1. (4) 计算 $A = \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$ 求 A^n

解：$A^2 = \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda & \\ & \lambda^2 & 2\lambda \\ & & \lambda^2 \end{pmatrix}$

$$A^3 = A \cdot A^2 = \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}\begin{pmatrix} \lambda^2 & 2\lambda & \\ & \lambda^2 & 2\lambda \\ & & \lambda^2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^3 & 3\lambda^2 & \\ & \lambda^3 & 3\lambda^2 \\ & & \lambda^3 \end{pmatrix}$$

假设 $A^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \\ & \lambda^n & n\lambda^{n-1} \\ & & \lambda^n \end{pmatrix}$

则 $A^{n+1} = A \cdot A^n$

$$= \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}\begin{pmatrix} \lambda^n & n\cdot\lambda^{n-1} & \\ & \lambda^n & n\lambda^{n-1} \\ & & \lambda^n \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^{n+1} & (n+1)\lambda^n & \\ & \lambda^{n+1} & (n+1)\lambda^n \\ & & \lambda^{n+1} \end{pmatrix}$$

How about A is of size $m\times m$ instead of 3×3

$\therefore A^n = \begin{pmatrix} \lambda^n & n\cdot\lambda^n & \\ & \lambda^n & n\lambda^n \\ & & \lambda^n \end{pmatrix}$ #

1.(5) 设 $A=\begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}^n$

解: $A^2=\begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}\cdot\begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}$

$$=\begin{pmatrix}\cos\alpha\cdot\cos\alpha-\sin\alpha\cdot\sin\alpha & \cos\alpha\cdot(-\sin\alpha)+(-\sin\alpha)\cdot\cos\alpha \\ \sin\alpha\cdot\cos\alpha+\cos\alpha\cdot\sin\alpha & \sin\alpha\cdot(-\sin\alpha)+\cos\alpha\cdot\cos\alpha\end{pmatrix}$$

$$=\begin{pmatrix}\cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha\end{pmatrix}$$

假设 $A^n=\begin{pmatrix}\cos(n\alpha) & -\sin(n\alpha) \\ \sin(n\alpha) & \cos(n\alpha)\end{pmatrix}$

则 $A^{n+1}=A\cdot A^n$

$$=\begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\cos(n\alpha) & -\sin(n\alpha) \\ \sin(n\alpha) & \cos(n\alpha)\end{pmatrix}$$

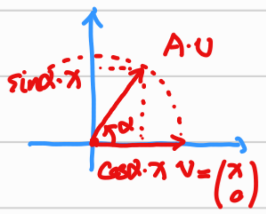
$$=\begin{pmatrix}\cos\alpha\cdot\cos(n\alpha)-\sin\alpha\cdot\sin(n\alpha) & \cos\alpha\cdot(-\sin(n\alpha))-\sin\alpha\cdot\cos(n\alpha) \\ \sin\alpha\cdot\cos(n\alpha)+\cos\alpha\cdot\sin(n\alpha) & \sin\alpha(-\sin(n\alpha))+\cos\alpha\cdot\cos(n\alpha)\end{pmatrix}$$

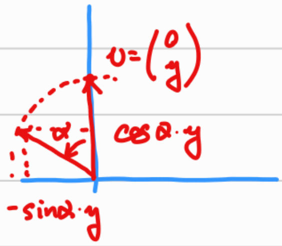
$$=\begin{pmatrix}\cos((n+1)\alpha) & -\sin((n+1)\alpha) \\ \sin((n+1)\alpha) & \cos((n+1)\alpha)\end{pmatrix}$$

得 $A^n=\begin{pmatrix}\cos(n\alpha) & \sin(n\alpha) \\ -\sin(n\alpha) & \cos(n\alpha)\end{pmatrix}$ #

Q: $A=\begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}$ 的几何意义?

e.g.





$\forall\ v=\begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{R}^2$

$$A\cdot v=\begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$=\begin{pmatrix}\cos\alpha\cdot x-\sin\alpha\cdot y \\ \sin\alpha\cdot x+\cos\alpha\cdot y\end{pmatrix}$$

相当于 v 逆时针旋转角度 α.

$\Rightarrow A^n\cdot v=\underbrace{A\cdot A\cdots\cdots A}_{n}\cdot v$ 旋转 n 次

$\Leftrightarrow$ 角度为 $n\alpha$

3 (1) 令 $A=\begin{pmatrix} & 1 & \\ & & 1 \\ -2 & & 1 \end{pmatrix}$

求 A^2, A^3 , $f(A)$ where $f(x)=x^3-3x^2-2x+2$

解: 直接计算 A^2, A^3

$f(A)=A^3-3\cdot A^2-2\cdot A+2\cdot E$ 易得 (略) #

4. 求与 $A=\begin{pmatrix}3 & 1\\ -2 & 2\end{pmatrix}$ 可交换的所有矩阵.

