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Acoustic Field in a Shallow, Stratified Ocean with a Poro-Elastic Seabed

In this paper we investigate the propagating solutions of the acoustic equation in a stratified shallow ocean with a poroelastic, semi-infinite seabed. The ocean-seabed system is first Hankel transformed. Then the method of transmutation is used to generate the transformed solutions in the water column; whereas, the modal seabed equations are solved using the computer algebra Macsyma. In the water column the transformed solutions are first decomposed using the Mittag-Leffler expansion for the discrete spectrum. Both the discrete and continuous spectrum are then used to give a spectral representation of the solution, from which we develop a numerical scheme. Some numerical examples are given to illustrate the method.

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1. Introduction

Underwater acoustics is becoming increasingly important in the case of shallow oceans because of the possibility of using it to study the seabeds in harbors and estuaries, and in the search for inhomogeneities in the sea-floor. The acoustical environment in a shallow ocean is much more complicated than that in a deep ocean, because of the loss of acoustic energy to the seabed and the coupling of the ocean-seabed system. For our purposes we choose to classify the seabeds into three types: reflecting, elastic, and poro-elastic. The reflecting seabed case has been investigated by GILBERT-XU in a sequence of papers [13–16, 26], where they considered both direct and inverse problems. Such studies might be considered as a model for ocean acoustics over a rock seabed. If the sediment is tightly packed, the seabed may be considered to be elastic. Such cases are investigated in [12, 7, 8]. A more complete model, however, would be poro-elastic seabed. A popular model for the poro-elastic case was given by BIOT [3, 4]. For further investigations concerning the poro-elastic model of the sediment see [18, 19, 24, 25]. The equations for a poroelastic acoustic system have been formally derived by BURRIDGE and KELLER [6], SANCHEZ-PALENCIA [23], and LEVY [22]. They showed using homogenization that if the system is homogeneous then one obtains the Biot equations. The idea to use the Biot model to describe the seabed in the case of ocean acoustics seems to go back to YAMAMOTO. BUCHANAN and GILBERT [5] constructed the Green's function for a finite homogeneous ocean with a finite homogeneous seabed.

In the present paper, in contrast to BUCHANAN-GILBERT, we assume that the ocean is stratified due to the variation in the temperature with the depth, that is, the index of refraction is depth-dependent; however, the poro-elastic seabed will be assumed to be homogeneous and semi-infinite. We want to construct the acoustic pressure field exicted by a time-harmonic point source in the ocean with a homogeneous, poroelastic seabed. This pressure field is the Green's function for the ocean-seabed system. This fundamental singularity is useful for solving various inverse problems of underwater acoustics, such as the unidentified object problem or the undetermined coefficient problem.

We construct the solution by using a Hankel transformation, a Gelfand-Levitan type transmutation, and the Mittag-Leffler decomposition. This combination of tricks would not be possible, however, were it not for a powerful mathematical software such as Macsyma. We then derive a spectral representation for the pressure, from which numerical computations can be performed.

In our work we find the following list of symbols useful:

r	range (the horizontal distance between the reference point an the point source)
z	depth (the distance between the ocean surface and the reference point)
h	depth of sea flor
t	time
P	pressure in the ocean
u_{zo}	vertical displacement in the ocean
c_0	reference sound speed in the ocean water ($c_0 \approx 1500$ m/s)
ϱ	the density of the ocean water ($\varrho \approx 1$ kg/m^3)
ω	angle frequency of the point source
z_0	depth of the point source
$n(z)$	the depth-dependent index of refraction in the ocean
u_r	radial displacement in the solid matrix of the seabed
u_z	vertical displacement in the solid matrix of the seabed
U_r	radial displacement of the fluid in the seabed
U_z	vertical displacement of the fluid in the seabed
$\{e_{rr}, e_{\varphi\varphi}, e_{zz}, e_{rz}\}$	solid strain tensor in seabed
ε	dilatation of the fluid in the seabed
e	dilatation of the solid in the seabed
$\{\sigma_{rr}, \sigma_{\varphi\varphi}, \sigma_{zz}, \sigma_{rz}\}$	solid strain tensor in seabed
σ	fluid stress (pressure) in the seabed
λ, μ	Lamé coefficients for solid in the seabed
R, Q	coefficients in the Biot model (explained later in text)
ϱ_{11}	solid density in the seabed
ϱ_{22}	fluid density in the seabed
ϱ_{12}	density coupling parameter in the Biot model
b	dissipation parameter in Biot model
$\hat{f}$	the Hankel transform of function $f(z)$
G, K	transmutation kernels

In addition, we will use auxiliary symbols to substitute for some mathematical expressions occurring in the derivations. In order not disrupt the flow of the paper these auxiliary symbols are in Appendix.

2. The governing equations

Since the media are taken to be uniform in the range direction, it is convenient for us to employ cylindrical coordinates. Furthermore, as we wish to obtain a representation for Green's function, which may be written, using cylindrical coordinates, in the form $\hat{G}(|r\,e^{i\theta} - \varrho\,e^{i\varphi}|, z, \zeta)$, we may suppress the azimuthal angle and merely use the coordinates (r, z), (ϱ, ζ). We assume that the shallow stratified ocean occupies the region

$$R_h := \{(0, \infty) \times (0, h)\},$$

and that the poro-elastic seabed is semi-infinite and occupies

$$R_{-\infty} := \{(0, \infty) \times (h, \infty)\}.$$

The interface of the two different media is the sea floor $\Gamma_0 := (0, \infty) \times \{z = h\}$. In this paper, we consider a time-harmonic acoustic pressure $P(r, z, t) := P(r, z)\,e^{-i\omega t}$ generated by a point source located at $(0, z_0)$. It is well known that $P(r, z)$ satisfies the non-homogeneous Helmholtz equation [1]

$$\Delta P + k_0^2 n^2(z)\,P = -\frac{\delta(r)\,\delta(z - z_0)}{2\pi r}, \qquad 0 < z < h, \tag{2.1}$$

where $\Delta := \frac{1}{r}\,\partial_r(r\,\partial_r) + \partial_z^2$ is the Laplacian operator, $k_0 := \omega/c_0$ a reference wave number, c_0 a reference sound speed in the ocean, and $n^2(z)$ the depth dependent index of refraction. As usual, δ is the Dirac measure.

The vertical displacement of fluid, denoted by $u_{zo}(r, z)\,e^{-i\omega t}$, is related to the pressure $P(r, z)$ by [20]

$$\partial_z P + \omega^2 \varrho u_{zo} = 0, \qquad 0 < z < h, \quad z_0. \tag{2.2}$$

The ocean environment is such that radiation and refraction occur in the ocean whereas transmission and reflection occur at the interface between the seabed and ocean. We adopt the Biot model to describe the acoustic motion in the poro-elastic seabed [3, 4]. Following Biot, we use $\mathbf{u} := (u_r, u_z)$ and $\mathbf{U} := (U_r, U_z)$ to denote solid and fluid displacements respectively. The strain tensor for the porous skeleton is defined as usual by

$$\begin{aligned} e_{rr} &= \partial_r u_r, & e_{\varphi\varphi} &= \frac{1}{r}\,u_r, \\ e_{zz} &= \partial_z u_z, & e_{rz} &= \frac{1}{2}\,(\partial_z u_r + \partial_r u_z). \end{aligned} \tag{2.3}$$

The fluid and solid dilatations are given by

$$\begin{aligned} \varepsilon &:= \nabla \cdot \mathbf{U} = \partial_r U_r + \frac{U_r}{r} + \partial_z U_z, \\ e &:= \nabla \cdot \mathbf{u} = \partial_r u_r + \frac{u_r}{r} + \partial_z u_z, \end{aligned} \tag{2.4}$$

respectively.

