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The LP/IP hybrid method for construction time-cost trade-off analysis

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Construction planners face the decisions of selecting appropriate resources, including crew sizes, equipment, methods and technologies, to perform the tasks of a construction project. In general, there is a trade-off between time and cost to complete a task – the less expensive the resources, the longer it takes. Using Critical Path Method (CPM) techniques, the overall project cost can be reduced by using less expensive resources for non-critical activities without impacting the duration. Furthermore, planners need to adjust the resource selections to shorten or lengthen the project duration. Finding the optimal decisions is difficult and time-consuming considering the numbers of permutations involved. For example, a CPM network with only eight activities, each with two options, will have 2^8 alternatives. For large problems, exhaustive enumeration is not economically feasible even with very fast computers. This paper presents a new algorithm using linear and integer programming to obtain optimal resource selections efficiently that optimize time/cost of a construction project.

Keywords: Time-cost trade-off, critical path method, optimization, construction planning.

Introduction

Critical Path Method (CPM) has been used in the construction industry for project scheduling and control since the 1960s. It provides a practical tool for planning and controlling construction projects. Many new algorithms and techniques have been developed to enhance the usefulness of CPM. Among these algorithms and techniques, time-cost trade-off analysis has been one of the most important enhancements for using CPM to plan and control construction projects. In general, there is a certain relationship between time and cost to complete the activities within a project. A contractor may choose different crew sizes, equipment and construction methods to complete the activities. These decisions will ultimately decide the duration and cost of a project. Since there may be hundreds of activities within a project, it is almost impossible to enumerate all possible combinations to identify the best decisions that can complete it in the shortest time and at the minimum cost.

This paper presents a new approach to assist construction planners in making time-cost trade-off decisions. This new approach, named the LP/IP hybrid method, is a combination of linear and integer programming (LP and IP). The LP/IP hybrid method first establishes a high quality lower bound of the time/cost relationship of a project using LP. Once the lower bound to the time-cost trade-off curve has been generated, IP is used to identify the exact solution for one specified duration. This avoids having to solve for the entire time-cost trade-off curve exactly using IP, which is computationally prohibitive for realistic construction projects. In addition, the lengthy and error-prone process of formulating linear and integer programming objective functions and constraints are replaced by a spreadsheet tool so that users can intuitively define time-cost and precedence relationships for activities. The LP/IP hybrid method along with the spreadsheet tool provides construction planners with an accurate and efficient means of making good time-cost decisions. A detailed description of the LP/IP hybrid method and spreadsheet tool is provided and an example is given to demonstrate the use of this new method.

Time and cost of construction projects

Time and cost are two main concerns of construction projects. In the construction industry, contractors usually use previous experience to estimate project duration and cost. Typically, a project is broken down into activities to which resources can be assigned and durations and costs estimated. The activities are linked according to work sequences to form a network. CPM techniques are used to analyse the network to identify critical path(s) and project duration. In general, the more resources assigned to an activity, the less time it takes to complete, but cost is usually higher. This trade-off between time and cost gives construction planners both challenges and opportunities to work out the best construction plan which optimizes time and cost to complete a project.

In addition, the project manager may need to accelerate a project to meet a deadline issued by the owner because of a delay of previous activities. Adjustments are needed to change the resource assignments to optimize the resource allocations that yield the accelerated project duration at a minimum cost. Since there are often too many permutations of resource allocations, there is a need to develop tools and methods to assist construction planners in making resource allocation decisions which optimize time and cost.

Existing techniques for construction time-cost trade-off analysis

Mathematical and heuristic methods are the two major approaches used to solve the time-cost trade-off problems in project scheduling. Mathematical methods convert the project time-cost trade-off problems to mathematical models and utilize linear programming, integer programming, or dynamic programming to solve the problem. Kelly (1961) formulated the time-cost trade-off problems using LP by assuming a linear relationship between time and cost for construction activities. This approach was limited to two options per activity, with a continuous linear variation between them. Meyer and Shaffer (1963) used IP to solve time-cost problems including both linear and discrete relationships within the same activity. However, the solution process is time consuming. Robinson (1975) developed a dynamic programming approach to solve time-cost trade-off problems which require special network relationships. Reda (1986) and Reda and Carr (1989) used mixed IP to solve time-cost trade-off problems within related activities.

The advantages of mathematical approaches include efficiency and accuracy. However, formulating constraints and objective function is time-consuming and error-prone. Besides, mathematical programming knowledge is necessary to formulate these mathematical models correctly. Very few construction planners are trained to perform this type of formulation, especially for large networks.

Heuristic methods are non-computer approaches which require less computational effort than mathematical methods. Examples of heuristic approaches include Fondahl's method (1961), Prager's structural model (1963), Siemens's effective cost slope model (1971), and Moselhi's structural stiffness method (1993). These heuristic methods provide a way to obtain good solutions but do not guarantee optimal solutions. In addition, the solutions offered by heuristic methods do not provide the range of possible solutions, making it difficult to experiment with different scenarios for what-if analysis.

Current practice of time-cost trade-off analysis

Although techniques for construction time-cost analysis have been available since the 1960s, the construction industry has not widely accepted these techniques in their day-to-day planning tasks. From personal experiences and from many open discussions with professionals in construction, we find that most construction engineers are unaware of the optimization techniques. Those who are use predominantly the heuristic approaches. The general consensus regarding the heuristics approaches is that 1) they take a long time to formulate, 2) the solutions are not always optimal, 3) it is difficult to use them to solve large/complex networks, and 4) they lack the flexibility to experiment with what-if scenarios. Most users limit their use to small networks, believing that it is impossible to utilize them on larger networks.

While some construction engineers believe the heuristics approaches are good enough for the construction industry, despite the approximations and imprecision, others are searching for new and improved methods. Some construction engineers, well trained in operations research and computer applications, are experimenting with new approaches using linear and integer programming to provide better solutions. The results are quite satisfactory and the time needed to solve large scale networks is significantly reduced with the use of computers and commercially available linear/integer programming solvers. However, the biggest complaint is that formulation requires considerable effort and is prone to errors. A network is rarely formulated, even by experienced construction engineers, without several revisions.

