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Fuzzy Models in Geotechnical Engineering and Construction Management

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Abstract: *This article is devoted to a variety of applications of* fuzzy models *in civil engineering, presenting the current work of a group of researchers at the University of Innsbruck. With* fuzzy methods *and* possibility theory *as an encompassing framework, the following areas are addressed*: *uncertainties in geotechnical engineering*, *fuzzy finite-element computation of a foundation raft*, *fuzzy dynamic systems*, *and processing uncertainty in project scheduling and cost planning*.

1 INTRODUCTION

Traditional engineering models are deterministic; sharp inputs are processed in a structurally determined system to produce a sharp output, which is taken as an approximate prediction of reality. To account for fluctuations, probability theory has been introduced. This shifts the emphasis from sharp input data to sharp probabilities of sharply defined events. Among engineers, there is increasing discomfort with the observed fact that predictions obtained in this way may deviate to such an extent from reality as to render them useless.

For example, a case study[17] concerning the design of a sheetpile wall, in which a number of reputable European engineering companies had agreed to participate, resulted in a corresponding number of seemingly precise predictions. These, however, differed drastically from each other and from the observed behavior of the completed structure.

Conspicuous differences between prediction and reality may arise in many areas in the course of the realization of an engineering structure: from site investigations and analysis to scheduling and cost planning. We should be clear about

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the facts that

- Models are approximations to and conjectures about reality.
- The input parameters are known only imprecisely.
- Probabilistic methods may fail to capture the information available about the deviations.

The inherent vagueness of modeling procedures can be traced to various reasons, e.g., lack of knowledge of boundary conditions, simplification in complex circumstances forcing a single parameter to cover a wider range of situations, lack of a precisely quantifiable definition of some verbally defined variable, and uncertainty about future dispositions. However, there is clearly no alternative to employing rational models in the three central activities of engineering: design, construction, and control. Rather, the engineer should face the limitations of the modeling process, put the range of imprecision into the open, and make it accessible to responsible assessment by all participants in the construction process. This will involve processing not only data but also the available objective and subjective information on the uncertainty.

We believe that fuzzy set theory provides a framework for accomplishing this task. The power of fuzzy set theory is that it allows a formalization of vague data, a representation of their fuzziness that can be entered into computations, and a possibility theoretic interpretation. Assigning degrees of possibility as a replacement of probability appears to be more adapted to formalizing expert knowledge, due to the relaxation of axioms. Working with fuzzy methods forces and allows the engineer to address the uncertainties, see and judge the possible range of outputs the fuzzy model predicts, and gain understanding of the possible behavior of the system, given the imperfect description formulated initially.

The purpose of this article is to demonstrate that fuzzy formulation and computation are possible in a number of engineering tasks ranging from geotechniques to dynamics to project planning. It presents the ongoing work of a group of researchers in construction management, strength of materials, mathematics, and numerics at the University of Innsbruck; we refer to references 4, 5, 9, and 11.

The plan of exposition is as follows. After a short account of the basic notions of fuzzy set theory, we address questions of modeling and uncertainties in geomechanics. This is followed by an investigation of the effect of fuzzy parameters in a raft foundation, the corresponding fuzzy finite-element computation, and a discussion of the interpretation. Next, we exhibit some fuzzy ideas on dynamic systems. Finally, we turn to project planning: scheduling and cost estimation. We show that suitable methods of presentation allow a clear exhibition of fuzziness in a network structure, providing means of control under risk. The fuzzy approach gives a lucid picture of the complexities involved when duration-dependent costs enter into network analysis. It explains why cost optimization is impossible but must be replaced by the search for a satisficing solution.

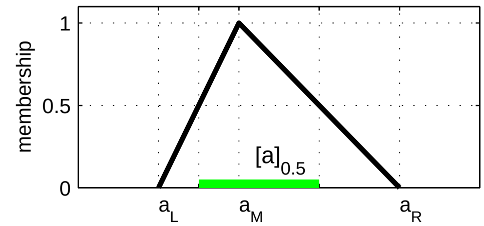


Fig. 1. Triangular fuzzy number.

A methodologic remark seems in order: We use fuzzy set theory to characterize vague data by intervals of variation, supplied with a valuation. This valuation provides the additional degrees of freedom needed to model uncertainty—to describe and interpret its effects. Numerically, the results are computed by repeated application of deterministic algorithms to all relevant data combinations. The increased computational effort currently can be handled in problems of moderate size.

2 FUZZY SETS

This section serves to briefly collect what we need from fuzzy set theory: definition of and computation with fuzzy quantities. Given a basic set X of discourse, a fuzzy subset A is characterized by (and can be identified with) its membership function $m_A(x)$, $0 \leq m_A(x) \leq 1$, defined for each element x of X. The value $m_A(x)$ can be interpreted as

- The membership degree of the element x belonging to A
- The degree of possibility that the variable A takes the value x

Introducing the α-level sets ($0 \leq \alpha \leq 1$)

$$[A]_\alpha = \{x \in X : m_A(x) \geq \alpha\} \tag{1}$$

we arrive at the interpretation mentioned in the introduction:

> The variable A fluctuates in the range $[A]_\alpha$ with possibility degree α.

A fuzzy real number a is defined as a fuzzy subset of the basic set X of real numbers with the property that each level set $[a]_\alpha$ is a compact interval, $0 < \alpha \leq 1$. See Figure 1 for an example of a triangular fuzzy real number a; depicted is the level set $[a]_{1/2}$.

In general, the graph of the membership function will be curved. However, as an approximation, it often suffices to work with polygonal fuzzy numbers. Their graphs are piecewise linear with corner points at $a_{L0} \leq a_{L1} \leq \cdots \leq a_{Ln} \leq$

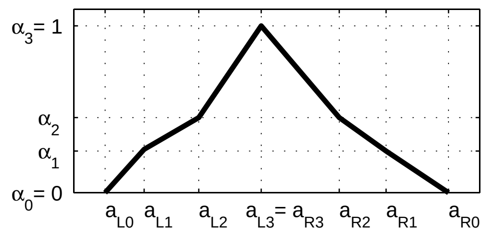


Fig. 2. Polygonal fuzzy number.

$a_{Rn} \leq \cdots a_{R1} \leq a_{R0}$ and corresponding levels $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = 1$ (see Fig. 2).

Polygonal fuzzy numbers are denoted by $a = \langle a_{L0}, a_{L1}, \ldots, a_{Ln}, a_{Rn}, \ldots, a_{R0}\rangle$. Frequently used special cases are triangular fuzzy numbers $\langle a_L, a_M, a_R\rangle$ as well as trapezoidal shapes. Compact intervals $[a_L, a_R]$ can be viewed as fuzzy numbers with rectangular membership function. Among the more general fuzzy quantities are fuzzy vectors, in case the basic set X is the n-dimensional Euclidean space, or even fuzzy functions, in case X is a space of functions.

In scientific computations, parameters have to be inserted in functions. Thus the necessity arises of evaluating functions on fuzzy numbers (or vectors). This is achieved by means of the Zadeh extension principle,[18] exhibited here for the case of a function $z = f(x, y)$ of two variables. If a, b are fuzzy numbers, so will be the result $f(a, b)$. Its fuzzy value is determined by prescribing its membership function:

$$m_{f(a,b)}(z) = \sup_{z=f(x,y)} \min\{m_a(x), m_b(y)\} \tag{2}$$

When using the possibilistic interpretation, the Zadeh extension principle is especially intuitive. In order to determine the membership degree of the dependent variable z, one considers all possible combinations (x, y) leading to $z = f(x, y)$. Each single combination arises with degree of possibility $\min\{m_a(x), m_b(y)\}$. For the final result, the maximal degree of possibility (the supremum) is decisive.

