Outline

Sec 2.0 引言

Sec 2.1 矩阵与其运算

Sec 2.2 矩阵的分块

Sec 2.3 矩阵的秩

Sec 2.4 矩阵的逆

Sec 2.5 初等矩阵

1st 初等矩阵 1

$r_i \leftrightarrow r_j$

$c_i \leftrightarrow c_j$

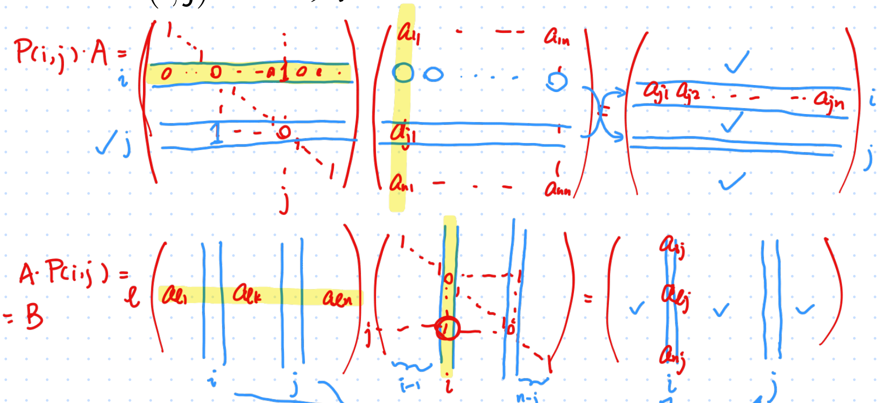
定义 5.1 (行对换)

$P(i,j)\cdot P(i,j) = E$

$$P(i,j) := \begin{array}{cc} & \begin{array}{cccccccccc} & & & i & & & j & & & \end{array} \\ \begin{array}{c} \\ \\ \\ i \\ \\ \\ j \\ \\ \\ \\ \end{array} & \begin{pmatrix} 1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 1 & & & & & & & \\ & & & 0 & & & 1 & & & \\ & & & & 1 & & & & & \\ & & & & & 1 & & & & \\ & & & 1 & & & 0 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 1 \end{pmatrix} \end{array}$$

1st 初等矩阵 2

- $P(i,j)A$: 通过交换 A 的第 i 行和第 j 行得到.
- $AP(i,j)$: …… 列 ……



$b_{li} = \sum_k a_{lk} \cdot P_{ki} = a_{lj}$.

2nd 初等矩阵 1

定义 5.2 (数乘行)

$$P(i(k)) := \begin{array}{cc} \begin{array}{ccccccc} & & & i & & & \end{array} & \\ \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & k & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} & \begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ \end{array} \end{array}$$

2nd 初等矩阵 2

- $P(i(k))A$：把 A 的第 i 行乘以 k 得到
- $AP(i(k))$：…… 列 ……

$$P(i(k))\cdot A = \begin{array}{c} \\ \\ i \\ \\ \\ \end{array}\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ 0 \cdots & & k & & 0 \cdots \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}\begin{pmatrix} a & & \\ & & \vdots \\ & & \boxed{}\ i \\ & & \vdots \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{i-1,1} & \cdots & a_{i-1,n} \\ k\cdot a_{i,1} & \cdots & ka_{i,n} \\ a_{i+1,1} & \cdots & a_{i+1,n} \\ \vdots & & \\ a_{n,1} & \cdots & a_{nn} \end{pmatrix}$$

$= B$

$$A\cdot P(i(k)) = \begin{pmatrix} \text{———} \\ \text{———} \\ \text{———} \\ \text{———} \\ \text{———} \end{pmatrix}\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & k & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} = \begin{pmatrix} \Big[& ka_{1,i} & \Big] \\ & \vdots & \\ & ka_{ni} & \end{pmatrix}$$

$\begin{array}{c} i \end{array}$

3rd 初等矩阵 1

定义 5.3 (行加行)

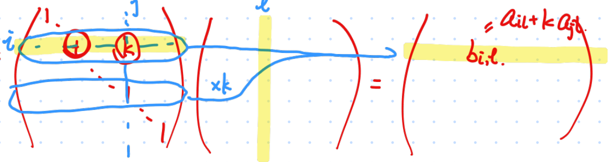
$$P(i,j(k)) := \begin{array}{cc} & \begin{array}{ccccccc} & & i & & j & & \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} & \begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 & \cdots & k && \\ &&& \ddots &&& \\ &&&& 1 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{pmatrix} \end{array}$$

