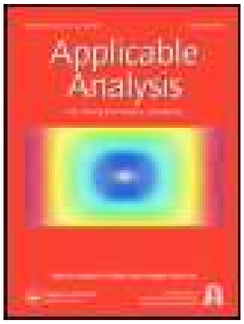

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Some Results on the Linearized Oscillation of the Odd-order Neutral Difference Equation

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In this article, we obtain some results on the linearized oscillation of the odd-order neutral difference equation

$$\Delta^m(x_n + p_n g(x_{n-k})) + q_n h(x_{n-l}) = 0, \quad n = 0, 1, 2, \ldots$$

where Δ is the forward difference operator, m is odd, $\{p_n\}, \{q_n\}$ are sequences of nonnegative real numbers, k, l are nonnegative integers, $g(x), h(x) \in C(R, R)$ with $xg(x) > 0$ for $x \neq 0$.

Keywords: Odd-order; Neutral difference equation; Linearized oscillation

AMS Subject Classifications: 39A10; 39A12

1. INTRODUCTION

Let Z denote the set of all integers. For $a, b \in Z$ with $a \leq b$, define $N(a) = \{a, a+1, \ldots\}$, $N = N(0)$, and $N(a, b) = \{a, a+1, \ldots, b\}$.

Consider the following neutral difference equation

$$\Delta^m(x_n + p_n g(x_{n-k})) + q_n h(x_{n-l}) = 0, \quad n \in N, \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$ and $\Delta^m x_n = \Delta(\Delta^{m-1})x_n$, m is odd, $\{p_n\}, \{q_n\}$ are sequences of nonnegative real numbers, $k \in N(1)$, $l \in N$, $g(x), h(x) \in C(R, R)$ with $xg(x) > 0$ for $x \neq 0$.

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Equation (1.1) was first considered by Brayton and Willoughby [1], from the numerical analysis point of view and can be regarded as the discrete analog of delay differential equation

$$\frac{d^m}{dt^m}(x(t)+p(t)g(x(t-\tau)))+q(t)h(x(t-\sigma))=0. \tag{1.2}$$

The linearized oscillations of Eq. (1.2) has been extensively investigated, see for example [2–5]. It is well known, that the properties of differential equations and their discrete analogs can be quite different. For example, every solution of the logistic equation

$$x'(t)=ax(t)\left(1-\frac{x(t)}{K}\right)$$

is monotonic, see Fig. 1. But its discrete analog

$$x(n+1)=bx(n)(1-x(n))$$

has a chaotic solution when $b=4$ [6]. See Fig. 2.

When $p_n\equiv 0$, the linearized oscillation of Eq. (1.1) has been developed in [7–9]. Roughly speaking, it has been proved that, under appropriate hypotheses, Eq. (1.1) has the same oscillatory character as an associated linear equation. However, there are no results for the linearized oscillation of neutral difference equations except [10–12]. In [10,11], we considered the linearized oscillation of Eq. (1.1) when $m=1$ and $xg(x)<0$ for $x\neq 0$. And in [12], we considered the linearized oscillation of Eq. (1.1) when m is even and $xg(x)<0$ for $x\neq 0$. Our goal in this article is to obtain some results

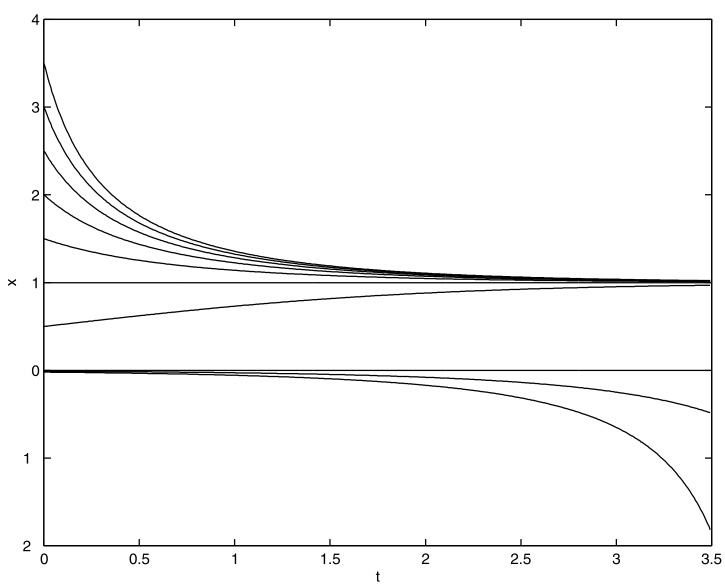

FIGURE 1 The behavior of the solutions of $x'(t)=x(t)(1-x(t))$. They are monotonic and $\lim_{t\to\infty}x(t)=0,1$ or $-\infty$.