Most practitioners in construction agree that time-cost trade-off is an important issue, yet acknowledge that it is commonly ignored because of the complexity involved in the analysis. Many express their wish to have an easy-to-use tool to interface directly with project planning systems, such as Primavera. With this vision in mind, the authors have developed a tool targeted toward practical use in construction. The research effort has focused on two critical issues: 1) more efficient algorithms that yield more accurate solutions, 2) easy-to-use tools for real practice. The following sections describe the result of this research: a new hybrid algorithm and prototype computerized tool named 'Optimum'.

The LP/IP hybrid method for construction time-cost analysis

Obtaining a good and nearly optimal solution with a reasonable amount of computational effort is the major motivating factor for this method. IP can find the exact optimal solution, but it is computationally intensive. The LP/IP hybrid method is a hybrid approach which uses 1) LP to generate a lower bound of the minimum direct cost curve efficiently and 2) IP to find the exact solution for a single specified duration. The following sections describe the formulation of the linear and integer programming models. Examples are given to demonstrate how to formulate the mathematical models.

The linear programming model – the convex hull method

In the LP model, the time-cost curve of each activity is identified as a curve connecting various options to complete the activity. These options can be either discrete or continuous depending upon the available resources/methods or simply a contractor's preference. For example, 'form and rebar' may be finished in 5–8 days as the crew size decreases from five to two persons. This relationship between activity options is fairly continuous. On the other hand, the discrete relationship between activity options may occur in activities with different methods, such as using a bucket or a pump for pouring concrete. It is also possible that the activity could have both types of relationships. Figure 1 shows an example activity whose options have both discrete and continuous time-cost relationships. It shows that the contractor can choose one option along segment AB or option C to complete the activity. Each option has its associated time and cost. Figure 2 shows an example of a continuous time-cost relationship. It represents an activity that can be completed at any combination of time and cost for the points on the curve. A piece-wise linear approximation is usually used to simplify calculations, so the time-cost relationship in Figure 2 can be considered as the linear segments consisting of A–B, B–C, C–D and D–E.

If the relationship among the options within the activity is linear, it is possible that the durations of the possible options are non-discrete and non-integer, such as 3.5 days or 3.6 days, which implies that the durations of the possible options within the activity are continuous. However, for construction applications, only integer durations are considered, such as days. Therefore, we define a time-cost curve as 'linear and continuous' when the durations on this curve are successive integers and the relationship between the options is linear.

Preprocessing

The convex-hull algorithm in this method generates the convex hull formed by the various options specified for each activity, for example, as shown in Figure 3. The convex hull is established by the following steps:

