Chapter 2 Random Variable Generation

Exercise 2.1

if for cdf F, $F^{-}(u) = \inf\{x, F(x) \leq u\}$

$\Rightarrow$ for $U \sim u[0,1]$, $P(F^{-}(U) < y) = P(\inf\{x, F(x) \leq u\} < y)$

$= P(F(y) \geqslant U) = F(y)$

hence if $U \sim u[0,1]$, $F^{-}(U)$ is distributed like X

Exercise 2.3

a. $E[U_i] = 0$, $Z = \sum_{i=1}^{12} U_i$

$\Rightarrow Var(U_i) = \frac{1}{12} \Rightarrow Var(Z) = 1$ Q.E.D

b.

```
nsim=10000
u1=runif(nsim);u2=runif(nsim);
X1=sqrt(-2*log(u1))*cos(2*pi*u2);X2=sqrt(-2*log(u1))*sin(2*pi*u2);
U=array(0,dim=c(nsim,1))
for(i in 1:nsim)U[i]=sum(runif(12,-.5,.5))
par(mfrow=c(1,2))
hist(X1);hist(U);
a=3
mean(X1>a);mean(U>a);
mean(rnorm(nsim)>a)
1-pnorm(a)
```

Exercise 2.5

in Accept-Reject algorithm, $U \leqslant f(Y)/Mg(Y)$

$\Rightarrow P(U \leqslant f(Y)/Mg(Y)) = \int_{-\infty}^{+\infty} \int_0^{f(Y)/Mg(Y)} du\, g(y)\, dy = \int_{-\infty}^{+\infty} \frac{f(y)}{Mg(y)} g(y)\, dy$

since $\int_{-\infty}^{+\infty} f(y)\, dy = 1$

$\Rightarrow P(U \leqslant f(Y)/Mg(Y)) = \frac{1}{M}$.

$P(U \leqslant \hat{f}(Y)/\tilde{M}\tilde{g}(Y)) = \int_{-\infty}^{+\infty} \int_0^{\hat{f}(Y)/\tilde{M}\tilde{g}(Y)} du\, g(y)\, dy = \int_{-\infty}^{+\infty} \frac{\tilde{f}(y)}{\tilde{M}\tilde{g}(y)} g(y)\, dy$

$= \int_{-\infty}^{+\infty} \frac{f(y)}{k\tilde{M}g(y)} g(y)\, dy = \frac{1}{k\tilde{M}}$

$\Rightarrow$ missing constant: $k = \dfrac{1}{\tilde{M} \cdot P(U \leqslant \hat{f}(Y)/\tilde{M}\tilde{g}(Y))}$

Q.E.D

Exercise 2.7

the ratio f/g is equal to $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}x^{\alpha-a}(1-x)^{\beta-b}$

and the ratio diverge at $x=0$ if $a\leq\alpha$, at $x=1$ if $b\leq\beta$

f/g is maximum if $\frac{1-x}{x}=\frac{\beta-b}{\alpha-a}$

Exercise 2.9

$P(X=x_0|\theta)=f(x_0|\theta)$

if X is discrete, $f(x_0|\theta)$ is the true probability

if X is continuous, the probability is zero

the distribution of θ is dependent on the event $X=x_0$ happening.

Exercise 2.11

a. plot a histogram and compare with binomial mass function

```
m<-5000
n=25;p=.2;
cp=pbinom(c(0:n),n,p)
X=array(0,c(nsim,1))
for(i in 1:nsim){
   u=runif(1)
   X[i]=sum(cp<u)
   }
hist(X,freq=F)
lines(1:n,dbinom(1:n,n,p),lwd=2)
```

the function of checking time:

```
MYbinom<-function(s0,n0,p0){
   cp=pbinom(c(0:n0),n0,p0)
   X=array(0,c(s0,1))
   for (i in 1:s0){
      u=runif(1)
      X[i]=sum(cp<u)
      }
   return(X)
   }
```

b. create the function Wait and Trans in R language:

```
Wait<-function(s0,alpha){
   U=array(0,c(s0,1))
   for (i in 1:s0){
      u=runif(1)
      while (u > alpha) u=runif(1)
      U[i]=u
      }
   return(U)
   }

Trans<-function(s0,alpha){
   U=array(0,c(s0,1))
   for (i in 1:s0) U[i]=alpha*runif(1)
   return(U)
   }
```

Exercise 2.13

$P(\alpha)$ distribution : $f(x|\alpha) = \alpha x^{-\alpha-1}$ in $(1, +\infty)$

$\Rightarrow$ cdf of $P(\alpha)$: $F(x) = 1 - x^{-\alpha}$, $x \in (1, +\infty)$

$\Rightarrow F^{-1}(U) = (1-U)^{-1/\alpha}$

hence it can be generated as $-1/\alpha$ power of a uniform variate

Q.E.D