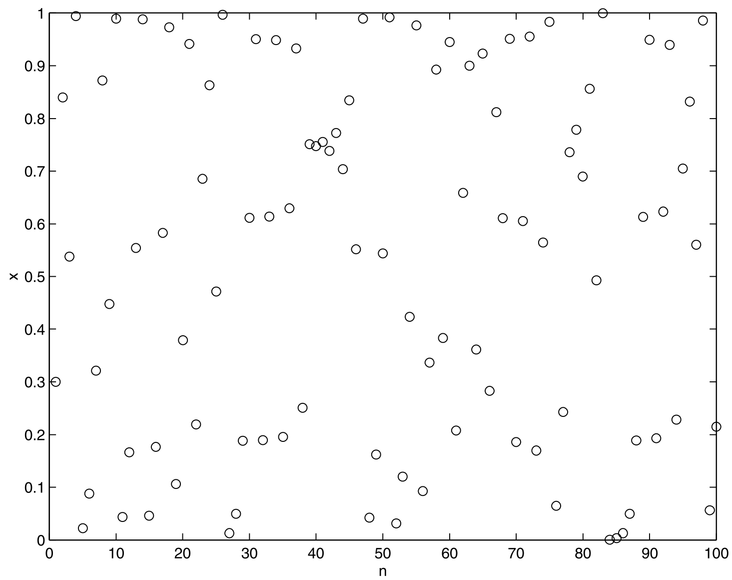

FIGURE 2 The chaos of $\{x_n\}$ of $x_{n+1} = 4x_n(1 - x_n)$, $x_0 = 0.3$.

on the linearized oscillation of Eq. (1.1) when $xg(x) > 0$ for $x \neq 0$. More precisely, we will assume that:

$$\limsup_{n\to\infty} p_n = p \in (1, \infty), \qquad \lim_{n\to\infty} q_n = q \in (0, \infty); \tag{1.3}$$

$$xg(x) > 0 \text{ for } x \neq 0, \quad \text{and} \quad \lim_{x\to 0} \frac{g(x)}{x} = 1; \tag{1.4}$$

$$xh(x) > 0 \text{ for } x \neq 0, \quad \lim_{x\to 0} \frac{h(x)}{x} = 1, \quad \text{and} \quad \liminf_{|x|\to\infty} |h(x)| > 0; \tag{1.5}$$

and

$$p \neq q \quad \text{when } k = l. \tag{1.6}$$

Under the above assumptions, we associate Eq. (1.1) with the linear difference equation of constant coefficients

$$\Delta^m (y_n + p y_{n-k}) + q y_{n-l} = 0, \quad n \in N. \tag{1.7}$$

Let $r = \max\{k, l\}$. By a solution of Eq. (1.1), we mean a sequence $\{x_n\}$ of real numbers which is defined for $n \in N(n_0 - r)$ and satisfies Eq. (1.1) for $n \in N(n_0)$ for some integer $n_0 \in N$. It is easy to see that under the initial conditions:

$$x_{n_0+j} = a_j, \quad j \in N(-r, m-1). \tag{1.8}$$

Eq. (1.1) has a unique solution satisfying (1.8). A solution $\{x_n\}$ of Eq. (1.1) is said to oscillate if for each $M \in N(n_0)$ there exists an $n \in N(M)$ such that $x_n x_{n+1} \leq 0$. Otherwise the solution is called nonoscillatory. For the general background on difference equation, one can refer to [6,13–15].

In Section 2, we establish some linearized oscillation theorems for Eq. (1.1). And in Section 3, we show that, under appropriate hypotheses, if Eq. (1.7) has a positive solution, so does Eq. (1.1).

2. LINEARIZED OSCILLATIONS

Denote $X = (0, \infty) \times (0, \infty)$ with the usual metric. Before giving the main theorem, we establish some lemmas which can be proved by a similar method to Lemma 2.2 in [10].

LEMMA 2.1 *Let Y be the subset of X consisting of all points (a, b) such that the difference equation*

$$\Delta^m(y_n + ay_{n-k}) + by_{n-l} = 0, \quad n \in N, \tag{2.1}$$

has a nonoscillatory solution, where m is odd and $k \in N(1)$, $l \in N$. If $k > l$, then $Y = X$; If $k < l$, then Y is closed in X.