1. Identify all options as 'active'.

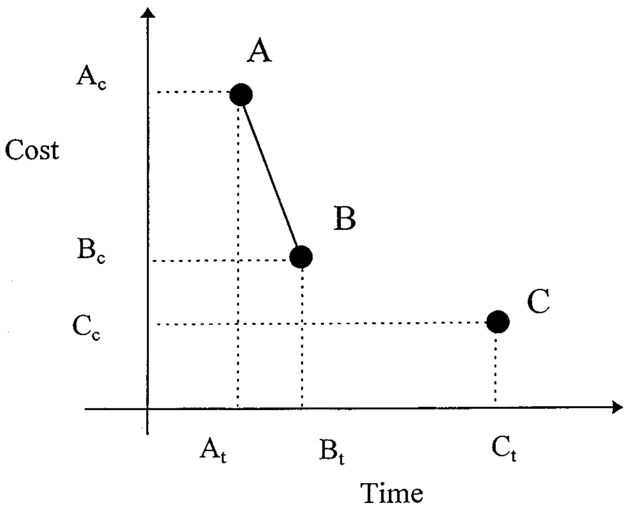

Figure 1 An activity with both discrete and continuous time-cost relationships

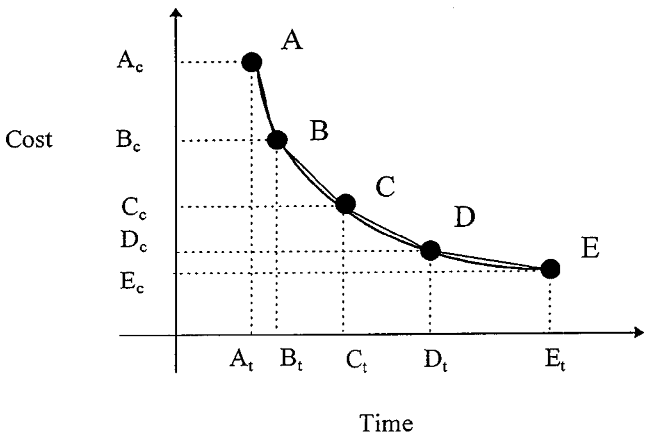

Figure 2 Piece-wise linear approximation of construction time-cost relationship

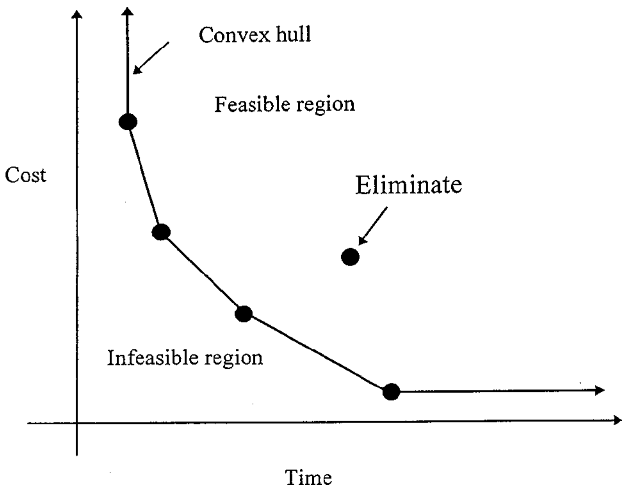

Figure 3 An example of convex hull

2. Arrange the active options within the activity in ascending duration order, and assign 1, 2,...n to i for the option i. (Associated costs and durations are $C_1, C_2,\ldots, C_n$ and $D_1, D_2,\ldots, D_n$, respectively, and n is the number of active options.)
   If $n = 1$, go to step 7.
3. Calculate the cost slopes M_i of all active options and deactivate the option $i+1$, if $M_i \geqslant 0$ ($i = 1, 2,\ldots, n-1$). When $M_i \geqslant 0$, option $i+1$ can be eliminated. Option i will always be more desirable than option $i+1$ because option i costs less and takes shorter time.
   The cost slope of option i is defined as
   $M_i = \dfrac{C_{i+1} - C_i}{D_{i+1} - D_i}$, for $i = 1, 2,\ldots, n-1$. (see Figure 4)

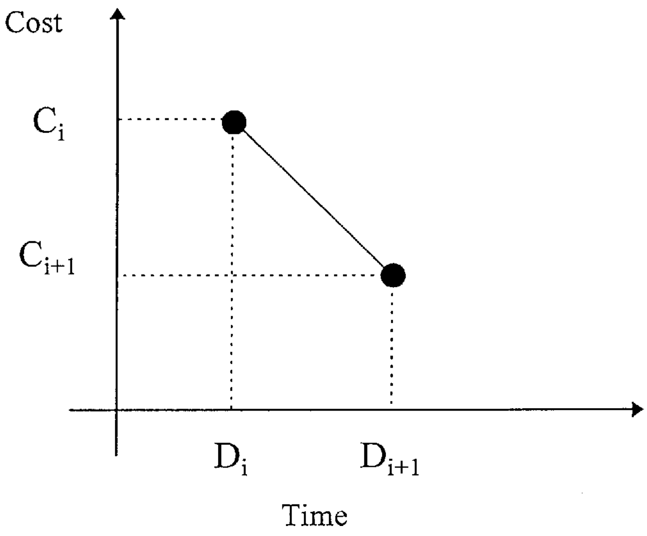

Figure 4 Cost slope of option i; $M_i = \dfrac{C_{i+1} - C_i}{D_{i+1} - D_i}$

   Renumber the active options $i = 1,2,\ldots,n$, where n is the new number of active options. If $n = 2$, go to step 6.
4. Recompute the cost slopes and compare M_i with M_{i+1} and deactivate option $i+1$ if $M_i > M_{i+1}$ ($i = 1, 2,\ldots, n-1$).
5. Repeat steps 2 to 4 until $M_i \leqslant M_{i+1}$ ($i = 1, 2,\ldots, n-1$). Renumber the active options $i = 1, 2,\ldots, n$, where n is the new number of active options.
6. Draw the straight lines which connect active options i and $i+1$, for $i = 1, 2,\ldots, n-1$, to form the convex hull.
7. Draw straight lines from the minimum duration and minimum cost points parallel to the Y-axis (cost) and X-axis (duration), respectively.

The objective function and constraints

Once the convex hull is established for each activity, a linear program can be formulated. The objective function and constraints of all activities contribute to the linear programming model of the time-cost trade-off problem and can be presented as follows:

Objective function:

$$\text{Minimize} \quad \sum_{i=1}^{n} C_i$$

Constraints:

Subject to

$S_i \geqslant 0$	$i = 1, 2, 3, \ldots n$
$S_i + D_i \leqslant D_{\max}$	$i = 1, 2, 3, \ldots n$
$S_a + D_a \leqslant S_b$	for each precedence $a \to b$
$C_i \geqslant M_{ij} D_i + B_{ij}$	$i = 1, 2, \ldots n, j = 1, 2, \ldots O_i$
$C_i \geqslant C_i^{\min}$	$i = 1, 2, 3, \ldots n.$
$D_i \geqslant D_i^{\min}$	$i = 1, 2, 3, \ldots n$

where

B_{ij}: The intercept of cost (Y-axis) for option j, activity i
C_i: Cost of activity i
$C_i^{\min}$: The minimum cost of the activity i
D_i: Duration of activity i
$D_{\max}$: The maximum allowable overall project duration
$D_i^{\min}$: The minimum duration of the activity i
M_{ij}: Slope of inequality constraint connecting adjacent active options pair j, activity i (see Figure 5).
n: The total number of the activities
O_i: Number of inequalities in time-cost constraint set for activity i
S_i: Start time of activity i

The objective function seeks to minimize the overall project cost subject to a set of constraints. The first three

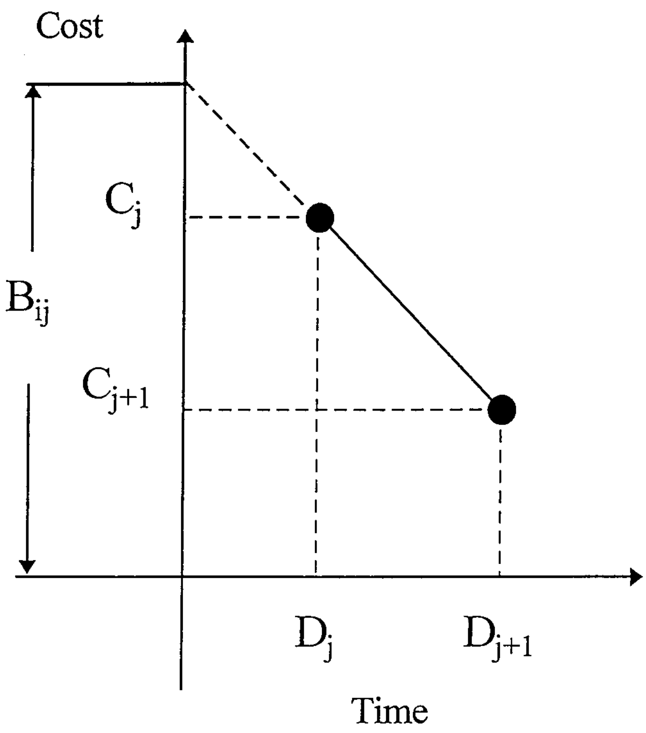

Figure 5 M_{ij} and B_{ij} of option *j*, activity *i*; $M_{ij} = \dfrac{C_{j+1} - C_j}{D_{j+1} - D_j}$

constraints are the precedence relationships of the network. The next three constraints form the convex hull of each activity. These six constraints model the time-cost trade-off problem.

