

1. 4) $\frac{3}{2}$

$$5) \left| \begin{array}{ccc} x & y & x \\ y & x & y \\ y & y & x \end{array} \right| \xrightarrow{\frac{r_1+r_2}{r_1+r_2}} \left| \begin{array}{ccc} x+2y & x+2y & x+2y \\ y & x & y \\ y & y & x \end{array} \right|$$

$$= (x+2y) \left| \begin{array}{ccc} 1 & 1 & 1 \\ y & x & y \\ y & y & x \end{array} \right|$$

$$\xrightarrow[r_3-y r_1]{r_2-y r_1} (x+2y) \left| \begin{array}{ccc} 1 & 1 & 1 \\ x-y & & \\ x-y & & \end{array} \right|$$

$$= (x+2y)(x-y)^2$$

2. 1) 略

$$2) \begin{vmatrix} b & a \\ 1 & e \\ d & c \end{vmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{vmatrix} 1 & e & f \\ b & a \\ d & c \end{vmatrix} = \begin{vmatrix} b & a \\ d & c \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

求 i, j s.t. $1, 7, i, 5, 2, j, 6$ 为偶.

解：记 τ_i 为 i th 数与其后的数构成逆序对数量.

注意到 i, j 为 3 或 4.

得 $\tau_1 \ \tau_2 \ \tau_3 \ \tau_4 \ \tau_5 \ \tau_6 \ \tau_7$

0 5 ? 2 0 0 0

分情况讨论

↳ 3 & 4 的小予 6

① $i=3 \ \& \ j=4$:

$$\tau_3 = 1 \Rightarrow \tau = 8$$

② $i=4 \ \& \ j=3$

$$\tau_3 = 2 \Rightarrow \tau = 9$$

综上 $i=3 \ \& \ j=4$.

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6 3). 求逆序数并判断奇偶性.

$$I = \underbrace{2n \cdot (2n-2) \cdots 4 \cdot 2}_n \cdot \underbrace{(2n-1)(2n-3) \cdots 3 \cdot 1}_n.$$

解: 记 \bar{I} 为 I 的逆.

$$\text{i.e. } \bar{I} = 1 \cdot 3 \cdots (2n-3)(2n-1) \times 2 \cdot 4 \cdots (2n-2) \cdot 2n.$$

在排列 \bar{I} 中任取一对数 (i, j)

若 i, j 均为奇或均为偶

则 (i, j) 不构成逆序

仅需考虑 i 为奇, j 为偶的情况

为简便记 $(2i-1, 2j)$ $i, j \in \{1, 2, \dots, n\}$

ith 奇数 j th 偶数

可见 $(2i-1, 2j)$ 构成逆序

$$\Leftrightarrow j < i$$

注意到所有偶数均在奇数后面.

\therefore ith 奇数 "产生" $(i-1)$ 个逆数.

$$\therefore \bar{I}$$
 的逆序数为 $\sum_{i=1}^n (i-1) = \frac{1}{2}n(n-1)$

回顾 Ex 5

\therefore 由 $T(I) + T(\bar{I}) = C_{2n}^2$ 得

$$T(I) = C_{2n}^2 - T(\bar{I})$$

$$= \frac{1}{2}2n(2n-1) - \frac{1}{2}n(n-1)$$

$$= \frac{1}{2}n(3n-1)$$

现判断 $\frac{1}{2}n(3n-1)$ 何时取偶数.

$$\text{若 } \frac{1}{2}n(3n-1) = 2m$$

$$12m+6-1$$

$$\Leftrightarrow n(3n-1) = 4m$$

$$12m+9-1$$

可见 n 含因子 4 时, $T(I)$ 为偶.

仅需考虑 $n = 4m+1, 4m+2, 4m+3$ 时.

• $n = 4m+1 \Rightarrow T(I) = (4m+1) \cdot 2(6m+1)$ 不含因子 4

• $n = 4m+2 \Rightarrow T(I) = 2(2m+2)(12m+5) \quad \dots \dots \dots$

• $n = 4m+3 \Rightarrow T(I) = (4m+3) \cdot 4(3m+2)$ 含因子 4.

