COEN 4890 HW#1 | Zhou Shen

1. Parameter Estimation

1) log-likelihood for $\{x_1, \cdots x_n\}$.

$l(\theta|x) = \log l(\theta|x)$, $l(\theta|x) = \log \prod_{i=1}^{n} f(x_i|\theta) = \sum_{i=1}^{n} l(\theta|x_i)$ +15

$P(x|\lambda) = \frac{\lambda^x}{x!} e^{-\lambda} \rightarrow \lambda = \theta$

$p(\lambda|D) = p(D|\lambda) \cdot p(\lambda)$ for (3)

$l(\theta|x) = \log \prod_{i=1}^{n} p(x_i|\theta) = \sum_{i=1}^{n} \log(\theta|x_i) = \log \frac{\lambda^{x_1}}{x_1!} e^{-\lambda} + \log \frac{\lambda^{x_2}}{x_2!} e^{-\lambda} + \cdots + \log \frac{\lambda^{x_n}}{x_n!} e^{-\lambda}$

For 1st element, $\log \frac{\lambda^{x_1}}{x_1!} e^{-\lambda} = \log \frac{\lambda^{x_1}}{x_1!} + \log e^{-\lambda} \Rightarrow \log \frac{\lambda^{x_1}}{x_1!} - \lambda \Rightarrow$ for different i, $\sum_{i=1}^{n} \log \frac{\lambda^{x_i}}{x_i!} - n\lambda$

↳ continue, $-\lambda + \log \frac{\lambda^{x_1}}{x_1!} = -\lambda + \log \lambda^{x_1} - \log x_1! = -\lambda + x_1 \log \lambda - \log x_1!$

$\Rightarrow -n\lambda + \sum_{i=1}^{n} x_i \log \lambda - \sum_{i=1}^{n} \log x_i! = -n\lambda + \log \lambda \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \log x_i!$ log-likelihood. +3

2) MLE

Find Maximum, set derivative to zero. $\frac{d}{d\theta} \ln P(D|\theta) = 0$.

$\frac{\partial \log L(\lambda)}{\partial \lambda} = -n + \frac{\sum_{i=1}^{n} x_i}{\lambda} - 0 \overset{\text{set}}{\equiv} 0$ multiply by λ, $-n\lambda + \sum_{i=1}^{n} x_i = 0 \Rightarrow \sum_{i=1}^{n} x_i = n\lambda$

$\Rightarrow \lambda = \frac{\sum_{i=1}^{n} x_i}{n}$ +4

3) Bayesian. Compute MAP for λ.

good prior distribution for λ: $p(\lambda|\alpha|\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$, $\lambda > 0$. Gamma distribution.

mode of λ: $(\alpha-1)/\beta$ for $\alpha > 1$

$p(\lambda|x) \propto p(\lambda) \cdot p(x|\lambda) \rightarrow$ priority · likelihood.

$p(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$, $\lambda > 0$

$p(x|\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda} \cdot e^{x_i}}{\lambda_i!} = \frac{\prod_{i=1}^{n} (\lambda^{x_i} \cdot e^{-\lambda})}{\prod_{i=1}^{n} \lambda_i!} = \frac{e^{-N\lambda} \cdot \prod_{i=1}^{n} \lambda^{x_i}}{\prod_{i=1}^{n} x_i!}$

$p(\lambda|x) \propto p(\lambda) \cdot p(x|\lambda) = \left(\frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \right) \cdot \left(\frac{e^{-N\lambda} \cdot \sum_{i=1}^{N} x_i \lambda}{\prod_{i=1}^{n} x_i!} \right)$

$= \frac{\beta^\alpha}{\Gamma(\alpha) \cdot \prod_{i=1}^{n} x_i!} \cdot \lambda^{\alpha - 1 - \sum_{i=1}^{N} x_i} \cdot e^{-(\beta+N)\lambda} \rightarrow$ Gamma distribution

$\frac{\partial p(\lambda|x)}{\partial \lambda} = \frac{\partial \left(\frac{\beta^\alpha}{\Gamma(\alpha) \cdot \prod_{i=1}^{n} x_i!} \right)}{\partial \lambda} \cdot \left(\frac{\alpha - 1}{\beta} \right)$

$\frac{\alpha \cdot 1}{\beta}$

$\alpha = \alpha + \sum_{i=1}^{n} x_i$

$\beta = \beta + N$

$\frac{\alpha + \sum_{i=1}^{N} x_i - 1}{\beta + N} = 0$

+8