Chapter 5
Reducible Markov Chains

5.1 Introduction

Reducible Markov chains describe systems that have particular states such that once we visit one of those states, we cannot visit other states. An example of systems that can be modeled by reducible Markov chains is games of chance where once the gambler is broke, the game stops and the casino either kicks him out or gives him some compensation (comp). The gambler moved from being in a state of play to being in a comp state and the game stops there. Another example of reducible Markov chains is studying the location of a fish swimming in the ocean. The fish is free to swim at any location as dictated by the currents, food, or presence of predators. Once the fish is caught in a net, it cannot escape and it has limited space where it can swim.

Consider the transition diagram in Fig. 5.1(a). Starting at any state, we are able to reach any other state in the diagram directly, in one step, or indirectly, through one or more intermediate states. Such a Markov chain is termed *irreducible Markov chain* for reasons that will be explained shortly. For example, starting at s_1, we can directly reach s_2 and we can indirectly reach s_3 through either of the intermediate s_2 or s_5. We encounter irreducible Markov chains in systems that can operate for long periods of time such as the state of the lineup at a bank, during business hours. The number of customers lined up changes all the time between zero and maximum. Another example is the state of buffer occupancy in a router or a switch. The buffer occupancy changes between being completely empty and being completely full depending on the arriving traffic pattern.

Consider now the transition diagram in Fig. 5.1(b). Starting from any state, we might not be able to reach other states in the diagram, directly or indirectly. Such a Markov chain is termed *reducible Markov chain* for reasons that will be explained shortly. For example, if we start at s_1, we can never reach any other state. If we start at state s_4, we can only reach state s_5. If we start at state s_3, we can reach all other states. We encounter reducible Markov chains in systems that have terminal conditions such as most games of chance like gambling. In that case, the player keeps on playing till she loses all her money or wins. In either cases, she leaves the game. Another example is the game of snakes and ladders where the player keeps

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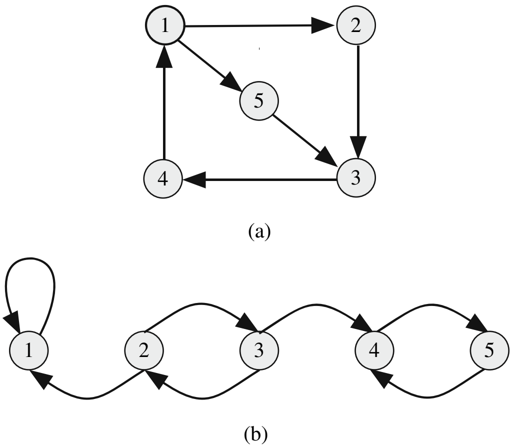

Fig. 5.1 State transition diagrams. (**a**) Irreducible Markov chain. (**b**) Reducible Markov chain

on playing but cannot go back to the starting position. Ultimately, the player reaches the final square and could not go back again to the game.

5.2 Definition of Reducible Markov Chain

The traditional way to define a reducible Markov chain is as follows.

A Markov chain is irreducible if there is some integer $k > 1$ such that all the elements of $\mathbf{P}^k$ are nonzero.

What is the value of k? No one seems to know; the only advice is to keep on multiplying till the conditions are satisfied or computation noise overwhelms us!

This chapter is dedicated to shed more light on this situation and introduce, for the first time, a simple and rigorous technique for identifying a reducible Markov chain. As a bonus, the states of the Markov chain will be simply classified too without too much effort on our part.

5.3 Closed and Transient States

We defined an irreducible (or regular) Markov chain as one in which every state is reachable from every other state either directly or indirectly. We also defined a reducible Markov chain as one in which some states cannot reach other states. Thus the states of a reducible Markov chain are divided into two sets: closed state (C) and transient state (T). Figure 5.2 shows the two sets of states and the directions of transitions between the two sets of states.

When the system is in T, it can make a transition to either T or C. However, once our system is in C, it can never make a transition to T again no matter how long we iterate. In other words, the probability of making a transition from a closed state to a transient state is exactly zero.

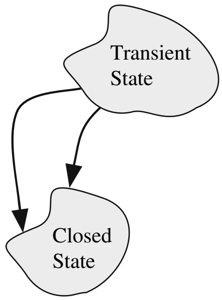

Fig. 5.2 Reducible Markov chain with two sets of states. There are no transitions from the closed states to the transient states as shown.

When C consists of only one state, then that state is called an *absorbing* state. When s_i is an absorbing state, we would have $p_{ii} = 1$. Thus inspection of the transition matrix quickly informs us of the presence of any absorbing states since the diagonal element for that state will be 1.

5.4 Transition Matrix of Reducible Markov Chains

Through proper state assignment, the transition matrix **P** for a reducible Markov chain could be partitioned into the *canonic form*

$$\mathbf{P} = \begin{bmatrix} \mathbf{C} & \mathbf{A} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \tag{5.1}$$

where

$\mathbf{C}$ = square column stochastic matrix
$\mathbf{A}$ = rectangular nonnegative matrix
$\mathbf{T}$ = square column substochastic matrix

Appendix D defines the meaning of nonnegative and substochastic matrices. The matrix **C** is a column stochastic matrix that can be studied separately from the rest of the transition matrix **P**. In fact, the eigenvalues and eigenvectors of **C** will be used to define the behavior of the Markov chain at equilibrium.

The states of the Markov chain are now partitioned into two mutually exclusive subsets as shown below.

C = set of closed states belonging to matrix **C**
T = set of transient states belonging to matrix **T**

The following equation explicitly shows the partitioning of the states into two sets, closed state C and transient state T.

