SIAM J. APPL. MATH.
Vol. 58, No. 3, pp. 875–903, June 1998

008

ON THE DETERMINATION OF RADIALLY DEPENDENT LAMÉ COEFFICIENTS*

ZHONGYAN LIN†

Abstract. In this paper, we prove that one can determine radially dependent Lamé coefficients λ and μ of an isotropic elastic ball by choosing only two correspondences in the Dirichlet-to-Neumann map, i.e., twice making measurements of displacement and stress on the surface of the ball. This is a one-dimensional, multiunknown, undetermined coefficient problem for a system of partial differential equations. We use a matrix form of the transmutation mapping for solving this problem. The procedure for proving the uniqueness based on transmutation can be viewed as a computational algorithm. Also the method we develop has the potential of solving other one-dimensional, multiunknown inverse problems.

Key words. inverse problem, elasticity, elliptic system, one dimension, Lamé coefficients, computer algebra, matrix, transmutation, moments, hyperbolic system

AMS subject classifications. 35R30, 73C02

PII. S0036139996303038

1. Introduction to the inverse problem and the main result.

The problems of determining the coefficients in partial differential equations are important and interesting inverse problems. The coefficients often describe the physical properties of the region to be detected. Most studies of undetermined coefficient problems involve a single unknown coefficient in a scalar equation such as $u_{tt} = u_{xx} + a(x)u(x)$ or $\text{div}(\gamma(\vec{x}) \bigtriangledown u) = 0$. Recently, inverse problems for systems of equations are being considered. In fact, for many problems in elastic and electromagnetic inhomogeneous media, only by restricting the coefficients do we get a single potential equation. For example, if only one coefficient is variable while others are constant, we can decouple the system of equations; or, on the other hand, when we set some coefficients to be zero, we may get a single acoustic-type equation. Otherwise we have to deal with systems of equations and identify simultaneously multiple coefficients.

Since for radially symmetric objects the inverse problems for a single potential equation have been well studied [12, 17, 6, 4], in this paper we consider an inverse problem for a system of equations with radially dependent coefficients. The problem is to determine the radially dependent Lamé coefficients of an elastic ball by the measurements of stress and displacement taken on the surface. To introduce this problem more precisely, first we quote from [1] the following formulation for the problem of determining variable elastic coefficients by boundary measurements.

Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary $\partial\Omega$ and filled with a linear, inhomogeneous elastic medium. The deformation of Ω due to the given displacement vector $\vec{f} \in C^\infty(\partial\Omega)$ is expressed by the following Dirichlet boundary value problem in terms of displacement vector $\vec{u} = \vec{u}(x)$:

$$L_C\vec{u} = \text{div}\sigma(\vec{u}) = 0, \quad \in \Omega, \tag{1.1}$$

$$\vec{u}\big|_{\partial\Omega} = \vec{f}, \tag{1.2}$$

*Received by the editors May 6, 1996; accepted for publication (in revised form) November 26, 1996; published electronically April 3, 1998. This research was supported by a fellowship from the Department of Mathematical Sciences, University of Delaware.

http://www.siam.org/journals/siap/58-3/30303.html

†Department of Mathematical Sciences, University of Delaware, Newark, DE 19716 (zlin@math.udel.edu).

where the strain tensor $\sigma(\vec{u}) = C(\frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^\top)$. When C satisfies the convexity condition, we can define the Dirichlet-to-Neumann map

$$\Lambda_C : C^\infty(\partial\Omega) \ni \vec{f} \to \sigma(\vec{u})\vec{\nu}\big|_{\partial\Omega} \in C^\infty(\partial\Omega),$$

where $\vec{u}$ is the solution of (1.1) and (1.2) and $\vec{\nu}$ is the outward normal to $\partial\Omega$. The inverse problem is to recover C from Λ_C.

The general case of this problem has been studied in [13, 14, 15]. Roughly speaking, the authors of those papers proved that in the case when $n \geq 3$, one can determine completely the Lamé coefficients $(\lambda(\vec{x}), \mu(\vec{x}))$ from Λ_C, and in the case of $n = 2$ one can determine either the boundary values and all derivatives of Lamé coefficients or their interior values if the coefficients are close to constants in an appropriate topology.

In this paper, we study the following special case of the problem. The domain Ω is a ball filled with isotropic, radially symmetric elastic medium. We want to recover the radially dependent Lamé coefficient λ and μ. However, we do not intend to use the whole Dirichlet-to-Neumann map Λ_c, and, moreover, we assume the elastic deformation is symmetric about the z-axis. The reason is the following. In the studies of undetermined coefficient problems, the uniqueness occurs if the number of parameters in the unknown coefficients agrees with that in the data. For the present problem, since the coefficients are only radially dependent, we expect that they can be determined by the boundary data containing one parameter. As the surface of a ball is two dimensional, the use of all of the boundary data for an actual three-dimensional deformation may cause overspecification. Therefore, for simplicity, we assume the incident wave or the external forces are symmetric about the z-axis so that the boundary data has one parameter. Furthermore, we will try to determine the coefficients using finitely many correspondences in Λ_c, instead of the whole map, which contains more than one parameter and is usually not available in practical applications.

Because of the geometry of the domain, a formulation in spherical coordinates (r, θ, ϕ) is more convenient. Denoting the displacements in the directions of increasing r and ϕ by u and v, respectively, the relations between displacements and stresses are given by Hooke's law [9]:

$$\tau_{rr} = \lambda U + 2\mu\frac{\partial u}{\partial r}, \tag{1.3}$$

$$\tau_{\phi\phi} = \lambda U + 2\mu\left(\frac{1}{r}\frac{\partial v}{\partial\phi} + \frac{u}{r}\right), \tag{1.4}$$

$$\tau_{\theta\theta} = \lambda U + 2\mu\left(\frac{u}{r} + \frac{\cot\phi}{r}v\right), \tag{1.5}$$

$$\tau_{r\phi} = \mu\left(\frac{1}{r}\frac{\partial u}{\partial\phi} + \frac{\partial v}{\partial r} - \frac{v}{r}\right), \tag{1.6}$$

where

$$U = \frac{\partial u}{\partial r} + \frac{2u}{r} + \frac{1}{r}\frac{\partial v}{\partial\phi} + \frac{\cot\phi}{r}v. \tag{1.7}$$

The equations of motion in spherical coordinates are [16]

$$\frac{\partial\tau_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\phi\phi}}{\partial\phi} + \frac{3}{r}\tau_{r\phi} + \frac{\cot\phi}{r}(\tau_{\phi\phi} - \tau_{\theta\theta}) = \rho\frac{\partial^2 v}{\partial t^2},$$
$$\frac{\partial\tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\phi}}{\partial\phi} + \frac{\cot\phi}{r}\tau_{r\phi} + \frac{1}{r}(2\tau_{rr} - \tau_{\phi\phi} - \tau_{\theta\theta}) = \rho\frac{\partial^2 u}{\partial t^2}.$$

Without loss of generality, we assume that the radius of the ball is one unit. The inverse problem of determining λ, μ, and ρ can be technically decomposed into the following two problems.

Problem 1. Determine λ and μ from the equilibrium equation of the steady case

$$\frac{\partial \tau_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{3}{r}\tau_{r\phi} + \frac{\cot\phi}{r}(\tau_{\phi\phi} - \tau_{\theta\theta}) = 0, \tag{1.8}$$

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\phi}}{\partial \phi} + \frac{\cot\phi}{r}\tau_{r\phi} + \frac{1}{r}(2\tau_{rr} - \tau_{\phi\phi} - \tau_{\theta\theta}) = 0, \tag{1.9}$$

using finitely many sets of boundary measurements of stress and displacement $\{u(1,\phi), v(1,\phi), \tau_{rr}(1,\phi), \tau_{r\phi}(1,\phi)\}$.

Problem 2. Determine ρ from the equations for time-harmonic vibration

$$\frac{\partial \tau_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{3}{r}\tau_{r\phi} + \frac{\cot\phi}{r}(\tau_{\phi\phi} - \tau_{\theta\theta}) = -\omega^2\rho v,$$
$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\phi}}{\partial \phi} + \frac{\cot\phi}{r}\tau_{r\phi} + \frac{1}{r}(2\tau_{rr} - \tau_{\phi\phi} - \tau_{\theta\theta}) = -\omega^2\rho u$$

assuming that λ and μ are given and by using appropriate boundary measurements.

In this paper we only consider Problem 1, which is an undetermined coefficient problem for an elliptic system. We use $\{u(1,\phi), v(1,\phi), \tau_{rr}(1,\phi), \tau_{r\phi}(1,\phi)\}$ because of two reasons. First, they are physically measurable quantities, namely, normal displacement, tangential displacement, normal load, and tangential load. Second, they relate to each other in a relatively simple manner.

For this problem, the questions we want to answer are (1) what type of data are good for the unique determination and (2) how many sets of boundary measurements $\{u(1,\phi), v(1,\phi), \tau_{rr}(1,\phi), \tau_{r\phi}(1,\phi)\}$ are sufficient. To see that question (1) is not trivial, let's consider an extremal case first. When all data in the list are zero, i.e., there is no elastic deformation, then we know nothing about λ and μ. Hence, uniqueness, if it exits, is conditional on the data in the list. An arbitrary data list does not guarantee uniqueness. Because of uniqueness theorems for the direct problem [10] we are allowed to control two and only two pieces of data in the list. Therefore, question (1) is actually how to conduct "experiments" (see Fig. 1.1). We want to answer question (2) because we are concerned with the cost of collecting data. To minimize the cost, we wish to use the least amount of data to achieve uniqueness and an efficient algorithm. The precise answers to these questions are contained in the main theorem of this paper.

To state the main results of this paper, first we need to prescribe the set of functions in which we look for a solution. Throughout this paper, we discuss the uniqueness of solution λ and μ in the set

$$\mathcal{H} = \left\{ (\lambda, \mu) \;\middle|\; \text{(i)}\;\; \lambda \in C[0,1],\;\; \mu \in C^2[0,1],\;\; \text{(ii)}\; \mu > 0, 3\lambda + 2\mu > 0 \right\}.$$

The restriction $\mu \in C^2[0,1]$ in (i) is convenient for our derivations, but it may be weakened. The restriction (ii) is the condition for the strong convexity. The convexity condition also implies

$$\lambda + \mu > 0, \quad \lambda + 2\mu > 0. \tag{1.10}$$

Second, we need the following definition.

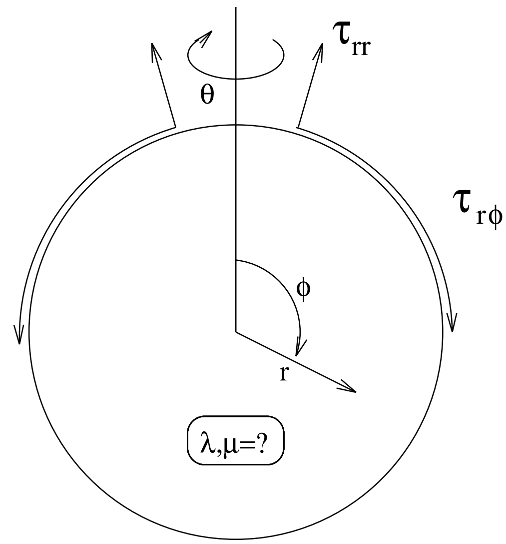

FIG. 1.1. *Problem illustration.*

DEFINITION 1.1. *We call the ordered list* $\{u, v, \tau_{rr}, \tau_{\phi\phi}, \tau_{\theta\theta}, \tau_{r\phi}\}$ *a nondestructive steady state (of the ball), corresponding to the Lamé coefficients* $\lambda, \mu \in \mathcal{H}$, *if*

(i) $u(r, \phi) \in C^1((0, 1]; H^1(0, \pi)), v(r, \phi) \in C^1((0, 1]; H_0^1(0, \pi))$,

(ii) $|u(r, \phi)|, |r\partial_r u(r, \phi)|, |v(r, \phi)|, |r\partial_r v(r, \phi)| < \infty$,

(iii) *equations* (1.3)–(1.9) *are satisfied in the weak sense.*

We don't use the usual energy space because the coefficients depend only on r, and, moreover, by the Sobolev embedding theorem the deformations in energy space do not guarantee the continuity or nondestructiveness. The equation $v(r, 0) = v(r, \pi) = 0$ from $v(r, \cdot) \in H_0^1(0, \pi)$ is also a requirement for nondestructiveness. This is because $v(r, 0) \neq 0$ or $v(r, \pi) \neq 0$ would mean that the ball is ruptured somewhere on the z-axis. According to this definition, $\tau_{rr}(1, \cdot)$, $\tau_{r\phi}(1, \cdot) \in L^2(0, \pi)$, which implies that we are allowed to use discontinuous boundary loads for convenience when designing experiments, but we are not allowed to hit the ball using concentrated forces.

Finally, let us denote by $\{P_n(x)\}_0^\infty$ the set of Legendre polynomials and use the notation

$$(f, g)_w := \int_0^\pi f(\phi)g(\phi)\sin\phi d\phi. \tag{1.11}$$

Now we are ready to state the main results concerning unique determinations.

THEOREM 1.2. (i) *If we are given the boundary data* $\{u(1, \phi), v(1, \phi), \tau_{rr}(1, \phi), \tau_{r\phi}(1, \phi\}$, $0 \le \phi \le \pi$, *of a nondestructive steady state, which satisfies*

$$\tau_{rr}(1, \phi) = 0 \tag{1.12}$$

or

$$\tau_{r\phi}(1, \phi) = 0 \tag{1.13}$$

and

$$\sum_{n \in \mathbb{M}} \frac{1}{n} = \infty, \tag{1.14}$$

where

$$\mathbb{M} := \left\{ n \in \mathbb{N} \;\middle|\; |(u(1,\phi), P_n(\cos\phi))_w| + \left| \left(v(1,\phi), \frac{d}{d\phi} P_n(\cos\phi) \right)_w \right| \neq 0 \right\}, \tag{1.15}$$

then we can determine $\mu(1)$ *uniquely.*

(ii) *If in addition to the data in* (i), *given the boundary data* $\{u(1,\phi), v(1,\phi), \tau_{rr}(1,\phi), \tau_{r\phi}(1,\phi)\}, 0 \le \phi \le 2\pi$, *of another nondestructive steady state, which satisfies*

$$\tau_{rr}(1,\phi) + 2\mu(1)\left(\frac{\partial v(1,\phi)}{\partial \phi} + \cot\phi\, v(1,\phi) \right) = 0 \tag{1.16}$$

or

$$\tau_{r\phi}(1,\phi) = 2\mu(1)\frac{\partial u(1,\phi)}{\partial \phi} \tag{1.17}$$

and still

$$\sum_{n\in\mathbb{M}} \frac{1}{n} = \infty,$$

then we can determine uniquely $\lambda(r)$ *and* $\mu(r)$ *for all* $r \in [0,1]$.

