习题课 Ch 2

1. (4) 计算 $A=\begin{pmatrix}\lambda & 1 & \\ & \lambda & 1\\ & & \lambda\end{pmatrix}$ 求 A^n

解：$A^2=\begin{pmatrix}\lambda & 1 & \\ & \lambda & 1\\ & & \lambda\end{pmatrix}\begin{pmatrix}\lambda & 1 & \\ & \lambda & 1\\ & & \lambda\end{pmatrix}=\begin{pmatrix}\lambda^2 & 2\lambda & 1\\ & \lambda^2 & 2\lambda\\ & & \lambda^2\end{pmatrix}$

$$A^3=A\cdot A^2=\begin{pmatrix}\lambda & 1 & \\ & \lambda & 1\\ & & \lambda\end{pmatrix}\begin{pmatrix}\lambda^2 & 2\lambda & 1\\ & \lambda^2 & 2\lambda\\ & & \lambda^2\end{pmatrix}$$
$$=\begin{pmatrix}\lambda^3 & 3\lambda^2 & 3\lambda\\ & \lambda^3 & 3\lambda^2\\ & & \lambda^3\end{pmatrix}$$

假设 $A^n=\begin{pmatrix}\lambda^n & n\lambda^{n-1} & \frac{1}{2}n(n-1)\cdot\lambda^{n-2}\\ & \lambda^n & n\cdot\lambda^{n-1}\\ & & \lambda^n\end{pmatrix}$

则 $A^{n+1}=A\cdot A^n$
$$=\begin{pmatrix}\lambda & 1 & \\ & \lambda & 1\\ & & \lambda\end{pmatrix}\begin{pmatrix}\lambda^n & n\cdot\lambda^{n-1} & \frac{1}{2}n(n-1)\lambda^{n-2}\\ & \lambda^n & n\cdot\lambda^{n-1}\\ & & \lambda^n\end{pmatrix}$$
$$=\begin{pmatrix}\lambda^{n+1} & (n+1)\lambda^n & \frac{1}{2}n(n-1)\lambda^{n-1}+n\cdot\lambda^{n-1}\\ & \lambda^{n+1} & n\cdot\lambda^n+\lambda^n\\ & & \lambda^{n+1}\end{pmatrix}$$
$$=\begin{pmatrix}\lambda^{n+1} & (n+1)\lambda^n & \frac{1}{2}(n+1)n\cdot\lambda^{n-1}\\ & \lambda^{n+1} & (n+1)\lambda^n\\ & & \lambda^{n+1}\end{pmatrix}$$

How about A is of size $m\times m$ instead of 3×3

得证 $A^n=\begin{pmatrix}\lambda^n & n\cdot\lambda^{n-1} & \frac{1}{2}n(n-1)\lambda^{n-2}\\ & \lambda^n & n\cdot\lambda^{n-1}\\ & & \lambda^n\end{pmatrix}$ #

1.(5) 设 $A=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}^n$

解: $A^2=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\cdot\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$

$$=\begin{pmatrix}\cos\alpha\cdot\cos\alpha-\sin\alpha\cdot\sin\alpha & \cos\alpha\cdot(-\sin\alpha)+(-\sin\alpha)\cdot\cos\alpha\\ \sin\alpha\cdot\cos\alpha+\cos\alpha\cdot\sin\alpha & \sin\alpha\cdot(-\sin\alpha)+\cos\alpha\cdot\cos\alpha\end{pmatrix}$$

$$=\begin{pmatrix}\cos2\alpha & -\sin2\alpha\\ \sin2\alpha & \cos2\alpha\end{pmatrix}$$

假设 $A^n=\begin{pmatrix}\cos(n\alpha) & -\sin(n\alpha)\\ \sin(n\alpha) & \cos(n\alpha)\end{pmatrix}$

则 $A^{n+1}=A\cdot A^n$

$$=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\cos(n\alpha) & -\sin(n\alpha)\\ \sin(n\alpha) & \cos(n\alpha)\end{pmatrix}$$