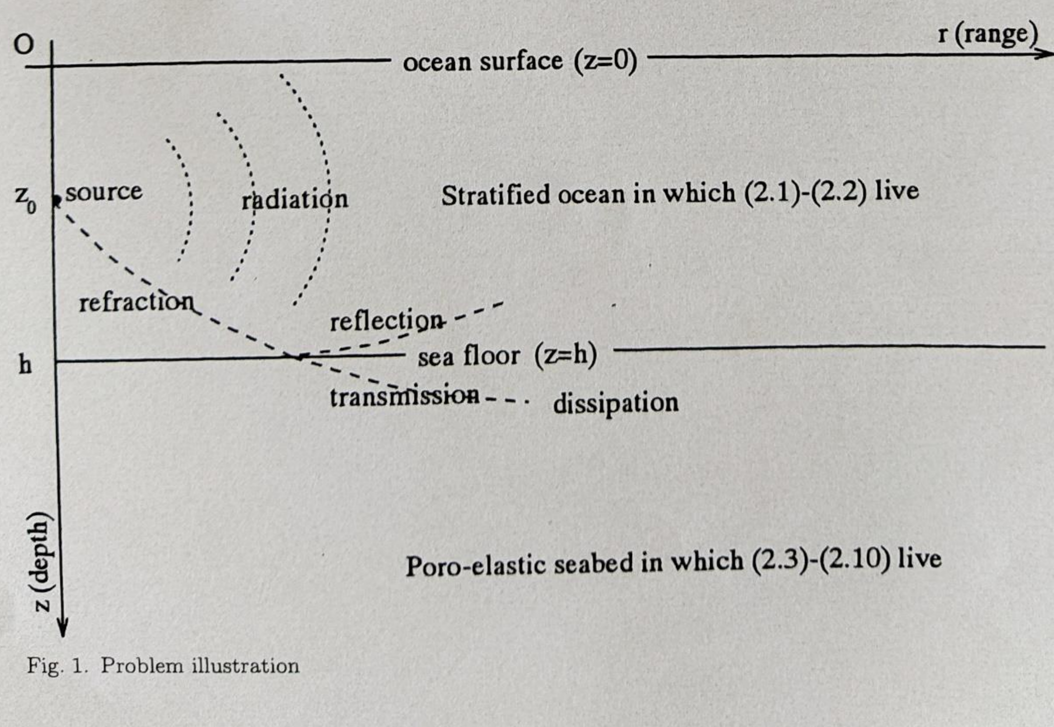

Fig. 1. Problem illustration

The Biot constitutive equations for an isotropic poroelastic material have been rigorously derived by BURRIDGE and KELLER [6] and SANCHEZ-PALENCIA [23]. For further discussion of the Biot model see that papers by DERSKI [9], KOWALASKI [21], and EHLERS and KUBIK [10]. For a homogeneous seabed these equations are as follows:

$$\sigma_{rr} = \lambda e + 2\mu e_{rr} + Q\varepsilon\,, \qquad \sigma_{\varphi\varphi} = \lambda e + 2\mu e_{\varphi\varphi} + Q\varepsilon\,,$$
$$\sigma_{zz} = \lambda e + 2\mu e_{zz} + Q\varepsilon\,, \qquad \sigma_{rz} = 2\mu e_{rz}\,, \qquad \sigma = Qe + R\varepsilon\,. \tag{2.5}$$

Here λ and μ are the Lamé coefficients of the porous skeleton, R is a Biot parameter measuring the pressure on the fluid required to force a certain volume of fluid into the sediment at constant volume, and Q measures the coupling of changes in the volume of the solid and fluid.

The equations of motion are then obtained from the conservation of momentum considerations to be

$$\partial_r\sigma_{rr} + \frac{1}{r}\,(\sigma_{rr} - \sigma_{\varphi\varphi}) + \partial_z\sigma_{rz} = \partial_{tt}(\varrho_{11}u_r + \varrho_{12}U_r) + b\,\partial_t(u_r - U_r)\,,$$
$$\partial_r\sigma = \partial_{tt}(\varrho_{12}u_r + \varrho_{22}U_r) + b\,\partial_t(u_r - U_r)\,,$$
$$\partial_r\sigma_{rz} + \frac{1}{r}\,\sigma_{rz} + \partial_z\sigma_{zz} = \partial_{tt}(\varrho_{11}u_z + \varrho_{12}U_z) + b\,\partial_t(u_z - U_z)\,,$$
$$\partial_z\sigma = \partial_{tt}(\varrho_{12}u_z + \varrho_{22}U_z) + b\,\partial_t(u_z - U_z)\,, \tag{2.6}$$

where ϱ_{11} and ϱ_{22} are density parameters for the solid and fluid, ϱ_{12} is a density coupling parameter, and b is a dissipation parameter. It is traditional to ignore gravity in considerations of acoustical phenomena. Displacement equations are obtrained by substituting the constitutive equations into the equations of motion.

By assuming time harmonic oscillations $u(r,\,z,\,t) := u(r,\,z)\,\mathrm{e}^{-i\omega t}$, etc., the displacement equations may be further simplified, using (2.4) to obtain [5]

$$\triangle((\lambda + 2\mu)\,e + Q\varepsilon) + p_{11}e + p_{12}\varepsilon = 0\,, \qquad h < z < \infty\,, \tag{2.7}$$
$$\triangle(Qe + R\varepsilon) + p_{12}e + p_{22}\varepsilon = 0\,, \qquad h < z < \infty\,, \tag{2.8}$$
$$\mu\,\triangle u_z + \mu\,\partial_z e + \partial_z(\lambda e) + \partial_z(Q\varepsilon) + p_{11}u_z + p_{12}U_z = 0\,, \qquad h < z < \infty\,, \tag{2.9}$$
$$\partial_z(Qe) + \partial_z(R\varepsilon) + p_{12}u_z + p_{22}U_z = 0\,, \qquad h < z < \infty\,. \tag{2.10}$$

Here p_{11}, p_{12}, and p_{22} are constants determined by the Biot's coefficients and the frequency ω (see the Appendix).

The following boundary conditions seem physically reasonable. At the ocean surface, it is customary to assume a pressure release condition

$$P(r,\,0) = 0\,. \tag{2.11}$$

At the ocean-sediment interface, we assume that no separation between the ocean and seabed occurs, i.e. the vertical displacements of the media should match here

$$u_{zo}(r,\,h^-) = u_z(r,\,h^+) = U_z(r,\,h^+)\,. \tag{2.12}$$

In this paper, we use the notation

$$g(h^+) = \lim_{z\to h,\,z>h} g(z)\,, \qquad g(h^-) = \lim_{z\to h,\,z<h} g(z)$$

to indicate the ocean and seabed side of the interface respectively.

Also we assume that there is no energy stored at the interface, namely, the stresses in two media should match at the interface. The normal stress in the ocean is just the acoustic pressure and the shear in ocean vanishes. Whereas in the seabed, the normal stress is $\sigma_{zz} + \sigma$, and only solid maintains a shear stress. Therefore, that the normal stresses must match at the interface leads to

$$P(r,\,h^-) = \sigma_{zz}(r,\,h^+) + \sigma(r,\,h^+) = (\lambda + Q)\,e(r,\,h^+) + 2\mu\,\partial_z u_z(r,\,h^+) + (Q + R)\,\varepsilon(r,\,h^+)\,, \tag{2.13}$$

and that the shear stresses match at the interface gives

$$\partial_z u_r(r,\,h^+) + \partial_r u_z(r,\,h^+) = 0\,. \tag{2.14}$$

We want emphasize that this Biot formulation of the seabed is valid only for a homogeneous seabed. The transition conditions seem reasonable and are widely accepted in the underwater acoustic literature [7, 8], but they have to date not been formally derived by the homogenization methodology. We are investigating this problem.

Finally, because of the radiation condition and also dissipation in the seabed, we assume the following asymptotic condition at infinity:

$$e_s,\ \varepsilon,\ u_r,\ u_z,\ U_r,\ U_z \to 0 \quad \text{as} \quad z \to \infty\,. \tag{2.15}$$

To eliminate u_r from the transmission conditions, we differentiate (2.4) with respect to z then substitute (2.14) into it and use (2.9) to obtain

$$2\mu\left(\partial_r^2+\frac{1}{r}\,\partial_r\right)u_z(r,h^+)+\partial_z((\lambda+2\mu)\,e(r,h^+)+Q\varepsilon(r,h^+))+p_{11}u_z(r,h^+)+p_{12}U_z(r,h^+)=0\,, \tag{2.16}$$

which then replaces (2.14).