Linear programming algorithms, such as the simplex method, can then be used to find the optimal solutions. Many commercially available LP software packages, such as Lindo, can perform the task very efficiently. The convex hull method, in conjunction with LP, establishes the lower bounds for time-cost relationships of a project. These lower bounds give construction planners a general idea of the project time-cost relationship. From these lower bounds, construction planners can select desired durations and find the exact solutions using IP as described later.

Discussion of the elimination in convex-hull algorithm

In the preprocessing of the convex hull algorithm, some options within activities are eliminated. Consider two consecutive options on the convex hull, A and B. These two options form the areas called zone 1 and zone 2, as shown in Figure 6. The options in zone 2, such as option D, are not suitable options because either A or B (or both) are cheaper and faster than option D and thus provide a better choice than D. Hence, it is reasonable to eliminate all options in zone 2. On the other hand, option C might be selected because neither options A nor B are both cheaper and faster than option C. In the LP model, an option along line segment AB will be selected instead

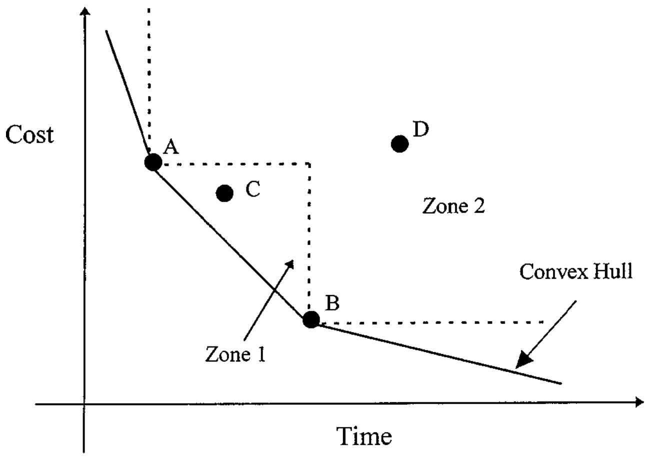

Figure 6 The elimination zones of the convex-hull algorithm

of option C. To sum up, options in zone 2 will never be selected in the exact solution and can safely be ignored. Options in zone 1 may be selected in the exact solution and are approximated in the LP formulation by line segment AB.

The integer programming model

To get an exact solution for the time-cost trade-off problem, IP must be used. Although it is computationally intensive and typically prohibitive for solving the entire problem, it often can be used to find one point on the time-cost trade-off curve once the convex hull algorithm has identified an approximated solution.

In the IP model (Patterson and Huber, 1974), the variable X_{ij} is assigned to the option *j* within activity *i*. For each activity *i*, the sum of the variables X_{ij} should be 1 because only one option will be selected as the optimal option for the activity *i*. This model can be described as the following integer program.

Objective function:

$$\text{Minimize} \quad \sum_{i=1}^{n} C_i$$

Constraints:

Subject to

$$S_i \geq 0 \qquad i = 1, 2, 3.........n$$
$$S_i + D_i \leq D_{\max} \qquad i = 1, 2, 3.........n$$
$$S_a + D_a \leq S_b \qquad \text{for each precedence } a \to b$$

$$\sum_{j=1}^{O_i} C_{ij} X_{ij} = C_i \text{ for all activity i}$$

$$\sum_{j-1}^{O_i} D_{ij} X_{ij} = \mathrm{D_i} \text{ for all activity } i$$

$$\sum_{i=1}^{O_i} X_{ij} = 1 \text{ for all activity } i$$

$$\forall X_{ij} \geqslant 0$$
$$\forall X_{ij} \leqslant 1$$
$$\forall X_{ij} \text{ are integer}$$

where

- C_i: Cost of activity i
- C_{ij}: The cost of option j, activity i
- D_i: Duration of activity i
- D_{ij}: The duration of option j, activity i
- n: The total number of the activities
- O_i: Number of inequalities in time-cost constraint set for activity i
- S_i: Start time of activity i
- X_{ij}: The assigned variable for option j, activity i

The first three constraints are the constraints of the network to set up the precedence relationships between the activities. The remaining constraints are used to make sure that only one option will be selected as the optimal option. If the relationships of the options within an activity are both linear and continuous, the points on the time cost-curve can be regarded as discrete points and each discrete point will be assigned an integer variable X_{ij} to generate the constraints for the IP model. Once the objective function and constraints are formulated, commercially available IP software can be used to find the optimal solutions.

Example of formulation

An example is used to demonstrate the formulations of the LP/IP hybrid method developed by the authors.

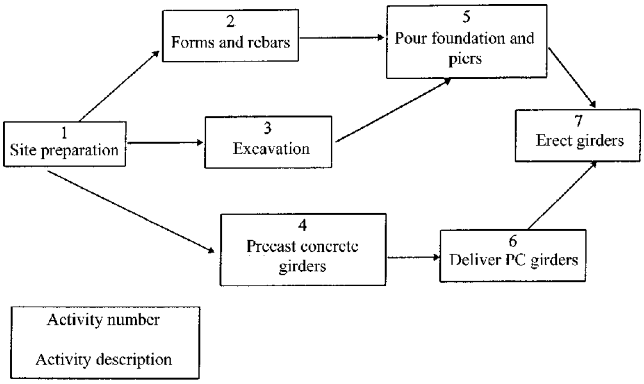

Figure 7 The network of example 1

Figure 7 shows an example construction project with seven activities (example 1). Table 1 shows the available options and their time and costs to complete the activities. The slope relationship column in Table 1 specifies whether an option is discrete, continuous, or any combination. For example, the time-cost relationship for activity 2 is shown graphically in Figure 8. This means that acitivity 2 can be completed 1) anywhere between 15 and 18 days, 2) in 20 days exactly, or 3) between 23 and 25 days. The objective function and the constraints for linear and integer programming can be formulated as follows.

