6.2 Volumes by Cross Section

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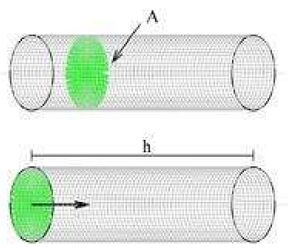
Remember how to compute the volume of a cylinder or prism using the cross-sectional area and length (height) of the object? If the cross-sectional area is known and constant along the height, the volume calculation is easy. But, what if the cross-sectional area changes in a known manner along the line that is the height, like it does for a cone or pyramid? How could a single method in calculus be used to determine the volume of either of these types of solids?

Volumes by Cross Section

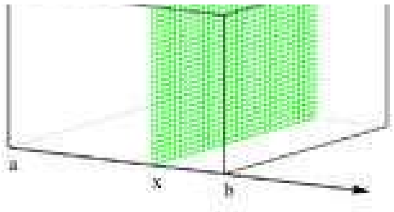
A circular cylinder can be generated by translating a circular disk along a line that is perpendicular to the disk. In other words, the cylinder can be generated by moving the cross-sectional area A (the disk) through a distance h. The resulting volume is called the **volume of solid** and it is defined to be

$$V = Ah.$$



[Figure 1]

ume of solid does not necessarily have to be circular. It can take any arbitrary shape. One useful way to find the volume is by a technique called “slicing”. To explain idea, suppose a solid S is positioned on the x-axis and extends from points to $x = b$.



[Figure 2]

Let $A(x)$ be the cross-sectional area of the solid at some arbitrary point x. Just like we did in calculating the definite integral in the previous chapter, divide the interval $[a, b]$ into n sub-intervals and with widths

$$\triangle x_1, \triangle x_2, \triangle x_3, \ldots, \triangle x_n.$$

Eventually, we get planes that cut the solid into n slices

$$S_1, S_2, S_3, \ldots, S_n.$$

Take one slice, S_k. We can approximate slice S_k to be a rectangular solid with thickness $\triangle x_k$ and cross-sectional area $A(x_k)$. Thus the volume V_k of the slice is approximately

$$V_k \approx A(x_k) \triangle x_k.$$

Therefore the volume V of the entire solid is approximately

$$\begin{aligned} V &= V_1 + V_2 + \ldots + V_n \\ &\approx \sum_{k=1}^{n} A(x_k) \triangle x_k. \end{aligned}$$

If we use the same argument to derive a formula to calculate the area under the curve, let us increase the number of slices in such a way that $\triangle x_k \to 0$. In this case, the slices become thinner and, as a result, our approximation will get better and better. That is,

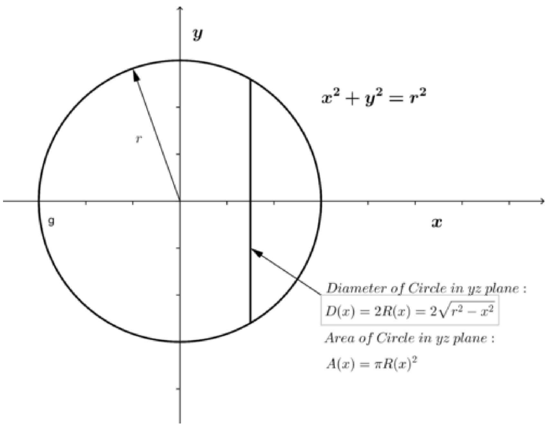
$$\triangle x \to 0 \qquad k=1$$

Notice that the right-hand side is just the definition of the definite integral. Thus

$$V = \lim_{\triangle x \to 0} = \sum_{k=1}^{n} A(x_k) \triangle x_k$$
$$= \int_a^b A(x) dx.$$

Now, let's derive a formula for the volume of a sphere with radius r centered on the point (0, 0, 0) whose cross section in the xy-plane is as shown using slices perpendicular to the x-axis to determine $A(x)$.

A cross-section of the sphere in the xy plane is shown.



