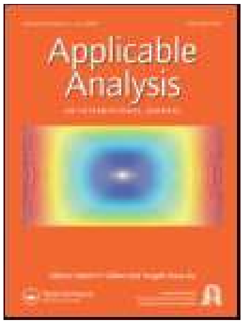

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The Fundamental Singularity in a Shallow Ocean with an Elastic Seabed*

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Abstract. In this paper, the problem of constructing the acoustic pressure field excited by a time-harmonic point source in a shallow ocean over an elastic seabed is considered. This is a transmission boundary value problem for a system of partial differential equations. Two different expansions for the *propagating solutions* are derived. The first expansion is obtained by using Hankel transformation. We symbolically compute the Hankel transform of the propagating solution and then perform inversion using *Mittag-Leffler* decomposition and other spectral analysis techniques. The second expansion is normal mode expansion. We find an appropriate inner product in a suitable function space, which permits us to compute the *generalized Fourier* coefficients. Numerical computations are presented to verify the orthogonality of the set of eigenfunctions with respect to this inner product, and to verify the equivalence of the two expansions.

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KEY WORDS: Acoustic, elastic, Hankel, Mittag Leffler decomposition, inner product

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1 Introduction

In this paper, we study wave propagation in a shallow ocean with an interactive seabed. In particular, we wish to construct the fundamental solution corresponding to an acoustic point source situated in the ocean, where the seabed is taken to be elastic. In many applications, an acoustic source may be idealized as a point source; this is particularly true when our prime concern is the far field. As is well known the knowledge of the fundamental solution, and in particular, the Green's function is useful for treating the direct scattering problem in a wave-guide.

Throughout this paper we assume that the ocean depth and the physical parameters are uniform in the range variable. Consequently, the Green's function due to a point source must be axially symmetric. for convenience, we will work with the cylindrical coordinates (r, θ, z), and without loss generality, we ignore the azimuthal angle θ and assume that the point source is at z_0 on the z axis. Secondly, we assume that the acoustic vibration is small

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enough that we can use the linear model. As a simplification we assume a depth variable sound-speed, but a constant water density ρ_w. The seabed is assumed to be isotropic and heterogeneous, namely the the Lamé coefficients and the density are functions of the depth z. In earlier papers, we investigated this problem when the seabed was homogeneously elastic. [5, 6].

Following is a list of the symbols we will use in this paper.

h — depth of sea floor

G — acoustic pressure in ocean excited by a point source.

u_{zo} — vertical displacement in ocean

c_0 — reference sound speed in the ocean water ($c_0 \approx 1500$m/s)

ρ_w — density of the ocean water ($\rho_w \approx 1000\, kg/m^3$)

ω — angular frequency of point source

z_0 — depth of point source

$n^2(z)$ — depth-dependent index of refraction in ocean

u_r — horizontal displacement of the seabed

u_z — vertical displacement of the seabed

$\{e_{rr}, e_{\phi\phi}, e_{zz}, e_{rz}\}$ — strain tensor of the seabed

e — dilatation of the seabed

$\{\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz}\}$ — strain tensor in seabed.

λ, μ — Lamé coefficients for seabed

ρ — density in elastic seabed

b — thickness of the seabed,

$\mathcal{H}_k f$ — the Hankel transform of function $f(z)$ of the k^{th} order.

2 Governing Equations

In a stratified ocean let $G(r, z)$ be the acoustic Green's function in the ocean, that is it is the acoustic pressure field excited by a time-harmonic point source at $(0, 0, z_0)$. Moreover, let $u_{zo}(r, z)$ be the corresponding vertical displacement. These functions must satisfy [1, 2]

$$(\partial_r + \frac{1}{r})\partial_r G + \partial_z^2 G + k_0^2 n^2(z) G + \frac{\delta(r)\,\delta(z - z_0)}{2\pi r} = 0, \tag{2.1}$$

$$\omega^2 \rho_w u_{zo} + \partial_z G = 0, \quad 0 < z < h, \tag{2.2}$$

where the second relation allows the computation of the displacement from the Green's function. Here $n^2(z)$ is the depth-dependent index of refraction, $k_0 = \omega / c_0$ the reference wave number, ω the frequency, c_0 the reference sound speed in water, and ρ_w the ocean water density. $\lambda = \lambda(z)$ and $\mu = \mu(z)$ are the depth-dependent Lamé coefficients of the seabed, and $\rho = \rho(z)$ is the depth-dependent density. We have the following Hooke's law relations between the displacements $\{u_r, u_z\}$ and the stresses $\{\tau_{rr}, \tau_{rz}, \tau_{\theta\theta}, \tau_{zz}\}$.

$$\begin{aligned} \tau_{rr} &= \lambda e + 2\mu \partial_r u_r, && (2.3) \\ \tau_{\theta\theta} &= \lambda e + \frac{2\mu}{r} u_r, && (2.4) \\ \tau_{zz} &= \lambda e + 2\mu \partial_z u_z, && (2.5) \\ \tau_{rz} &= \mu(\partial_z u_r + \partial_r u_z), && (2.6) \end{aligned}$$

where

$$e = \partial_r u_r + \frac{1}{r} u_r + \partial_z u_z \tag{2.7}$$

is the dilatation. The equations of motion are given by

$$\partial_r \tau_{rr} + \partial_z \tau_{rz} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) + \rho\omega^2 u_r = 0, \tag{2.8}$$

$$\partial_r \tau_{rz} + \partial_z \tau_{zz} + \frac{1}{r}\tau_{rz} + \rho\omega^2 u_z = 0, \tag{2.9}$$

where $\frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) = 2\mu\partial_r(\frac{u_r}{r})$ in view of (2.3)-(2.4).

We now turn to the boundary and transition conditions. At the ocean surface, as the air is sound-soft relative to the water, it is customary to assume the pressure release condition

$$G(r, 0) = 0. \tag{2.10}$$

At the ocean-seabed interface, we assume that there is no separation between the two media, i.e. the vertical displacements of the two media match

$$u_{zo}(r, h^-) = u_z(r, h^+). \tag{2.11}$$

Also we assume that there is no energy stored at the interface, namely, the normal stresses in two media should also match at the interface. Note that in water, the stresses are not dependent on direction, and are equal to the pressure; hence,

$$G(r, h^-) = \tau_{zz}(r, h^+). \tag{2.12}$$

As water can not sustain shear, the interface is a shear-free boundary for the elastic seabed,

$$\tau_{zr}(r, h^+) = 0. \tag{2.13}$$

We assume that the elastic seabed lies on a layer of rigid rock

$$\partial_z \tau_{zz}(r, b^-) \;=\; u_z(r, b^-) \;=\; 0. \tag{2.14}$$

Besides the above conditions, we use a radiation condition which does not permit an incoming wave. The out-going radiation condition for a wave guide is not as convenient as that for the whole space for which we have Sommerfeld condition, or in the electromagnetic case, the Silver-Mueller condition. This is because the far-field in a waveguide is composed of direct radiation and reflected waves (which results in a modification of the phase angle). To accomplish this we take $H_0^{(1)}(\sqrt{\xi}r)$, instead of $H_0^{(2)}(\sqrt{\xi}r)$, to solve the separated (Bessel) equation

$$(\frac{d}{dr} + \frac{1}{r})\frac{dy}{dr} + \xi y = \frac{4\delta(r)}{2\pi r}. \tag{2.15}$$

This implicit radiation condition essentially allows us to perform a generalized Hankel transformation. Here *generalized* means the functions to be transformed are not required to belong to $L^2(0, \infty)$ just as $\mathcal{F}[1] = \delta(\omega)$ in the case of Fourier transforms. We can clearly express the radiation condition by expanding the solution in the far-field, [9].

