

Numerical Algorithm Based on Transmutation for Solving Inverse Wave Equation

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Abstract—This paper develops algorithms for solving an undetermined coefficient problem for a wave equation. The algorithms are based on an integral representation for the solution to the wave equation obtained by using transmutation. The convergence of the algorithm is studied and numerical experiments are performed. © 2004 Elsevier Ltd. All rights reserved.

1. INTRODUCTION

In this paper, we consider the one-dimensional inverse problem for wave equation: given $\{b(x), c(x), g(t), u_0(t), f(x, t)\}$ to find $\{q(x), u(x, t)\}$, such that

$$u_{tt} = u_{xx} - q(x)u + f(x, t), \quad x, t > 0, \quad x + t < l, \quad (1.1)$$

$$u(x, 0) = b(x), \quad 0 < x < l, \quad (1.2)$$

$$u_t(x, 0) = c(x), \quad 0 < x < l, \quad (1.3)$$

$$u_x(0, t) = g(t), \quad 0 < t < l, \quad (1.4)$$

$$u(0, t) = u_0(t), \quad 0 < t < l. \quad (1.5)$$

Many research papers in the 1980s discussed this problem for various special cases of (b, c, g, u_0, f) , such as $c(x) = b(x) = f(x, t) = 0$, $g(t) = f(x, t) = 0$, or $g(t) = \delta(t)$, etc., [1–8]. In the later 1990s, we still see the related research on various boundary conditions for the wave equation

$$\eta(x)u_{tt} - (\eta(x)u_x)_x = 0,$$

which can be transformed to (1.1), [9,10].

In this paper, we want to study this problem in a uniform manner by developing a “comprehensive” transmutation representation of $u(0, t)$. As we will see, many existing results can easily be derived from this representation. Also, we can study the unknown source problem.

For the sake of simplicity, we assume $q \in C$, $u \in C^2$ when we derive the formulas. Similarly, assume $b \in C^2$, $c \in C^1$, $g \in C^1$, $f \in C$. However, as we will see, in many cases, the assumption can be weakened.

2. TRANSMUTATION METHOD

There are various methods for investigating undetermined coefficient problems [5,11]. All of them are based on a complete understanding of the associated forward problems. In fact, the key to the success of solving an inverse problem is how to develop a representation of the solution to the direct problem that relates in an explicit manner to the boundary data and the unknown coefficient.

Among various constructive methods for solving forward problems, transmutation is a method which provides a solution in especially good form for the investigation of the corresponding inverse problem.

Transmutation is a method for constructing a solution of a differential equation in terms of a solution to a related equation, similar to the equation to be considered. Usually this is done by means of an integral operator [12,13]. While transmuting one equation into another equation, solution of the first equation transmute into the solution of second equation, moreover, spectral properties of the first equation go over to those of the second equation. The most well-known transmutation is a Gelfand-Levitan type transmutation [14], which is responsible for the solutions of many one-dimensional inverse problems associated with a single scalar equation [15–18]. In [19], we used transmutation to solve an inverse problem for a system of equations. The transmutation method has two significant virtues. First, the proof of uniqueness based on transmutation forms an algorithm for computation. Second, transmutation frequently gives a tip to how an inverse problem is ill-posed.

In [14], the original Gelfand-Levitan transmutation is used to represent the solution of

$$y''(x) - (q(x) - c)y(x) = 0, \quad x > 0, \quad (2.1)$$

$$y(0) = 1, \quad y'(0) = 0, \quad (2.2)$$

by transmuting $\cos \sqrt{c}x$, i.e., the solution of the equation with $q = 0$. A representation of this transmutation is given by

$$y(x) = \cos \sqrt{c}x + \int_0^x K(x, s) \cos \sqrt{c}s \, ds. \quad (2.3)$$

The inverse transmutation is

$$\cos \sqrt{c}x = y(x) + \int_0^x R(x, s)y(s) \, ds. \quad (2.4)$$

Here, the kernel K satisfies

$$K_{xx}(x, s) - K_{ss}(x, s) - q(x)K(x, s) = 0, \quad 0 < s < x, \quad (2.5)$$

$$K_s(x, 0) = 0, \quad x > 0, \quad (2.6)$$

$$K(x, x) = \frac{1}{2} \int_0^x q(s) \, ds, \quad (2.7)$$

whereas the kernel of inverse transmutation, R , satisfies

$$R_{xx}(x, s) - R_{ss}(x, s) + q(s)R(x, s) = 0, \quad 0 < s < x, \quad (2.8)$$

$$R_s(x, 0) = 0, \quad x > 0, \quad (2.9)$$

$$R(x, x) = -\frac{1}{2} \int_0^x q(s) \, ds. \quad (2.10)$$

As we will see, the critical feature in determining these kernels is the Goursat condition (2.7) and (2.10), which usually leads to a system of Volterra integral equations.