[Figure 3]

Cross-sections through the sphere that are perpendicular to the x-axis are circles. Since the equation of the circle in the xy-plane is $x^2 + y^2 = r^2$, the radius of the circle in the plane perpendicular to the x-axis is $y = \sqrt{r^2 - x^2}$, which means the area of each circle is

$$A(x) = \pi y^2 = \pi(r^2 - x^2).$$

If we look at the area slices from $x = 0$ to $x = r$, the volume of the sphere is given by

$$
\begin{aligned}
V &= \int_{a} A(x)dx \\
&= 2\int_{0}^{r} \pi(r^2 - x^2)dx \\
&= 2\pi\left[r^2 x - \frac{x^3}{3}\right]_0^r \\
&= 2\pi\left[\frac{2r^3}{3}\right]
\end{aligned}
$$

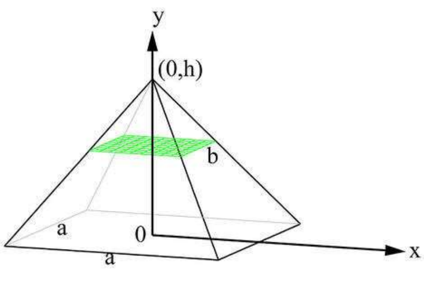
$$V = \int_{a}^{b} A(x)dx$$

$$V = \frac{4}{3}\pi r^3$$

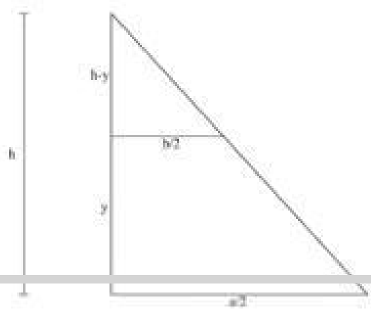
This is the formula we expect to see for the volume of a sphere, $V = \frac{4}{3}\pi r^3$.

The above derivation used a cross-section perpendicular to the x-axis, and therefore integration along the x-axis. The next example uses integration along the y-axis.

Now, let's derive a formula for the volume of a pyramid whose base is a square of sides a and whose height (altitude) is h.



[Figure 4]



[Figure 5]

any arbitrary square, then, by similar triangles (Figure 7b),

$$\frac{\frac{1}{2}b}{\frac{1}{2}a} = \frac{h-y}{h},$$
$$b = \frac{a}{h}(h-y).$$

Since the cross-sectional area at y is $A(y) = b^2$,

$$A(y) = b^2 = \frac{a^2}{h^2}(h-y)^2.$$

Using the volume formula,

$$\begin{aligned} V &= \int_c^d A(y)dy \\ &= \int_0^h \frac{a^2}{h^2}(h-y)^2 dy \\ &= \frac{a^2}{h^2}\int_0^h (h-y)^2 dy. \\ &= \frac{a^2}{h^2}\left[-\frac{1}{3}(h-y)^3\right]_0^h && \ldots \text{Using } u-\text{substitution of } u = h-y \text{ and } du = -dy \\ V &= \frac{1}{3}a^2 h && \text{to evaluate the integral.} \end{aligned}$$

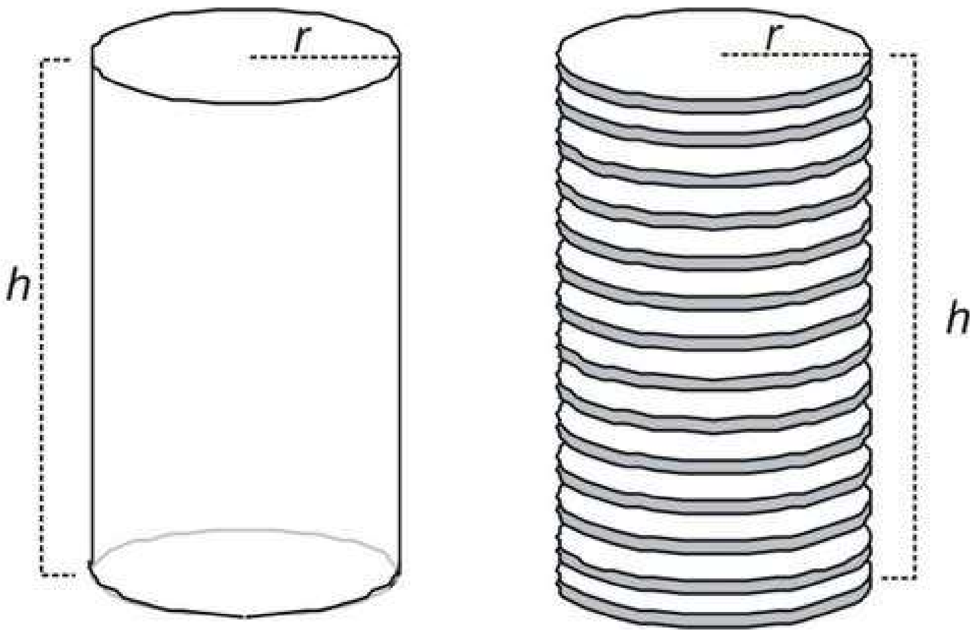
Therefore the volume of the pyramid is $V = \frac{1}{3}a^2 h$, which agrees with the standard formula.