LEMMA 2.2 *If $k = l$, $(a, b) \in X$ and $a > b$, then Eq. (1.7) has a positive solution.*

THEOREM 2.1 *Assume that (1.3)–(1.6) hold. If every solution of Eq. (1.7) oscillates, then every solution of Eq. (1.1) also oscillates.*

Proof Assume, for the sake of contradiction, that Eq. (1.1) has a nonoscillatory solution $\{x_n\}_{n \in N(n_0 - r)}$. We will assume that $\{x_n\}$ is eventually positive. The case where $\{x_n\}$ is eventually negative is similar and the proof is omitted. Let $\bar{n} \in N(n_0 - r)$ be a large integer such that

$$x_n > 0, \qquad q_n > 0, \quad \text{for } n \in N(\bar{n} - r).$$

Set

$$z_n = x_n + p_n g(x_{n-k}) \tag{2.2}$$

then

$$z_n > 0 \quad \text{for } n \in N(\bar{n}),$$

and

$$\Delta^m z_n = -q_n h(x_{n-l}) < 0 \quad \text{for } n \in N(\bar{n}).$$

Therefore, $\{\Delta^{m-1} z_n\}$ is decreasing and there exists a nonnegative real number z^* such that

$$\lim_{n \to \infty} z_n = z^*.$$

This implies that

$$\lim_{n\to\infty} q_n h(x_{n-l}) = -\lim_{n\to\infty} \Delta(\Delta^{m-1} z_n) = 0.$$

By (1.3), we know $\lim_{n\to\infty} h(x_n) = 0$. And in view of (1.5), we see that

$$\lim_{n\to\infty} x_n = 0, \qquad \lim_{n\to\infty} \Delta^i z_n = 0, \quad i = 0, 1, \ldots, m-1. \tag{2.3}$$

Denote

$$P_n = \frac{p_n g(x_{n-k})}{x_{n-k}}, \qquad Q_n = \frac{q_n h(x_{n-l})}{x_{n-l}}, \quad \text{for } n \in N(\bar{n}). \tag{2.4}$$

then

$$\limsup_{n\to\infty} P_n = \limsup_{n\to\infty} p_n \lim_{x\to 0} \frac{g(x)}{x} = p, \qquad \lim_{n\to\infty} Q_n = \lim_{n\to\infty} q_n \lim_{x\to 0} \frac{h(x)}{x} = q. \tag{2.5}$$

and Eq. (1.1) becomes

$$\Delta^m (x_n + P_n x_{n-k}) + Q_n x_{n-l} = 0. \tag{2.6}$$

Using (2.2), (2.6) reduces to

$$\Delta^m z_n + P_{n-l} \frac{Q_n}{Q_{n-k}} \Delta^m z_{n-k} + Q_n z_{n-l} = 0, \qquad n \in N(\bar{n} + l). \tag{2.7}$$

For every $\epsilon \in (0, q)$, by (2.5), there exists an integer $m_1 \in N(\bar{n} + l)$ such that

$$P_{n-l} \frac{Q_n}{Q_{n-k}} < p + \epsilon, \qquad Q_n > q - \epsilon \quad \text{for } n \in N(m_1 - k).$$

By (2.7), we have

$$\Delta^m z_n + (p+\epsilon) \Delta^m z_{n-k} + (q-\epsilon) z_{n-l} \le 0, \quad n \in N(m_1 - k). \tag{2.8}$$

Set

$$w_n = \frac{-\Delta^m z_n - (p+\epsilon)\Delta^m z_{n-k}}{z_{n-l}} \ge q - \epsilon, \quad n \in N(m_1 - k). \tag{2.9}$$

Then

$$\Delta^m z_n + (p+\epsilon) \Delta^m z_{n-k} + w_n z_{n-l} = 0, \quad n \in N(m_1 - k). \tag{2.10}$$

Consider two possible cases.

Case 1 $m > 1$.