3. REPRESENTATION OF BOUNDARY DATA

By using transmutation, we are able to develop the following representation formula which can be used to formulate the inverse problem associated with given data

$$\begin{aligned} u_0(t) = & b(t) + \int_0^t R(t, \tau) b(\tau) d\tau + \int_0^t \left[c(\xi) + \int_0^\xi R(\xi, \tau) c(\tau) d\tau \right] d\xi \\ & + \int_0^t \int_0^{t-\eta} \left[f(\xi, \eta) + \int_0^\xi R(\xi, \tau) f(\tau, \eta) d\tau \right] d\xi d\eta \\ & - \int_0^t g(\eta) d\eta - \int_0^t R(\xi, 0) \int_0^{t-\xi} g(\eta) d\eta d\xi. \end{aligned} \quad (3.1)$$

From this expression, we can clearly see how each piece of boundary-initial data contributes to $u(0, t)$. It also separates the boundary-initial data from the coefficient $q(x)$ because the kernel $R(x, s)$ depends on $q(x)$ only. This equation together with (2.8)–(2.10) constitute an inverse problem for (q, R) , which is equivalent to the original problem, but it is easier than the original one because of the Goursat condition. In fact, after we convert (2.8)–(2.10) into two integral equations using the d'Alembert formula, the problem becomes a system of Volterra integral equations, hence, the uniqueness of solution can be obtained very easily and computations can be performed efficiently as it is a causal problem.

To derive (3.1), let $\epsilon > 0$ and $L = l + \epsilon$, and extend $u(x, t)$ sufficiently smoothly to the domain $D = x + t \geq l, x \leq L$ so that

$$u(L, t) = 0. \quad (3.2)$$

In a like manner, extend $b(x)$ to $(l, L]$ by $b(x) = u(x, 0)$, $c(x) = u_t(x, 0)$, and $f(x, t)$ to D so that the wave equation still holds, i.e.,

$$f(x, t) = u_{tt}(x, t) - u_{xx}(x, t) + q(x)u(x, t).$$

Suppose $\{k_n, \phi_n\}$ is an eigenvalue/eigenfunction pair for the Sturm-Liouville eigenvalue problem

$$-\phi_n'' + q(x)\phi_n = k_n^2 \phi_n, \quad (3.3)$$

$$\phi_n'(0) = \phi_n(L) = 0, \quad (3.4)$$

$$\phi_n(0) = 1, \quad (\text{normalization}). \quad (3.5)$$

Denote by $v_n(t)$, the Fourier coefficients of $v(\cdot, t)$, namely,

$$v_n(t) := \int_0^L v(x, t) \phi_n(x) dx.$$

In particular, when v is u ,

$$u_n(t) := \int_0^L u(x, t) \phi_n(x) dx. \quad (3.6)$$

Taking derivative $\frac{d^2}{dt^2}$ on both sides of (3.6), we get

$$u_n''(t) = \int_0^L u_{tt}(x, t) \phi_n(x) dx.$$

By using equation (1.1) to replace u_{tt} , integrating by parts, and making use of boundary conditions (1.4), (3.2), (3.4), and (3.5), we have

$$\begin{aligned} u_n''(t) &= \int_0^L (u_{xx}(x, t) - q(x)u + f(x, t)) \phi_n(x) dx \\ &= [u_x(x, t) \phi_n(x) - u(x, t) \phi_n'(x)] \Big|_{x=0}^{x=L} \\ &\quad + \int_0^L (u(x, t) (\phi_n''(x) - q(x)\phi_n(x)) + f(x, t) \phi_n(x)) dx \\ &= -k_n^2 u_n - g(t) + f_n(t). \end{aligned} \quad (3.7)$$

The initial conditions (1.2) and (1.3) imply

$$u_n(0) = b_n, \quad (3.8)$$

$$u'_n(0) = c_n. \quad (3.9)$$

By solving (3.7)–(3.9) for $u_n(t)$, we obtain

$$u_n(t) = b_n \cos k_n t + c_n \int_0^t \cos k_n \xi \, d\xi + \int_0^t f_n(\eta) \int_0^{t-\eta} \cos k_n \xi \, d\xi \, d\eta - \int_0^t g(\eta) \int_0^{t-\eta} \cos k_n \xi \, d\xi \, d\eta.$$

By using the Gelfand-Levitan transmutation (2.4), we obtain

$$\cos k_n x = \phi_n(x) + \int_0^x R(x, \tau) \phi_n(\tau) \, d\tau, \quad 0 \leq x \leq L. \quad (3.10)$$