Cavalieri's Principle

You may recall from geometry that two triangles with congruent bases and the same height have the same area regardless of what the triangle looks like. This arises from the 2-dimensional case of **Cavalieri's Principle**. The 2-dimensional case states: Two regions between two parallel lines have the same area if and only if every line parallel to the two bounding lines intersects the two regions with line segments of equal length. Watch the first 1 minute and 30 seconds of the following video for a good visual explanation of the 2-dimensional case.

There is also a 3-dimensional case of Cavalieri's Principle which applies to volume. The 3-dimensional case of Cavalieri's Principle states: **The volumes of two objects are equal if the areas of their corresponding cross-sections are in all cases equal.** Two cross-sections correspond if they are intersections of the figure with planes equidistant from a section of the two figures with a plane that is parallel to the plane the base is on. If the areas for both of those cross-sections are the same and that same result holds true for any plane between the base planes, then the two figures have the same volume.

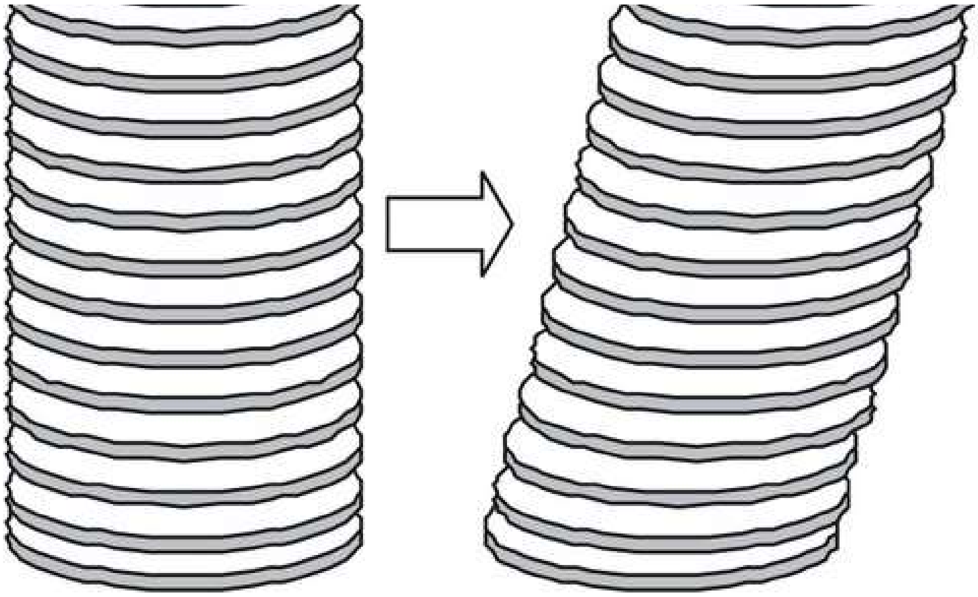
Visually, take a stack of coins such as in the image below.



[Figure 6]

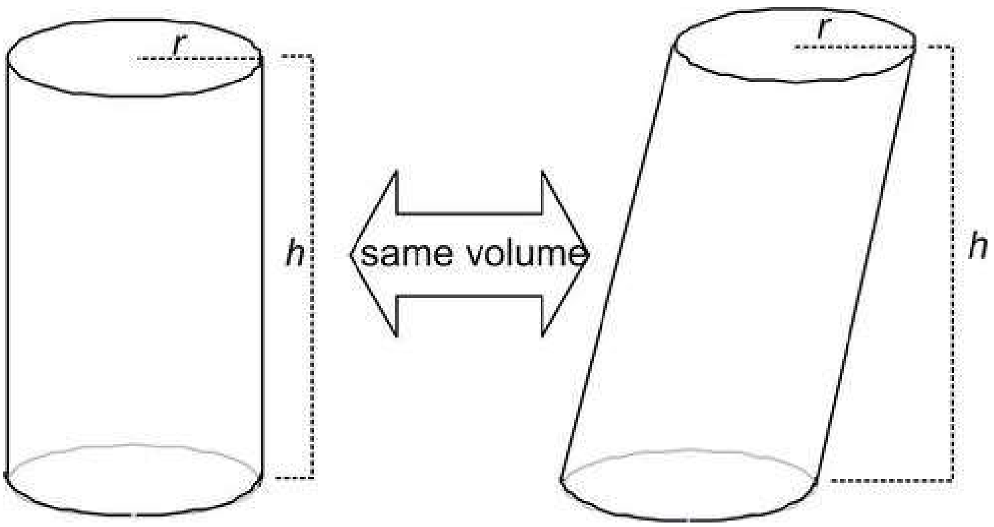
The stack is a cylinder and it is easy to calculate the volume because we have the formula: $V = \pi r^2 h$.

