正交矩阵

向量空间 $(V;+,\cdot)$ ⟶ 内积空间

内积: $(\cdot,\cdot): V\times V \to \mathbb{R}$

内积空间 $\begin{cases} \text{长度: } |\alpha| := \sqrt{(\alpha,\alpha)} \\ \text{夹角: } \langle\alpha,\beta\rangle = \arccos\dfrac{(\alpha,\beta)}{|\alpha|\cdot|\beta|} \end{cases}$

$\langle\alpha,\beta\rangle$: $\begin{cases} 0 \text{ or } \pi: \text{共线} \\ \frac{\pi}{2}: \text{垂直（正交）} \end{cases}$

定义 3.7

$n\times n$ 实矩阵 A 被称为 正交矩阵 如果

$$A^TA = E.$$

α,β 正交 $\Leftrightarrow (\alpha,\beta)=0$

基 $\{\alpha_1,\cdots,\alpha_n\}$

↓

正交基 $\{\alpha_i\}$ $(\alpha_i,\alpha_j) = \begin{cases} 0 & i\neq j \\ |\alpha_i|^2 & i=j \end{cases}$

↓ 单位化

单位正交基 $\{\alpha_i\}$ $(\alpha_i,\alpha_j) = \delta_{ij} = \begin{cases} 0 & i\neq j \\ 1 & i=j \end{cases}$

⟷ 正交矩阵 $A=(\alpha_1,\ldots,\alpha_n)$

如何构造单位正交基.

$\alpha_1 \to \beta_1 \to \gamma_1 = \beta_1/|\beta_1|$

$\alpha_2 \to \beta_2 = \alpha_2 - (\alpha_2,\gamma_1)\gamma_1 \rightsquigarrow \gamma_2 = \dfrac{\beta_2}{|\beta_2|}$

$\vdots$

$\alpha_n \to \beta_n = \alpha_n - \sum_{i=1}^{n-1}(\alpha_n,\gamma_i)\gamma_i \rightsquigarrow \gamma_n = \dfrac{\beta_n}{|\beta_n|}$

正交矩阵 A 满足以下性质:

- $A^{-1} = A^T$
- A^{-1} 和 A^T 也是正交矩阵
- $|A| = \pm 1$ $\quad |A|^2 = |A|\cdot|A| = |A^T|\cdot|A| = |A^TA| = |E| = 1.$
- 正交矩阵的乘积也是正交

$(A\cdot B)^T(A\cdot B)$

$= B^T\underbrace{A\cdot A}_{=E}B = E.$

定理 3.3

定理 3.3

记 n 阶实矩阵 $A = (\alpha_1, \ldots, \alpha_n)$. A 是正交矩阵, 当且仅当, $\alpha_1, \ldots, \alpha_n$ 为 $\mathbb{R}^n$ 的一组标准正交基.

$A = (\alpha_1, \ldots, \alpha_n)$. 是正交.

$\Leftrightarrow \quad A^T A = E$

$$\underset{n\times 1}{\begin{pmatrix} \alpha_1^T \\ \vdots \\ \alpha_n^T \end{pmatrix}} \underset{1\times n}{(\alpha_1 \cdots \alpha_n)} = \begin{pmatrix} \alpha_1^T\alpha_1 & \cdots & \alpha_1^T\alpha_n \\ \vdots & \alpha_i^T\alpha_j & \vdots \\ \alpha_n^T\alpha_1 & \cdots & \alpha_n^T\alpha_n \end{pmatrix}_{n\times n} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$\Leftrightarrow (\alpha_i, \alpha_j) = \alpha_i^T\alpha_j = \delta_{ij}$

$\Leftrightarrow \{\alpha_i\}$ 单位正交

正交 $A = (\alpha_1, \ldots, \alpha_n) = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \Leftrightarrow$ 正交 $A^T = (\beta_1^T, \ldots, \beta_n^T)$

$\Updownarrow$

$\{\beta_j^T\}$ 正交

$\Leftrightarrow$ 行向量 $\{\alpha_i\}$ 正交

$\Leftrightarrow$ 列向量 $\{\beta_j\}$ 正交

定理 3.4

$A\in\mathbb{R}^{n\times n}\subseteq\mathbb{C}^{n\times n}$

代数基本定理

$|\lambda E-A|=0$

有 n 个根(可能复数,重根)

Thm3.4: 实对称 A 存在 n 个实特征值 (重根)

定理 3.4

实对称矩阵的特征值均为实数

证明

$\because A\in\mathbb{R}^{n\times n}\subseteq\mathbb{C}^{n\times n}$

$\therefore |\lambda E-A|=0$ 总有解.

