Quasi-trees and geodesic trees

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0 Abstract

This paper study the quasisymmetric uniformization problem for metric trees.

1 Introduction

1.1 Preliminaries

An important question in geometric analysis is whether a given metric space is geometrically equivalent to a model space in a natural way.

- (X, d_X) and (Y, d_Y): metric spaces
- $f : X \to Y$ is said to be **quasisymmetric** if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ (playing the role of a control function for distortion) s.t.

$$\frac{d_Y(f(x), f(y))}{d_Y(f(y), f(z))} \leq \eta(\frac{d_X(x, y)}{d_X(y, z)}).$$

- Two metric spaces X and Y are quasisymmetrically equivalent if there exists a quasisymmetry $f : X \to Y$.
- Note that bi-Lipschitz is stronger than quasisymmetric. Check that

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq C^2 \frac{d_X(x, y)}{d_X(x, z)}.$$

- Quasisymmetric uniformization problem: what conditions when a given metric space X from a class of spaces is quasisymmetrically equivalent to some model space Y.

1.2 Tukia-Vaisala theorem

- **metric arc** J : a metric space J homeomorphic to the unit interval $[0, 1]$
- **quasi-arc**: it is quasisymmetrically equivalent to $[0, 1]$

- **bounded turning**: (X,d) is K bounded turning if there exists $K \geq 1$ s.t. for all $x, y \in X$ there exists a compact connected set $E \subset X$ containing x, y s.t. $\text{diam}(E) \leq Kd(x,y)$.

- **doubling**: (X,d) is called doubling if there exists $N \in \mathbb{N}$ (called the **doubling constant** of X) s.t. each ball $B(R)$ can be covered by N or fewer balls $B(R/2)$.

Tukia-Vaisala theorem: a metric arc is a quasi-arc iff it is doubling and of bounded turning. (In order words, one can "straighten out" such an arc (which may well have Hausdorff dimension > 1) to the interior $[0,1]$ by a quasisymmetry.)

1.3 Theorem 1.1

This paper study the quasisymmetric uniformization problem for metric tress.

- **metric tree** (sometimes called dendrites): a compact connected and locally connected metric space (T,d) that contains at least two distinct points and there is no closed curve (that is, for any $x, y \in T$, there is a unique arc connecting x and y in T).

- $[x,y]$: the arc connecting x and y in T

- an arc $[x,y]$ is **degenerate** if $x = y$

- **quasi-tree** : a tree is doubling and of bounded turning (inspired by T-V theorem)

Question 1 *Can all arcs in a quasi-tree be out straightened out simultaneously by a quasisymmetry?*

Theorem 1.1 *Every quasi-tree is quasisymmetrically equivalent to a geodesic tree.*

Idea of theorem 1.1 is defining a geodesic metric ϱ on (T,d) s.t. the identity map $\text{id}_T : (T,d) \to (T,\varrho)$ is a quasisymmetry.

1. (Section 5) Choose a sequence of decompositions $\{X^n\}$ of T into subtrees.

2. (Section 6) Assign a weight $w(X^n)$ to each X^n.

3. (Section 7) Define a distance ϱ_n on T w.r.t. $w(X^n)$.

4. (Lemma 7.3) The limit $\varrho := \lim_n \varrho_n$ exists and (Lemma 7.6) it defines a geodesic metric on T.

5. (Proposition 7.7 (i)) $\mathrm{diam}_\varrho(X) \asymp w(X)$.

6. The metric ϱ is a "conformal" deformation of d on T controlled by weight $w(X)$ near each tile X.

7. (Lemma 8.2) The map $\mathrm{id}_T : (T, d) \to (T, \varrho)$ is a quasisymmetry.

The main difficulty: how to define the decompositions X^n?

1.4 Theorem 1.2

Question 2 *How small we can make the quasisymmetric image of a quasi-tree?*

The **conformal dimension** of a metric space X, denoted by $\mathrm{confdim}(X)$, is the infimum of all Hausdorff dimensions of metric spaces Y that are quasisymmetrically equivalent to X.

Theorem 1.2 *The conformal dimension of a quasi-tree is 1.*

Idea of theorem 1.2: The construction of ϱ involves a parameter $\epsilon_0 > 0$. Show that, if ϵ_0 close to 0, then the Hausdorff dimension of (T, ϱ) is close to 1.

1.5 Organization

- Section 2: review some basic topological facts about trees.
- Section 3: prove a general fact that is of independent interest: we show that if on an arc some points cast a "shadow" satisfying suitable conditions, then one can always find a "place in the sun".
- Section 4: introduce the concept of a (β, γ) good double point at scale $\Delta > 0$.
- Section 5: define the subdivisions of T into tiles.
- Section 6: define weights of tiles.
- Section 7: define the metric ϱ and show that it is a geodesic metric.

- Section 9: prove theorem 1.2.
- Section 10: conclude with remarks and open problems.