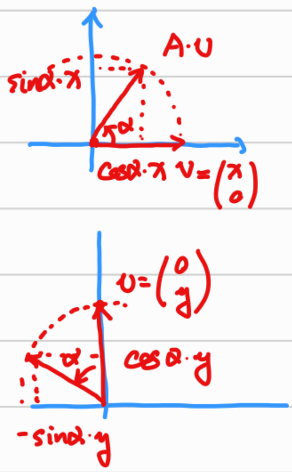
$$=\begin{pmatrix}\cos\alpha\cdot\cos(n\alpha)-\sin\alpha\cdot\sin(n\alpha) & \cos\alpha\cdot(-\sin(n\alpha))-\sin\alpha\cdot\cos(n\alpha)\\ \sin\alpha\cdot\cos(n\alpha)+\cos\alpha\cdot\sin(n\alpha) & \sin\alpha(-\sin(n\alpha))+\cos\alpha\cdot\cos(n\alpha)\end{pmatrix}$$

$$=\begin{pmatrix}\cos((n+1)\alpha) & -\sin((n+1)\alpha)\\ \sin((n+1)\alpha) & \cos((n+1)\alpha)\end{pmatrix}$$

得 $A^n=\begin{pmatrix}\cos(n\alpha) & \sin(n\alpha)\\ -\sin(n\alpha) & \cos(n\alpha)\end{pmatrix}$ #

Q: $A=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$ 的几何意义?

e.g.



$\forall\ v=\begin{pmatrix}x\\ y\end{pmatrix}\in\mathbb{R}^2$

$$A\cdot v=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$=\begin{pmatrix}\cos\alpha\cdot x-\sin\alpha\cdot y\\ \sin\alpha\cdot x+\cos\alpha\cdot y\end{pmatrix}$$

相当于 v 逆时针旋转角度 α.

$\Rightarrow A^n\cdot v=\underbrace{A\cdot A\cdots\cdots A}_{n}\cdot v$ 旋转 n 次

$\Leftrightarrow$ 角度为 $n\alpha$

3 (1) 令 $A=\begin{pmatrix} & 1 & \\ & & 1 \\ -2 & & 1 \end{pmatrix}$

求 A^2, A^3, $f(A)$ where $f(x)=x^3-3x^2-2x+2$

解：直接计算 A^2, A^3

$f(A)=A^3-3\cdot A^2-2\cdot A+2\cdot E$ 易得 (略) #

4. 求与 $A=\begin{pmatrix}3 & 1\\ -2 & 2\end{pmatrix}$ 可交换的所有矩阵.

解：设 $B=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$ 与 A 可交换

计算 $A\cdot B=\begin{pmatrix}3 & 1\\ -2 & 2\end{pmatrix}\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}=\begin{pmatrix}3b_{11}+b_{21} & 3b_{12}+b_{22}\\ -2b_{11}+2b_{21} & -2b_{12}+2b_{22}\end{pmatrix}$

$B\cdot A=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\begin{pmatrix}3 & 1\\ -2 & 2\end{pmatrix}=\begin{pmatrix}3b_{11}-2b_{12} & b_{11}+2b_{12}\\ 3b_{21}-2b_{22} & b_{21}+2b_{22}\end{pmatrix}$

由假设 B 与 A 可交换 i.e. $A\cdot B=B\cdot A$

得 $\begin{cases}3b_{11}+b_{21}=3b_{11}-2b_{12}\\ 3b_{12}+b_{22}=b_{11}+2b_{12}\\ -2b_{11}+2b_{21}=3b_{21}-2b_{22}\\ -2b_{12}+2b_{22}=b_{21}+2b_{22}\end{cases}$ $\quad -b_{11}+b_{22}=-x$

$\Leftrightarrow \begin{pmatrix} & 2 & 1 & \\ -1 & 1 & & 1\\ -2 & & -1 & 2\\ & -2 & -1 & \end{pmatrix}\begin{pmatrix}b_{11}\\ b_{12}\\ b_{21}\\ b_{22}\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

记作 $C\cdot b=0$

不妨取 $b_{12}=x$，则 $b_{21}=-2x$ $\quad$ 1 3 -1 -1

取 $b_{11}=y$，则 $b_{22}=y-x$

∴ 可与 A 交换的矩阵有形式

$\begin{pmatrix}y & x\\ -2x & y-x\end{pmatrix}$ $\quad x, y\in\mathbb{R}$ $\quad$ #

8. 若 $A=\frac{1}{2}(B+E)$

求证 $A^2=A$ iff $B^2=E$

Pf. $A^2=A$

$\Leftrightarrow \left(\frac{1}{2}(B+E)\right)^2=\frac{1}{2}(B+E)$

$\Leftrightarrow B^2+2B+E=2B+2E$

$\Leftrightarrow B^2=E$ #

11. 求证：任一方阵可表为对称矩阵与反对称矩阵之和.

Pf. 对 $\forall$ 方阵 $A \in \mathbb{R}^{n\times n}$

记 $B = \frac{1}{2}(A+A^T)$

$C = \frac{1}{2}(A-A^T)$

可验证 $B^T = B$ & $C^T = -C$

i.e. B 对称 & C 反对称

得 $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$ #

12\. 设 A, B 为对称阵矩.