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} C & T \end{array} \\ \begin{array}{c} C \\ T \end{array} & \begin{bmatrix} \mathbf{C} & \mathbf{A} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \end{array} \tag{5.2}$$

Example 5.1 The given transition matrix represents a reducible Markov chain.

$$\mathbf{P} = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \left[\begin{array}{cccc} 0.8 & 0 & 0.1 & 0.1 \\ 0 & 0.5 & 0 & 0.2 \\ 0.2 & 0.2 & 0.9 & 0 \\ 0 & 0.3 & 0 & 0.7 \end{array} \right] \end{array}$$

where the states are indicated around **P** for illustration. Rearrange the rows and columns to express the matrix in the canonic form in (5.1) or (5.2) and identify the matrices **C**, **A**, and **T**. Verify the assertions that **C** is column stochastic, **A** is nonnegative, and **T** is column substochastic.

After exploring a few possible transitions starting from any initial state, we see that if we arrange the states in the order 1, 3, 2, 4 then the following state matrix is obtained

$$\mathbf{P} = \begin{array}{c} \\ s_1 \\ s_3 \\ s_2 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \left[\begin{array}{cccc} 0.8 & 0.1 & 0 & 0.1 \\ 0.2 & 0.9 & 0.2 & 0 \\ 0 & 0 & 0.5 & 0.2 \\ 0 & 0 & 0.3 & 0.7 \end{array} \right] \end{array}$$

We see that the matrix exhibits the reducible Markov chain structure and matrices **C**, **A**, and **T** are

$$\mathbf{C} = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0.1 \\ 0.2 & 0 \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$$

The sum of each column of **C** is exactly 1, which indicates that it is column stochastic. The sum of columns of **T** is less than 1, which indicates that it is column substochastic.

The set of closed states is $C = \{1,\ 3\}$ and the set of transient states is $T = \{2,\ 4\}$.

Starting in state s_2 or s_4, we will ultimately go to states s_1 and s_3. Once we are there, we can never go back to state s_2 or s_4 because we entered the closed states. ■

Example 5.2 Consider the reducible Markov chain of the previous example. Assume that the system was initially in state s_3. Find the distribution vector at 20 time step intervals.

We do not have to rearrange the transition matrix to do this example. We have

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0 & 0.1 & 0.1 \\ 0 & 0.5 & 0 & 0.2 \\ 0.2 & 0.2 & 0.9 & 0 \\ 0 & 0.3 & 0 & 0.7 \end{bmatrix}$$

The initial distribution vector is

$$\mathbf{s} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^t$$

The distribution vector at 20 time step intervals is

$$\mathbf{s}(20) = \begin{bmatrix} 0.3208 & 0.0206 & 0.6211 & 0.0375 \end{bmatrix}^t$$
$$\mathbf{s}(40) = \begin{bmatrix} 0.3327 & 0.0011 & 0.6642 & 0.0020 \end{bmatrix}^t$$
$$\mathbf{s}(60) = \begin{bmatrix} 0.3333 & 0.0001 & 0.6665 & 0.0001 \end{bmatrix}^t$$
$$\mathbf{s}(80) = \begin{bmatrix} 0.3333 & 0.0000 & 0.6667 & 0.0000 \end{bmatrix}^t$$

We note that after 80 time steps, the probability of being in the transient state s_2 or s_4 is nil. The system will definitely be in the closed set composed of states s_1 and s_3. ■

5.5 Composite Reducible Markov Chains

In the general case, the reducible Markov chain could be composed of two or more sets of closed states. Figure 5.3 shows a reducible Markov chain with two sets of closed states. If the system is in the transient state T, it can move to either sets of closed states, C_1 or C_2. However, if the system is in state C_1, it cannot move to T or C_2. Similarly, if the system is in state C_2, it cannot move to T or C_1. In that case, the canonic form for the transition matrix $\mathbf{P}$ for a reducible Markov chain could be expanded into several subsets of noncommunicating closed states

$$\mathbf{P} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} & \mathbf{A}_1 \\ \mathbf{0} & \mathbf{C}_2 & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{T} \end{bmatrix} \tag{5.3}$$

where

$$\begin{aligned} \mathbf{C}_1 \text{ and } \mathbf{C}_2 &= \text{square column stochastic matrices} \\ \mathbf{A}_1 \text{ and } \mathbf{A}_2 &= \text{rectangular nonnegative matrices} \\ \mathbf{T} &= \text{square column substochastic matrix} \end{aligned}$$

Since the transition matrix contains two-column stochastic matrices $\mathbf{C}_1$ and $\mathbf{C}_2$, we expect to get two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1$ also. And we will be getting

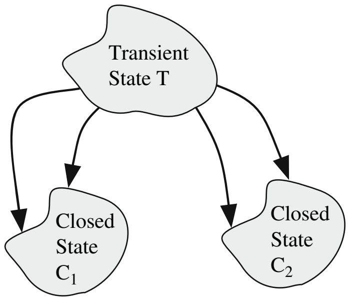

Fig. 5.3 A reducible Markov chain with two sets of closed states

two possible steady-state distributions based on the initial value of the distribution vector $\mathbf{s}(0)$—more on that in Sections 5.7 and 5.9.

The states of the Markov chain are now divided into three mutually exclusive sets as shown below.

1. C_1 = set of closed states belonging to matrix $\mathbf{C}_1$
2. C_2 = set of closed states belonging to matrix $\mathbf{C}_2$
3. T = set of transient states belonging to matrix $\mathbf{T}$

The following equation explicitly shows the partitioning of the states.

$$\mathbf{P} = \begin{array}{c} \\ C_1 \\ C_2 \\ T \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & T \end{array} \\ \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} & \mathbf{A}_1 \\ \mathbf{0} & \mathbf{C}_2 & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{T} \end{bmatrix} \end{array} \tag{5.4}$$

Notice from the structure of $\mathbf{P}$ in (5.4) that if we were in the first set of closed state C_1, then we cannot escape that set to visit C_2 or T. Similarly, if we were in the second set of closed state C_2, then we cannot escape that set to visit C_1 or T. On the other hand, if we were in the set of transient states T, then we can not stay in that set since we will ultimately fall into C_1 or C_2.