1.6 Notation

- $a \asymp b$: there is a constant $C \geq 1$ s.t. $C^{-1}a \leq b \leq Ca$.
- $\mathbb{N}$ and $\mathbb{N}_0$
- $\#X$: the cardinality of a set X.
- open ball $B_d(a,r) := \{x \in X : d(a,x) < r\}$ and closed ball $\bar{B}_d(a,r)$

For $A, B \subset X$

- $\mathrm{diam}_d(A)$: diameter of A w.r.t. d
- $\bar{A}$: the closure of A in X
- $\mathrm{int}(A)$: the interior of A in X
- $\mathrm{dist}_d(A,B) := \inf\{d(x,y) : x \in A, y \in B\}$

1.7 Something about Quasisymmetry

1.7.1 Quasisymmetric map and quasiconformal map

- If f is quasisymmetric, then it is quasiconformal.

(Check the f image every small circle.)

- If f is quasiconformal in $B(r)$, then it is quasisymmetric in $B(r/2)$.

(Proof ?)

1.7.2 Quasicircle curve and Jordan curve

A Jordan curve cuts a sphere into two John domains iff it is quasicircle.

- A Jordan curve is a plain curve which is topologically equivalent to a homeomorphic image of a circle.
- John domain is a region in the Riemann sphere satisfies that every point can be reach from a fixed base point by a flexible cone at a definite angle at its vertex.
- A quasicircle is the image of a circle under a quasiconformal map of a plane onto itself.

2 Auxiliary facts

- $S \subset X$ is s **separated** for some $s > 0$ if $d(x, y) > s$ for all $x, y \in S$.
- S is **maximal** s separated set if it is not contained in a strictly larger subset which is also s separated.
- If X is compact, then s separated set is finite.
- The space (X, d) is **doubling** iff $\forall \lambda \in (0, 1)$, $\exists N' = N'(\lambda, N)$ where N is the doubling number of X s.t. if $s > 0$ and $S \subset X$ is a λs separated set contained in a ball $B(x, s)$ with $x \in X$, then S contains at most N' points.

(*if.* Assume that S is maximal λS separated set. Then any $x \in X$ is contained in $B(y, \lambda s)$ for some $y \in S$. We have $\{B(y, \lambda s)\}$ covers $B(x, s)$ and its number is bounded above by N'.

only if. Assume that the number of S is unbounded. It is easy to show that (X, d) is not doubling.)

- Doubling is a bi-Lipschitz invariant property.

(Let f is a L bi-Lipschitz map from (X, d) to (Y, d'). Assume (X, d) is doubling. Need to show that (Y, d) is also doubling.

For any unit ball $B_{d'}(y, 1)$, there exists a ball $B_d(x, K)$ s.t. $B_{d'}(y, 1) \subset f(B_d(x, K))$. Since (X, d) is doubling, there are bounded number of balls $B_d(x', \frac{1}{2K})$ covering $B_d(x, K)$. Note that the f image of each $B_d(x', \frac{1}{2K})$ is contained in some $B_{d'}(y', \frac{1}{2})$. Then there are also bounded number of $B_{d'}(y', \frac{1}{2})$ covering $B_{d'}(x, 1)$.)

- An **arc** $J \subset X$: homeomorphic to $[0, 1] \subset \mathbb{R}$
- The **endpoints** ∂J of J
- The **interior points** of J: $\mathrm{int}(J) := J \setminus \partial J$

Lemma 2.1 *Let (J, d) be an arc and $n \geq 2$. Then we can decompose J into n non-overlapping subacrcs of equal diameter $\Delta \geq \frac{1}{n} diam(J)$.*

Proof. A decomposition of J into n non-overlapping subarcs of equal dimeter exists. (Lemma 2 of [Mel1]) Denote their diameter by Δ. We have $\mathrm{diam}(J) \leq n\Delta$ by the triangle inequality.

- A tree T and two points $x, y \in T$
- $[x, y]$: the unique (closed) arc joining x and y
- $(x, y]$, $[x, y)$, (x, y)
- A subtree X of T if X equipped with the restriction of d is also a tree. X is a subtree of T iff X contains at least two points and is closed and connected.

Lemma 2.2 *Let (T, d) be a tree and $V \subset T$ be a finite set. Then the following statements are true:*

1. *$x, y \in T \setminus V$ lie in the same component of $T \setminus V$ iff $[x, y] \cap V = \emptyset$*
2. *If U is component of $T \setminus V$, then U is open and $\bar{U}$ is a subtree of T with $\partial \bar{U} \subset \partial U \subset V$.*
3. *If U and W are two distinct components of $T \setminus V$, then $\bar{U}$ and $\bar{W}$ have at most one point in common. Such a common point belongs to V, and is a boundary point of both $\bar{U}$ and $\bar{W}$.*

Proof of 1). Since V is finite, it is closed. Then $T \setminus V$ is open. Since T is locally path-connected (Lemma 3.1 of [BT18]), each component U of $T \setminus V$ is open and path-connected. If x and y are in U, then there exists a path γ connecting x and y which stays in U. Since T is a tree, we have $[x, y] \subset \gamma \subset U$. Then $[x, y] \cap U = \emptyset$.

