-7-

Hackenbush

All things by immortal power,
Near or far,
Hiddenly
To each other linkéd are.
That thou canst not stir a flower
Without troubling of a star.

Francis Thompson, *The Mistress of Vision*.

In this chapter we'll tell you what we know about Hackenbush (except for infinite and loopy varieties that you'll find in Chapter 11) but first we'd better warn you that the arguments are rather long. For those who are eager to skip on, some of the remarks about flower gardens are repeated in Chapter 8, so that you won't need to read this chapter to understand anything else in the book.

Green Hackenbush

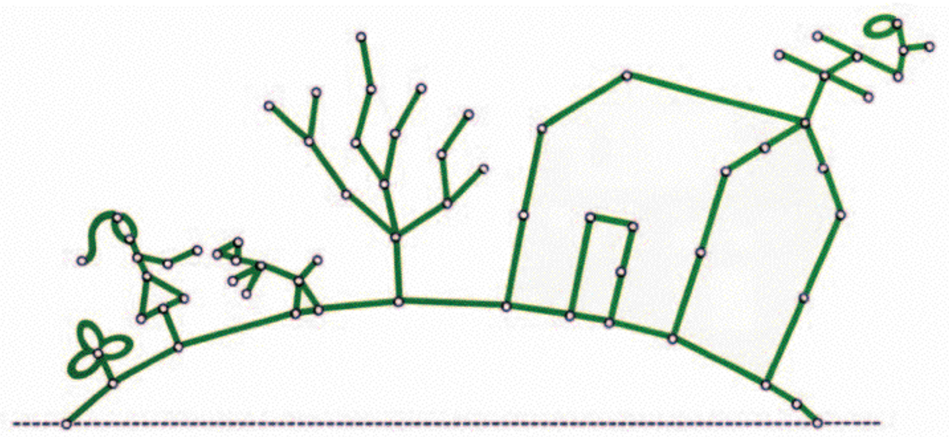
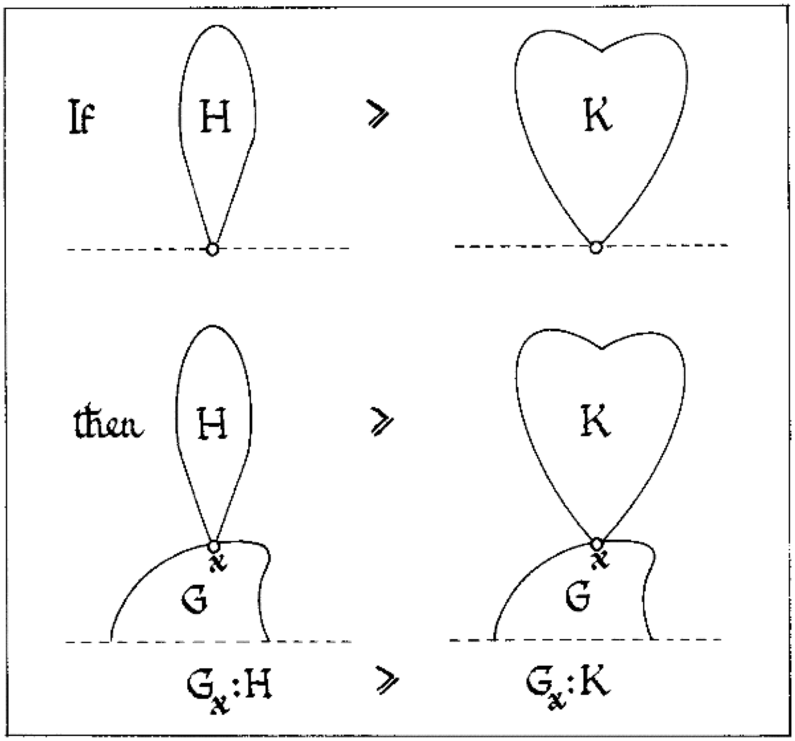


Figure 1. A Green Hackenbush Bridge.

In a totally grEen Hackenbush picture such as Fig. 1, any edge may be chopped by Either player, after which any edges no longer connected to the ground disappear.

Here there's a complete theory. First we observe that the Snakes-in-the-Grass argument of Chapter 2 shows that Hackenbush pictures made only of green strings are directly equivalent to Nim.