By substituting (3.10) into (3.9), we have

$$\begin{aligned} u_n(t) = & b_n \phi_n(t) + \int_0^t R(t, \tau) b_n \phi_n(\tau) \, d\tau + \int_0^t \left[c_n \phi_n(\xi) + \int_0^\xi R(\xi, \tau) c_n \phi_n(\tau) \, d\tau \right] d\xi \\ & + \int_0^t \int_0^{t-\eta} \left[f_n(\eta) \phi_n(\xi) + \int_0^\xi R(\xi, \tau) f_n(\eta) \phi_n(\tau) \, d\tau \right] d\xi \, d\eta \\ & - \int_0^t g(\eta) \int_0^{t-\eta} \left(\phi_n(\xi) + \int_0^\xi R(\xi, \tau) \phi_n(\tau) \, d\tau \right) d\xi \, d\eta. \end{aligned} \quad (3.11)$$

By using the fact that the Sturm-Liouville eigenfunction expansion converges at $x = 0$, we conclude

$$u(0, t) = \sum \rho_n u_n(t). \quad (3.12)$$

By substituting (3.11) into (3.12)

$$\begin{aligned} u(0, t) = & \sum \rho_n b_n \phi_n(t) + \int_0^t R(t, \tau) \sum \rho_n b_n \phi_n(\tau) \, d\tau \\ & + \int_0^t \left[\sum \rho_n c_n \phi_n(\xi) + \int_0^\xi R(\xi, \tau) \sum \rho_n c_n \phi_n(\tau) \, d\tau \right] d\xi \\ & + \int_0^t \int_0^{t-\eta} \left(\sum \rho_n f_n(\eta) \phi_n(\xi) + \int_0^\xi R(\xi, \tau) \sum \rho_n f_n(\eta) \phi_n(\tau) \, d\tau \right) d\xi \, d\eta \\ & - \int_0^t g(\eta) \int_0^{t-\eta} \left(\sum \rho_n \phi_n(\xi) + \int_0^\xi R(\xi, \tau) \sum \rho_n \phi_n(\tau) \, d\tau \right) d\xi \, d\eta. \end{aligned} \quad (3.13)$$

Using the convergence theorem of expansions concerning Sturm-Liouville eigenfunctions [20], we have

$$\begin{aligned} \sum \rho_n b_n \phi_n(x) &= b(x), & x \in [0, L), \quad b \in C^{1+}[1, L), \\ \sum \rho_n c_n \phi_n(\cdot) &= c(\cdot), & \text{weakly for } c \in L(1, L), \\ \sum \rho_n f_n(t) \phi_n(\cdot) &= f(\cdot, t), & \text{weakly for } f(\cdot, t) \in L(0, L). \end{aligned} \quad (3.14)$$

Also, note that

$$\delta_n := \int_0^L \phi_n(x) \delta(x) \, dx = \phi(0) = 1.$$

Hence,

$$\sum \rho_n \phi_n(\cdot) = \delta(\cdot), \quad \text{weakly.}$$

By substituting these expressions into (3.13i), we obtain

$$\begin{aligned}
 u(0, t) = & b(t) + \int_0^t R(t, \tau) b(\tau) d\tau \\
 & + \int_0^t \left[c(\xi) + \int_0^\xi R(\xi, \tau) c(\tau) d\tau \right] d\xi \\
 & + \int_0^t \int_0^{t-\eta} \left[f(\xi, \eta) + \int_0^\xi R(\xi, \tau) f(\tau, \eta) d\tau \right] d\xi d\eta \\
 & - \int_0^t g(\eta) \int_0^{t-\eta} \left(\delta(\xi) + \int_0^\xi R(\xi, \tau) \delta(\tau) d\tau \right) d\xi d\eta.
 \end{aligned} \tag{3.15}$$

The last item can be reduced to the form

$$- \int_0^t g(\eta) d\eta - \int_0^t g(\eta) \int_0^{t-\eta} R(\xi, 0) d\xi d\eta = - \int_0^t g(\eta) d\eta - \int_0^t R(\xi, 0) \int_0^{t-\xi} g(\eta) d\eta d\xi, \tag{3.16}$$

which implies (3.1).

4. UNIQUENESS AND SOLVABILITY OF THE INVERSE PROBLEM

From the integral representation (3.1), we can study the uniqueness of the inverse problem. For example, the following sufficient conditions for the uniqueness can be discovered immediately.

THEOREM 4.1.

- A. If any one of the following is true, then $u_0(t)$, $0 \leq t \leq T$, determines $q(x)$, $0 \leq x \leq T$:
- Case 1. $b(x) \neq 0$ in $[0, T)$ (see [5]);
 - Case 2. $b(x) \equiv 0$ and $c(x) \neq 0$ (see [5]);
 - Case 3. $b(x) \equiv c(x) \equiv 0$, $f(x, 0) \neq 0$.
- B. When $b \equiv c \equiv f \equiv 0$, $g(t)$ is not identical to 0 in any $(0, \epsilon)$, then $u_0(t)$, $0 \leq t \leq 2T$, determines $q(x)$, $0 \leq x \leq T$.

Part B also may be found in [6]. But here, we point out the minimum data required. If we consider the system in more detail, we will find weaker conditions, for example, Condition 1 can be weakened to: the zeros of $b(x)$ do not accumulate in $[0, T)$ and $|b(x)| + |c(x)| \neq 0$.