Example 5.3 You play a coin tossing game with a friend. The probability that one player winning \$1 is p, and the probability that he loses \$1 is $q = 1 - p$. Assume the combined assets of both players is \$6 and the game ends when one of the players is broke. Define a Markov chain whose state s_i means that you have \$$i$ and construct the transition matrix. If the Markov chain is reducible, identify the closed and transient states and rearrange the matrix to conform to the structure of (5.3) or (5.4).

Since this is a gambling game, we suspect that we have a reducible Markov chain with closed states where one player is the winner and the other is the loser.

A player could have \$0, \$1, $\cdots$, or \$6. Therefore, the transition matrix is of dimension 7×7 as shown in (5.5).

$$\mathbf{P} = \begin{bmatrix} 1 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & p & 0 & q & 0 & 0 & 0 \\ 0 & 0 & p & 0 & q & 0 & 0 \\ 0 & 0 & 0 & p & 0 & q & 0 \\ 0 & 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p & 1 \end{bmatrix} \tag{5.5}$$

Notice that states 0 and 6 are absorbing states since $p_{00} = p_{66} = 1$. The set $T = \{ s_1, \ s_2, \ \cdots, \ c_5 \}$ is the set of transient states. We could rearrange our transition matrix such that states s_0 and s_6 are adjacent as shown below.

$$\mathbf{P} = \begin{array}{cc} & \begin{array}{ccccccc} s_0 & s_6 & s_1 & s_2 & s_3 & s_4 & s_5 \end{array} \\ \begin{array}{c} s_0 \\ s_6 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{array} & \left[\begin{array}{ccccccc} 1 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & p & 0 & q & 0 & 0 \\ 0 & 0 & 0 & p & 0 & q & 0 \\ 0 & 0 & 0 & 0 & p & 0 & q \\ 0 & 0 & 0 & 0 & 0 & p & 0 \end{array} \right] \end{array}$$

We have added spaces between the elements of the matrix to show the outline of the component matrices $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{T}$. In that case, each closed matrix corresponds to a single absorbing state (s_0 and s_6), while the transient states correspond to a 5×5 matrix. ■

5.6 Transient Analysis

We might want to know how a reducible Markov chain varies with time n since this leads to useful results such as the probability of visiting a certain state at any given time value. In other words, we want to find $\mathbf{s}(n)$ from the expression

$$\mathbf{s}(n) = \mathbf{P}^n \ \mathbf{s}(0) \tag{5.6}$$

Without loss of generality we assume the reducible transition matrix to be given in the form

$$\mathbf{P} = \begin{bmatrix} \mathbf{C} & \mathbf{A} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \tag{5.7}$$

After n time steps the transition matrix of a reducible Markov chain will still be reducible and will have the form

$$\mathbf{P}^n = \begin{bmatrix} \mathbf{C}^n & \mathbf{Y}^n \\ \mathbf{0} & \mathbf{T}^n \end{bmatrix} \tag{5.8}$$

where matrix $\mathbf{Y}^n$ is given by

$$\mathbf{Y}^n = \sum_{i=0}^{n-1} \mathbf{C}^{n-i-1}\,\mathbf{A}\,\mathbf{T}^i \tag{5.9}$$

We can always find $\mathbf{C}^n$ and $\mathbf{T}^n$ using the techniques discussed in Chapter 3 such as diagonalization, finding the Jordan canonic form, or even repeated multiplications. The stochastic matrix $\mathbf{C}^n$ can be expressed in terms of its eigenvalues using (3.80) on page 94.

$$\mathbf{C}^n = \mathbf{C}_1 + \lambda_2^n \mathbf{C}_2 + \lambda_3^n \mathbf{C}_3 + \cdots \tag{5.10}$$

where it was assumed that $\mathbf{C}_1$ is the expansion matrix corresponding to the eigenvalue $\lambda_1 = 1$ and $\mathbf{C}$ is assumed to be of dimension $m_c \times m_c$. Similarly, the substochastic matrix $\mathbf{T}^n$ can be expressed in terms of its eigenvalues using (3.80) on page 94.

$$\mathbf{T}^n = \lambda_1^n \mathbf{T}_1 + \lambda_2^n \mathbf{T}_2 + \lambda_3^n \mathbf{T}_3 + \cdots \tag{5.11}$$

We should note here that all the magnitudes of the eigenvalues in the above equation are less than unity. Equation (5.9) can then be expressed in the form

$$\mathbf{Y}^n = \sum_{j=1}^{m} \mathbf{C}_j\ \mathbf{A} \sum_{i=0}^{n-1} \lambda_j^{n-i-1}\,\mathbf{T}^i \tag{5.12}$$

After some algebraic manipulations, we arrive at the form

$$\mathbf{Y}^n = \sum_{j=1}^{m} \lambda_j^{n-1} \mathbf{C}_j\ \mathbf{A} \left[\mathbf{I} - \left(\frac{\mathbf{T}}{\lambda_j} \right)^n \right] \left(\mathbf{I} - \frac{\mathbf{T}}{\lambda_j} \right)^{-1} \tag{5.13}$$