3. Hankel transforms

Two popular methods for solving propagating-field problems with range-independent coefficients are the separation of variables technique and Hankel transformations. Separation of variables leads to an eigenvalue problem which results in a solution expressed as a Fourier expansion in the eigenfunctions. This method has the disadvantage that the Fourier coefficients are technically difficult to compute, as has been shown even in the case of an elastic seabed [12]. Moreover, the expansion may suffer from the inconvenience of there being a continuous spectrum. Because of this we have opted for the Hankel transform method. However, Hankel transforms may also be complicated in the case of non-constant coefficients and inhomogeneous terms in the ocean pressure equation (3.2). We are able to cope this with difficulty, however, by using a modified Gelfand-Levitan type transmutation.

Denote the Hankel transform of $f(r,z)$ with parameter $\sqrt{\xi}$ by $\hat{f}(z)$, namely,

$$\hat{f}(z)=\int_0^\infty rJ_0(\sqrt{\xi}\,r)\,f(r,z)\,\mathrm{d}r$$

for any function f involved. Also we denote $\dfrac{\mathrm{d}f}{\mathrm{d}z}$ by $f'(z)$.

Transforming the boundary-transition problem above yields

$$\hat{P}(0)=0\,, \tag{3.1}$$

$$\hat{P}''(z)+(k_0^2n^2(z)-\xi)\,\hat{P}(z)=-\frac{\delta(z-z_0)}{2\pi}\,,\qquad 0<z<h\,, \tag{3.2}$$

$$\hat{P}'(z)+\omega^2\varrho\hat{u}_{zo}(z)=0\,,\qquad 0<z<h,\ z\neq z_0\,, \tag{3.3}$$

$$\hat{u}_{zo}(h^-)=\hat{u}_z(h^+)=\hat{U}_z(h^+)\,, \tag{3.4}$$

$$\hat{P}(h^-)=(\lambda+Q)\,\hat{e}(h^+)+2\mu\hat{u}_z'(h^+)+(Q+R)\,\hat{\varepsilon}(h^+)\,, \tag{3.5}$$

$$-2\xi\mu\hat{u}_z(h^+)+(\lambda+2\mu)\,\hat{e}'(h^+)+Q\varepsilon'(h^+)+p_{11}u_z(h^+)+p_{12}\hat{U}_z(h^+)=0\,, \tag{3.6}$$

$$(\lambda+2\mu)\,(\hat{e}''(z)-\xi\hat{e}(z))+Q(\hat{e}''(z)-\xi\hat{e}(z))+p_{11}\hat{e}(z)+p_{12}\hat{e}(z)=0\,,\qquad h<z<\infty\,, \tag{3.7}$$

$$(Q(\hat{e}''(z)-\xi\hat{e}(z))+R(\hat{\varepsilon}(z)-\xi\hat{\varepsilon}(z))+p_{12}\hat{\varepsilon}(z)+p_{22}\hat{\varepsilon}(z)=0\,,\qquad h<z<\infty\,, \tag{3.8}$$

$$\mu(\hat{u}_z''(z)-\xi\hat{u}_z(z))+\mu\hat{e}'(z)+\lambda\hat{e}'(z)+Q\hat{\varepsilon}'(z)+p_{11}\hat{u}_z(z)+p_{12}\hat{U}_z(z)=0\,,\qquad h<z<\infty\,, \tag{3.9}$$

$$Q\hat{e}'(z)+R\hat{\varepsilon}'(z)+p_{12}\hat{u}_z(z)+p_{22}\hat{U}_z(z)=0\,,\qquad h<z<\infty\,. \tag{3.10}$$

Asymptotic condition at infinity:

$$\hat{e},\ \hat{\varepsilon},\ \hat{u}_z,\ \hat{U}_z\to 0\quad\text{as}\quad z\to\infty\,. \tag{3.11}$$

4. Solving the transformed equation

To construct the inverse Hankel transform, we need explicit expressions in ξ for the Hankel transformed functions. A modified form of the Gelfand-Levitan transmutation may be used for this purpose. The original version of the Gelfand-Levitan transmutation is used only for homogeneous differential equations $y''(z)-(c+q(z))\,y(z)=0$; whereas, the equation of interest here is inhomogeneous. Hence we are required to find a suitable modification for the Gelfand-Levitan transmutation.

Lemma: Let $p(z)$ be a C^1 function, and $G(z,s;p(\cdot))$ be the solution to the following Goursat problem for the Gelfand-Levitan equation

$$G_{zz}(z,s)-G_{ss}(z,s)-p(z)\,G(z,s)=0\,,\qquad 0<s<z\,, \tag{4.1}$$

$$G(z,0)=0\,, \tag{4.2}$$

$$2\,\frac{\mathrm{d}}{\mathrm{d}z}\,G(z,z)=p(z)\,. \tag{4.3}$$

Then the solutions to the following initial value problem:

$$y''(z) - (p(z) + c)\, y(z) = \delta(z - z_0)\,, \qquad z > 0\,, \tag{4.4}$$

$$y(0) = 0 \tag{4.5}$$

can be written as

$$y(z) = y'(0)\left(\frac{\sinh(\sqrt{c}\,z)}{\sqrt{c}} + \int_0^z G(z,\, s;\, p)\,\frac{\sinh(\sqrt{c}\,s}{\sqrt{c}}\,\mathrm{d}s\right)$$

$$+ H(z - z_0)\left(\frac{\sinh(\sqrt{c}(z - z_0))}{\sqrt{c}} + \int_0^{z - z_0} G(z - z_0,\, s,\, p(\cdot + z_0))\,\frac{\sinh(\sqrt{c}\,s)}{\sqrt{c}}\,ds\right), \tag{4.6}$$

where $H(\cdot)$ is the Heaviside function, i.e. if $x \geq 0$, $H(x) = 1$ else $H(x) = 0$.

Proof: Denote the right hand side of expression (4.6) by y_1. Using a similar computation to that in [11] it may be shown that y_1 satisfies the boundary condition and also the differential equation for all $z > 0$ except at the source point z_0. Furthermore, y_1 is continuous for all z and the jump in the first derivative $y_1'(z)$ is 1 at z_0. This completes the proof.

Remark: We can use (4.6) to provide representations of solutions for the equation with general homogeneous terms. Hence we can also solve the problem with a source distribution.

Let us return to the original problem. The equation (3.2) can be rewritten as

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}(-2\pi\hat{P}(z)) - (k_0^2(1 - n^2(z)) + (\xi - k_0^2))\,(-2\pi\hat{P}(z)) = \delta(z - z_0)\,, \qquad 0 < z < h\,,$$

$$\hat{P}(0) = 0\,.$$

By applying the transmutation formula (4.6),

$$\hat{P}(z) = \hat{P}'(0)\left(\frac{\sinh(x_k z)}{x_k} + \int_0^z G(z,\, s)\,\frac{\sinh(x_k s)}{x_k}\,\mathrm{d}s\right)$$

$$- \frac{H(z - z_0)}{2\pi}\left(\frac{\sinh(x_k(z - z_0))}{x_k} + \int_0^{z - z_0} K(z - z_0,\, s)\,\frac{\sinh(x_k s)}{x_k}\,\mathrm{d}s\right), \tag{4.7}$$

where

$$x_k := \sqrt{\xi - k_0^2}\,,$$

$$G(z,\, s) := G(z,\, s,\, k_0^2(1 - n^2(\,\cdot\,)))\,,$$

$$K(z,\, s) := G(z,\, s,\, k_0^2(1 - n^2(z_0 + \cdot)))\,.$$

In particular, $G(z,\, s) = K(z,\, s) = 0$ when $n(z) \equiv 1$, which corresponds to the homogeneous ocean.