The linear programming formulation

Objective function:

$$\text{Minimize} \quad \sum_{i=1}^{7} C_i$$

Constraints:

Subject to

$$S_1 + D_1 \leqslant S_2$$
$$S_1 + D_1 \leqslant S_3$$
$$S_1 + D_1 \leqslant S_4$$
$$S_2 + D_2 \leqslant S_5$$
$$S_3 + D_3 \leqslant S_5$$
$$S_4 + D_4 \leqslant S_6$$
$$S_5 + D_5 \leqslant S_7$$
$$S_6 + D_6 \leqslant S_7$$
$$S_7 + D_7 \leqslant D_d$$
$$\forall S_i \geqslant 0$$
$$i = 1, 2, \ldots, 7$$
$$C_1 \geqslant -1100\,D_1 + 38\,400$$
$$C_1 \geqslant 12\,000$$
$$D_1 \geqslant 14$$
$$C_2 \geqslant -240\,D_2 + 6600$$
$$C_2 \geqslant -160\,D_2 + 5000$$
$$C_2 \geqslant 1000$$
$$D_2 \geqslant 15$$
$$C_3 \geqslant -71.429\,D_3 + 5571.44$$
$$C_3 \geqslant -72.727\,D_3 + 5599.99$$
$$C_3 \geqslant 3200$$
$$D_3 \geqslant 15$$
$$C_4 \geqslant -2500\,D_4 + 75\,000$$
$$C_4 \geqslant -1250\,\mathrm{D}_4 + 55\,000$$
$$C_4 \geqslant 30\,000$$
$$D_4 \geqslant 12$$
$$C_5 \geqslant -1250\,\mathrm{D}_5 + 47\,500$$
$$C_5 \geqslant 10\,000$$
$$D_5 \geqslant 22$$
$$C_6 \geqslant -2200\,D_6 + 70\,800$$
$$C_6 \geqslant 18\,000$$
$$D_6 \geqslant 14$$
$$C_7 \geqslant -1000\,\mathrm{D}_7 + 39\,000$$

Table 1 The activity options of example 1 – 'Optimum' data input for the example network

Activity description (A)	Activity number (B)	Option (C)	Duration (D)	Cost (E)	Slope relationship (F)
Site preparation	1	Crew 1 + Equip 1	14	23 000	D
	1	Crew 2 + Equip 2	20	18 000	D
	1	Crew 3 + Equip 3	24	12 000	D
Forms and rebar	2	Method 1	15	3000	C
	2	Method 2	18	2400	
	2	Method 3	20	1800	D
	2	Method 4	23	1500	C
	2	Method 5	25	1000	
Excavation	3	Equipment 1	15	4500	D
	3	Equipment 2	22	4000	D
	3	Equipment 3	33	3200	D
Precast concrete girder	4	Method 1	12	45 000	D
	4	Method 2	16	35 000	D
	4	Method 3	20	30 000	D
Pour foundation and piers	5	Method 1	22	20 000	D
	5	Method 2	24	17 500	D
	5	Method 3	28	15 000	D
	5	Method 4	30	10 000	D
Deliver PC girders	6	Railroad	14	40 000	D
	6	Truck	18	32 000	D
	6	Barge	24	18 000	D
Erect girders	7	Crane 1 + Crew 1	9	30 000	D
	7	Crane 2 + Crew 2	15	24 000	D
	7	Crane 3 + Crew 3	18	22 000	D

$C_7 \geqslant -666.67\, D_7 + 34\,000.05$
$C_7 \geqslant 22\,000$
$D_7 \geqslant 9$
D_d: A desired project duration or the duration of the critical path

The first ten constraints describe the network relationship and the other constraints are related to the activity options. In activity 1, the option with duration of 20 and cost of \$18 000 is eliminated because it is above the convex hull. In activity 2, the options between durations of 1) 14–18 days and 2) 23 and 24 days are eliminated because they are also above the convex hull. The other activity's options are inspected by the above method. The above constraints are used to solve the example network by LP.

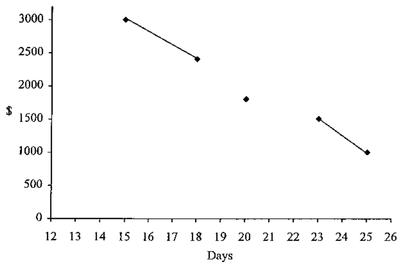

Figure 8 The time-cost relationship of activity 2

The integer programming formulation

Objective function:

Minimize $\sum_{i=1}^{7} C_i$

Constraints:
Subject to

$S_1 + D_1 \leqslant S_2$
$S_1 + D_1 \leqslant S_3$
$S_1 + D_1 \leqslant S_4$
$S_2 + D_2 \leqslant S_5$
$S_3 + D_3 \leqslant S_5$
$S_4 + D_4 \leqslant S_6$
$S_5 + D_5 \leqslant S_7$
$S_6 + D_6 \leqslant S_7$
$S_7 + D_7 \leqslant D_d$
$\forall S_i \leqslant 0$
$i = 1, 2, \ldots, 7$
$C_1 = 23\,000X_{11} + 18\,000X_{12} + 12\,000X_{13}$
$D_1 = 14X_{11} + 20X_{12} + 24X_{13}$