This uniqueness theorem actually describes a process for determining the Lamé coefficients. After determining $\mu(1)$ from the data of the first experiment, we can do the second experiment—note that $\mu(1)$ appears in (1.16) and (1.17)—and using all the data, we determine both λ and μ in the whole interval $[0,1]$. Equations (1.12)–(1.17) show us how to choose boundary loads. For example, (1.12) and (1.13) mean releasing the normal load and tangential load, respectively. As will be seen later on, (1.12) and (1.13) are only two representatives of a class of selection schemes, while (1.16) and (1.17) are only two representatives of another class of selection schemes. The requirement (1.14) can be fulfilled easily. For instance, if $u(1,\phi)$ or $v(1,\phi)$ is a nonconstant piecewise linear function of ϕ, (1.14) is satisfied. Also it is easy to see that (1.14) fails when $u(1,\phi)$ and $v(1,\phi)$ are polynomials in the circular functions.

We outline the proof of Theorem 1.2 and the rest of the paper. In section 2, we discuss the representations of the corresponding forward problem. First we reduce the system of partial differential equations to a family of systems of ordinary differential equations by expanding all quantities involved using Legendre polynomials. Then we use computer algebra to simplify the system into a form to which the transmutation method applies [2]. Finally we will show that if the boundary data satisfies any of (1.12), (1.13), (1.16), and (1.17), the solutions to the system can have integral representations in which the kernels $\mathbf{k}_i(r,s)$ satisfy hyperbolic equations with a Goursat condition. In section 3, using the representations in section 2, we can first determine $\mu(1)$. Then we decompose the original problem into three subproblems. The first subproblem will be to recover functions $\mathbf{k}_i(1,s)$ from their moments. Condition (1.14) can be seen to be sufficient and necessary for the uniqueness of this recovery. The second subproblem is an undetermined coefficient problem for the system of hyperbolic equations that $\mathbf{k}(r,s)$ satisfies. The third subproblem is a simple inverse problem for a system of ordinary differential equations. Finally, we proof the uniqueness of the last two subproblems. We also argue why a single list of data is not enough for the uniqueness or an algorithmic method. The proof of Theorem 1.2 can be viewed as a procedure for developing an algorithm.

At the end of this section, we mention that for Problem 2, when $\mu \equiv 0$, the system of equations is reduced to an acoustic equation

$$\rho\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\left(\frac{1}{\rho}\frac{\partial P}{\partial r}\right)+\frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial P}{\partial\phi}\right)+\frac{\omega^2\rho}{\lambda}P=0.$$

Some inverse problems related to or similar to this equation have been solved [12, 6, 8]. Moreover, inverse acoustic scattering theory is discussed in detail in [3].

2. The transmutation representation for solution of the direct problem.

2.1. General bounded solutions. In order to solve the inverse problem, we have to construct the solution of the direct problem for any instance of the Lamé coefficients. The solution we construct should satisfy the requirements for a nondestructive steady state.

To take the advantage of the symmetry of u and v about the z-axis and the geometry of the ball, we use expansions associated with Legendre polynomials $\{P_n(x)\}_{n=0}^{\infty}$. To this end, we introduce the following notation:

$$\mathcal{D}f(\phi):=\frac{df(\phi)}{d\phi}\quad\text{(in the sense that } (\mathcal{D}f,g)_w=(f,\mathcal{D}g)_w \text{ for } g\in C^\infty), \tag{2.1}$$

$$\mathcal{S}f(\phi):=\mathcal{D}f(\phi)+\cot\phi\, f(\phi)=\mathcal{D}(\sin\phi\, f(\phi))/\sin\phi, \tag{2.2}$$

$$p_n(\phi):=P_n(\cos\phi)\quad\text{for } \phi\in[0,\pi].$$

Then it follows from the properties of Legendre polynomials [11] that

$$\|p_n\|_{2,w}=\sqrt{\frac{2}{2n+1}}, \tag{2.3}$$

$$(p_n,p_m)_w=0\quad\text{if } m\neq n, \tag{2.4}$$

$$\mathcal{D}(\sin\phi\mathcal{D}p_n(\phi))+n(n+1)\sin\phi\, p_n(\phi)=0\quad\text{or } \mathcal{S}\mathcal{D}p_n+n(n+1)p_n=0. \tag{2.5}$$

Using integration by parts and (2.5) we get

$$(\mathcal{S}f,p_n)_w=\int_0^\pi \mathcal{D}(f(\phi)\sin\phi)p_n(\phi)\,d\phi=-(f,\mathcal{D}p_n)_w, \tag{2.6}$$

$$(\mathcal{D}f,\mathcal{D}p_n)_w=-\int_0^\pi f(\phi)\mathcal{D}(\sin\phi\mathcal{D}p_n(\phi))\,d\phi=n(n+1)(f,p_n)_w \tag{2.7}$$

for $f\in H^1(0,\pi)$. These formulas allow us to shift the differential operator $\mathcal{D}$ or $\mathcal{S}$ from u and v to the test functions that are chosen to be p_n and $\mathcal{D}p_n$, respectively, while we use the weak form of equations (1.8) and (1.9). Denote by

$$\begin{aligned}
u_n(r)&:=n(u(r,\cdot),p_n)_w,\\
v_n(r)&:=(v(r,\cdot),\mathcal{D}p_n)_w,\\
U_n(r)&:=n(U(r,\cdot),p_n)_w,\\
\tau_{rr}^{(n)}(r)&:=n(\tau_{rr}(r,\cdot),p_n)_w,\\
\tau_{r\phi}^{(n)}(r)&:=(\tau_{r\phi}(r,\cdot),\mathcal{D}p_n)_w.
\end{aligned}$$

By the requirements for a nondestructive steady state, we have

$$|u_n(r)|,|v_n(r)|,|r\tau_{rr}^{(n)}|,|r\tau_{rr}^{(n)}|<\infty. \tag{2.8}$$

Using relations (1.7), (1.3), (1.6), the definitions above, and (2.6)–(2.7), we can find the relations among $\{u_n(r), v_n(r), U_n(r), \tau_{rr}^{(n)}(r), \tau_{r\phi}^{(n)}(r)\}$:

$$U_n(r) = u_n'(r) + \frac{2}{r}u_n(r) + \frac{n}{r}(\mathcal{S}v(r,\cdot), p_n) = u_n'(r) + \frac{2}{r}u_n(r) - \frac{n}{r}v_n(r), \tag{2.9}$$

$$\tau_{rr}^{(n)}(r) = \lambda U_n(r) + 2\mu u_n'(r) = \lambda\left(u_n'(r) + \frac{2}{r}u_n(r) - \frac{n}{r}v_n(r)\right) + 2\mu u_n'(r), \tag{2.10}$$

$$\begin{aligned}\tau_{r\phi}^{(n)}(r) &= \mu\left(\frac{1}{r}(\mathcal{D}\,u(r,\cdot), \mathcal{D}p_n)_w + v_n'(r) - \frac{1}{r}v_n(r)\right)\\ &= \mu\left(\frac{n(n+1)}{r}(u(r,\cdot), p_n)_w + v_n'(r) - \frac{1}{r}v_n(r)\right)\\ &= \mu\left(\frac{(n+1)u_n(r)}{r} + v_n'(r) - \frac{v_n(r)}{r}\right).\end{aligned} \tag{2.11}$$

From (1.4), (1.5), by using (2.7), $v(r,\cdot) \in H_0^1(0,1)$, and (2.5), we have

$$\begin{aligned}&(\mathcal{D}\tau_{\phi\phi}(r,\cdot) + \cot(\cdot)[\tau_{\phi\phi}(r,\cdot) - \tau_{\theta\theta}(r,\cdot)],\, \mathcal{D}p_n)_w\\ &= n(n+1)(\tau_{\phi\phi}(r,\cdot),\, p_n)_w + \frac{2\mu}{r}\int_0^\pi \cot\phi\,(\mathcal{D}v(r,\phi) - \cot\phi\, v(r,\phi))\mathcal{D}p_n(\phi)\sin\phi\, d\phi\\ &= (n+1)\left(\lambda U_n(r) + \frac{2\mu}{r}u_n(r)\right) + \frac{2\mu}{r}\int_0^\pi (\mathcal{D}v(r,\phi)[n(n+1)\sin\phi p_n(\phi)\\ &\quad + \cos\phi\mathcal{D}p_n(\phi)] - \cos\phi\cot\phi\, v(r,\phi)\mathcal{D}p_n(\phi))\, d\phi\\ &= (n+1)\left(\lambda U_n(r) + \frac{2\mu}{r}u_n(r)\right) + v(r,\phi)[n(n+1)\sin\phi p_n(\phi) + \cos\phi\mathcal{D}p_n(\phi)]\,|_0^\pi\\ &\quad -\frac{2\mu}{r}\int_0^\pi v(r,\phi)\,(\mathcal{D}[n(n+1)\sin\phi p_n(\phi) + \cos\phi\mathcal{D}p_n(\phi)] + \cos\phi\cot\phi\mathcal{D}p_n(\phi))\, d\phi\\ &= (n+1)\left(\lambda U_n(r) + \frac{2\mu}{r}u_n(r)\right) - \frac{2\mu}{r}\int_0^\pi v(r,\phi)[n(n+1) - 1]\mathcal{D}p_n(\phi)\sin\phi d\phi\\ &= (n+1)\left(\lambda U_n(r) + \frac{2\mu}{r}u_n(r)\right) - \frac{2[n(n+1) - 1]\mu}{r}v_n(r).\end{aligned}$$

Hence, by taking the inner product of the both sides of (1.8) with $\mathcal{D}p_n$, we obtain

$$\left(\frac{d}{dr} + \frac{3}{r}\right)\tau_{r\phi}^{(n)}(r) + \frac{n+1}{r}\left(\lambda\left(u_n' + \frac{2u_n}{r} - \frac{nv_n}{r}\right) + 2\mu\left(\frac{u_n}{r} - \frac{nv_n}{r}\right)\right) + 2\mu\frac{v_n}{r^2} = 0. \tag{2.12}$$

Next we use (1.4)–(1.5) and the operator $\mathcal{S}$ to rewrite (1.9) as

$$\frac{\partial\tau_{rr}(r,\phi)}{\partial r} + \frac{1}{r}\mathcal{S}\tau_{r\phi}(r,\phi) + \frac{2\mu}{r}\left(2\frac{\partial u(r,\phi)}{\partial r} - \frac{2}{r}u(r,\phi) - \frac{1}{r}\mathcal{S}v(r,\phi)\right) = 0.$$

By taking the inner product of both sides of this equation with np_n and using (2.6), we obtain

$$\frac{d}{dr}\tau_{rr}^{(n)}(r) - \frac{n}{r}\tau_{r\phi}^{(n)}(r) + \frac{2\mu}{r}\left(2u_n'(r) - \frac{2}{r}u_n(r) + \frac{n}{r}v_n(r)\right) = 0. \tag{2.13}$$

Therefore, for any nondestructive steady state, the sequence of lists of functions $\{u_n(r), v_n(r), \tau_{rr}^{(n)}(r), \tau_{r\phi}^{(n)}(r)\}_{n=1}^\infty$ satisfies (2.8) and (2.10)–(2.13).

Next we are going to "solve" the system of equations (2.10)–(2.13) for $\{u_n(r), v_n(r), \tau_{rr}^{(n)}(r), \tau_{r\phi}^{(n)}(r)\}$. Unlike the case of constant Lamé coefficients in [9], the system of equations (2.10)–(2.13) cannot be decoupled in general for variable λ and μ. To construct the solutions, we need to simplify the equations and eliminate n^2, which corresponds to the second derivative with respect to ϕ. By using computer algebra,[1] we find that the following two intermediate variables simplify the equations:

$$y_n(r) := -r\tau_{rr}^{(n)}(r) + 2n\mu v_n(r) - 4\mu u_n(r), \tag{2.14}$$

$$z_n(r) := r\tau_{r\phi}^{(n)}(r) - 2(n+1)\mu u_n(r) + 2(r\mu)' v_n(r). \tag{2.15}$$

Substituting these into (2.10)–(2.13) yields

$$\begin{pmatrix} u_n(r) \\ y_n(r) \\ v_n(r) \\ z_n(r) \end{pmatrix}' + \frac{1}{r}\begin{pmatrix} 2 & \frac{1}{\lambda+2\mu} & -n & 0 \\ 4r\mu' & -1 & 0 & -n \\ -n-1 & 0 & 1+\frac{2r\mu'}{\mu} & -\frac{1}{\mu} \\ 0 & -n-1 & \frac{4(r\mu')^2}{\mu} - 2r(r\mu')' & -\frac{2r\mu'}{\mu} \end{pmatrix}\begin{pmatrix} u_n(r) \\ y_n(r) \\ v_n(r) \\ z_n(r) \end{pmatrix} = 0. \tag{2.16}$$

To make use of the special form of the matrix, let

$$\mathbf{B}(r) := \begin{pmatrix} 2 & \frac{1}{\lambda+2\mu} \\ 4r\mu' & -1 \end{pmatrix}, \quad \mathbf{D}(r) := \begin{pmatrix} 1+\frac{2r\mu'}{\mu} & -\frac{1}{\mu} \\ \frac{4(r\mu')^2}{\mu} - 2r(r\mu')' & -\frac{2r\mu'}{\mu} \end{pmatrix}, \tag{2.17}$$
$$\mathbf{u}(r) = \{u_n(r), y_n(r)\}^\top, \quad \mathbf{v}(r) = \{v_n(r), z_n(r)\}^\top.$$