求证 $A \cdot B$ 对称 并 A, B 可交换.

Pf. $A \cdot B$ 对称,

$$\Leftrightarrow (A\cdot B)^T = A \cdot B$$

$$\Leftrightarrow B^T \cdot A^T = A \cdot B$$

$$\xLeftrightarrow[B^T=B]{A^T=A} B \cdot A = A \cdot B$$

$\Leftrightarrow$ A, B 可交换 #

13. 设 $A \in \mathbb{R}^{n\times n}$

s.t. $Ax = 0 \quad \forall x \in \mathbb{R}^n$

求证 $A = 0$

Pf. 取 $v_i = (0,\dots,0,\overset{\text{ith 坐标}}{1},0,\dots,0)^T \quad i=1,\dots,n$

则 $A\cdot v_i = (a_{1i},\dots\dots,a_{ni})^T \underset{\text{(条件)}}{=} 0 \quad i=1,\dots,n$

$\Rightarrow a_{1i}=0,\dots,a_{ni}=0$, for $i=1,\dots,n$

i.e. $A=0$ #

Pf 2. use rank

14. 2) 用初等行变换化成行梯形矩阵

$$A = \begin{pmatrix} -3 & 1 & -3 & 0 & 5 \\ 4 & 3 & 2 & 3 & 0 \\ 6 & -1 & -5 & 0 & -7 \\ 2 & 5 & 1 & 4 & 1 \end{pmatrix}$$

解：$A \xrightarrow{r_1+r_4} \begin{pmatrix} 1 & 6 & -2 & 4 & 6 \\ & - & - & & \\ - & - & & - & \\ - & - & & - & \end{pmatrix}$

$\xrightarrow[r_4-2r_1]{\substack{r_2-4r_1 \\ r_3-6r_1}} \cdots\cdots$ （略） #

15 6) 计算 $r(A)$；若 A 满秩 则求 A^{-1}

$$\begin{pmatrix} 9 & -12 & 7 & 18 \\ & 1 & 1 & 1 \\ 3 & -12 & 1 & \\ -1 & 4 & & 1 \end{pmatrix}$$

解. 计算 $r(A)$：$A \longrightarrow \begin{pmatrix} 1 & -4 & & -1 \\ 3 & -12 & 1 & \\ 9 & -12 & 7 & 18 \\ & 1 & 1 & 1 \end{pmatrix}$

$$\longrightarrow \begin{pmatrix} 1 & -4 & & -1 \\ & & & 3 \\ & 24 & & 9 \\ & 1 & 1 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & -4 & & -1 \\ & 1 & 1 & 1 \\ & & -24 & -15 \\ & & & 3 \end{pmatrix}$$

可见 $r(A) = 4$.

计算 A^{-1} （略） #.

16. 设 $A_n = \begin{pmatrix} 0 & a_1 & & & \\ & & a_2 & & \\ & & & \ddots & \\ & & & & a_{n-1} \\ a_n & & & & 0 \end{pmatrix}$ where $a_i \neq 0$

求 A_n^{-1}

解1: 记 $P = P(1,2)\cdot P(2,3)\cdot\cdots\cdot P(n,n-1)$

则 $P\cdot A_n = \mathrm{diag}(a_1,\ldots,a_n)$

$\Rightarrow (P\cdot A_n)^{-1} = \mathrm{diag}(a_1^{-1},\ldots,a_n^{-1})$

另一方面 $(P\cdot A_n)^{-1} = A_n^{-1}\cdot P^{-1}$

$\therefore A_n^{-1} = \mathrm{diag}(a_1^{-1},\ldots,a_n^{-1})\cdot P$

$$\begin{pmatrix} & & & a_1^{-1} \\ a_2^{-1} & & & \\ & \ddots & & \\ & & a_n^{-1} & \end{pmatrix}$$ #

解2: 待定系数直接求 A_n^{-1} (略) #

17 求矩阵 X s.t.

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ & 1 & & \vdots \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix} \cdot X = \begin{pmatrix} 2 & 1 & & & 0 \\ 1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2 & 1 \\ 0 & & & 1 & 2 \end{pmatrix}$$

解：记 $A \cdot X = B$

其中 A, B 均为 $n \times n$ 矩阵

STEP 1. 确定 X 的形状

- 要矩阵乘法成立，则 X 的行数为 n
- X 的列数 $=$ B 的列数 $= n$.

STEP 2. 求解 X 的值.