$$\sum_{j=1}^{3} X_{1j} = 1$$

$C_2 = 3000X_{21} + 2800X_{22} + 2600X_{23} + 2400X_{24} + 1800X_{25} + 1500X_{26} + 1250X_{27} + 1000X_{28}$
$D_2 + 15X_{21} + 16X_{22} + 17X_{23} + 18X_{24} + 20X_{25}$

$$+ 23X_{26} + 24X_{27} + 25X_{28}$$

$$\sum_{j=1}^{8} X_{2j} = 1$$

$$C_3 = 4500X_{31} + 4000X_{32} + 3200X_{33}$$
$$D_3 = 15X_{31} + 22X_{32} + 22X_{33}$$

$$\sum_{j=1}^{3} X_{3j} = 1$$

$$C_4 = 45\,000X_{41} + 35\,000X_{42} + 30\,000X_{43}$$
$$D_3 = 12X_{41} + 16X_{42} + 20X_{43}$$

$$\sum_{j=1}^{3} X_{4j} = 1$$

$$C_5 = 20\,000X_{51} + 17\,500X_{52} + 15\,000X_{53} + 10\,000X_{54}$$
$$D_5 = 22X_{51} + 24X_{52} + 28X_{53} + 30X_{54}$$

$$\sum_{j=1}^{4} X_{5j} = 1$$

$$C_6 = 40\,000X_{61} + 32\,000X_{62} + 18\,000X_{63}$$
$$D_3 = 14X_{61} + 18X_{62} + 24X_{63}$$

$$\sum_{j=1}^{3} X_{6j} = 1$$

$$C_7 = 30\,000X_{71} + 24\,000X_{72} + 22\,000X_{73}$$
$$D_7 = 9X_{71} + 15X_{72} + 18X_{73}$$

$$\sum_{j=1}^{3} X_{7j} = 1$$

$\forall X_{ij} \geqslant 0$
$\forall X_{ij} \leqslant 1$
$\forall X_{ij}$ are integer
D_d: A desired project duration or the duration of the critical path

In the IP model, the variables X_{ij} are assigned to each option within all activities. In activity 2, the options with duration of 16, 17 and 24 days are created because the slope relationship of this activity are continuous between durations of 1) 15 and 18 days, and 2) 23 and 25 days, respectively.

Computer implementation – the optimum system

The LP/IP hybrid method provides an efficient means of solving construction time-cost trade-off problems; however, the process of formulating constraints and objective functions for linear and integer programming is time-consuming and error-prone. A prototype program, 'Optimum', was developed by the authors using macros and templates of Microsoft® Excel 5.0 for Windows, a popular commercial spreadsheet program. The development of Optimum enhances the usefulness of the LP/IP hybrid algorithm by providing a user-friendly template for data entry. Users can enter time and cost relationships for activity options (either continuous or discrete) and network precedence relationships similar to defining CPM networks. Optimum then automatically formulates objective functions and constraints for linear and integer programming. Graphics are used to show the results of time-cost analysis. Users can perform what-if analysis by entering the desired duration to find the optimal strategies which minimize cost. The following paragraphs show how to use Optimum to solve the example network discussed earlier.

The process of using Optimum includes: 1) entering data for network and activity options, 2) finding the LP solutions (lower bounds), 3) setting the desired duration, 4) using IP to find the exact solutions, and 5) plotting the total cost curve.

The input form

1) Data input

The data required to run Optimum are shown in Tables 1 and 2. Column A, activity description, of the input worksheet is used to identify activities of a project. This description is needed only once for each activity. Column B, activity number, is the activity index. The activity numbers are repeated for different options in the same activity. The order of this index is not important; however, this index is required for each option, and cannot be avoided. Column C is option. This column allows the user to label each option which will later be used to identify which options are optimal. Columns D and E are the duration and cost for each option, respectively. Column F indicates the slope relationship of two successive options. The letter C in this column implies the options between the current option and the next successive option in the input form are continuous, and the letter D indicates that the option is discrete.

Table 2 'Optimum' data input for the example network

Cost unit (G)	Duration unit (H)	(I)	Predecessor (J)	Successor (K)	(L)	Increment (M)	Indirect cost rate (N)
$	Days		1	2		1	1000
			1	3			
			1	4			
			2	5			
			3	5			
			4	6			
			5	7			
			6	7			

Columns G and H are the units of cost and duration, respectively. The user can customize the units of cost and duration for their schedules. They could be $, $1000, minutes, days, or months depending upon the user's needs.

Columns J and K are used to define the precedence relationships among activities. In this example, activity 5 cannot start until activity 2 and activity 3 are completed. Column M is the increment of duration for the LP solution. For example, 1 means that the LP solver will solve the problem every single day between the longest and the shortest feasible duration and plot the lower bound. Column N is the indirect cost rate which indicates the indirect cost per time unit.

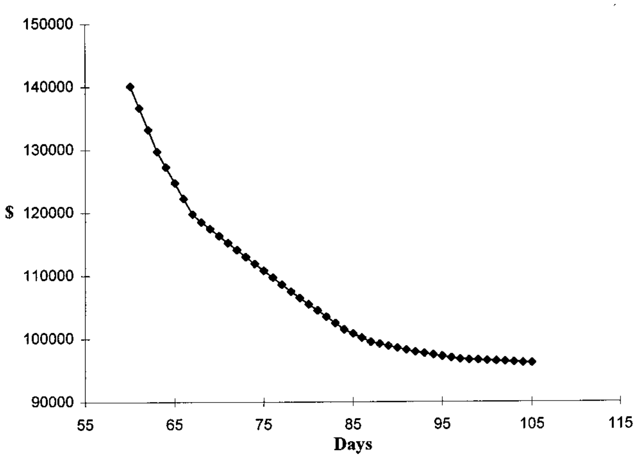

Figure 9 Direct cost curve (LP approach) (convex hull algorithm)

2) Find the LP solutions

The LP solutions provide the lower bounds of project time-cost curve. Figure 9 shows the chart of the direct cost curve generated by LP which required 5 min to complete.