By (2.3), (2.10), we get

$$\Delta z_n + (p+\epsilon)\Delta z_{n-k} + \alpha_n = 0, \quad n \in N(m_1 - k), \tag{2.11}$$

where

$$\alpha_n = \sum_{i=n}^{\infty} \frac{(i-n+1)\cdots(i-n+m-2)}{(m-2)!} w_i z_{i-l}.$$

Thus

$$\begin{aligned} z_{n-k} &= \frac{1}{p+\epsilon}\sum_{j=0}^{\infty}(-p-\epsilon)^{-j}\sum_{i=0}^{\infty}\alpha_{n+jk+i} \\ &= \frac{1}{p+\epsilon}\sum_{j=0}^{\infty}(-p-\epsilon)^{-j}\sum_{i=jk}^{\infty}\alpha_{n+i} \\ &= \frac{1}{p+\epsilon}\sum_{i=0}^{\infty}\alpha_{n+i}\sum_{j=0}^{[i/k]}(-p-\epsilon)^{-j} \\ &= \frac{1}{1+p+\epsilon}\sum_{i=0}^{\infty}\left(1-(-p-\epsilon)^{-[i/k]-1}\right)\alpha_{n+i}, \quad n \in N(m_1 - k), \end{aligned}$$

where $[\cdot]$ denotes the greatest integer function. So by (2.9), for $n \in N(m_1)$, we get

$$\begin{aligned} z_n &= \frac{1}{1+p+\epsilon}\sum_{i=k}^{\infty}\left(1-(-p-\epsilon)^{-[i/k]}\right) \\ &\quad \times \sum_{j=i}^{\infty}\frac{(j-i+1)\cdots(j-i+m-2)}{(m-2)!} w_{n+j} z_{n+j-l} \\ &\geq \frac{q-\epsilon}{1+p+\epsilon}\sum_{i=k}^{\infty}\left(1-(-p-\epsilon)^{-[i/k]}\right)\sum_{j=i}^{\infty}\frac{(j-i+1)\cdots(j-i+m-2)}{(m-2)!} z_{n+j-l}. \end{aligned} \tag{2.12}$$

Since every solution of Eq. (1.7) oscillates, by Lemma 2.1, we see that $k \leq l$.

If $k < l$, we consider the Banach Space $l_{\infty}^{m_1-l}$ of all real sequences $a = \{a_n\}$ where $n \in N(m_1 - l)$ with the sup norm $\|a\| = \sup_{n \in N(m_1-l)} |a_n|$, we define a subset S in $l_{\infty}^{m_1-l}$ as:

$$S = \{a \in l_{\infty}^{m_1-l}\colon a_n = z_n,\ n \in N(m_1 - l, m_1) \text{ and } 0 \leq a_n \leq z_n,\ n \in N(m_1 + l)\}.$$

And define a partial order on $l_{\infty}^{m_1-l}$ in the usual way, that is

$$a, b \in l_{\infty}^{m_1-l}, \quad a \leq b \text{ means that } a_n \leq b_n \quad \text{for } n \in N(m_1 - l).$$

Obviously, for every subset A of S both $\inf A$ and $\sup A$ exist in S. Now we define an operator $T : S \to l_\infty^{m_1-l}$ as follows: for $a \in S$, set $Ta = b$ where

$$b_n = z_n \quad \text{for } n \in N(m_1 - l, m_1),$$

and for $n \in N(m_1 + 1)$,

$$b_n = \frac{q-\epsilon}{1+p+\epsilon}\sum_{i=k}^{\infty}\left(1-(-p-\epsilon)^{-[i/k]}\right)\sum_{j=i}^{\infty}\frac{(j-i+1)\cdots(j-i+m-2)}{(m-2)!}a_{n+j-l}.$$

It follows from (2.12) that

$$\begin{aligned} b_n &\le \frac{q-\epsilon}{1+p+\epsilon}\sum_{i=k}^{\infty}\left(1-(-p-\epsilon)^{-[i/k]}\right)\sum_{j=i}^{\infty}\frac{(j-i+1)\cdots(j-i+m-2)}{(m-2)!}z_{n+j-l} \\ &\le z_n, \end{aligned}$$

for $n \in N(m_1 + 1)$. So $TS \subset S$, and clearly T is an increasing mapping. By Knaster's fixed point theorem [16] there is $a \in S$ such that

$$\begin{aligned} a_n = {} & \frac{q-\epsilon}{1+p+\epsilon}\sum_{i=k}^{\infty}\left(1-(-p-\epsilon)^{-[i/k]}\right) \\ & \times \sum_{j=i}^{\infty}\frac{(j-i+1)\cdots(j-i+m-2)}{(m-2)!}a_{n+j-l}, \quad n \in N(m_1+1). \end{aligned} \tag{2.13}$$

Since $a_n = z_n > 0$ for $n \in N(m_1 - l, m_1)$ and $k < l$, (2.13) implies $a_n > 0$ for $n \in N(m_1 + 1)$, and

$$\Delta^m a_n + (p+\epsilon)\Delta^m a_{n-k} + (q-\epsilon)a_{n-k} = 0, \quad n \in N(m_1 + k + 1), \tag{2.14}$$

this implies $(p+\epsilon, q-\epsilon) \in Y$, since ϵ is arbitrary, we see that $(p, q) \in Y$ by Lemma 2.1, which contradicts with the assumption that every solution of Eq. (1.7) oscillates.