An essential computational tool is interval analysis on each α-level set: Indeed, in case f is a continuous function, the α-level set of $f(a, b)$ is obtained by evaluating the image of the α-level sets of a, b under the function f:

$$[f(a, b)]_\alpha = f([a]_\alpha, [b]_\alpha) \tag{3}$$

In case f is an arithmetic operation, this leads to simple formulas. For example, addition $f(a, b) = a+b$ of triangular or polygonal fuzzy numbers is achieved by simply adding the abscissas of their corner points:

$$\langle a_L, a_M, a_R\rangle + \langle b_L, b_M, b_R\rangle = \langle a_L + b_L, a_M + b_M, a_R + b_R\rangle$$

the result being again a triangular or polygonal fuzzy number. In the case of subtraction $f(a, b) = a - b$, a crosswise interchange is required:

$$\langle a_L, a_M, a_R\rangle - \langle b_L, b_M, b_R\rangle = \langle a_L - b_R, a_M - b_M, a_R - b_L\rangle$$

For general operations f, the resulting membership function can be curved; the bounds of the α-level sets are obtained by solving an optimization problem.

In network planning, the need arises to compare two or more fuzzy numbers. Here we face the difficulty that there is no total order on fuzzy real numbers; i.e., it cannot be decided in general whether $a \leq b$ or $b \leq a$ (in contrast to the case of usual real numbers). A closer look at network planning shows that we actually need to compute $\max(a, b)$ at nodes where two paths meet. This can be done by applying the Zadeh extension principle to the function $f(a, b) = \max(a, b)$. In the case of triangular or polygonal fuzzy numbers, we approximate the result by taking the maximal value at each corner point of the membership function (in order to avoid the introduction of additional levels):

$$\max(a, b) = \langle \max(a_L, b_L), \max(a_M, b_M), \max(a_R, b_R)\rangle$$

and correspondingly for the minimum needed in the backward computation.

3 AN APPLICATION OF FUZZY SET THEORY IN GEOTECHNICAL ENGINEERING

3.1 Preliminary remarks

In foundation engineering, the design of mat foundations is just an ordinary task, and it is certainly far from being classed as a special geotechnical job. At first glance, one might be tempted to assume that the result of a raft design with well-defined structural input data will fall into a narrow band of solutions. However, whereas the loading, the material parameter of the foundation structure, and the requirements the foundation should meet, like allowable differential settlement, are well defined, there is generally a substantial lack of information concerning the soil. In some cases this deficiency of information serves as an excuse for simplified modeling, and it is often argued that it is not worth using a possibly more appropriate and expensive model for analysis of the problem at hand. On the other hand, even the most sophisticated model will not guarantee that the solution based on it will correctly predict all desired aspects of the foundation response. Hence the decision about which model will be appropriate with respect to the importance of the project and the available soil data will very much depend on the expertise of the geotechnical engineer. Irrespective of both the vagueness of input data and the uncertainties with respect to the model, in today's practice the engineer will come up with crisp results. Here fuzzy set theory opens new opportunities to reflect on the lack of information and uncertainties in the results of engineering computations in a rational manner. In this section we will demonstrate the application of fuzzy set theory to the analysis of a foundation raft.

3.2 Material modeling

Constitutive equations or inequalities describe the response of materials. To specify a certain class of material with desired accuracy, a sufficient number of such relations is required. The purpose of engineering constitutive models is not to give a mirror image of realistic material behavior but rather to describe the main mechanical properties that are important for the design of constructions.

Constitutive relations consist of equations and inequalities that contain the basic principles of continuum mechanics (conservation of mass, momentum and energy) and of material theory (principle of frame invariance, etc.). They must be satisfied exactly by the constitutive law. Since these principles are not sufficient for the determination of all the variables involved in the material model, additional assumptions are required. These assumptions are based on the interpretation of material test data, and thus they introduce an amount of uncertainty into the material model. For example, elasticity, plasticity, and damage characterize properties of idealized material behavior. In reality, the existence of an elastic domain, the transition from elasticity to plasticity, or the initiation of damage cannot be defined exactly. The theory of fuzzy sets offers a framework to model these various uncertainities in a consistent manner.[6]

General constitutive relations are tensor-valued functions of stress, strain, strain rate, and additional variables that describe the mechanical state of the material. In order to simplify the constitutive law, material parameters are introduced, replacing complicated functions of the variables by one single constant. As a consequence, the material properties described by the function that is replaced by the constant will be captured by the numerical value of one single material parameter. Thus material parameters of simple models have to cover a larger bandwidth of values than material parameters of more refined models.

An additional source of uncertainity arises when the numerical values for material parameters are determined by carrying out material tests. Usually the constants introduced into the model are identified by minimizing a measure of error between theoretical (calculated) and experimental (measured) variables. Due to the mathematical structure of most nonlinear constitutive equations, a global minimum of error is rarely found. In reality, several equivalent solutions exist. Further, considering that experimental data are burdened by scatter and errors, the results gained by the calibrated model contain some additional uncertainty.

3.3 Numerical model

It should be emphasized very clearly that using fuzzy set theory will in no way release the engineer from his or her job of establishing a correct mechanical model and using adequate numerical techniques to tackle the problem. It is common sense that soil behavior is far from being adequately described as linear elastic; however, for the analysis of raft foundations, it is usually justified to assume linear elastic soil response within the range of working loads. Although yielding eventually might occur in some confined regions of the soil, it usually remains small compared with the overall area of the soil-structure interface. In current engineering practice, the Winkler model is probably most widely used to analyze rafts on elastic foundations. This is based much more on the fact that this model is very easy to handle and inexpensive in terms of both discretization and computation cost, rather than due to its mechanical merits. More reliable results are obtained by using a continuum model. Some of these models may even account for the increasing stiffness of the soil with depth.

In this example the soil is assumed to be adequately modeled as an isotropic nonhomogeneous elastic medium. We therefore will adopt a continuum model with a power variation of Young's modulus E with depth like the one proposed by Booker et al.[2]; i.e., the elastic constants of the soil medium are given by

$$\begin{aligned} E(z) &= E_1 z^{\rho}, \qquad 0 \le \rho \le 1 \\ \nu &= \text{const.} \end{aligned} \tag{4}$$

where E_1 is a constant that determines Young's modulus at the depth $z = 1$, ρ is referred to as the *nonhomogeneity parameter*, and ν denotes Poisson's ratio. Equation (4) could be used for sand deposits, where E is likely to vary nonlinearly with the overburden pressure. The raft, which also is assumed to behave elastically, is modeled by finite elements.

Numerically, we are combining the finite-element procedure with a boundary-element solution. One of the main tasks is to determine the displacements of the nonhomogeneous elastic half-space caused by a constant surface traction acting on an arbitrarily shaped area. In the context of the applied algorithm, the shape of the loaded area depends on the shape of the finite element. Assuming that the boundary of the loaded domain Ω can be approximated by a polygon consisting of k segments, as shown in Figure 3, the vertical displacement u_z of a point $(x, y, 0)$ on the soil surface due to a vertical traction p_z is given by

$$u_z(x, y) = \frac{Bp_z}{E_1} \sum_{i=1}^{k} \int_{L_i} \frac{un_\xi + vn_\eta}{(\rho - 1)R^{\rho+1}} ds \tag{5}$$

In this equation, B is a function of ρ and ν; u, v, and R

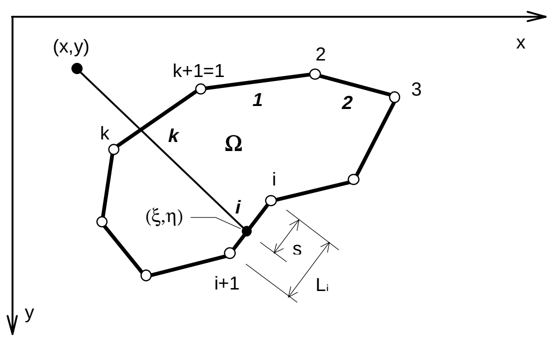


Fig. 3. Polygonal approximation of the loaded area.