Then (2.16) can be rewritten as

$$r\mathbf{u}'(r) - n\mathbf{v}(r) + \mathbf{B}(r)\mathbf{u}(r) = \mathbf{0}, \tag{2.18}$$

$$r\mathbf{v}'(r) - (n+1)\mathbf{u}(r) + \mathbf{D}(r)\mathbf{v}(r) = \mathbf{0}. \tag{2.19}$$

By eliminating $\mathbf{v}(r)$ from these two equations to obtain a system of differential equations of the second order, we obtain

$$r(r\mathbf{u}'(r))' + (\mathbf{B}(r)\mathbf{u}(r))' + \mathbf{D}(r)(r\mathbf{u}'(r) + \mathbf{B}(r)\mathbf{u}(r)) + n(n+1)\mathbf{u}(r) = 0.$$

The similarity of this equation to the equation $y''(x) - q(x)y(x) - c^2y(x) = 0$ suggests that the transmutation method may apply. The transmutation approach is to represent the solution of an equation or a system, through an integral, in terms of the solution of another equation or system which is simpler but of the same type as the original equation or system. To see this, let us recall the Gelfand–Levitan transmutation in [7]. The solution to

$$y''(x) - q(x)y - c^2y(x) = 0, \quad y(0) = 1, y'(0) = 0 \tag{2.20}$$

was represented as

$$y(x) = \cos cx + \int_0^x K(x,s)\cos cs ds, \tag{2.21}$$

where $K(x,s)$, the so-called Gelfand–Levitan kernel, satisfies a hyperbolic equation and the Goursat condition. This representation is responsible for the solutions of

[1]All calculations in this paper have been done with Macsyma.

many one-dimensional inverse problems and inverse spectral problems. Although it does not apply directly to the present problem, we use the idea behind this formula and the derivation in [7]; that is, to find a solution y_1 to a simpler equation of the same type—note that $\cos cx$ is nothing but the solution of (2.20) with $q = 0$—compose a candidate for the solution based on y_1 as in (2.21), and then substitute it into the original equation to find out what the kernel should be. All transmutations that have been used so far are for scalar equations, but now we generalize this method to deal with a system of equations. In the present case, the corresponding simpler system is, of course, the system with constant coefficients. The general bounded solution of the system with constant coefficients can be found easily by using computer algebra to be

$$\begin{aligned}
u_n &= c_1 r^{n-1} + c_2 r^{n+1},\\
y_n &= \frac{2\mu(2n+3)(\lambda+2\mu)c_2}{n\lambda+(n-2)\mu} r^{n+1},\\
v_n &= u_n + \frac{c_1}{n} r^{n-1} + \frac{(3\lambda+7\mu)c_2}{n\lambda+(n-2)\mu} r^{n+1},\\
z_n &= y_n.
\end{aligned}$$

The generic term in this solution is r^n. Hence, we compose solution candidates based on r^n. The following expressions are found to be a suitable transmutation representation for (2.16) and require short calculations.

$$\mathbf{u}(r) = a_n \left[\mathbf{l}(r) r^n + \int_0^r \mathbf{k}_1(r,s) s^{n-1} ds \right], \tag{2.22}$$

$$\mathbf{v}(r) = a_n \left[\mathbf{l}(r) r^n + \int_0^r \mathbf{k}_2(r,s) s^{n-1} ds \right], \tag{2.23}$$

where $\mathbf{l}(r) = [l_1(r), l_2(r)]^\top$, $\mathbf{k}_1(r,s) = [k_{11}(r,s), k_{12}(r,s)]^\top$, and $\mathbf{k}_2(r,s) = [k_{21}(r,s), k_{22}(r,s)]^\top$ are vector functions to be determined.

By differentiating $\mathbf{u}(r)$ with respect to r, we have

$$r\mathbf{u}'(r) = a_n \left[n\mathbf{l}(r) r^n + r\mathbf{l}'(r) r^n + \mathbf{k}_1(r,r) r^n + \int_0^r r \frac{\partial}{\partial r} \mathbf{k}_1(r,s) s^{n-1} ds \right].$$

By using integration by parts and assuming

$$\lim_{s\to 0} s^n \mathbf{k}_i(r,s) = \mathbf{0}, \tag{2.24}$$

which will be proved later on, we get

$$-n\mathbf{v}(r) = a_n \left[-n\mathbf{l}(r) r^n - \mathbf{k}_2(r,r) r^n + \int_0^r s \frac{\partial}{\partial s} \mathbf{k}_2(r,s) s^{n-1} ds \right].$$

Similarly,

$$\begin{aligned}
r\mathbf{v}'(r) &= a_n \left[n\mathbf{l}(r) r^n + r\mathbf{l}'(r) r^n + \mathbf{k}_2(r,r) r^n + \int_0^r r \frac{\partial}{\partial r} \mathbf{k}_2(r,s) s^{n-1} ds \right],\\
-n\mathbf{u}(r) &= a_n \left[-n\mathbf{l}(r) r^n - \mathbf{k}_1(r,r) r^n + \int_0^r s \frac{\partial}{\partial s} \mathbf{k}_1(r,s) s^{n-1} ds \right].
\end{aligned}$$

Substituting these expressions into equations (2.18) and (2.19), we find that $\mathbf{l}(r)$ and $\mathbf{k}_i(r,s)$ should satisfy the following equations:

$$r\mathbf{l}'(r) + \mathbf{B}(r)\mathbf{l}(r) + \mathbf{k}_1(r,r) - \mathbf{k}_2(r,r) = \mathbf{0}, \tag{2.25}$$

$$r\mathbf{l}'(r) + (\mathbf{D}(r) - \mathbf{I})\mathbf{l}(r) + \mathbf{k}_2(r,r) - \mathbf{k}_1(r,r) = \mathbf{0}, \tag{2.26}$$

$$r\frac{\partial}{\partial r}\mathbf{k}_1(r,s) + s\frac{\partial}{\partial s}\mathbf{k}_2(r,s) + \mathbf{B}(r)\mathbf{k}_1(r,s) = \mathbf{0}, \tag{2.27}$$

$$r\frac{\partial}{\partial r}\mathbf{k}_2(r,s) + s\frac{\partial}{\partial s}\mathbf{k}_1(r,s) - \mathbf{k}_1(r,s) + \mathbf{D}(r)\mathbf{k}_2(r,s) = \mathbf{0}. \tag{2.28}$$

Let

$$\mathbf{m}_1 = \mathbf{k}_1(r,s) + \mathbf{k}_2(r,s), \tag{2.29}$$

$$\mathbf{m}_2 = \mathbf{k}_1(r,s) - \mathbf{k}_2(r,s), \tag{2.30}$$

$$\mathbf{E}(r) = \frac{1}{2}(\mathbf{B}(r) + \mathbf{D}(r) - \mathbf{I}) = \begin{pmatrix} 1 + \frac{r\mu'}{\mu} & \frac{1}{2}\left(\frac{1}{\lambda+2\mu} - \frac{1}{\mu}\right) \\ 2r\mu' + \frac{2(r\mu')^2}{\mu} - r(r\mu')' & -1 - \frac{r\mu'}{\mu} \end{pmatrix}, \tag{2.31}$$

$$\mathbf{F}(r) = \frac{1}{2}(\mathbf{B}(r) - \mathbf{D}(r) + \mathbf{I}) = \begin{pmatrix} 1 - \frac{r\mu'}{\mu} & \frac{1}{2}\left(\frac{1}{\lambda+2\mu} + \frac{1}{\mu}\right) \\ 2r\mu' - \frac{2(r\mu')^2}{\mu} + r(r\mu')' & \frac{r\mu'}{\mu} \end{pmatrix}. \tag{2.32}$$

By adding (2.25) to (2.26), we get

$$r\mathbf{l}'(r) + \mathbf{E}(r)\mathbf{l}(r) = \mathbf{0}. \tag{2.33}$$

Subtracting (2.26) from (2.25) yields

$$\mathbf{m}_2(r,r) + \mathbf{F}(r)\mathbf{l}(r) = \mathbf{0}. \tag{2.34}$$

Similarly, we can manipulate equations (2.27) and (2.28) into

$$\left(r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}\right)\mathbf{m}_1(r,s) + \mathbf{E}(r)\mathbf{m}_1(r,s) + (\mathbf{F}(r) - \mathbf{I})\mathbf{m}_2(r,s) = \mathbf{0}, \tag{2.35}$$

$$\left(r\frac{\partial}{\partial r} - s\frac{\partial}{\partial s}\right)\mathbf{m}_2(r,s) + \mathbf{F}(r)\mathbf{m}_1(r,s) + (\mathbf{E}(r) + \mathbf{I})\mathbf{m}_2(r,s) = \mathbf{0}. \tag{2.36}$$

To assert (2.22) and (2.23), we need to show that (2.33)–(2.36) are solvable for $\mathbf{l}(r)$ and $\mathbf{m}_i(r,s)$, and the requirement (2.24) can be fulfilled. In fact, the solvability is almost obvious because (2.33) is a system of linear ordinary differential equations and (2.34)–(2.36) is a system of hyperbolic equations with a Goursat condition. What we really have to consider is the asymptotic behavior of $\mathbf{m}_i(r,s)$ as $s \to 0$, which concerns (2.24).

Lemma 2.1. *If $(\lambda,\mu) \in \mathcal{H}$, the system of ordinary differential equations* (2.33) *is solvable in $\mathbf{C}^1(0,1]$ for any initial condition $\mathbf{l}(1) = [a,b]^\top$ and the following estimates hold:*

$$|l_1(r)| \le \frac{C(|a|+|b|)}{r}, \tag{2.37}$$

$$|l_2(r)| \le C(|a|+|b|), \tag{2.38}$$

where the constants $C = C(\mu)$.

Proof. By multiplying equation (2.33) on the left by the matrix

$$\mathbf{T}(r) := \begin{pmatrix} r\mu & 0 \\ -\frac{\mu'}{\mu} & \frac{1}{r\mu} \end{pmatrix}, \tag{2.39}$$

we get

$$\begin{aligned} &\mathbf{T}(r)r\mathbf{l}'(r) + \mathbf{T}(r)\mathbf{E}(r)\mathbf{l}(r) \\ &= r(\mathbf{T}(r)\mathbf{l}(r))' + [(\mathbf{T}(r)\mathbf{E}(r) - r\mathbf{T}'(r))\mathbf{T}^{-1}(r)](\mathbf{T}(r)\mathbf{l}(r)) = \mathbf{0}. \end{aligned} \tag{2.40}$$

From (2.31) we can evaluate the matrix $\mathbf{Q}(r) := (\mathbf{T}(r)\mathbf{E}(r) - r\mathbf{T}'(r))\mathbf{T}^{-1}(r)$ to

$$\mathbf{Q}(r) = -E_{12}\begin{pmatrix} -r\mu' & -(r\mu)^2 \\ (\frac{\mu'}{\mu})^2 & r\mu' \end{pmatrix}. \tag{2.41}$$

Let

$$[l_1^*(r) \quad l_2^*(r)]^\top := \mathbf{T}(r)\mathbf{l}(r) = \left[r\mu l_1(r) \quad \frac{1}{\mu}(-\mu' l_1(r) + l_2(r)/r)\right]^\top.$$

Then (2.40) becomes

$$r\begin{pmatrix} l_1^*(r) \\ l_2^*(r) \end{pmatrix}' + \mathbf{Q}(r)\begin{pmatrix} l_1^*(r) \\ l_2^*(r) \end{pmatrix} = \mathbf{0}.$$

By converting this system to a system of integral equations and using the Gronwall inequality we can obtain

$$\begin{aligned} |l_1^*(r)| &\le C(|a| + |b|), \\ |l_2^*(r)| &\le C(|a| + |b|)(1 - \ln r). \end{aligned}$$

Consequently,

$$\begin{aligned} |l_1(r)| &= |l_1^*(r)/(r\mu)| \le C(|a| + |b|)/r, \\ |l_2(r)| &= |r\mu l_2^*(r) + r\mu' l_1(r)| \le C(|a| + |b|). \qquad \square \end{aligned}$$

To state the second lemma, we denote

$$D_2 := \{(r, s) \,|\, 0 < s \le r \le 1\}$$

and we say that for any matrix or vector $\boldsymbol{\Xi}$ and domain D involved, $\boldsymbol{\Xi} \in \mathbf{C}^k(D)$ if all components or entries of $\boldsymbol{\Xi}$ are in $C^k(D_a)$, where D_a is any closed subdomain of D.