综上所述, $n = 4m$ 或 $4m+3$ 时 $T(I)$ 为偶; 否则为奇 #

8. 左多项式 $f(x) = \begin{vmatrix} x & 7 & 3 & -1 \\ 1 & 4 & x & 0 \\ 0 & x & -1 & 5 \\ 2 & 1 & 2 & 3 \end{vmatrix}$ 中 x^2 的系数.

解 1. 直接计算行列式 $f(x)$, 略.

解 2. 行列式定义为 $\sum_{(j_1, j_2, j_3, j_4)} (-1)^{\tau(j_1, \dots, j_4)} a_{1j_1} a_{2j_2} a_{3j_3} a_{4j_4}$

含 x^2 的项 $\Leftrightarrow a_{ij_i}$ 中有且仅有两项为 x .

可得 (j_1, j_2, j_3, j_4) 的可能组合是 C_3^2 种.

- (1, 3, 4, 2)

- (1, 4, 2, 3)

- (4, 3, 2, 1)

$$\begin{aligned} \therefore \text{含 } x^2 \text{ 的项为} & (-1)^{\tau(1, 3, 4, 2)} \cdot x \cdot x \cdot 5 \cdot 1 \\ & + (-1)^{\tau(1, 4, 2, 3)} \cdot x \cdot 0 \cdot x \cdot 2 \\ & + (-1)^{\tau(4, 3, 2, 1)} \cdot -1 \cdot x \cdot x \cdot 2 \\ & = 3 \cdot x^2 \end{aligned}$$

$\therefore x^2$ 项的系数为 3

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10. 4) 求

$$\begin{vmatrix} a_{11} & 0 & \cdots & \cdots & 0 \\ 0 & a_{22} & \cdots & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & a_{ij} & \vdots \\ 0 & a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix}$$

解 1: 按 1st 行展开. 得

$$|\dots| = a_{11} \cdot \begin{vmatrix} 0 & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \text{记为 } D.$$

其中, D 只有一项 $(-1)^{\tau(n-1, n-2, \dots, 2, 1)} \cdot a_{2n} \cdot a_{3,n-1} \cdots a_{n-1,3} \cdot a_{n,2}$

$$= (-1)^{C_{n-1}^2} \cdot a_{2n} \cdots a_{n,2}$$
$$\therefore |\dots| = (-1)^{\frac{(n-1)(n-2)}{2}} \cdot a_{11} \cdot a_{2n} \cdots a_{n,2}$$

解 2: 先通过 $(n-1)$ 次互换, 将 1st 行 换至 最后一行. 得.

$$|\dots| = (-1)^{n-1} \times \begin{vmatrix} 0 & & a_{11} \\ & \ddots & a_{2,n} \\ a_{n,2} & \cdots & 0 \end{vmatrix}$$

$$= (-1)^{n-1} \times (-1)^{C_n^2} \cdot a_{11} \cdot a_{2,n} \cdots a_{n,2}$$

= ... #

11. 6) 求

$$\begin{vmatrix} x & x & \cdots & x & a \\ & & & a & x \\ a & & & & x \\ & & & & \vdots \\ & & & & x \\ & & & & n \end{vmatrix}$$

回顾爪型
行列式解

$n=1$ 时的表达

式到底是什么?

还是 $|a|$?

暂且不答.

解: Case $n=1$: ?

$$\text{Case } n=2. \quad \begin{vmatrix} x & a \\ a & x \end{vmatrix} = x^2 - a^2$$

Case $n>2$.

• Subcase $a=0$

$$|\dots| = \begin{vmatrix} x & \cdots & x & 0 \\ & & & x \\ 0 & & & \vdots \\ & & & x \end{vmatrix} = 0 \quad (\because \text{1st 行展开})$$

• Subcase $a \neq 0$

进行初等列变换 $C_n - \frac{x}{a} \cdot C_j, \quad j=1, \dots, n-1$

得 $\begin{vmatrix} x & \cdots & x & a \cdot (n-1) \cdot \frac{x^2}{a} \\ & & & a & 0 \\ a & & & & 0 \end{vmatrix}$

$$= (-1)^{C_n^2} \cdot \left(a \cdot (n-1) \cdot \frac{x^2}{a} \right) \cdot a^{n-1}$$