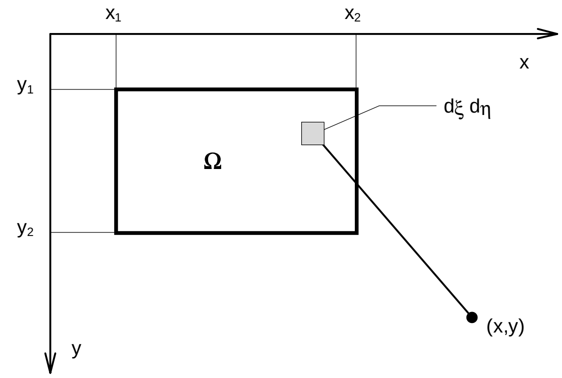


Fig. 4. Uniformly distributed load over a rectangular area.

measure the distance between source point and field point; and n_ξ and n_η are the direction cosines of the actual segment. The integral in Eq. (5) can be evaluated numerically in a standard manner by applying Gauss-Legendre quadrature. Solutions for other displacement components due to loading in any direction are given in ref. 15.

For rectangular-shaped elements, the displacement u_z can be found without performing any numerical integration at all, making the procedure quite efficient. In the case of vertical loading within the domain Ω, u_z is given by

$$u_z(x, y) = \frac{Bp_z}{E_1}\lambda \tag{6}$$

where λ is a function of (x, y) and is determined by evaluating κ for the boundaries of the loaded rectangle (see Fig. 4); that is,

$$\begin{aligned}\lambda = {} & \kappa(x - x_1, y - y_1) - \kappa(x - x_1, y - y_2) \\ & + \kappa(x - x_2, y - y_2) - \kappa(x - x_2, y - y_1)\end{aligned} \tag{7}$$

κ is given by

$$\kappa = \frac{v \operatorname{sgn}(u)\, |v|^{-\rho}\, \mathrm{B}_\zeta(a, b) + u \operatorname{sgn}(v)\, |u|^{-\rho}\, \mathrm{B}_{(1-\zeta)}(a, b)}{2(1-\rho)} \tag{8}$$

where $\mathrm{B}_\zeta(a, b)$ is the incomplete beta function, its parameters given by

$$\zeta = \frac{u^2}{u^2 + v^2} \qquad a = \frac{1}{2} \qquad b = \frac{\rho}{2} \tag{9}$$

For a more detailed derivation of all displacement components due to vertical and horizontal loading, the reader is referred to ref. 16.

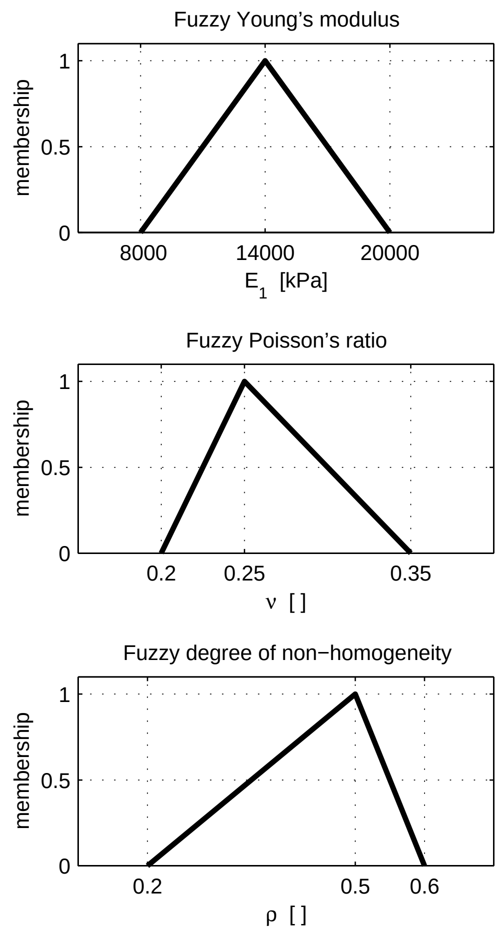


Fig. 5. Membership functions of $\tilde{E}_1$, $\tilde{\nu}$, and $\tilde{\rho}$.

3.4 Numerical example

A rectangular raft with a flexural rigidity of 20460 kN·m and an aspect ratio of 8 to 4 m on granular soil is considered. The elastic constants E_1 and ν and the exponent ρ determining the degree of nonhomogeneity of the soil are assumed to be fuzzy numbers $\tilde{E}_1$, $\tilde{\nu}$, and $\tilde{\rho}$ shown in Figure 5. The raft is subjected to a uniform loading of 100 kPa. The interface is assumed to be smooth; i.e. no shear stresses are transmitted between raft and soil.

3.5 Calculation of the fuzzy solution

The numerical algorithm for crisp data was presented in subsection 3.3. Our task is to perform it with fuzzy data, producing a fuzzy output. Thus we shall have to evaluate functions $f_{(x,y)}(\tilde{E}_1, \tilde{\nu}, \tilde{\rho})$, where

- f represents an output quantity such as displacement, stress, or bending moment.
- (x, y) is a point of the raft under consideration.
- $\tilde{E}_1$, $\tilde{\nu}$, and $\tilde{\rho}$ are the fuzzy parameters.

As shown in Section 2, we can reduce this to a calculation of the image

$$[\, f_{(x,y)}(\tilde{E}_1, \tilde{\nu}, \tilde{\rho})]_\alpha = f_{(x,y)}([\tilde{E}_1]_\alpha, [\tilde{\nu}]_\alpha, [\tilde{\rho}]_\alpha) \tag{10}$$

using the α-cuts. In general, the function $f_{(x,y)}$ does not depend monotonically on the parameters E_1, ν, and ρ, so it is not sufficient to use only the bounds of the α-cuts for computing the bounds of $[\, f_{(x,y)}(\tilde{E}_1, \tilde{\nu}, \tilde{\rho})]_\alpha$. Instead of this, we have to solve a global optimization problem. This has to be done for each desired point (x, y)—and also for each desired output quantity f.

To reduce the computional effort, we approximate the function $f_{(x,y)}$ on a three-dimensional grid. The gridpoints are given by the Cartesian product $S_{E_1} \times S_\nu \times S_\rho$, where S_{E_1}, S_ν, and S_ρ are discretizations of intervals that include the supports of the fuzzy parameters. For the preceding example, the following sets are taken:

$$S_{E_1} = \{8000, 10000, 12000, \ldots, 20000\} \tag{11}$$

$$S_\nu = \{0.2, 0.25, 0.3, 0.35, 0.4\} \tag{12}$$

$$S_\rho = \{0.0, 0.1, 0.2, \ldots, 0.9\} \tag{13}$$

The finite-element computation for a crisp triple $(E_1, \nu, \rho) \in S_{E_1} \times S_\nu \times S_\rho$ using the preceding method results in $f_{(x,y)}(E_1, \nu, \rho)$ for all desired points (x, y) and for all output quantities f. Doing this for all $(E_1, \nu, \rho) \in S_{E_1} \times S_\nu \times S_\rho$, we get an approximation $\hat{f}_{(x,y)}$ of $f_{(x,y)}$ using linear interpolation. Using this approximation, it is easy to compute approximative bounds of $[\, f_{(x,y)}(\tilde{E}_1, \tilde{\nu}, \tilde{\rho})]_\alpha$ for all points (x, y).

After performing this for all α-cuts (e.g., $\alpha \in \{0, 0.2, 0.4, \ldots, 1\}$), we get a fuzzy number $f_{(x,y)}(\tilde{E}_1, \tilde{\nu}, \tilde{\rho})$ for each point (x, y), e.g., for each point on a grid on the raft.

Caution: We have to treat each fuzzy number $f_{(x,y)}(\tilde{E}_1, \tilde{\nu}, \tilde{\rho})$ separately from fuzzy numbers at other points because we have neglected the interactions. Thus, in general, fuzzy numbers $f_{(x',y')}(\tilde{E}_1, \tilde{\nu}, \tilde{\rho})$ at additional points (x', y') may not be calculated by interpolation, but interpolation can be used at the stage of the crisp finite-element computation.