3 Hankel Transforms and Normal Modal Expansions

It is convenient to denote several Hankel transforms which appear in our work

$$\hat{G}(k^2, z) := \mathcal{H}_0[G(\cdot, z)](k) = \int_0^\infty rJ_0(kr)G(r,z)dr, \quad z \in (0,h) \tag{3.1}$$

$$\hat{u}_r(k^2, z) := \frac{1}{k}\mathcal{H}_1[u_r(\cdot, z)] = \frac{1}{k}\int_0^\infty rJ_1(kr)u_r(r,z)dr, \quad z \in (h,b) \tag{3.2}$$

$$\hat{u}_z(k^2, z) := \mathcal{H}_0[u_z(\cdot, z)] = \int_0^\infty rJ_0(kr)u_z(r,z)dr, \quad z \in (h,b). \tag{3.3}$$

A rigorous mathematical analysis of the Hankel transform of the displacement vector is quite technical, especially for the behavior of these functions in the neighborhood of infinity (see the discussion in the Appendix of [2]); hence, we provide instead some heuristic arguments below. One of the arguments dealing with existence of these transforms is to view the wave number as a limit of a complex quantity in the upper-half plane, i.e. $k_0 = \lim_{\epsilon \to 0^+}(k_0 + \epsilon i)$. The argument for the use of complex wave numbers with positive imaginary part is that this provides viscous damping in the media [4], pp333. With damping, the oscillation exponentially decreases with range. A small imaginary part for k provides enough damping so that we can apply the Hankel transformation to the range-dependent displacements and stresses without concerning ourselves over their decay at ∞. In fact, the formula we have derived for the Hankel inversion will take this consideration into account. With this in mind, we can use the following three formulas that are based on the differentiation properties of Bessel functions.

$$\mathcal{H}_0[(\partial_r + \frac{1}{r})f] = k\mathcal{H}_1[f], \quad \mathcal{H}_1[\partial_r f] = -k\mathcal{H}_0[f] \tag{3.4}$$

$$\mathcal{H}_0((\partial_r + \frac{1}{r})\partial_r f) = -k^2\mathcal{H}_0[f]. \tag{3.5}$$

Applying (3.4), (3.5) to (2.1)-(2.2), we obtain

$$\partial_{zz}G(\hat{k^2}, z) + (k_0^2 n^2(z) - k^2)\hat{G} + \frac{\delta(z - z_0)}{2\pi} = 0, \tag{3.6}$$

$$\omega^2 \rho_w \hat{u}_{zo} + \partial_z \hat{G} = 0, \quad 0 < z < h. \tag{3.7}$$

Applying (3.4) to (2.7), (2.3), (2.5)-(2.6), we get

$$\begin{aligned}
\mathcal{H}_0[e] &= k^2\hat{u}_r + \partial_z\hat{u}_z, &(3.8)\\
\mathcal{H}_0[\tau_{zz}] &= \lambda(k^2\hat{u}_r + \partial_z\hat{u}_z) + 2\mu\partial_z\hat{u}_z &(3.9)\\
\mathcal{H}_1[\tau_{rz}] &= \mu k(\partial_z\hat{u}_r - \hat{u}_z), &(3.10)\\
\mathcal{H}_0[\tau_{rr}] &= (\lambda + 2\mu)k^2\hat{u}_r + \lambda\partial_z\hat{u}_z - 2\mu\mathcal{H}_0[\frac{u_r}{r}]. &(3.11)
\end{aligned}$$

Letting $\mathcal{H}_1$ act on (2.8), $\mathcal{H}_0$ on (2.9) and applying (3.4) to them , we obtain

$$-k\mathcal{H}_0[\tau_{rr}] + \partial_z\mathcal{H}_1[\tau_{rz}] - 2k\mu\mathcal{H}_0[\frac{u_r}{r}] + \omega^2\rho k\hat{u}_r = 0. \tag{3.12}$$

$$k\mathcal{H}_1[\tau_{rz}] + \partial_z\mathcal{H}_0[\tau_{zz}] + \omega^2\rho\hat{u}_z = 0 \tag{3.13}$$

Inserting (3.9)-(3.11) into (3.12)-(3.13) yields

$$\partial_z(\mu\partial_z\hat{u}_r) + [\rho\omega^2 - (\lambda+2\mu)k^2]\hat{u}_r - \mu'\hat{u}_z - (\lambda+\mu)\partial_z\hat{u}_z = 0, \tag{3.14}$$
$$\partial_z((\lambda+2\mu)\partial_z\hat{u}_z) + (\rho\omega^2 - \mu k^2)\hat{u}_z + k^2\lambda'\hat{u}_r + k^2(\lambda+\mu)\partial_z\hat{u}_r = 0. \tag{3.15}$$

Similarly, the boundary transmission conditions are transformed into

$$\hat{G}|_{z=0} = 0, \tag{3.16}$$
$$\hat{u}_{zo}|_{z=h^-} = \hat{u}_z|_{z=h^+}, \tag{3.17}$$
$$\hat{G}|_{z=h^-} = [\lambda(k^2\hat{u}_r + \partial\hat{u}_z) + 2\mu\partial_z\hat{u}_z]\Big|_{z=h^+}, \tag{3.18}$$
$$[\partial_z\hat{u}_r - \hat{u}_z]\Big|_{z=h^+} = 0, \tag{3.19}$$
$$\partial_z\hat{u}_r\Big|_{z=b} = \hat{u}_z\Big|_{z=b} = 0. \tag{3.20}$$

Let us call (3.6), (3.7), (3.14)-(3.20) the Hankel transformed ocean-seabed system. In the case of constant coefficients, we can solve it explicitly. If the coefficients are variable, we can construct a solution to this system and compute it numerically.

We also may find $G(r,z)$ using the normal mode expansion. As is well known, using the separation of variables

$$G(r,z) = g(z)\,H_0^{(1)}(kr),\ u_r(r,z) = k\alpha(z)\,H_1^{(1)}(kr),$$
$$u_z(r,z) = \beta(z)\,H_1^{(1)}(kr) \tag{3.21}$$

for the non-source (homogeneous) equation

$$(\partial_r + \frac{1}{r})\partial_r G + \partial_z^2 G + k_0^2 G = 0$$

coupled with the elasticity equations (2.3)-(2.9) leads to a system that is identical to the Hankel transformed system modulo the term $\delta(z-z_0)$. This new system provides an eigenvalue problem, whose eigenvalues are nothing but the poles of the solution of the transformed equations. For each eigenvalue, the corresponding eigenfunction is called a normal mode solution, or simply a mode. In [?], an eigenvalue problem for slightly different boundary conditions at the bottom $z = b$ was solved numerically, but a procedure for constructing the Fourier coefficients was not given.

We wish to stress that we employ two ways to obtain the Green function, namely using Hankel transforms and normal mode expansions. Moreover, the systems of ordinary differential equations used by these two methods are almost identical. Nevertheless, our methods for solving the two system are quite different. For the Hankel transformed system, we use certain standard integration techniques to perform the inversion; whereas, for the eigenvalue problem, we have to first show the completeness of the set of eigenfunctions in appropriate function spaces. Next we need to find a suitable inner-product under which the set of eigenfunctions is orthogonal, in order to construct the Fourier coefficients of $G(r,z)$. That is the required spectral analysis associated with the separated system must be developed.

4 Hankel Inversion in Constant Coefficient Case

When the coefficients of the ocean-seabed system are constant, the Hankel transformed equations can be solved exactly. Hence, we can perform the Hankel inversion of the solution in a number of steps.

Let us consider the case where $n^2(z) \equiv 1$, and λ, μ, ρ are constants. Then we have the constant compressional and shear wave speeds which are given by

$$c_l = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \text{ and } \quad c_t = \sqrt{\frac{\mu}{\rho}};$$

$$k_l = \frac{\omega}{c_l}, \text{ and } \quad k_t = \frac{\omega}{c_t}$$

where K_l and k_t are the compression and shear wave numbers, respectively. In this case, the Hankel transformed ocean-seabed system (3.6), (3.7), (3.14)-(3.20) can be solved explicitly by using computer algebra. In particular, we have

$$\begin{aligned} \hat{G}(\xi, z, z_0) \quad &= \quad \frac{1}{2\pi}\left(\frac{\hat{L}(\xi, h - z_0)}{\hat{L}(\xi, h)}\frac{\sinh(\sqrt{\xi - k_0^2}z)}{\sqrt{\xi - k_0^2}}\right. \\ &\left. -H(z - z_0)\frac{\sinh(\sqrt{\xi - k_0^2}(z - z_0))}{\sqrt{\xi - k_0^2}}\right), \quad z \in [0, h], z \neq z_0, \end{aligned} \tag{4.1}$$

where $H(\cdot)$ is the Heaviside function, and $\hat{L}(\xi, z)$ is given by

$$\hat{L}(\xi, z) = \hat{A}\cosh\left(\sqrt{\xi - k_0^2}z\right) + \hat{B}\frac{\sinh\left(\sqrt{\xi - k_0^2}z\right)}{\sqrt{\xi - k_0^2}}, \tag{4.2}$$

with

$$\begin{aligned} \hat{A}(\xi) \quad &= \quad (k_t^2 - 2\xi)^2 \cosh\left(\sqrt{\xi - k_l^2}(b - h)\right)\frac{\sinh\left(\sqrt{\xi - k_t^2}(b - h)\right)}{\sqrt{\xi - k_t^2}} \\ &- \quad 4\xi\,(\xi - k_l^2)\frac{\sinh\left(\sqrt{\xi - k_l^2}(b - h)\right)}{\sqrt{\xi - k_l^2}}\cosh\left(\sqrt{\xi - k_t^2}(b - h)\right) \end{aligned} \tag{4.3}$$

$$\hat{B}(\xi) \quad = \quad \sigma k_t^4(\xi - k_l^2)\frac{\sinh\left(\sqrt{\xi - k_l^2}(b - h)\right)}{\sqrt{\xi - k_l^2}}\frac{\sinh\left(\sqrt{\xi - k_t^2}(b - h)\right)}{\sqrt{\xi - k_t^2}}. \tag{4.4}$$