In the case that any condition in Part A is satisfied, the inverse problem is well-posed. In the case of Part B, the inverse problem may be well-posed or ill-posed depending on the behavior of $g(t)$.

Now we give a proof of Case 1 that is based on (3.1) and different from Romanov's proof in [5]. Differentiating the both sides of (3.1), we get

$$\begin{aligned}
 u'_0(t) = & b'(t) + R(t, t)b(t) + \int_0^t R_t(t, \tau)b(\tau) d\tau + c(t) + \int_0^t R(t, \tau)c(\tau) d\tau \\
 & + \int_0^t \left[f(\eta, t-\eta) + \int_0^{t-\eta} R(t-\eta, \tau)f(\tau, \eta) d\tau \right] d\eta \\
 & - g(t) - \int_0^t R(\xi, 0)g(t-\xi) d\xi, t \in [0, T).
 \end{aligned} \tag{4.1}$$

Suppose (q_1, R_1) and (q_2, R_2) are two different solutions. Let $\bar{R} = R_1 - R_2$, $\bar{q} = q_1 - q_2$. The equation above implies

$$\begin{aligned}
 0 = & \bar{R}(t, t)b(t) + \int_0^t \bar{R}_t(t, \tau)b(\tau) d\tau + \int_0^t \bar{R}(t, \tau)c(\tau) d\tau \\
 & + \int_0^t \int_0^{t-\eta} \bar{R}(t-\eta, \tau)f(\tau, \eta) d\tau d\eta - \int_0^t \bar{R}(\xi, 0)g(t-\xi) d\xi.
 \end{aligned} \tag{4.2}$$

Differentiating the both sides of (4.2) yields

$$\begin{aligned} 0 = & -\bar{q}(t)b(t) + \int_0^t [\bar{R}_t(t, \tau)b''(\tau) - (R_1q_1 - R_2q_2)b(\tau)] d\tau \\ & + c(t)\bar{R}(t, t) + \int_0^t \bar{R}_t(t, \tau)c(\tau) d\tau \\ & + \int_0^t \bar{R}(\eta, \eta)f(t - \eta, \eta) d\eta + \int_0^t \int_0^{t-\eta} \bar{R}_t(t - \eta, \tau)f(\tau, \eta) d\tau d\eta \\ & - \bar{R}(t, 0)g(0) - \int_0^t \bar{R}(\xi, 0)g'(t - \xi) d\xi. \end{aligned} \quad (4.3)$$

On the other hand, from (2.8)–(2.10), we can estimate on the stability [17], namely

$$\begin{aligned} |\bar{R}_x(x, s)| & \leq C \max_{0 < \xi < (x+s)/2} |\bar{q}(\xi)|, \\ |\bar{R}(x, s)| & \leq C \int_0^{(x+s)/2} |\bar{q}(\xi)| d\xi. \end{aligned} \quad (4.4)$$

Using these estimates in (4.3), we have

$$|b(t)\bar{q}(t)| \leq C \int_0^t \max_{0 \leq \xi \leq \eta} \bar{q}(\xi) d\eta.$$

Since $b(t) \neq 0$, it follows that

$$\left| b(t) \max_{0 \leq \xi \leq t} \bar{q}(\xi) \right| \leq C \int_0^t \max_{0 \leq \xi \leq \eta} \bar{q}(\xi) d\eta,$$

from which we conclude $\bar{q} = 0$.

Similarly, we can prove Cases 2 and 3 in the theorem above, by first converting each of them into a system of Volterra equations.

Now let us prove Part B of Theorem 4.1. Denote

$$r(x) = R(x, 0), \quad x \in (0, T). \quad (4.5)$$

Substituting the conditions into (3.1), we get

$$(r * g)(t) + u'_0(t) + g(t) = 0, \quad t \in [0, T]. \quad (4.6)$$

The condition that $g(t)$ does not vanish in $(0, \epsilon)$ guarantees the unique solvability of the equation for the unknown $r(x)$, $x \in (0, T)$, [21]. On the other hand, by using the d'Alembert formula on (2.5)–(2.7), we have

$$R(s, s) = \frac{r(2s)}{2} + \frac{1}{2} \int_0^s \int_\eta^{2s-\eta} q(\eta) R(\xi, \eta) d\xi d\eta, \quad s \in \left(0, \frac{T}{2}\right).$$

By differentiating both sides and using the Goursat (2.10) on the right-hand side, we have

$$\frac{-q(s)}{2} = r'(2s) + \int_0^s q(\eta) R(2s - \eta, \eta) d\eta \quad (4.7)$$

and

$$q_1(s) - q_2(s) = 2 \int_0^s [q_1(\eta) R_1(2s - \eta, \eta) - q_2(\eta) R_2(2s - \eta, \eta)] d\eta.$$

Using (4.4) once again, we can get

$$\bar{q}(s) < C \int_0^t \bar{q}(\xi) d\xi,$$

and it follows that $\bar{q}(s) = 0$, $s \in (0, T)$.

