1 习题

设随机变量 X 的概率密度 $f(x)$ 满足

$$\begin{cases} f(1+x) & = f(1-x) \\ \int_0^2 f(x)dx & = 0.6 \end{cases}$$

试求 $P(X<0)=?$

解:

$$\begin{aligned}
\int_2^{+\infty} f(x)dx &= \int_1^{+\infty} f(1+x)\,d(1+x) \\
&= \int_1^{+\infty} f(1-x)\,dx \\
&= -\int_1^{+\infty} f(1-x)\,d(1-x) \\
&= -\int_0^{-\infty} f(x)\,dx \\
&= \int_{-\infty}^{0} f(x)\,dx
\end{aligned}$$

$$\begin{aligned}
\therefore \int_{-\infty}^{0} f(x)dx &= \frac{1}{2} \times 2 \cdot \int_{-\infty}^{0} f(x)dx \\
&= \frac{1}{2}\left(\int_{-\infty}^{0} f(x)dx + \int_2^{+\infty} f(x)dx\right) \\
&= \frac{1}{2}\left(1 - \int_0^2 f(x)dx\right) \\
&= \frac{1}{2}(1-0.6) = 0.2 \quad \#
\end{aligned}$$

2 习题

设随机变量 X 的概率密度

$$f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

试求随机变量 $Y = e^X$ 的概率密度 $f_Y(y)$。

解:

分布函数 $F_Y(y) = P(Y \leq y) = P(e^X \leq y)$

对 $y \leq 0$, $\{X \in \mathbb{R} \mid e^X \leq y\} = \phi$

$\Rightarrow F_Y(y) = 0$

对 $y > 0$, $F_Y(y) = P(X \leq \log y)$

$$= \int_{-\infty}^{\log y} f_X(x)\,dx$$

$$= \int_0^{\log y} e^{-x}\,dx$$

$$= -e^{-x}\Big|_0^{\log y} = 1 - \frac{1}{y}$$

i.e. $F_Y(y) = \begin{cases} 1 - \frac{1}{y}, & y > 0 \\ 0, & y \leq 0 \end{cases}$

分布密度函数

$$f_Y(y) = \begin{cases} \frac{d}{dy} F_Y(y) = \frac{1}{y^2}, & y > 0 \\ 0, & y \leq 0 \end{cases}$$

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3 习题

求证:

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = 1.$$

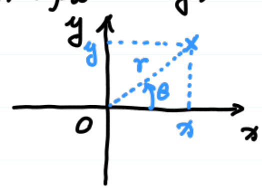
Pf: 由 $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2} d\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)$

$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-x^2}dx$

可见只需证 $\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$

考虑 $\left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)\cdot\left(\int_{-\infty}^{+\infty} e^{-y^2}dy\right)$

$= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}dxdy$



$\begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \end{cases}$

$= \int_0^{2\pi}\int_0^{+\infty} e^{-r^2} r\,dr\,d\theta$

$\begin{cases} dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta \\ \quad = \cos\theta dr - r\cdot\sin\theta\, d\theta \\ dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}\cdot d\theta \\ \quad = \sin\theta\, dr + r\cdot\cos\theta\, d\theta \end{cases}$

$= \frac{1}{2}\int_0^{2\pi} e^{-r^2}\Big|_0^{+\infty} d\theta$

$= \pi$

$\therefore \int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$ #

$\Rightarrow dx\wedge dy = (\cos\theta dr - r\sin\theta d\theta)$
$\wedge(\sin\theta dr + r\cos\theta d\theta)$
$= r(\cos\theta^2 + \sin^2\theta)dr\wedge d\theta$
$= r\cdot dr\wedge d\theta$

$\Rightarrow dxdy = |dx\wedge dy| = r\,dr\,d\theta$