For convenience set $\sigma = \frac{\rho_w}{\rho_s}$ and define

$$\frac{\sinh(\Xi t)}{\Xi} := \begin{cases} \frac{\sin(\nu t)}{\nu}, & \text{if } \Xi = \nu i \\ t, & \text{if } \Xi = 0 \end{cases}.$$

We shall represent the Green's function in terms of a Hankel-inversion by

$$G(r, z) = \int_0^\infty k\,\hat{G}(\xi, z)J_0(kr)dk = \frac{1}{2}\int_0^\infty \hat{G}(\xi, z)J_0(\sqrt{\xi}r)d\xi.$$

To get a far-field representation, we need the Mittag-Leffler decomposition of $\hat{G}(\xi, h)$. To this end, first we continue $\hat{G}(\cdot, z)$ to the complex ξ-plane. For $\Xi = |\Xi|e^{\theta i}$, $-\pi < \theta \leq \pi$, we choose the branch $\sqrt{\Xi} = \sqrt{|\Xi|}e^{\theta i/2}$. Note both $\cosh\sqrt{\Xi}z$ and $\sinh\sqrt{\Xi}z/\sqrt{\Xi}$ are continuous across the branch cut. Hence $\hat{A}$, $\hat{B}$ and $\hat{L}(\cdot, z)$ are entire functions of ξ, and consequently, $\hat{G}$ is a meromorphic function.

Let $\{\xi_j\}_{j=0}^{\infty}$ be the simple zeros of $\hat{L}(\cdot, h)$ and $\{\xi_j^*\}_{j=0}^{N_p}$ be the multiple zeros. To examine the distributions of zeros of $\hat{L}(\xi, z)$, first we note that

$$\frac{(k_t^2 - 2\xi)^2}{\sqrt{\xi - k_t^2}} - 4\xi\sqrt{\xi - k_l^2} = 2(k_l^2 - k_t^2)\sqrt{\xi} + O(1), \quad \text{for } |\xi| \text{ large.} \tag{4.5}$$

When $Re(\sqrt{\xi - k_l^2})$, $Re(\sqrt{\xi - k_t^2})$ and $Re(\sqrt{\xi - k_0^2})$ are large, which is equivalent to saying $Re(\xi)$ is large when $z > 0$, we have

$$\begin{aligned} \hat{L}(\xi, z) &\sim \left(\frac{(k_t^2 - 2\xi)^2}{\sqrt{\xi - k_t^2}} - 4\xi\sqrt{\xi - k_l^2} + \sigma k_t^4 \frac{\sqrt{\xi - k_l^2}}{\sqrt{\xi - k_t^2}\sqrt{\xi - k_0^2}}\right) \\ &\quad \times \exp(\sqrt{\xi - k_l^2}(b - h) + \sqrt{\xi - k_t^2}(b - h) + \sqrt{\xi - k_0^2}z) \\ &\sim 2(k_l^2 - k_t^2)\sqrt{\xi}\exp\left((\sqrt{\xi - k_l^2} + \sqrt{\xi - k_t^2})(b - h) + \sqrt{\xi - k_0^2}z\right), \end{aligned} \tag{4.6}$$

which implies that there is a $C(z)$ such that $|\hat{L}(\xi, z)| \neq 0$ for $Re(\sqrt{\xi}) > C(z)$. In particular,

$$Re(\sqrt{\xi_j}), Re(\sqrt{\xi_j^*}) < C(h) \tag{4.7}$$

which defines a region in which the zeros are distributed. To examine the asymptotic behavior of $\{\xi_j\}$, $\{\xi_j^*\}$, let us assume that $\{\xi_j\}$, $\{\xi_j^*\}$ has been ordered so that if $Re(\xi_j) > Re(\xi_l)$ then $j < l$. Now restrict the range to $Re(\xi) < a$. Then all sheet functions involved are bounded by some $C_1(a)$. Starting with the expression of $\hat{A}$ and then adding and subtracting a term and using (4.5), we have

$$\begin{aligned} \hat{A}(\xi) &= (2(k_l^2 - k_t^2)\sqrt{\xi} + O(1)) \cosh\left(\sqrt{\xi - k_l^2}(b - h)\right) \sinh\left(\sqrt{\xi - k_t^2}(b - h)\right) \\ &\quad + 4\xi\sqrt{\xi - k_l^2}\sinh\left((\sqrt{\xi - k_t^2} - \sqrt{\xi - k_l^2})(b - h)\right) \\ &= (2(k_l^2 - k_t^2)\sqrt{\xi} + O(1)) \cosh\left(\sqrt{\xi - k_l^2}(b - h)\right) \sinh\left(\sqrt{\xi - k_t^2}(b - h)\right) \\ &\quad + 4\xi\sqrt{\xi - k_l^2}\sinh\left(\frac{k_l^2 - k_t^2}{\sqrt{\xi - k_t^2} + \sqrt{\xi - k_l^2}}(b - h)\right) \\ &= (2(k_l^2 - k_t^2)\sqrt{\xi} + O(1)) \cosh\left(\sqrt{\xi - k_l^2}(b - h)\right) \sinh\left(\sqrt{\xi - k_t^2}(b - h)\right) \\ &\quad + 4\xi\sqrt{\xi - k_l^2}\left(\frac{(k_l^2 - k_t^2)(b - h)}{\sqrt{\xi - k_t^2} + \sqrt{\xi - k_l^2}}(1 + O(\frac{1}{(\sqrt{\xi - k_t^2} + \sqrt{\xi - k_l^2}})^3\right) \\ &= 2(k_l^2 - k_t^2)(b - h)\xi + \sqrt{\xi}O_a(1). \end{aligned} \tag{4.8}$$

It is easy to see that $\hat{B}(\xi)$ is bounded. Hence,

$$\hat{L}(\xi, z) = 2(k_l^2 - k_t^2)(b-h)\xi \cosh(\sqrt{\xi - k_0^2} z) + \sqrt{\xi} O_a(1), \quad \text{for } Re(\sqrt{\xi}) < a. \tag{4.9}$$

Using this estimate, we conclude that the $\{\xi_j\}$ are close to the zeros of $\cosh(\sqrt{\xi - k_0^2} h)$, i.e,

$$\xi_j \sim k_0^2 - \left(\frac{1}{h}(j + \frac{1}{2})\pi\right)^2. \tag{4.10}$$

Moreover, from the estimate (4.9), we can show that there are at most finitely many complex zeros and multiple zeros of $\hat{L}(\xi, h)$. To see this, first note that

$$\hat{G}(\bar{\xi}, z) = \hat{G}(\bar{\xi}, z), \quad \hat{L}(\bar{\xi}, z) = \hat{L}(\bar{\xi}, z),$$

which implies that the conjugate of a zero is also a zero. Now let us consider the contour O_j consisting of $Re(\xi) = k_0^2 - (j\pi/h)^2$ $Re(\xi) = k_0^2 - ((j+1)\pi/h)^2$ $Re(\sqrt{\xi - k_0^2}) = C(h)$ for large j. From (4.9), we have , for ξ on O_j,

$$\begin{aligned} & |\hat{L}(\xi, h) - 2(k_l^2 - k_t^2)(b-h)\xi \cosh\left(\sqrt{\xi - k_0^2} h\right)| \\ = \; & |\sqrt{\xi} O_{C(h)}(1)| \\ < \; & |2(k_l^2 - k_t^2)(b-h)\xi \cosh\left(\sqrt{\xi - k_0^2} h\right)|. \end{aligned} \tag{4.11}$$

By Rouché's theorem, we conclude that $\hat{L}(\xi, h)$ has as the same number of zeros as $2(k_l^2 - k_t^2)(b-h)\xi \cosh\left(\sqrt{\xi - k_0^2} h\right)$. Obviously, the latter has only one zero inside O_j, i.e, $k_0^2 - \left(\frac{1}{h}(j + \frac{1}{2})\pi\right)^2$; hence, $\hat{L}(\xi, h)$ also has just one there. This is the unique zero of $\hat{L}(\xi, h)$ inside O_j, so it is a simple zero, and, consequently, must be real. Otherwise, its conjugate would also be inside O_j, because O_j is symmetric about the real axis and this would contradict Rouché's theorem. We note that $\cup O_j$ covers the region in which, for large j, lie the $\{\xi_j\}$. Therefore, for sufficiently large j, ξ_j can only be real and simple. In fact, we will show later on that it is not likely for $\hat{L}(\xi, h)$ to have complex zeros in the whole ξ-plane, precisely because the set of frequencies at which this happens is countable.