3) Set the desired duration

The user may choose a desired region on the time-cost curve generated by the previous step to find the exact

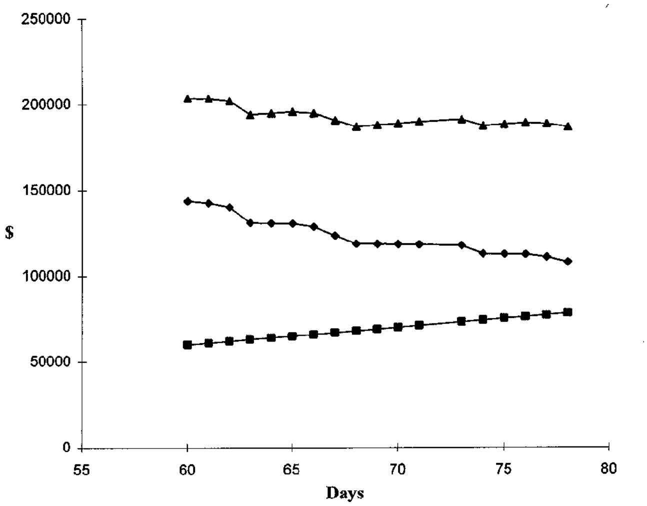

Figure 10 The total cost curve (IP approach). ◆ Direct cost; ■ Indirect cost; ▲ Total cost

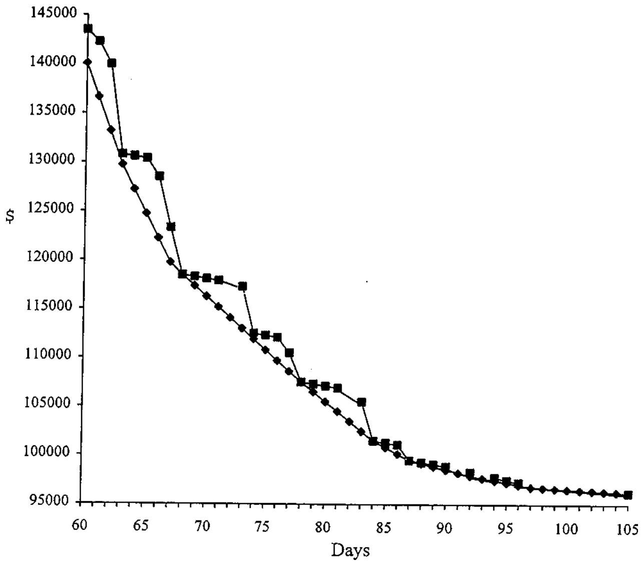

Figure 11 The comparison of convex hull algorithm (◆) and exact solution (■)

Table 3 The result of desired duration

Activity number (B)	Option (C)	Start (D)	Duration (E)	Cost (F)	End (G)	Total cost (H)	Total duration (I)
1	Crew 3 + Equip 3	0	24	12 000	24	107 500	78
2	Method 1	24	15	3000	39		
3	Equipment 1	24	15	4500	39		Desired duration
4	Method 3	24	20	30 000	44		78
5	Method 4	39	30	10 000	69		
6	Barge	44	24	18 000	68		
7	Crane 1 + Crew 1	69	9	30 000	78		

solution. In Table 3, a duration of 78 is entered to solve the optimal selections for each activity.

4) Use IP to solve the exact solutions

The exact solutions can be generated by IP. As mentioned earlier, it provides the exact solutions; however, it takes much longer than LP. In this simple example, it takes approximately 30 min to find all exact solutions on a 486 PC, but it only takes about 1 min to find one exact solution. Table 4 shows the results of the IP. The direct cost column shows the time-cost relationship for direct costs. The indirect cost in this example is assumed to be $1000/day for the duration of the project.

5) Plot the total cost curve

The total cost curve can also be generated by Optimum if the user has defined the indirect cost rate. The total cost curve of the described example is shown in Figure 10.

The efficiency of LP/IP hybrid algorithm

Comparing the solutions obtained by convex hull algorithm with the exact solutions, the convex hull solutions bound the exact solution from below, as shown in Figure 11. The time required to solve the entire time-cost trade-off curve based on the LP/IP algorithm is 5 min using Optimum on a 486 PC. The solution by IP requires 30 min. Another example (example 2, see Figures 12, 13 and Table 5 for network relationship, time-cost trade-off curve, and activity options, respectively) with 18 activities emphasizes the time savings provided by the proposed method. The time required to obtain the entire time-cost trade-off curve by the LP/IP algorithm is approximately 15 min, in contrast to 17 h required by the IP model. It is obvious that the proposed algorithm has the advantage of saving time and computing effort. The convex hull algorithm gives very close solutions to the exact solutions as shown in Figure 13. It also provides information about the project cost and a detailed schedule of the desired project duration. Therefore, the LP/IP hybrid approach provides an efficient way to obtain an optimal solution to the time-cost trade-off problem.

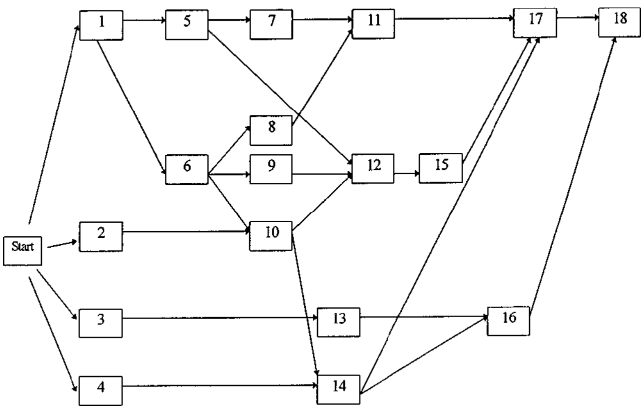

Figure 12 The network of example 2. Numbers given are activity ID

Future development

Future developments of Optimum focuses on three areas: 1) develop an interface to link existing project schedule applications, such as Microsoft Project and

