Outline

1st 初等矩阵 1

定义 5.1 (行对换)

初等变换 $\longleftrightarrow$ 初等矩阵乘法

行　　　左..

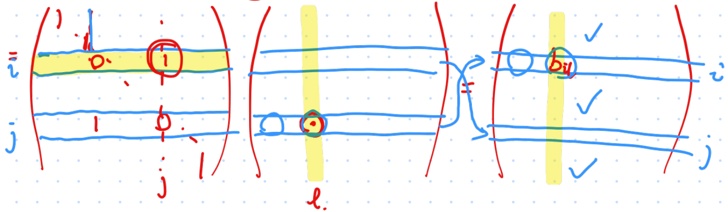
列　　　右..

$$
P(i,j) := \begin{array}{c}
\\ \\ \\ i \\ \\ \\ j \\ \\ \\ \\
\end{array}
\begin{pmatrix}
1 & & & & & & & & & \\
& \ddots & & & & & & & & \\
& & 1 & & & & & & & \\
& & & 0 & & & 1 & & & \\
& & & & 1 & & & & & \\
& & & & & 1 & & & & \\
& & & 1 & & & 0 & & & \\
& & & & & & & 1 & & \\
& & & & & & & & \ddots & \\
& & & & & & & & & 1
\end{pmatrix}
$$

1st 初等矩阵 2

- $P(i,j)A$: 通过交换 A 的第 i 行和第 j 行得到.
- $AP(i,j)$: ……列……

$P(i,j)\cdot A =$



$$b_{il} = \sum_k P_{ik} \cdot A_{kl} = A_{jl}$$

$$A \cdot P(i,j) = \Big(\underbrace{P(i,j)^T}_{=P(i,j)} A^T\Big)^T$$

2nd 初等矩阵 1

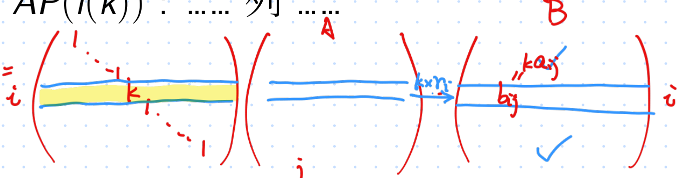
定义 5.2 (数乘行)

$$P(i(k)) := \begin{array}{cc} & \begin{matrix} & & & i & & & \end{matrix} \\ i & \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & k & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

2nd 初等矩阵 2

- $P(i(k))A$：把 A 的第 i 行乘以 k 得到
- $AP(i(k))$：…… 列 ……

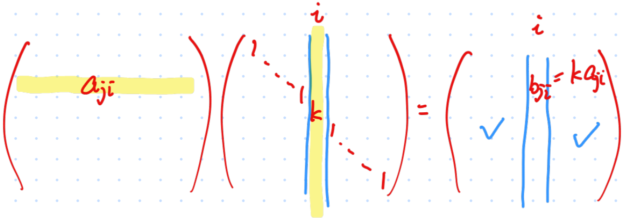
$P(i(k)) \cdot A =$



$= B$

$$b_{ij} = \sum_k P_{ik} a_{kj} = P_{ii} \cdot a_{ij} = k \cdot a_{ij}$$

$A \cdot P(i(k)) =$



3rd 初等矩阵 1

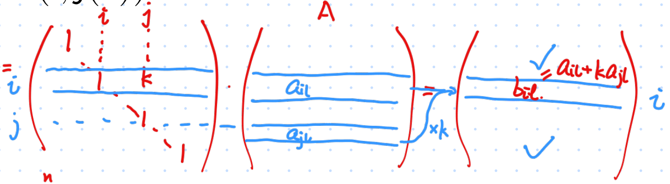
定义 5.3 (行加行)

$$P(i,j(k)) := \begin{array}{cc} & \begin{array}{ccccccc} & & & & j & & \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ \\ \\ \\ \end{array} & \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & k & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

3rd 初等矩阵 2

- $P(i,j(k))A$: k 乘以 A 的第 j 行, 加到第 i 行
- $AP(i,j(k))$: …… 列 … 列

$P(i,j(k))\cdot A =$

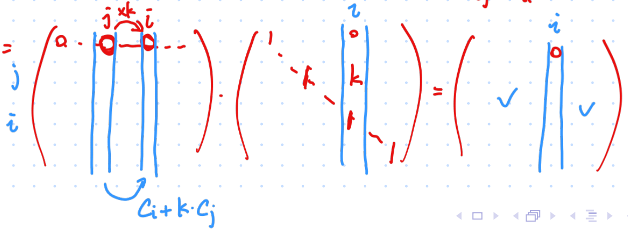


$$b_{il} = \sum_{m=1}^{n} P_{im}\cdot a_{ml} = P_{ii}a_{il} + P_{ij}a_{jl}$$

$$= 1\cdot a_{il} + k\cdot a_{jl}$$

$k\cdot a_{lj} + a_{ll}$

$A\cdot P(i,j(k)) =$



定理 5.1

定理 5.1 (初等变换的矩阵表示)