3rd 初等矩阵 2

$r_i + k \cdot r_j$

- $P(i,j(k))A$: k 乘以 A 的第 j 行, 加到第 i 行
- $AP(i,j(k))$: …… 列 … 列

$P(i,j(k)) \cdot A =$

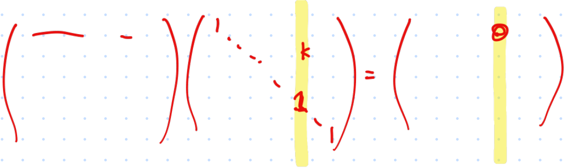


$= B$

$$b_{i,l} = \sum_{k=1}^{n} P_{ik} \cdot a_{kl} = P_{ii} \cdot a_{il} + P_{ij} \cdot a_{jl}$$

$$= a_{il} + k\,a_{jl}$$

$A \cdot P(i,j(k)) =$



定理 5.1

定理 5.1 (初等变换的矩阵表示)

- 左乘: 作用于行
- 右乘: 作用于列

$A^T_{i,j} = A_{j,i}$

$A^T_{j,i}$

证明

P.A 作用于行.

$$\boxed{AP = ((AP)^T)^T = (\underline{P^T A^T})^T.}$$

$P(i,j)^T = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} = P(i,j)$

$A \cdot P(i,j) = (P(i,j) \cdot A^T)^T$

$A \cdot P(i(k)) = (P(i(k)) \cdot A^T)^T$

$P(i(k))^T = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & k & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = P(i(k))$

$P(i,j(k))^T = \begin{pmatrix} 1 & & & \\ & \ddots & k & \\ & & \ddots & \\ & & & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & k & \ddots & \\ & & & 1 \end{pmatrix} = P(j,i(k))$

例题 5.1

已知 $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$, 求 $P(3,1(2))A$, $AP(2,3)$, $P(3(3))A$.

$$P(3,1(2))\cdot A = \begin{pmatrix} - & - & - \\ - & - & - \\ 4 & 2 & 5 \end{pmatrix}$$

$$A\cdot P(2,3) = \begin{pmatrix} \vdots & 2 & 2 \\ \vdots & -2 & 1 \\ \vdots & 1 & -2 \end{pmatrix}$$

$$P(3(3))A = \begin{pmatrix} - & - & - \\ - & - & - \\ 6 & -6 & 3 \end{pmatrix}$$

初等矩阵的逆

逆 A^{-1}: $A^{-1} \cdot A = A \cdot A^{-1} = E$

- $P(i,j)^{-1} = P(i,j)$ ✓
- $P(i(k))^{-1} = P(i(k^{-1}))$ ✓
- $P(i,j(k))^{-1} = P(i,\underline{j(-k)})$ ✓

初等矩阵的应用

矩阵的初等变换 $=$ 左乘或者右乘初等矩阵
接下来, 我们将以初等矩阵的乘法重新描述关于初等变换的结果.

定理 5.2

回顾定理 3.2: 对秩为 $r(A) = r$ 的矩阵 A, 通过有限次初等变换化为

初等行变换		初等矩阵左乘
$r_i \leftrightarrow r_j$	$\longleftrightarrow$	$P(i,j)\cdot A$
kr_i	$\longleftrightarrow$	$P(i(k))\cdot A$
$r_i + kr_j$	$\longleftrightarrow$	$P(i,j(k))\cdot A$

$$\boxed{\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}}$$

定理 5.2

对秩为 $r(A) = r$ 的 $m \times n$ 矩阵 A, 存在有限个

- m 阶初等矩阵 $P_1, \ldots, P_s$
- n 阶初等矩阵 $Q_1, \ldots, Q_t$

s.t.

$$\underbrace{P_s \cdots P_1}_{\text{行}} A \underbrace{Q_1 \cdots Q_t}_{\text{列}} = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}.$$

推论 5.1

A^{-1}存在 $\Leftrightarrow |A| \neq 0 \Leftrightarrow$ 满秩

进一步地, 如果 A 可逆, 则只需行变换即可.

推论 5.1 $P_s \cdots P_1 A Q_1 \cdots Q_t = E$

对于可逆矩阵 A, 存在有限初等变换 $P_1, \ldots, P_m$, s.t.

$$P_m \ldots P_1 A = E.$$

进而有

$$\boxed{A^{-1} = P_m \ldots P_1.}$$

$A \cdot x = b$

$\Updownarrow$

$$\left\{\begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = b_n \end{array}\right. \rightsquigarrow \left\{\begin{array}{l} x_1 = ? \\ x_2 = ? \\ \vdots \\ x_n = ? \end{array}\right.$$

$E \cdot x = \boxed{A^{-1}} \cdot b$

$\Updownarrow$

$$\left\{\begin{array}{l} x_1 = ? \\ x_2 = ? \\ \vdots \\ x_n = ? \end{array}\right.$$

