第k小元素的期望

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不妨令x=1，则第k小元素的期望值为：

$$\int_0^1 \left(\sum_{i=0}^{k-1} \binom{n}{i} t^i (1-t)^{n-i}\right) dt$$

$$= \int_0^1 \left(\sum_{i=0}^{k-1} \binom{n}{i} (1-t)^i t^{n-i}\right) dt$$

$$= \sum_{i=0}^{k-1} \binom{n}{i} \int_0^1 (1-t)^i t^{n-i} dt$$

$$= \sum_{i=0}^{k-1} \binom{n}{i} \int_0^1 \sum_{j=0}^{i} \binom{i}{j} (-t)^j\, t^{n-i} dt$$

$$= \sum_{i=0}^{k-1} \binom{n}{i} \sum_{j=0}^{i} \binom{i}{j} (-1)^j \int_0^1 t^{n-i+j} dt$$

$$= \sum_{0\le j\le i<k} \binom{n}{i} \binom{i}{j} \frac{(-1)^j}{n-i+j+1}$$

$$= \sum_{0\le j\le i<k} \binom{n}{i} \binom{i}{i-j} \frac{(-1)^j}{n-i+j+1}$$

$$= \sum_{0\le j\le i<k} \binom{n}{i-j} \binom{n-(i-j)}{i-(i-j)} \frac{(-1)^j}{n-i+j+1}$$

- 设$d = i - j$：

$$= \sum_{0\le d\le k} \binom{n}{d} \frac{\sum_{i=d}^{k-1} \binom{n-d}{i-d} (-1)^{i-d}}{n-d+1}$$

$$= \sum_{0\le d\le k} \frac{\binom{n}{d}}{n-d+1} \sum_{i=0}^{k-d-1} \binom{n-d}{i} (-1)^i$$

- 设$s = n - d, m = n - k + 1$

$$= \sum_{m\le s\le n} \frac{\binom{n}{s}}{s+1} \sum_{i=0}^{s-m} \binom{s}{i} (-1)^i$$

- 该式记为$f(m)$。

要证$f(n-k+1) = \frac{k}{n+1}$，差分后转化为$f(m) - f(m+1) = \frac{1}{n+1}$。

而不难发现$f(m) - f(m+1) = \sum_{m\le s\le n} \frac{\binom{n}{s}\binom{s}{m}(-1)^{s-m}}{s+1}$。

转化为证明$g(m) = \sum_{m\le s\le n} \frac{\binom{n}{s}\binom{s}{m}(-1)^{s-m}}{s+1} = \frac{1}{n+1}$，其中$m < n$。

再对g差分：$g(m) - g(m-1)$

- $= \sum_{m\le s\le n} \frac{\binom{n}{s}\left(\binom{s}{m} - \left(-\binom{s}{m-1}\right)\right)(-1)^{s-m}}{s+1} - \frac{\binom{n}{m}}{n-m+1}$
- $= \sum_{m\le s\le n} \frac{\binom{n}{s}\binom{s+1}{m}(-1)^{s-m}}{s+1} - \frac{\binom{n}{m}}{n-m+1}$
- $= \left(\sum_{m\le s<n} \frac{\binom{n}{s+1}\binom{s+1}{m}(-1)^{s-m}}{n-s} + (-1)^{n-m}\frac{\binom{n+1}{m}}{n+1}\right) - \frac{\binom{n}{m}}{n-m+1}$
- $= \left(-\sum_{m<s\le n} \frac{\binom{n}{s}\binom{s}{m}(-1)^{s-m}}{n-s+1} + (-1)^{n-m}\frac{\binom{n}{m}}{n-m+1}\right) - \frac{\binom{n}{m}}{n-m+1}$
- $= -\sum_{m<s\le n} \frac{\binom{n}{s}\binom{s}{m}(-1)^{s-m}}{n-s+1} - \left(1+(-1)^{n-m+1}\right)\frac{\binom{n}{m}}{n-m+1}$
- $= -\sum_{m\le s\le n} \frac{\binom{n}{s}\binom{s}{m}(-1)^{s-m}\left(1+[s=m]\cdot(-1)^{n-m+1}\right)}{n-s+1} (\#1)$

要证$\#1 = 0$。

设多项式c满足$c(t) = \frac{1+[t=n-m]\cdot(-1)^{n-m+1}}{t+1} (\forall \mathbf{t} \in \mathbb{Z} \cap [\mathbf{m}, \mathbf{n}])$，则：

- $\#1 = -\sum_{m\le s\le n} \binom{n}{s}\binom{s}{m}(-1)^{s-m}c(n-s)$
- $= \binom{n}{m}\sum_{t=0}^{n-m}\binom{n-m}{t}(-1)^t c(t)$
- $= \binom{n}{m}\Delta^{n-m}c(0)$

如果$\deg c < n-m$，则该式=0。

即证：用$\left\{\left(t, \frac{1}{t+1}\right)\right\}_{t=0\ldots n-m-1}$这n-m个点拟合出的多项式$\hat{c}(t)$，符合$\hat{c}(n-m) = \frac{1+(-1)^{n-m+1}}{n-m+1}$。

记$d = n-m-1$，由拉格朗日插值：$\hat{c}(t) = \sum_{i=0}^{d}\frac{\prod_{j\ne i,0\le j\le d}\frac{t-j}{i-j}}{i+1}$，转化为证明：

- $\sum_{i=0}^{d}\frac{1}{i+1}\prod_{j\ne i,0\le j\le d}\frac{d+1-j}{i-j} = \frac{1+(-1)^d}{d+2}$
- $\sum_{i=0}^{d}\frac{d+2}{i+1}\prod_{j\ne i,0\le j\le d}\frac{d+1-j}{i-j} = 1+(-1)^d$
- $\sum_{i=0}^{d}\prod_{j\ne i,-1\le j\le d}\frac{d+1-j}{i-j} = 1+(-1)^d$
- $\sum_{i=0}^{d}\frac{\frac{(d+2)!}{(d+1-i)}}{(i+1)!\,(d-i)!\,(-1)^{d-i}} = 1+(-1)^d$
- $(-1)^d\sum_{i=0}^{d}\binom{d+2}{i+1}(-1)^i = 1+(-1)^d$
- $(-1)^d\left((1-1)^{d+2} - (-1)^{-1}\binom{d+2}{0} - (-1)^{d+1}\binom{d+2}{d+2}\right) = 1+(-1)^d$
- $(-1)^d + 1 = 1+(-1)^d$，QED。