Now as $\hat{G}(\xi, z)$ has a Laurent expansion at each of its poles, ξ_j or ξ_j, we construct a function by summing up the principle parts of all these Laurent expansions as

$$\hat{G}_p(\xi, z) = \sum_{j=0}^{\infty} \frac{c_j(z)}{\xi - \xi_j} + \sum_{j=0}^{N_p} \sum_{l=1}^{m_j} \frac{c_{jl}^*(z)}{(\xi - \xi_j^*)^l}, \tag{4.12}$$

where the coefficients $c_j(z)$ are the residues of $\hat{G}(\xi, z, z_0)$ at the ξ_j. From (4.12) by using the residue formula, we obtain

$$c_j = \frac{\hat{L}(\xi_j, h - z_0)}{2\pi \partial_\xi \hat{L}(\xi_j, h)} \frac{\sinh(\sqrt{\xi_j - k_0^2} z)}{\sqrt{\xi_j - k_0^2}}. \tag{4.13}$$

The asymptotic behavior of $\hat{L}(\cdot, h)$ and ξ with respect to large j, which is given namely by (4.9) and (4.10), implies

$$c_j \sim \frac{\cosh((\sqrt{\xi_j - k_0^2}(h - z_0)}{\sinh(\sqrt{\xi_j - k_0^2} h) \frac{h}{2\sqrt{\xi_j - k_0^2 z)}}} \frac{\sinh(\sqrt{\xi_j - k_0^2} z)}{(\sqrt{\xi_j - k_0^2}} = O(1), \quad \text{as } j \to \infty, \tag{4.14}$$

from which we conclude the convergence of the series. Hence, by the Mittag-Leffler decomposition, the function $f(\xi) := \hat{G}(\xi, z) - \hat{G}_p(\xi, z)$ is an entire function in the ξ-plane. We are going to show $f(z) \equiv 0$. It follows from (4.6) and (4.1) that as $Re(\sqrt{\xi})$ is larger than $C(z)$ and $C(h - z_0)$,

$$\hat{G}(\xi, z) \sim \begin{cases} \frac{1}{2\pi\sqrt{\xi - k_0^2}} exp(\sqrt{\xi - k_0^2}(-|z - z_0|)), & \text{if } z < z_0, \\ O(1), & \text{if } z > z_0 \end{cases} . \tag{4.15}$$

The estimate (4.7) also implies $\hat{G}_p$ is bounded for $Re(\sqrt{\xi} > C(h)$. Consequently, $f(\xi)$ is bounded for $Re(\sqrt{x\xi}) \geq a := max(C(z), C(h - z_0), C(h))$ On the other hand, it is easy to see that for all sufficiently large j, on the segment S_j:$Re(\xi) = -(j\pi/h)$, $Re(\sqrt{x}) < a$, $\hat{L} = 2(k_t^2 - k_l^2)(b - h)\xi \cosh(\sqrt{\xi - k_0^2}h) + O_a(1)\sqrt{\xi} = O(\xi)$ from which we can see $\hat{G}(\xi, h)$ is bounded on S_j uniformly in j. In a similar way we show that $\hat{G}_p(\xi, h)$ is also uniformly bounded, and consequently so is $f(z)$. Now we argue in a similar fashion as in the proof of Liouville's theorem, where $f'(z)$ is represented by a Cauchy integral over the contour given by $Re(\sqrt{\xi})$ connected by S_j. We allow j to approach ∞ and conclude $f'(z) \equiv 0$, which implies $f(z)$ is a constant. But since $\lim_{\xi \to +\infty} \hat{G}(\xi, h) = \lim_{\xi \to +\infty} \hat{G}_p(\xi, h) = 0$, it follows that $f(z) \equiv 0$.

Therefore, we have the expansion $\hat{G}(\xi, h) = \hat{G}_p(\xi, h)$, and from which we need to discuss the following set of Hankel inversion formulae:

(1) For complex or negative ξ_j^- or ξ_j^{*-}, the Hankel inversion of the term $1/(\xi - \xi_j^-)$ or $1/(\xi - \xi_j^{*-})$ exists in the usual sense, in particular, if $\xi_j^- < 0$

$$\mathcal{H}_0^{-1}\left(\frac{1}{\xi - \xi_j^-}\right) = \int_0^\infty \frac{kJ_0(kr)}{k^2 - \xi_j} dk = \frac{1}{2}\int_0^\infty \frac{J_0(\sqrt{\xi_j^-}r)}{\xi - \xi_j^-} d\xi = K_0(\sqrt{-\xi_j^-}r),$$

where $K_0(\cdot)$ is the second kind of modified Bessel function of zero order [8]. The inversion of the $1/(\xi - \xi_j^{*-})^{d_j}$ can obtained by differentiating the above $d_j - 1$ times with respect to ξ_j^-; the result is then expressed in terms of the $K_n, n = 0, 1, \cdots, d_j - 1$. Since the $K_n(x)$ decrease exponentially, these negative zeros actually don't make a significant contribution to the far-field.

(2) For positive zeros ξ_j^+, the Hankel inversion of $1/(\xi - \xi_j^+)$ exists and

$$\mathcal{H}_0^{-1}\left(\frac{1}{\xi - \xi_j^+}\right) = \frac{1}{2}\int_0^\infty \frac{J_0(\sqrt{\xi}r)}{\xi - \xi_j^+} d\xi = \int_0^\infty \frac{kJ_0(kr)}{k^2 - \xi_j^+} dk = \frac{\pi i}{2} H_0^{(1)}(\sqrt{\xi_j^+}r). \tag{4.16}$$

This formula is well-known [8], and is just what is needed to construct the fundamental singular solution of the two-dimensional Helmholtz equation. The Cauchy principle value integral only gives the imaginary part of $H_0^{(1)}(\sqrt{\xi_j^+}r)$, namely, the Bessel function of the second kind Y_0, while the real part J_0 must then be added to satisfy the radiation condition. This formula holds in the sense $k_0 = \lim_{\epsilon_1 \to 0^+}(k_0 + \epsilon_1)$. Another point of view of this formula is that it is nothing but the inversion of Hankel transform of Bessel's equation (2.15). The terms arising from the inversion of the $\frac{c_j}{\xi - \xi_j^+}$ constitute the main contribution to the far-field. The zeros of analytic functions are isolated; moreover, since they are real and bounded above by k_0^2, the positive zeros, $\{\xi_j^+\}$, are finite in number.

(3) For positive ξ_j^{*+}, the Hankel inversion of $1/(\xi - \xi_j^{*+})^{d_j}$ does not exist because the integral $\int_0^\infty \frac{kJ_0(kr)}{(k^2 - \xi_j^{*+})^{d_j}} dk$ does not exist in any sense.

(4) If the $\xi_j = 0$ or $\xi_j^* = 0$, the resulting terms $1/\xi$ or $1/\xi^{d_j}$ have no Hankel inversion for the integrals $\int_0^\infty \frac{kJ_0(kr)}{k^2}dk$ and $\int_0^\infty \frac{kJ_0(kr)}{k^{2d_j}}dk$ do not exist in any sense.

