作业六

题目：已知有数据集 $\mathrm{D} = \{\mathrm{x_i}\}_{\mathrm{i=1}}^{\mathrm{n}}$，$x_i \in \mathbb{R}$，$X \sim \mathcal{N}(\mu, \sigma^2)$，均值$\mu \sim \mathcal{N}\left(\mu_0, \sigma_0^2\right)$，有 $\mathrm{p}(\mu) = \frac{1}{\sigma_0\sqrt{2\pi}} exp\left\{-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2\right\}$，则似然 $\mathrm{p}(D|\mu) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}$，根据贝叶斯公式，后验概率 $\mathrm{p}(\mu|D) \propto \mathrm{p}(\mu)\mathrm{p}(D|\mu) \propto \exp\left[-\frac{1}{2\sigma_n^2}(\mu-\mu_n)^2\right]$，其中$\mu_n = \frac{\sigma^2}{n\sigma_0^2+\sigma^2}\mu_0 + \frac{\sum_i x_i\sigma_0^2}{n\sigma_0^2+\sigma^2}$，$\frac{1}{\sigma_n^2} = \frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}$，即 $\mathrm{p}(\mu|D) = \mathcal{N}(\mu_n, \sigma_n^2)$。

证明 ： 验证$P(\mathrm{x}|D)$是关于 x 的高斯分布

$$\mathrm{P(x|D)} = \int P(x, \mathrm{\mu|D})\, d\mathrm{\mu} = \mathcal{N}(\mathrm{x}|\mu_x, \sigma_x^2)$$

提示 ： 最终 $\mathrm{P(x|D)} = A\frac{1}{\sqrt{\sigma^2+\sigma_n^2}\cdot\sqrt{2\pi}} \cdot exp\left\{-\frac{(x-\mu_n)^2}{2(\sigma^2+\sigma_n^2)}\right\}$

$$= A\mathcal{N}(\mathrm{x}|\mu_x, \sigma_x^2)$$

其中 A 为常数，$\mu_x = \mu_n$ ，$\sigma_x^2 = \sigma^2 + \sigma_n^2$

(答案仅供参考)

证明：

$P(x|D) = \int P(x, \mu|D)\, d\mu$

$= \int P(x|\mu)P(\mu|D)\, d\mu$

$= \int N(\mu|\sigma^2)N(\mu_n|\sigma_n^2)\, d\mu$

$$= \int \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2\sigma^2}(x-\mu)^2\}\ \frac{1}{\sigma_n\sqrt{2\pi}} \exp\{-\frac{1}{2\sigma_n^2}(\mu-\mu_n)^2\}\, d\mu$$

$$= \frac{1}{2\pi\sigma\sigma_n} \exp\{-\frac{1}{2}(\frac{\mu_n^2}{\sigma_n^2} + \frac{x^2}{\sigma^2})\} \int \exp\{-\frac{1}{2}(\frac{1}{\sigma^2} + \frac{1}{\sigma_n^2})\,\mu^2 + (\frac{x}{\sigma^2} + \frac{\mu_n}{\sigma_n^2})\,\mu\}\, d\mu$$

$$= \frac{1}{2\pi\sigma\sigma_n} \exp\{-\frac{1}{2}(\frac{\mu_n^2}{\sigma_n^2} + \frac{x^2}{\sigma^2} + \frac{(\sigma_n^2 x + \sigma^2\mu_n)^2}{\sigma_n^2\sigma(\sigma^2 + \sigma_n^2)})\}$$

$$\bullet \int \exp\{-\frac{\sigma^2+\sigma_n^2}{2\sigma_n^2\sigma^2}(\mu - \frac{\sigma^2\mu_n + \sigma_n^2 x}{\sigma^2+\sigma_n^2})^2\}\, d\mu$$

$$= \frac{1}{2\pi\sigma\sigma_n} \exp\{-\frac{1}{2(\sigma^2+\sigma_n^2)}(x-\mu_n)^2\}$$

$$\bullet \int \exp\{-\frac{\sigma^2+\sigma_n^2}{2\sigma_n^2\sigma^2}(\mu - \frac{\sigma^2\mu_n + \sigma_n^2 x}{\sigma^2+\sigma_n^2})^2\}\, d\mu$$

对于 $\int \exp\{-\frac{\sigma^2+\sigma_n^2}{2\sigma_n^2\sigma^2}(\mu - \frac{\sigma^2\mu_n+\sigma_n^2 x}{\sigma^2+\sigma_n^2})^2\}\, d\mu$，由积分公式$\int \exp\{-x^2\}\, dx = \sqrt{\pi}$，用换元法，可得：

$$\int \exp\{-\frac{\sigma^2+\sigma_n^2}{2\sigma_n^2\sigma^2}(\mu - \frac{\sigma^2\mu_n + \sigma_n^2 x}{\sigma^2+\sigma_n^2})^2\}\, d\mu = \frac{\sqrt{2\pi}\sigma_n\sigma}{\sqrt{\sigma_n^2+\sigma^2}}$$

故原式

$$= \frac{1}{2\pi\sigma\sigma_n} \exp\{-\frac{1}{2(\sigma^2+\sigma_n^2)}(x-\mu_n)^2\} \bullet \frac{\sqrt{2\pi}\sigma_n\sigma}{\sqrt{\sigma_n^2+\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2+\sigma_n^2}} \exp\{-\frac{1}{2(\sigma^2+\sigma_n^2)}(x-\mu_n)^2\}$$

所以 P(x|D)是关于 x 的高斯分布

其中$\mu_x = \mu_n$，$\sigma_x^2 = \sigma^2 + \sigma_n^2$