3.6 Visualizing the fuzzy solution

3.6.1 *At single points* (x, y). To represent the value of a single fuzzy quantity at a single point, it suffices to plot the membership function. See Figure 6, where the fuzzy bending moment $\tilde{M}_x$ at the center of the raft is shown.

3.6.2 *Along a section*. Along the section we plot at each point, for which we have calculated the fuzzy output, the function values indicating the degree of membership by a gray scale. White represents 0, light gray represents lower and dark gray higher degrees of membership, and finally, black represents 1. See Figure 7, where the fuzzy bending moment $\tilde{M}_x$ on a section in x direction through the middle of the raft is plotted. Taking the membership values of the fuzzy bending moment in Figure 7 at the location $x = 4$ leads to Figure 6.

3.6.3 *On the whole raft*. To get an overall picture of the stresses or bending moments, these quantities f are visualized on the whole raft using contour plots. Usually areas such as $A = \{(x, y) : c_1 \leq f_{(x,y)}(E_1, \nu, \rho) \leq c_2\}$ for fixed E_1, ν, and ρ are colored to indicate lower or higher values of f.

Here we use a fuzzy extension of this visualizing concept. Let $\tilde{f}(x, y) := f_{(x,y)}(\tilde{E}_1, \tilde{\nu}, \tilde{\rho})$ be the fuzzy value at an arbitrary point (x, y) on the raft. We define the degree of membership of the fuzzy value $\tilde{f}(x, y)$ to the crisp interval

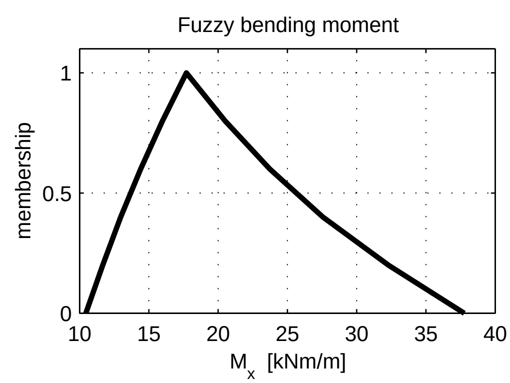


Fig. 6. Membership function of $\tilde{M}_x$.

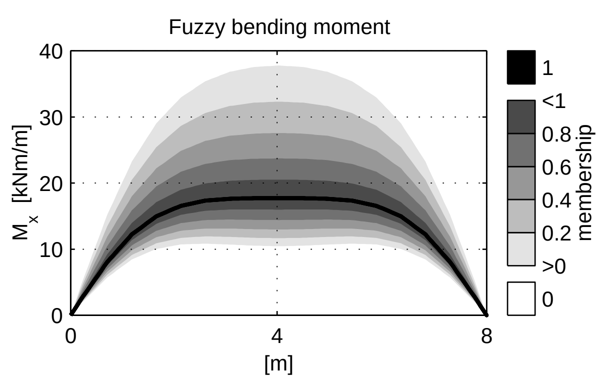


Fig. 7. Fuzzy bending moment $\tilde{M}_x$ along a section.

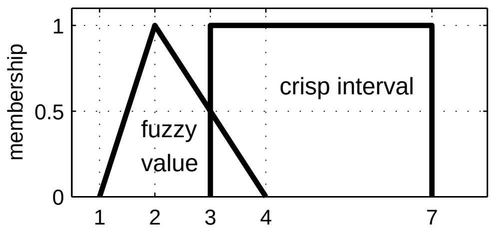


Fig. 8. Fuzzy element of a crisp interval.

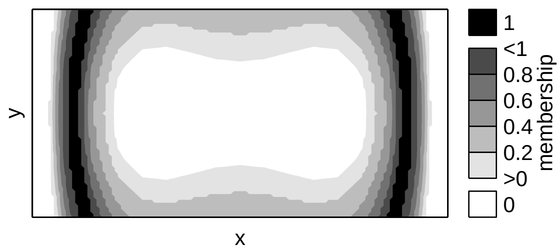


Fig. 9. $\tilde{M}_x \in [7.5, 10]$ kN·m/m.

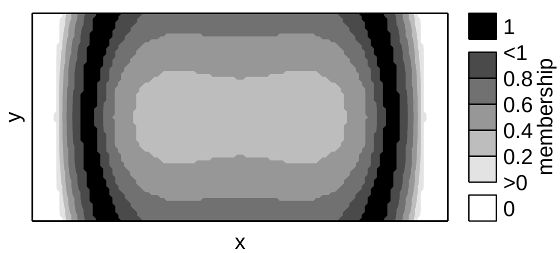


Fig. 10. $\tilde{M}_x \in [10, 12.5]$ kN·m/m.

$C = [c_1, c_2]$ by

$$m_C[\tilde{f}(x, y)] = \sup_{a \in C} m_{\tilde{f}(x,y)}(a) \tag{14}$$

Refer to Figure 8, showing an example with $m_C[\tilde{f}(x, y)] = 0.5$.

Performing this for all (x, y) on a grid on the raft, we get a fuzzy set or area that itself is visualized by a contour plot. Figures 9 and 10 show the degree of membership of the fuzzy bending moment $\tilde{M}_x$ to the interval [7.5, 10] and [10, 12.5] kN·m/m, respectively.

Remark: The areas plotted black in Figures 7, 9, and 10 represent the output with membership degree 1. This corresponds to the crisp output M_x for $E_1 = 14000$ kPa, $\nu = 0.25$, and $\rho = 0.5$; see Figure 5. In Figure 6, the value $M_x = 17.71$ kN·m/m with membership degree 1 also represents the crisp solution using the preceding parameters. This shows

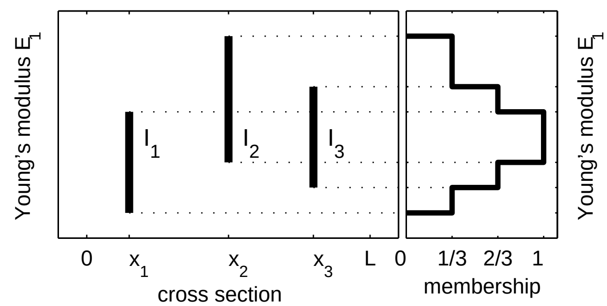


Fig. 11. Determination of fuzzy number $\tilde{E}_1$.

the great loss of information if uncertainties are neglected and the analysis is done just using crisp soil parameters.

3.7 Determining membership functions

In the numerical example of subsection 3.4, membership functions of the fuzzy parameters $\tilde{E}_1$, $\tilde{\nu}$, and $\tilde{\rho}$ have been taken as triangular fuzzy numbers, for the sake of exposition. In practical applications, the membership functions can be constructed from sample data obtained from site investigation and laboratory testing as well as geotechnical judgment provided by the engineer. Various methods for fuzzy data aggregation have been suggested in the literature (see, for example, refs. 1 and 3). We are going to elaborate on one of the approaches, say, for the case of Young's modulus E_1. Assume that in a cross section parallel to the x axis of the part of soil in question of length L, three measurements have been taken at points x_1, x_2, and x_3, yielding three intervals I_1, I_2, and I_3 for the parameter E_1. If no additional information on the distribution of E_1 over the length L is given, one might define the membership function $m_{\tilde{E}_1}(E_1')$ of the fuzzy parameter $\tilde{E}_1$, at a given value E_1', as one-third the number of intervals I_j in which E_1' is contained. Thus $m_{\tilde{E}_1}(E_1') = 1$ if E_1' appears as a value in all measurement intervals I_1, I_2, and I_3, $m_{\tilde{E}_1}(E_1') = \frac{2}{3}$ if E_1' appears in two of the intervals, and so on (see Fig. 11).

On the other hand, in case more geotechnical information on the approximate distribution of E_1 is available or can be estimated from further indications, one might describe the conjectured distribution over the length L by a band in the (x, E_1) plane. A piecewise linear extrapolation is sketched in Figure 12.