不妨设 λ_1

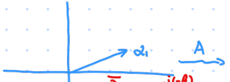
s.t. $A\alpha_1=\lambda_1\alpha_1$

反证: 假设 $\lambda_1\notin\mathbb{R}$

$A\bar{\alpha}_1=\bar{A}\bar{\alpha}_1$ ($A\in\mathbb{R}^{n\times n}$)

$=\overline{(A\alpha_1)}=\overline{\lambda_1\alpha_1}$

$=\bar{\lambda}_1\cdot\bar{\alpha}_1$

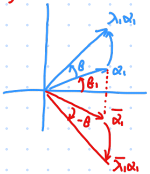


$\bar{\lambda}_1=r\cdot e^{i(-\theta)}$

记 $\lambda_1=r\cdot e^{i\theta}$

拉伸 旋转

$\left(x+iy=\sqrt{x^2+y^2}\left(\frac{x}{\sqrt{\cdots}}+i\cdot\frac{y}{\sqrt{\cdots}}\right)=r\cdot(\cos\theta+i\cdot\sin\theta)\right)$



α_1 是 λ_1 的特征向量

$\Leftrightarrow$ $k\alpha_1$ 也是 λ_1 ...

$(a+ib)(a-ib)=$

特别地 $k=\frac{\alpha_1^T\bar{\alpha}_1}{\alpha_1^T\alpha_1}=\frac{|\alpha_1|^2}{\alpha_1^T\cdot\alpha_1}$

$\Rightarrow k\alpha_1=\frac{|\alpha_1|^2}{\alpha_1^T\cdot\alpha_1}\cdot\alpha_1$

记 $\alpha_1=r_1\cdot e^{i\theta_1}$

$=\frac{r_1^2}{r_1^2\cdot e^{i(2\theta_1)}}\cdot r_1\cdot e^{i\theta_1}$

$=r_1\cdot e^{i(-\theta_1)}=\bar{\alpha}_1$

$(\alpha_1,\bar{\alpha}_1)_A=\alpha_1^TA\bar{\alpha}_1=\alpha_1^T\bar{\lambda}_1\bar{\alpha}_1=\bar{\lambda}_1\alpha_1^T\bar{\alpha}_1=\bar{\lambda}_1\cdot|\alpha_1|^2$

$\parallel$ A对称 $\parallel$

$(\bar{\alpha}_1,\alpha_1)_A=\bar{\alpha}_1^TA\alpha_1=\bar{\alpha}_1\cdot\lambda_1\alpha_1=\lambda_1|\alpha_1|^2$

$\Rightarrow \lambda_1=\bar{\lambda}_1\Rightarrow\lambda_1\in\mathbb{R}$.

定理 3.5

定理 3.5

实对称矩阵中, 属于不同特征值的特征向量必正交

回顾: A 特征值 $\lambda_i \neq \lambda_j$

$\Rightarrow$ 特征向量 α_i, α_j 线性无关

Pf: $(\alpha_i, \alpha_j)_A = \alpha_i^T A \alpha_j = \alpha_i^T \lambda_j \alpha_j = \lambda_j \underline{\alpha_i^T \alpha_j}$

$\parallel$ (A 对称)

$(\alpha_j, \alpha_i)_A = \text{- - - - - - - - - -} = \lambda_i \underline{\alpha_j^T \alpha_i}$

$\Rightarrow \lambda_j (\alpha_i, \alpha_j) = \lambda_i (\alpha_i, \alpha_j)$

又 $\lambda_i \neq \lambda_j$

$\therefore (\alpha_i, \alpha_j) = 0$ if $i \neq j$.

定理 3.6

对任意 n 阶实对阵矩阵 A, 则存在正交矩阵 T, s.t. $T^{-1}AT$ 为对角矩阵.

