

1. 解下列线性方程组

$$(1) \left\{ \begin{array}{l} x_1 + 2x_2 - 5x_3 + 4x_4 + x_5 = 4 \\ 3x_1 + 7x_2 - x_3 - 3x_4 + 2x_5 = 10 \\ -x_2 - 13x_3 - 2x_4 + x_5 = -14 \\ x_3 - 16x_4 + 2x_5 = -11 \\ 2x_4 + 5x_5 = 12 \end{array} \right.$$

解：考虑增广矩阵

$$\left(\begin{array}{ccccc|c} 1 & 2 & -5 & 4 & 1 & 4 \\ 3 & 7 & -1 & -3 & 2 & 10 \\ -1 & -13 & -2 & 1 & -14 \\ 1 & -16 & 2 & -11 & \\ 2 & 5 & & 12 & \end{array} \right)$$

$$\xrightarrow{r_2 - 3r_1} \left(\begin{array}{ccccc|c} 1 & 2 & -5 & 4 & 1 & 4 \\ 1 & 14 & -15 & -1 & -2 & \\ - & - & - & - & - & \end{array} \right)$$

$$\xrightarrow{r_3 + r_2} \left(\begin{array}{ccccc|c} & & & & & - \\ & & & & & - \\ 1 & -17 & 0 & & & -16 \\ & \dots & \dots & & & \dots \\ & & & & & \end{array} \right)$$

$$\xrightarrow{r_4 - r_3} \left(\begin{array}{ccccc|c} & & & & & - \\ & & & & & - \\ & & & & & - \\ 1 & 2 & 5 & & & \\ & \dots & \dots & & & \end{array} \right)$$

$$\xrightarrow{r_5 - 2r_4} \left(\begin{array}{ccccc|c} 1 & 2 & -5 & 4 & 1 & 4 \\ 1 & 14 & -15 & -1 & -2 & \\ 1 & -17 & 0 & & & -16 \\ 1 & 2 & 5 & & & \\ 1 & 2 & & & & \end{array} \right)$$

得 $(x_1, x_2, x_3, x_4, x_5) = (1, 1, 1, 1, 2)$

2. 设 $\beta = (1, 2, 1, 1)$

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

求 β 由 α_i 的线性表示.

解: 求解方程

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

考虑增广矩阵

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 2 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 \end{array} \right)$$

$$\xrightarrow{\begin{array}{l} r_i - r_1 \\ -\frac{1}{2}r_i, i=2,3,4 \end{array}} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{array} \right)$$

$$\xrightarrow{\begin{array}{l} r_2 \leftrightarrow r_3 \\ r_3 \leftrightarrow r_4 \end{array}} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -\frac{1}{2} \end{array} \right)$$

$$\xrightarrow{r_3 - r_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -\frac{1}{2} \end{array} \right)$$

$$\xrightarrow{\begin{array}{l} r_3 - r_4 \\ \frac{1}{2}r_4 \end{array}} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{4} & 0 \end{array} \right)$$

得 $(k_1, k_2, k_3, k_4) = (\frac{5}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})$

$\Rightarrow \beta = \frac{5}{4}\alpha_1 + \frac{1}{4}\alpha_2 - \frac{1}{4}\alpha_3 - \frac{1}{4}\alpha_4$ #

3. 设 $\alpha_1 = (3, -1, 1)$

$$\alpha_2 = (1, 1, 2)$$

$$\alpha_3 = (1, -3, -3)$$

$$\alpha_4 = (4, 0, 5)$$

(1) 求证 $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ 线性相关

Pf. 考虑 $k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4 = 0$

i.e.
$$\begin{pmatrix} 3 & 1 & 1 & 4 \\ -1 & 1 & -3 & 0 \\ 1 & 2 & -3 & 5 \end{pmatrix} \cdot \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

消元法
$$\begin{pmatrix} 1 & 2 & -3 & 5 \\ 1 & -2 & \frac{5}{3} & -\frac{8}{3} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = 0$$

可取 k_3 为自由未知量

$$k_1 = -k_3$$

$$\begin{cases} k_1 = -k_3 \\ k_2 = 2k_3 \\ k_4 = 0 \end{cases}$$

$$k_2 = 2k_3$$

不妨取 $k_3 = 1$ 得

$$-\alpha_1 + 2\alpha_2 + \alpha_3 = 0$$

$\therefore \alpha_1, \alpha_2, \alpha_3, \alpha_4$ 线性相关.

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只消 $\alpha_1, \alpha_2, \alpha_3$
线性相关.

(2) 求证 $\alpha_1, \alpha_2, \alpha_4$ 线性无关

Pf: 方程 $k_1\alpha_1 + k_2\alpha_2 + k_4\alpha_4 = 0$

对应 (1) 中方程的系数矩阵去掉 3rd 行

得消元后矩阵

$$\begin{pmatrix} 1 & 2 & 5 \\ 1 & \frac{5}{3} & -\frac{8}{3} \end{pmatrix}$$

\therefore 方程只有零解

$\therefore \alpha_1, \alpha_2, \alpha_4$ 线性无关.