Conversely, omitted.

Proof of 2). $\bar{U}$ is a subtree: only need to prove it contains more than one point which follows from the fact that T has no isolated points.

$\partial \bar{U} \subset \partial U$: Why not $=$? $\partial A = \bar{A} \setminus A$.

$\partial U \subset T$: for any $x \in \partial U$, then x does not belong to $T \setminus V$. Then $x \in V$ since $x \in T$.

Proof of 3). Assume that $\bar{U}$ and $\bar{V}$ has two common points. Let γ_U and γ_V be two arcs connecting u and v in $\bar{U}$ and $\bar{V}$ respectively. Then $\gamma_U \cup \gamma_V$ is a closed curve in T. Contradiction arises.

- $p \in T$ and U a component of $T \setminus \{p\}$
- A **branch** of p in T is a subtree $B := \bar{U} = U \cup \{p\}$.
- There are only countable many distinct branches B of p. Only finite of these branches can have a diameter exceeding a given positive number. This is given by Lemma 3.8 of [BT18]. (It needs assumption that T is compact.)

(*Proof.* Assume there are infinite B_n with $\mathrm{diam}(B_n) > \delta > 0$. Take $x_n \in B_n$ s.t. $d(x_i, p) > \delta/2$. Since T is compact, there is a acumulation point x of $\{x_n\}_n$. Let N be the $\delta/4$ neighborhood of x. Any $x_i, x_j \in N$ satisfies $d(x_i, p) \geq \delta/4 > 0$. So x_i and x_j are in the same component containing N. However they are in different components.)

- It implies that we can label the branches B_n s.t. for all $n \in \mathbb{N}$

$$\mathrm{diam}(B_n) \geq \mathrm{diam}(B_{n+1}).$$

- If p has precisely two branches, then p is called **double point** of T and the diameter of the smallest branch is denoted by $D_T(p) := \mathrm{diam}(B_2)$.

- If p has at least three branches, then p is called a **branch point** of T and the diameter of the third largest branch is denoted by $H_T(p) := \mathrm{diam}(B_3)$.

Lemma 2.3. (A criterion how to detect branch points) A tree (T, d) and $b, x_1, x_2, x_3 \in T$. Arcs $[x_i, b)$ are pairwise disjoint. Then x_i lie in different components of $T \setminus \{b\}$ and b is a branch point of T.

- A tree (T, d) is K bounded turning with $K \geq 1$ iff $\mathrm{diam}[x, y] \leq Kd(x, y)$. (Take $[x, y]$ as the compact set in the definition.)

- The **diameter distance** on T is defined as $\mathrm{dd}(x, y) := \mathrm{diam}[x, y]$.

Lemma 2.4. (Properties of dd)

1. dd is a metric on T.
2. For each arc $J \subset T$ we have $\mathrm{diam}_{dd}(J) = \mathrm{diam}(J)$.
3. (T, dd) is of 1 bounded turning.
4. (T, d) is K boundaed turning iff $id_T : (T, d) \to (T, dd)$ is K bi-Lipschitz.

Proof of 1). Check the followings.

1. $dd(x, y) = \mathrm{diam}[x, y] \geq 0$ and $\mathrm{diam}[x, y] = 0$ iff $x = y$.
2. $dd(x, y) = \mathrm{diam}[x, y] = dd(y, x)$.
3.

$$\begin{aligned} dd(x,z) &= \mathrm{diam}[x,z] \le \mathrm{diam}[x,y] \cup [y,z] \\ &\le \mathrm{diam}[x,y] + \mathrm{diam}[y,z] = dd(x,y) + dd(y,z). \end{aligned}$$

Proof of 2).

1. $\ge$: Since $dd \ge d$, we have $\mathrm{diam}_d d \ge \mathrm{diam}$.
2. $\le$: for any $x, y \in J$, we have $[x,y] \subset J$. Then $dd(x,y) = \mathrm{diam}[x,y] \le \mathrm{diam}J$. Then $\mathrm{diam}_{dd}(J) = \sup_{x,y \in J} dd(x,y) \le \mathrm{diam}J$.

Proof of 3). By 2), $dd(x,y) = \mathrm{diam}[x,y] = \mathrm{diam}_{dd}[x,y]$.

Proof of 4). if: K bounded turning implies that $dd(x,y) = \mathrm{diam}[x,y] \le Kd(x,y)$. Combining $d \le dd$, we have id_T is K Lips.

only if: $\mathrm{diam}[x,y] = dd(x,y) \le Kd(x,y)$.

- A metric d is called a **diameter metric** if $d(x,y) = \mathrm{diam}[x,y]$. Then $d = dd$.

Suppose (T,d) is quasi-tree. Then Lemma 2.4 implies that (T,dd) is bi-Lipschitz equivalent to (T,d). Moreover, (T,dd) is of 1 bounded turning. Note that doubling is a bi-Lipschitz invariant property. So (T,dd) is still a quasi-tree.

Assumption: (T,d) carries a diameter metric.