To determine the membership degree $m_{\tilde{E}_1}(E_1')$ of a given value E_1', the length of the broken horizontal at height E_1' cut out by the boundaries of the shaded band is measured and divided by L. For simplification, the result can be approximated by a triangular or trapezoidal fuzzy number. For computational purposes, one could place a rectangular sub-

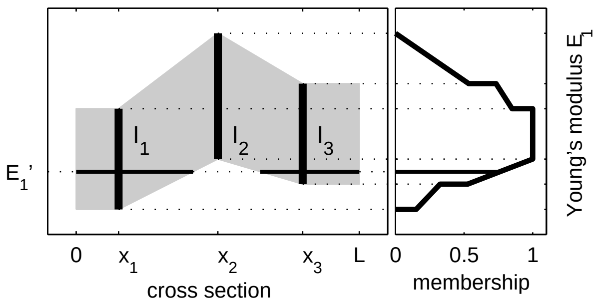


Fig. 12. Determination of fuzzy number $\tilde{E}_1$.

division on the (x, E_1) plane and just count the number of rectangles met by the shaded region at each height. Here we have presented the procedure in a one-dimensional case; it extends in an obvious fashion to the two- and three-dimensional situations.

At this point an important modeling question arises. Namely, in the fuzzy finite-element model we work with a constant, albeit fuzzy parameter $\tilde{E}_1$. On the other hand, measurements of E_1 have exhibited fluctuations in certain intervals I_j. The question is the meaning of the measurements in the context of the model. There are two possibilities:

1. The parameter E_1 is approximately constant in reality, and the variations I_j arise from measurement uncertainties. In this case, the use of a constant, fuzzy $\tilde{E}_1$ is fully justified. The membership function simply reflects our information on the measurement uncertainties.
2. The parameter E_1 is not constant, and the fluctuations in our measurements indicate physical variations over length L. In this case, the actual displacements and stresses might not be covered by a computation using a fuzzy constant $\tilde{E}_1$, even on α-level zero. This problem has to be faced by any modeling procedure, fuzzy or not. It concerns the uncertainty or lack of knowledge about the actual functional dependence of the parameter E_1 on the location along length L, which, in our model, is absorbed in the fuzziness of the assumed constant $\tilde{E}_1$. Preliminary results of a comparative case study underway show that linear variations of E_1 are covered by fuzzy constants, but more drastic fluctuations might lead to bending moments numerically outside the fuzzy range yet still qualitatively predicted.

4 FUZZY DIFFERENTIAL EQUATIONS

In this section we report on work in progress on dynamic problems. We are concerned with models described by systems of differential equations

$$\begin{aligned} \frac{dx(t)}{dt} &= F[t, x(t), b] \\ x(0) &= a \end{aligned} \tag{15}$$

Here t denotes time, and $x = (x_1, \ldots, x_n)$ is the state vector of the system. The function F is assumed to be known formally but may contain fuzzy parameters $b = (b_1, \ldots, b_m)$; the initial state $x(0) = a$ could be a fuzzy vector as well. Typically, these systems arise in structural mechanics, the vector $x(t)$ comprising displacements and velocities of the nodes of the structure. Further engineering applications include dynamic problems in continuum mechanics (subjected to a spatial discretization), problems of heat conduction described by Newton's law, or the time evolution of Markovian probabilities in queueing, maintenance, and reliability models.

The parameters and data in problem (15) being given by fuzzy vectors, the solution $x(t)$ at any point of time t will be a fuzzy vector as well, and the trajectory of the system $t \to x(t)$ in state space will be a fuzzy function. Our method of computation is by means of the Zadeh extension principle. Assuming that the initial value problem (15) can be solved uniquely when the data and parameters (a, b) are usual numbers, we can assert that the solution is given as a continuous function of the data and parameters:

$$x(t) = L_t(a, b) \tag{16}$$

It is this function L_t to which we apply the extension principle. Thus the α-level set of the fuzzy solution $x(t)$ at time t is given by

$$[x(t)]_\alpha = L_t([a]_\alpha, [b]_\alpha) \tag{17}$$

This shows that the solution concept above produces information of engineering interest; if the data and parameters fluctuate in the sets $[a]_\alpha, [b]_\alpha$ with degree of possibility α, then the state of the system at time t is confined to the level set $[x(t)]_\alpha$. This way the assessment of the variations of the input parameters is faithfully processed and reflected in the fuzzy output. In addition, $x(t) = L_t(a, b)$ can be seen as the unique fuzzy solution corresponding to a fuzzy solution concept for system (15), obtaining further mathematical justification this way.[11]

As a simple illustrative example, consider the displacement $x(t)$ of a mass m under the influence of a linear spring with stiffness k. It is described by the second-order differential equation

$$m\frac{d^2x(t)}{dt^2} + kx(t) = 0 \tag{18}$$

Assuming that parameters m and k are fuzzy while the initial data $x(0) = 0$ and $dx(0)/dt = 1$ are known precisely, the fuzzy solution is given by

$$x(t) = L_t(m, k) = \frac{k}{m}\sin\left(\frac{m}{k}t\right) \tag{19}$$

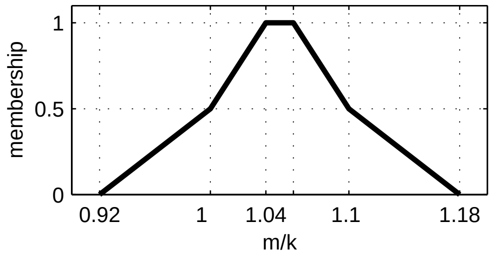


Fig. 13. Fuzzy parameter m/k.

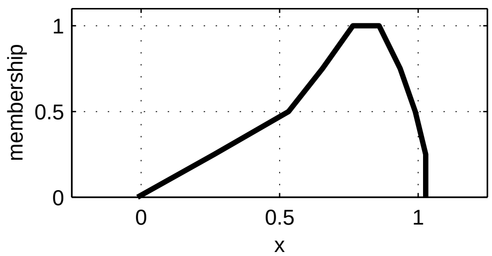


Fig. 14. Fuzzy displacement $x(8)$.

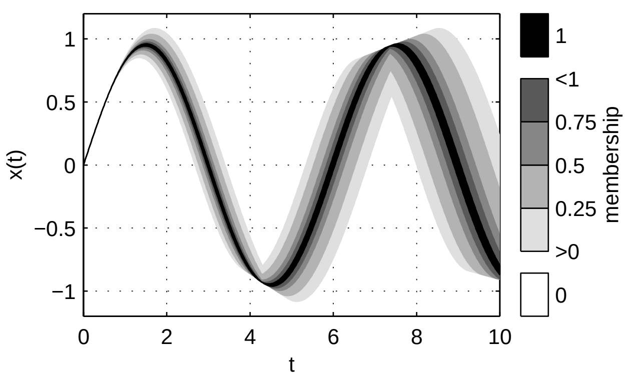


Fig. 15. Fuzzy trajectory.

Figure 13 depicts a certain fuzzy parameter m/k; Figures 14 and 15 depict the corresponding fuzzy displacement $x(8)$ at time $t = 8$ (polygonal approximation), respectively, α-level sets for part of the fuzzy trajectory.

The computation proceeds as follows: First, trajectories $t \to L_t(m, k)$ are computed corresponding to an array of values m/k. Next, the bounding curves of the level trajectories $[t \to x(t)]_\alpha$ are constructed as envelopes of the trajectories arising from parameter values in level set $[m/k]_\alpha$. Finally, the membership function of $x(t)$ at fixed points of time t is obtained by projection. More efficient numerical algorithms involving interpolation, evolution properties, and piecewise monotonicity of the membership functions are currently being developed.[12]

As a final observation, we note that from the fuzzy trajectory as a primary object, various secondary quantities of interest can be computed, e.g., maximal displacement at level α, resonance frequency, spectral coefficients, and so on.

5 FUZZY DATA ANALYSIS IN PROJECT PLANNING AND CONSTRUCTION MANAGEMENT

Fuzzy methods can be a valuable help in network planning. They enable the engineer to incorporate his or her information on the uncertainties of the available data and his or her assessment of future conditions. They provide a tool for monitoring and control and do not, after all, give the false impression of precision in the project schedule that can never be kept up in reality.