LEMMA 2.2. *If* $(\lambda, \mu) \in \mathcal{H}$*, the hyperbolic system of equations* (2.35)–(2.36) *with the Goursat condition* (2.34) *and the imposed initial condition*

$$\mathbf{m}_1(1, s) = \mathbf{J}(s)\mathbf{m}_2(1, s) + \mathbf{a}(s), \tag{2.42}$$

where $(\mathbf{J}(s))_{2\times 2}$ *and* $(\mathbf{a}(r))_{2\times 1}$ *are matrices in* $\mathbf{C}^1[0, 1]$*, has a unique weak solution in* $\mathbf{C}(D_2)$*, and the solution has a finite order of singularity at* $s = 0$*.*

Proof. By integrating (2.35) and (2.36) along the characteristic lines, we get

$$\mathbf{m}_1(r, s) = \int_s^{s/r} \frac{1}{y}[\mathbf{E}\mathbf{m}_1 + (\mathbf{F} - \mathbf{I})\mathbf{m}_2]\left(\frac{yr}{s}, y\right) dy + [\mathbf{J}\mathbf{m}_2 + \mathbf{a}]\left(\frac{s}{r}\right), \tag{2.43}$$

$$\mathbf{m}_2(r, s) = -\int_s^{\sqrt{rs}} \frac{1}{y}[\mathbf{F}\mathbf{m}_1 + (\mathbf{E} + \mathbf{I})\mathbf{m}_2]\left(\frac{rs}{y}, y\right) dy - [\mathbf{F}\mathbf{l}](\sqrt{rs}). \tag{2.44}$$

This system of integral equations is obviously solvable in $\mathbf{C}(D_2)$. To look at the asymptotic behavior of the solution as $s \to 0$, let

$$w(t) := \max_{1\le i,j\le 2, t\le s\le r\le 1} |m_{ij}(r,s)| \quad \text{for } t \in (0,1],$$

$$\Omega_1 := \max_{0\le r\le 1, 1\le i\le 2} [E_{i1} + E_{i2} + F_{i1} + F_{i2}] + 1, \tag{2.45}$$

$$\Omega_2 := \max_{0\le r\le 1, 1\le i\le 2} |J_{i1}| + |J_{i2}|. \tag{2.46}$$

Then from (2.44), (2.37), and (2.38), we have

$$\begin{aligned} |m_{21}(r,s)|, |m_{22}(r,s)| &= \Omega_1 \int_s^{\sqrt{rs}} \frac{1}{y} w(y)dy + C_1 s^{-1} \\ &\le \Omega_1 \int_s^1 \frac{1}{y} w(y)dy + C_1 s^{-1}. \end{aligned} \tag{2.47}$$

From (2.42) and (2.47) we have

$$\begin{aligned} |m_{11}(r,s)|, |m_{12}(r,s)| &\le \Omega_1 \int_{rs}^{s/r} \frac{1}{y} w(y)dy + \Omega_2 \left[\Omega_1 \int_s^1 \frac{1}{y} w(y)dy + C_1 s^{-1} \right] \\ &\quad + \max\{|a_1(s/r)|, |a_2(s/r)|\} \\ &\le \Omega_1(1+\Omega_2) \int_s^1 \frac{1}{y} w(y)dy + C_2 s^{-1}. \end{aligned}$$

Therefore,

$$w(s) \le \Omega_1(1+\Omega_2) \int_s^1 \frac{1}{y} w(y)dy + C_3 s^{-1}.$$

By solving this inequality for $w(s)$, we get

$$|m_{ij}(r,s)| \le w(s) \le C_3 s^{-\Omega_1(1+\Omega_2)}. \tag{2.48}$$

Hence, the order of singularity of $\mathbf{m}_j(r,s)$ at $s=0$ is finite. $\square$

The estimate (2.48) implies that when $n > \Omega_1(1+\Omega_2)$, the requirement (2.24) is fulfilled. By summarizing the above, we have Theorem 2.3.

Theorem 2.3 (transmutation). *If* $(\lambda,\mu) \in \mathcal{H}$, (2.22)–(2.23) *represent a bounded solution associated with* (2.42) *of the system of equation* (2.16) *for* $n > \Omega_1(1+\Omega_2)$, *where the* Ω_i *are defined by* (2.45) *and* (2.46).

It seems that the result $\mathbf{m}_i(r,s) \in \mathbf{C}(D_2)$ we obtained in Lemma 2.2 is not enough for the derivations following (2.22) and (2.23) where we differentiated $\mathbf{k}$. This difficult can be coped with by using the continuity arguments similar to those in [7].

One of the virtues of this transmutation representation is that we can choose $\mathbf{l}(1)$, $\mathbf{J}(r)$, and $\mathbf{a}$ with different goals in mind. For example, by choosing these parameters appropriately, we can find two linear independent solutions in the form of (2.22)–(2.23) that form the basis of bounded solutions.

We call the number n_1 the limitation number of a transmutation if n_1 is the minimum number such that the transmutation representation is valid for all $n > n_1$. Unfortunately, the estimate of the limitation number $\Omega_1(\Omega_2+1)$ depends on the sizes of λ and μ, which are unknown in the inverse problem. This results from the rough estimation made in the proof Lemma 2.2. An upper bound of Ω_1 can be derived from (2.31) and (2.32); namely,

$$\max\left\{1+\max\frac{1}{\mu}, 2\max_{0<r\le 1}|2r\mu'| + \max_{0<r\le 1}\left|(r\mu')^2 - \frac{r(r\mu')'}{\mu}\right|\right\} + 2\max_{0<\le 1}\left|\frac{r\mu'}{\mu}\right| + 2. \tag{2.49}$$

Nevertheless, this constraint turns out not to be an obstacle for proving the uniqueness.

2.2. The schemes for selecting the surface loads. To solve the inverse problem, we have to decide how to enact the forces on the surface of the ball in order that the elastic deformation brings out enough information about the interior of the ball onto the surface. Our strategy is to choose boundary loads such that the solution to the boundary value problem (2.16) has a transmutation representation.

As we want to specify the conditions explicitly in terms of the physically measurable quantities, first we need the expressions of displacements and stresses over the boundary. If the solution is really in the form of (2.22)–(2.23), by substituting $r = 1$ we have

$$u_n(1) = a_n\left[l_1(1) + \int_0^1 k_{11}(1,s)s^{n-1}ds\right], \tag{2.50}$$

$$v_n(1) = a_n\left[l_1(1) + \int_0^1 k_{21}(1,s)s^{n-1}ds\right]. \tag{2.51}$$

By using (2.14) and (2.15), we have

$$\begin{aligned}\tau_{rr}^{(n)}(1) &= -y_n(1) + 2n\mu(1)v_n(1) - 4\mu(1)u_n(1)\\ &= a_n(1)\left[-l_2(1) - \int_0^1 k_{12}(1,s)s^{n-1}ds + 2n\mu(1)\left(l_1(1) + \int_0^1 k_{21}(1,s)s^{n-1}ds\right)\right.\\ &\left.-\ 4\mu(1)\left(l_1(1) + \int_0^1 k_{11}(1,s)s^{n-1}ds\right)\right],\end{aligned} \tag{2.52}$$

$$\begin{aligned}\tau_{r\phi}^{(n)}(1) &= z_n(1) + 2(n+1)\mu(1)u_n(1) - 2(\mu(1)+\mu'(1))v_n(1)\\ &= a_n\left[l_2(1) + \int_0^1 k_{22}(1,s)s^{n-1}ds + 2(n+1)\mu(1)\left(l_1(1) + \int_0^1 k_{11}(1,s)s^{n-1}ds\right)\right.\\ &\left.-\ 2(\mu(1)+\mu'(1))\left(l_1(1) + \int_0^1 k_{21}(1,s)s^{n-1}ds\right)\right].\end{aligned} \tag{2.53}$$

We have the following two types of selection schemes.

Type (I). Control the loads so that $l_1(1) = 0$, whereas $l_2(1) \neq 0$. This is feasible, for example, as was listed in Theorem 1.2 and will be shown later on; if (1.12) or (1.13) holds, then $\mathbf{l}(1) = [0, 1]$.

Type (II). Control the loads so that $l_1(1) \neq 0$. As was listed in Theorem 1.2 and will be shown later on, if (1.16) or (1.17) hold, then $l_1(1) \neq 0$.

We will work out the details for these particular selection schemes, from which we can see how general schemes of type (I) and (II) may be used. They are based on the following corollaries of Theorem 2.3. To state and prove them, first we need a reduced energy identity. To this end, we put (2.10)–(2.13) into matrix form:

$$\frac{d}{dr}\begin{pmatrix} r\tau_{rr}^{(n)}(r)\\ r\tau_{r\phi}^{(n)}(r)\\ ru_n(r)\\ rv_n(r)\end{pmatrix} = \mathbf{A}\begin{pmatrix} u_n(r)\\ v_n(r)\\ \tau_{rr}^{(n)}(r)\\ \tau_{r\phi}^{(n)}(r)\end{pmatrix}, \tag{2.54}$$

where

$$\mathbf{A} = \begin{pmatrix} \frac{4\,\mu\,(3\,\lambda+2\,\mu)}{r\,(\lambda+2\,\mu)} & -\frac{2\,\mu\,n\,(3\,\lambda+2\,\mu)}{r\,(\lambda+2\,\mu)} & 1-\frac{4\,\mu}{\lambda+2\,\mu} & n \\ -\frac{2\,\mu\,(n+1)\,(3\,\lambda+2\,\mu)}{r\,(\lambda+2\,\mu)} & \frac{2\,\mu\left(2\,n^2\,\lambda+2\,n\,\lambda-\lambda+2\,\mu\,n^2+2\,\mu\,n-2\,\mu\right)}{r(\lambda+2\,\mu)} & -\frac{(n+1)\,\lambda}{\lambda+2\,\mu} & -2 \\ 1-\frac{2\,\lambda}{\lambda+2\,\mu} & \frac{n\,\lambda}{\lambda+2\,\mu} & \frac{r}{\lambda+2\,\mu} & 0 \\ -(n+1) & 2 & 0 & \frac{r}{\mu} \end{pmatrix}.$$

By multiplying (2.54) on the left by the row vector

$$\{(n+1)ru_n(r), nrv_n(r), (n+1)r\tau_{rr}^{(n)}(r), nr\tau_{r\phi}^{(n)}(r)\},$$

integrating the product over $[0,1]$, and using (2.8), we can obtain the desired energy identity

$$\begin{aligned} &(n+1)u_n(1)\tau_{rr}^{(n)}(1) + nv_n(1)\tau_{r\phi}^{(n)}(1) \\ &= 2(n+1)\int_0^1 \frac{\mu\,(3\lambda+2\mu)}{\lambda+2\mu}[u_n(r)\ v_n(r)]\begin{pmatrix} 2 & -n \\ -n & \frac{2n^2(\lambda+\mu)}{3\lambda+2\mu} - \frac{n(\lambda+2\mu)}{(n+1)(3\lambda+2\mu)} \end{pmatrix}\begin{pmatrix} u_n(r) \\ v_n(r) \end{pmatrix} dr \\ &\quad + \int_0^1 \left(\frac{n+1}{\lambda+2\,\mu}[r\tau_{rr}^{(n)}(r)]^2 + \frac{n}{\mu}[r\tau_{r\phi}^{(n)}(r)]^2 \right) dr. \end{aligned}$$

Here the determinant of the 2×2 matrix is $\frac{n(n-1)(n+2)(\lambda+2\mu)}{(n+1)(3\lambda+2\mu)}$, which shows the matrix is positive definite because of the convexity condition. The left-hand side of the identity corresponds to the work done by the boundary loads and the right-hand side corresponds to the strain energy. The significance of this reduced energy identity is that the results concerning the uniqueness for the forward problem in the present case are the same as those in the case of constant coefficients, e.g., [10]. In particular, certain combinations of boundary data such as $\{u(1,\phi), v(1,\phi)\}$, $\{u(1,\phi), \tau_{r\phi}(1,\phi)\}$, and $\{v(1,\phi), \tau_{rr}(1,\phi)\}$ uniquely determine the solution to the forward problem.

COROLLARY 2.4. *If $v(1,\phi)$ is given and $\tau_{rr}(1,\phi)=0$, then for sufficiently large n, the unique, bounded solution to system* (2.16)*, which satisfies the given boundary conditions, can be represented in the form of* (2.22)–(2.23) *with $a_n = 2\mu(1)nv_n(1)$ and the initial conditions for $\mathbf{l}(r)$ and $\mathbf{m}_i(r,s)$ given by*

$$\mathbf{l}(1) = [0,1]^\top, \tag{2.55}$$

$$\mathbf{m}_1(1,s) = \begin{pmatrix} 1 & 0 \\ -8\mu(1) & -1 \end{pmatrix}\mathbf{m}_2(1,s) + \begin{pmatrix} \frac{1}{\mu(1)} \\ -4 \end{pmatrix}. \tag{2.56}$$

Proof. By Theorem 2.3 and the uniqueness of the solution, we only need to show that the specified choice of $a_n, \mathbf{J}$, and $\mathbf{a}$ makes the boundary conditions $\tau_{rr}^{(n)}(1)=0$ be satisfied, and $v_n(1)$ obtained by substitution $r=1$ in (2.23) makes the boundary conditions agree with the given $v_n(1)$.

By using the definitions of $\mathbf{m}_i$, i.e., (2.29)–(2.30) and writing out the components $\mathbf{k}_i(1,s)$, we have

$$\mathbf{m}_1(1,s) = \begin{pmatrix} k_{11}(1,s)+k_{21}(1,s) \\ k_{12}(1,s)+k_{22}(1,s) \end{pmatrix}, \quad \mathbf{m}_2(1,s) = \begin{pmatrix} k_{11}(1,s)-k_{21}(1,s) \\ k_{12}(1,s)-k_{22}(1,s) \end{pmatrix}. \tag{2.57}$$

Substituting (2.57) into (2.56), we have

$$\begin{pmatrix} k_{11}(1,s)+k_{21}(1,s) \\ k_{12}(1,s)+k_{22}(1,s) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -8\mu(1) & -1 \end{pmatrix}\begin{pmatrix} k_{11}(1,s)-k_{21}(1,s) \\ k_{12}(1,s)-k_{22}(1,s) \end{pmatrix} + \begin{pmatrix} \frac{1}{\mu(1)} \\ -4 \end{pmatrix},$$

which can be simplified to

$$k_{21} = \frac{1}{2\mu(1)}, \tag{2.58}$$

$$k_{12}(1,s) + 4\mu(1)k_{11}(1,s) = 0. \tag{2.59}$$

In other words, (2.56) is nothing more than a matrix form of the last two equations. To check if the boundary conditions are satisfied, we substitute $a_n = 2\mu(1)nv_n(1)$, (2.55), and (2.58) into (2.51) and (2.52). Using (2.59), we get

$$\begin{aligned}
v_n(1) &= 2\mu(1)nv_n(1)\int_0^1 \frac{1}{2\mu(1)}s^{n-1}ds = v_n(1),\\
\tau_{rr}^{(n)}(1) &= 2\mu(1)nv_n(1)\left[-1 - \int_0^1 k_{12}(1,s)s^{n-1}ds + 2n\mu(1)\int_0^1 \frac{1}{2\mu(1)}(1,s)s^{n-1}ds\right.\\
&\qquad \left. -4\mu(1)\int_0^1 k_{11}(1,s)s^{n-1}ds\right]\\
&= -2\mu(1)nv_n(1)\int_0^1 (k_{12}(1,s) + 4\mu(1)k_{11})s^{n-1}ds = 0. \qquad \square
\end{aligned}$$

To see a general condition for (2.16) to have the solution in the form of (2.22)–(2.23) with $l_1(1) = 0$, let us examine the last equation without "$= 0$" at the end. By dividing the equation by $2\mu(1)nv_n(1)$, we have

$$\frac{\tau_{rr}^{(n)}(1)}{2\mu(1)nv_n(1)} = -\int_0^1 (k_{12}(1,s) + 4\mu(1)k_{11}(1,s))s^{n-1}ds.$$

We see that it is not necessary for $\tau_{rr}^{(n)}(1)/(2\mu(1)nv_n(1))$ and $k_{12}(1,s)+4\mu(1)k_{11}(1,s)$ to be zero. Instead, it suffices for $\{\tau_{rr}^{(n)}(1)/(2\mu(1)nv_n(1))\}_{n=1}^{\infty}$ to be the set of moments of some function in $C[0,1]$. For example, for arbitrary c, when $\tau_{rr}^{(n)}(1) = 2c\mu(1)v_n(1)$, we have $k_{12}(1,s) + 4\mu(1)k_{11} = -c$. Hence condition (1.12), namely, $\tau_{rr}^{(n)}(1) = 0$, is the simplest case when $c = 0$.