Assumption: $\mathrm{diam}(T) = 1$. (Rescale metric if needed)

3 Sun and shadow

In this section we will prove the following proposition, which allow us to find double points in T that stay away from the branch points of T in a geometrically controlled manner.

Proposition 3.1 *There exists a constant $\gamma = \gamma(N) > 0$ with the following property: if $\Delta > 0$ and $J \subset T$ is an arc with diam$(J) \ge \Delta$, then there exists a double point $x \in J$ of T s.t.*

$$d(x,b) \ge \gamma \cdot \min\{H_T(b), \Delta\}$$

for all branch points $b \in T$.

To prove this proposition, we require two auxiliary facts.

Lemma 3.1 *Let $(J.d)$ be a metric arc with a diameter metric d, $J' \subset J$ and $A \subset J$ be a set with $\#(A \cap J') \leq M$. Then there exists an arc $I \subset J'$ s.t.*

$$diam(I) = \frac{1}{6M} diam(J')$$

and

$$dist(I, A \cup \partial J') \geq \frac{1}{6M} diam(J).$$

Proof. By Lemma 2.1, there exists a decomposition $\{J_i | i = 1, ..., 2M\}$ of J s.t. $\mathrm{diam} J_i = \Delta \geq \frac{1}{2M} \mathrm{diam} J'$. By $\#(A \cap J') \leq M$, there exists J_i does not intersect A. Take the middle subarc I of this arc of length $\frac{1}{6M} \mathrm{diam} J'$. Then the distance between I and the boundary of this arc is larger $\frac{1}{3}\Delta = \frac{1}{6M} \mathrm{diam} J$.

Lemma 3.2 *(Ein Platz an der Sonne) Let (J, d) be a diameter metric arc and $S : J \to [0, diam(J)]$ be a function. Suppose there is a constant M s.t. for all subarcs $I \subset J$ we have*

$$\#\{p \in I : S(p) \geq diam(I)\} \leq M. \tag{3.2}$$

There exists a constant $\sigma = \sigma(M) > 0$ and a point $x \in J$ s.t. $d(x, p) \geq \sigma S(p)$ for all $p \in J$.

In other words, the set $J \setminus \cup_{p \in J} B(p, \sigma S(p))$ is non-empty. If we think of each point $p \in J$ with $S(p) > 0$ as "casting a shadow" of radius $\sigma S(p)$ around p, then the lemma says that the union of all shadows does not cover J, and so there is a "place in the sun".

Proof. WLOG, assume that $\mathrm{diam} J = 1$. Let $\lambda := /frac16M$ and

$$A_n := \{p \in A : S(p) \geq \lambda^n\}.$$

Then $A := \cup_n A_n$ is all points in J with $S(p) > 0$.

Init: Set $J_0 := J$. Then $\mathrm{diam} J_0 = 1 = \lambda^0$.

$n+1$ step: Assume that J_n satisfies $\mathrm{diam} J_n = \lambda^n$. We have $A_n \cap J_n$ is

$$\{p \in J_n : S(p) \geq \lambda^n = \mathrm{diam} J_n\}.$$

By the condition (3.2), we have $\#(A_n \cap J_n) \leq M$. By Lemma 3.2, we can take a subarc $J_{n+1} \subset J_n$ satisfies

$$\mathrm{diam} J_{n+1} = \frac{1}{6M} \mathrm{diam} J_n = \lambda \mathrm{diam} J_n = \lambda^{n+1}$$

and

$$\mathrm{dist}(J_{n+1}, A_n) \geq \frac{1}{6M}\mathrm{diam}J_n = \lambda^{n+1}.$$

Then $\cap_n J_n \neq \emptyset$. Pick x from it. For p with $S(p) > 0$, there exists an n with $\lambda^n \leq S(p) < \lambda^{n-1}$ which implies that $p \in A_n$. Since $x \in J_{n+1}$, we have

$$d(x,p) \geq \mathrm{dist}(x, A_n) \geq \lambda^{n+1} > \lambda^2 S(p).$$

Take $\sigma = \lambda^2 = \frac{1}{36M^2}$, then x is a point as desired.

We close this section by proving Proposition 4.1.

Proof of Proposition 3.2

Let $\Delta > 0$ and $J = [u,v]$ with $\mathrm{diam}J \geq \Delta$. Define a function $S : J \to [0, \mathrm{diam}J]$ as follows:

- $S(p) = \Delta$ if $p = u$ or v
- $S(p) = \min\{H_T(p), \Delta\}$ if p is a branch point
- $S(p) = 0$ otherwise

Claim. (Condition of Lemma 3.3) There exists a constant $M = M(N)$ s.t. for all arcs $I \subset J$ we have

$$\#\{p \in I : S(p) \geq \mathrm{diam}I\} \leq M.$$

By Lemma 3.3, there exists a constant σ and $x \in J$ s.t. for all $p \in J$ we have

$$d(x,p) \geq \sigma S(p).$$

Assume that $0 < \sigma \leq 1$. (so that $\sigma^2 \leq \sigma$)

Let $\gamma := \frac{\sigma^2}{2}$. Check that x is a desired point for proposition. Let $b \in T$ be a branch point and r be the first point in $J \cap [b,x]$. Consider two cases depending on the location of r.