To determine $\hat{P}'(0)$ we need the representation of $\hat{P}(h)$ and $\hat{P}'(h)$ to substitute in the transmission conditions. Using (4.7) and (3.3), we obtain

$$\hat{P}(h^-) = \hat{P}'(0)\,T_1(\xi) - T_2(\xi)\,, \tag{4.8}$$

$$\omega^2 \varrho \hat{u}_{z_0}(h^-) = -\hat{P}'(h^-) = -\hat{P}'(0)\,T_3(\xi) + T_4(\xi)\,, \tag{4.9}$$

where

$$T_1(\xi) = \frac{\sinh(x_k h)}{x_k} + \int_0^h G(h,\, s)\,\frac{\sinh(x_k s)}{x_k}\,\mathrm{d}s\,,$$

$$T_2(\xi) = \frac{1}{2\pi}\left(\frac{\sinh(x_k h - z_0)}{x_k} + \int_0^{h - z_0} K(h - z_0,\, s)\,\frac{\sinh(x_k s)}{x_k}\,\mathrm{d}s\right),$$

$$T_3(\xi) = \cosh(x_k h) + G(h,\, h)\,\frac{\sinh(x_k h)}{x_k} + \int_0^h G_z(h,\, s)\,\frac{\sinh(x_k s)}{x_k}\,\mathrm{d}s\,,$$

We now solve the system of equations (3.7)–(3.10) in the seabed using the asymptotic conditions (3.11) for $\hat{e}(z)$, $\hat{\varepsilon}(z)$, $\hat{u}_z(z)$, and $\hat{U}_z(z)$ using the computer algebra system Macsyma, which provides analytical expressions for $\hat{e}(z)$, $\hat{\varepsilon}(z)$, $\hat{u}_z(z)$, and $\hat{U}_z(z)$ containing three parameters C_1, C_2, C_3.

$$\hat{e}(z) = C_2\,\mathrm{e}^{-\sqrt{\xi-\beta}\,(z-h)} + C_1\,\mathrm{e}^{-\sqrt{\xi-\alpha}\,(z-h)}\,,$$

$$\hat{\varepsilon}(z) = \frac{C_1(\alpha - q_{11})\,\mathrm{e}^{-\sqrt{\xi-\alpha}\,(z-h)}}{q_{12}} + \frac{C_2(\beta - q_{11})\,\mathrm{e}^{-\sqrt{\xi-\beta}\,(z-h)}}{q_{12}}\,,$$

$$\hat{u}_z(z) = -\frac{C_1(q_{32}\alpha - q_{11}q_{32} + q_{12}q_{31})\sqrt{\xi-\alpha}\,\mathrm{e}^{-\sqrt{\xi-\alpha}\,(z-h)}}{q_{12}(\alpha - q_{33})} - \frac{C_2(q_{32}\beta - q_{11}q_{32} + q_{12}q_{31})\sqrt{\xi-\beta}\,\mathrm{e}^{-\sqrt{\xi-\beta}\,(z-h)}}{q_{12}(\beta - q_{33})} + C_3\,\mathrm{e}^{-\sqrt{\xi-\gamma}\,(z-h)}\,,$$

$$\hat{U}_z(z) = -\frac{1}{p_{22}}\left(p_{12}u_z + R\,\frac{\mathrm{d}\varepsilon}{\mathrm{d}z} + R\,\frac{\mathrm{d}e}{\mathrm{d}z}\right), \qquad -\frac{\pi}{2} < \arg\left(\sqrt{\xi-\alpha}\right),\ \arg\left(\sqrt{\xi-\beta}\right),\ \arg\left(\sqrt{\xi-\gamma}\right) \le \frac{\pi}{2}\,,$$

where the constants α, β, γ, $\{q_{ij}\}$ are uniquely expressed in terms of the physical coefficients (see the Appendix). The three different terms expressed as square roots correspond to the first and second compressional wave, and shear wave of Biot. In order to obtain the boundary condition for the ocean acoustic wave on the seabed, these seabed solutions are substituted, using (4.8)–(4.9), into the transmission conditions (3.4)–(3.6). This leads to a system of linear, algebraic equations for $\{C_1, C_2, C_3\hat{P}'(h)\}$, namely

$$\begin{pmatrix} w_{1,1,1}x_\alpha & w_{1,1,2}x_\beta & w_{0,1,3} & 0 \\ w_{1,2,1}x_\alpha & w_{1,2,2}x_\beta & w_{0,2,3} & \dfrac{T_3}{\varrho\omega^2} \\ w_{2,3,1}x_\alpha^3 + w_{0,3,1} & w_{2,3,2}x_\beta^2 + w_{0,3,2} & w_{1,3,3}x_\gamma & -T_1 \\ w_{3,4,1}x_\alpha^3 + w_{1,4,1}x_\alpha & w_{3,4,2}x_\beta^3 + w_{1,4,2}x_\beta & w_{2,4,3}x_\gamma^2 + w_{0,4,3} & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ \hat{P}'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{T_3}{\varrho\omega^2} \\ -T_2 \\ 0 \end{pmatrix},$$

where

$$x_\alpha = \sqrt{\xi-\alpha}\,, \qquad x_\beta = \sqrt{\xi-\beta}\,, \qquad x_\gamma = \sqrt{\xi-\gamma}\,,$$

and $\{w_{ikj}\}$ are displayed in the Appendix. By solving this system we obtain

$$\hat{P}'(0) = \frac{B(\xi)}{A(\xi)}\,, \tag{4.10}$$

where

$$A(\xi) := s_1(\xi)\,T_3(\xi) + s_2(\xi)\,T_1(\xi)\,, \tag{4.11}$$

$$B(\xi) := s_1(\xi)\,T_4(\xi) + s_2(\xi)\,T_2(\xi)\,, \tag{4.12}$$

and

$$s_1(\xi) = \det\begin{pmatrix} w_{1,1,1}x_\alpha & w_{1,1,2}x_\beta & w_{0,1,3} \\ w_{2,3,1}x_\alpha^2 + w_{0,3,1} & w_{2,3,2}x_\beta^2 + w_{0,3,2} & w_{1,3,3}x_\gamma \\ w_{3,4,1}x_\alpha^3 + w_{1,4,1}x_\alpha & w_{3,4,2}x_\beta^3 + w_{1,4,2}x_\beta & w_{2,4,3}x_\gamma^2 + w_{0,4,3} \end{pmatrix},$$

$$s_2(\xi) = \varrho\omega^2\det\begin{pmatrix} w_{1,1,1}x_\alpha & w_{1,1,2}x_\beta & w_{0,1,3} \\ w_{1,2,1}x_\alpha & w_{1,2,2}x_\beta & w_{0,2,3} \\ w_{3,4,1}x_\alpha^3 + w_{1,4,1}x_\alpha & w_{3,4,2}x_\beta^3 + w_{1,4,2}x_\beta & w_{2,4,3}x_\gamma^2 + w_{0,4,3} \end{pmatrix}.$$

By substituting (4.10) into (4.7) we obtain the explicit expression of the Hankel transformed pressure in the ocean. In particular, when $n(z) \equiv 1$, we can find that $\hat{P}$ is given by

$$\hat{P} = \frac{1}{2\pi}\,\frac{\sinh\left(x_k(h - |z - z_0|) + \theta\right) - \sinh\left(x_k(h - (z + z_0)) + \theta\right)}{\cosh\left(x_k h + \theta\right)}\,, \tag{4.13}$$

where $\theta = \theta(\xi) := \dfrac{1}{2}\ln\left(\dfrac{s_1 + s_2}{s_1 - s_2}\right)$.

5. The Hankel inversion

The Hankel inversion formula provides a representation for $P(r, z)$, namely

$$P(r, z) = \int_0^\infty \sqrt{\xi}\, J_0(r\sqrt{\xi})\, \hat{P}(z)\, d\sqrt{\xi}. \tag{5.1}$$

Since the explicit expression of $\hat{P}$ is given by (4.7) and (4.10), one may directly use numerical integration to evaluate (5.1) when $A(\xi) \neq 0$ for $\xi \in (0, \infty)$. However, the following method based on the Mittag-Leffler expansion is computationally more stable.