To solve the inverse problem, we need to determine $\mathbf{k}_i(1,s)$ from the knowledge of $\{u_n(1), v_n(1), \tau_{rr}^{(n)}(1), \tau_{r\phi}^{(n)}(1)\}$. By substituting $r = 1$ in (2.22), (2.23), we obtain the expressions of the moments of $\mathbf{k}_i(1,s)$; in particular,

$$\begin{aligned}
a_n\int_0^1 k_{11}(1,s)s^{n-1}ds &= u_n(1) - a_n l_1(1),\\
a_n\int_0^1 k_{22}(1,s)s^{n-1}ds &= z_n(1) - a_n l_2(1).
\end{aligned}$$

Under the assumptions of Corollary 2.4, by substituting the determined $a_n, l_1(1)$ and $l_2(1)$ into the equations above, also using (2.15), (2.58)–(2.59), we obtain

$$nv_n(1)\int_0^1 2\mu(1)k_{11}(1,s)s^{n-1}ds = u_n(1), \tag{2.60}$$

$$k_{12}(1,s) = -4\mu(1)k_{11}(1,s), \tag{2.61}$$

$$k_{21}(1,s) = \frac{1}{2\mu(1)}, \tag{2.62}$$

$$\begin{aligned}
&nv_n(1)\int_0^1 2\mu(1)k_{22}(1,s)s^{n-1}ds \\
&\quad = \tau_{r\phi}^{(n)}(1) - 2(n+1)\mu(1)u_n(1) + 2v_n(1)(\mu(1)+\mu'(1)) - 2\mu(1)nv_n(1).
\end{aligned} \tag{2.63}$$

COROLLARY 2.5. *If $u_n(1)$ is given and $\tau_{r\phi}(1,\phi) = 0$, then for sufficiently large n, the unique, bounded solution to the system of equation* (2.16) *satisfying the given boundary conditions can be represented in the form* (2.22)–(2.23) *with $a_n = -2\mu(1)(n+1)u_n(1)$ and the initial conditions for $\mathbf{l}(r)$ and $\mathbf{m}_i(r,s)$ given by*

$$\mathbf{l}(1) = [0,1]^\top, \tag{2.64}$$

$$\mathbf{m}_2(1,s) = \begin{pmatrix} -1 & 0 \\ -4(\mu(1)+\mu'(1)) & 1 \end{pmatrix} \mathbf{m}_1(1,s) + \frac{1}{\mu}\begin{pmatrix} -s \\ 2(\mu(1)+\mu'(1))s \end{pmatrix}. \tag{2.65}$$

Proof. Similarly, from (2.57), (2.65) can be seen to be nothing more than a matrix form of the following two equations:

$$k_{11}(1,s) = -\frac{s}{2\mu(1)}, \tag{2.66}$$

$$k_{22}(1,s) - 2(\mu(1)+\mu'(1))k_{21}(1,s) = 0. \tag{2.67}$$

Substituting $a_n = -2\mu(1)(n+1)u_n(1)$, (2.64), and (2.66) into (2.50) and (2.53), and using (2.67), we have

$$u_n(1) = -2\mu(1)(n+1)u_n(1)\int_0^1 \frac{-s}{2\mu(1)} s^{n-1}ds = u_n(1),$$

$$\begin{aligned}\tau_{r\phi}^{(n)}(1) &= -2\mu(1)(n+1)\,u_n(1)\left[1 + \int_0^1 k_{22}(1,s)s^{n-1}ds\right.\\ &\qquad \left. +2(n+1)\mu(1)\int_0^1 \frac{-s}{2\mu(1)}s^{n-1}ds - 2(\mu(1)+\mu'(1))\int_0^1 k_{21}(1,s)s^{n-1}ds\right]\\ &= -2\mu(1)(n+1)\,u_n(1)\int_0^1 [k_{22}(1,s) - 2(\mu(1)+\mu'(1))k_{21}(1,s)]s^{n-1}ds = 0.\end{aligned}$$ ∎

Under the assumption of Corollary 2.5, by substituting $r = 1$ in (2.22), (2.23), (2.14) and using the results from the proof, we have the following equations concerning the determination of $\mathbf{k}_i(1,s)$:

$$k_{11}(1,s) = -\frac{s}{2\mu(1)}, \tag{2.68}$$

$$\begin{aligned}&2\mu(1)(n+1)u_n(1)\int_0^1 k_{12}(1,s)s^{n-1}ds\\ &\quad = \tau_{rr}^{(n)}(1) - 2n\mu(1)v_n(1) + \frac{4\mu(1)(n+1)u_n(1)}{n+1} - 2\mu(1)(n+1)u_n(1),\end{aligned} \tag{2.69}$$

$$2\mu(1)(n+1)u_n(1)\int_0^1 k_{21}(1,s)s^{n-1}ds = v_n(1), \tag{2.70}$$

$$k_{22}(1,s) = 2(\mu(1)+\mu'(1))k_{21}(1,s). \tag{2.71}$$

COROLLARY 2.6. *If $\{v(1,\phi), \tau_{rr}(1,\phi)\}$ are given and* (1.16) *holds, then for sufficiently large n, the unique, bounded solution to the system of equation* (2.16) *satisfying the given boundary conditions can be represented in the form of* (2.22)–(2.23) *with $a_n = v_n(1)$ and the initial conditions for $\mathbf{l}(r)$ and $\mathbf{m}_i(1,s)$ given by*

$$\mathbf{l}(1) = [1, -4\mu(1)]^\top, \tag{2.72}$$

$$\mathbf{m}_1(1,s) = \begin{pmatrix} 1 & 0 \\ -8\mu(1) & -1 \end{pmatrix} \mathbf{m}_2(1,s). \tag{2.73}$$

Proof. First, we rewrite (1.16) by using the operator $\mathcal{S}$ defined by (2.2):

$$\tau_{rr}(1,\phi) + 2\mu(1)\mathcal{S}v(1,\phi) = 0,$$

which leads to

$$\tau_{rr}^{(n)}(1) = -n(2\mu(1)\mathcal{S}v(1,\cdot), p_n)_w = 2n\mu(1)(v(1,\cdot), \mathcal{S}p_n)_w = 2\mu(1)nv_n(1). \tag{2.74}$$

Using (2.57) again, we can see (2.73) to be the same as

$$k_{21}(1,s) = 0, \tag{2.75}$$

$$k_{12}(1,s) + 4\mu(1)k_{11}(1,s) = 0. \tag{2.76}$$

Substituting $a_n = v_n(1)$, (2.72), (2.75), and (2.76) into (2.51) and (2.52), we get

$$\begin{aligned} &v_n(1) = v_n(1), \\ &\tau_{rr}^{(n)}(1) = v_n(1)\left[4\mu(1) - \int_0^1 k_{12}(1,s)s^{n-1}ds \right. \\ &\left. +2n\mu(1) - 4\mu(1)\left(1 + \int_0^1 k_{11}(1,s)s^{n-1}ds\right)\right] = 2n\mu(1)v_n(1). \end{aligned}$$

The last equation is true because of (2.74). ▯

Under the assumption of Corollary 2.6, by substituting $r = 1$ in (2.22), (2.23), (2.15), and using the results from the proof, we have

$$v_n(1)\int_0^1 k_{11}(1,s)s^{n-1}ds = u_n(1) - v_n(1), \tag{2.77}$$

$$k_{12}(1,s) = -4\mu(1)k_{11}(1,s), \tag{2.78}$$

$$k_{21}(1,s) \equiv 0, \tag{2.79}$$

$$\begin{aligned} &v_n(1)\int_0^1 k_{22}(1,s)s^{n-1}ds \\ &= \tau_{r\phi}^{(n)}(1) - 2(n+1)\mu(1)u_n(1) + (6\mu(1) + 2\mu'(1))v_n(1). \end{aligned} \tag{2.80}$$

COROLLARY 2.7. *If $\{u(1,\phi), \tau_{r\phi}(1,\phi)\}$ are given and (1.17) holds, then for sufficiently large n, the unique, bounded solution to the system (2.16) satisfying the given boundary conditions can be represented in the form of (2.22)–(2.23) with $a_n = u_n(1)$ and the initial conditions for $\mathbf{l}(r)$ and $\mathbf{m}_i(r,s)$ given by*

$$\mathbf{l}(1) = [1, 2(\mu(1) + \mu'(1))]^\top, \tag{2.81}$$

$$\mathbf{m}_2(1,s) = \begin{pmatrix} -1 & 0 \\ -4(\mu(1) + \mu'(1)) & 1 \end{pmatrix} \mathbf{m}_1(1,s). \tag{2.82}$$

Proof. First, from (1.17) and (2.7) we have

$$\begin{aligned} \tau_{r\phi}^{(n)}(1) &= 2\mu(1)(\mathcal{D}u(1,\cdot), \mathcal{D}p_n)_w \\ &= 2\mu(1)(n+1)n(u(1,\cdot), p_n)_w = 2(n+1)\mu(1)u_n(1). \end{aligned} \tag{2.83}$$

And (2.82) is equivalent to

$$\begin{aligned} &k_{11} = 0, \\ &k_{22}(1,s) - 2(\mu(1) + \mu'(1))k_{21}(1,s) = 0. \end{aligned}$$

Substituting $a_n = u_n(1)$, (2.81), and the previous two equations into (2.50) and (2.53), we have

$$
\begin{aligned}
u_n(1) &= u_n(1),\\
\tau_{r\phi}^{(n)} &= u_n(1)\left[2(\mu(1)+\mu'(1)) + \int_0^1 k_{22}(1,s)s^{n-1}ds + 2(n+1)\mu(1)\right.\\
&\quad \left. -2(\mu(1)+\mu'(1))\left(1+\int_0^1 k_{21}(1,s)s^{n-1}ds\right)\right]\\
&\quad = 2(n+1)\mu(1)u_n(1).
\end{aligned}
$$

The previous equation is true because of (2.83).

Under the assumption of Corollary 2.7, by substituting $r = 1$ in (2.22), (2.23), (2.14), and using the results from the proof, we have

$$
k_{11}(1,s) = 0, \tag{2.84}
$$

$$
\begin{aligned}
&u_n(1)\int_0^1 k_{12}(1,s)s^{n-1}ds \\
&= -\tau_{rr}^{(n)}(1) + 2n\mu(1)v_n(1) - (6\mu(1)+2\mu'(1))u_n(1),
\end{aligned} \tag{2.85}
$$

$$
u_n(1)\int_0^1 k_{21}(1,s)s^{n-1}ds = v_n(1) - u_n(1), \tag{2.86}
$$

$$
k_{22}(1,s) = 2(\mu(1)+\mu'(1))k_{21}(1,s). \tag{2.87}
$$

A sufficient condition for (2.16) to have the solution of the form (2.22)–(2.23) with $l_1(1) \neq 0$ can be seen to be

$$
\tau_{rr}^{(n)}(1) = (2\mu(1)n + q_n)v_n(1), \quad q_n = O(1), \tag{2.88}
$$

or

$$
\tau_{r\phi}^{(n)}(1) = (2\mu(1)n + q_n)u_n(1). \tag{2.89}
$$

These conditions are the generalizations of (1.16) and (1.17), respectively.

Finally, if we let n_1 be the maximum limitation number of the transmutations associated with the boundary conditions in the corollaries above, then from the proofs of Lemma 2.2 and Corollary 2.4–2.7, we have the estimate $n_1 \leq N$, where

$$
N = N(\|\mu\|_{C^2[0,1]}, \|(1/\mu)\|_{C[0,1]}, \|\lambda\|_{C[0,1]}) = \Omega_1(\bar{\Omega}_2 + 1) \tag{2.90}
$$

with the Ω_1 bounded by (2.49) and $\bar{\Omega}_2 = 8\|\mu\|_{C^1[0,1]} + 1$.

2.3. Numerical verification. Since the transmutation representation (2.22)–(2.23) is crucial to our further discussions about the inverse problem, we verify it numerically. Let us use $\lambda = 1+\sin r$, $\mu = (1+r)/2$, and initial conditions (2.72)–(2.73) to do a computational example. We first numerically compute $\mathbf{k}_i(r,s)$ at $(r_i, s_j), j \leq i = 0,1,2,\ldots,$ where $r_0 = 1, r_1 = e^{1-\delta}$, $r_{i+1} = r_1 r_i$, $s_j = r_j$; then compute the integrals in (2.22)–(2.23) to get an approximate solution $\mathbf{x}_\delta$ to (2.16). Finally, we compute the maximum of the relative errors

$$
Er(\delta) := \max_{0<r_i<1} \left\| r_i \frac{d}{dr}\mathbf{x}_\delta(r_i) + \tilde{\mathbf{A}}_n(r_i)\mathbf{x}_\delta(r_i) \right\| / \left\| r_i \frac{d}{dr}\mathbf{x}_\delta(r_i) \right\|,
$$

where $\tilde{\mathbf{A}}_n$ is the matrix in (2.16), $||\cdot||$ is l_1 norm for the vector, and d/dr is understood to be the central difference ratio.