This can be written in the form

$$
\begin{aligned}
\mathbf{Y}^n \quad = \quad & \mathbf{C}_1\mathbf{A}\,(\mathbf{I}-\mathbf{T})^{-1}\left[\mathbf{I}-\mathbf{T}^n\right]+ \\
& \lambda_2^{n-1}\mathbf{C}_2\mathbf{A}\left(\mathbf{I}-\frac{1}{\lambda_2}\mathbf{T}\right)^{-1}\left[\mathbf{I}-\frac{1}{\lambda_2^n}\mathbf{T}^n\right]+ \\
& \lambda_3^{n-1}\mathbf{C}_3\mathbf{A}\left(\mathbf{I}-\frac{1}{\lambda_3}\mathbf{T}\right)^{-1}\left[\mathbf{I}-\frac{1}{\lambda_3^n}\mathbf{T}^n\right]+\cdots
\end{aligned}
\tag{5.14}
$$

If some of the eigenvalues of $\mathbf{C}$ are repeated, then the above formula has to be modified as explained in Section 3.14 on page 103. Problem 5.25 discusses this situation.

Example 5.4 A reducible Markov chain has the transition matrix

$$
\mathbf{P} = \begin{bmatrix} 0.5 & 0.3 & 0.1 & 0.3 & 0.1 \\ 0.5 & 0.7 & 0.2 & 0.1 & 0.3 \\ 0 & 0 & 0.2 & 0.2 & 0.1 \\ 0 & 0 & 0.1 & 0.3 & 0.1 \\ 0 & 0 & 0.4 & 0.1 & 0.4 \end{bmatrix}
$$

Find the value of $\mathbf{P}^{20}$ and from that find the probability of making the following transitions:

(a) From s_3 to s_2.
(b) From s_2 to s_2.
(c) From s_4 to s_1.
(d) From s_3 to s_4.

The components of the transition matrix are

$$
\begin{aligned}
\mathbf{C} &= \begin{bmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{bmatrix} \\
\mathbf{A} &= \begin{bmatrix} 0.1 & 0.3 & 0.1 \\ 0.2 & 0.1 & 0.3 \end{bmatrix} \\
\mathbf{T} &= \begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.1 & 0.3 & 0.1 \\ 0.4 & 0.1 & 0.4 \end{bmatrix}
\end{aligned}
$$

We use the MATLAB function `EIGPOWERS`, which we developed to expand matrix $\mathbf{C}$ in terms of its eigenpowers, and we have $\lambda_1 = 1$ and $\lambda_2 = 0.2$. The corresponding matrices according to (3.80) are

$$\mathbf{C}_1 = \begin{bmatrix} 0.375 & 0.375 \\ 0.625 & 0.625 \end{bmatrix}$$
$$\mathbf{C}_2 = \begin{bmatrix} 0.625 & -0.375 \\ -0.625 & 0.375 \end{bmatrix}$$

We could now use (5.13) to find $\mathbf{P}^{20}$ but instead we use repeated multiplication here

$$\mathbf{P}^{20} = \begin{bmatrix} 0.375 & 0.375 & 0.375 & 0.375 & 0.375 \\ 0.625 & 0.625 & 0.625 & 0.625 & 0.625 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) $p_{32} = 0.625$
(b) $p_{22} = 0.625$
(c) $p_{14} = 0.375$
(d) $p_{43} = 0$

■

Example 5.5 Find an expression for the transition matrix at times $n = 4$ and $n = 20$ for the reducible Markov chain characterized by the transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.3 & 0.3 & 0.3 & 0.2 \\ 0.1 & 0.7 & 0.2 & 0.1 & 0.3 \\ 0 & 0 & 0.2 & 0.2 & 0.1 \\ 0 & 0 & 0.1 & 0.3 & 0.1 \\ 0 & 0 & 0.2 & 0.1 & 0.2 \end{bmatrix}$$

The components of the transition matrix are

$$\mathbf{C} = \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix}$$
$$\mathbf{A} = \begin{bmatrix} 0.3 & 0.1 & 0.3 \\ 0.2 & 0.1 & 0.3 \end{bmatrix}$$
$$\mathbf{T} = \begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.3 & 0.3 & 0.1 \\ 0.2 & 0.1 & 0.4 \end{bmatrix}$$

$\mathbf{C}^n$ is expressed in terms of its eigenvalues as

$$\mathbf{C}^n = \lambda_1^n \mathbf{C}_1 + \lambda_2^n \mathbf{C}_2$$

where $\lambda_1 = 1$ and $\lambda_2 = 0.6$ and

$$\mathbf{C}_1 = \begin{bmatrix} 0.75 & 0.75 \\ 0.25 & 0.25 \end{bmatrix}$$
$$\mathbf{C}_2 = \begin{bmatrix} 0.25 & -0.75 \\ -0.25 & 0.75 \end{bmatrix}$$

At any time instant n the matrix $\mathbf{Y}^n$ has the value

$$\begin{aligned} \mathbf{Y}^n = {} & \mathbf{C}_1\mathbf{A}\left(\mathbf{I}-\mathbf{T}\right)^{-1}\left[\mathbf{I}-\mathbf{T}^n\right] + \\ & (0.6)^{n-1}\mathbf{C}_2\mathbf{A}\left(\mathbf{I}-\frac{1}{0.6}\mathbf{T}\right)^{-1}\left[\mathbf{I}-\frac{1}{0.6^n}\mathbf{T}^n\right] \end{aligned}$$

By substituting $n = 4$, we get

$$\mathbf{P}^4 = \begin{bmatrix} 0.7824 & 0.6528 & 0.6292 & 0.5564 & 0.6318 \\ 0.2176 & 0.3472 & 0.2947 & 0.3055 & 0.3061 \\ 0 & 0 & 0.0221 & 0.0400 & 0.0180 \\ 0 & 0 & 0.0220 & 0.0401 & 0.0180 \\ 0 & 0 & 0.0320 & 0.0580 & 0.0261 \end{bmatrix}$$