A 是可对角化 $(A \sim \Lambda)$

$\exists$ 可逆 $P = \{\alpha_1, \dots, \alpha_n\}$

s.t. $P^{-1}AP = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$

Pf: 矩阵 $A_0 = A$

是某个线性映射 $f: \mathbb{R}^n \to \mathbb{R}$

在 $\{e_1, \dots, e_n\}$ 下的表示

目标:构造 $\{\alpha_1, \dots, \alpha_n\}$

s.t. $\begin{cases} \alpha_i \text{ 是 } A \text{ 的特征向量} \\ (\alpha_i, \alpha_j) = \delta_{ij} \end{cases}$

STEP 1: α_1 存在的

($A \in \mathbb{R}^{n\times n} \subseteq \mathbb{C}^{n\times n}$

总有 $\lambda_1 \in \mathbb{C}$ 特征值

又 $\lambda_1 \in \mathbb{R}$ $\therefore \exists \alpha_1 \in \mathbb{R}^n$ s.t. $A\alpha_1 = \lambda_1 \alpha_1$)

$\because$ α_1 顶多 $\{e_i\}$ 的一个向量线性相关

$\therefore$ 不妨设 α_1 和 $\{e_2, \dots, e_n\}$ 线性无关

$\therefore \{\alpha_1, e_2, \dots, e_n\}$

$\xrightarrow{\text{单位正交化}} \{\alpha_1, e_2^{(1)}, \dots, e_n^{(1)}\}$

$\rightsquigarrow T_1 = (\alpha_1, e_2^{(1)}, \dots, e_n^{(1)})$

$$T_1^{-1}AT_1 = T_1^TAT_1 = \begin{pmatrix} \alpha_1^T \\ e_2^{(1)T} \\ \vdots \\ e_n^{(1)T} \end{pmatrix} A (\alpha_1\ e_2^{(1)} \cdots e_n^{(1)})$$

$$= \left(\begin{array}{c|c} \alpha_1^TA\alpha_1 & \alpha_1^TAe_2^{(1)} \cdots \alpha_1^TAe_n^{(1)} \\ \hline e_2^{(1)T}A\alpha_1 & \\ \vdots & A_2 \\ e_n^{(1)T}A\alpha_1 & \end{array}\right)$$

$e_2^{(1)T}A\alpha_1 = \lambda_1 \underline{e_2^{(1)T}\alpha_1} = 0$

$$(\text{续}) = \left(\begin{array}{c|ccc} \lambda_1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A_2 & \\ 0 & & & \end{array}\right)$$

Claim: A_2 还是实对称

STEP 2: $T_2 = \left(\begin{array}{c|c} 1 & \\ \hline & T_2' \end{array}\right)$

$T_2T_1 = (\alpha_1, \alpha_2, e_3^{(2)}, \dots, e_n^{(3)})$

STEP i: $T_i = \begin{pmatrix} E_{i-1} & \\ & T_i' \end{pmatrix}$

STEP n:

$T = T_n \cdots\cdots T_2 \cdot T_1$

$$\Rightarrow T^{-1}AT = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

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例题 3.3

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设

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$

求正交矩阵 T, 使得 $T^{-1}AT$ 为对角矩阵.

STEP 1: 求特征值 λ

$|\lambda E - A| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0$$

- $\lambda = -1$.
- $\lambda = 5$

STEP 2. 求特征向量

- $\lambda = -1$.

$x_1 + x_2 + x_3 = 0$

$\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \alpha_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

- $\lambda = 5$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

STEP 3. 对 $\{\alpha_1, \alpha_2, \alpha_3\}$ 单位正交化.

- $\beta_1 = \alpha_1 \xrightarrow{\text{单位化}} \gamma_1 = \dfrac{\beta_1}{|\beta_1|} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$

- $\beta_2 = \alpha_2 - (\alpha_2, \gamma_1) \cdot \gamma_1 \rightsquigarrow \gamma_2 = \dfrac{\beta_2}{|\beta_2|}$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \underbrace{(-1, 0, 1) \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}}_{\frac{1}{\sqrt{2}}} \cdot \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \cdots$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

$\beta_3 = \alpha_3 - (\alpha_3, \gamma_1)\gamma_1 - (\alpha_3, \gamma_2) \cdot \gamma_2$

$\gamma_3 = \dfrac{\beta_3}{|\beta_3|}$

$T = (\gamma_1, \gamma_2, \gamma_3)$

例题 3.4

例题 3.4

求证: 如果实对称 $A \sim B$, 那么存在正交矩阵 T s.t.

$$T^{-1}AT = B.$$