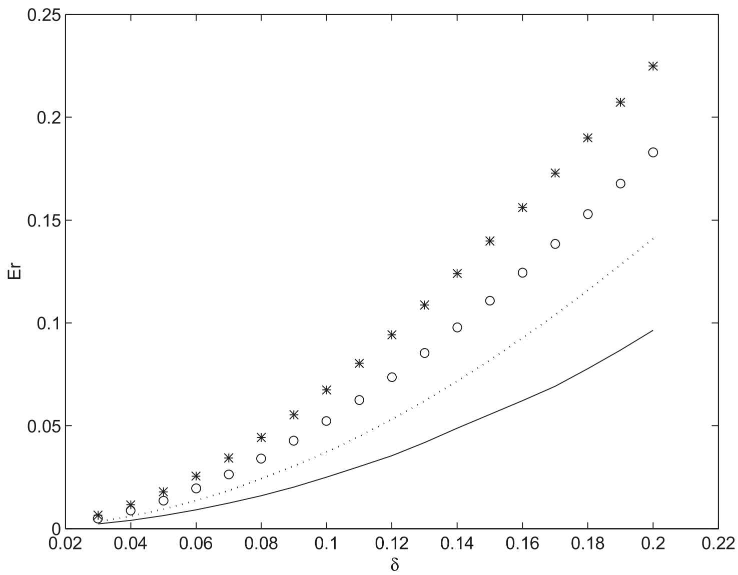

FIG. 2.1. *The plot of Er against δ for $n = 2$(solid), $n = 3$(dot), $n = 4$(circle), and $n = 5$(star).*

To verify that the transmutation formula is correct, we want to show by computations that

$$\lim_{\delta \to 0} Er(\delta) = 0.$$

The result, illustrated in Fig. 2.1, shows that the transmutation representation holds for $n > 1$. The convergent rate appears quadratic because we have used linear interpolations.

3. The uniqueness of the Lamé coefficients. Although this section mainly concerns the proof of Theorem 1.2, we slightly extend our discussions in order to form a procedure for solving the problem.

First, to deal with the limitation number of the transmutation, we introduce the set of function pairs

$$\mathcal{H}_\beta = \left\{ (\lambda, \mu) \in \mathcal{H} \,\middle|\, N(\|\mu\|_{C^2[0,1]}, \|(1/\mu)\|_{C[0,1]}, \|\lambda\|_{C[0,1]}) \le \beta \right\}$$

for arbitrary $\beta \in (0, \infty)$, where N was defined by (2.90). To show that (λ,μ) is unique in $\mathcal{H}$, it suffices to show that (λ, μ) is unique in $\mathcal{H}_\beta$ for an arbitrary $\beta \in (0, \infty)$ since $\mathcal{H} = \cup \mathcal{H}_\beta$. Therefore, we can restrict (λ, μ) to lie in $\mathcal{H}_\beta$.

We prove Theorem 1.2(i) first. Suppose we are given the list of boundary data $\{u(1, \phi), v(1, \phi), \tau_{rr}(1, \phi), \tau_{r\phi}(1, \phi)\}$ of a nondestructive steady state, which satisfies (1.14) with $\tau_{rr}(1, \phi) = 0$. Then there is a family of $\{u_n(r), y_n(r), v_n(r), z_n(r)\}_{n=1}^{\infty}$ satisfying the system of ordinary differential equations (2.16). By Corollary 2.4, there is a sequence $\{a_n\}_{n=1}^{\infty}$ and vector functions $\mathbf{l}$, $\mathbf{k}_1$, and $\mathbf{k}_2$ and a_n such that (2.22) and (2.23) hold for $n \ge \beta$. While these vector functions satisfy (2.25)–(2.28), (2.55), $\mathbf{k}_1(1, s)$ and $\mathbf{k}_2(1, s)$ have to satisfy the additional equations (2.60)–(2.63). Now let

$$\mathbb{M}_u := \left\{ n \in \mathbb{N} \;\middle|\; n > \beta, u_n(1) \neq 0 \right\}, \tag{3.1}$$

$$\mathbb{M}_v := \left\{ n \in \mathbb{N} \;\middle|\; n > \beta, v_n(1) \neq 0 \right\}. \tag{3.2}$$

Then the set $\mathbb{M}$, which was defined by (1.15), can be written as

$$\mathbb{M} = \mathbb{M}_f \cup \mathbb{M}_u \cup \mathbb{M}_v, \tag{3.3}$$

where $\mathbb{M}_f \subset \{1, 2, 3, \ldots, \beta\}$ is a finite set. On the other hand, (2.60) implies that $\mathbb{M}_u \subset \mathbb{M}_v$. Therefore, $\mathbb{M} = \mathbb{M}_f \cup \mathbb{M}_v$ and the condition (1.14) is equivalent to

$$\sum_{n \in \mathbb{M}_v} \frac{1}{n} = \infty, \tag{3.4}$$

which implies that $\mathbb{M}_v$ is an infinite set. To determine $\mu(1)$, first we need to show

$$\lim_{n \in \mathbb{M}_v, n \to \infty} \frac{u_n}{v_n} \neq -1.$$

To this end, we multiply both sides of (2.60) by $\frac{n+\beta+1}{n v_n(1)}$,

$$\int_0^1 2\mu(1) k_{11}(1,s) s^{\beta+1} n s^{n-1} ds = \frac{n+\beta+1}{n} \frac{u_n(1)}{v_n(1)} \quad \text{for } n \in \mathbb{M}_v. \tag{3.5}$$

Denote $k_{11}(1,s)s^{\beta+1}$ by $f(s)$. By the proof of Lemma 2.2 and the assumption $(\lambda, \mu) \in \mathcal{H}_\beta$, $f(s) \in C[0,1]$ and $\lim_{s \to 0} f(s) = 0$. We rewrite (3.5) using $f(s)$ and change the variable of integration. This gives

$$\int_0^1 2\mu(1) f(t^{1/n}) dt = \frac{n+\beta+1}{n} \frac{u_n(1)}{v_n(1)} \quad \text{for } n \in \mathbb{M}_v.$$

Because of the properties of $f(s)$,

$$\lim_{n \in \mathbb{M}_v, n \to \infty} \int_0^1 2\mu(1) f(t^{1/n}) dt = 2\mu(1) f(1) = 2\mu(1) k_{11}(1,1).$$

Hence we have

$$\lim_{n \in \mathbb{M}_v, n \to \infty} \frac{u_n(1)}{v_n(1)} = 2\mu(1) k_{11}(1,1). \tag{3.6}$$

Next we need to find an expression of $k_{11}(1,1)$. To this end, let us set $r = 1$ in (2.34). This gives

$$\mathbf{m}_2(1,1) = -\mathbf{F}(1)\mathbf{l}(1).$$

By using the definition of $\mathbf{m}_2$ and the expression of $\mathbf{F}$, we have

$$\begin{pmatrix} k_{11}(1,1) - k_{21}(1,1) \\ k_{12}(1,1) - k_{22}(1,1) \end{pmatrix} = -\begin{pmatrix} F_{11}(1) & \frac{1}{2}\left(\frac{1}{\lambda+2\mu} + \frac{1}{\mu(1)}\right) \\ F_{21}(1) & F_{22}(1) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

from which it follows that

$$k_{11}(1,1) - k_{21}(1,1) = -\frac{1}{2}\left(\frac{1}{\lambda(1) + 2\mu(1)} + \frac{1}{\mu(1)}\right).$$

On the other hand, (2.62) gives $k_{21}(1,s) = \frac{1}{2\mu(1)}$. Hence

$$k_{11}(1,1) = -\frac{1}{2(\lambda(1) + 2\mu(1))}.$$

Substituting this into (3.6) and using (1.10) we obtain

$$\lim_{n\in\mathbb{M}_v, n\to\infty} \frac{u_n(1)}{v_n(1)} = -\frac{\mu(1)}{\lambda(1)+2\mu(1)} = -1 + \frac{\lambda(1)+\mu(1)}{\lambda(1)+2\mu(1)} > -1. \tag{3.7}$$

Now by dividing both sides of (2.63) by $nv_n(1)$, we obtain that for $n \in \mathbb{M}_v$,

$$\int_0^1 2\mu(1)k_{22}(1,s)s^{n-1}ds = \frac{\tau_{r\phi}^{(n)}(1)}{nv_n(1)} - 2\mu(1)\frac{(n+1)u_n(1)}{nv_n(1)} + 2\frac{\mu(1)+\mu'(1)}{n} - 2\mu(1).$$

By letting n approaching ∞, we obtain

$$0 = \lim_{n\in\mathbb{M}_v, n\to\infty} \frac{\tau_{r\phi}^{(n)}(1)}{nv_n(1)} - 2\mu(1)\left(\lim_{n\in\mathbb{M}_v, n\to\infty} \frac{u_n(1)}{v_n(1)} + 1\right). \tag{3.8}$$

Taking (3.7) into account, we obtain

$$\mu(1) = \frac{1}{2}\frac{\lim_{n\in\mathbb{M}_v, n\to\infty} \frac{\tau_{r\phi}^{(n)}(1)}{nv_n(1)}}{\lim_{n\in\mathbb{M}_v, n\to\infty} \frac{u_n(1)}{v_n(1)} + 1}, \tag{3.9}$$

which can be simplified to

$$\mu(1) = \lim_{n\in\mathbb{M}_v, n\to\infty} \frac{\tau_{r\phi}^{(n)}(1)}{2n(v_n(1)+u_n(1))}. \tag{3.10}$$

Similarly, if the given boundary data satisfies $\tau_{r\phi}(1,\phi) = 0$, then we make use of Corollary 2.5 and (2.68)–(2.71). In fact, we have $\mathbb{M}_v \subset \mathbb{M}_u$ in this case, and from (2.70) we can obtain

$$\lim_{n\in\mathbb{M}_u, n\to\infty} \frac{v_n(1)}{u_n(1)} = -2\mu k_{21}(1,1).$$

Also by using (2.34) and (2.68), we can show that this limit cannot be -1. Finally, by using (2.69), we obtain

$$\mu(1) = \lim_{n\in\mathbb{M}_u, n\to\infty} \frac{\tau_{rr}^{(n)}(1)}{2n(v_n(1)+u_n(1))}. \tag{3.11}$$

This completes the proof of Theorem 1.2(i).

Remark. (i) We actually don't need condition (1.14) for determining $\mu(1)$, but we only need $\mathbb{M}$ to be an infinite set. (ii) (3.7) shows that $\lambda(1)$ also can be determined. (iii) In the case of constant coefficients and $\tau_{rr}(1,\phi) = 0$, a straightforward computation shows

$$\mu = \frac{\tau_{r\phi}^{(n)}(1)}{2(n+1)(u_n(1)+v_n(1))} \quad \text{for all } n \geq 1.$$

Let us go on to prove Theorem 1.2(ii). First we note that (2.60)–(2.63) or (2.68)–(2.71) provide us with much more than (3.10) or (3.11). To see this, let us consider the case $\tau_{rr}(1,\phi)=0$ first. Equations (2.60) and (2.63) can be rewritten as

$$\int_0^1 (k_{11}(1,s)s^\beta)s^{n-1-\beta}ds = \frac{u_n(1)}{2\mu(1)nv_n(1)} \qquad \text{for } n\in\mathbb{M}_v,$$
$$\int_0^1 \left(k_{22}(1,s)-\frac{\mu'(1)}{\mu(1)}\right)s^\beta s^{n-1-\beta}ds$$
$$= \frac{\tau_{r\phi}^{(n)}(1)-2(n+1)\mu(1)u_n(1)}{2\mu(1)nv_n(1)}+\frac{1}{n}-1 \qquad \text{for } n\in\mathbb{M}_v.$$

Hence the $(n-1-\beta)$th moments of the functions $k_{11}(1,s)s^\beta$ and $(k_{22}(1,s)s^{n-1}-\frac{\mu'(1)}{\mu(1)})s^\beta$, $n\in\mathbb{M}_v$, are determined by the given boundary data. Now let us recall Müntz's theorem [5]: if $\mathbb{M}_1$ is a set of natural numbers, then the set of functions $\{x^n\}_{n\in\mathbb{M}_1}$ is complete in $L^1[0,1]$ if and only if

$$\sum_{n\in\mathbb{M}_1}\frac{1}{n}=\infty. \tag{3.12}$$

Müntz's theorem applies to the present problem because $k_{11}(1,s)s^\beta\in L^1[0,1]$, $(k_{22}(1,s)s^{n-1}-\frac{\mu'(1)}{\mu(1)})s^\beta\in L^1[0,1]$ and by (3.4),

$$\sum_{n\in\mathbb{M}_v}\frac{1}{n-\beta}=\sum_{n\in\mathbb{M}_v}\frac{1}{n}=\infty.$$

Hence $k_{11}(1,s)$ and $k_{22}(1,s)s^{n-1}-\frac{\mu'(1)}{\mu(1)}$ are determined by the list of boundary data. Consequently, $k_{21}(1,s)$ and $k_{12}(1,s)$ can be determined by (2.61) and (2.62). Similarly, if $\tau_{r\phi}(1,\phi)=0$, by using (2.68)–(2.71) we can determine $k_{11}(1,s),k_{12}(1,s),k_{21}(1,s)$ completely and leave $k_{22}(1,s)=2(\mu(1)+\mu'(1))k_{21}(1,s)$. Hence $k_{ij}(1,s)$ can be determined completely once $\mu'(1)$ is known.

To determine $\mu'(1)$, we have to make use of the second list of boundary data, which is associated with the selection schemes of type (II). Corollaries 2.6 and 2.7 show the validity of the transmutation representation of $\{u_n(r),y_n(r),v_n(r),z_n(r)\}$ for $n>\beta$. The moments of the corresponding $\mathbf{k}_i(1,s)$ are given by (2.77)–(2.80) and (2.84)–(2.87). If the scheme of type (II) (1.16) is adopted, then by taking the limits of the both sides of (2.80) as $n\to\infty$, we get

$$\mu'(1)=-3\mu(1)+\lim_{n\in\mathbb{M}_v,n\to\infty}\frac{2(n+1)\mu(1)u_n(1)-\tau_{r\phi}^{(n)}(1)}{2v_n(1)}.$$

If type (II) (1.17) is adopted, similarly, using (2.85), we obtain

$$\mu'(1)=-3\mu(1)+\lim_{n\in\mathbb{M}_u,n\to\infty}\frac{2n\mu(1)v_n(1)-\tau_{rr}^{(n)}(1)}{2u_n(1)}.$$

Therefore, $\mu'(1)$ can be determined and $\mathbf{l}(1)$ is also known by (2.72) or (2.81). Note that $l_1(1)=1$ implies that $u_n(1)=v_n(1+O(1/n))$ and as a result of this,

$$\sum_{n\in\mathbb{M}}\frac{1}{n}=\sum_{n\in\mathbb{M}_u}\frac{1}{n}=\sum_{n\in\mathbb{M}_v}\frac{1}{n}=\infty.$$

Therefore, the application of Müntz's theorem yields the determinations of all $k_{ij}(1,s)$ for both type (II) and type (I) selection schemes.