From the proof, we can also see that the solvability of the inverse problem in the case of Part B is equivalent to that of the integral equation (4.6), which is a Volterra equation of the first type. Results of this type equation can be found in [22].

5. NUMERICAL ALGORITHMS

In this section, let us consider an algorithm for solving the inverse problem in the case of Part B in the theorem. As shown in the previous section, the original inverse problem can be solved in two steps: solving the integral equation (4.6) for $r(x)$ and then the inverse problem (2.5)–(2.7), (4.5) for (q, R) .

The integral equation in the form of (4.6) has been studied by many people, see [22–24]. In particular, when $g(t) = \delta(t)$, i.e, $g(t)$ is a pulse, (4.6) degenerates to

$$r(t) = -u'_0(t).$$

This particular type of inverse problem was considered in many papers, [2,8,11].

Let us concentrate on the last inverse problem, i.e, (2.5)–(2.7), (4.5).

In order to discretize the equations, we notice that the boundary condition (2.6) allows us to extend the $R(x, s), q(s)$ and equation (2.5) to the domain $s < 0$ smoothly and evenly about s so that $R(x, -\epsilon) = R(x, \delta)$ and $q(-\delta) = q(\delta)$.

Denote by \tilde{X} the numerical approximate of X for $X = R, q, r, \dots$. By integrating the equation over the square S with the corner points $(x_1 + \epsilon, s_1)$, $(x_1 - \epsilon, s_1)$, $(x_1, s_1 - \epsilon)$, $(x_1, s_1 + \epsilon)$, we obtain

$$\begin{aligned} & R(x_1 + \epsilon, s_1) + R(x_1 - \epsilon, s_1) - R(x_1, s_1 - \epsilon) \\ & - R(x_1, s_1 + \epsilon) + \frac{1}{2} \iint_S q(s) R(x, s) dx ds = 0, |s_1| < x_1. \end{aligned} \quad (5.1)$$

There are various schemes for approximating the integral. The simplest is to use $[q(s_1)R(x_1, s_1) + O(\epsilon)]\epsilon^2$, which yields the difference equation

$$\tilde{R}(x + \epsilon, s) + \tilde{R}(x - \epsilon, s) - \tilde{R}(x, s - \epsilon) - \tilde{R}(x, s + \epsilon) + \epsilon^2 \tilde{q}(s) \tilde{R}(x, s) = 0, |s| < x.$$

However, it turns out that this difference equation, which can be also obtained by applying the central difference on equation (2.5), does not offer a convergent algorithm. To have a more accurate approximation, we use the average value of $q(s)R(x, s)$ at the four corner points of the square, i.e,

$$\frac{1}{4} (q(s_1) R(x_1 + \epsilon, s_1) + q(s_1) R(x_1 - \epsilon, s_1) + q(s_1 - \epsilon) R(x_1, s_1 - \epsilon) + q(s_1 + \epsilon) R(x_1, s_1 + \epsilon)) 2\epsilon^2.$$

We then get the following difference equation

$$\begin{aligned} & \left(1 + \frac{\epsilon^2}{4} \tilde{q}(s)\right) (\tilde{R}(x + \epsilon, s) + \tilde{R}(x - \epsilon, s)) - \left(1 - \frac{\epsilon^2}{4} \tilde{q}(s - \epsilon)\right) \tilde{R}(x, s - \epsilon) \\ & - \left(1 - \frac{\epsilon^2}{4} \tilde{q}(s + \epsilon)\right) \tilde{R}(x, s + \epsilon) = 0. \end{aligned} \quad (5.2)$$

Setting $s = 0$ in the equation above and using $\tilde{R}(x, -\epsilon) = \tilde{R}(x, \epsilon)$, we get

$$\left(1 + \frac{\epsilon^2}{4} \tilde{q}(0)\right) (\tilde{R}(x + \epsilon, 0) + \tilde{R}(x - \epsilon, 0)) - 2 \left(1 - \frac{\epsilon^2}{4} \tilde{q}(\epsilon)\right) \tilde{R}(x, \epsilon) = 0. \quad (5.3)$$

The boundary condition (2.7) can be approximated by

$$\tilde{R}(x + \epsilon, x + \epsilon) - \tilde{R}(x, x) + \frac{\epsilon}{4} (\tilde{q}(x + \epsilon) + \tilde{q}(x)) = 0. \quad (5.4)$$

Denoting by

$$\tilde{q}[i] := \frac{\epsilon^2}{4} \tilde{q}(i\epsilon), \quad r[i] := r(2i\epsilon), \quad \tilde{R}[i, j] := \tilde{R}((i + j)\epsilon, (i - j)\epsilon),$$

from equations (4.7), (4.5), (5.2)–(5.4), we obtain the following formulas to solve for (\tilde{R}, \tilde{q}) .