By substituting $n = 20$, we get

$$\mathbf{P}^{20} = \begin{bmatrix} 0.7500 & 0.7500 & 0.7500 & 0.7500 & 0.7500 \\ 0.2500 & 0.2500 & 0.2500 & 0.2500 & 0.2500 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that all the columns of $\mathbf{P}^{20}$ are identical, which indicates that the steady-state distribution vector is independent of its initial value. ■

5.7 Reducible Markov Chains at Steady-State

Assume we have a reducible Markov chain with transition matrix $\mathbf{P}$ that is expressed in the canonic form

$$\mathbf{P} = \begin{bmatrix} \mathbf{C} & \mathbf{A} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \tag{5.15}$$

According to (5.8), after n time steps the transition matrix will have the form

$$\mathbf{P}^n = \begin{bmatrix} \mathbf{C}^n & \mathbf{Y}^n \\ \mathbf{0} & \mathbf{T}^n \end{bmatrix} \tag{5.16}$$

where matrix $\mathbf{Y}^n$ is given by

$$\mathbf{Y}^n = \sum_{i=0}^{n-1} \mathbf{C}^{n-i-1}\,\mathbf{A}\,\mathbf{T}^i \tag{5.17}$$

To see how $\mathbf{P}^n$ will be like when $n \to \infty$, we express the matrices $\mathbf{C}$ and $\mathbf{T}$ in terms of their eigenvalues as in (5.10) and (5.11).

When $n \to \infty$, matrix $\mathbf{Y}^\infty$ becomes

$$\mathbf{Y}^\infty = \mathbf{C}^\infty\,\mathbf{A} \sum_{i=0}^{\infty} \mathbf{T}^i \tag{5.18}$$

$$= \mathbf{C}_1\ \mathbf{A}\ (\mathbf{I} - \mathbf{T})^{-1} \tag{5.19}$$

where $\mathbf{I}$ is the unit matrix whose dimensions match that of $\mathbf{T}$.

We used the following matrix identity to derive the above equation

$$(\mathbf{I} - \mathbf{T})^{-1} = \sum_{i=0}^{\infty} \mathbf{T}^i \tag{5.20}$$

Finally, we can write the steady-state expression for the transition matrix of a reducible Markov chain as

$$\mathbf{P}^\infty = \begin{bmatrix} \mathbf{C}_1 & \mathbf{Y}^\infty \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{5.21}$$

$$= \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_1\mathbf{A}\,(\mathbf{I} - \mathbf{T})^{-1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{5.22}$$

The above matrix is column stochastic since it represents a transition matrix. We can prove that the columns of the matrix $\mathbf{C}_1\mathbf{A}\,(\mathbf{I} - \mathbf{T})^{-1}$ are all identical and equal to the columns of $\mathbf{C}_1$. This is left as an exercise (see Problem 5.16). Since all the columns of $\mathbf{P}$ at steady-state are equal, all we have to do to find $\mathbf{P}^\infty$ is to find one column only of $\mathbf{C}_1$. The following examples show this.

Example 5.6 Find the steady-state transition matrix for the reducible Markov chain characterized by the transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0.4 & 0 & 0.3 & 0.1 \\ 0.2 & 0.6 & 0.2 & 0.2 & 0.3 \\ 0 & 0 & 0.2 & 0.2 & 0.1 \\ 0 & 0 & 0 & 0.3 & 0.1 \\ 0 & 0 & 0.6 & 0 & 0.4 \end{bmatrix}$$

The components of the transition matrix are

$$\begin{aligned}
\mathbf{C} &= \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix} \\
\mathbf{A} &= \begin{bmatrix} 0 & 0.3 & 0.1 \\ 0.2 & 0.2 & 0.3 \end{bmatrix} \\
\mathbf{T} &= \begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0 & 0.3 & 0.1 \\ 0.6 & 0 & 0.4 \end{bmatrix}
\end{aligned}$$

The steady-state value of $\mathbf{C}$ is

$$\mathbf{C}^{\infty} = \mathbf{C}_1 = \begin{bmatrix} 0.6667 & 0.6667 \\ 0.3333 & 0.3333 \end{bmatrix}$$

The matrix $\mathbf{Y}^{\infty}$ has the value

$$\mathbf{Y}^{\infty} = \mathbf{C}_1 \ \mathbf{A} \ (\mathbf{I} - \mathbf{T})^{-1} = \begin{bmatrix} 0.6667 & 0.6667 \\ 0.3333 & 0.3333 \end{bmatrix}$$

Thus the steady state value of $\mathbf{P}$ is

$$\mathbf{P}^{\infty} = \begin{bmatrix} 0.6667 & 0.6667 & 0.6667 & 0.6667 & 0.6667 \\ 0.3333 & 0.3333 & 0.3333 & 0.3333 & 0.3333 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first thing we notice about the steady-state value of the transition matrix is that all columns are identical. This is exactly the same property for the transition matrix of an irreducible Markov chain. The second observation we can make about the transition matrix at steady-state is that there is no possibility of moving to a transient state irrespective of the value of the initial distribution vector. The third observation we can make is that no matter what the initial distribution vector was, we will always wind up in the same steady-state distribution. ■

Example 5.7 Find the steady-state transition matrix for the reducible Markov chain characterized by the transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.3 & 0.1 & 0.3 & 0.1 \\ 0.5 & 0.7 & 0.2 & 0.1 & 0.3 \\ 0 & 0 & 1 & 0.2 & 0.1 \\ 0 & 0 & 0 & 0.3 & 0.1 \\ 0 & 0 & 0 & 0.1 & 0.4 \end{bmatrix}$$