Therefore, if and only if there are no positive multiple zeros of $\hat{L}(\cdot, h)$ and $\hat{L}(0, h) \neq 0$, can we invert $\hat{G}(\xi, z)$. In particular, if there are no $\{\xi_j^*\}$ terms, we obtain the following concise formula

$$\begin{aligned} G(r,z) &= = \sum_{j=0}^{\infty} \frac{\hat{L}(\xi_j, h-z_0)}{\partial_\xi \hat{L}(\xi_j, h)} \frac{\sinh(\sqrt{\xi_j - k_0^2}z)}{\sqrt{\xi_j - k_0^2}} \frac{i}{4\pi i} \int_0^\infty \frac{J_0(\sqrt{\xi}r)}{\xi - \xi_j} d\xi \\ &= \frac{i}{4} \sum_{j=0}^{\infty} \frac{\hat{L}(\xi_j, h-z_0)}{\partial_\xi \hat{L}(\xi_j, h)} \frac{\sinh(\sqrt{\xi_j - k_0^2}z)}{\sqrt{\xi_j - k_0^2}} H_0^{(1)}(\sqrt{\xi}r), \end{aligned} \tag{4.17}$$

where

$$H_0^{(1)}(\sqrt{\xi}r) = \frac{2}{\pi i} K_0(\sqrt{-\xi}r), \text{ for } \xi < 0.$$

By expanding $\hat{L}(\xi_j, h - z_0)$ and using $\hat{L}(\xi_j, h) = 0$, we have

$$\begin{aligned} &\hat{L}(\xi_j, h-z_0) = \hat{A}(\xi_j) \cosh\left(\sqrt{\xi_j - k_0^2}(h-z_0)\right) + \hat{B}(\xi_j) \frac{\sinh\left(\sqrt{\xi_j - k_0^2}(h-z_0)\right)}{\sqrt{\xi_j - k_0^2}} \\ &= \hat{L}(\xi, h) \cosh(\sqrt{\xi_j - k_0^2}z_0) \\ &- \left(\hat{A}(\xi_j)(\xi_j - k_0^2) \frac{\sinh(\sqrt{\xi_j - k_0^2}h)}{\sqrt{\xi_j - k_0^2}} + \hat{B}(\xi_j) \cosh(\sqrt{\xi_j - k_0^2}h) \right) \frac{\sinh(\sqrt{\xi_j - k_0^2}z_0)}{\sqrt{\xi_j - k_0^2}} \\ &= \left(\hat{A}(\xi_j)(\xi_j - k_0^2) \frac{\sinh(\sqrt{\xi_j - k_0^2}h)}{\sqrt{\xi_j - k_0^2}} + \hat{B}(\xi_j) \cosh(\sqrt{\xi_j - k_0^2}h) \right) \frac{\sinh(\sqrt{\xi_j - k_0^2}z_0)}{\sqrt{\xi_j - k_0^2}} \end{aligned}$$

we have,

$$G(r,z) = \frac{i}{4} \sum_{j=0}^{\infty} a_j \frac{\sinh(\sqrt{\xi_j - k_0^2}z_0)}{\sqrt{\xi_j - k_0^2}} \frac{\sinh(\sqrt{\xi_j - k_0^2}z)}{\sqrt{\xi_j - k_0^2}} H_0^{(1)}(\sqrt{\xi}r) \tag{4.18}$$

where

$$a_j = \frac{-1}{\partial_\xi \hat{L}(\xi_j, h)} \left(\hat{A}(\xi_j)(\xi_j - k_0^2) \frac{\sinh(\sqrt{\xi_j - k_0^2}h)}{\sqrt{\xi_j - k_0^2}} + \hat{B}(\xi_j) \cosh(\sqrt{\xi_j - k_0^2}h) \right) \tag{4.19}$$

does not depend on the variables r, z, z_0. Hence (4.18) exhibits the symmetry between z and z_0 in $G(r, z)$. In general, as the summation above has a only a finite number of terms, we rewrite this as

$$G(r,z) = \frac{i}{4} \sum_{\xi_j > 0} a_j \frac{\sinh(\sqrt{\xi_j - k_0^2}z_0)}{\sqrt{\xi_j - k_0^2}} \frac{\sinh(\sqrt{\xi_j - k_0^2}z)}{\sqrt{\xi_j - k_0^2}} H_0^{(1)}(\sqrt{\xi}r) + O(e^{-\epsilon_1 r})$$

We may summarize the above discussion with the following existence theorem for the propagating solutions.

Theorem For the frequencies ω such that the following hold

$$\begin{aligned} E_1 &:= \hat{L}(0,h) = -k_t^3 \sin(k_t(b-h))\cos(k_l(b-h))\cos(k_0 h) \\ &+\frac{\beta k_l}{k_0}\sin(k_l(b-h))\sin(k_0 h)) \neq 0, \\ E_2 &:= |\hat{L}(\xi,h)| + |\frac{\partial}{\partial\xi}\hat{L}(\xi,h)| \neq 0 \text{ for } \xi > 0, \end{aligned} \tag{4.20}$$

the Green's function, $G(r,z)$, exists.

In the case $\sigma = 0$, i.e. the completely-reflecting seabed, the first condition should be replaced by

$$\cos(k_0 h) \neq 0$$

because the others terms will be canceled by the terms in $\hat{L}(\xi, h - z_0)$.

We call the frequencies failing to satisfy the above conditions the exceptional frequencies, and these are denoted by $\{\omega_j\}$. It is easy to see that the set of exceptional frequencies is countable, for $\hat{L}$ is also an analytic function of ω if we replace k_j by ω/c_j for $j = 0, t, l$.

These exceptional frequencies form the discrete spectrum of the Helmholtz equation. They are the natural frequencies of the ocean-seabed system. While they are to be avoided for the direct scattering problem, they have significant utility for the problem of identifying physical parameters of the seabed such as h, b, μ, λ, ρ. This may be accomplished by making use of resonance. Note that in the term

$$\frac{\hat{L}(\xi_j, h - z_0)}{\partial_\xi \hat{L}(\xi_j, h)} \frac{\sinh(\sqrt{\xi_j - k_0^2}z)}{\sqrt{\xi_j - k_0^2}} H_0^{(1)}(\sqrt{\xi_j}r)$$

when $\partial_\xi \hat{L}(\xi_j, h)$ or ξ_j is very small, which means the frequency is very close to one of natural frequencies ,the term becomes the dominant term in the summation. Hence, because of the resonant phenomenon it is easy to observe the corresponding natural frequency. If sufficiently many natural frequencies are known, say $\{\omega_j\}_{j=1}^N$, $N > 6$, then it is possible to use them to solve $E_1 = E_2 = 0$ backwards for $\lambda, \mu, h, b, \beta$, etc. As the implementation details involves spectral analysis and signal processing techniques we do not pursue this topic.

Finally, we verify our result for the completely-reflecting seabed. In this case, we take $\beta = \rho_w/\rho_s = 0$; hence, $B = 0$ and

$$\frac{\hat{L}(\xi, h - z_0)}{\hat{L}(\xi, h)} = \frac{\cosh(\sqrt{\xi - k_0^2}(h - z_0))}{\cosh(\sqrt{\xi - k_0^2}h)} = \frac{\cos(\sqrt{k_0^2 - \xi}(h - z_0))}{\cos(\sqrt{k_0^2 - \xi}h)}.$$

Since the zeros of $\cos(\sqrt{k_0^2 - \xi}h)$, ξ_j satisfy

$$\sqrt{k_0^2 - \xi_j}h = (j + \frac{1}{2})\pi,$$

and

$$a_j = \frac{2(k_0^2 - \xi_j)}{h}$$

it follows from (4.18) that

$$G(r,z)=\frac{i}{2h}\sum_{j=-\infty}^{\infty}\sin[(j+\frac{1}{2})\frac{\pi z_0}{h})\sin((j+\frac{1}{2})\frac{\pi z}{h})\,H_0^{(1)}(\sqrt{k_0^2-((j+\frac{1}{2})\pi/h)^2}r).$$

which is identical the representation in Ahluwalia and Keller's [2].

We will perform numerical computations for this constant coefficient case in a subsequent paper together with the near field approximation when we need it to solve a scattering problem there.

5 Construction of the Solution to the Hankel Transformed Ocean Seabed System with variable coefficients and Hankel Inversion

We use the undetermined coefficient method to construct solution of the Hankel transformed ocean-seabed system. To this end, we denote by $X(k^2,z,c)$ the solution to the initial value problem

$$X''(z)+(k_0^2n^2(z)-k^2)X(z)=0,\quad z\in(0,h) \tag{5.1}$$

$$X(c)=0,\quad X'(c)=1. \tag{5.2}$$

Because $\hat{G}(k^2,z)$ satisfies (3.6)-(3.16), it must be expressed as

$$\hat{G}(k^2,z)=\frac{1}{2\pi}[C_1X(k^2,z,0)-H(z-z_0)X(k^2,z,z_0)] \tag{5.3}$$

for some C_1. Next, denote by $\{Y_1(k^2,z),Z_1(k^2,z)\}$ the solution to the initial value problem

$$(\mu Y')'+[\rho\omega^2-(\lambda+2\mu)k^2]Y-\mu' Z-(\lambda+\mu)Z'=0, \tag{5.4}$$

$$((\lambda+2\mu)Z')'+(\rho\omega^2-\mu k^2)Z+k^2\lambda' Y+k^2(\lambda+\mu)Y=0. \tag{5.5}$$

$$Y'(b)=Z(b)=0, \tag{5.6}$$

$$Y(b)=0,\quad Z'(b)=1 \tag{5.7}$$

and $\{Y_2(k^2,z),Z_2(k^2,z)\}$ the solution to (5.4)-(5.6) and

$$Y'(b)=0,\quad Z(b)=1 \tag{5.8}$$

Then the solution $\{\hat{u}_r,\hat{u}_z\}$ to (3.14), (3.15), (3.20) can be written as

$$\hat{u}_r = \frac{1}{2\pi\omega^2\rho_w}[C_2Y_1(k^2,z)+C_3Y_2(k^2,z)] \tag{5.9}$$

$$\hat{u}_z = \frac{1}{2\pi\omega^2\rho_w}[C_2Z_1(k^2,z)+C_3Z_2(k^2,z)] \tag{5.10}$$