由 $|A| = 1$ 可得 A 可逆

有 $X = A^{-1} \cdot B$

Claim $A_n^{-1} = \begin{pmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & \ddots & \ddots \\ & & 1 & -1 \\ & & & 1 \end{pmatrix}$ where $A_n = \begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \vdots \\ & & 1 \end{pmatrix}$

(解1 $(A_n \mid E)$

$$\xrightarrow{r_1 - r_2} \left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 1 & -1 & & \\ & 1 & \cdots & 1 & & 1 & & \\ & & \ddots & \vdots & & & \ddots & \\ & & & 1 & & & & 1 \end{array}\right) \xrightarrow{r_2 - r_3} \cdots$$

$$\xrightarrow{r_{n-1} - r_n} \left(\begin{array}{ccc|cccc} 1 & & & 1 & -1 & & \\ & 1 & & & 1 & -1 & \\ & & \ddots & & & \ddots & \ddots \\ & & & & & 1 & -1 \\ & & 1 & & & & 1 \end{array}\right) \quad \checkmark \ )$$

(解2：数学归纳法

$n = 1$： $A_0 = (1) \Rightarrow A_0^{-1} = (1)$

假设 $n = N$ 时 $A_N^{-1} = \begin{pmatrix} 1 & -1 & & \\ & 1 & & \\ & & \ddots & -1 \\ & & & 1 \end{pmatrix}$

则 A_{N+1} 满足

$$A_{N+1} \cdot A_{N+1}^{-1} = \left(\begin{array}{c|c} A_N^{-1} & \begin{matrix} a^{-1}_{1,N+1} \\ a^{-1}_{N,N+1} \end{matrix} \\ \hline a^{-1}_{N+1,1} \cdots a^{-1}_{N+1,N} & a_{N+1,N+1} \end{array}\right) \cdot \begin{pmatrix} A_N & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \\ & 1 \end{pmatrix} = \begin{pmatrix} E_N & \\ & 1 \end{pmatrix} = E$$

得
- $E_{N+1,i} = 0$ for $i = 1, \dots, N$
  $\Rightarrow a^{-1}_{N+1,i} = 0$
- $E_{N+1,N+1} = 1 \Rightarrow a_{N+1,N+1} = 1$.
- $E_{N,N+1} = 0 \Rightarrow 1 \cdot 1 + a^{-1}_{N,N+1} \cdot 1 = 0 \Rightarrow a^{-1}_{N,N+1} = -1$

17(续)

$\circ\ E_{i,N+1}=0 \Rightarrow 1\cdot 1+(-1)\cdot 1+a^{-1}_{1,N+1}\cdot 1=0$

$\Rightarrow a^{-1}_{1,N+1}=0$

综上所述，得证 $\checkmark$)

继续计算 $X=A^{-1}\cdot B$

$\circ$ 1st row: $X_{1,\cdot}=(1,-1,0\cdots 0)\cdot\begin{pmatrix}2&1&0&0&\cdots&0\\1&2&1&0&\cdots&0\\ & & & & & \end{pmatrix}$

$=(1\ -1\ -1,0\cdots 0)$

$\circ$ last row: $X_{n,\cdot}=(0,\cdots 0,1)\begin{pmatrix}\vdots& &\vdots&\vdots&\vdots\\ \vdots& &\vdots&\vdots&\vdots\\0& &0&1&2\end{pmatrix}=(0,\cdots,0,1,2)$

$\circ$ ith row: $X_{i,\cdot}=\begin{array}{c}\begin{matrix} & & & & i & i+1 & & \end{matrix}\\(0,\ldots,0,1,-1,0\cdots 0)\end{array}\begin{pmatrix}\ddots&\ddots&\ddots& & \\ &1&2&1&0\\ & &1&2&1\\ & & &\ddots&\ddots\end{pmatrix}$

$i=2,\ldots,n-1$

$=\begin{array}{c}\begin{matrix} & & & i-1 & i & i+1 & i+2 & & \end{matrix}\\(0,\ldots,0,1,1,-1,-1,0,\ldots 0)\end{array}$

综上所述，$X=\begin{pmatrix}1&-1&-1& & & & \\1&1&-1&-1& & & \\ &1&1&-1&-1& & \\ & &\ddots&\ddots&\ddots&\ddots& \\ & & &1&1&-1&-1\\ & & & &1&1&-1\\ & & & & &1&2\end{pmatrix}$

#

21. 求 $A=\begin{pmatrix} E_k & B \\ & E_l \end{pmatrix}$ 的逆

解：记 $A^{-1}=\begin{pmatrix} A_{11}^{-1} & A_{12}^{-1} \\ A_{21}^{-1} & A_{22}^{-1} \end{pmatrix}$