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5. 试证：向量组 $\alpha_1, \dots, \alpha_s$

$$\text{与向量组 } \beta_1 = \alpha_2 + \alpha_3 + \dots + \alpha_{s-1} + \alpha_s \\ \beta_1 = \alpha_1 + \alpha_3 + \dots + \alpha_{s-1} + \alpha_s$$

$$\beta_1 = \alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_{s-1}$$

等价。

Pf. 要证 $\{\alpha_i\}$ 与 $\{\beta_i\}$ 等价

即证它们可以相互线性表示。

• $\{\beta_i\}$ 可被 $\{\alpha_i\}$ 线性表示。

从 β_i 的表达式可知

• $\{\alpha_i\}$ 可被 $\{\beta_i\}$ 线性表示。

可验证 $\beta_1 + \dots + \beta_n = (n-1)(\alpha_1 + \dots + \alpha_n)$

$$\therefore \alpha_i = (\alpha_1 + \dots + \alpha_n) - (\alpha_1 + \dots + \alpha_{i-1} + \alpha_i + \dots + \alpha_n)$$

$$= \frac{1}{n-1}(\beta_1 + \dots + \beta_n) - \beta_i \quad \#$$

Recall

Ch.1 [5] 4.3

10. 在 \mathbb{R}^3 中的两组基 $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

 $(\eta_1, \eta_2, \eta_3) = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & -1 \end{pmatrix}$

(1) 求 $\{\varepsilon_3\}$ 到 $\{\eta_3\}$ 的过渡矩阵

解：即求矩阵 A s.t. $(\eta_1, \eta_2, \eta_3) = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \cdot A$.

记为增广矩阵 $(\varepsilon_1 \ \varepsilon_2 \ \varepsilon_3 \mid \eta_1 \ \eta_2 \ \eta_3)$

i.e.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 3 & -1 & \end{array} \right)$$

消元

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & -1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 1 & \frac{1}{2} & -\frac{1}{2} & \end{array} \right)$$

得

$$\left(\begin{array}{ccc} \frac{1}{2} & 2 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{array} \right) \text{ 为 } \{\varepsilon_3\} \rightarrow \{\eta_3\} \text{ 过渡矩阵}$$

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(2) 求向量 $\xi = (3, 5, 0)$ 在 $\{\eta_3\}$ 下的坐标.

Rmk: 此处

是的坐标

默认是在

标准基 $\{\varepsilon_3\}$ 下

解：解方程 $k_1 \eta_1 + k_2 \eta_2 + k_3 \eta_3 = \xi$

其增广矩阵

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 1 & 1 & 1 & 5 \\ 0 & 3 & -1 & 0 \end{array} \right)$$

消元

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 1 & -1 & -2 & -2 \\ 1 & & 3 & \end{array} \right)$$

得 ξ 在 $\{\eta_3\}$ 下的坐标为

$$(k_1, k_2, k_3) = (1, 1, 3)$$

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附 1. 已知 $(\alpha_1, \alpha_2, \alpha_3, \beta) = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 4 & 7 & 1 & 10 \\ 0 & 1 & -1 & b \\ 2 & 3 & a & 4 \end{pmatrix}$

1) 求 a, b 的值 s.t. β 不能由 $\{\alpha_i\}$ 线性表示,

解: β 不能由 $\{\alpha_i\}$ 线性表示,

$$\Leftrightarrow k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = \beta \text{ 无解.}$$

考虑增广矩阵

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 4 & 7 & 1 & 10 \\ 0 & 1 & -1 & b \\ 2 & 3 & a & 4 \end{array} \right)$$

消元

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & a-1 & 0 & 0 \\ 0 & b-2 & & \end{array} \right)$$

$\therefore b-2 \neq 0$ 时方程无解

$\therefore b \neq 2$ 时, β 不能被 $\{\alpha_i\}$ 线性表示 并

Rank:

$\{\alpha_1, \alpha_2, \alpha_3, \beta\}$ 线性无关

$\Leftrightarrow \beta$ 不能被 $\{\alpha_i\}$ 线性表示

e.g. $\{\alpha_1, \dots, \beta\}$ 线性无关

$$\Leftrightarrow a \neq 1 \& b \neq 2$$

$$\Leftrightarrow b \neq 2$$

$\Leftrightarrow \beta$ 不能被 $\{\alpha_i\}$...

2) 求 a, b 的值 s.t. β 可被 $\{\alpha_i\}$ 线性表示.