Basic tools in fuzzy network planning and its application to engineering projects have been developed in ref. 9. In this section we shall elaborate on three additional topics of practical interest: possibilistic modeling of geologic data in a tunneling project, aids for project monitoring, and questions of time/cost optimization.

5.1 Time estimate for a tunneling project: An example

Data for this example come from a preliminary investigation at the site of a projected road tunnel at the German-Austrian border in geologically challenging terrain. We contrast our approach[7] with a previous study[13] in which the probabilistic PERT technique had been employed. The total extent of the tunnel of approximately 1250 m was divided into 16 sections, according to geologic criteria. For each section, a geologist had provided a verbal description plus estimated percentages of the rock classes to be expected. For example, section 2, of 340 m length, was classified as 80% slightly fractured rock and 20% fractured rock. A deterministic engineering estimate yielded driving times for each section and rock class. For example, completion of section 2 was estimated at 60 days, provided only slightly fractured rock was encountered and at 81 days under conditions of fractured rock. In the PERT analysis, the duration of each section was interpreted as a discrete random variable with elementary probabilities defined by the given percentages. For example, this way the expected duration for section 2 was computed to $0.8 \cdot 60 + 0.2 \cdot 81 \approx 64$ with standard deviation ~ 8.4. In section 8, risk of tunnel failure was presumed, and in sections 7 and 9, risk of water inrush was presumed. In the PERT analysis, the corresponding delays were modeled in a similar probabilistic way.

Tunneling durations as triangular fuzzy numbers. We argue that possibility theory provides a viable alternative. The

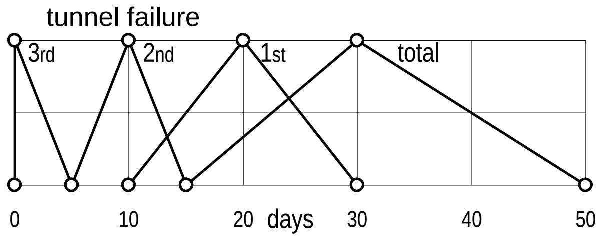


Fig. 16. Delay caused by tunnel failure.

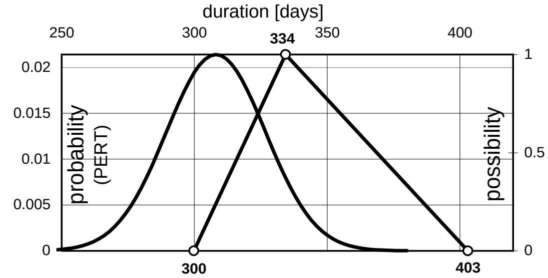


Fig. 17. Probability/possibility distribution of total duration.

key to our approach is the interpretation of the percentages as fuzzy ratios. In our treatment of the tunneling project, we assigned membership degree 1 to the ratios proposed by the geologist, i.e., 80:20 in section 2, for example. In the subsequent analysis, we estimated the ratios defining the bounds for the domain of membership degree zero, i.e., in section 2, 90:10 and 50:50. This resulted in the triangular fuzzy number $\langle 62, 64, 70 \rangle$ for the tunneling duration in section 2 and similarly for the other 15 sections. Risk of tunnel failure in section 8 was analyzed separately. In particular, the possible occurrence of one major tunnel failure and up to two minor ones was taken into account, with corresponding delays modeled by triangular fuzzy numbers, as shown in Figure 16.

The sum of these prognoses for possible delays was added to the driving duration of section 2. Further delays due to water inrush in sections 7 and 9 were estimated in a similar way, and the respective durations were modified accordingly.

We would like to point out that the fuzzy approach allows the incorporation of information going beyond probability distributions. For example, at each transition of the rock classes, individual delays are due to change in the cross-sectional area of the tunnel, change in equipment, safety regulations to be observed, and so on. The planning engineer can assess these individual circumstances from previous experience, from specific enterprise data, and from discussions with experts involved in the construction process. The information gathered in this way can be subsumed under a formulation by means of fuzzy numbers.

Total duration. The linear structure of the tunneling process is described by a serial network consisting of one path only and 16 nodes for the sections. The total project duration is obtained by simply adding the individual durations, resulting here in a triangular fuzzy number. In contrast to this, in the probabilistic approach, expectation values and variances are added, assuming stochastic independence of the individual activities. As is customary in the PERT technique, the total duration is assumed to be normally distributed. The two results are contrasted in Figure 17.

The deviation of the central value (membership degree 1) in the fuzzy approach from the expectation value in the probabilistic approach is mainly a consequence of different handling of the exceptional risks in sections 7, 8, and 9. We emphasize that the meaning of the two graphs is totally different; the determining nodes of the triangular fuzzy number directly reflect the risk assessment performed in the analysis, the height of the curve representing the estimated degree of possibility associated with each duration. On the other hand, the area under the probability distribution curve defines the probability that the total duration appears in a certain interval. In view of the various artificial stochastic hypotheses entering in the PERT algorithm, it is questionable whether it allows a direct interpretation relevant for managerial decisions.

As with most construction projects, the underlying uncertainties are not of a statistical nature. The percentages provided by the geologist are neither samples from a large number of completed tunnels, nor are they statistical averages from a large number of exploratory borings along the prospective tunnel route. They are nothing but subjective estimates based on expertise. Thus it appears more appropriate to translate them into a possibilistic rather than a probabilistic formulation.

5.2 Aids for monitoring

In serial networks such as, for example, a rise in tunneling projects, a fuzzy time/velocity diagram may be taken as a monitoring device for the unfolding of the construction. It simply describes the fuzzy point of time when a certain position along the tunnel route should be reached. The diagram in Figure 18 reflects all uncertainties taken into account when the project starts.

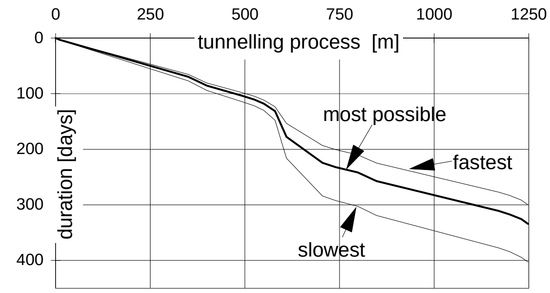


Fig. 18. Time/velocity diagram.

As construction progresses, the uncertainties are narrowed down step by step. The diagram can be actualized by a simple cancelation of the design uncertainties, once a definite state has been reached.

In a branching network, a time/velocity diagram might not contain enough information. Here it is essential to assess the criticality of activities or branches. In deterministic network planning, the slack time for each activity is computed by a backward pass through the graph from the desired completion date, and criticality means slack zero. As explained in detail in ref. 9, the backward pass with fuzzy durations no longer yields the slack of each activity but rather its critical potential. We prefer to call the fuzzily computed slack *range of uncertainty*. Negative range of uncertainty implies a certain possibility that delay of the completion date may be caused at the respective activity. Numerically, the *critical potential* is defined as the possibility degree of zero range of uncertainty.

This opens the way for enhancing the network presentation by shading (or coloring) areas of different criticality in the network (see Figure 19, where an example of a project plan for a sewage plant is presented).

Such a presentation uncovers and emphasizes the uncertainties and can help the construction manager to assess rapidly which activities may become critical. In the course of realization of the project, the diagram can be updated continually, thereby recording shifts in criticality and making it possible to recognize trends early. This is the central objective of monitoring and control, and it is the basis for taking adequate measures in order to avert developments endangering the timely completion of the project or its economic success. As opposed to deterministic planning methods, the project uncertainties do not disappear in the "black box" of an algorithm. The fuzzy network representation may aid all persons concerned with the project in strategic considerations.

5.3 Construction time and cost

In this subsection we address the following: Is it possible to design and implement a cost-optimal project plan? There are many parameters to be varied:

1. We can change the overall method of construction.
2. Given a construction method, we can change its internal structure, the temporal and causal interdependencies of the individual activities.
3. Given a fixed project structure, we can vary the costs of individual activities by acceleration, deceleration, and resource modification.

