Pricing of multi-asset out-performance options using Monte Carlo simulation

A multi-asset option's payoff depends on the prices of multiple risky assets. Assume that under the risk neutral probability measure Q the prices of these underlying assets are all geometric Brownian motions as governed by

$$S_t^{(i)} = S_0^{(i)} \exp\left\{\left(r - \frac{\sigma_i^2}{2}\right)t + \sigma_i W_t^{(i)}\right\}, i = 1, \cdots, d,$$

where $\mathbf{W} = (W^{(1)}, W^{(2)}, \cdots, W^{(d)})$ is a d-dimensional Brownian motion with covariance matrix $\Sigma = [\sigma_{ij}]$ such that $\sigma_{ii} = 1$ for all i. Consider an out-performance option with maturity T and terminal payoff

$$\left(\max\left\{S_T^{(1)}, \cdots, S_T^{(d)}\right\} - K\right)^+,$$

where K is the strike price. This assignment aims to use the Monte Carlo simulation method to estimate the price of this out-performance option.

The price of the multi-asset option is the expected value of the discounted terminal payoff:

$$v = E_Q\left[e^{-rT}\left(\max\left\{S_T^{(1)}, \cdots, S_T^{(d)}\right\} - K\right)^+\right].$$

In order to estimate v, we need to generate samples of $(S_T^{(1)}, \cdots, S_T^{(d)})$, or equivalently, those of $(W^{(1)}, \cdots, W^{(d)})$, which is a jointly normal random vector with mean 0 and covariance matrix $T\Sigma$.

Pseudocode:

find a matrix A such that $AA' = \Sigma$ through the Cholesky factorization for $k = 1, \cdots, d$

generate independent samples $Z_1, \cdots, Z_d$ from $N(0, 1)$

set $\mathbf{Z} = (Z_1, \cdots, Z_d)'$

set $\mathbf{Y} = A\mathbf{Z}$

for $k = 1, \cdots, d$

set $S_k = S_0^{(k)} \exp\left\{(r - \sigma_k^2/2)T + \sigma_k\sqrt{T}Y_k\right\}$

set $H_i = e^{-rT}\left(\max\left\{S_T^{(1)}, \cdots S_T^{(d)}\right\} - K\right)^+$

compute the estimate $\hat{v} = \frac{1}{n}(H_1 + \cdots + H_n)$, where n is the number of simulation runs.
compute the standard error S.E. $= \sqrt{\frac{1}{n(n-1)}\left(\sum_{i=1}^n H_i^2 - n\hat{v}^2\right)}$.

In your numerical calculation, take $d = 4$. The parameters are given by

$$S_0^{(1)} = 45, S_0^{(2)} = 50, S_0^{(3)} = 45, S_0^{(4)} = 55, r = 0.02, T = 0.5,$$

$$\sigma_1 = \sigma_2 = \sigma_3 = 0.1, \sigma_4 = 0.2, \Sigma = \begin{bmatrix} 1.0 & 0.3 & -0.2 & 0.4 \\ 0.3 & 1.0 & -0.3 & 0.1 \\ -0.2 & -0.3 & 1.0 & 0.5 \\ 0.4 & 0.1 & 0.5 & 1.0 \end{bmatrix}.$$

As a useful hint, the Cholesky factorization of Σ is found to be

$$A = \begin{bmatrix} 1.0000 & 0 & 0 & 0 \\ 0.3000 & 0.9539 & 0 & 0 \\ -0.2000 & -0.2516 & 0.9469 & 0 \\ 0.4000 & -0.0210 & 0.6069 & 0.6864 \end{bmatrix},$$

which satisfies $AA' = \Sigma$.

The following table of numerical results serve as benchmark reference.

	Sample size $n = 2500$			Sample size $n = 10000$		
Strick price K	50	55	60	50	55	60
M.C. Estimate	6.9439	3.5159	1.5406	7.0714	3.4197	1.4730
Standard Error	0.1250	0.1009	0.0711	0.0624	0.0501	0.0346

Work elements

1. Develop the subroutine to compute the Cholesky factorization of Σ. Verify the numerical result presented above in matrix A.

2. Generate the standard normal distribution using three different methods (i) inverse transform method, (ii) acceptance-rejection method, (iii) Box-Muller method. Compare the estimates based on these three methods.

3. In all subsequent calculations, fix one choice of your standard normal distribution generator that is considered to be the best. Examine the convergence of the estimates of the option price with respect to the number of sample size n. Try with $n = 2500, 5000, 10000, 20000, 40000$ and 80000. Plot the estimates of option value against $\log n$ and deduce the order of convergence with respect to n.

4. Examine the ratio of standard error to estimate (which is known as the empirical relative error) with respect to the strike price K. Try $K = 50, 55, 60, \cdots, 80$. Plot the empirical relative error with respect to the strike price.

5. Apply the antithetic variate method and examine the impact of this simple variance reduction technique in achieving better convergence.

6. Bonus: Explore the use of the other more sophisticated sampling method like stratification to achieve better variance reduction.