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12. 3) 求

$$\begin{vmatrix} a & 2 & 3 & \dots & n \\ 1 & a+1 & 3 & & n \\ 1 & 2 & a+2 & & n \\ \vdots & & & \ddots & \vdots \\ 1 & 2 & 3 & \dots & a+n-1 \end{vmatrix}$$

解：进行初等行变换 $r_i - r_1 \quad i=2, \dots, n$

得爪型

$$\begin{vmatrix} a & 2 & 3 & \dots & n \\ 1-a & a-1 & & & \\ 1-a & & a-1 & & \\ \vdots & & & \ddots & \\ 1-a & & & & a-1 \end{vmatrix}$$

进行初等列变换 $C_i + C_j \quad j=2, \dots, n$

得上三角

$$\begin{vmatrix} a+(2+3+\dots+n) & 2 & 3 & \dots & n \\ a-1 & & & & \\ & & a-1 & & \\ & & & \ddots & \\ & & & & a-1 \end{vmatrix}$$

$$\therefore | \dots | = \left(a + \frac{(n+2)(n-1)}{2} \right) \cdot (a-1)^{n-1}$$

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12 4) 求

$$\begin{vmatrix} a+b & ab \\ 1 & a+b \end{vmatrix} \quad \ddots \quad \begin{vmatrix} a+b & ab \\ 1 & a+b \end{vmatrix}$$

数学归纳法

解：记

$$D_m = \begin{vmatrix} a+b & ab \\ 1 & a+b \end{vmatrix} \quad \ddots \quad \begin{vmatrix} a+b & ab \\ 1 & a+b \end{vmatrix}_m$$

$$\boxed{m=1}, D_1 = |a+b| = a+b$$

$$\boxed{m=2}, D_2 = \begin{vmatrix} a+b & ab \\ 1 & a+b \end{vmatrix} = (a+b)^2 - ab = a^2 + ab + b^2$$

考虑 $m=n+1$ 情况

$$\text{假设 } D_m = \sum_{i=0}^n a^i \cdot b^{m-i} \quad \forall m=1, \dots, n-1 \text{ 成立}$$

$\rightarrow D_{n+1}$ 试² 展开.

$$D_{n+1} = (a+b) \cdot D_n + (-1) \cdot ab \cdot \left(\begin{array}{cccccc} a+b & ab & & & & \\ a+b & a+b & & & & \\ & & a+b & & & \\ & & & a+b & ab & \\ & & & a+b & a+b & \end{array} \right)_{n-2}$$

$$= (a+b) \cdot D_n - ab \cdot D_{n-1}$$

(假设)

$$= (a+b) \cdot \sum_{i=0}^n a^i \cdot b^{n-i} - ab \cdot \sum_{i=0}^{n-1} a^i \cdot b^{n-1-i}$$

$$= \sum_{i=0}^n (a^{i+1}b^{n-i} + a^i b^{n-1-i}) - \sum_{i=0}^{n-1} a^{i+1}b^{n-1-i+1}$$

$$= ab^n + \dots + a^n b + a^{n+1}$$

$$+ b^{n+1} + ab^n + \dots + a^n b$$

$$- ab^n - \dots - a^n b$$

$$= b^{n+1} + ab^n + \dots + a^n b + a^{n+1}$$

$$= \sum_{i=0}^{n+1} a^i \cdot b^{n-i}$$

#

16. 求证

$$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix} = 0$$

证明：进行 $C_4 - C_3, C_3 - C_2, C_2 - C_1$ 得

$$\begin{vmatrix} a^2 & 2a+1 & 2a+3 & 2a+5 \\ b^2 & 2b+1 & 2b+3 & 2b+5 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

进行 $C_4 - C_3, C_3 - C_2$ 得

$$\begin{vmatrix} a^2 & 2a+1 & 2 & 2 \\ \vdots & \vdots & 2 & 2 \\ \vdots & \vdots & 2 & 2 \\ \vdots & \vdots & 2 & 2 \end{vmatrix}$$

后两列相等，行列式为零
并.