$$\tilde{q}[0] = -\frac{\epsilon^2 r'(0)}{2}, \quad (5.5)$$

$$\tilde{R}[i, i] = r[i], \quad i \geq 0, \quad (5.6)$$

$$\tilde{R}[i, i-1] = \frac{1 + \tilde{q}[0]}{2(1 - \tilde{q}[1])} (\tilde{R}[i-1, i-1] + \tilde{R}[i, i]), \quad i \geq 1, \quad (5.7)$$

$$(1 + \tilde{q}[i-j-1]) (\tilde{R}[i, j+1] + \tilde{R}[i-1, j]) - (1 - \tilde{q}[i-j]) \tilde{R}[i, j] - (1 - \tilde{q}[i-j-2]) \tilde{R}[i-1, j+1], \quad i-1 > j \geq 0, \quad (5.8)$$

$$\tilde{q}[i] + \tilde{q}[i-1] + \epsilon (\tilde{R}[i, 0] - \tilde{R}[i-1, 0]) = 0, \quad i > 0. \quad (5.9)$$

In particular, by setting $i = 1$ in (5.7) and (5.8), we get

$$\tilde{R}[1, 0] = \frac{1 + \tilde{q}[0]}{2(1 - \tilde{q}[1])} r[1], \quad (5.10)$$

$$\tilde{q}[1] + \tilde{q}[0] + \epsilon \tilde{R}[1, 0] = 0, \quad (5.11)$$

which solves $\tilde{q}[1]$ and $\tilde{R}[1, 0]$. Having found $\tilde{q}[0]$ and $\tilde{q}[1]$, we have the following explicit scheme to find $\tilde{q}[i]$, for $i > 1$. After the assignment (5.6), we compute $\tilde{R}[i, i-1]$ from (5.7), and for $j = i-2, i-3, \dots, 3, 2, 1$, compute $\tilde{R}[i, j]$ from (5.8), finally, we solve (5.8) and (5.7) with $j = 0$ for $\tilde{q}[i]$ and $\tilde{R}[i, 0]$. Note that for each i , when we compute $\tilde{q}[i]$, we only need to use $\tilde{R}[i, j]$ and $\tilde{R}[i-1, j]$, $j = i, i-1, \dots, 2, 1, 0$. This suggests us that for an efficient implementation, we use two one-dimensional arrays to replace the two-dimensional array \tilde{R} .

Formula (5.7)–(5.9) can also be used to solve the direct Goursat problem.

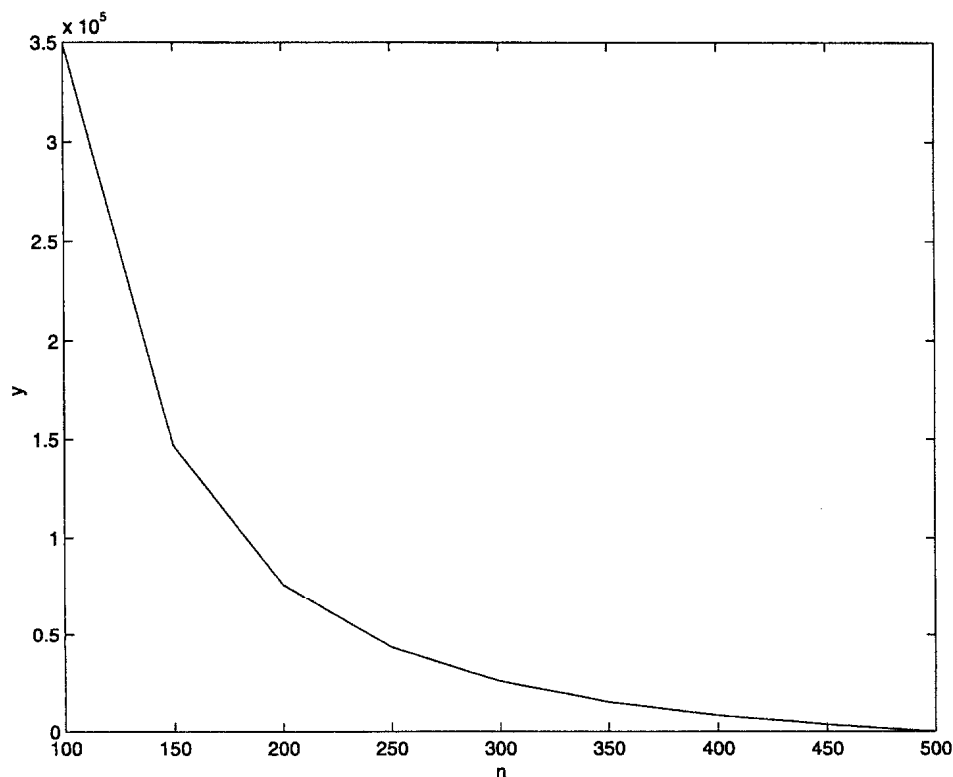


Figure 1. Convergence rate: $y = q_n(0.5) - q_{500}(0.5)$, where $q_n(0.5)$ is the computed value of $q(0.5)$ using $1/n$ as step size. $r(x) = \sin(x)$.