If you push the coins so that they are not all perfectly aligned like in the image below, it becomes more difficult to calculate the volume just by looking at the shape.



[Figure 7]

However, since no coins were taken from or added to the stack, the overall number of coins is still the same and thus the volume of the object is the same as it was when it was a regular cylinder.



[Figure 8]

Cavalieri's principle can be proven by finding the volume by cross section. The volume by cross section method takes the area of all of the slices of the shape and adds them together to find the total volume. For two shapes, if the corresponding slices have the same area then the sum will be the same and the shapes will have the same volume.

Examples

Example 1

its height (length).

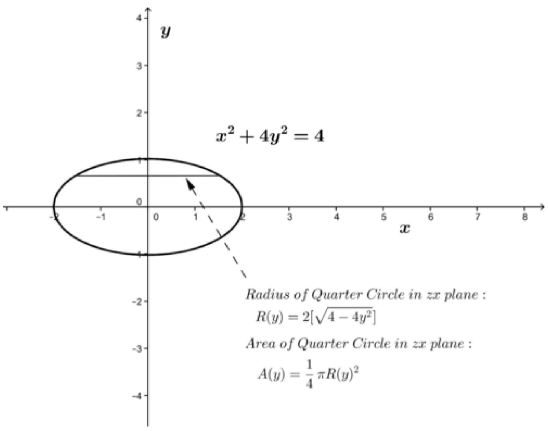
This concept presented a method to find the volume by:

1. Sketching the solid and a cross-section in a plane perpendicular to its height (length)
2. Finding an expression for the cross-sectional area A (volume slice Adx) in the direction of the height (length).
3. Determining the limits of integration along the direction of the height (length).
4. Integrating to find the volume

Example 2

A solid has as it's base the region in the xy-plane defined by the ellipse $x^2 + 4y^2 = 4$. Every cross-section by a plane perpendicular to the y-axis is a quarter circle, with radius in the base. Find the volume of the solid.

The cross section of the elliptical base in the xy plane is shown.



[Figure 9]

Each cross-section perpendicular to the y-axis through the ellipse is a quarter-circle with a radius equal to the line through the ellipse. The expression for the radius can be determined by solving the equation of the ellipse for x:

$x = \sqrt{4-4y^2} = 2\sqrt{1-y^2}$, which gives the portion of the ellipse in Quadrant I.

The radius of any quarter-circle section in the upper half of the ellipse is:

This means that the area of any quarter-circle section in the upper half of the ellipse is

$$A(y) = \frac{1}{4} \cdot \pi r(y)^2 = \frac{1}{4} \cdot \pi \left[4\sqrt{1-y^2}\right]^2, \ or$$
$$A(y) = 4\pi(1-y^2)$$

The volume of the solid can now be evaluated as follows:

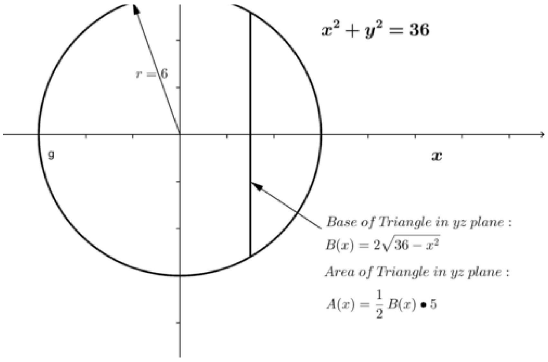
$$V = 2\int_c^d A(y)dy \ldots \text{ Account for the upper and lower portions of the ellipse}$$
$$= 2\int_0^1 4\pi(1-y^2)dy$$
$$= 8\pi\left[y - \frac{y^3}{3}\right]_0^1$$
$$V = \frac{16}{3}\pi$$

The volume of the solid is $\frac{16}{3}\pi$ cubic units.