17. 计算

D_n

$$\left| \begin{array}{cccccc} x^n & x^{n-1} & \cdots & x^2 & x^1 & 1 \\ (x+1)^n & (x+1)^{n-1} & & & x+1 & \\ \vdots & \vdots & & & \vdots & \\ (x+n-1)^n & \vdots & & & \vdots & \\ (x+n)^n & (x+n)^{n-1} & & & x+n & 1 \end{array} \right|$$

解：进行 $C_i - (x+n) \cdot C_{i+1}$ $i=1, \dots, n$

$$\left| \begin{array}{cccc} -n \cdot x^{n-1} & -n \cdot x^{n-2} & -n & 1 \\ -(n-1) \cdot (x+1)^{n-1} & -(n-1) \cdot (x+1)^{n-2} & -(n-1) & 1 \\ \vdots & \vdots & \vdots & \vdots \\ -1 \cdot (x+n-1)^{n-1} & -1 \cdot (x+n-1)^{n-2} & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right|$$

$$\left| \begin{array}{ccccc} -n \cdot x^{n-1} & \cdots & \cdots & \cdots & -n \\ -(n-1) \cdot (x+1)^{n-1} & & & & -(n-1) \\ \vdots & & & & \vdots \\ -1 \cdot (x+n-1)^{n-1} & \cdots & \cdots & \cdots & 1 \end{array} \right|$$

$$= \prod_{i=1}^n (-i) \cdot \left| \begin{array}{ccccc} x^{n-1} & \cdots & \cdots & \cdots & 1 \\ (x+1)^{n-1} & & & & 1 \\ \vdots & & & & \vdots \\ (x+n-1)^{n-1} & \cdots & \cdots & \cdots & 1 \end{array} \right|$$

$$= (-1)^n \cdot n! \times D_{n-1}$$

$$\text{又 } D_0 = 1$$

$$\begin{aligned} \therefore D_n &= (-1)^n \cdot n! \cdot D_{n-1} \\ &= (-1)^n \cdot n! \times (-1)^{n-1} (n-1)! \times \cdots \times 1 \\ &= (-1)^{\frac{n(n-1+\dots+1)}{2}} \times \prod_{i=1}^n i! \\ &= (-1)^{\frac{1}{2}n(n+1)} \cdot \prod_{i=1}^n i! \end{aligned}$$

18. | 解方程

$$\left(\begin{array}{cccc} 3 & 1 & 2 & 0 \\ 0 & 5 & 1 & 4 \\ 4 & 0 & 2 & 1 \\ 2 & 1 & 0 & 1 \end{array} \right) \left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right) = \left(\begin{array}{c} 0 \\ 1 \\ 0 \\ 1 \end{array} \right)$$

解：克莱默法则 (略).

19. 用数学归纳法

求证

$$\left| \begin{array}{cccc} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \\ \hline c_{11} & \cdots & c_{1n} & b_{11} & \cdots & b_{1m} \\ \vdots & \cdots & \vdots & b_{m1} & \cdots & b_{mm} \end{array} \right| = \left| \begin{array}{c} a_{11} \cdots a_{1n} \\ \vdots \\ a_{m1} \cdots a_{mn} \end{array} \right| \cdot \left| \begin{array}{c} b_{11} \cdots b_{1m} \\ \vdots \\ b_{m1} \cdots b_{mm} \end{array} \right|$$

证明：记 $D_{n,m} = \begin{pmatrix} A_n & C \\ C^T & B_m \end{pmatrix}$ 其中 $A_n = \begin{pmatrix} a_{11} \cdots a_{1n} \\ \vdots \\ a_{m1} \cdots a_{mn} \end{pmatrix}$, $B_m = \begin{pmatrix} b_{11} \cdots b_{1m} \\ \vdots \\ b_{m1} \cdots b_{mm} \end{pmatrix}$

对 $n=0$ 或 $m=0$ 时，显然 $|D_{n,m}| = |A_n| \cdot |B_m|$.

假设 $(n,m)=(N,M)$ 时有 $|D_{n,m}| = |A_n| \cdot |B_m|$.

现证 $(n,m)=(N+1,M)$ 也成立.