Case 1. $r \in B(u, \sigma\Delta/2) \cup B(v, \sigma\Delta/2)$. Assume that $r \in B(u, \sigma\Delta/2)$. We have

$$\begin{aligned}
d(x,b) &= \mathrm{diam}[x,b] \geq d(x,r) \\
&\geq d(x,u) - d(r,u) \\
&\geq \sigma S(u) - \sigma\Delta/2 \quad (\text{choice of } x \text{ and } r \in B(u, \Delta/2)) \\
&= \sigma\Delta/2 \quad (\text{definite of } S) \\
&\geq \sigma \min\{H_T(b), \Delta\}/2 \\
&\geq \gamma \min\{H_T(b), \Delta\} \quad (\text{definition of } \gamma \text{ and } 0 < \sigma < 1).
\end{aligned}$$

Case 2. $r \notin B(u, \sigma\Delta/2) \cup B(v, \sigma\Delta/2)$. See Figure 3 of the paper. There exists a component U of $T \setminus \{b\}$ disjoint from J and satisfies

$$\mathrm{diam} U \geq H_T(b).$$

Let V_1 be a component of $T \setminus \{r\}$ containing U. Hence

$$\mathrm{diam} V_1 \geq \mathrm{diam} U \geq H_T(b).$$

Let V_2 and V_3 be the components containing $[u, r)$ and $[v, r)$ respectively. We have

$$\begin{aligned} \mathrm{diam} V_2 &\geq \mathrm{diam}[u, r) \geq d(u, r) \\ &\geq \sigma\Delta/2 \quad \text{(assumption of case)} \end{aligned}$$

and $\mathrm{diam} V_3 \geq \sigma\Delta/2$. Then

$$\begin{aligned} H_T(r) &\geq \min\{\mathrm{diam} V_i : i = 1, 2, 3\} \\ &\geq \min\{H_T(b), \sigma\Delta/2\} \\ &\geq \frac{\sigma}{2} \min\{H_T(b), \Delta\}. \end{aligned}$$

It follows that

$$\begin{aligned} d(x, b) &\geq d(x, r) \\ &\geq \sigma S(r) \quad \text{(choice of } x\text{)} \\ &= \sigma \min\{H_T(r), \Delta\} \quad \text{(def of } S\text{)} \\ &\geq \frac{\sigma^2}{2} \min\{H_T(b), \Delta\} \quad \text{(result above)} \\ &= \gamma \min\{H_T(b), \Delta\}, \end{aligned}$$

as desired.

It remains to check to x is a double point. Indeed, x is not a branch point since $d(x, b) > 0$ for all branch b.

Here is the proof of the claim in the proof Proposition 3.1: There exists $M = M(N)$ s.t.

$$\#\{p \in I : S(p) \geq \mathrm{diam} I\} \geq M.$$

Proof of Claim. Fix $I \subset J$. Let $R := \{p \in \mathrm{int} I : S(p) \geq \rho\}$ where $\rho := \mathrm{diam} I > 0$. By the definition of S, each $r \in R$ is a branch point and there exists a large component U_r of $T \setminus \{r\}$ s.t. $U_r \cap I = \emptyset$ and diam$U_r \geq H_T(r)$. Let $\mathcal{U}_R := \{U_r : r \in R\}$. It is sufficient to give a upper bound for $\#\mathcal{U}_R$.

Let B_I be a $\frac{3\rho}{2}$ ball containing I of diameter ρ. We will define pairwise disjoint ball B_r containing $r \in R$ of diameter $\rho/4$. Then $\#\{B_r\} = \#R$ is bounded by the double number N. It is sufficient to construct such B_r.

Consider arbitrary $r \in R$. Then

$$\begin{aligned} H_T(r) &\geq S(r) \quad \text{(def of } S) \\ &\geq \rho > 0 \quad \text{(def of } R). \end{aligned}$$

Take U_r be a component of $T \setminus \{r\}$ and of diameter $\mathrm{diam}(U_r) \geq H_T(r) \geq \rho$. Take $q_r \in U_R$ s.t.

$$d(q_r, x) = \rho/2.$$

Check that $B_{q_r}(\rho/2)$ and $B_{q_{r'}}(\rho/2)$ are disjoint. It follows from

$$\begin{aligned} d(q_r, q_{r'}) &= diam[q_r, q_{r'}] \\ &> d(q_r, r) + d(r' + q_{r'}) = \rho. \end{aligned}$$

4 Good double points

In this section we introduce the concept of a "good" double point of T. Attached to this concept are certain numerical parameters. The goal of this section is to show that with appropriate choices of these parameters, one can use a maximal set V of good double points to obtain a decomposition of T with some desired geometric properties (see Proposition 4.2).

Fix a scale $0 < \Delta \leq \mathrm{diam}(T) = 1$.