It is clear right away that possibilities (1) and (2) allow an infinity of variations, depending on the inventiveness of the designing engineer. No single solution can be guaranteed to be optimal; an absolutely optimal construction method or project structure simply does not exist. For the discussion to follow, we therefore concentrate on point (3), which already features all difficulties (see ref. 8).

Thus we assume a fixed network structure chosen for the project plan. We want to optimize costs by changing the duration of the activities. As a precondition, we must know the effects of resource modification on the time/cost relation of the activities. This is a major source of uncertainty and will be discussed below. We first have a look at the standard deterministic approach to this optimization problem. The basic assumption is that for each of the activities A_i, $i = 1, \ldots, n$, the time/cost relation is known. We denote by $C_i(D_i)$ the cost of activity A_i when performed at duration D_i. Further, the duration D_i of activity A_i ranges between certain upper and lower bounds:

$$D_{iL} \leq D_i \leq D_{iR} \tag{20}$$

Each choice of duration $D_1, \ldots, D_n$ for the activities results in a total duration T of the project, determined by the activities on the corresponding critical paths. The smallest and largest possible project durations $T_{\min}$ and $T_{\max}$ are obtained by performing all activities in minimal time D_{iL} or maximal time D_{iR}. There is an external time limit on the total duration T, given by the required deadline T_e and the commencement date T_b; T must be smaller than the difference $T_e - T_b$. This results in the constraint on the total duration:

$$T_L \leq T \leq T_R \tag{21}$$

where $T_L = T_{\min}$ and T_R is the smaller of the two values $T_{\max}$ and $T_e - T_b$. The standard optimization proceeds in two steps.

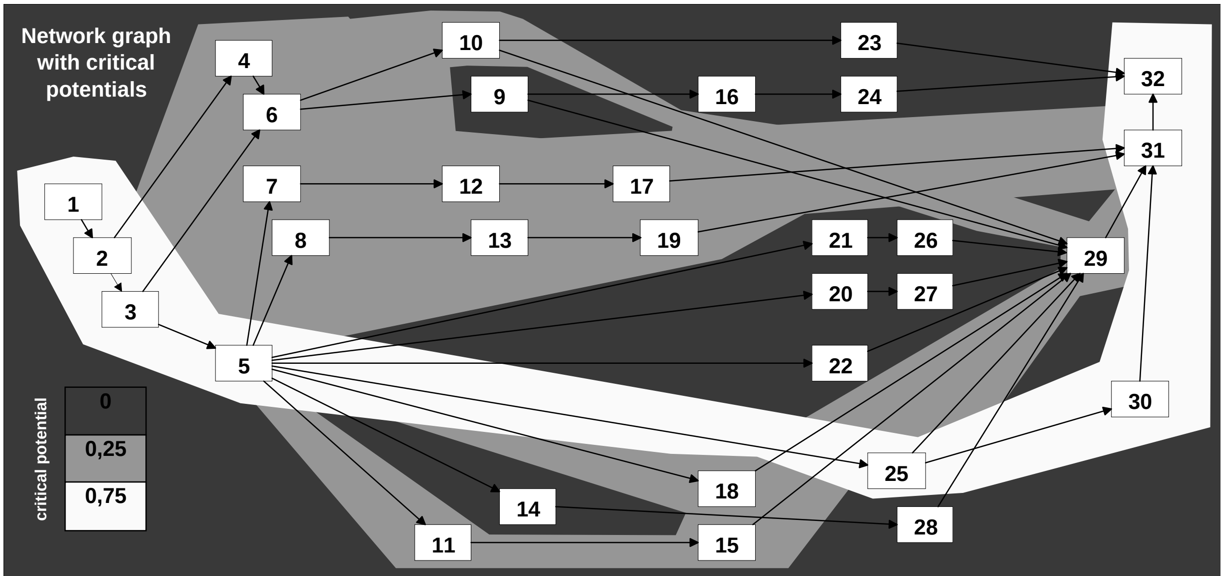
Step 1: *Cost optimization at fixed project duration T*. The duration T is attained by many different combinations of individual durations $D_1, \ldots, D_n$. For each combination, we get a corresponding total cost

$$C_T(D_1, \ldots, D_n) = \sum C_i(D_i) \tag{22}$$

The objective is to minimize $C_T(D_1, \ldots, D_n)$ subject to constraint (20). This results in the minimal cost $C(T)$ at fixed project duration T.

Step 2: *Optimizing the total project duration*. In this step we simply choose the project duration T^*, subject to constraint (21), such that the corresponding total cost $C(T^*)$ is the least among all minimized costs $C(T)$.

Under various assumptions on the time/cost relationships $C_i(D_i)$, this standard optimization procedure has been extensively dealt with in the literature (see, for example, ref. 10). It seems we have solved the problem. However, we shall see in the simple example below that the combinatorial structure of the possibilities leading to optimal duration T^* becomes tremendously complex with increasing size of the project. The construction manager would be required to control the construction process in such a way that each activity runs precisely the duration ultimately producing minimal costs



Sewage plant (duration in months)

job	description	duration	crit.pot.	job	description	duration	crit.pot.	job	description	duration	crit.pot.
1	start	0	0.83	11	screening plant, grit chamber, part 1	4	0.83	22	Archimedian screw pump station	5	0
2	site installations, river divertion	1,**1**,2	0.83	12	primary settling tank, part 2	5	0.26	23	digestion towers finishes	4	0
3	bridge, access	2	0.83	13	activated sludge tank, part 2	5	0.83	24	management building finishes, part 2	5	0
4	deep foundation digestion towers, part 1	1.5,**2**,3	0.26	14	gas plant, part 1	3	0.28	25	small structures, canalization, part 1	8,**9**,11	0.83
5	filtering well, dewatering	2,**2**,4	0.83	15	screening plant, grit chamber, part 2	4	0.28	26	secondary settling tank 2	3	0
6	deep foundation digestion towers, part 2	1.5,**2**,3	0.28	16	management building finishes, part 1	5	0.28	27	secondary sludge thickener	2	0
7	primary settling tank, part 1	1,**1**,1.5	0.28	17	primary settling tank, part 3	7.5,**8**,9	0.28	28	gas plant, part 2	4	0.06
8	activated sludge tank, part 1	1,**1**,1.5	0.28	18	engine house	7.5,**8**,9.5	0.34	29	asphalt, recultivation, fence, part 1	5.5,**6**,7	0.56
9	management building carcass	6.5,**7**,8	0.83	19	activated sludge tank, part 3	7.5,**8**,9	0.28	30	small structures, canalization, part 2	7	0.83
10	digestion towers carcass	7.5,**8**,9	0.83	20	primary sludge thickener	3	0	31	asphalt, recultivation, fence, part 2	5	0.83
				21	secondary settling tank 1	3	0	32	end	0	0.83

Fig. 19. Fuzzy network.

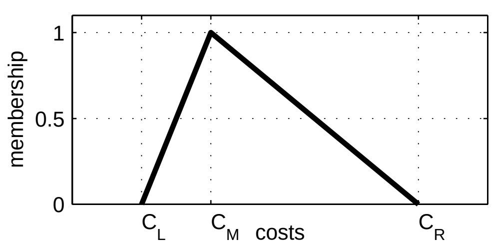


Fig. 20. Membership function for cost of activity at fixed duration.

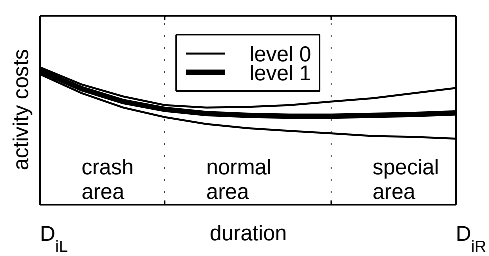


Fig. 21. Fuzzy time/cost dependence.

$C(T^*)$. This is impossible. Therefore, the information obtained by the standard optimization procedure is useless.