- 左乘: 作用于行
- 右乘: 作用于列

$P^T \cdot A^T$

证明 $P \cdot A$ 作用行初等变换

$$AP = ((AP)^T)^T = \boxed{(\boxed{P^T}\boxed{A^T})^T}.$$

$$P(i,j)^T = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & & 1 \\ & & 1 & & 0 \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}^T = P(i,j)$$

$$P(i,j(k))^T = \begin{array}{c} i \\ j \end{array}\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & k & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & k & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}\begin{array}{c} i \\ j \end{array} = P(j,i(k))$$

$$P(i(k))^T = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & k & \\ & & & \ddots \\ & & & & 1 \end{pmatrix}^T = P(i(k))$$

例题 5.1

已知 $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$, 求 $P(3,1(2))A$, $AP(2,3)$, $P(3(3))A$.

$$P(3,1(2))\cdot A = \begin{pmatrix} \cdot & - & \\ - & - & \cdot \\ 4 & 2 & 5 \end{pmatrix}$$

$$A\cdot P(2,3) = \begin{pmatrix} \vdots & 2 & 2 \\ \vdots & -2 & 1 \\ \vdots & 1 & -2 \end{pmatrix}$$

$$P(3(3))\cdot A = \begin{pmatrix} - & - & \\ - & - & \\ 6 & -6 & 3 \end{pmatrix}$$

初等矩阵的逆

$$\underbrace{P(i,j)^{-1}\cdot \boxed{P(i,j)\cdot A}}_{E} = A$$

- $P(i,j)^{-1} = P(i,j)$ ✓
- $\underline{P(i(k))}^{-1} = \underline{P(i(k^{-1}))}$ ✓
- $\underline{P(i,j(k))}^{-1} = \underline{P(i,j(-k))}$ ✓

初等矩阵的应用

矩阵的初等变换 = 左乘或者右乘初等矩阵

接下来，我们将以初等矩阵的乘法重新描述关于初等变换的结果.

初等行变换	$\longleftrightarrow$	左乘初等矩阵
$r_i \leftrightarrow r_j$	$\longleftrightarrow$	$P(i,j)\times\cdot$
kr_i	$\longleftrightarrow$	$P(i(k))\times\cdot$
$r_i + kr_j$	$\longleftrightarrow$	$P(i,j(k))\times\cdot$
‥列‥		右‥‥.

定理 5.2

标准形　有限次初等变换

$A \sim \begin{pmatrix} E_r & \\ & 0 \end{pmatrix}$

回顾定理 3.2: 对秩为 $r(A) = r$ 的矩阵 A, 通过有限次初等变换化为

$$\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}$$

定理 5.2

$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \xrightarrow{r_2 - r_1} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \xrightarrow{r_1 + 2r_2} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \xrightarrow{-r_2} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

对秩为 $r(A) = r$ 的 $m \times n$ 矩阵 A, 存在有限个

- m 阶初等矩阵 $P_1, \ldots, P_s$
- n 阶初等矩阵 $Q_1, \ldots, Q_t$

$P_1 = P(2,1(-1))$

$P_2 = P(1,2(2))$

$P_3 = P(2(-1))$

$P_3 P_2 P_1 A$

$= E_2$

s.t.

$$\underbrace{P_s \ldots P_1}_{行} A \underbrace{Q_1 \ldots Q_t}_{列} = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}.$$

推论 5.1

A^{-1} 存在 $\Leftrightarrow |A| \neq 0 \Leftrightarrow$ A 满秩

$r(A)=n$

进一步地, 如果 A 可逆, 则只需行变换即可.

推论 5.1

$\xrightarrow{\text{Thm 5.3}}$ $P_s \cdots P_1 A \boxed{Q_1 \cdots Q_t} = E_n$

对于可逆矩阵 A, 存在有限初等变换 $P_1, \ldots, P_m$, s.t.

$$P_m \ldots P_1 A = E.$$

进而有

$$A^{-1} = P_m \ldots P_1.$$

推论 5.1 证明

只需证明一个初等矩阵的右乘, 等价于某个初等矩阵的左乘. 对三类初等矩阵 Q, 验证

$$Q\,AQ\,Q^{-1} = Q\,E\,Q^{-1} \Longrightarrow Q(AQ)Q^{-1} = E \Longrightarrow QA = E.$$

由定理 5.2 得

$$\begin{aligned} E &= P_s \dots P_1 A Q_1 \dots Q_t \\ &= Q_1 \dots Q_t P_s \dots P_1 A \end{aligned}$$

m - - - - - 1

此时, 我们可得

$$A^{-1} = P_m \dots P_1.$$

推论 5.2

将推论 5.1 中的行变换换为列变换:

推论 5.2

……

$$AQ_1 \dots Q_m = E.$$

由此可得 $A^{-1} = Q_1 \dots Q_m$.

