I. Continuous Growth Models (Differential Equations)

1. Single Species (Logistic Growth) Equation:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)$$

- **x(t):** population at time t
- **r:** intrinsic growth rate
- **K:** carrying capacity

Behavior: - S-shaped (sigmoid) curve - Starts exponential, slows as x approaches K - Stable equilibrium at x = K

Analytical Solution:

$$x(t) = \frac{K}{1 + \left(\frac{K - x_0}{x_0}\right) e^{-rt}}$$

2. Multiple Species (Coupled ODEs) Example with 3 species (Carrots C, Rabbits R, Wolves W):

$$\begin{aligned}
\frac{dC}{dt} &= r_C C\left(1 - \frac{C}{K_C}\right) - aCR \\
\frac{dR}{dt} &= r_R R\left(1 - \frac{R}{K_R}\right) + bCR - cRW \\
\frac{dW}{dt} &= r_W W\left(1 - \frac{W}{K_W}\right) + dRW
\end{aligned}$$

- Interactions between species are modeled as additional terms
- Can produce cycles, oscillations, or stable states

Visualization Tools: - Time-series plots - 2D or 3D phase plots

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II. Discrete Growth Models (Difference Equations)

1. Single Species (Logistic Map) Equation:

$$x_{n+1} = rx_n(1 - x_n)$$

- $x \in [0, 1]$ normalized
- $r \in [0, 4]$ typically

Behavior: - For $r < 3$: stable fixed point - $r \to 3.0+$: period-doubling bifurcations - $r >\sim 3.57$: chaos

Visualization: - Bifurcation diagram (r on x-axis, long-term x on y-axis)

2. Multiple Species (Coupled Difference Equations) Example:

$$
\begin{aligned}
C_{n+1} &= C_n + r_C C_n \left(1 - \frac{C_n}{K_C}\right) - aC_n R_n \\
R_{n+1} &= R_n + r_R R_n \left(1 - \frac{R_n}{K_R}\right) + bC_n R_n - cR_n W_n \\
W_{n+1} &= W_n + r_W W_n \left(1 - \frac{W_n}{K_W}\right) + dR_n W_n
\end{aligned}
$$

- Same structure as continuous model, but evaluated at discrete steps
- Requires simulation (no closed-form solution)

Behavior: - Can mimic continuous system - Sensitive to time step size and parameter tuning

Visualization: - Time-step charts - Bifurcation surfaces (higher dimensional)

III. Comparison Summary

Feature	Continuous Model	Discrete Model
Time	Continuous (real-valued t)	Discrete (integer steps n)
Math	Differential equations (ODEs)	Difference equations (maps)
Predictability (1 species)	Stable, smooth	Can be chaotic
Behavior (multi-species)	Cycles, equilibrium, oscillations	Same, but can show numerical artifacts
Analytical solutions	Often possible for 1 species	Rare, mostly simulated
Visualization	S-curves, phase space	Bifurcation diagrams, iterations
Common uses	Biology, ecology, physics	Chaos theory, computation, population

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IV. Key Takeaways

- Use **continuous models** for smooth growth and biologically accurate long-term behavior.
- Use **discrete models** when modeling generation-based or algorithmic dynamics.
- Chaos and bifurcations appear naturally in discrete logistic maps.

- Coupled systems in both continuous and discrete forms can model ecosystems with interacting species.
- Visualization is essential to understand behavior in nonlinear systems.