Consider double point $x \in T$ with property that both components of $T \setminus \{x\}$ are large, that is,

$$D_T(x) \geq \beta\Delta \tag{4.1}$$

for some constant $\beta \geq 1$.

Proposition 4.1 *There is a constant $\beta = \beta(N) \geq 1$ s.t. the following statement is true:*

if $V \subset T$ is a set of double points of T that are Δ separated and satisfy $D_T(x) \geq \beta\Delta$, then either

i) for each component X of $T \setminus V$ we have

$$diam(X) \leq 3\beta\Delta$$

or

ii) there is an arc $I \subset T$ with

$$diam(I) \geq \Delta \text{ and } dist(I, V) \geq \Delta$$

and so that $D_T(x) \geq \beta\Delta$ for each double point $x \in I$ of T.

Proposition 3.1 implies that each arc $I \subset T$ contains double points of T. So in case 2 of the previous statement, we can add a double point of V and get a new set of double points which still satisfies the conditions of Proposition 4.1. This implies that for a maximal set in the proposition, statement i) will always be true.

(Proof of Proposition 4.1) Since T is doubling, there exists a constant $N' = N'(N)$ s.t. any ball $B(6r)$ contains at most N' r separated points. We set

$$\beta = 6N'.$$

Let V be a Δ separated set of double points satisfying (4.1) i.e.

$$D_T(v) \geq \beta\Delta.$$

We check all components X of $T \setminus V$ for i) or ii). If all X satisfy i), then the proposition is proved. It is sufficient to consider the component X which do not satisfies i), that is,

$$\text{diam}X > 3\beta\Delta.$$

Let $x, z \in X$ with $d(x, z) > 3\beta\Delta$ and $J := [x, z]$. Take J' be the "middle" subarc of J as in Lemma 3.2, that is,

$$\text{diam}J' = \beta\Delta$$

and

$$\text{dist}(J', \partial J) \geq \beta\Delta.$$

Then any double point $x \in J'$ satisfies (4.1), that is,

$$D_T(x) \geq \beta\Delta.$$

It remains to find an subarc $I \subset J'$ satisfying $\text{diam}I \geq \Delta$ and $\text{dist}(I, V) \geq \Delta$.

Fix $a \in J'$ as a base point. Let $v \in V$ with $\text{dist}(v, J') < \Delta$. (If there is no such v, then all subarc I satisfies 2nd condition of ii), that is $\text{dist}(I, V) \geq \Delta$.)

Let $r_v \in J'$ be the first point from v to a and

$$R := \{r_v : v \in V, \text{dist}(v, J') < \Delta\}.$$

Claim: $\#R \le N'$.

Applying Claim to Lemma 3.2, we have there exists a subarc $I \subset J'$ with

$$\mathrm{diam} I = \frac{1}{6N'} \mathrm{diam} J' = \Delta (\text{then 1st cond of ii) follows})$$

and

$$\mathrm{dist}(I, R \cup \partial J') \ge \frac{1}{6N'} \mathrm{diam} J' = \Delta.$$

Check that $\mathrm{dist}(I, V) \ge \Delta$. Consider $v \in V$.

Case 1. $\mathrm{dist}(v, J') \ge \Delta$: then

$$\mathrm{dist}(v, I) \ge \mathrm{dist}(v, J') \ge \Delta.$$

Case 2. $\mathrm{dist}(v, J') < \Delta$: then

$$\mathrm{dist}(v, I) \ge \mathrm{dist}(r_v, I) \ge \Delta.$$

So I is a desired subarc.

Proof of the claim above: $\#R \le N'$.

For any $v \in V$ corresponding to $r_v \in R$, there is a component U of $T \subset \{v\}$ which does not contain J'. Then

$$\begin{aligned} \mathrm{diam} U &\ge D_T(v) \quad (\text{def of } D_T) \\ &\ge \beta\Delta \quad (V \text{ satisfies (4.1)}). \end{aligned}$$

Take $q \in U$ with $d(q, v) = \frac{\beta\Delta}{2}$. Denote $v_r := v$ and $q_r := q$. Then

$$\begin{aligned} d(q_r, a) &\le d(q_r, v_r) + \mathrm{dist}(v_r, J') + \mathrm{diam} J' \\ &\le \frac{\beta\Delta}{2} + \Delta + \beta\Delta \\ &\quad (\text{def of } q_r,\ v_r \in V \text{ and } J') \\ &\le 3\beta\Delta \quad (\beta = 6N' \text{ is large}) \end{aligned}$$

It follows that $q_r \in B(a, 3\beta\Delta)$.

For any $r \ne r' \in R$, we have

$$d(q_r, q_{r'}) = \mathrm{diam}[q_r, q_{r'}] \ge d(q_r, v_r) \ge \frac{\beta\Delta}{2}.$$

Then $Q := \{q_r : r \in R\}$ is a $\frac{\beta\Delta}{2}$ separated set in $B(a, 3\beta\Delta)$. There exits an upper bound for $\#R = \#Q$ since T is doubling.