并写出此时表达式

解: 即求 a, b s.t. 题1) 方程有非零解

得 $b=2$ 时 β 能被 $\{\alpha_i\}$ 线性表示

解方程

$$\left(\begin{array}{ccc} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & a-1 & 0 \end{array} \right) \cdot \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

当 $a-1 \neq 0$ 时, 得 $(k_1, k_2, k_3) = (-1, 2, 0)$

$$\text{i.e. } a \neq 1 \text{ 时, } \beta = -\alpha_1 + \alpha_2$$

当 $a-1 = 0$ 时, k_3 为自由未知量

$$\text{i.e. } a = 1 \text{ 时, } \beta = (-1-2k_3)\alpha_1 + (2+k_3)\alpha_2 + k_3\alpha_3$$

附 2.

设矩阵通过有限次行变换矩阵 A 变到

求证：A 与 B 的列向量有相同的线性关系

如何从几何
来理解

Pf. 记 A, B 的列向量分别为 $\{\alpha_i\}$ 和 $\{\beta_i\}$

要证 $\{\alpha_i\}$ 与 $\{\beta_i\}$ 有相同线性关系

即证，对 $\forall (k_1, \dots, k_n) \in F^n$ 有

$$k_1\alpha_1 + \dots + k_n\alpha_n = 0$$

$$\Leftrightarrow k_1\beta_1 + \dots + k_n\beta_n = 0$$

已知初等变换不改变方程的解集

由 A, B 相差有限初等变换得

$$\{k \mid A \cdot k = 0\} = \{k \mid B \cdot k = 0\}$$

$\therefore k = (k_1, \dots, k_n)$ 满足 $A \cdot k = 0$

$\Leftrightarrow k$ 满足 $B \cdot k = 0$

$$\text{又 } A \cdot k = k_1\alpha_1 + \dots + k_n\alpha_n$$

$$B \cdot k = k_1\beta_1 + \dots + k_n\beta_n$$

$\therefore \{\alpha_i\}$ 与 $\{\beta_i\}$ 有相同线性关系 #

附 3. 设 A 为 $n \times n$ 方阵 满足

◦ $A^k \alpha = 0$ 有解 $\alpha = \alpha$

◦ $A^{k-1} \alpha \neq 0$

求证： $\{\alpha, A\alpha, \dots, A^{k-1}\alpha\}$ 线性无关

Pf. 考虑方程 $k_0\alpha + k_1(A\alpha) + \dots + k_{k-1}(A^{k-1}\alpha) = 0$

同时乘 A^{k-1} 得 $k_0(A^{k-1}\alpha) + k_1(A^k\alpha) + \dots + k_{k-1}(A^{2k-2}\alpha) = 0$

$$= k_0(A^{k-1}\alpha) + k_1(A^k\alpha) + k_2(A \cdot A^k\alpha)$$

$$+ \dots + k_{k-1}(A^{k-2} \cdot A^k\alpha)$$

$$\underline{A^k\alpha = 0} \quad k_0(A^{k-1}\alpha)$$

由 $A^{k-1}\alpha \neq 0$ 得 $k_0 = 0$

类似地，乘 A^{k-2} 得 $k_1 = 0$

\vdots A 得 $k_{k-1} = 0$

$\therefore k_0\alpha + k_1(A\alpha) + \dots + k_{k-1}(A^{k-1}\alpha) = 0$ 只有零解

$\therefore \{\alpha, A\alpha, \dots, A^{k-1}\alpha\}$ 线性无关 #

附 4. 设 $A \in m \times n$, $B \in n \times m$ ($m < n$)

(What if
 $n \leq m$)

求证: $A \cdot B = E \Rightarrow B$ 的列向量组线性无关

Pf 1: 记 B 的列向量组为 $\{\beta_1, \dots, \beta_m\}$

反证, 假设 $\{\beta_i\}$ 线性相关

i.e. 存在不全为零 k_i s.t. $k_1\beta_1 + \dots + k_m\beta_m = 0$

$$\therefore A \cdot B \begin{pmatrix} k_1 \\ \vdots \\ k_m \end{pmatrix} = A \cdot (\beta_1, \dots, \beta_m) \begin{pmatrix} k_1 \\ \vdots \\ k_m \end{pmatrix} = A \cdot \vec{o} = \vec{o}$$

与 " $A \cdot B = E$ " 矛盾

$$\hookrightarrow (\because A \cdot B \cdot \begin{pmatrix} k_1 \\ \vdots \\ k_m \end{pmatrix} = E \cdot \begin{pmatrix} k_1 \\ \vdots \\ k_m \end{pmatrix} = \begin{pmatrix} k_1 \\ \vdots \\ k_m \end{pmatrix} \neq 0) \quad \#$$

回顾

Pf 2: $\because \text{rank}(AB) = \text{rank}(E) = m$

又 $\text{rank}(AB) \leq \text{rank}(B) \leq m$

\therefore 上述不等式中 " \leq " 取 " $=$ "

$\Rightarrow \text{rank}(B) = m$

$\therefore B$ 的列秩 $= \text{rank}(B) = m$

i.e. $\{\beta_i\}$ 极大线性无关组的向量个数为 m

$\therefore \{\beta_i\}$ 线性无关

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