Next we use a very important tool, applicable not only in Green Hackenbush, but in Hackenbush more generally, called the **Colon Principle.**



THE COLON PRINCIPLE

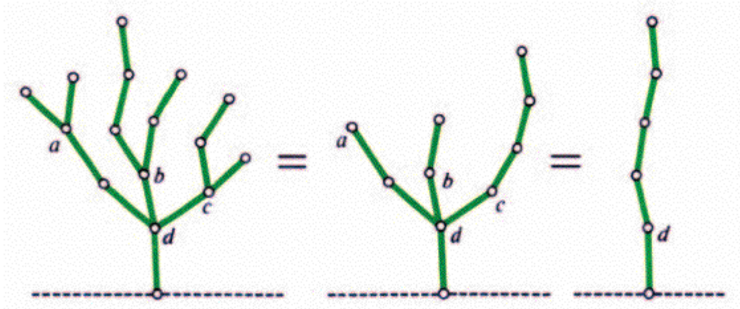
You can easily prove this by playing the difference of the two lower games. In particular,

$$\boxed{\text{if } H = K, \text{ then } G_x : H = G_x : K}$$

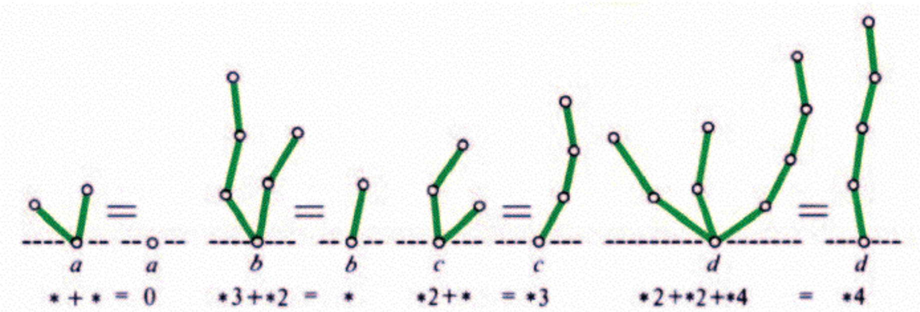
For a formal definition of $G : H$ see the Extras.

Green Trees

Green trees can now be evaluated using only the Colon Principle. For example, the tree



has value $*5$ because



You can see that the Colon Principle often allows you to do your additions at some distance above the ground.

Observe that two kinds of addition are needed here. When moving down a branch towards the ground the nim-value is increased by adding 1 in the schoolbook way, $+1$, but when several branches join at a node, their values are added in the nim way, $\overset{*}{+}$. But because *both* types of addition have the properties

$$\text{odd plus odd} = \text{even plus even} = \text{even},$$
$$\text{odd plus even} = \text{even plus odd} = \text{odd},$$

you can see that

The nim-value of any sum of green trees has the same parity as the total number of edges.

THE PARITY PRINCIPLE

The Fusion Principle below will show that this extends to all Green Hackenbush pictures.

Fusion

You **fuse** two nodes of a picture by bringing them together into a single one. Any edge joining them gets bent into a loop at the resulting node, If you fuse x and y in Fig. 2(a) you get Fig. 2(b); if you fused x and z, you'd get Fig. 2(c).

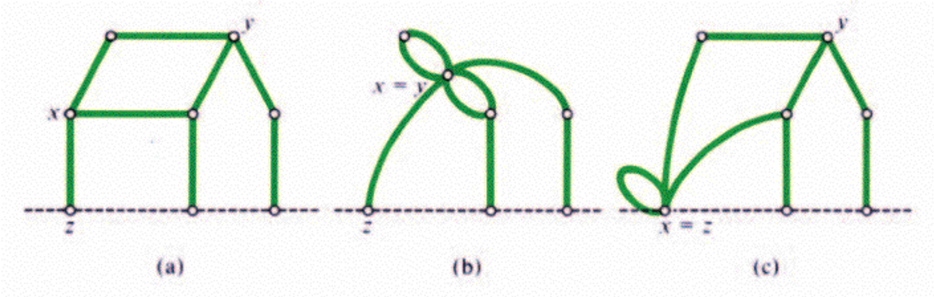


Figure 2. Fusion to your house!