Example 3

The base of a solid is in the xy-plane, and is defined by the equation $x^2 + y^2 = 36$. Each cross-section defined by a plane perpendicular to the x-axis is an isosceles triangle, with height 5 inches. The side of the triangle that is not congruent to the other two is the base in the xy-plane. Find the volume of the solid.

The base of the solid in the xy-plane is shown below.



[Figure 10]

The equation $x^2 + y^2 = 36$ is a circle of radius 6. Each cross-section perpendicular to the x -axis through the circle is an isosceles triangle whose base is determined to be:

$$B(x) = 2[\sqrt{36 - x^2}], \text{ from } x = 0 \text{ to } x = 6.$$

This means that the area of any isosceles section in Quadrants I & IV of the circle is

$$A(x) = \frac{1}{2} \cdot B(x) \cdot 5 = 5\sqrt{36 - x^2}.$$

The volume of the solid can now be evaluated as follows:

$$V = 2\int_a^b A(x)dx \ldots \text{Account for the left portions of the circle (Quadrants II and III),}$$
$$\text{and right portions (Quadrants I and IV).}$$
$$= 2\int_0^6 5\sqrt{36 - x^2}dx \ldots \text{The integral can be evaluated by (1) using a graphing calculator}$$
$$\text{OR}$$
$$= 10\int_0^6 \sqrt{36 - x^2}dx \ldots \text{(2) by first making the following variable substitution and then evaluating :}$$
$$V = 2\int_a^b A(x)dx \ldots\ldots x = 6\sin u, dx = 6\cos u du, \text{ and integration limits } u = 0 \text{ to } u = \frac{\pi}{2}$$
$$V = 10\int_0^{\frac{\pi}{2}} 36\cos^2 u du = 360\int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2u)du = 180\left[u + \frac{\sin 2u}{u}\right]_0^{\frac{\pi}{2}}$$
$$V = 90\pi\ in^3$$
$$V = 282.7\ in^3$$

The volume of the solid is 90π cubic units.

Review

1. Find the volume of a pyramid whose base is a square with sides of length 20 and whose height is 15.
2. Find the volume of a cone of height 4 and base diameter 10.
3. A 12 by 15 swimming pool has a depth that changes along its longer side as the function $f(x) = 6 + 2\cos\left(\frac{\pi}{20}x\right)$. Find its volume.
4. Use the method of slicing to find the volume of a pyramid of height two whose base is an equilateral triangle with sides of length two.
5. Use the method of slicing to find the volume of an object of length 5 whose cross sections are triangles of height 4 and width given by $w(x) = 1 + x$, $0 \le x \le 5$.
6. There is a solid lying on the Cartesian plane between $x = 0$ and $x = 1$ whose cross sectional area is given by the function $A(x) = x^3 + x$. What is its volume?
7. There is a solid lying on the Cartesian plane between $x = 5$ and $x = 6$ whose cross sectional area is given by the function $A(x) = \frac{1}{x}$. What is its volume?

volume?

9. There is a solid lying on the Cartesian plane between $x = 2$ and $x = 4$ whose cross sectional area is given by the function $A(x) = \frac{\ln(x)}{x}$. What is its volume?
10. There is a solid lying on the Cartesian plane between $x = 10$ and $x = 20$ whose cross sectional area is given by the function $A(x) = \sin(x)\sqrt{\cos(x) + 1}$. What is its volume?
11. State Cavalieri's principle.
12. Consider a cross sectional cut of a sphere, taken parallel to and y units above the sphere's "equator." What is the radius of this cut, in terms of R, the radius of the sphere, and y?
13. Say that a cylinder of height h is drilled into a sphere along a line that intersects the sphere's center. What is the radius of this cylinder, in terms of R, the radius of the sphere, and h?
14. Say that a cylinder of height h is drilled into a sphere along a line that intersects the sphere's center. What is the volume of what remains of the sphere? What term does this surprisingly not rely on? This is commonly known as the napkin ring problem.
15. How does the above napkin ring result relate to Cavalieri's principle?

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.2.

Image Attributions