Inserting (5.3), (5.9) and (5.10) into (3.17)-(3.19), we obtain a linear system for C_1, C_2 and C_3

$$\mathbf{A}\{C_1,C_2,C_3\}^\top=\mathbf{b},$$

where the matrix $\mathbf{A}$ and vector $\mathbf{b}$ are

$$\mathbf{A}=\begin{pmatrix} \partial_z X(\xi,h,0) & Z_1(\xi,h) & (h)Z_2(\xi,h)) \\ X(\xi,h,0) & t_1\xi Y_1(\xi,h)+t_2\partial_z Z_1(\xi,h) & t_1\xi Y_2(\xi,h)+t_2\partial_z Z_2(\xi,h) \\ 0 & Y_1(\xi,z)-Z_1(\xi,h) & Y_2(\xi,z)-Z_2(\xi,h)] \end{pmatrix} \tag{5.11}$$

$$\mathbf{b} = \begin{pmatrix} \partial_z X(\xi, h, z_0) \\ X(\xi, h, z_0) \\ 0 \end{pmatrix}, \tag{5.12}$$

and

$$\xi = k^2, \quad t_1 = -\frac{\lambda(h)}{\omega^2 \rho_w}, \quad t_2 = -\frac{\lambda(h) + 2\mu(h)}{\omega^2 \rho_w}.$$

Therefore we can represent $\hat{L}(\xi, c)$ as

$$\hat{L}(\xi, c) = \begin{pmatrix} \partial_z X(\xi, h, h-c) & A_{12} & A_{13} \\ X(\xi, h, h-c) & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{pmatrix}. \tag{5.13}$$

Then we have $\hat{L}(\xi, h) = |\mathbf{A}|$ and $C_1 = \frac{\hat{L}(\xi, h-z_0)}{\hat{L}(\xi, h)}$. The substitution of C_1 into (5.3) yields

$$\hat{G}(\xi, z) = \frac{1}{2\pi}\left(\frac{\hat{L}(\xi, h-z_0)}{\hat{L}(\xi, h)} X(\xi, z, 0) - H(z - z_0) X(\xi, z, z_0)\right). \tag{5.14}$$

Now we can perform the same Hankel inversion procedure as in the previous section, but the asymptotic analysis is even more difficult. A rigorous proof, however, might be based on transmutation theory. We do not do this here, but we will establish the usual theorems concerning the existence of propagating solutions.

Theorem 2.2.2 For frequencies ω such that

$$E_1 := \hat{L}(0, h) \neq 0, \tag{5.15}$$

$$E_2 := |\hat{L}(\xi, h)| + |\frac{\partial}{\partial \xi}\hat{L}(\xi, h)| \neq 0 \text{ for } \xi > 0, \tag{5.16}$$

the propagating solution $G(r, z)$ exists, and

$$G(r, z, z_0) = \frac{i}{4}\sum_{j=0}^{M} \frac{\hat{L}(\xi_j, h-z_0)}{\partial_\xi \hat{L}(\xi_j, h)} X(\xi_j, z, 0) H_0^{(1)}(\sqrt{\xi_j} r) + O(e^{-\epsilon_2 r}). \tag{5.17}$$

In particular, if $\hat{L}(\xi, h)$ has no multiple zeros, then

$$G(r, z, z_0) = \frac{i}{4}\sum_{j=0}^{\infty} \frac{\hat{L}(\xi_j, h-z_0)}{\partial_\xi \hat{L}(\xi_j, h)} X(\xi_j, z, 0) H_0^{(1)}(\sqrt{\xi_j} r). \tag{5.18}$$

The numerical implementation for constructing $G(r, z)$ based on this analytical representation of the solution is clear. We solve numerically the two ordinary differential systems to construct the function $\hat{L}(\xi, c)$. We then use a brute-force search for the poles and a Newton iteration to find the $\{\xi_j\}$ and the coefficients $\{\frac{\hat{L}(\xi_j, h-z_0)}{\partial_\xi \hat{L}(\xi_j, h)}\}$.

If we use the normal mode expansion method, it is clear that the characteristic equation is nothing more than $\hat{L}(\xi, h) = |\mathbf{A}| = 0$. Therefore, the eigenvalues are just the poles of $\hat{G}(\xi, z)$.

6 Appropriate Inner Products and Construction of the Fourier Coefficients

Suppose we used the normal mode expansion only, then we would not know the Fourier coefficients of the expansion to be $\frac{\hat{L}(\xi_j, h-z_0)}{\partial_\xi \hat{L}(\xi_j, h)}$. Now we try to construct the coefficients by studying the eigenvalue problem. The important thing we have to do is to construct an appropriate inner-product under which the set of eigen-functions is orthogonal. Only in this way, can we find the Fourier coefficients of the normal-mode expansion of $G(r,z)$.

Let $\{\xi_n\}$ be the set of eigenvalues and $\{(g_n, \alpha_n, \beta_n)\}$ be the corresponding eigenfunctions normalized by $g_n'(0) = 1$. Then from (3.14)-(3.15), we have

$$g_n'' + (k_0^2 n^2(z) - \xi_n) g_n = 0, \quad \text{for } 0 < z < h, \tag{6.1}$$

$$(\mu \alpha_n')' + [\rho\omega^2 - (\lambda + 2\mu)\xi_n]\alpha_n - \mu' \beta_n - (\lambda + \mu)\beta_n' = 0,\ h < z < b \tag{6.2}$$

$$((\lambda + 2\mu)\beta_n')' + (\rho\omega^2 - \mu\xi_n^2)\beta_n + \xi_n \lambda' \alpha_n + \xi_n(\lambda + \mu)\alpha_n' = 0.\ h < z < b \tag{6.3}$$

$$g_n(0) = 0, \tag{6.4}$$

$$g_n'(h) + \omega^2 \rho_w \beta_n(h) = 0, \tag{6.5}$$

$$g_n(h) = \xi_n \lambda(h)\alpha_n(h) + (\lambda(h) + 2\mu(h))\beta_n'(h), \tag{6.6}$$

$$\alpha_n'(h) - \beta_n(h) = 0, \tag{6.7}$$

$$\alpha_n'(b) = \beta_n(b) = 0. \tag{6.8}$$

For each equation above, we have a conjugate form and also we can change the index from n to m. Let us use the shorter notation $\int_b^a f(z)dz = \int_b^a f$. Multiplying (6.1) by $\bar{g}_m$, integrating by parts over $[0, h]$ and using (6.4) and (6.5), we obtain

$$-\omega^2 \rho_w \beta_n(h)\bar{g}_m(h) - \int_0^h g_n' \bar{g}_m' - \xi_n \int_0^h g_n \bar{g}_m(z) + k_0^2 \int_0^h n^2 g_n \bar{g}_m = 0. \tag{6.9}$$

By multiplying (6.2) by $\bar{\alpha}_m$ and integrating by parts over $[h, b]$ and using (6.8) and (6.7), we have

$$\begin{aligned} &- \int_h^b \mu \alpha_n' \bar{\alpha}_m' + \int_h^b \rho\omega^2 \alpha_n \bar{\alpha}_m - \xi_n \int_h^b (\lambda + 2\mu)\alpha_n \bar{\alpha}_m \\ &+ \int_h^b (\mu \beta_n \bar{\alpha}_m' - \lambda \beta_n' \bar{\alpha}_m) = 0. \end{aligned} \tag{6.10}$$

By multiplying (6.3) by $\bar{\beta}_m$ and performing the same procedure as the above, but making use of (6.6), we obtain

$$\begin{aligned} &-g_n(h)\bar{\beta}_m(h) - \int_h^b (\lambda + 2\mu)\beta_n' \bar{\beta}_m' + \int_h^b (\rho\omega^2 - \mu\xi_n)\beta_n \bar{\beta}_m \\ &+\xi_n \int_h^b (\mu \alpha_n' \bar{\beta}_m - \lambda \alpha_n \bar{\beta}_m') = 0. \end{aligned} \tag{6.11}$$