Green Hackenbush is completely solved by

THE FUSION PRINCIPLE:

> you can fuse all the nodes in
> any cycle of a Green Hackenbush
> picture without changing its value,

and the fact that a loop at any node has the same effect as a twig there. For example the girl

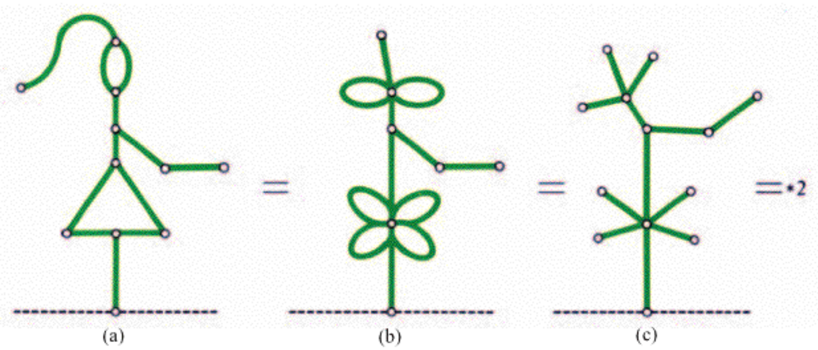


Figure 3. Sizing Up a Green Girl.

of Fig. 3(a) becomes the green shrub of Fig. 3(b) when we've fused the four nodes of her skirt and the two nodes of her head, and this becomes a tree (Fig. 3(c)) on replacing the leaves by twigs. The Colon Principle then shows the tree, and therefore the girl, to have value $*2$.

Proving The Fusion Principle

It will take us quite a long time to prove this principle. An alternative, but equally long, proof using mating functions and the Welter function (see Chapter 15) will be found in ONAG. The proof here has the advantage of explicitly constructing the winning move. We'll omit some purely arithmetic computations which are needed for the proof, but not to find the winning move.

If there's any counter-example to the Principle, choose one with the smallest number n of edges, and among counter-examples with n edges, choose one, G, say, with the smallest number of nodes (so there can be no legal fusion of any two nodes of G).

First, *G can only have one ground node* since it never affects play to fuse all ground nodes.

Next, *G can contain no pair of nodes a, b, connected by three or more edge-disjoint paths* for otherwise the game H, obtained by fusing a and b, would have to have a different nim-value and so there would be a winning move in $G + H$. Whichever of G and H this move is in, respond with the corresponding move in the other, reaching a game $G' + H'$. But since G' and H' have at most $n - 1$ edges, we can fuse any cycles in them to single points without affecting their values and because there is still a cycle containing both a and b, we see that $G' + H' = 0$, dismissing the supposed winning move from $G + H$.

No cycle of G can exclude the ground for if G had such a cycle C, consider the position G' which would remain after Hackenbush moves chopping all edges of C. Then G' can't contain two distinct nodes of C, for these would be connected, in G, by three edge-disjoint paths (two in C and one in G'). So G' contains only one node, x, of C, and G looks like Fig. 4(a). Now if x were the ground we could apply the Fusion Principle to fuse all nodes of the smaller graph (Fig. 4(b)) and the Colon Principle allows us to fuse these nodes *off* the ground.

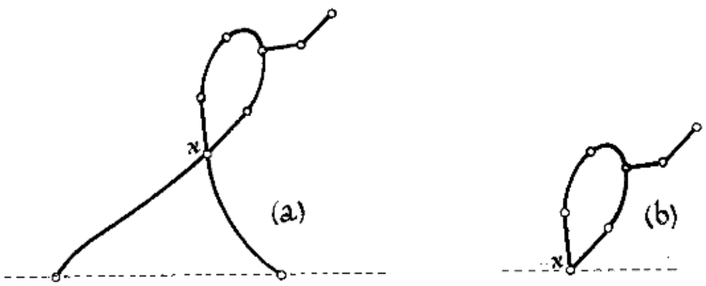


Figure 4. Pulling Cycles to the Ground.

Finally, *G contains only one cycle which includes the ground* for otherwise it would be the sum of smaller graphs, since nodes from distinct cycles can't be joined by other paths. But we could now apply the Fusion Priciple to the smaller graphs.

