equilibrium selection/refinement (similar to SPE, but more robust)



Comparison with Cutoff-Simplified UG

When we simplify the full game using cutoff strategies, the basin of the lowest offer NE decreases significantly!

- This is good news: more realistic representation of the game leads to results that better align with empirical observations.



Part I Summary

1. In the Ultimatum Game, cutoff strategies are a simplification of normal-form strategies that better represent how people think about the game
2. Combining this with evolutionary (learning) dynamics yields a novel explanation for altruistic behavior in this game
3. For larger games the effects of this simplification are even stronger

Part II: Generalizing to Other Games

Choosing the right notion of complexity

This is a tricky issue — what we consider complex depends on many aspects of the game/decision:

- Game tree structure (P2 faces many parallel decisions — only one is realized)
- Set of choices at the decisions (P2 faces the same two actions: Accept and Reject)
- Payoffs (P2 can order the decisions based on P1 offer or Accept payoff)
- Etc.

Side note: Symmetric Cutoff Strategies

Define a **symmetric cutoff** strategy as follows: First choose a cutoff value x, then *either* accept offers above x and reject offers below x (regular cutoff strategy), *or* reject offers above x and accept offers below x (inverted cutoff strategy).

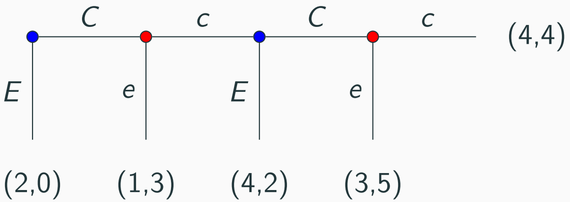
- Inverted cutoff strategies are just as "simple" as regular cutoff strategies
- Using symmetric cutoff strategies to simplify UG still results in a smaller basin for the lowest offer (i.e., more altruism)
- However, this difference is marginal — only 1-2% instead of 20-30% for regular cutoff strategies

A Very Different Example: The Centipede Game

Two players take turns passing a pile of coins (initially 2) to each other. At each turn, a player can either

1. (E)nd the game, receiving $x/2+1$ coins while the other player receives $x/2-1$; or
2. (C)ontinue the game, in which case two more coins are added to the pile.

(Assume the game ends after $2n$ turns) Extensive form:



Centipede Game (continued)

- Equilibrium analysis: Player 1 should end the game immediately (this the unique NE set *and* SPE!)
- Similar story to UG: Game theory tells us to be selfish, but we often don't actually do this

Normal-form* strategies in this game: $\{$E, CE, CCE, CCCE,$\ldots\}$
How can we simplify this strategy set? A few possibilities:

- Choose strategies by length (shortness)
- Choose strategies close to the start or end (salience)
- Choose a representative strategy out of every few adjacent ones

Salient Strategies

If we eliminate the middle strategies (perhaps because it's easier to lose track of where we are in the game)...

	e	ce	c^9e	c^{10}
E	(2,0)	(2,0)	(2,0)	(2,0)
CE	(1,3)	(4,2)	(4,2)	(4,2)
C^9E	(1,3)	(3,5)	(20,18)	(20,18)
C^{10}	(1,3)	(3,5)	(19,21)	(20,20)

- A second (Pareto-superior) NE emerges
- Explanation: Backward induction is a step-by-step reasoning process; grouping decisions into chunks invalidates this process
- This outcome is also similar to some observed behavior in experiments

Final Thoughts

The ideal long-term goal of this project is to capture some formal notion of human behavior in much more complex games (chess, Go, poker).

- There already exist informal notions 'human play' vs 'AI play' in these games
- Human play involves following general (intuitive) heuristics that guide our moves
    - "Don't let your pieces get captured for free" (Chess)
    - "Play moves that surround territory" (Go)
    - "Play aggressively" (Poker)
- AI play will often violate these general rules (and thus seem more random)

Final Thoughts (continued)

There are many similarities between these informal heuristic-based strategies and the “simple” strategies used in the previous examples:

- They both significantly reduce the total number of strategies from the normal form game
- They eliminate many strategies that seem random or cognitively costly
- Simplifying games using these strategies does not seem to yield worse results (and may sometimes yield *better* ones)

Immediate next steps: Get a better understanding of how *people* actually simplify various kinds of decisions — this is crucial for pinpointing the right kinds of theoretical models.