In what follows let us use the notation

$$A_{nm} = -\int_0^h g_n' \bar{g}_m' + k_0^2 \int_0^h n^2 g_n \bar{g}_m,$$

$$B_{nm} = \int_0^h g_n \bar{g}_m,$$

$$
\begin{aligned}
C_{nm} &:= \int_h^b (-\mu\alpha_n'\bar{\alpha}_m' + \rho\omega^2\alpha_n\bar{\alpha}_m),\\
D_{nm} &:= \int_h^b (\lambda+2\mu)\alpha_n\bar{\alpha}_m,\\
E_{nm} &:= \int_h^b (\mu\beta_n\bar{\alpha}_m' - \lambda\beta_n'\bar{\alpha}_m),\\
F_{nm} &:= \int_h^b (-(\lambda+2\mu)\beta_n'\bar{\beta}_m' + \rho\omega^2\beta_n\bar{\beta}_m),\\
G_{nm} &:= \int_h^b \mu\beta_n\bar{\beta}_m,\\
H_{nm} &= \beta_n(h)\bar{g}_m(h).
\end{aligned}
$$

Note that all of these quantities except E_{nm} and H_{nm} satisfy the symmetry relation

$$\bar{\Xi}_{nm} = \Xi_{mn}, \quad \Xi_{nn} \text{ is real.} \tag{6.12}$$

Then the relations (6.9)-(6.11) become

$$-\omega^2\rho_w H_{nm} + A_{nm} - \xi_n B_{nm} = 0, \tag{6.13}$$
$$C_{nm} - \xi_n D_{nm} - E_{nm} = 0, \tag{6.14}$$
$$F_{nm} - \xi_n G_{nm} + \xi_n \bar{E}_{mn} - \bar{H}_{mn} = 0. \tag{6.15}$$

Now let us interchange the indices n and m, and then conjugate each term and use (6.12) to obtain

$$-\omega^2\rho_w \bar{H}_{mn} + A_{nm} - \bar{\xi}_m B_{nm} = 0, \tag{6.16}$$
$$C_{nm} - \bar{\xi}_m D_{nm} - \bar{E}_{mn} = 0, \tag{6.17}$$
$$F_{nm} - \bar{\xi}_m G_{nm} + \bar{\xi}_m E_{nm} - H_{nm} = 0. \tag{6.18}$$

By combining (6.13)-(6.18) using Maple, we can obtain

$$(\xi_n - \bar{\xi}_m)T(m,n) = 0, \tag{6.19}$$

where

$$
\begin{aligned}
T(m,n) &= B_{nm} + \omega^2\rho_w G_{nm} - \omega^2\rho_w C_{nm}\\
&= \int_0^h g_n\bar{g}_m + \omega^2\rho_w \int_h^{b+h} \mu\beta_n\bar{\beta}_m + \omega^2\rho_w \int_h^{b+h} (\mu\alpha_n'\bar{\alpha}_m' - \rho\omega^2\alpha_n\bar{\alpha}_m).
\end{aligned} \tag{6.20}
$$

Therefore if $T(n,n) \neq 0$, then ξ_n is real. But even though it is uncertain whether this condition is true for an arbitrary frequency, we are sure it is true for small frequencies, while for $\omega = 0$ it is obvious. The answer is that $T(n,n) \neq 0$ for all n except for all but a countable number of frequencies. The reason is that for each $n, T(n,n)$ is an analytic function of ω , and, moreover, $T(n,n) \neq 0$ when $\omega = 0$, for all n. This means that since $T(n,n)$ is a nontrivial analytic function of ω, there are at most a countable number of ω for which $T(n,n)$ is zero. A rigorous proof of this requires several complex variable theory because we are dealing a multiple-parameter spectral analysis problem. That is the exceptional frequencies are constructed from the time domain spectrum which are coupled to the spatial eigenvalues $\{\xi_n\}$ [3, 7].

Let us suppose that ω is such that $T(n,n) \neq 0$, for all n. Then $T(m,n)$ can be used as an inner-product, because according to (6.20), $T(m,n) = 0$ for all $m \neq n$.

Now we are going to construct the Fourier coefficients $\{F_n\}$ assuming that $G(r,z)$ has normal mode expansion and $T(n,n) \neq 0$ for all n. Using (3.21), we have

$$G(r,z) = \sum_{n=0}^{\infty} F_n g_n(z) H(\sqrt{\xi_n} r), \quad 0 < z < h, \tag{6.21}$$

$$u_r(r,z) = \sum_{n=0}^{\infty} F_n \sqrt{\xi_n} \alpha_n(z) H(\sqrt{\xi_n} r), \quad h < z < b+h, \tag{6.22}$$

$$u_z(r,z) = \sum_{n=0}^{\infty} F_n \beta_n(z) H(\sqrt{\xi_n} r), \quad h < z < b. \tag{6.23}$$

We substitute (6.21) into the equation (2.1), using (6.1) and the differential equation for the Hankel functions

$$(\partial_{rr} + \frac{1}{r}\partial_r + \xi_n) H_0(\sqrt{\xi_n} r) = \frac{4i\delta(r)}{2\pi r}, \tag{6.24}$$

we get

$$\sum_{n=1}^{\infty} F_n g_n(z) = \frac{i}{4}\delta(z - z_0), \quad 0 < z < h.$$

Since the source is located at $(r = 0, z = z_0)$ in the ocean, it follows that

$$\lim_{r \to 0} u_r(r,z) = 0, \quad |u_z(r,z)| < \infty, \quad h < z < b+h$$

Using (6.22)-(6.23), by making use of the asymptotic behavior of Hankel functions

$$H_1(\sqrt{\xi_n} x) = -\frac{2i}{\pi\sqrt{\xi_n} x} + o(x), \quad H_0(\sqrt{\xi_n} x) = \frac{2i}{\pi} \ln x + O(1), \quad \text{as } x \to 0,$$

we get

$$u_r = \sum_{n=1}^{\infty} F_n \sqrt{\xi_n} \alpha_n(z) \left(-\frac{2i}{\pi\sqrt{\xi_n} r}\right) + o(r). \qquad u_z = \sum_{n=1}^{\infty} F_n \beta_n(z) \frac{2i}{\pi} \ln r + O(1)$$

These imply

$$\sum_{n=1}^{\infty} F_n \alpha_n(z) = 0, \quad h < z < b, \tag{6.25}$$

$$\sum_{n=1}^{\infty} F_n \beta_n(z) = 0, \quad h < z < b. \tag{6.26}$$

These two equalities also can be derived by inserting (6.22)- (6.23) into the homogeneous equations (2.8)-(2.9). Consequently,

$$\sum_{n=1}^{\infty} F_n \, \alpha_n'(z) = 0. \tag{6.27}$$

By multiplying (6) by $g_m(z)$ and integrating over $[0,h]$, then multiplying (6.26) by $\omega^2 \rho_w \beta_m(z)$ and integrating over $[h,b]$, then multiplying (6.25) by
$-\omega^2 \rho_w \rho(z) \omega^2 \alpha_m(z)$ and integrating over $[h,b]$, then multiplying (6.27) by

$\omega^2\rho_w\mu\alpha'_m(z)$ and integrating over $[h, b]$, and using (6.20) and the orthogonality relation $T(m,n) = T(m,m)\delta(m,n)$, we get

$$F_m T(m,m) = \frac{i}{4} g_m(z_0).$$

Therefore

$$F_n = \frac{i}{4}\frac{g_n(z_0)}{T(n,n)}. \tag{6.28}$$

Consequently,

$$G(r,z,z_0) = \frac{i}{4}\sum_{j=0}^{\infty}\frac{1}{T(n,n)} g_n(z_0) g_n(z) H_0^{(1)}(\sqrt{\xi_n} r). \tag{6.29}$$

We note that this Fourier coefficient expression has a quite different form from that in (5.18). The question which arises naturally, is if the two types of expansions are equivalent. We are not able to prove the equivalence by the third way at this stage. We expect that numerical computations using this expression for the Fourier coefficients will return more accurate results than those in (5.18), since numerical integration is much more stable than the numerical differentiation. In lieu of a proof, we test them numerically. In fact, it is good to have two formulas to verify the correctness of our computations.

7 Numerical Verifications

A good way for verifying the formulas and the computationally obtained eigenvalues is to evaluate the matrix $\{\frac{|T(m,n)|}{\sqrt{|T(m,m)\,T(n,n)|}}\}$. This should be the identity matrix because of the orthogonality of the eigenfunctions, if all computations were completely accurate. The numerical approximations will result in very small, nonzero, off-diagonal terms if there is no mistake in our formal derivation. We also compute the Fourier coefficients in (5.18), i.e, $\frac{\hat{L}(\xi_n, h-z_0)}{\partial_\xi L(\xi_n, h)}$, and those in (6.29), namely, $\frac{g_n(z_0)}{T(n,n)}$, which should agree.