Time/cost analysis. We turn to fuzzy modeling of the duration-dependent costs of a single activity A_i. The costs $C_i(D_i)$ required to complete this activity in a certain time D_i are described by a triangular fuzzy number; see Figure 20.

The planning engineer might first arrive at the central value C_M with membership degree 1 by employing the standard deterministic computations from construction management data. Then a risk analysis might provide the lowest possible costs C_L and a largest bound C_R for the estimated costs. Of course, further subdivision of risk levels can provide a refined analysis (this was carried through, for example, in ref. 9), but as a first approximation, a triangular fuzzy number may satisfactorily reflect the cost fluctuations under risk.

Next we discuss the cost distribution as the duration D_i of activity A_i varies in its bounds D_{iL}, D_{iR}. An example of such a diagram is shown in Figure 21, exhibiting 0-level and 1-level curves of the fuzzy cost $C_i(D_i)$.

We can distinguish three regimes of the time/cost dependence:

1. *Normal area*. This is the normal range for completing the activity. Costs for equipment, labor, and material and costs for site overhead are balanced.

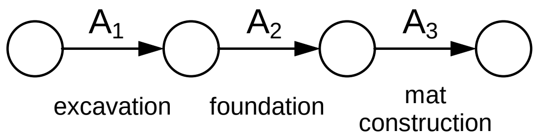


Fig. 22. Serial network.

2. *Crash area*. The acceleration of the activity causes the costs for equipment, labor, and material to predominate. The increase in capacities may cause interference and unintended obstructions, thereby further raising costs uneconomically. In any case, a progressive cost development is to be expected when the activity duration is pushed to its lower limit.
3. *Special area*. The largest uncertainties arise in this range. Costs for maintaining the construction site at minimum capacity may be high or low, depending very much on the specific project and circumstances.

We point out that the diagram in Figure 21 depicts but one of the many time/cost relationships that can arise. They typically are nonlinear but may even have discontinuities. For example, to achieve a certain acceleration, the construction process may have to be changed, additional machinery may be required, or shift work may have to be introduced.

Example. As an illustrative example, we consider a simple serial network consisting of the three activities: excavation (A_1), foundation (A_2), and mat construction (A_3); see Figure 22. The durations D_i of each activity vary in an interval $[D_{iL}, D_{iR}]$, and thus can be described by a rectangular fuzzy number. We choose $[D_{1L}, D_{1R}] = [3, 8], [D_{2L}, D_{2R}] = [D_{3L}, D_{3R}] = [7, 12]$. For each activity, a time/cost relationship as in Figure 21 is assumed. The resulting total duration is the sum of the three duration intervals and thus may vary in the interval $[17, 32]$. For computational simplicity, we allow only integer values for each duration. Following the pattern of the deterministic optimization algorithm, we first choose a fixed duration T in the interval $[17, 32]$ and determine the combinations of individual durations D_1, D_2, and D_3 summing up to T. Each combination requires a cost of $C_T(D_1, D_2, D_3) = C_1(D_1) + C_2(D_2) + C_3(D_3)$, represented by a triangular number. Superposition of these triangular numbers shows the cost variations that can arise if the total project duration is T ($T = 29$ in Figure 23). The envelope of these triangular numbers can be approximated by a trapezoidal number; see Figure 24.

We note that the boundaries C_{ML}, C_{MR} of the central plateau arise already from the combinatorial possibilities when all activities run at deterministic costs C_{iM} (membership degree 1). Thus the shaded region can be considered as an indicator of the economic risk the designing engineer has to face. To assess the risk of economic failure, the engineer should examine the combinations of activity durations leading up to the characterizing values C_{ML}, C_{MR}, C_R, respectively. The number of these combinations grows fast with the project size. Thus, in practice, this is an impossible task. One must be content with the indications extractable from the diagram and estimates from a detailed study of a few extremal cases.

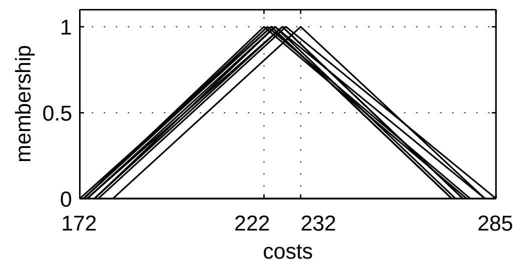


Fig. 23. Fuzzy cost variations at duration $T = 29$.

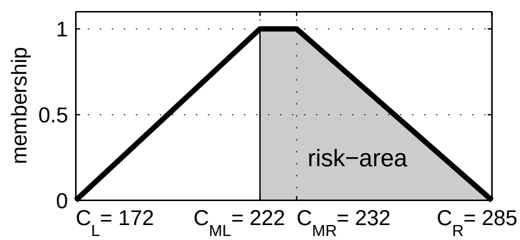


Fig. 24. Risk assessment of range of costs.

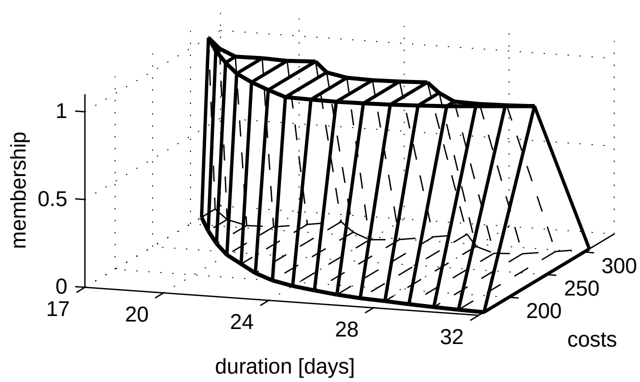


Fig. 25. Time/cost/possibility diagram.

Three-dimensional presentation. To each attainable total duration T in the interval $[T_L, T_R]$ there are corresponding costs described by a trapezoidal fuzzy number as in Figure 24. We can collect these in a three-dimensional diagram; see Figure 25.

The height of the resulting surface over a point (T, C) shows the degree of possibility that the project duration is T with project costs C. The trapezoidal number in Figure 24 arises as a cross section of the surface at fixed duration T. The plateau area of possibility degree 1 embraces the time/cost combinations attained when all activities run at deterministic cost C_{iM}. This three-dimensional graph puts in evidence the domain in which cost and duration of the project may vary when the duration of each single activity has been modeled by a rectangular fuzzy number, single costs by a triangular fuzzy number, and the time/cost relation by a diagram as in Figure 21. Accelerating or decelerating single activities will move the result within the boundaries of this domain.

Final remark. We realize from these considerations that in a time/cost analysis with fuzzy data a large variety of possible results with different membership degrees arises, already in a simple example involving three activities only. In view of this observation, it is clear that the goal of classic optimization—the search for and the implementation of a cost-optimal project plan—cannot be achieved. Not only is it legitimate from a modeling perspective to assume that cost and duration data are fuzzy, but in real life construction projects the only type of data available are fuzzy data. We may conclude that with respect to cost and duration of construction projects, one cannot strive for optimization but rather should attempt to achieve a reasonable solution within the limitations of the given risks and uncertainties.

These considerations also show that textbook strategies to accomplish a certain result do not exist, but every measure and its effects have to be evaluated in each specific situation. Frequently heard statements such as “Reduction of the construction period will reduce costs” have no validity, with the possible exception of specific projects where they may have resulted from a thorough investigation of the determining factors. It is essential that the designing engineer consider the data and project uncertainties from the earliest planning phase onwards so as to have a firm basis for the assessment of the risk of economic failure.

As the economist H. A. Simon[14] puts it:

> . . . exact solutions to the larger optimization problems of the real world are simply not within reach or sight. In the face of this complexity, the real-world business firm turns to procedures that find good enough answers to questions whose best answers are unknowable. Thus normative microeconomics, by showing real-world optimization to be impossible, demonstrates that economic man is in fact a satisficer, a person who accepts “good enough” alternatives, not because he prefers less to more but because he has no choice.

ACKNOWLEDGMENT

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