∘ $A^{-1}\cdot A=\begin{pmatrix} A_{11}^{-1} & A_{12}^{-1} \\ A_{21}^{-1} & A_{22}^{-1} \end{pmatrix}\begin{pmatrix} E & B \\ & E \end{pmatrix}=\begin{pmatrix} E & \\ & E \end{pmatrix}$

$\Rightarrow \begin{cases} A_{11}^{-1}=E \\ A_{11}^{-1}\cdot B+A_{12}^{-1}\cdot E=0 \end{cases} \Rightarrow \begin{cases} A_{11}^{-1}=E \\ A_{12}^{-1}=-B \end{cases}$

∘ $A\cdot A^{-1}=\begin{pmatrix} E & B \\ & E \end{pmatrix}\cdot\begin{pmatrix} E & B \\ A_{21}^{-1} & A_{22}^{-1} \end{pmatrix}=\begin{pmatrix} E & \\ & E \end{pmatrix}$

$\Rightarrow \begin{cases} E\cdot A_{21}^{-1}=0 \\ E\cdot A_{22}^{-1}=E \end{cases} \Rightarrow \begin{cases} A_{21}^{-1}=0 \\ A_{22}^{-1}=E \end{cases}$

综上所述，得 $A^{-1}=\begin{pmatrix} E_k & -B \\ & E_l \end{pmatrix}$ #

22. 对 $D = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$ where A, B, C 同阶
A, B 可逆

求 D^{-1}.

解: 记 $D^{-1} = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$

- $D \cdot D^{-1} = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix}\begin{pmatrix} X & Y \\ Z & T \end{pmatrix} = \begin{pmatrix} E & \\ & E \end{pmatrix}$

$\Rightarrow \begin{cases} A \cdot Z = E \\ A \cdot T = 0 \end{cases} \xRightarrow{A\text{可逆}} \begin{cases} Z = A^{-1} \\ T = 0 \end{cases}$

- $D^{-1} \cdot D = \begin{pmatrix} X & Y \\ A^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & A \\ B & C \end{pmatrix} = \begin{pmatrix} E & \\ & E \end{pmatrix}$

$\Rightarrow \begin{cases} Y \cdot B = E \xRightarrow{B\text{可逆}} Y = B^{-1} \\ XA + YC = 0 \Longrightarrow X = -B^{-1}CA^{-1} \end{cases}$

综上所述 $D^{-1} = \begin{pmatrix} -B^{-1}CA^{-1} & B^{-1} \\ A^{-1} & 0 \end{pmatrix}$

#

23 如果 $A^k=0$

求证 $(E-A)^{-1}=E+A+A^2+\dots+A^{k-1}$.

Q: 怎样的 A s.t. $E-A$ 可逆?

Pf. 验证 $(E+A+A^2+\dots+A^{k-1})(E-A)$

$$\begin{aligned}&= E+A+A^2+\dots+A^{k-1}\\ &\quad -A-A^2-\dots-A^{k-1}-A^k\\ &= E-A^k\\ &\overset{A^k=0}{=\!=} E\end{aligned}$$

类似可验证 $(E-A)(E+A+\dots+A^{k-1})=E$ #

24. 设 A 为 $n\ (n\geq 2)$ 阶方阵

求证 $|A^*| = |A|^{n-1}$

Pf. case $|A|\neq 0$

则 A^{-1} 存在 $\Rightarrow A^* = |A|\cdot A^{-1}$

$\Rightarrow |A^*| = \big|\, |A|\cdot A^{-1} \big|$

$= \big|\, |A|\cdot E \big| \cdot |A^{-1}|$

$= |A|^n \cdot |A|^{-1} = |A|^{n-1}$

case $|A| = 0$

考察 $(A^*)^T$ 的可逆性.

取 $a = (a_{11}, a_{12}, \dots, a_{1n})^T$

假设 $a\neq 0$ (否则选其他行, 若选不到, 则 $A=0$, 即证 $|A^*|=0$)

则 $$(A^*)^T\cdot a = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}\begin{pmatrix} a_{11} \\ \vdots \\ a_{1n} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{n} a_{1j}A_{ij} \end{pmatrix}$$

$$\xlongequal{\text{Thm 5.2}} \begin{pmatrix} |A| \\ 0 \\ \vdots \\ 0 \end{pmatrix} \xlongequal{|A|=0} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

由 $(A^*)^T\cdot a = 0$ & $a\neq 0$ 得.

$(A^*)^T$ 不可逆

$\therefore |A^*| = |(A^*)^T| = 0$ #