Obviously, $\hat{P}(z)$ can be extended to the complex ξ-plane with three branch cuts

$$l_\alpha := \{\xi \mid -\infty < \mathrm{Re}\,(\xi) < \mathrm{Re}(\alpha),\ \mathrm{Im}\,(\xi) = \mathrm{Im}\,(\alpha)\},$$
$$l_\beta := \{\xi \mid -\infty < \mathrm{Re}\,(\xi) < \mathrm{Re}(\beta),\ \mathrm{Im}\,(\xi) = \mathrm{Im}\,(\beta)\},$$
$$l_\gamma := \{\xi \mid -\infty < \mathrm{Re}\,(\xi) < \mathrm{Re}(\gamma),\ \mathrm{Im}\,(\xi) = \mathrm{Im}\,(\gamma)\}.$$

From the explicit expression of $\hat{P}(z)$, we can see that it is analytic everywhere on the cut ξ-plane except for the zeros of $A(\xi)$ (Note that $\sqrt{\xi - k_0^2}$ does not cause discontinuity, because $\cos(\varkappa_k z)$ and $\dfrac{\sin(\varkappa_k z)}{\varkappa_k}$ are analytic functions in the entire ξ-plane).

Denote the set of poles of $(B/A)(\xi)$ by $\{\xi_n\}_{n=1}^\infty$, and assume that the poles are all simple and that none of them lies on the cuts. The frequencies that make $\hat{P}(z)$ fail to satisfy these assumptions are called exceptional frequency values. In the case of $n(z) \equiv 0$, we can see from (4.13) that the $\{\xi_n\}$ are simple and satisfy

$$\sqrt{\xi_n - k_0^2}\, h = (n + \tfrac{1}{2})\, \pi i - \theta(\xi_n). \tag{5.2}$$

From the explicit expressions of s_i, T_j, we observe that

$$s_2/s_1 = O(|\xi|^{-1/2}), \tag{5.3}$$

$$A(\xi) = s_1(\cosh(\varkappa_k h) + O\left(\frac{1}{|\xi|}\right) \sinh(\varkappa_k h)), \quad \text{as} \quad |\xi| \to \infty, \tag{5.4}$$

$$B(\xi) = \frac{1}{2\pi}\, s_1(\cosh(\varkappa_k(h - z_0)) + O\left(\frac{1}{|\xi|}\right) \sinh(\varkappa_k(h - z_0))), \quad \text{as} \quad |\xi| \to \infty, \tag{5.5}$$

hence, there is an integer n_0 such that

$$\sqrt{\xi_n - k_0^2} = \left(n + n_0 + \frac{1}{2}\right) \frac{\pi i}{h} + O\left(\frac{1}{n^2}\right), \quad \text{as} \quad n \to \infty, \tag{5.6}$$

$$\begin{aligned} A'(\xi_n) &= s_1(\xi_n)\, T_3'(\xi_n) + s_1'(\xi_n)\, T_3(\xi_n) + \varrho_0 \omega^2 (T_1'(\xi_n)\, s_2(\xi_n) + T_1(\xi_n)\, s_2'(\xi_n)) \\ &= s_1(\xi_n)\, O\left(\frac{1}{n}\right). \end{aligned} \tag{5.7}$$

First we want to strip the principal part from $\hat{P}(z)$. Let us define

$$V(\xi, z) := \frac{\sinh(\sqrt{\xi - k_0^2}\, z)}{\sqrt{\xi - k_0^2}} + \int_0^z G(z, s)\, \frac{\sinh(\sqrt{\xi - k_0^2}\, s)}{\sqrt{\xi - k_0^2}}\, d\xi. \tag{5.8}$$

The principal part of $\hat{P}(z)$ is

$$\hat{P}_1 = \sum_{n=1}^\infty \mathrm{Res}_{\xi = \xi_n} \hat{P}(z)\, \frac{1}{\xi - \xi_n} = \sum_{n=1}^\infty \frac{B(\xi_n)}{A'(\xi_n)}\, \frac{V(\xi_n, z)}{\xi - \xi_n}. \tag{5.9}$$

From the estimates (5.5) and (5.7) we know that $\{B(\xi_n)\, V(\xi_n, z)/A'(\xi_n)\}$ are bounded; hence, by taking (5.6) into account, we know that $\hat{P}_1$ is convergent and analytic over the complex plane except for the zeros $\{\xi_n\}$. Therefore $\hat{P} - \hat{P}_1$ is analytic and bounded on the cut ξ-plane. To remove the jump component from $\hat{P}(z) - \hat{P}_1(z)$, we define

$$\hat{P}_2 := \frac{1}{2\pi i} \left(\int_{l_\alpha} + \int_{l_\beta} + \int_{l_\gamma} \right) \frac{[\hat{P} - \hat{P}_1]^*(\eta)}{\eta - \xi}\, d\eta,$$

where $[f]^*(\xi)$ denotes the jump in the function $f(\xi)$ across the cut, i.e.,

$$[f]^*(\xi) = f(\xi + 0i) - f(\xi - 0i),$$

and $\int\limits_{l_\alpha} = \int\limits_{-\infty+\mathrm{Im}(\alpha)i}^{\alpha}$. Since only s_i has a jump, we have

$$[\hat{P} - \hat{P}_1]^*(\xi) = [\hat{P}'(0)]^*(\xi)\, V(\xi, z) = \left[\frac{B}{A}\right]^*(\xi)\, V(\xi, z)\,.$$

By the definition of $\hat{P}_2$, we know that $\hat{P}_2$ is analytic on the cut plane and the Plemjl formula gives

$$[\hat{P}_2]^* = [\hat{P} - \hat{P}_1]^*\,,$$

which implies that $\hat{P}_3 := \hat{P} - \hat{P}_1 - \hat{P}_2$ is analytic and bounded function on the *whole* ξ-plane, hence, a constant C by Liouville theorem. Since $\lim\limits_{|\xi|\to\infty} \hat{P} = \lim\limits_{|\xi|\to\infty} \hat{P}_1 = \lim\limits_{|\xi|\to\infty} \hat{P}_2 = 0$, it follows that $C = 0$, therefore,

$$\hat{P} = \hat{P}_1 + \hat{P}_2 = \sum_{n=1}^{\infty} \frac{B(\xi_n)}{A'(\xi_n)}\, V(\xi_n, z)\, \frac{1}{\xi - \xi_n} + \frac{1}{2\pi i} \left(\int\limits_{l_\alpha} + \int\limits_{l_\beta} + \int\limits_{l_\gamma}\right) \left[\frac{B}{A}\right]^*(\eta)\, V(\eta, z)\, \frac{1}{\eta - \xi}\, \mathrm{d}\eta\,.$$

By substituting this into (5.1) and using the identity [20]

$$\frac{\pi i}{2}\, H_0^{(1)}(\sqrt{\varkappa}r) = \int\limits_0^\infty \frac{J_0(rk)\, k\, \mathrm{d}k}{k^2 - \varkappa} = \int\limits_0^\infty \frac{J_0(r\sqrt{\xi})\, \sqrt{\xi}\, \mathrm{d}\sqrt{\xi}}{\xi - \varkappa}\,, \qquad \mathrm{Im}\,(\sqrt{\varkappa}) > 0$$

we get the representation of the acoustic pressure in the ocean.

$$\begin{aligned} P(r, z) &= \sum_{n=1}^{\infty} \frac{B(\xi_n)}{A'(\xi_n)}\, V(\xi_n, z) \int\limits_0^\infty \frac{\sqrt{\xi}\, J_0(r\sqrt{\xi})}{\xi - \xi_n}\, \mathrm{d}\sqrt{\xi} \\ &\quad + \frac{1}{2\pi i} \left(\int\limits_{l_\alpha} + \int\limits_{l_\beta} + \int\limits_{l_\gamma}\right) \left[\frac{B}{A}\right]^*(\eta)\, V(\eta, z) \int\limits_0^\infty \frac{\sqrt{\xi}\, J_0(r\sqrt{\xi})}{\eta - \xi}\, \mathrm{d}\sqrt{\xi}\, \mathrm{d}\eta \\ &= \frac{\pi i}{2} \sum_{n=1}^{\infty} \frac{B(\xi_n)}{A'(\xi_n)}\, V(\xi_n, z)\, H_0^{(1)}(\sqrt{\xi_n} r) - \frac{1}{4} \int\limits_{l_\alpha \cup l_\beta \cup l_\gamma} \left[\frac{B}{A}\right]^*(\xi)\, V(\xi, z)\, H_0^{(1)}(\sqrt{\xi} r)\, \mathrm{d}\xi\,, \end{aligned} \tag{5.10}$$

where $A(\xi)$ and $B(\xi)$ are defined in (4.10), $V(\xi, z)$ is defined as (5.8), and $H_0^{(1)}(\sqrt{\varkappa}r)$ is the Hankel function of the first kind and order 1, which is outgoing, i.e., $0 < \arg(\sqrt{\varkappa}) < \pi$. It has integral representation

$$H_0^{(1)}(s) = \frac{2}{\Gamma(\frac{1}{2})} \sqrt{\frac{2}{\pi}}\, \mathrm{e}^{i(s-\pi/4)} \int\limits_0^\infty \mathrm{e}^{-u^2} (s + iu^2/2)^{-1/2}\, \mathrm{d}u \tag{5.11}$$

and asymptotic behavior

$$H_0^{(1)}(s) = o(\mathrm{e}^{-\mathrm{Im}(s)}) \quad \text{as} \quad |s| \to \infty\,. \tag{5.12}$$