The components of the transition matrix are

$$\mathbf{C} = \begin{bmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{bmatrix}$$
$$\mathbf{A} = \begin{bmatrix} 0.1 & 0.3 & 0.1 \\ 0.2 & 0.1 & 0.3 \end{bmatrix}$$
$$\mathbf{T} = \begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.1 & 0.3 & 0.1 \\ 0.4 & 0.1 & 0.4 \end{bmatrix}$$

The steady-state value of $\mathbf{C}$ is

$$\mathbf{C}^\infty = \mathbf{C}_1 = \begin{bmatrix} 0.375 & 0.375 \\ 0.625 & 0.625 \end{bmatrix}$$

The matrix $\mathbf{Y}^\infty$ has the value

$$\mathbf{Y}^\infty = \mathbf{C}_1\ \mathbf{A}\ (\mathbf{I} - \mathbf{T})^{-1} = \begin{bmatrix} 0.375 & 0.375 & 0.375 \\ 0.625 & 0.625 & 0.625 \end{bmatrix}$$

Thus the steady-state value of $\mathbf{P}$ is

$$\mathbf{P}^\infty = \begin{bmatrix} 0.375 & 0.375 & 0.375 & 0.375 & 0.375 \\ 0.625 & 0.625 & 0.625 & 0.625 & 0.625 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

■

5.8 Reducible Composite Markov Chains at Steady-State

In this section, we will study the steady-state behavior of reducible composite Markov chains. In the general case, the reducible Markov chain could be composed of two or more closed states. Figure 5.4 shows a reducible Markov chain with two sets of closed states. If the system is in the transient state T, it can move to either sets of closed states, C_1 or C_2. However, if the system is in state C_1, it cannot move to T or C_2. Similarly, if the system is in state C_2, it can not move to T or C_1.

Assume the transition matrix is given by the canonic form

$$\mathbf{P} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} & \mathbf{A}_1 \\ \mathbf{0} & \mathbf{C}_2 & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{T} \end{bmatrix} \tag{5.23}$$

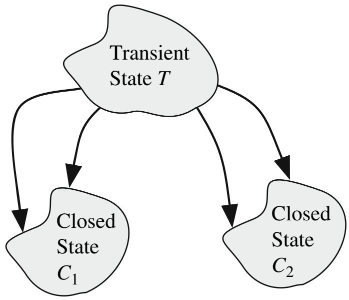

Fig. 5.4 A reducible Markov chain with two sets of closed states

where

$$\begin{aligned} \mathbf{C}_1 \text{ and } \mathbf{C}_2 &= \text{square column stochastic matrices} \\ \mathbf{A}_1 \text{ and } \mathbf{A}_2 &= \text{rectangular nonnegative matrices} \\ \mathbf{T} &= \text{square column substochastic matrix} \end{aligned}$$

It is easy to verify that the steady-state transition matrix for such a system will be

$$\mathbf{P}^{\infty} = \begin{bmatrix} \mathbf{C}'_1 & \mathbf{0} & \mathbf{Y}_1 \\ \mathbf{0} & \mathbf{C}''_1 & \mathbf{Y}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{5.24}$$

where

$$\mathbf{C}'_1 = \mathbf{C}_1^{\infty} \tag{5.25}$$
$$\mathbf{C}''_1 = \mathbf{C}_2^{\infty} \tag{5.26}$$
$$\mathbf{Y}_1 = \mathbf{C}'_1\mathbf{A}_1 (\mathbf{I} - \mathbf{T})^{-1} \tag{5.27}$$
$$\mathbf{Y}_2 = \mathbf{C}''_1\mathbf{A}_2 (\mathbf{I} - \mathbf{T})^{-1} \tag{5.28}$$

Essentially, $\mathbf{C}'_1$ is the matrix that is associated with $\lambda = 1$ in the expansion of $\mathbf{C}_1$ in terms of its eigenvalues. The same also applies to $\mathbf{C}''_1$, which is the matrix that is associated with $\lambda = 1$ in the expansion of $\mathbf{C}_2$ in terms of its eigenvalues.

We observe that each column of matrix $\mathbf{Y}_1$ is a scaled copy of the columns of $\mathbf{C}'_1$. Also, the sum of each column of $\mathbf{Y}_1$ is lesser than one. We can make the same observations about matrix $\mathbf{Y}_2$.

Example 5.8 The given transition matrix corresponds to a composite reducible Markov chain.

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.3 & 0 & 0 & 0.1 & 0.4 \\ 0.5 & 0.7 & 0 & 0 & 0.3 & 0.1 \\ 0 & 0 & 0.2 & 0.7 & 0.1 & 0.2 \\ 0 & 0 & 0.8 & 0.3 & 0.1 & 0.1 \\ 0 & 0 & 0 & 0 & 0.1 & 0.2 \\ 0 & 0 & 0 & 0 & 0.3 & 0 \end{bmatrix}$$

Find its eigenvalues and eigenvectors then find the steady-state distribution vector.