$P^{-1}PAP = P^{-1}EP = E$

$= AP$

推论 5.3

推论 5.3

$$r(A^T) = r(A).$$

证明

若 $PAQ = \begin{pmatrix} E_r & \\ & 0 \end{pmatrix}$, 则 $Q^T A^T P^T = \begin{pmatrix} E_r^T & \\ & 0 \end{pmatrix}$

$= (PAQ)^T$

$= \begin{pmatrix} E_r & \\ & 0 \end{pmatrix}^T$

初等行变换求逆矩阵

利用行变换表达 A^{-1}:

$$
\begin{aligned}
& P_m \cdots P_1 \left(A \,\middle|\, E \right) \\
A^{-1} = \ & = \left(P_m \cdots P_1 A \,\middle|\, P_m \cdots P_1 \right) \\
& = \left(E \,\middle|\, A^{-1} \right)
\end{aligned}
$$

$$
(A \mid E) = \begin{array}{c}
\left(\begin{array}{ccc|ccc}
a_{11} & \cdots & a_{1n} & 1 & & \\
\vdots & & \vdots & & \ddots & \\
\vdots & & \vdots & & & \ddots \\
a_{n1} & \cdots & a_{nn} & & & 1
\end{array}\right)
\end{array}
\begin{array}{l} \\ i \\ j \\ \\ \end{array}
$$

$$
\begin{cases}
a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\
\vdots \\
a_{n1}x_1 + \cdots + a_{nn}x_n = b_n
\end{cases}
$$

$\Updownarrow$

$A \cdot x = b$ 解 $x = A^{-1} b$

$\Updownarrow$

$(A \cdot x \mid E \cdot b)$

$\Updownarrow$

$(A \cdot x \mid E \cdot b)$

$(P_m \cdots P_1 \cdot A \cdot x \mid P_m \cdots P_1 \cdot E \cdot b)$

$(E \cdot x \mid P_m \cdots P_1 \cdot b)$

$P \cdot (A \mid E) = (PA \mid PE)$

$P(i,j)$

$P(i(k))$

$P(i,j(k))$

$$
\begin{cases}
x_1 = \cdots \\
x_2 = \cdots \\
\\
x_n = \cdots
\end{cases}
$$

例题 5.2

例题 5.2

求以下矩阵的逆

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 4 & 2 \\ 5 & -3 & 1 \end{pmatrix}$$

$$\begin{array}{c} 23 \\ 3 \\ \hline 69 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & & \\ 1 & 4 & 2 & & 1 & \\ 5 & -3 & 1 & & & 1 \end{array}\right)$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & & 1 & \\ 2 & 1 & -1 & 1 & & \\ 5 & -3 & 1 & & & 1 \end{array}\right)$$

$$\xrightarrow[r_3-5r_1]{r_2-2r_1} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & & 1 & \\ & -7 & -5 & 1 & -2 & \\ & -23 & -9 & & -5 & 1 \end{array}\right)$$

$$\xrightarrow{r_2 \times 3} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & & 1 & \\ & -21 & -15 & 3 & -6 & \\ & -23 & -9 & & -5 & 1 \end{array}\right)$$

$$\xrightarrow{r_2 - r_3} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & & 1 & \\ & 2 & -6 & 3 & -1 & \\ & -23 & -9 & & -5 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2} r_2} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & & 1 & \\ & 1 & -3 & \frac{3}{2} & -\frac{1}{2} & \\ & -23 & -9 & & -5 & 1 \end{array}\right)$$

$$\xrightarrow[r_3+23r_2]{r_1-4r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 14 & -6 & 3 & \\ & 1 & -3 & \frac{3}{2} & -\frac{1}{2} & \\ & & -78 & \frac{69}{2} & -\frac{33}{2} & 1 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{78} r_3} \left(\begin{array}{ccc|ccc} 1 & & 14 & -6 & 3 & \\ & 1 & -3 & \frac{3}{2} & -\frac{1}{2} & \\ & & 1 & -\frac{69}{156} & -\frac{33}{156} & -\frac{1}{78} \end{array}\right)$$

例题 5.2 解 1

$$\left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & & \\ 1 & 4 & 2 & & 1 & \\ 5 & -3 & 1 & & & 1 \end{array}\right)$$

初等列变换求逆矩阵

$$
\begin{aligned}
&\left(\frac{A}{E}\right) Q_1 \dots Q_m \\
=&\left(\frac{AQ_1 \dots Q_m}{Q_1 \dots Q_m}\right) \\
=&\left(\frac{E}{A^{-1}}\right)
\end{aligned}
$$

例题 5.2 解 2