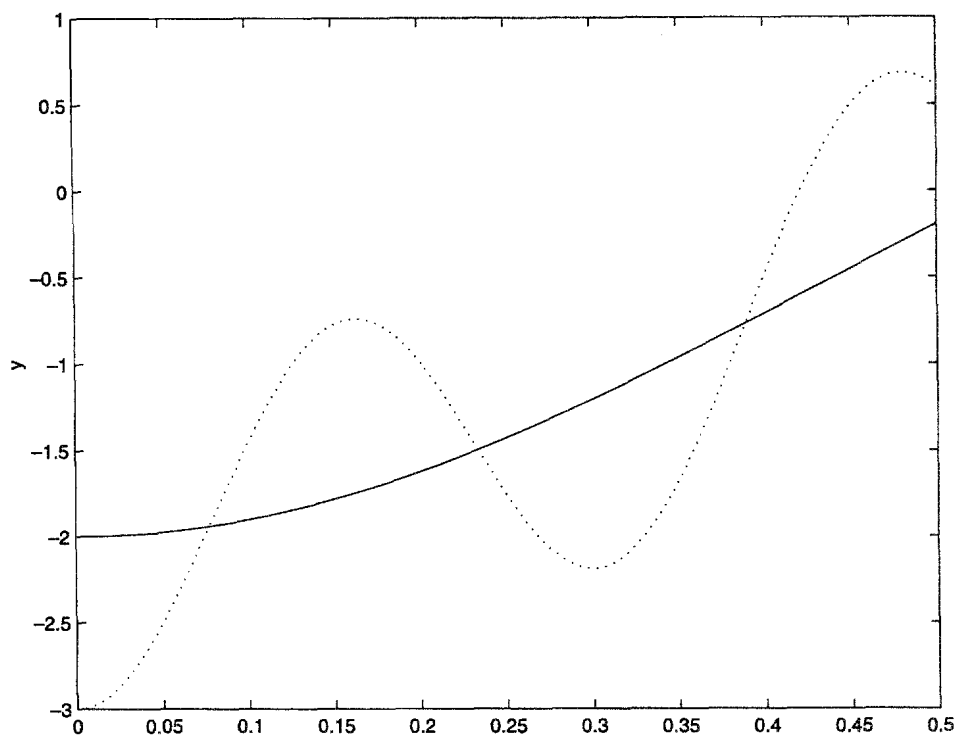


Figure 2. The dependency of $q(x)$ on $r(x)$. Solid: the computed $q(x)$ when $r(x) = \sin(x)$, dotted: when $r(x) = \sin(x) + 0.05 \sin(10x)$.

6. A DISCUSSION CONCERNING THE INVERSE SOURCE PROBLEM

We conclude with a discussion about the inverse source problem. The inverse source problem where one seeks the source $f(x, t)$ in the format $f(x, t) = F(x)$ or $f(x, t) = F_1(t)$ is of practical interest [25]. By using (3.1), we can study some of these problems. Suppose $f(x, t) = F(x)$ and we want to find $F(x)$ from other boundary and initial data. From (3.1), we get

$$\int_0^t \int_0^{t-\eta} \left[F(\xi) + \int_0^\xi R(\xi, \tau) F(\tau) d\tau \right] d\xi d\eta = G(t), \quad (6.1)$$

where

$$\begin{aligned} G(t) = & u_0(t) - b(t) + \int_0^t R(t, \tau) b(\tau) d\tau \\ & - \int_0^t \left[c(\xi) + \int_0^\xi R(\xi, \tau) c(\tau) d\tau \right] d\xi + \int_0^t g(\eta) d\eta + \int_0^t R(\xi, 0) \int_0^{t-\xi} g(\eta) d\eta d\xi, \end{aligned} \quad (6.2)$$

and can be evaluated from the given boundary data and $q(x)$. By differentiating (6.1) with respect to t twice, we get

$$F(t) + \int_0^t R(t, \tau) F(\tau) d\tau = G''(t). \quad (6.3)$$

By using the relation between R and K in [14], we can get a formal solution of $F(\cdot)$

$$F(t) = G''(t) + \int_0^t K(t, s) G''(s) ds. \quad (6.4)$$

Similarly, we can consider the inverse source problem in another case such as $F(x, t) = F_1(t)$, $F(x, t) = \delta(t)F(x)$, $F(x, t) = \delta(x)F_1(t)$, etc.