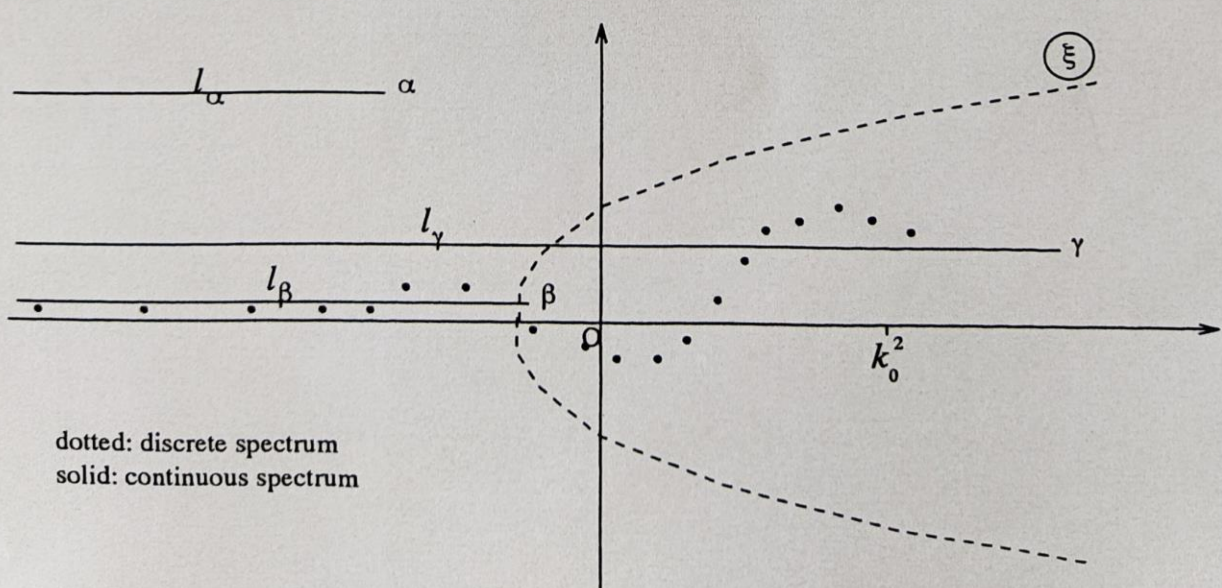

Fig. 2. The configuration of the spectrum on the ξ-plane

Using (5.12) and (5.3)–(5.7), we can the get the truncation error

$$\sum_{\mathrm{Re}(\xi_n) < -N^2} \frac{B(\xi_n)}{A'(\xi_n)} V(\xi_n, z) H_0^{(1)}(\sqrt{\xi_n} r) = o(\mathrm{e}^{-Nr}),$$

$$\int\limits_{(l_\alpha \cup l_\beta \cup l_\gamma) \cap \{\mathrm{Re}(\xi) < -N^2\}} \left[\frac{B}{A}\right]^* (\xi) V(\xi, z) H_0^{(1)}(\sqrt{\xi} r) \, \mathrm{d}\xi = o(\mathrm{e}^{-Nr}).$$

This shows the series and the inegral in (5.10) are convergent and that the spectrum having a large negative real part make no contribution to the far-field.

Remark: (5.10) is the spectral representation of the oceanic pressure. The spectral set Λ consists of a discrete spectral $\{\xi_n\}_{n=1}^{\infty}$ and a continuous spectral set l_α, l_β, l_γ. We may call $B(\xi_n)/A'(\xi_n)$ and $[B(\xi)/A(\xi)]^*$ the spectral function.

6. Numerical scheme and example

To evaluate the representation for $P(r, z)$, first we have to solve the Goursat problem for the transmutation kernel $G(z, s; p(\cdot))$. For this purpose we use a modified finite difference method based on Green's formula; indeed, from (4.1)–(4.2), we have

$$G(z + \triangle z, s) + G(z - \triangle z, s) - G(z, t + \triangle z) - G(z, s - \triangle z) = \frac{1}{2} \int\limits_{\diamond} p(\tau) G(\tau, \eta) \, \mathrm{d}\tau \, \mathrm{d}\eta, \tag{6.1}$$

$$\begin{aligned} G_z(z, s) = {} & \frac{1}{4}\left(p\left(\frac{z+s}{2}\right) - p\left(\frac{z-s}{2}\right)\right) + \frac{1}{2}\left(-\int\limits_{(z-s)/2}^{z-s} (p(\tau)) G(\tau, z - s - \tau) \, \mathrm{d}\tau \right. \\ & \left. + \int\limits_{z-s}^{z} p(\tau) G(\tau, -z + s + \tau) \, \mathrm{d}\tau + \int\limits_{(z+s)/2}^{z} p(\tau) G(\tau, z + s - \tau) \, \mathrm{d}\tau \right), \end{aligned} \tag{6.2}$$

where $\diamond$ is the square with corners $(z + \triangle z, s)$, $(z - \triangle z, s)$, $(z, s + \triangle z)$, and $(z, s - \triangle z)$, and linearly interpolating the integrand in (6.1), we have

$$\begin{aligned} & (1 - (p(z + \triangle z)) \triangle z^2/6) G(z + \triangle z, s) + (1 - (a(z - \triangle z)) \triangle z^2/6) G(z - \triangle z, s) \\ & - (1 + (p(z)) \triangle z^2/3)(G(z, s + \triangle z) + G(z, s - \triangle z)) \approx 0. \end{aligned} \tag{6.3}$$

By using this and boundary conditions (4.2)–(4.3), we have an explicit scheme to solve for $G(z, t)$, and since we use linear interpolation, we can expect the error to be order $(\triangle z)^2$. Having found $G(z, t)$, instead of using finite differences to evaluate $G_z(h, t)$, we can use (6.2) to compute G_z and achieve second order accuracy.

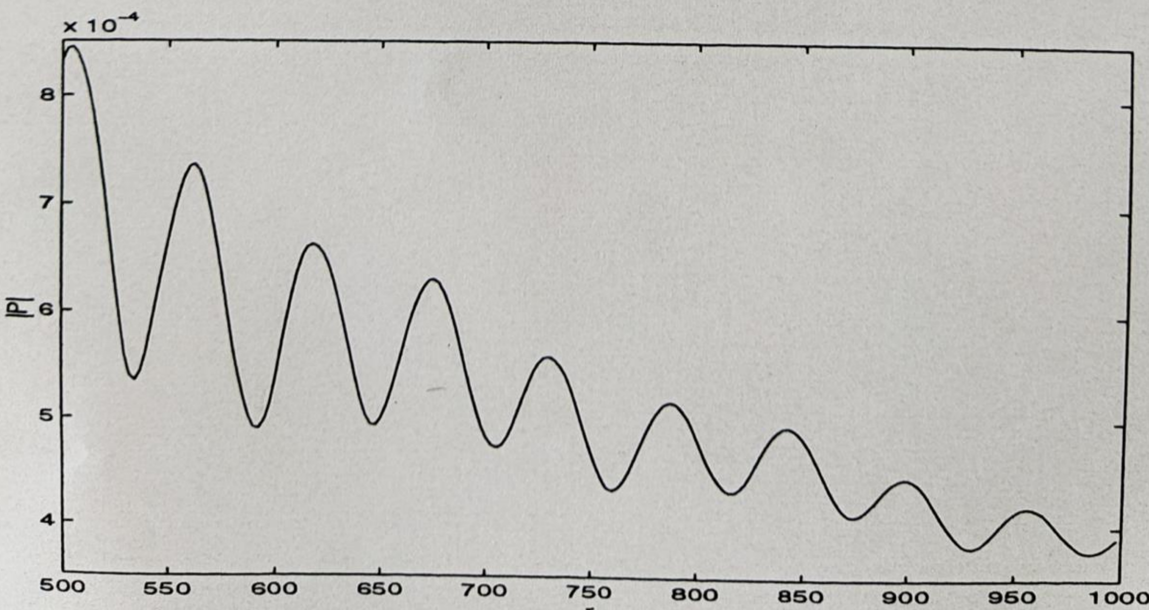

Fig. 3. The plot of oceanic pressure amplitude ($z = z_0$, $500 < r < 10000$)