In addition of (4.1), we want to choose double points $x \in T$ that are separated from the branch points of T in a controlled way:

$$d(x,b) \geq \gamma \cdot \min\{H_T(b), \Delta\} \tag{4.2}$$

for all branch points b, where $\gamma = \gamma(N)$ in Proposition 3.1. A double point $x \in T$ is called (β, γ) **good** at scale Δ if it satisfies (4.1) and (4.2).

Proposition 4.2 *(Proposition 4.1 restrict to good doulbe point)*

Let $\beta = \beta(N) \geq 1$ in Proposition 4.1, $\gamma = \gamma(N) > 0$ in Proposition 3.1 and $0 < \Delta < 1$. If $V \subset T$ is a maximal Δ separated set of (β, γ) good double points at scale Δ, then

$$\mathit{diam}(X) \leq 3\beta\Delta$$

for each component X of $T \setminus V$.

Again, the "maximal" part is easy to get. However V may be empty. In this case, the proposition showed that $1 = \text{diam}X \leq 3\beta\Delta$. Set $\Delta < \frac{1}{3\beta}$ to avoid this.

Assume that there is a component X of $T \setminus V$ with $\text{diam}X > 3\beta\Delta$. By Proposition 4.1, there is a subarc I satisfies ii) of Proposition 4.1, that is,

$$\text{diam}I \geq \Delta,$$

$$\text{dist}(I, V) \geq \Delta$$

$$D_T(x) \geq \beta\Delta \text{ for all double point } x \in I.$$

By Proposition 3.1, we find a double point $x \in I$ s.t.

$$d(x,b) \geq \gamma \min\{H_T(b), \Delta\} \text{ for all branch point } b \in T.$$

Then x satisfies (4.1) and (4.2). Therefore, x is a (β, γ) good double point at scale Δ.

Point x also satisfies

$$\text{dist}(x, V) \geq \text{dist}(I, V) \geq \Delta \quad (\text{def of } I).$$

Hence $V' := V \cup \{x\}$ is also a Δ separated (β, γ) good doulbe point at scale Δ. It contradicts the maximality of V

5 Subdividing the tree

We want to subdivide our quasi-tree T. As before, assume that T is equipped with a diameter metric d and $\text{diam}(T) = 1$. Fix $\beta \geq 1$ and $\gamma > 0$ depending on doubling constant N of T and a small constant $0 < \delta < 1/(3\beta)$.

5.1 Vertices and tiles

We will inductively construct set $V^n \subset T$ s.t.

$$V^1 \subset V^2 \subset ... \tag{5.1}$$

where each V^n is a maximal δ^n separated set consisting of (β, γ) good double points at scale δ^n. Each set V^n is finite since T is compact.

- Each point $v \in V^n$ is called n **vertex**.
- The closure of a component of $T \setminus V^n$ is called an n **tile**.
- X^n : the set of n tiles.

Lemma 5.1 (topological properties of vertices and tiles)

1. Each n tile X is a subtree of T with $\partial X \subset V^n$.
2. For $X \in X^n$ and $x \in V^n$, X is contained in the closure of one of the two components of $T \setminus \{v\}$ and disjoint from the other component of $T \setminus \{v\}$.
3. $\partial X \neq \emptyset$ for $X \in X^n$.
4. $X \neq Y \in X^n$ have at most one point in common. Such a common point of X and Y is an n vertex and a boundary point of both X and Y.
5. Each n vertex v is contained in precisely two distinct $X, Y \in X^n$.
6. X^n is a finite set.
7. Each $X' \in X^{n+1}$ is contained in a unique $X \in X^n$.
8. Each n tile $X = \cup_{\{X' \in X^n : X' \subset X\}} X'$.
9. If v is n vertex and $X \in X^n$ containing v, then $v \in \partial X$. Moreover, there exists precisely one $(n+1)$ tile $X' \subset X$ containing v.
10. If $X \in X^n$ and $\partial X = \{v\} \subset V^n$ is a singleton set, then $X = \bar{W}$ where W is a component of $T \setminus \{v\}$.

Here are some metric properties of vertices and tiles. For $X \in X^n$, we have

- V^n consists of δ^n separated points that is

$$d(u,v) \geq \delta^n \tag{5.2}$$

for all $u, v \in V^n$.

- δ^n separated and Proposition 4.2 implies that

$$\delta^n \leq \operatorname{diam}(X) \leq 3\beta\delta^n. \tag{5.3}$$

- We have good separation of n tiles in the following sense. If $X^n, Y^n \in \mathbf{X}^n$ are disjoint n tiles, then

$$\operatorname{dist}(X^n, Y^n) \geq \delta^n. \tag{5.4}$$

- The component of $T \setminus \{v\}$ are large in the following sense:

$$D_T(v) \geq \beta\delta^n \tag{5.5}$$

since each $v \in V^n$ is a (β, γ) good double point at scale δ^n.

- Each n vertex v stays away from the branch points of T in a controlled way:

$$d(v,b) \geq \gamma \min\{H_T(b), \delta^n\} \tag{5.6}$$

since (4.2) for each banch point.

For convenience, set $V^0 := \emptyset$ and $X^0 := T$.