The components of the transition matrix are

$$\mathbf{C}_1 = \begin{bmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{bmatrix}$$

$$\mathbf{C}_2 = \begin{bmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{bmatrix}$$

$$\mathbf{A}_1 = \begin{bmatrix} 0.2 & 0.4 \\ 0.2 & 0.1 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0 \end{bmatrix}$$

The steady-state value of $\mathbf{C}_1$ is

$$\mathbf{C}'_1 = \begin{bmatrix} 0.375 & 0.375 \\ 0.625 & 0.625 \end{bmatrix}$$

The matrix $\mathbf{Y}_1$ has the value

$$\mathbf{Y}_1 = \mathbf{C}'_1 \, \mathbf{A}_1 \, (\mathbf{I} - \mathbf{T})^{-1} = \begin{bmatrix} 0.2455 & 0.2366 \\ 0.4092 & 0.3943 \end{bmatrix}$$

The steady-state value of $\mathbf{C}_2$ is

$$\mathbf{C}''_1 = \begin{bmatrix} 0.4667 & 0.4667 \\ 0.5333 & 0.5333 \end{bmatrix}$$

The matrix $\mathbf{Y}_2$ has the value

$$\mathbf{Y}_2 = \mathbf{C}''_1 \, \mathbf{A}_2 \, (\mathbf{I} - \mathbf{T})^{-1} = \begin{bmatrix} 0.1611 & 0.1722 \\ 0.1841 & 0.1968 \end{bmatrix}$$

Thus the steady-state value of $\mathbf{P}$ is

$$\mathbf{P}^{\infty} = \begin{bmatrix} 0.375 & 0.375 & 0 & 0 & 0.2455 & 0.2366 \\ 0.625 & 0.625 & 0 & 0 & 0.4092 & 0.3943 \\ 0 & 0 & 0.4667 & 0.4667 & 0.1611 & 0.1722 \\ 0 & 0 & 0.5333 & 0.5333 & 0.1841 & 0.1968 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We notice that all columns are identical for the closed state matrices. However, the columns for the matrices corresponding to the transient states ($\mathbf{Y}_1$ and $\mathbf{Y}_2$) are not. The second observation we can make about the transition matrix at steady-state is that there is no possibility of moving to a transient state irrespective of the value of the initial distribution vector. The third observation we can make is that no matter what the initial distribution vector was, we will always wind up in the same steady-state distribution. ∎

5.9 Identifying Reducible Markov Chains

We saw above that reducible Markov chains have a transition matrix that can be expressed, by proper reordering of the states, into the canonic form

$$\mathbf{P} = \begin{bmatrix} \mathbf{C} & \mathbf{A} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \tag{5.29}$$

This rearranged matrix allowed us to determine the closed and transient states. We want to show in this section how to easily identify a reducible Markov chain and how to find its closed and transient states without having to rearrange the matrix. The following theorem helps us to determine if our Markov chain is reducible or not by observing the structure of its eigenvector corresponding to the eigenvalue $\lambda = 1$.

Theorem 5.1 *Let* $\mathbf{P}$ *be the transition matrix of a Markov chain whose eigenvalue* $\lambda = 1$ *corresponds to an eigenvector* $\mathbf{s}$. *Then this chain is reducible if and only if* $\mathbf{s}$ *has one or more zero elements.*

Proof. We start by assuming that the eigenvector $\mathbf{s}$ has k nonzero elements and $m - k$ zero elements where m is the number of rows and columns of $\mathbf{P}$. Without loss of generality we can write $\mathbf{s}$ in the canonic form

$$\mathbf{s} = \begin{bmatrix} \mathbf{a} \\ \mathbf{0} \end{bmatrix} \tag{5.30}$$

where the vector $\mathbf{a}$ has k elements none of which is zero such that $0 < k < m$. Partition $\mathbf{P}$ into the form

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \tag{5.31}$$

where $\mathbf{A}$ is a square $k \times k$ matrix, $\mathbf{D}$ is a square $(m - k) \times (m - k)$ matrix, and the other two matrices are rectangular with the proper dimensions. Since $\mathbf{s}$ is the eigenvector corresponding to $\lambda = 1$, we can write

$$\mathbf{P}\,\mathbf{s} = \mathbf{s} \tag{5.32}$$

or

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{a} \\ \mathbf{0} \end{bmatrix} \tag{5.33}$$

This equation results in

$$\mathbf{A}\,\mathbf{a} = \mathbf{a} \tag{5.34}$$

and

$$\mathbf{C}\,\mathbf{a} = \mathbf{0} \tag{5.35}$$

Having $\mathbf{a} = \mathbf{0}$ is contrary to our assumptions. Since the above two equations are valid for any nonzero value of $\mathbf{a}$, we conclude that $\mathbf{A}$ is column stochastic and $\mathbf{C} = \mathbf{0}$.

Thus the transition matrix reduces to the form

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \tag{5.36}$$

This is the general canonic form for a reducible Markov chain and this completes one part of the proof.

Now let us assume that $\mathbf{P}$ corresponds to a reducible Markov chain. In that case, we can write $\mathbf{P}$ in the canonic form

$$\mathbf{P} = \begin{bmatrix} \mathbf{C} & \mathbf{A} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \tag{5.37}$$

There are two cases to consider here: $\mathbf{A} = \mathbf{0}$ and $\mathbf{A} \neq \mathbf{0}$.

Case 1: $\mathbf{A} = \mathbf{0}$

This is the case when we have

$$\mathbf{P} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \tag{5.38}$$

We have in reality two independent and noncommunicating Markov systems. Assume vector $\mathbf{s}$ is the distribution vector associated with the unity eigenvalue for matrix $\mathbf{C}$. In that case, we can express $\mathbf{s}$ as

$$\mathbf{s} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix}^t \tag{5.39}$$

and $\mathbf{s}$ satisfies the equations

$$\begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \tag{5.40}$$

We can write

$$\mathbf{C\,a} = \mathbf{a} \tag{5.41}$$

$$\mathbf{T\,b} = \mathbf{b} \tag{5.42}$$

The first equation indicates that **a** is an eigenvector of **C** and it should be a valid distribution vector. Since the sum of the components of **a** must be unity, the sum of the components of **b** must be zero, which is possible only when

$$\mathbf{b} = \mathbf{0} \tag{5.43}$$

This completes the second part of the proof for Case 1.