By summarizing the discussions above, we have actually obtained the uniqueness for the following first subproblem that resulted from the decomposition of the inverse problem.

Subproblem 1. Given $\{u(1), v(1), \tau_{rr}(1), \tau_{r\phi}(1)\}$ that satisfy (1.14) and (1.12) or (1.13) or (1.16) or (1.17), find the corresponding $\mathbf{k}_i(1,s)$.

After we find $\mathbf{k}_i(1,s)$ or, equivalently, $\mathbf{m}_i(1,s)$, the inverse problem of determining (λ,μ) using a single list of boundary data is reduced to the following inverse problem for a system of hyperbolic equations.

Given $[a,b]^\top$, $\mathbf{h}_1(s)$, and $\mathbf{h}_2(s)$, find λ and μ such that

$$r\mathbf{l}'(r) + \mathbf{E}(r)\mathbf{l}(r) = \mathbf{0},\ 0<r<1, \tag{3.13}$$

$$\mathbf{l}(1) = [a,b]^\top, \tag{3.14}$$

$$\left(r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}\right)\mathbf{m}_1(r,s) + \mathbf{E}(r)\mathbf{m}_1(r,s) + (\mathbf{F}(r)-\mathbf{I})\mathbf{m}_2(r,s) = \mathbf{0},\ 0<s<r<1, \tag{3.15}$$

$$\left(r\frac{\partial}{\partial r} - s\frac{\partial}{\partial s}\right)\mathbf{m}_2(r,s) + \mathbf{F}(r)\mathbf{m}_1(r,s) + (\mathbf{E}(r)+\mathbf{I})\mathbf{m}_2(r,s) = \mathbf{0},\ 0<s<r<1, \tag{3.16}$$

$$\mathbf{m}_1(1,s) = \mathbf{h}_1(s),\ 0<s\le 1, \tag{3.17}$$

$$\mathbf{m}_2(1,s) = \mathbf{h}_2(s),\ 0<s\le 1, \tag{3.18}$$

$$\mathbf{m}_2(r,r) + \mathbf{F}(r)\mathbf{l}(r) = \mathbf{0},\ 0<r\le 1. \tag{3.19}$$

We may view these equations as an iteration procedure for finding λ and μ. For an approximate solution $\{\lambda,\mu\}$, we solve the initial value problem (3.13)–(3.14) for $\mathbf{l}(r)$, we solve the Cauchy problem (3.15)–(3.18) for $\mathbf{m}_i(r,s)$, and finally we update λ and μ by solving (3.19) for μ'' and $\frac{1}{\lambda+2\mu}$:

$$\frac{1}{\lambda+2\mu} \leftarrow \frac{1}{\mu} - \frac{2}{l_2(r)}\left[\left(1-\frac{r\mu'}{\mu}\right)l_1(r) + m_{21}(r,r,)\right],$$
$$r(r\mu')' \leftarrow 2r\mu'(r\mu-\mu) - \frac{1}{l_1(r)}[r\mu' l_2(r) + \mu m_{22}(r,r)].$$

However, $l_1(r)$ and $l_2(r)$ serve as the denominators in this iteration. As a result, the iterations in general will not converge unless we can make $l_1(r)$, $l_2(r)\neq 0$ for all $r\in(0,1]$. It turns out this can be done only by using an inconvenient restriction, namely, $E_{21}\le 0$, on the unknown coefficients. This is why we have to use two lists of boundary data or repeat the experiment, even if we were given $\mu(1)$ and $\mu'(1)$.

We use superscripts (I) and (II) to distinguish all quantities corresponding to two different types of selection schemes. Denoting by

$$c = \begin{cases} -4\mu(1) & \text{if (1.16) is adopted,} \\ 2(\mu(1)+\mu'(1) & \text{if (1.17) is adopted,}\end{cases}$$

from (2.72) and (2.81), we have $\mathbf{l}^{(\mathrm{II})}(1) = [1,c]^\top$.

We build the following matrices:

$$\mathbf{L}(r) := [\mathbf{l}^{(\mathrm{I})}(r)\ \mathbf{l}^{(\mathrm{II})}(r)]_{2\times 2}, \mathbf{Y} := \begin{pmatrix} 0 & 1 \\ 1 & c\end{pmatrix},$$
$$\mathbf{M}_i(r,s) := [\mathbf{m}_i^{(\mathrm{I})}(r)\ \mathbf{m}_i^{(\mathrm{II})}(r,s)]_{2\times 2}, \mathbf{H}_i(s) := [\mathbf{h}_i^{(\mathrm{I})}(r)\ \mathbf{h}_i^{(\mathrm{II})}(s)]_{2\times 2}, \quad i=1,2.$$

We note that the use of two different types of selection schemes results in the obvious existence of $\mathbf{Y}^{-1}$, which is crucial to the further decomposition of the inverse problem.

Now we can reformulate the inverse problem for hyperbolic equations.

Given two matrices of functions $\mathbf{H}_1(s)$ and $\mathbf{H}_1(s)$, find λ and μ and 2×2 matrices $\mathbf{L}(r)$, $\mathbf{M}_1(r,s)$, $\mathbf{M}_2(r,s)$ such that

(3.20) $r\mathbf{L}'(r) + \mathbf{E}(r)\mathbf{L}(r) = \mathbf{0}$, $0 < r < 1$,

(3.21) $\mathbf{L}(1) = \mathbf{Y}$,

(3.22) $\left(r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}\right)\mathbf{M}_1(r,s) + \mathbf{E}(r)\mathbf{M}_1(r,s) + (\mathbf{F} - \mathbf{I})\mathbf{M}_2(r,s) = \mathbf{0}$, $0 < s < r < 1$,

(3.23) $\left(r\frac{\partial}{\partial r} - s\frac{\partial}{\partial s}\right)\mathbf{M}_2(r,s) + \mathbf{F}(r)\mathbf{M}_1(r,s) + (\mathbf{E} + \mathbf{I})\mathbf{M}_2(r,s) = \mathbf{0}$, $0 < s < r < 1$,

(3.24) $\mathbf{M}_1(1,s) = \mathbf{H}_1(s)$, $0 < s \le 1$,

(3.25) $\mathbf{M}_2(1,s) = \mathbf{H}_2(s)$, $0 < s \le< 1$,

(3.26) $\mathbf{M}_2(r,r) + \mathbf{F}(r)\mathbf{L}(r) = \mathbf{0}.\ 0 < r < 1.$

We solve this problem in two steps.

Step 1. First we note that $\mathbf{L}(r)$ is comprised of two linear independent solutions of the differential equation $r\mathbf{y}' + \mathbf{E}\mathbf{y} = \mathbf{0}$ because $\mathbf{L}(1) = \mathbf{Y}$ is nonsingular. Hence $\mathbf{L}^{-1}(r)$ exits for all $r \in (0,1]$, which allows us to denote

$$\mathbf{G}(r) := \mathbf{L}^{-1}(r)\mathbf{F}(r)\mathbf{L}(r),\ \ \mathbf{P}_1(r) := \mathbf{L}^{-1}(r)\mathbf{M}_1(r),\ \ \mathbf{P}_2(r) := \mathbf{L}^{-1}(r)\mathbf{M}_2(r). \tag{3.27}$$

From $\mathbf{L}(r)\mathbf{L}^{-1}(r) = \mathbf{I}$ and (3.20) it follows that

$$r\frac{d}{dr}\mathbf{L}^{-1}(r) = -\mathbf{L}^{-1}(r)r\mathbf{L}'(r)\mathbf{L}^{-1}(r) = \mathbf{L}^{-1}(r)\mathbf{E}(r). \tag{3.28}$$

By multiplying (3.22) on the left by $\mathbf{L}^{-1}(r)$ and using (3.27) and (3.28), we have

$$\begin{aligned}\mathbf{0} &= \mathbf{L}^{-1}(r)\left(r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}\right)\mathbf{M}_1(r,s) + \mathbf{L}^{-1}(r)\mathbf{E}(r)\mathbf{M}_1(r,s) + \mathbf{L}^{-1}(r)(\mathbf{F}(r) - \mathbf{I})\mathbf{M}_2(r,s)\\ &= \mathbf{L}^{-1}(r)r\frac{\partial}{\partial r}\mathbf{M}_1(r,s) + \left[r\frac{d}{dr}\mathbf{L}^{-1}(r)\right]\mathbf{M}_1(r,s)\\ &\quad + s\frac{\partial}{\partial s}(\mathbf{L}^{-1}(r)\mathbf{M}_1(r,s)) + (\mathbf{G}(r) - \mathbf{I})\mathbf{P}_2(r)\\ &= \left(r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}\right)\mathbf{P}_1(r,s) + (\mathbf{G}(r) - \mathbf{I})\mathbf{P}_2(r,s), \quad 0 < s < r < 1.\end{aligned}$$

By multiplying (3.23) on the left by $r\mathbf{L}^{-1}(r)$ and using (3.27) and (3.28), we get

$$\left(r\frac{\partial}{\partial r} - s\frac{\partial}{\partial s}\right)(r\mathbf{P}_2(r,s)) + r\mathbf{G}(r)\mathbf{P}_1(r,s) = \mathbf{0}, \quad 0 < s < r < 1.$$

Therefore, we have the second subproblem.

Subproblem 2. Given two 2×2 matrices of functions $\mathbf{H}_1(r)$ and $\mathbf{H}_2(r)$, find 2×2 matrices of functions $\mathbf{P}_1(r,s)$, $\mathbf{P}_2(r,s)$, and $\mathbf{G}(r)$ such that

$$\left(r\frac{\partial}{\partial r}+s\frac{\partial}{\partial s}\right)\mathbf{P}_1(r,s)+(\mathbf{G}(r)-\mathbf{I})\mathbf{P}_2(r,s)=\mathbf{0},\quad 0<s<r<1, \tag{3.29}$$

$$\left(r\frac{\partial}{\partial r}-s\frac{\partial}{\partial s}\right)(r\mathbf{P}_2(r,s))+r\mathbf{G}(r)\mathbf{P}_1(r,s)=\mathbf{0},\quad 0<s<r<1, \tag{3.30}$$

$$\mathbf{P}_1(1,s)=\mathbf{Y}^{-1}\mathbf{H}_1(s), \tag{3.31}$$

$$\mathbf{P}_2(1,s)=\mathbf{Y}^{-1}\mathbf{H}_2(s), \tag{3.32}$$

$$\mathbf{P}_2(r,r)+\mathbf{G}(r)=\mathbf{0}. \tag{3.33}$$

We postpone the proof of the following lemma until later in this paper.

LEMMA 3.1. *Subproblem* 2 *has at most one solution* $(\mathbf{P}_1(r,s),\mathbf{P}_2(r,s),\mathbf{G}(r))\in \mathbf{C}(D_2)\times\mathbf{C}(D_2)\times\mathbf{C}(0,1]$.

Step 2. Having found $\mathbf{G}(r)$, we have the last subproblem; namely, given $\mathbf{G}(r)$, find $\mathbf{L}(r)$, λ, and μ such that (3.20), (3.21), and

$$\mathbf{L}^{-1}(r)\mathbf{F}(r)\mathbf{L}(r)=\mathbf{G}(r) \tag{3.34}$$

are satisfied.

We can simplify this problem by using the matrix $\mathbf{T}$ and $\mathbf{Q}$ introduced by (2.39) and (2.41) in the previous section. To this end, let $\mathbf{Z}(r):=\mathbf{T}(r)\mathbf{L}$. By multiplying (3.20) on the left by $\mathbf{T}(r)$ and using (2.40)–(2.41), we get

$$r\mathbf{Z}'(r)+\mathbf{Q}(r)\mathbf{Z}(r)=\mathbf{0},$$

where

$$\mathbf{Q}(r)=\eta\begin{pmatrix}-r\mu' & -(r\mu)^2\\ (\frac{\mu'}{\mu})^2 & r\mu'\end{pmatrix} \tag{3.35}$$

and

$$\eta:=-E_{12}=\frac{1}{2}\left(\frac{1}{\mu}-\frac{1}{\lambda+2\mu}\right)=\frac{1}{2}\frac{\lambda+\mu}{\mu(\lambda+2\mu)}>0. \tag{3.36}$$

By multiplying (3.34) on the right by $\mathbf{Z}^{-1}(r)$ and on the left by $\mathbf{Z}(r)$, we get

$$\mathbf{T}(r)\mathbf{F}(r)\mathbf{T}^{-1}(r)=\mathbf{Z}(r)\mathbf{G}(r)\mathbf{Z}^{-1}(r).$$

Evaluating the matrix product on the left-hand side of the last equation, we obtain

$$\mathbf{W}(r):=\mathbf{T}(r)\mathbf{F}(r)\mathbf{T}^{-1}(r)=\begin{pmatrix}1-r\mu'\eta & (r\mu)^2(\frac{1}{\mu}-\eta)\\ \frac{\mu'+(r\mu')'}{r\mu^2}+\frac{(\mu')^2(\eta\mu-1)}{\mu^3} & r\mu'\eta\end{pmatrix}. \tag{3.37}$$

Hence we have the following formulation.

Subproblem 3. Given $\mathbf{G}(r)$, $\mu(1)$, and $\mu'(1)$, find $\mathbf{Z}(r)$, η, and μ such that

$$r\mathbf{Z}'(r)+\mathbf{Q}(r)\mathbf{Z}(r)=\mathbf{0}, \tag{3.38}$$

$$\mathbf{Z}(1)=\mathbf{T}(1)\mathbf{Y}=\begin{pmatrix}1/\mu(1) & 0\\ c\mu(1)-\mu'(1)/\mu(1) & \mu(1)\end{pmatrix}, \tag{3.39}$$

$$\mathbf{S}(r):=\mathbf{Z}(r)\mathbf{G}(r)\mathbf{Z}^{-1}(r)=\mathbf{W}(r), \tag{3.40}$$

where $\mathbf{Q}(r)$ and $\mathbf{W}(r)$ are defined as in (3.35) and (3.37), respectively.