We can now see that G must look like a bridge (Fig. 5, though officially we should identify the two ground nodes) in which, by the Colon Principle, we can suppose that the edges not in the bridge form at most one string at each node.

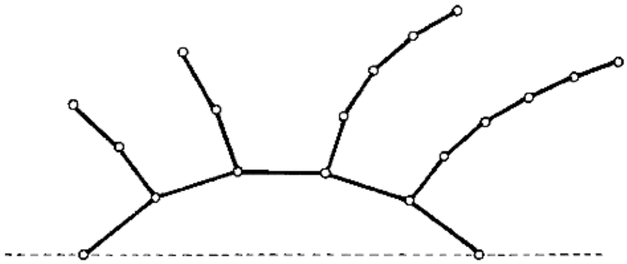


Figure 5. What a Minimal Criminal Looks Like.

The number of edges in the bridge (its **span-length**) *is odd.* If a bridge has an even span-length, consider the sum (Fig. 6) of this bridge with copies of all of its strings. Removing any edge of the bridge in this is bad, because the resulting nim-value is odd by the Parity Principle. A symmetry strategy therefore shows that Fig. 6 has value 0 and the Fusion Principle applies.

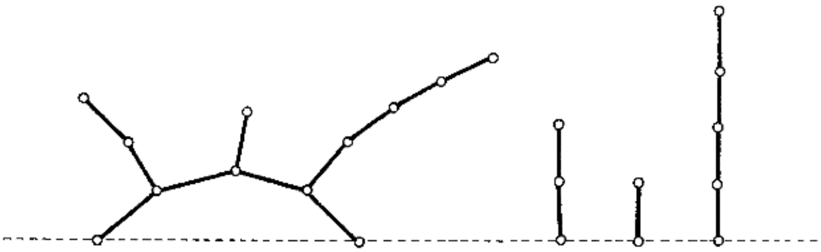


Figure 6. An Even Span Bridge with Copies of its Strings.

The Fusion Principle for a bridge of *odd* span-length asserts that its value is found by adding $*$ to the sum of its strings. So we must show that Fig. 7 has value $*$.

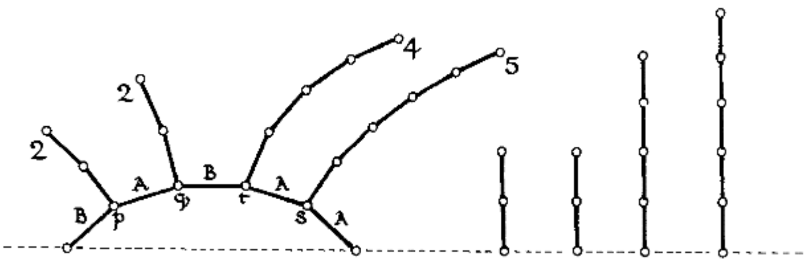


Figure 7. An Odd Span Bridge with Copies of its Strings.

Certainly *no option has value* $*$, because moves in the bridge lead to even nim-values by the Parity Principle, and moves in the strings can be reversed to $*$ by responding with their images (after which the Fusion Principle will apply to the smaller picture.

It will therefore suffice to find an option of value 0. To do this, label the bridge edges with A or B, giving adjacent edges the *same* label if there is an *odd* string between them, and different lables if there's an *even* string between. The edges with the (*Devil's*) label which occurs an *even* number of times (B occurs twice in Fig. 7) are bad moves since each of them can be seen to lead to a sum of two trees and several strings where the nim-value of the sum is congruent to 2, mod 4, and therefore non-zero. However any of the (*odd* number of) edges with the other (*Godd's*) label leads to a sum with nim-value congruent to 0, mod 4. To find a good bridge move among these, we reduce the graph to a simpler one by shrinking any edge with a Devil's label to a single point, and halving all string-lengths (rounding *down* if they're odd). It can be shown that this reduction also halves the nim-value. Applying it to Fig. 7 leads to the simpler Fig. 8 because 2 halves to 1, 2 ‡ 4 halves to 3 and 5 halves to 2. A similar labelling splits the bridge edges into an even (Devil's) number labelled D and an odd (Godd's) number labelled C. Since in our case there is only one C edge, it is the winning move in Fig. 8, and the corresponding edge (between the 5 string and the ground) wins in Fig. 7.