数学归纳
的推导链

$n \backslash m$	0	1	\cdots	M
0	✓	✓	\cdots	✓
1	✓	✓	\cdots	✓
\vdots	\vdots	\vdots	\vdots	\vdots
N	✓	✓	\cdots	✓

显而易见

$$|D_{N+1,M}| = \left| \begin{array}{cc} a_{11} \cdots a_{1,N+1} \\ \vdots \\ a_{N+1,1} \cdots a_{N+1,N+1} \\ \hline C & B_M \end{array} \right|$$

$$\frac{\text{行列式的} \quad \sum_{j=1}^{N+1}}{\text{线性性质}} \left| \begin{array}{cccc} 0 & \cdots & 0 & a_{1,j} & 0 & \cdots & 0 \\ a_{2,1} & \cdots & a_{2,j-1} & a_{2,j} & a_{2,j+1} & \cdots & a_{2,N+1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{N+1,1} & \cdots & -a_{N+1,j} & - & -a_{N+1,1} & & \end{array} \right| \quad C \quad B_M$$

$$\frac{\text{适当} \quad \sum_{j=1}^{N+1} (-1)^{j-1}}{\text{交错}} \left| \begin{array}{cc} a_{1j} & 0 \\ a_{2,j} & A_{1,j} \\ \vdots & \vdots \\ a_{N+1,j} & A_{1,j} \\ \hline C & B_M \end{array} \right|$$

$$= \sum_{j=1}^{N+1} (-1)^{j-1} a_{1j} \left| \begin{array}{cc} A_{1j} & 0 \\ \vdots & \vdots \\ C' & B_M \end{array} \right|$$

$$\stackrel{\text{假设}}{=} \sum_{j=1}^{N+1} (-1)^{j-1} a_{1j} |A_{1j}| \cdot |B_M|$$

$$\stackrel{\text{1st行}}{\text{展开式}} |A_{N+1}| \cdot |B_M|$$

#

21. 设 $a_1, \dots, a_n \in \mathbb{R}$ 互不相同

$b_1, \dots, b_n \in \mathbb{R}$.

用克拉默法则证明

已知实系数多项式

$$f(x) = C_{n-1}x^{n-1} + \dots + C_1x + C_0$$

$$\text{s.t. } f(a_i) = b_i, i=1, \dots, n.$$

证明：代入 $f(a_i) = b_i$ 得方程组。

$$\begin{pmatrix} a_1^{n-1} & \cdots & a_1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \vdots & \vdots \\ a_n^{n-1} & \cdots & a_n & 1 \end{pmatrix} \begin{pmatrix} C_{n-1} \\ \vdots \\ C_0 \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

要证存在唯一解，仅需证行列式

$$D_n = \begin{vmatrix} a_1^{n-1} & \cdots & a_1 & 1 \\ \vdots & & \vdots & \vdots \\ a_n^{n-1} & \cdots & a_n & 1 \end{vmatrix} \neq 0$$

数学归纳法

$$\circ n=1 \text{ 时 } D_1 = |1| \neq 0$$

$$\circ n=N+1 \text{ 时}$$

$$D_{N+1} = \begin{vmatrix} a_1^N & \cdots & a_1 & 1 \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ a_{N+1}^N & \cdots & a_{N+1} & 1 \end{vmatrix}$$

$$\xrightarrow[i=1, \dots, N]{C_i - a_1 C_{i+1}} \begin{vmatrix} (a_1 - a_N) a_1^{N-1} & \cdots & a_1 - a_N & 1 \\ \vdots & & \vdots & \vdots \\ (a_{N-1} - a_N) a_{N-1}^{N-1} & \cdots & a_{N-1} - a_N & 1 \\ 0 & \cdots & 0 & 1 \end{vmatrix}$$

$$\frac{\#(N+1) \times (-1)^{N+1+N+1}}{\text{展开式}} \cdot \begin{vmatrix} (a_1 - a_{N+1}) a_1^{N-1} & \cdots & a_1 - a_N & 1 \\ \vdots & & \vdots & \vdots \\ (a_{N+1} - a_{N+1}) a_{N+1}^{N-1} & \cdots & a_{N+1} - a_{N+1} & 1 \end{vmatrix}$$

$$\frac{\text{每行提出公因子}}{\prod_{i=1}^N (a_i - a_{N+1})} D_N \neq 0 \hookrightarrow a_i \neq a_{N+1} \& D_N \neq 0$$