For an arbitrarily given $\mathbf{G}(r)$, Subproblem 3 may be overspecified since there are four scalar equations in (3.40), while there are only two independent unknown coefficients η and μ. However, for $\mathbf{G}(r)$ solving the previous problem, it must bring in some compatibility. The profound implication behind this overspecification is that for the original problem a single list of data is not enough to provide a convergent algorithm, while two lists of data seem to be too many! As we are only concerned with the unique determination, this overspecification is not bad news for us. For simplicity, we only use

$$S_{12} = W_{12}, \quad S_{22} = W_{22};$$

i.e.,

$$\eta = \frac{1}{\mu} - \frac{S_{12}}{(r\mu)^2}, \tag{3.41}$$

$$\mu' = \frac{S_{22}}{r\eta}. \tag{3.42}$$

We postpone the proof of the following lemma until later in this paper.

LEMMA 3.2. *If $\mu(1)$ and $\mu'(1)$ are given, then Subproblem* 3 *has at most one solution $\{\mathbf{Z}, \eta, \mu\}$ in $\mathbf{C}^1(0,1] \times C[0,1] \times C^2[0,1]$.*

Having found η and μ, we compute λ from (3.36):

$$\lambda = -\frac{\mu(4\eta\mu - 1)}{2\eta\mu - 1}. \tag{3.43}$$

The discussions above provide an algorithm, namely, sequentially solving Subproblem 1, Subproblem 2, and Subproblem 3. A detailed algorithm for solving Subproblem 1, i.e., recovering $\mathbf{k}_i(1,s)$ from their partial moments, can be found in [6]. As will be shown later, Subproblems 2 and 3 can be transformed into systems of Volterra integral equations of the second kind. Among these subproblems, Subproblem 2 is well posed, and Subproblem 3 is overspecified, but its regulated solution will be stable. Therefore, the ill-posedness of this problem is no more serious than that of recovering a function from its moments.

Remark. Instead of solving Subproblems 2 and 3 sequentially, we could solve (3.19)–(3.24) directly. However, those equations involve $\mu''(r)$, while the subproblems do not involve $\mu''(r)$. Because of this, the assumption $\mu \in C^2[0,1]$ is not essential to the present method.

The remainder of this paper is devoted to the proof of Lemmas 3.1 and 3.2. The common method is to transform differential equations into integral equations.

Proof of Lemma 3.1. By integrating (3.29) and (3.30) along the characteristic lines, we get

$$\mathbf{P}_1(r,s) = \int_r^1 \frac{1}{x}(\mathbf{G}(x) - \mathbf{I})\mathbf{P}_2\left(x, \frac{xs}{r}\right) dx + \mathbf{Y}\mathbf{H}_1\frac{s}{r}, \tag{3.44}$$

$$\mathbf{P}_2(r,s) = \frac{1}{r}\left[\int_r^1 \mathbf{G}(x)\mathbf{P}_1\left(x, \frac{rs}{x}\right) dx + \mathbf{Y}\mathbf{H}_2(rs)\right]. \tag{3.45}$$

Suppose $\{\mathbf{P}_1(r,s), \mathbf{P}_2(r,s), \mathbf{G}(r)\}$ and $\{\tilde{\mathbf{P}}_1(r,s), \tilde{\mathbf{P}}_2(r,s), \tilde{\mathbf{G}}(r)\}$ are two solutions. Let us introduce the following notation.

$\bar{X} = X - \tilde{X}$ for any variable X involved,

$\|\boldsymbol{\Xi}\| := \max_{1\le i,j\le 2} |\Xi_{ij}|$ for any 2×2 matrix $\boldsymbol{\Xi}$,

$C_6 = C_6(\epsilon) := \max_{\epsilon\le rs, s\le r\le 1, 1\le k\le 2}\{\|\mathbf{P}_k(r,s)\|, \|\tilde{\mathbf{P}}_k(r,s)\|, \|\mathbf{G}(r)\|, \|\tilde{\mathbf{G}}(r)\|\}$,
$0<\epsilon<1$,

$p_k(r,\epsilon) := \max_{\epsilon/r\le s\le r} \|\bar{\mathbf{P}}_k(r,s)\|$ for any $r\in(0,1], k=1,2$,

$g(r) := \|\bar{\mathbf{G}}(r)\|$ for any $r\in(0,1]$.

From (3.44)–(3.45) we have

$$\bar{\mathbf{P}}_1(r,s) = \int_r^1 \frac{1}{x}\left[(\mathbf{G}(x)-\mathbf{I})\bar{\mathbf{P}}_2\left(x,\frac{xs}{r}\right) + \bar{\mathbf{G}}(x)\tilde{\mathbf{P}}_2\left(x,\frac{xs}{r}\right)\right]dx, \tag{3.46}$$

$$\bar{\mathbf{P}}_2(r,s) = \frac{1}{r}\int_r^1 \left[\mathbf{G}(x)\bar{\mathbf{P}}_1\left(x,\frac{rs}{x}\right) + \bar{\mathbf{G}}(x)\tilde{\mathbf{P}}_1\left(x,\frac{rs}{x}\right)\right]dx. \tag{3.47}$$

For any fixed $\epsilon>0$, let $s\in[\epsilon/r, r], r\in(0,\sqrt{\epsilon})$. From (3.46)–(3.47) we have

$$p_1(r,\epsilon) \le (2C_6+1)\int_r^1 \frac{1}{x}p_2(x,\epsilon)dx + C_6\int_r^1 \frac{1}{x}g(x)dx, \tag{3.48}$$

$$p_2(r,\epsilon) \le 2\frac{C_6}{r}\int_r^1 p_1(x,\epsilon)dx + \frac{C_6}{r}\int_r^1 g(x)dx. \tag{3.49}$$

By solving inequalities (3.48) and (3.49), we get

$$p_2(r,\epsilon) \le C_7\int_r^1 g(x)dx$$

from which by using (3.33) we get

$$g(r) = \|\bar{\mathbf{G}}(r)\| = \|\bar{\mathbf{P}}_2(r,r)\| \le \|p_2(r,\epsilon) \le C_7\int_r^1 g(x)dx.$$

Hence $g(r)\equiv 0$, which leads to $\bar{\mathbf{G}}(r)\equiv \mathbf{0}$ since ϵ is arbitrary. This completes the proof. ☐

Proof of Lemma 3.2. Suppose $\{\mathbf{Z}(r),\eta,\mu\}$ and $\{\tilde{\mathbf{Z}}(r),\tilde{\eta},\tilde{\mu}\}$ are two solutions; then there are corresponding $\mathbf{Q}(r)$ and $\tilde{\mathbf{Q}}(r)$ and $\mathbf{S}(r)$ and $\tilde{\mathbf{S}}(r)$. From (3.38), (3.39) we get

$$\bar{\mathbf{Z}}(r) = -\int_r^1[\mathbf{Q}(s)\bar{\mathbf{Z}}(s) + \bar{\mathbf{Q}}(s)\tilde{\mathbf{Z}}(s)]ds/s. \tag{3.50}$$

For any fixed $\epsilon>0$, let $r\in(\epsilon,1]$; then from the last equation, we have

$$\|\bar{\mathbf{Z}}(r)\| \le 2\int_r^1[\|\mathbf{Q}(s)\|\,\|\bar{\mathbf{Z}}(s)\| + \|\bar{\mathbf{Q}}(s)\|\,\|\tilde{\mathbf{Z}}(s)\|]ds/s. \tag{3.51}$$

By solving this inequality for $\|\mathbf{Z}(r)\|$ we have

$$\|\bar{\mathbf{Z}}(r)\| \le C_8\int_r^1 \|\bar{\mathbf{Q}}(s)\|]ds, \tag{3.52}$$

where C_8 depends on $\eta, \mu, \bar{\eta}, \bar{\mu}$, and ϵ. By observing the explicit entries of $\mathbf{Q}(r)$ in (3.35), we can see that

$$\|\bar{\mathbf{Q}}(r)\| \le C_9[|\bar{\eta}(r)| + |\bar{\mu}(r)| + |\bar{\mu}'(r)|],$$

from which and the fact $\bar{\mu}(1) = 0$ we have

$$\|\bar{\mathbf{Q}}(r)\| \le C_9\left[|\bar{\eta}(r)| + |\bar{\mu}'(r)| + \int_r^1 (|\bar{\mu}'(s)|)ds\right].$$

Substituting this into (3.52) we get

$$\|\bar{\mathbf{Z}}(r)\| \le C_{10}\int_r^1 (|\bar{\eta}(s)| + |\bar{\mu}'(s)|)ds. \tag{3.53}$$

From the definition of $\mathbf{S}$, we have

$$\begin{aligned}\bar{\mathbf{S}}(r) &= \mathbf{Z}(r)\mathbf{G}(r)\mathbf{Z}^{-1}(r) - \tilde{\mathbf{Z}}(r)\mathbf{G}(r)\tilde{\mathbf{Z}}^{-1}(r)\\ &= (\bar{\mathbf{Z}}(r)\mathbf{G}(r) - \tilde{\mathbf{S}}\bar{\mathbf{Z}}(r))\mathbf{Z}^{-1}(r),\end{aligned}$$

which leads to

$$\|\bar{\mathbf{S}}(r)\| \le C_{11}\|\bar{\mathbf{Z}}(r)\|.$$

By inserting (3.53) into the last inequality we get

$$\|\bar{\mathbf{S}}(r)\| \le C_{12}\int_r^1 (|\bar{\eta}(s)| + |\bar{\mu}'(s)|)ds.$$

In particular,

$$|\bar{S}_{12}(r)|, |\bar{S}_{22}(r)| \le C_{12}\int_r^1 (|\bar{\eta}(s)| + |\bar{\mu}'(s)|)ds. \tag{3.54}$$

On the other hand, by using (3.41) and (3.42) we have

$$\bar{\eta}(r) = \frac{1}{\mu} - \frac{1}{\tilde{\mu}} + \frac{\tilde{S}_{12}}{(r^2\tilde{\mu})^2} - \frac{S_{12}}{(r^2\mu)^2}, \tag{3.55}$$

$$\bar{\mu}' = \frac{S_{22}}{r\eta} - \frac{\tilde{S}_{22}}{r\tilde{\eta}}. \tag{3.56}$$

It follows from (3.54) and (3.55) that

$$|\bar{\eta}(r)| \le C_{13}\int_r^1 (|\bar{\eta}(s)| + |\bar{\mu}'(s)|)ds. \tag{3.57}$$

From (3.54), (3.56), and (3.57), we have

$$|\bar{\mu}'(r)| \le C_{14}\int_r^1 (|\bar{\eta}(s)| + |\bar{\mu}'(s)|)ds.$$

By adding the last two inequalities, we get

$$|\bar{\eta}(r)| + |\bar{\mu}'(r)| \le (C_{13} + C_{14})\int_r^1 (|\bar{\eta}(s)| + |\bar{\mu}'(s)|)ds.$$

Therefore,

$$|\bar{\eta}(r)| + |\bar{\mu}'(r)| \equiv 0;$$

consequently, $\bar{\eta}(r) = \bar{\mu}(r) \equiv 0$ since ϵ is arbitrary. This completes the proof. $\square$

Acknowledgment. The author sincerely thanks Professor Robert P. Gilbert for his stimulating discussions and help and thanks the editor and two referees for their careful reading and many valuable suggestions, which led to a significant improvement over the first version.

REFERENCES

[1] M. Akamatsu, G. Nakamura, and S. Steinberg, *Identification of Lamé coefficients from boundary observations*, Inverse Problem, 7 (1991), pp. 335–354.

[2] R. W. Carroll, *Transmutation and Operator Differential Equations*, Math. Studies 37, North–Holland, New York, 1979.

[3] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Verlag, Berlin, New York, 1992.

[4] B. Chaderjian, *A uniqueness theorem for a Lossy inverse problem in reflection seismology*, SIAM J. Appl. Math., 54 (1994), pp. 1224–1249.

[5] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. I, Intersciences Publishers, New York, 1952, p. 202.

[6] R. P. Gilbert and Z. Lin, *On the conditions for uniqueness and existence of the solution to an acoustic inverse problem:* I *theory*, J. Comput. Acoust., 1 (1993), pp. 229–247.

[7] I. M. Gelfand and B. M. Levitan, *On the determination of a differential equation from its spectral function*, Amer. Math. Soc. Transl., Ser. 2, I (1995), pp. 253–304.

[8] R. P. Gilbert and Z. Lin, *An acoustic inverse problem: Numerical experiment*, J. Comput. Acoust., 3 (1995), pp. 229–240.

[9] K. Graff, *Wave Motion in Elastic Solids*, Ohio State University Press, Columbus, OH, 1975, p. 402.

[10] V. D. Kupradze, *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*, North–Holland, Amsterdam, 1979.

[11] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons, New York, 1989. p. 180.

[12] J. R. McLaughlin, P. L. Polyakov, and P. E. Sacks, *Reconstruction of a spherically symmetric speed*, SIAM J. Appl. Math., 54 (1994), pp. 1203–1223.

[13] G. Nakamura and G. Uhlmann, *Global uniqueness for an inverse boundary problem arising in elasticity*, Invent. Math., 118 (1994), pp. 454–474.

[14] G. Nakamura and G. Uhlmann, *Identification of Lamé parameters by boundary measurements*, Amer. J. Math, 115 (1993), pp. 1167–1187.

[15] G. Nakamura and G. Uhlmann, *Inverse problem at the boundary for elastic medium*, SIAM J. Math. Anal., 26 (1995), pp. 263–279.

[16] A. S. Saada, *Elasticity Theory and Applications*, Pergamon Press Inc., Elmsford, NY, 1974, p. 180.

[17] J. Sylvester, *A convergent layer stripping algorithm for the radially symmetric impedance tomography problem*, Comm. Partial Differential Equations, 17 (1992), pp. 1955–1994.