Lagrangian Mechanics

Theoretical Mechanics

1 Calculus of Variations

The **calculus of variations**, is concerned with the extremals of functions whose domain is an infinite-dimensional space: the space of curves. Such functions are called **functionals**.

1.1 Variations and Extremals

Definitions of **differentiable** and **differential**: the functional Φ is *differentiable* if $\Phi(\gamma+h)-\Phi(\gamma) = F + R$ where F depends linearly on h. The linear part $F(h)$ is the *differential* of Φ.

Example: A curve γ in (t, x)-plane: $\gamma = \{(t, x) : x = x(t), t_0 \leq t \leq t_1\}$.

1.2 The Euler-Lagrange Equation

Definition: the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{1}$$

is the **Euler-Lagrange equation** of the functional

$$\Phi = \int_{t_0}^{t_1} L(x, \dot{x}, t)\mathrm{d}t. \tag{2}$$

Theorem: the curve γ is an extremal of the functional $\Phi(\gamma) = \int_{t_0}^{t_1} L(x, \dot{x}, t)\mathrm{d}t$ on the space of curves joining (t_0, x_0) and (t_1, x_1) if and only if the Euler-Lagrange equation is satisfied along γ.

1.3 Hamilton's equations

1.3.1 Equivalence of Lagrange's and Hamilton's equations

1.3.2 Hamilton's function and energy

Lagrangian has the usual form $L = T - U$, where T is a quadratic form with respect to $\dot{\mathbf{q}}$

$$T = \frac{1}{2}a_{ij}\dot{q}_i\dot{q}_j, \text{ where } a_{ij} = a_{ij}(\mathbf{q}, t) \text{ and } U = U(\mathbf{q}) \tag{3}$$

Theorem: The Hamiltonian H is the total energy $H = T + U$.

Lemma: The values of a quadratic form $f(x)$ and of its Legendre transform $g(p)$ coincide at corresponding points: $f(x) = g(p)$.

Proof of the lemma: for a quadratic form $f(x) = \frac{1}{2}\sum_{i,j} a_{ij}x_ix_j$ which is a homogeneous function of degree 2, it satisfies $f(\lambda x) = \lambda^2 f(x)$ and $x \cdot \nabla f(x) = 2f(x)$ (for a homogeneous function $f(x)$ of degree n, $\frac{\mathrm{d}}{\mathrm{d}\lambda}f(\lambda x) = n\lambda^{n-1}f(x)$, by the chain rule $\frac{\mathrm{d}}{\mathrm{d}\lambda}f(\lambda x) = \frac{\mathrm{d}}{\mathrm{d}(\lambda x)}f(\lambda x) \cdot \frac{\mathrm{d}(\lambda x)}{\mathrm{d}\lambda} = \nabla f(\lambda x) \cdot x$, so $\nabla f(\lambda x) \cdot x = n\lambda^{n-1}f(x)$, letting $\lambda = 1$ and we get $x \cdot \nabla f(x) = nf(x)$, which is the **Euler's Theorem** on homogeneous functions), so if f is strictly convex, $g(p) = px - f(x) = \frac{\partial f(x)}{\partial x}x - f(x) = 2f(x) - f(x) = f(x)$.

Proof of the theorem: $H = p\dot{q} - L = \frac{\partial L}{\partial \dot{q}}\dot{q} - L$, since $L = T(q, \dot{q}, t) - U(q)$, applying Euler's theorem and we get $H = \frac{\partial T}{\partial \dot{q}}\dot{q} - (T - U) = 2T - (T - U) = T + U$.

Corollary (the law of conservation):

1.4 Liouville's Theorem

Definition: **Phase space** is the $2n$-dimensional space with coordinates $p_1, \ldots, p_n; q_1, \ldots, q_n$. **Phase flow** is the one-parameter group of transformations of phase space

$$g^t : (\mathbf{p}(0), \mathbf{q}(0)) \to (\mathbf{p}(t), \mathbf{q}(t)) \tag{4}$$

where $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are solutions of Hamilton's system of equations.

Theorem 1: The phase flow preserves volume: for any region D, the volume V satisfies $V(g^t D) = V(D)$.