(1) $\log P(D)$

$= \log \prod_{i=1}^{n} P(x_i, y_i; \theta)$

$= \log \prod_{i=1}^{n} P(x_i;\theta) P(x_i|y_i;\theta)$

$= \log \prod_{i=1}^{n} \pi^{y_i}(1-\pi)^{1-y_i} + \log \prod_{i=1}^{n} (N(x_i|\mu_1,\Sigma_1))^{I\{y_i=1\}} \cdot [N(x_i|\mu_0,\Sigma_0))^{I\{y_i=0\}}$

$= \sum_{i=1}^{n} y_i \log\pi + \sum_{i=1}^{n}(1-y_i)\log(1-\pi) + \log\prod_{i=1}^{n}(N(x_i|\mu_1,\Sigma_1))^{I\{y_i=1\}} + \log\prod_{i=1}^{n}(N(x_i|\mu_0,\Sigma_0))^{I\{y_i=0\}}$

$= n_1 \log\pi + n_0 \log(1-\pi)$

$+ \sum_{i=1}^{n} I\{y_i=1\} \log (2\pi)^{-\frac{d}{2}} \det(\Sigma_1)^{-\frac{1}{2}} \exp\{-\frac{1}{2}(x_i-\mu_1)^T\Sigma_1^{-1}(x_i-\mu_1)\}$

$+ \sum_{i=1}^{n} I\{y_i=0\} \log (2\pi)^{-\frac{d}{2}} \det(\Sigma_0)^{-\frac{1}{2}} \exp\{-\frac{1}{2}(x_i-\mu_0)^T\Sigma_0^{-1}(x_i-\mu_0)\}$

$\frac{\partial \log P(D)}{\partial \pi} = \frac{n_1}{\pi} - \frac{n_0}{1-\pi} = 0 \Rightarrow \pi = \frac{n_1}{n_0+n_1} = \frac{n_1}{n}$

$\frac{\partial \log P(D)}{\partial \mu_1} = \sum_{i=1}^{n} I\{y_i=1\} \frac{\partial\left(-\frac{1}{2}(x_i-\mu_1)^T\Sigma_1^{-1}(x_i-\mu_1)\right)}{\partial \mu_1} \Rightarrow \mu_1 = \frac{1}{n_1}\sum_{i=1}^{n} I\{y_i=1\} \cdot x_i$

$= \sum_{i=1}^{n} I\{y_i=1\} (-1)\times(-\frac{1}{2})\times 2\Sigma_1^{-1}(x_i-\mu_1)$

$= \sum_{i=1}^{n} I\{y_i=1\} \Sigma_1^{-1}(x_i-\mu_1)$

$= 0$

$\frac{\partial \log P(D)}{\partial \Sigma_1} = \sum_{i=1}^{n} I\{y_i=1\} \cdot \left(\frac{\partial \log \det(\Sigma_1)^{-\frac{1}{2}}}{\partial \Sigma_1} + \frac{\partial \log \exp\{-\frac{1}{2}(x_i-\mu_1)^T\Sigma_1^{-1}(x_i-\mu_1)\}}{\partial \Sigma_1} \right)$

$= \sum_{i=1}^{n} I\{y_i=1\} \left(-\frac{1}{2}(\Sigma_1^{-1})^T + \frac{1}{2}(\Sigma_1^{-1})^T(x_i-\mu_1)(x_i-\mu_1)^T(\Sigma_1^{-1})^T \right)$

$= 0$

$\Rightarrow n_1(\Sigma_1^{-1})^T = \sum_{i=1}^{n} I\{y_i=1\} (\Sigma_1^{-1})^T(x_i-\mu_1)(x_i-\mu_1)^T(\Sigma_1^{-1})^T$

$\Rightarrow n_1(\Sigma_1)^T = \sum_{i=1}^{n} I\{y_i=1\}(x_i-\mu_1)(x_i-\mu_1)^T$

$\Rightarrow \Sigma_1 = \frac{1}{n_1}\sum_{i=1}^{n} I\{y_i=1\}(x_i-\mu_1)(x_i-\mu_1)^T$

同理可求 μ_0, Σ_0

综上: $\pi = \frac{n_1}{n}$, $\mu_1 = \frac{1}{n_1}\sum_{i=1}^{n} I\{y_i=1\} \cdot x_i$, $\Sigma_1 = \frac{1}{n_1}\sum_{i=1}^{n} I\{y_i=1\}(x_i-\mu_1)(x_i-\mu_1)^T$

$\mu_0 = \frac{1}{n_0}\sum_{i=1}^{n} I\{y_i=0\} x_i$ $\Sigma_0 = \frac{1}{n_0}\sum_{i=1}^{n} I\{y_i=0\}(x_i-\mu_0)(x_i-\mu_0)^T$