Example 2.1 We use the following experimental data to check our solution.

$$\begin{aligned}
&h = 30\,(m), \quad \rho_w = 1000\,(kg/m^3), \quad c_0 = 1500\,(m/s),\\
&n^2(z) = 1 - 0.05z/h,\\
&b = 40\,(m)\\
&\rho(z) = 3000(1 + 0.05(z-h)/(b-h))\,(kg/m^3),\\
&\mu(z) = 4.8\times 10^{10}(1 + 0.05(z-h)/(b-h))\,(kg\cdot m/s^2/m^2),\\
&\lambda(z) = 5.1\times 10^{10}(1 + 0.05(z-h)/(b-h))\,(kg\cdot m/s^2/m^2),\\
&\omega = 600(1/s)
\end{aligned}$$

This set of data suggests that the compressional wave speed is around $7000\,m/s$ and shear wave speed is around $4000\,m/s$. We only search the positive eigenvalues which correspond to the propagating modes.

The matrix $\{\frac{T(m,n)}{\sqrt{|T(m,m)T(n,n)|}}\}$ is as follows.

$$\begin{pmatrix}
1.00000 & 0.00344 & 0.00288 & 0.00281\\
0.00344 & 1.00000 & 0.00160 & 0.00116\\
0.00288 & 0.00160 & 1.00000 & 0.00117\\
0.00281 & 0.00116 & 0.00117 & 1.00000
\end{pmatrix}$$

Table 1: The comparison of Fourier Coefficients

n	ξ_n	$\frac{\hat{L}(\xi_n,h)}{\partial_\xi(\xi_n,h)}$ in (5.18)	$\frac{g_n(z_0)}{T(n,n)}$ in (6.29)
1	2.719182e-02	-1.717591e-02	-1.725774e-02
2	9.176573e-02	-1.203690e-02	-1.205485e-02
3	1.353713e-01	7.586275e-03	7.610731e-03
4	1.569599e-01	2.312212e-03	2.319580e-03

The Figure (7.1) shows the procedure of searching for eigenvalues. We use the combination of a *brute-force* search and the bisection method. It turns out that computing $\partial_\xi \hat{L}(\xi, h)$ is very expensive, so we approximate it by central differences. The eigenfunctions g_n are plotted in Figure (7.1), $\alpha_n(z)$ in Figure (7.2), and $\beta_n(z)$ in Figure (7.3). Finally in order to see the effect of the interaction with the seabed on the far-field, we compare an elastic seabed with the totally-reflecting seabed, using the following data as an input: Ocean as in Example 2.1; Elastic seabed with constant $c_l = 7000\,m/s$, $c_t = 4000\,m/s$ and $b = 10\,m$; Frequency $\omega = 600\,1/s$; Depth of the source $z_0 = 15\,m$; Range $r = 2000\,m$. Figure (7.5) shows this comparison between the case of a totally-reflecting seabed with an elastic seabed. We can see from the far-field that the effect of the seabed interaction is significant.

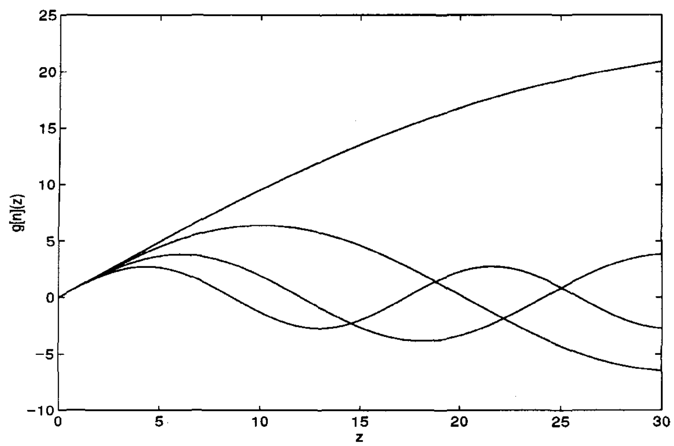

Figure 7.1: Eigenfunctions $g_n(z), n = 1, 2, 3, 4$ in the ocean

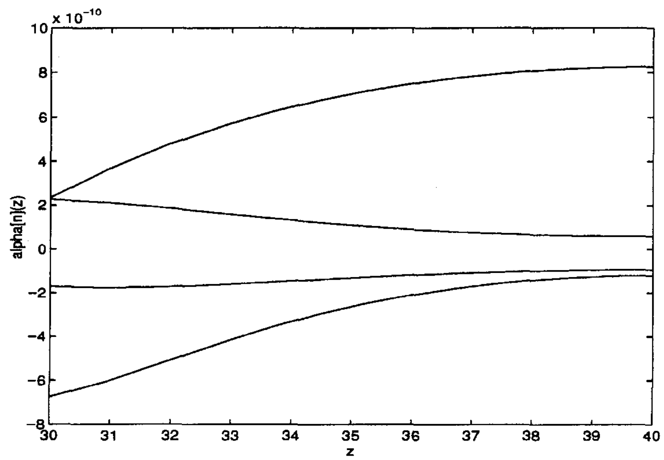

Figure 7.2: Eigenfunctions $\alpha_n(z)$ in the seabed

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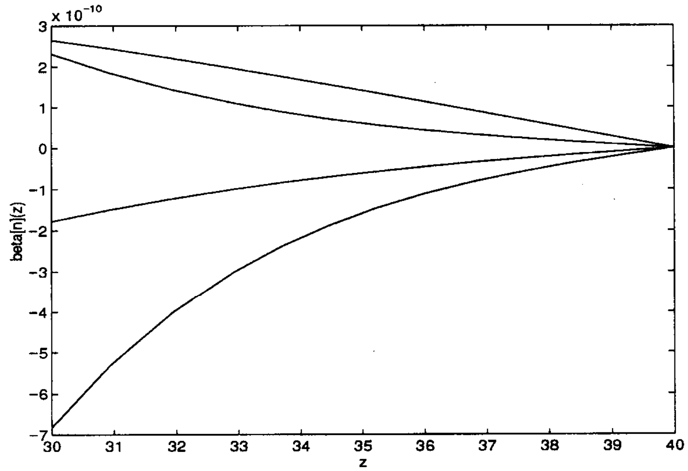

Figure 7.3: Eigenfunctions $\beta_n(z)$ in the seabed

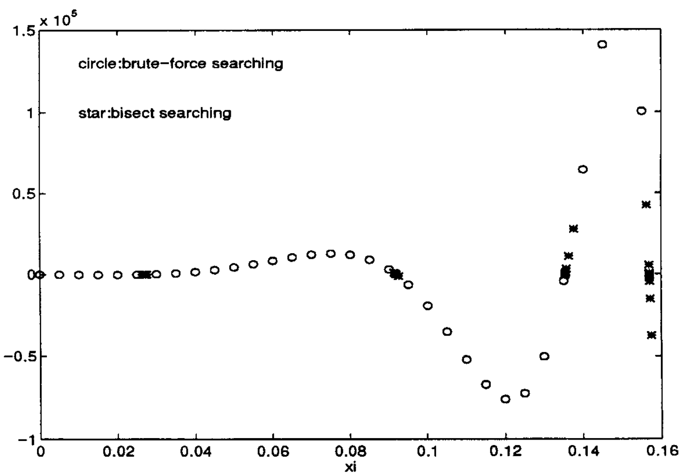

Figure 7.4: The computed values of the function $\hat{L}(\xi, h)$ during the brute-force searching and bisect searching

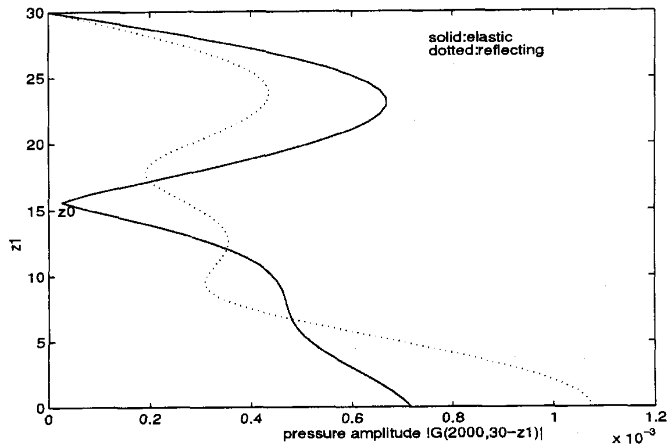

Figure 7.5: Comparison of the Pressure in the case of a totally Reflecting Seabed with that of an Elastic Seabed.