REFERENCES

1. T. Aktonsun, Inverse scattering in 1-D nonhomogeneous media and recovery of the wave speed, *J. Math. Phys.* **33**, 1395–1420, (1992).
2. K. Bube and R. Burridge, The one-dimensional inverse problem of relection seismology, *SIAM Reviews* **25**, 497–559, (1983).
3. R. Burridge, The Gelfand-Levitan, the Marchenko, and the Gopinath-Sondhi integral equations of inverse scattering theory, regarded in the context of inverse impulse-response problems, *Wave Motion* **2**, 305–323, (1980).
4. M.M. Lavrengeev, B.G. Romanov and S.P. Thithasky, *The Ill-posed Problems in Mathematical Physics and Analysis*, pp. 55–77, Science Press, Moscow, (1980).
5. V.G. Romanov, *Inverse Problems of Mathematical Physics*, Vista Science Press, (1986).
6. W.W. Symes, Impedance profile inversion via the first transport equation, *Journal of Mathematical Analysis and Application* **94**, 435–453, (1983).
7. W.W. Symes, On the relation between coefficient and boundary values for solutions of Webster's horn equation, *SIAM J. Math. Anal.* **17**, 1400–1420, (1986).
8. G.Q. Xie, A new iterative method for solving the coefficient inverse problem of the wave equation, *Comm. Pure and Appl. Math.* **XXXIX**, 307–322, (1986).
9. J. Liu and Y. Wang, On uniqueness of an inverse problem for a 1-D wave equation from transmission data, *SIAM J. Appl. Math.* **57** (1), 195–204, (February 1997).
10. P. Sacks and Rakesh, Impedance inversion from transmission data for the wave equation, *Wave Motion* **24**, 263–274, (1996).
11. S.I. Kabanikhin and A. Lorenzi, *Identification Problems of Wave Phenomena: Theory and Numerics (Inverse and Ill-Posed Problems)*, Coronet Books, (1999).
12. R.W. Carroll, *Transmutation and Operator Differential Equations*, *Math. Studies* 37, North Holland, New York, (1979).
13. R.W. Carroll, *Transmutation, Scattering Theory and Special Functions*, *Mathematics Studies* 69, North Holland, New York, (1982).
14. I.M. Gelfand and B.M. Levitan, On the determination of a differential equation from its spectral function, *Amer. Math. Soc. Trans., Ser. 2* **I**, 253–304, (1955).
15. J.R. Cannon and W. Rundell, An inverse problem for an elliptic partial differential equation, *J. Math. Anal. Appl.* **126**, 328–340, (1987).
16. R.P. Gilbert and Z. Lin, On the conditions for uniqueness and existence of the solution to an acoustic inverse problem: I Theory, *Journal of Computational Acoustics* **1** (2), 229–247, (1993).
17. R.P. Gilbert and Z. Lin, An acoustic inverse problem: Numerical experiment, *Journal of Computational Acoustics* **3** (3), 229–240, (1995).
18. R.P. Gilbert and D.H. Wood, A transmutation approach to underwater sound propagation, *Wave Motion* **8** (5), 383–397, (1986).
19. Z. Lin, On the determination of radially dependent Lamé coefficients, *SIAM J. Appl. Math.* **58**, (1998).
20. E.C. Titchmarsh, *Eigenfunction Expansions Associated with Second Different Equations, Part I*, pp. 1–20, Oxford Univ. Press, London, (1962).
21. K. Yosida, *Functional Analysis*, pp. 70–80, Springer-Verlag, (1978).
22. N.K. Karapetyants, A.A. Kilbas and M. Saigo, On the solution of nonlinear Volterra convolution equation with power nonlinearity, *Journal of Integral Equations and Applications* **8** (4), (Fall 1996).
23. J.D. Lambert, *Computational Methods in Ordinary Differential Equations*, John Wiley, (1973).
24. P.H.M. Wolkenfelt, The construction of reducible quadrature rules for Volterra integral and integro-differential equations, *IMA J. Numer. Anal.* **2**, 131–152, (1982).
25. H. Ammari, G. Bao and J.L. Fleming, An inverse source problem for Maxwell's equations in magnetoencephalography, *SIAM J. Appl. Math.* **62** (4), 1369–1382, (2002).
26. J.D. Achenbach, *Wave Propagation in Elastic Solids*, p. 85, p. 310, North-Holland, (1976).
27. A. Fang and Z. Lin, The determination of radially dependent conductivity coefficient, *Applicable Analysis* **48**, 51–62, (1993).
28. E. Kreyszig, *Introductory Functional Analysis with Applications*, p. 180.
29. H.T. Lau, *A Numerical Library in C for Scientists and Engineers*, CRC Press, (1995).
30. J.R. McLaughlin, P.L. Polyakov and P.E. Sacks, Reconstruction of a spherically symmetric speed, *SIAM J. Appl. Math.* **54** (5), 1203–1223, (October 1994).
31. H.A. Schenck, Improved integral formulation for acoustic radiation problems, *J. Acoust. Soc. Amer.* **44**, 41–58, (1968).