推论 5.1 证明

只需证明一个初等矩阵的右乘, 等价于某个初等矩阵的左乘. 对三类初等矩阵 Q, 验证

$Q\cdot$ $\cdot Q^{-1}$

$$(AQ) = E \implies Q(A\boxed{Q)Q^{-1}} = E \implies \underline{QA = E}.$$

由定理 5.2 得 $Q_t P_s \cdots P_1 \cdot A \boxed{Q_1 \cdot Q_{t-1}} = E$

$$\begin{aligned} E &= P_s \dots P_1 A Q_1 \dots Q_t \\ &= \boxed{Q_1 \dots Q_t}\boxed{P_s \dots P_1} A \end{aligned}$$

此时, 我们可得

$$A^{-1} = P_m \dots P_1.$$

推论 5.2

将推论 5.1 中的行变换换为列变换:

推论 5.2

……

$$AQ_1 \dots Q_m = E.$$

由此可得 $A^{-1} = Q_1 \dots Q_m$.

推论 5.3

推论 5.3

$$r(A^T) = r(A).$$

证明

若 $PAQ = \begin{pmatrix} E_r & \\ & 0 \end{pmatrix}$, 则 $Q^T A^T P^T = \begin{pmatrix} E_r^T & \\ & 0 \end{pmatrix}$

$= (PAQ)^T$

初等行变换求逆矩阵

利用行变换表达 A^{-1}:

$\boxed{P_m \cdots P_1} A = E$

$$
\begin{aligned}
& P_m \dots P_1 (A \mid E) \\
= & (P_m \dots P_1 A \mid P_m \dots P_1) \\
= & (E \mid A^{-1})
\end{aligned}
$$

$$P_i \cdot \boxed{(A \mid E)}$$

$$= P(i,j) \times \left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1n} & 1 & & \\ \vdots & & \vdots & & \ddots & \\ a_{n1} & \cdots & a_{nn} & & & 1 \end{array}\right)$$

$$= (P(i,j) \cdot A \mid P(i,j) \cdot E)$$

$$Ax = E \cdot b$$

$$\Updownarrow$$

$$P_i\,(A \cdot x \mid E \cdot b)$$

$$\xrightarrow{P_i} (P_i A \cdot x \mid P_i \cdot E \cdot b)$$

$$\vdots$$

$$\rightsquigarrow (\underbrace{P_m \cdots P_1 A}_{E} x \mid \underbrace{P_m \cdots P_1 \cdot E} \cdot b)$$

$$\rightsquigarrow (E \cdot x \mid A^{-1} \cdot b)$$

$$\Updownarrow \quad x = A^{-1} \cdot b$$

例题 5.2

例题 5.2

求以下矩阵的逆

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 4 & 2 \\ 5 & -3 & 1 \end{pmatrix}$$

$$(A|E) = \left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & & \\ 1 & 4 & 2 & & 1 & \\ 5 & -3 & 1 & & & 1 \end{array}\right)$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & & 1 & \\ 2 & 1 & -1 & 1 & & \\ 5 & -3 & 1 & & & 1 \end{array}\right)$$

$$\xrightarrow[r_3 - 5r_1]{r_2 - 2r_1} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & & 1 & \\ & -7 & -5 & 1 & -2 & \\ & -23 & -9 & & -5 & 1 \end{array}\right)$$

$$\xrightarrow[-r_3]{-\frac{1}{7}r_2} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & & 1 & \\ & 1 & \frac{5}{7} & -\frac{1}{7} & \frac{2}{7} & \\ & 23 & 9 & & 5 & -1 \end{array}\right)$$

$$\xrightarrow[r_3 - 23r_2]{r_1 - 4r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{6}{7} & \frac{4}{7} & -\frac{1}{7} & \\ & 1 & \frac{5}{7} & -\frac{1}{7} & \frac{2}{7} & \\ & 0 & -\frac{52}{7} & \frac{23}{7} & -\frac{11}{7} & -1 \end{array}\right)$$

$$\rightsquigarrow \cdots$$

例题 5.2 解 1

$$\left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & & \\ 1 & 4 & 2 & & 1 & \\ 5 & -3 & 1 & & & 1 \end{array}\right)$$

初等列变换求逆矩阵

$$\left(\begin{array}{ccc} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \\ \hline 1 & & \\ & \ddots & \\ & & 1 \end{array}\right) \cdot Q$$

$$= \left(\frac{Q \cdot A}{QE}\right)$$

$$\left(\frac{A}{E}\right) Q_1 \ldots Q_m$$

$$= \left(\frac{AQ_1 \ldots Q_m}{Q_1 \ldots Q_m}\right)$$

$$= \left(\frac{E}{A^{-1}}\right)$$

例题 5.2 解 2