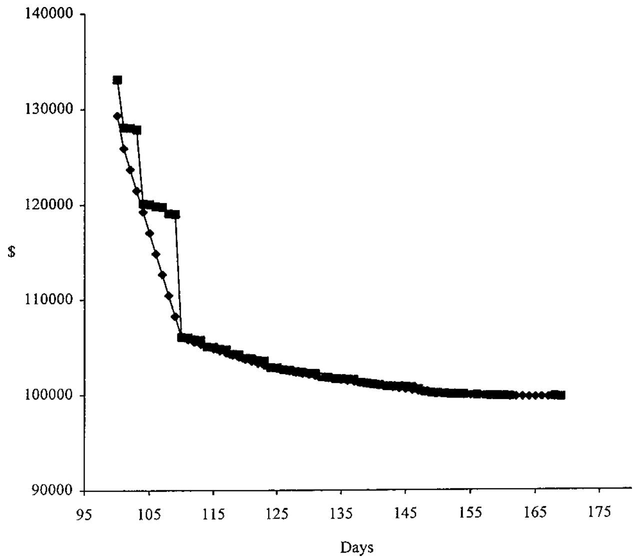

Figure 13 The comparison of convex hull algorithm (◆) and exact solution (■) of example 2

Table 4 The solutions of the example network (IP approach)

Duration (A)	Direct cost (B)	Total cost (C)	Duration (D)	Direct cost (E)	Total cost (F)
60	143 500	203 500	79	107 300	186 300
61	142 300	203 300	80	107 100	187 100
62	140 000	202 000	81	106 900	187 900
63	130 800	193 800	83	105 500	188 500
64	130 600	194 600	84	101 500	185 500
65	130 400	195 400	85	101 300	186 300
66	128 500	194 500	86	101 100	187 100
67	123 300	190 300	87	99 500	186 500
68	118 500	186 500	88	99 300	187 300
69	118 300	187 300	89	99 100	188 100
70	118 100	188 100	90	98 900	188 900
71	117 900	188 900	92	98 300	190 300
73	117 300	190 300	94	97 800	191 800
74	112 500	186 500	95	97 500	192 500
75	112 300	187 300	96	97 250	193 250
76	112 100	188 100	97	97 000	194 000
77	110 500	187 500	105	96 200	201 200
78	107 500	185 500			

Table 5 The activity options of example 2

AD	A	Option	D	C	SR	AD	A	Option	D	C	SR
A1	1	A1O1	14	2200	D		9	A9O3	20	180	D
	1	A1O2	15	2150	D		9	A9O4	23	150	C
	1	A1O3	16	2000	D		9	A9O5	25	100	D
	1	A1O4	21	2080	C	A10	10	A10O1	15	450	D
	1	A1O4	24	1200	D		10	A10O2	22	400	D
A2	2	A2O1	15	3000	C		10	A10O3	33	320	D
	2	A2O2	18	2400		A11	11	A11O1	12	450	D
	2	A2O3	20	1800	D		11	A11O2	16	350	D
	2	A2O4	23	1500	C		11	A11O3	20	300	D
	2	A2O5	25	1000	D	A12	12	A12O1	22	2000	D
A3	3	A3O1	15	4500	D		12	A12O2	24	1750	C
	3	A3O2	22	4000	D		12	A12O3	28	1500	D
	3	A3O3	33	3200	D		12	A12O4	30	1000	D
A4	4	A4O1	12	45 000	D	A13	13	A13O1	14	4000	D
	4	A4O2	16	35 000	D		13	A13O2	18	3200	D
	4	A4O3	20	30 000	D		13	A13O3	24	1800	D
A5	5	A5O1	22	20 000	D	A14	14	A14O1	9	3000	C
	5	A5O2	24	17 500	C		14	A14O2	15	2400	D
	5	A5O3	28	15 000	D		14	A14O3	18	2200	D
	5	A5O4	30	10 000	D	A15	15	A15O1	12	4500	D
A6	6	A6O1	14	40 000	D		15	A15O2	16	3500	D
	6	A6O2	18	32 000	D	A16	16	A16O1	20	3000	D
	6	A6O3	24	18 000	D		16	A16O2	22	2000	D
A7	7	A7O1	9	30 000	C		16	A16O3	24	1750	C
	7	A7O2	15	24 000	D		16	A16O4	28	1500	D
	7	A7O3	18	22 000	D		16	A16O5	30	1000	D
A8	8	A8O1	14	220	D	A17	17	A17O1	14	4000	D
	8	A8O2	15	215	D		17	A17O2	18	3200	D
	8	A8O3	16	200	D		17	A17O3	24	1800	D
	8	A8O4	21	208	D	A18	18	A18O1	9	3000	D
	8	A8O5	24	120	D		18	A18O2	15	2400	C
A9	9	A9O1	15	300	C		18	A18O3	18	2200	D
	9	A9O2	18	240							

AD: Activity description; A: Activity; D: Duration; C: Cost; SR: Slope relationship.

Primavera, with the Optimum input worksheet. This is so that schedule activity description, activity ID, duration, cost and precedence relationships can be directly imported into Optimum; 2) develop automatic generation of options and their costs and durations from historical databases or other commercially available cost data, such as RS Means; and 3) develop built-in (customized) linear/integer programming solvers to replace the less efficient general purpose solvers.

Conclusion

The time-cost trade-off problem for construction project scheduling has been investigated since the 1960s. Existing solutions can be classified into two categories, heuristic and mathematical approaches. Heuristic approaches provide good solutions, but do not guarantee optimal solutions. Mathematical approaches provide better solutions, however the process of formulating the objective function and constraints is complex and error-prone. This paper presents a new algorithm to provide an easy-to-use tool to solve time-cost trade-off problems using mathematical models. This method takes advantage of LP and the convex hull method for efficiency, and IP to find the exact solutions. This hybrid method, along with the spreadsheet tool, provides the construction planner with an efficient means of analysing time-cost trade-off decisions.

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