To find ξ_n, we use Newton's method to solve equation $A(\xi)/B(\xi) = 0$. A by-product of this computation is that we obtain the the Fourier coefficients $B(\xi_n)/A'(\xi_n)$. To find a suitable initial guess on ξ-plane, we may solve (5.2) by iteration, namely

$$\xi_n^{(0)} = k_0^2 \left(\frac{n+\frac{1}{2}}{h}\right)^2, \qquad \xi_n^{(j+1)} = k_0^2 + \frac{1}{h^2}\left(\left(n+\frac{1}{2}\right)\pi i - \theta(\xi_n^{(j)})\right)^2.$$

Since in general, $n(z) \approx 1$, these solutions will provide good approximations to the roots of $A/B = 0$.

If only the far-field is concerned, we can draw a contour around the positive real axis in the ξ-plane as shown in Figure 2. Let the contour be taken as the parabola

$$\sqrt{\xi} = d\,, \quad \text{i.e.,} \quad x = g(y) := \left(\frac{y}{2d}\right)^2 - d^2\,, \qquad \xi = x + yi\,.$$

It can be seen that for any $\xi = x + yi$, if $x < g(y)$ then $H(\sqrt{\xi}r) \sim O(\mathrm{e}^{-dr})$. We only have to compute the spectrum falling inside this contour and the corresponding spectral function, i.e., the Fourier coefficients $B(\xi_n)/A'(\xi_n)$ and $[B(\xi)/A(\xi)]^*$. Then we can construct the far-field by using (5.10). We can expect that the error is order of e^{-dr}.

Let us illustrate the algorithm by an example. We use the following physical constants and functions, where the Biot coefficients λ, R, ϱ_{11}, ϱ_{22}, and b are computed using the formulas given in BUCHANAN-GILBERT [5] from a table in [2] (values in SI-units):

$$\omega = 600\,, \qquad h = 30\,, \qquad z_0 = 15\,, \qquad \varrho = 1\,000\,, \qquad c_0 = 1\,500\,,$$
$$\lambda = 0.226\,521\,178 \times 10^{10} + 2\,486\,771.3i\,, \qquad \mu = 74\,000\,000 - 4\,700\,000i\,,$$
$$R = 0.852\,479\,81 \times 10^9 - 1\,187 - 563i\,, \qquad Q = 0.138\,880\,50 \times 10^{10} + 27\,709.803i\,,$$
$$\varrho_{11} = 1\,775.2\,, \qquad \varrho_{12} = -95.000\,, \qquad \varrho_{22} = 475\,,$$
$$b = 37\,113 - 0.141\,56 \times 10^6 i\,, \qquad n^2(z) = 1 - 0.01z/h\,.$$

The computation shows that

$$\alpha = -4.148\,6 + 0.846\,28i\,, \qquad \beta = -.087\,784 + 0.009\,091\,1i,, \qquad \gamma = 7.284\; - .249\,90i\,.$$

We choose $d = 1.5k_0$. Within this contour a total 5 discrete spectral points are contained.

n	ξ_n	$B(\xi_n)/A'(\xi_n)$	$\lvert A(\xi_n)/B(\xi_n)\rvert$
1	$0.150\,59 - 0.361\,34\text{E-}03i$	$0.938\,96\text{E-}03 + 0.388\,56\text{E-}04i$	0.115 58E-13
2	$0.122\,69 - 0.125\,04\text{E-}02i$	$0.462\,26\text{E-}03 - 0.778\,80\text{E-}04i$	0.226 05E-13
3	$0.119\,87\text{E-}01 - 0.177\,34\text{E-}02i$	$-0.182\,26\text{E-}02 + 0.135\,17\text{E-}03i$	0.671 60E-08
4	$0.763\,99\text{E-}01 + 0.198\,64\text{E-}02i$	$-0.261\,68\text{E-}02 - 0.709\,23\text{E-}04i$	0.752 13E-14
5	$-0.687\,58\text{E-}01 + 0.414\,90\text{E-}03i$	$0.337\,97\text{E-}02 - 0.206\,80\text{E-}03i$	0.254 92E-12

The continuous spectra contained within this contour are the segments

$$\{\xi \mid \operatorname{Im}(\xi) = \operatorname{Im}(\beta),\ g(\operatorname{Im}(\beta)) < \operatorname{Re}(\xi) < \operatorname{Re}(\beta)\}\,,$$
$$\{\xi \mid \operatorname{Im}(\xi) = \operatorname{Im}(\gamma),\ g(\operatorname{Im}(\gamma)) < \operatorname{Re}(\xi) < \operatorname{Re}(\gamma)\}\,.$$

Then we can compute the spectral function $\left[\frac{B}{A}\right]^*(x + \operatorname{Im}(\beta))$, $g(\operatorname{Im}(\beta)) < x < \operatorname{Re}(\beta)$, and $\left[\frac{B}{A}\right]^*(x + \operatorname{Im}(\gamma))$, $g(\operatorname{Im}(\gamma)) < x < \operatorname{Re}(\gamma)$. From the spectral data we are able to compute the far-field pressure using

$$\begin{aligned} P_{far} \approx{}& \frac{\pi i}{2}\sum_{n=1}^{5}\frac{B(\xi_n)}{A'(\xi_n)}\,V(\xi_n, z)\,H_0^{(1)}(\sqrt{\xi_n}r) \\ &- \frac{1}{4}\int\limits_{g(\operatorname{Im}(\beta))}^{\operatorname{Re}(\beta)}\left[\frac{B}{A}\right]^*(x + \operatorname{Im}(\beta))\,V(x + i\operatorname{Im}(\beta), z)\,H_0^{(1)}(\sqrt{x + i\operatorname{Im}(\beta)}\,r)\,\mathrm{d}x \\ &- \frac{1}{4}\int\limits_{g(\operatorname{Im}(\gamma))}^{\operatorname{Re}(\gamma)}\left[\frac{B}{A}\right]^*(x + \operatorname{Im}(\gamma))\,V(x + i\operatorname{Im}(\gamma), z)\,H_0^{(1)}(\sqrt{x + i\operatorname{Im}(\gamma)}\,r)\,\mathrm{d}x\,. \end{aligned}$$

Figure 3 and 4 are plots of the pressure amplitude in the far-field. Figure 4 shows that in the very far-field that $|P| \sim C\,\mathrm{e}^{-\varepsilon r}$, i.e., the sound intensity in db is essentially proportional to the range. Here $-\varepsilon$ is shown as the slope of the line in the figure. We find that $\varepsilon \approx \operatorname{Im}(\sqrt{\xi_1}) = 0.000\,534\,4i$, where ξ_1 is the pole closest to k_0^2, and its corre-