If $k = l$, by (2.12), we have

$$z_n \ge \frac{q-\epsilon}{1+p+\epsilon}\left(1-(-p-\epsilon)^{-1}\right)z_n,$$

which implies $q - \epsilon < p + \epsilon$. Since ϵ is arbitrary, we know that

$$q \le p. \tag{2.15}$$

By (1.6), we see that $q < p$. So by Lemma 2.2, Eq. (1.7) has a positive solution. This is a contradiction.

Case 2 $m = 1$.

In this case, replacing α_n in case 1 by $w_n z_{n-l}$, we can similarly prove that Eq. (1.7) has a positive solution which is impossible.

The proof of Theorem 2.1 is complete by combining Cases 1 and 2.

3. EXISTENCE OF POSITIVE SOLUTIONS

Consider the neutral difference equation

$$\Delta^m(x_n + px_{n-k}) + q_n h(x_{n-l}) = 0, \quad n \in N, \tag{3.1}$$

where $m \in N$ is odd and $\{q_n\}$ is a sequences of nonnegative real numbers,

$$k,\ l \in N, \quad l > k \geq 1, \quad p > 1, \quad h(x) \in C(R, R). \tag{3.2}$$

The next theorem is a partial converse of Theorem 2.1.

THEOREM 3.1 *Assume* (3.2) *holds and there exist positive constants* q, δ *and a positive integer* k^*, *such that*

$$0 \leq q_n \leq q \quad \textit{for } n \in N(k^*); \tag{3.3}$$

and

$$0 \leq h(x) \leq x \ \textit{ for } x \in [0, \delta] \quad \textit{or} \quad 0 \geq h(x) \geq x \ \textit{ for } x \in [-\delta, 0]. \tag{3.4}$$

Suppose $h(x)$ *is nondecreasing in* $x \in [-\delta, \delta]$, *if Eq.* (1.7) *has a nonoscillatory solution, then Eq.* (3.1) *has also a nonoscillatory solution.*

Proof Assume that $m > 1$ and $0 \leq h(x) \leq x$ for $x \in [0, \delta]$. The case where $m = 1$ or $0 \geq h(x) \geq x$ for $x \in [-\delta, 0]$ is similar and the proof will be omitted. Since Eq. (1.7) has a nonoscillatory solution, then the characteristic equation of Eq. (1.7)

$$(\lambda - 1)^m \big(1 + p\lambda^{-k}\big) + q\lambda^{-l} = 0$$

has a positive root λ_0. Obviously, $\lambda_0 < 1$. Set $y_n = \lambda_0^n$, then there exists a $m^* \in N(k^* + l)$ such that

$$0 < y_n \leq \delta \quad \text{for } n \in N(m^* - l).$$

Clearly

$$y_n = \frac{q}{1+p} \sum_{i=k}^{\infty} \big(1 - (-p)^{-[i/k]}\big) \sum_{j=i}^{\infty} \frac{(j-i+1)\cdots(j-i+m-2)}{(m-2)!} y_{n+j-l}$$

this implies

$$y_n \geq \frac{1}{1+p} \sum_{i=k}^{\infty} \big(1 - (-p)^{-[i/k]}\big) \sum_{j=i}^{\infty} \frac{(j-i+1)\cdots(j-i+m-2)}{(m-2)!} q_{n+j} h(y_{n+j-l}) \tag{3.5}$$

By a similar fashion to the proof of Case 1 in Theorem 2.1, it is easy to see that Eq. (3.1) has a positive solution and the proof is complete.

By combining Theorems 2.1 and 3.1, we obtain the following necessary and sufficient condition for the oscillation of all solutions of Eq. (3.1).

THEOREM 3.2 *Assume that* (1.5), (3.2), (3.4) *hold and*

$$0 \leq q_n \leq q = \lim_{n\to\infty} q_n,$$

$h(x)$ *is nondecreasing in* $[-\delta, \delta]$. *Then every solution of Eq.* (3.1) *oscillates if and only if every solution of Eq.* (1.7) *oscillates.*

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