1 习题

假设随机变量 $X_1, \ldots, X_4$ 相互独立且同分布，$P(X_i = 1) = 0.6$, $P(X_i = 0) = 0.4$。求行列式 $\begin{vmatrix} X_1 & X_2 \\ X_3 & X_4 \end{vmatrix}$ 的概率分布。

解：注意到 $\begin{vmatrix} X_1 & X_2 \\ X_3 & X_4 \end{vmatrix} = X_1X_4 - X_2X_3$

先计算 $X_iX_j \sim \begin{pmatrix} 0 & 1 \\ 1-0.6^2 & 0.6^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0.64 & 0.36 \end{pmatrix}$

再计算

$$\begin{aligned} P(X_1X_4 - X_2X_3 = -1) &= P(X_1X_4 = 0, X_2X_3 = 1) \\ &= P(X_1X_4 = 0) \cdot P(X_2X_3 = 1) \\ &= 0.64 \times 0.36 = 0.2304 \end{aligned}$$

$$\begin{aligned} P(X_1X_4 - X_2X_3 = 1) &= P(X_1X_4 = 1, X_2X_3 = 0) \\ &= \cdots = 0.2304 \end{aligned}$$

$$\begin{aligned} P(X_1X_4 - X_2X_3 = 0) &= 1 - P(\ldots = -1) - P(\cdots = 1) \\ &= 0.5392 \end{aligned}$$

$$\therefore X_1X_4 - X_2X_3 \sim \begin{pmatrix} -1 & 0 & 1 \\ 0.2304 & 0.5392 & 0.2304 \end{pmatrix}$$

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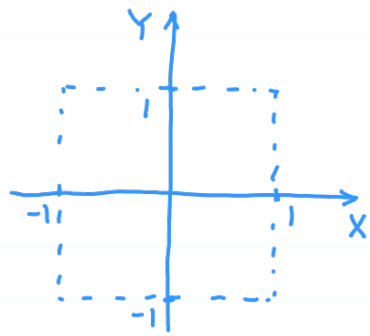
2 习题

设二维随机变量 (X, Y) 的联合概率密度函数为

$$f(x,y)=\begin{cases}\frac{1+xy}{4}, & |x|<1, |y|<1\\ 0, & \text{otherwise}\end{cases}$$

求证：X 与 Y 不独立，但 X^2 与 Y^2 独立。



Pf.

1. $f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \int_{-1}^{1} \frac{1+xy}{4}\,dy$

$= \frac{1}{4}\int_{-1}^{1} dy + \frac{x}{4}\cdot\int_{-1}^{1} y\,dy$

$= \frac{1}{2}$

$f_Y(y) = \cdots = \frac{1}{2}$

$\therefore\ f(x,y) \neq f_X(x)\cdot f_Y(y)$

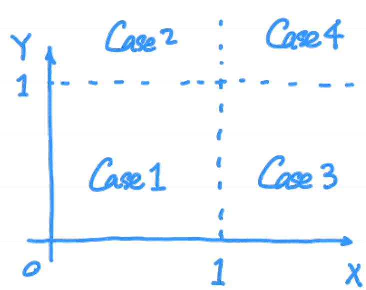
$\Rightarrow$ X, Y 不独立.

2. 记 $X' = X^2$, $Y' = Y^2$

分布函数 $F_{X'Y'}(x',y') = P(X' \le x', Y' \le y')$

$= P(X^2 \le x', Y^2 \le y')$

$= \begin{cases} 0, & x'<0 \text{ 或 } y'<0 \\ P(-\sqrt{x'} \le X \le \sqrt{x'}, -\sqrt{y'} \le Y \le \sqrt{y'}), & x' \ge 0 \text{ 且 } y' \ge 0 \end{cases}$



Case 1. $0 \le x' \le 1$, $0 \le y' \le 1$

$F_{X'Y'}(x',y') = \int_{-\sqrt{x'}}^{\sqrt{x'}} \int_{-\sqrt{y'}}^{\sqrt{y'}} \frac{1+xy}{4}\,dy\,dx$

$= \sqrt{x'y'}$

Case 2. $0 \le x' \le 1$, $1 < y'$

$F_{X'Y'}\cdots = \int_{-\sqrt{x'}}^{\sqrt{x'}} \int_{-1}^{1} \frac{1+xy}{4}\,dy\,dx = \sqrt{x'}$

Case 3. $1 \le x'$, $0 \le y' \le 1$

$F_{X'Y'}\cdots = \int_{-1}^{1} \int_{-\sqrt{y'}}^{\sqrt{y'}} \frac{1+xy}{4}\,dy\,dx = \sqrt{y'}$

Case 4. $1 \le x'$, $1 \le y'$

$F_{X'Y'}\cdots = 1$

可得 $F_{X'}(x') = \begin{cases} 0 & x'<0 \\ \sqrt{x'} & 0 \le x' \le 1 \\ 1 & 1 < x' \end{cases}$ ..

可见 $F_{X'Y'}(x',y') = F_{X'}(x')\cdot F_{Y'}(y')$

$\therefore$ X', Y' 独立 i.e. X^2, Y^2 独立 #