5.2 Chains

- n **chain** is a sequence P of n tiles $X_1, ..., X_r$ with $X_i \cap X_{i+1} \neq \emptyset$. And r is called the **length** of P.
- P **joins** x and y if $x \in X_1$ and $y \in X_r$.
- P is **simple** if $X_i \cap X_j = \emptyset$ for $|i-j| \geq 2$. The tiles in a simple chain P are all distinct.
- P is a simple n chain joining x and y if ...
- $|P| := \cup_{i=1}^r X_i$
- P **contains** x if $x \in |P|$
- Q is a **subchain** of P: Q is obtained by deleting some tiles in P while keeping the order of the remaining tiles.

Lemma 5.2

1. There exists a unique simple n chain P joining x and y
2. If both P and P' joins x and y and P is simple, then $|P| \subset |P'|$.

By 2) of Lemma 5.2, we denote by P^n_{xy} the unique simple n chain joining x and y.

Lemma 5.3 (construct simple $(n+1)$ chains from simple n chain) Let $P = (X_1, ..., X_r)$ be a simple n chain joining x and y. Let v_i be the unique vertex in $X_{i-1} \cap X_i$, $x_0 = x$ and $x_r = y$. Denote by P'_i the simple $(n+1)$ chain joining v_{i-1} and v_i. The followings are true:

1. P'_i consists of $X' \in \mathbf{X}^{(n+1)}$ with $X' \cap [v_{i-1}, v_i] \neq \emptyset$.
2. The simple $(n+1)$ chain P' joining x and y is obtained by concatenating $P'_1, ..., P'_r$.

5.3 Choosing δ

We now choose the parameter $0 < \delta < 1/(3\beta)$ used in the definition of vertices and tiles so that $(n+1)$ tiles are contained in n tiles in a "controlled way".

Lemma 5.4 (number of $(n+1)$ tiles) If $0 < \delta < 1/(3\beta)$ only depending on N, then the following hold for all n:

1. Each $X \in \mathbf{X}^n$ contains at least three $(n+1)$ tiles.
2. If n vertices $u \neq v$, then the simple $(n+1)$ chain joining u and v has length ≥ 3.

The first statement implies that there are at least three 1 tiles. The second statement implies each $(n+1)$ tile X' contains at most one n vertex.

Lemma 5.5 (location of $(n+1)$ vertices) Let δ in Lemma 5.4. Let X be an n tile, $u \in \partial X \subset V^n$ and $X' \subset X$ be the unique $(n+1)$ tile containing u. Then there exists an $(n+1)$ vertex $u' \in \partial X' \setminus \{u\}$ s.t. $[u, u'] \subset [u, v]$ for all $v \in \partial X \setminus \{u\}$.

For the rest of the paper, we fix $0 < \delta < 1/(3\beta)$ s.t. Lemma 5.4 and Lemma 5.5 are true. As we see from the proofs, it is enough to take $\delta = \frac{1}{2}\min\{1/(9\beta), \gamma/(3\beta)\}$. Then δ depends only on N. The sets of vertices V^n and tiles X^n are fixed from now on since they depend on δ.

Lemma 5.6 Let X be an n tile and $u \neq v \in \partial X \subset V^n$. Then the simple $(n+1)$ chain $P^{(n+1)}_{uv}$ joining u to v consists precisely of all $(n+1)$

tiles $X' \subset X$ with $X' \cap [u,v] \neq \emptyset$. Moreover, $P_{uv}^{(n+1)}$ does not contain any point $w \in \partial X$ distinct from u and v.

Lemma 5.7 (uniform control for the local combinatorics of tiles) There is a constant $K \in \mathbb{N}$ s.t. the following statements hold for each $X \in \mathbf{X}^n$:

1. There are at most K n tiles that intersect X.
2. There are at most K $(n+1)$ tiles contained in X.

6 Weights and main vertices of tiles

We will now define *weight* of tiles. Later they will be used to construct our desired geodesic metric ϱ.

- The w **length** of an n chain $P = (X_1, ..., X_r)$ is defined by

$$\operatorname{length}_w(P) := \sum_{i=1}^{r} w(X_i).$$

7 Construction of the geodesic metric

Based on weights defined in the previous section, we define a new metric ϱ on (T, d).

8 Quasisymmetry

In this section, we prove Theorem 1.1 by showing that (T, d) is quasisymmetrically equivalent to (T, ϱ).

9 Lowering the Hausdorff dimension

In this section we will prove Theorem 1.2.

10 Remarks and open problems

Question 3 *For every quasi-tree T, is there a quasisymmetric embedding $\phi : T \to \mathbb{C}$ with good geometric properties?*

- *e.g. s.t.* $\phi(T)$ *is quasi-convex w.r.t.* $d_{\mathbb{R}^2}$*. (Then it is geodesic if equipped with its internal path metric) and* $\mathbb{C} \setminus T$ *is a nice domain (e.g. a John domain)*

Question 4 *How about quasi-trees which are not doubling?*

Question 5 *How about quasi-graphs?*