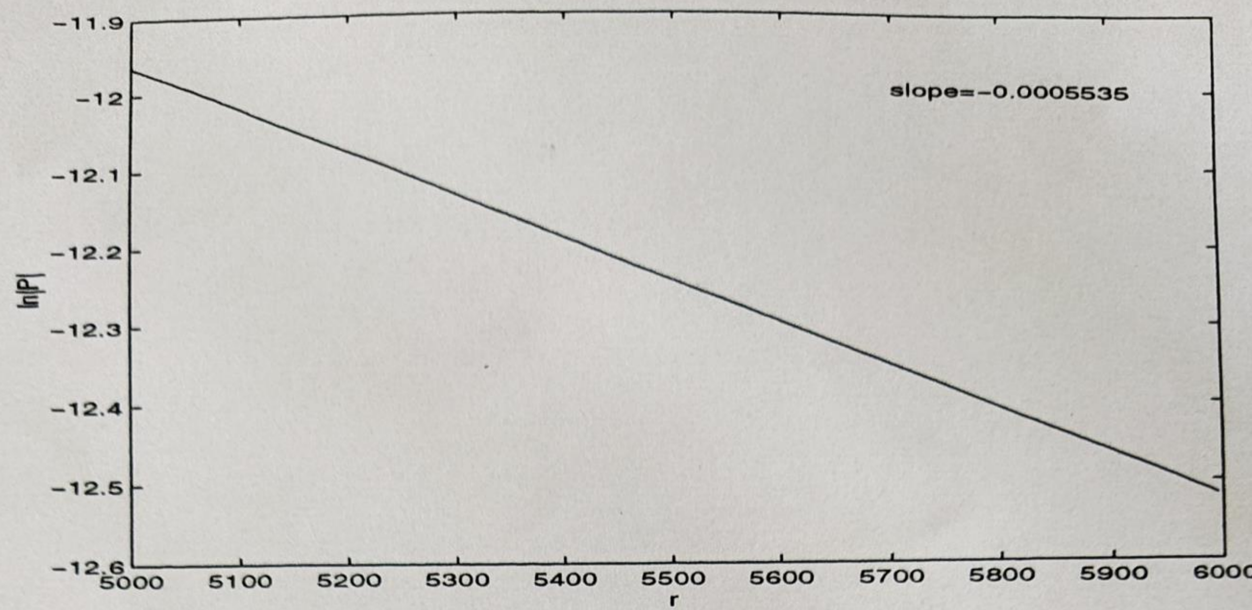

Fig. 4. The plot of oceanic pressure amplitude ($z = h$, $5000 < r < 6000$)

sponding mode is the dominant one. Due to the dissipation term, the poro-elastic seabed eventually absorbs the energy. This is a remarkable difference between finite, non-dissipative elastic sediments and infinite poro-elastic sediments.

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Appendix: A list of abbreviations

Here we list all abbreviations we used in the derivations. They are expressed in terms of the frequency ω and the coefficients in Biot model, i.e., λ, μ, ϱ_{11}, ϱ_{12}, ϱ_{22}, Q, R, b.

$$p_{11} = \omega^2 \varrho_{11} - i\omega b\,, \qquad p_{12} = \omega^2 \varrho_{12} + i\omega b\,, \qquad p_{22} = \omega^2 \varrho_{22} - i\omega b\,,$$

$$q_{11} = -\frac{p_{11}R - p_{12}Q}{R\lambda + 2\mu R - Q^2}\,, \qquad q_{12} - \frac{p_{12}R - p_{22}Q}{R\lambda + 2\mu R - Q^2}\,,$$

$$q_{21} = -\frac{p_{12}\lambda + 2p_{12}\mu - p_{11}Q}{R\lambda + 2\mu R - Q^2}\,, \qquad q_{22} - \frac{p_{22}\lambda + 2p_{22}\mu - p_{12}Q}{R\lambda + 2\mu R - Q^2}\,,$$

$$\alpha = -\frac{\sqrt{q_{22}^2 - 2q_{11}q_{22} + 4q_{12}q_{21} + q_{11}^2} - q_{22} - q_{11}}{2}\,, \qquad \beta = \frac{\sqrt{q_{22}^2 - 2q_{11}q_{22} + 4q_{12}q_{21} + q_{11}^2} + q_{22} + q_{11}}{2}\,,$$

$$q_{31} = \frac{p_{22}\lambda + 2p_{22}\mu - p_{12}Q}{p_{22}\mu}\,, \qquad q_{32} = -\frac{p_{12}R - p_{22}Q}{p_{22}\mu}\,, \qquad q_{33} = \frac{p_{11}p_{22} - p_{12}^2}{p_{22}\mu}\,, \qquad \gamma = q_{33}\,,$$

$$w_{1,1,1} = \frac{(\alpha^2 + (-q_{33} + q_{12} - q_{11})\,\alpha + (q_{11} - q_{12})\,q_{33})\,R + (p_{22} + p_{12})\,(q_{32}\alpha + q_{12}q_{31} - q_{11}q_{32})}{q_{12}p_{22}\alpha - q_{12}p_{22}q_{33}}\,,$$

$$w_{1,1,2} = \frac{(\beta^2 + (-q_{33} + q_{12} - q_{11})\,\beta + (q_{11} - q_{12})\,q_{33})\,R + (p_{22} + p_{12})\,(q_{32}\beta + q_{12}q_{31} - q_{11}q_{32})}{q_{12}p_{22}\beta - q_{12}p_{22}q_{33}}\,,$$

$$w_{0,1,3} = -\frac{p_{22} + p_{12}}{p_{22}}\,, \quad w_{1,2,1} = -\frac{q_{32}\alpha - q_{11}q_{32} + q_{12}q_{31}}{q_{12}\alpha - q_{12}q_{33}}\,, \quad w_{1,2,2} = -\frac{q_{32}\beta - q_{11}q_{32} + q_{12}q_{31}}{q_{12}\beta - q_{12}q_{33}}\,, \quad w_{0,2,3} = 1\,,$$

$$w_{2,3,1} = \frac{(2q_{32}\alpha - 2q_{11}q_{32} + 2q_{12}q_{31})\,\mu}{q_{12}\alpha - q_{12}q_{33}} \qquad w_{0,3,1} = \frac{q_{12}\lambda + (\alpha - q_{11})\,R + (\alpha + q_{12} - q_{11})\,Q}{q_{12}}\,,$$

$$w_{2,3,2} = \frac{(2q_{32}\beta - 2q_{11}q_{32} + 2q_{12}q_{31})\,\mu}{q_{12}\beta - q_{12}q_{33}}\,, \qquad w_{0,3,2} = \frac{q_{12}\lambda + (\beta - q_{11})\,R + (\beta + q_{12} - q_{11})\,Q}{q_{12}}\,,$$

$$w_{1,3,3} = -2\mu\,, \qquad w_{3,4,1} = \frac{(2q_{32}\alpha - 2q_{11}q_{32} + 2q_{12}q_{31})\,\mu}{q_{12}\alpha - q_{12}q_{33}}\,,$$

$$w_{1,4,1} = -\frac{(q_{12}\alpha - q_{12}q_{33})\,\lambda + (-2q_{32}\alpha^2 + (2q_{11}q_{32} - 2q_{12}q_{31} + 2q_{12})\,\alpha - 2q_{12}q_{33})\,\mu}{q_{12}\alpha - q_{12}q_{33}}\,,$$
$$-\frac{(\alpha^2 + (-q_{33} - q_{11})\,\alpha + q_{11}q_{33})\,Q + (p_{12} + p_{11})\,(q_{32}\alpha - q_{11}q_{32} + q_{12}q_{31})}{q_{12}\alpha - q_{12}q_{33}}\,,$$

$$w_{3,4,2} = \frac{(2q_{32}\beta - 2q_{11}q_{32} + 2q_{12}q_{31})\,\mu}{q_{12}\beta - q_{12}q_{33}}\,,$$

$$w_{1,4,2} = -\frac{(q_{12}\beta - q_{12}q_{33})\,\lambda + (-2q_{32}\beta^2 + (2q_{11}q_{32} - 2q_{12}q_{31} + 2q_{12})\,\beta - 2q_{12}q_{33})\,\mu}{q_{12}\beta - q_{12}q_{33}}\,,$$
$$-\frac{(\beta^2 + (-q_{33} - q_{11})\,\beta + q_{11}q_{33})\,Q + (p_{12} + p_{11})\,(q_{32}\beta - q_{11}q_{32} + q_{12}q_{31})}{q_{12}\beta - q_{12}q_{33}}\,,$$

$$w_{2,4,3} = -2\mu\,, \quad q_{0,4,3} = -2q_{33}\mu + p_{12} + p_{11}\,.$$