解：设 $B=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$ 与 A 可交换

计算 $A\cdot B=\begin{pmatrix}3 & 1\\ -2 & 2\end{pmatrix}\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}=\begin{pmatrix}3b_{11}+b_{21} & 3b_{12}+b_{22}\\ -2b_{11}+2b_{21} & -2b_{12}+2b_{22}\end{pmatrix}$

$B\cdot A=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\begin{pmatrix}3 & 1\\ -2 & 2\end{pmatrix}=\begin{pmatrix}3b_{11}-2b_{12} & b_{11}+2b_{12}\\ 3b_{21}-2b_{22} & b_{21}+2b_{22}\end{pmatrix}$

由假设 B与A可交换 i.e. $A\cdot B=B\cdot A$

得 $\begin{cases}3b_{11}+b_{21}=3b_{11}-2b_{12}\\ 3b_{12}+b_{22}=b_{11}+2b_{12}\\ -2b_{11}+2b_{21}=3b_{21}-2b_{22}\\ -2b_{12}+2b_{22}=b_{21}+2b_{22}\end{cases}$ $\qquad -b_{11}+b_{22}=-x$

$\Leftrightarrow \begin{pmatrix} & 2 & 1 & \\ -1 & 1 & & 1\\ -2 & & -1 & 2\\ & -2 & -1 & \end{pmatrix}\begin{pmatrix}b_{11}\\ b_{12}\\ b_{21}\\ b_{22}\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

记作 $C\cdot b=0$

不妨取 $b_{12}=x$，则 $b_{21}=-2x$ $\qquad 1 \quad 3 \quad -1 \quad -1$

取 $b_{11}=y$，则 $b_{22}=y-x$

∴ 可与A交换的矩阵有形式

$\begin{pmatrix}y & x\\ -2x & y-x\end{pmatrix}$ $\qquad x,y\in\mathbb{R}$ $\qquad$ #

8. 若 $A=\frac{1}{2}(B+E)$

求证 $A^2=A$ iff $B^2=E$

Pf. $A^2=A$

$\Leftrightarrow \left(\frac{1}{2}(B+E)\right)^2=\frac{1}{2}(B+E)$

$\Leftrightarrow B^2+2B+E=2B+2E$

$\Leftrightarrow B^2=E$ #

11. 求证：任一方阵可表为对称矩阵与反对称矩阵之和.

Pf. 对 $\forall$ 方阵 $A \in \mathbb{R}^{n\times n}$

记 $B = \frac{1}{2}(A+A^T)$

$C = \frac{1}{2}(A-A^T)$

可验证 $B^T = B$ & $C^T = -C$

i.e. B 对称 & C 反对称

得 $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$ #

12. 设 A, B 为对称阵矩.

求证 $A \cdot B$ 对称 iff A, B 可交换.

Pf. $A \cdot B$ 对称,

$$\Leftrightarrow (A \cdot B)^T = A \cdot B$$

$$\Leftrightarrow B^T \cdot A^T = A \cdot B$$

$$\xLeftrightarrow[B^T = B]{A^T = A} B \cdot A = A \cdot B$$

$\Leftrightarrow$ A, B 可交换 #

13. 设 $A \in \mathbb{R}^{n\times n}$

s.t. $Ax = 0 \quad \forall\, x \in \mathbb{R}^n$

求证 $A = 0$

Pf. 取 $v_i = (0,\dots,0,\overset{\text{ith 坐标}}{1},0,\dots,0)^T \quad i=1,\dots,n$

则 $A\cdot v_i = (a_{1i},\dots\dots,a_{ni})^T = 0 \quad i=1,\dots,n$ (条件)

$\Rightarrow a_{1i}=0,\dots\dots,a_{ni}=0$, for $i=1,\dots,n$

i.e. $A=0$ #

14. 2) 用初等行变换化成阶梯形矩阵

$$A = \begin{pmatrix} -3 & 1 & -3 & 0 & 5 \\ 4 & 3 & 2 & 3 & 0 \\ 6 & -1 & -5 & 0 & -7 \\ 2 & 5 & 1 & 4 & 1 \end{pmatrix}$$

解：$A \xrightarrow{r_1+r_4} \begin{pmatrix} 1 & 6 & -2 & 4 & 6 \\ & & & & \\ & & & & \\ & & & & \end{pmatrix}$

$\xrightarrow[r_4-2r_1]{\substack{r_2-4r_1 \\ r_3-6r_1}} \cdots\cdots$ （略） #

15 6) 计算 $r(A)$；若 A 满秩 则求 A^{-1}

$$\begin{pmatrix} 9 & -12 & 7 & 18 \\ & 1 & 1 & 1 \\ 3 & -12 & 1 & \\ -1 & 4 & & 1 \end{pmatrix}$$

解. 计算 $r(A)$：$A \longrightarrow \begin{pmatrix} 1 & -4 & & -1 \\ 3 & -12 & 1 & \\ 9 & -12 & 7 & 18 \\ & 1 & 1 & 1 \end{pmatrix}$

$$\longrightarrow \begin{pmatrix} 1 & -4 & & -1 \\ & & & 3 \\ & 24 & & 9 \\ & 1 & 1 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & -4 & & -1 \\ & 1 & 1 & 1 \\ & & -24 & -15 \\ & & & 3 \end{pmatrix}$$

可见 $r(A)=4$.

计算 A^{-1} （略） #.

16. 设 $A_n = \begin{pmatrix} 0 & a_1 & & & \\ & & a_2 & & \\ & & & \ddots & \\ & & & & a_{n-1} \\ a_n & & & & 0 \end{pmatrix}$ where $a_i \neq 0$

求 A_n^{-1}

解1: 记 $P = P(1,2) \cdot P(2,3) \cdot \cdots \cdot P(n,n-1)$

则 $P \cdot A_n = diag(a_1, \ldots, a_n)$

$\Rightarrow (P \cdot A_n)^{-1} = diag(a_1^{-1}, \ldots, a_n^{-1})$

另一方面 $(P \cdot A_n)^{-1} = A_n^{-1} \cdot P^{-1}$

$\therefore A_n^{-1} = diag(a_1^{-1}, \ldots, a_n^{-1}) \cdot P$

$$\begin{pmatrix} & & & a_1^{-1} \\ a_2^{-1} & & & \\ & \ddots & & \\ & & a_n^{-1} & \end{pmatrix}$$

#

解2: 待定系数直接求 A_n^{-1} (略) #