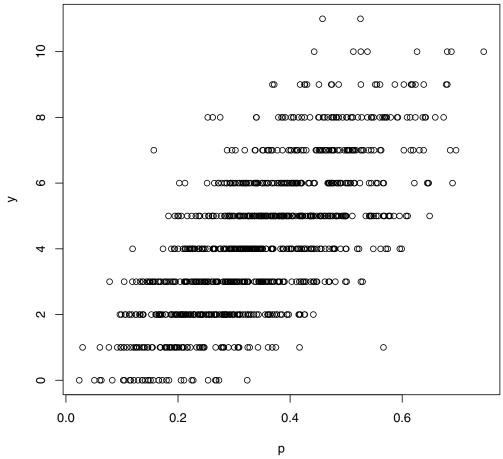

Fig. 13.9 Scatterplot of simulated sample of (p, y) from Gibbs sampling algorithm for random coin example.

```
> table(sim.values[ ,"y"])
  0   1   2   3   4   5   6   7   8   9  10  11
 47  92 135 148 168 152  85  81  55  22  11   4
```

In the sample of 1000 draws of y, we observed 168 fours, so $P(y = 4) \approx 168/1000 = 0.168$. Other properties of the marginal density of y, such as the mean and standard deviation, can be found by computing summaries of the sample of simulated draws of y.

13.6 Further Reading

Gentle [20] provides a general description of Monte Carlo methods. Chib and Greenberg [9] gives an introduction to the Metropolis-Hastings algorithm and Casella and George [8] give some basic illustrations of Gibbs sampling. Albert [1] provides illustrations of MCMC algorithms using R code.

Exercises

13.1 (Late to class?). Suppose the travel times for a particular student from home to school are normally distributed with mean 20 minutes and standard deviation 4 minutes. Each day during a five-day school week she leaves home

30 minutes before class. For each of the following problems, write a short Monte Carlo simulation function to compute the probability or expectation of interest.

a. Find the expected total traveling time of the student to school for a five-day week. Find the simulation estimate and give the standard error for the simulation estimate.
b. Find the probability that the student is late for at least one class in the five-day week. Find the simulation estimate of the probability and the corresponding standard error.
c. On average, what will be the longest travel time to school during the five-day week? Again find the simulation estimate and the standard error.

13.2 (Confidence interval for a normal mean based on sample quantiles). Suppose one obtains a normally distributed sample of size $n = 20$ but only records values of the sample median M and the first and third quartiles Q_1 and Q_3.

a. Using a sample of size $n = 20$ from the standard normal distribution, simulate the sampling distribution of the statistic

$$S = \frac{M}{Q_3 - Q_1}.$$

Store the simulated values of S in a vector.
b. Find two values, s_1, s_2, that bracket the middle 90% probability of the distribution of S.
c. For a sample of size $n = 20$ from a normal distribution with mean μ and standard deviation σ, it can be shown that

$$P\left(s_1 < \frac{M - \mu}{Q_3 - Q_1} < s_2\right) = 0.90.$$

Using this result, construct a 90% confidence interval for the mean μ
d. In a sample of 20, we observe $(Q_1, M, Q_3) = (37.8, 51.3, 58.2)$. Using your work in parts (b) and (c), find a 90% confidence interval for the mean μ.

13.3 (Comparing variance estimators). Suppose one is taking a sample $y_1, ..., y_n$ from a normal distribution with mean μ and variance σ^2.

a. It is well known that the sample variance

$$S = \frac{\sum_{j=1}^{n}(y_j - \bar{y})^2}{n-1}$$

is an unbiased estimator of σ^2. To confirm this, assume $n = 5$ and perform a simulation experiment to compute the bias of the sample variance S.

b. Consider the alternative variance estimator

$$S_c = \frac{\sum_{j=1}^{n}(y_j - \bar{y})^2}{c},$$

where c is a constant. Suppose one is interested in finding the estimator S_c that makes the mean squared error

$$MSE = E\left[(S_c - \sigma^2)^2\right]$$

as small as possible. Again assume $n = 5$ and use a simulation experiment to compute the mean squared error of the estimators S_3, S_5, S_7, S_9 and find the choice of c (among $\{3, 5, 7, 9\}$) that minimizes the MSE.

13.4 (Evaluating the "plus four" confidence interval). A modern method for a confidence interval for a proportion is the "plus-four" interval described in Agresti and Coull [2]. One first adds 4 imaginary observations to the data, two successes and two failures, and then apply the Wald interval to the adjusted sample. Let $\tilde{n} = n + 4$ denoted the adjusted sample size and $\tilde{p} = (y + 2)/\tilde{n}$ denotes the adjusted sample proportion. Then the "plus-four" interval is given by

$$INT_{Plus-four} = \left(\tilde{p} - z\sqrt{\frac{\tilde{p}(1-\tilde{p})}{\tilde{n}}}, \hat{p} + z\sqrt{\frac{\tilde{p}(1-\tilde{p})}{\tilde{n}}}\right),$$

where z denote the corresponding $1-(1-\gamma)/2$ percentile for a standard normal variable.

By a Monte Carlo simulation, compute the probability of coverage of the plus-four interval for values of the proportion p between 0.001 and 0.999. Contrast the probability of coverage of the plus-four interval with the Wald interval when the nominal coverage level is $\gamma = 0.90$. Does the plus-four interval have a 90% coverage probability for all values of p?

13.5 (Metropolis-Hastings algorithm for the poly-Cauchy distribution). Suppose that a random variable y is distributed according to the poly-Cauchy density

$$g(y) = \prod_{i=1}^{n} \frac{1}{\pi(1 + (y - a_i)^2},$$

where $a = (a_1, ..., a_n)$ is a vector of real-valued parameters. Suppose that $n = 6$ and $a = (1, 2, 2, 6, 7, 8)$.

a. Write a function to compute the log density of y. (It may be helpful to use the function `dcauchy` that computes the Cauchy density.)
b. Use the function `metrop.hasting.rw` to take a simulated sample of size 10,000 from the density of y. Experiment with different choices of the standard deviation C. Investigate the effect of the choice of C on the acceptance rate, and the mixing of the chain over the probability density.

c. Using the simulated sample from a "good" choice of C, approximate the probability $P(6 < Y < 8)$.

13.6 (Gibbs sampling for a Poisson/gamma model). Suppose the vector of random variables (X, Y) has the joint density function

$$f(x,y) = \frac{x^{a+y-1} e^{-(1+b)x} b^a}{y! \Gamma(a)}, \; x > 0, y = 0, 1, 2, ...$$

and we wish to simulate from this joint density.

a. Show that the conditional density $f(x|y)$ has a gamma density and identify the shape and rate parameters of this density.
b. Show that the conditional density $f(y|x)$ has a Poisson density.
c. Write a R function to implement Gibbs sampling when the constants are given by $a = 1$ and $b = 1$.
d. Using your R function, run 1000 cycles of the Gibbs sampler and from the output, display (say, by a histogram) the marginal probability mass function of Y and compute $E(Y)$.