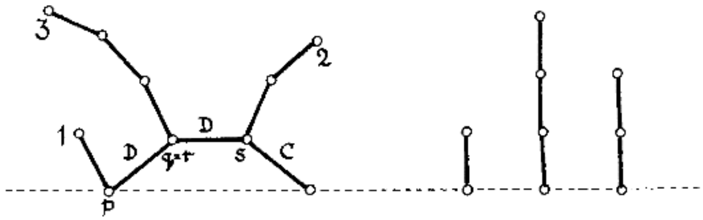


Figure 8. Half of Figure 7.

A More Complicated Picture

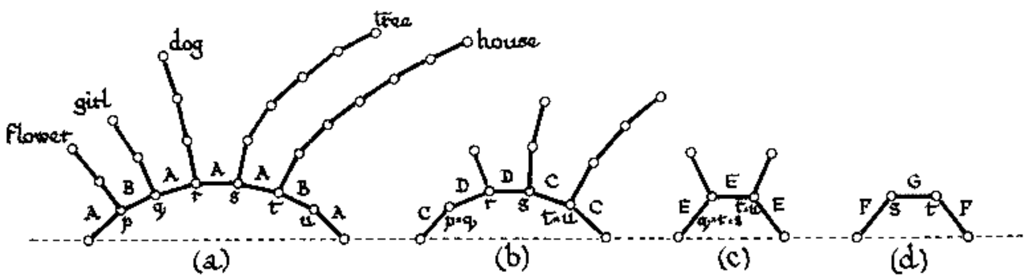


Figure 9. Simplifying and Halving Figure 1.

We'll find a winning move in our opening picture, Fig. 1. When we fuse the cycles contained in or under the girl, dog and house, and evaluate the various pieces we get Fig. 9(a). The halving process leads successively to Figs. 9(b), 9(c) and 9(d). The only good move in this last is the centre span of the reduced bridge. This corresponds to the edge between the tree and the house in Fig. 1.

Since edges on grounded cycles tend to split the picture up too quickly, the reader who wishes to bamboozle his opponents will verify that that there are 17 other good moves in Fig. 1: the bird's tail, the top left branch of the T.V. antenna, any of the four pieces of foundation under the house, the lowest twig on the (right of the) tree, the dog's tail, his face, either hind leg, either part of the girl's head and any of the four parts of her skirt.

Green Hackenbush can be applied to the theory of

Impartial Maundy Cake

Impartial Maundy Cake which is played like ordinary Maundy Cake (Chapter 2) except that either player may divide the cake in either direction. Since the game is impartial, any even number of identical cakes cancel, while an odd number have the same value as a single cake.

If α and β are the numbers of *odd* prime divisors of a and b, counted with mutiplicities, we shall say that an a by b cake has type

$$D(\alpha,\beta) \quad \text{or} \quad E(\alpha,\beta)$$

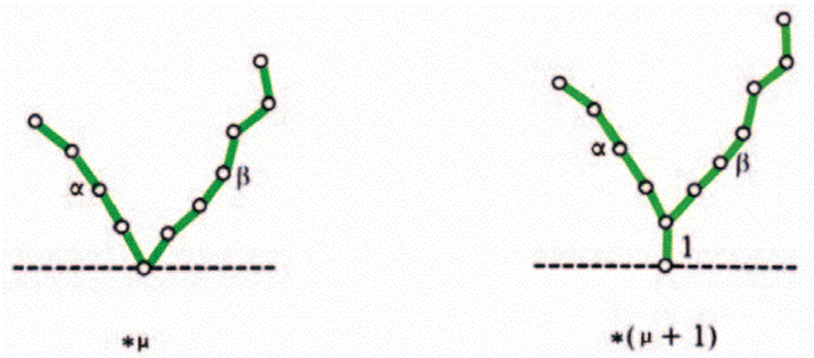
according as ab is

odd or even.

The moves that produce an odd number of cakes correspond just to reductions of α or β. However, a move that produces an even number of cakes is necessarily from a type $E(\alpha,\beta)$ cake and gives value 0, so

$$D(\alpha,\beta) \quad \text{and} \quad E(\alpha,\beta)$$

have the same values as the Green Hackenbush positions



where $\mu = \alpha \ddagger \beta$.

We shall discuss Many-Way Maundy Cake in the Extras.