(2) $\log\frac{P(y_1|x;\theta)}{P(y_0|x;\theta)}=\log\frac{P(y_1;\theta)P(x|y_1;\theta)}{P(y_0;\theta)P(x|y_0;\theta)}=\log\frac{\pi\cdot N(x|\mu_1,\Sigma_1)}{(1-\pi)\cdot N(x|\mu_0,\Sigma_0)}=\log\frac{\pi}{1-\pi}+\log\frac{N(x|\mu_1,\Sigma_1)}{N(x|\mu_0,\Sigma_0)}$

$\log\frac{\pi}{1-\pi}=\log\frac{\frac{n_1}{n}}{\frac{n_0}{n}}=\log\frac{n_1}{n_0}$

$$\log\frac{N(x|\mu_1,\Sigma_1)}{N(x|\mu_0,\Sigma_0)}=\log\frac{(2\pi)^{-\frac{d}{2}}\det(\Sigma_1)^{-\frac{1}{2}}\exp\{-\frac{1}{2}(x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1)\}}{(2\pi)^{-\frac{d}{2}}\det(\Sigma_0)^{-\frac{1}{2}}\exp\{-\frac{1}{2}(x-\mu_0)^T\Sigma_0^{-1}(x-\mu_0)\}}$$

$$=\log\frac{\det(\Sigma_1)^{-\frac{1}{2}}}{\det(\Sigma_0)^{-\frac{1}{2}}}+\log\exp\{-\frac{1}{2}[(x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1)-(x-\mu_0)^T\Sigma_0^{-1}(x-\mu_0)]\}$$

其中: $(x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1)$

$=x^T\Sigma_1^{-1}(x-\mu_1)-\mu_1^T\Sigma_1^{-1}(x-\mu_1)$

$=x^T\Sigma_1^{-1}x-x^T\Sigma_1^{-1}\mu_1-\mu_1^T\Sigma_1^{-1}x+\mu_1^T\Sigma_1^{-1}\mu_1$

$=x^T\Sigma_1^{-1}x-2\mu_1^T\Sigma_1^{-1}x+\mu_1^T\Sigma_1^{-1}\mu_1$

同理: $(x-\mu_0)^T\Sigma_0^{-1}(x-\mu_0)$

$=x^T\Sigma_0^{-1}x-2\mu_0^T\Sigma_0^{-1}x+\mu_0^T\Sigma_0^{-1}\mu_0$

$$\log\frac{N(x|\mu_1,\Sigma_1)}{N(x|\mu_0,\Sigma_0)}=-\frac{1}{2}(x^T\Sigma_1^{-1}x-2\mu_1^T\Sigma_1^{-1}x+\mu_1^T\Sigma_1^{-1}\mu_1-x^T\Sigma_0^{-1}x+2\mu_0^T\Sigma_0^{-1}x-\mu_0^T\Sigma_0^{-1}\mu_0)\log e+\log\frac{\det(\Sigma_1)^{-\frac{1}{2}}}{\det(\Sigma_0)^{-\frac{1}{2}}}$$

$$=(-\frac{1}{2}x^T\Sigma_1^{-1}x+\mu_1^T\Sigma_1^{-1}x-\frac{1}{2}\mu_1^T\Sigma_1^{-1}\mu_1+\frac{1}{2}x^T\Sigma_0^{-1}x-\mu_0^T\Sigma_0^{-1}x+\frac{1}{2}\mu_0^T\Sigma_0^{-1}\mu_0)\log e+\log\frac{\det(\Sigma_1)^{-\frac{1}{2}}}{\det(\Sigma_0)^{-\frac{1}{2}}}$$

假设 $\Sigma_1=\Sigma_2=\Sigma$

则 $\log\frac{N(x|\mu_1,\Sigma)}{N(x|\mu_0,\Sigma_0)}=\log^e(\mu_1^T-\mu_0^T)\Sigma^{-1}x+\frac{1}{2}\log^e(\mu_0^T\Sigma^{-1}\mu_0-\mu_1^T\Sigma^{-1}\mu_1)$

所以: $\log\frac{P(y_1|x;\theta)}{P(y_0|x;\theta)}=\log^e(\mu_1-\mu_0)^T\Sigma^{-1}x+\frac{1}{2}\log^e(\mu_0^T\Sigma^{-1}\mu_0-\mu_1^T\Sigma^{-1}\mu_1)+\log\frac{n_1}{n_0}$

$=W^Tx+b$

即: $W=\log^e(\Sigma^{-1})^T(\mu_1-\mu_0)$

$b=\frac{1}{2}\log^e(\mu_0^T\Sigma^{-1}\mu_0-\mu_1^T\Sigma^{-1}\mu_1)+\log\frac{n_1}{n_0}$