The same is true for the eigenvector corresponding to unity eigenvalue for matrix **T**. In that case, **a** will be null and **b** will be the valid distribution vector. Either way, this completes the second part of the proof for Case 1.

Case 2: $\mathbf{A} \neq \mathbf{0}$
This is the case when we have

$$\mathbf{P} = \begin{bmatrix} \mathbf{C} & \mathbf{A} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \tag{5.44}$$

In that case, **T** is a substochastic matrix and $\mathbf{T}^\infty = 0$. Now for large time values $(n \to \infty)$ we have

$$\mathbf{P}^\infty = \begin{bmatrix} \mathbf{X} \\ \mathbf{0} \end{bmatrix} \tag{5.45}$$

But we can also write

$$\mathbf{P}^\infty\,\mathbf{s} = \mathbf{s} \tag{5.46}$$

We partition **s** into the form

$$\mathbf{s} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \tag{5.47}$$

Substitute the above equation into (5.46) to get

$$\begin{bmatrix} \mathbf{X} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \tag{5.48}$$

And we get the two equations

$$\mathbf{X\,a} = \mathbf{a} \tag{5.49}$$

and

$$\mathbf{b} = \mathbf{0} \tag{5.50}$$

Thus the distribution vector corresponding to the eigenvalue $\lambda = 1$ will have the form

$$\mathbf{s} = \begin{bmatrix} \mathbf{a} \\ \mathbf{0} \end{bmatrix} \tag{5.51}$$

This completes the proof of the theorem for Case 2.

■

Example 5.9 Prove that the given transition matrix corresponds to a reducible Markov chain.

$$\mathbf{P} = \begin{bmatrix} 0 & 0.7 & 0.1 & 0 & 0 \\ 0.3 & 0 & 0.1 & 0 & 0 \\ 0.1 & 0.1 & 0.2 & 0 & 0 \\ 0.4 & 0 & 0.1 & 0.6 & 0.7 \\ 0.2 & 0.2 & 0.5 & 0.4 & 0.3 \end{bmatrix}$$

We calculate the eigenvalues and eigenvectors for the transition matrix. The distribution vector associated with the eigenvalue $\lambda = 1$ is

$$\mathbf{s} = \begin{bmatrix} 0 & 0 & 0 & 0.6364 & 0.3636 \end{bmatrix}^t$$

Since we have zero elements, we conclude that we have a reducible Markov chain.

5.9.1 Determining Closed and Transient States

Now that we know how to recognize a reducible Markov chain, we need to know how to recognize its closed and transient states. The following theorem provides the answer.

Theorem 5.2 *Let* $\mathbf{P}$ *be the transition matrix of a reducible Markov chain whose eigenvalue* $\lambda = 1$ *corresponds to an eigenvector* $\mathbf{s}$*. The closed states of the chain correspond to the nonzero elements of* $\mathbf{s}$ *and the transient states of the chain correspond to the zero elements of* $\mathbf{s}$.

Proof. Since we are dealing with a reducible Markov chain, then without loss of generality, the transition matrix can be arranged in the canonic form

$$\mathbf{P} = \begin{bmatrix} \mathbf{C} & \mathbf{A} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \tag{5.52}$$

where it is assumed that $\mathbf{C}$ is a $k \times k$ matrix and $\mathbf{T}$ is a $(m-k) \times (m-k)$ matrix. The first k states correspond to closed states and the last $m-k$ states correspond to transient states.

Assume the eigenvector $\mathbf{s}$ is expressed in the form

$$\mathbf{s} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}^t \tag{5.53}$$

where some of the elements of $\mathbf{s}$ are zero according to Theorem 5.1. Since this is the eigenvector corresponding to unity eigenvalue, we must have

$$\begin{bmatrix} \mathbf{C} & \mathbf{A} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \tag{5.54}$$

And we get the two equations

$$\mathbf{C}\,\mathbf{a} + \mathbf{A}\,\mathbf{b} = \mathbf{a} \tag{5.55}$$

and

$$\mathbf{T}\,\mathbf{b} = \mathbf{b} \tag{5.56}$$

The above equation seems to indicate that $\mathbf{T}$ has an eigenvector $\mathbf{b}$ with unity eigenvalue. However, this is a contradiction since $\mathbf{T}$ is column substochastic and it cannot have a unity eigenvalue. The absolute values of all the eigenvalues of $\mathbf{T}$ are less than unity [1]. For such a matrix, we say that its spectral radius cannot equal unity.[1] The above equation is satisfied only if

$$\mathbf{b} = \mathbf{0} \tag{5.57}$$

In that case, (5.55) becomes

$$\mathbf{C}\,\mathbf{a} = \mathbf{a} \tag{5.58}$$

Thus the eigenvector $\mathbf{s}$ will have the form

$$\mathbf{s} = \begin{bmatrix} \mathbf{a} \\ \mathbf{0} \end{bmatrix}^t \tag{5.59}$$

where $\mathbf{a}$ is a k-distribution vector corresponding to unity eigenvalue of $\mathbf{C}$.

We can therefore associate the closed states with the nonzero components of $\mathbf{s}$ and associate the transient states with the zero components of $\mathbf{s}$.

---

[1] Spectral radius equals the largest absolute value of the eigenvalues of a matrix.

So far we have proven that **s** has the form given in (5.59). We must prove now that *all* the components of **a** are nonzero. This will allow us to state with certainty that any zero component of **s** belongs solely to a transient state.

We prove this by proving that a contradiction results if **a** is assumed to have one or more zero components in it. Assume that **a** has one or more zero components. We have proven, however, that **a** satisfies the equation

$$\mathbf{C}\,\mathbf{a} = \mathbf{a} \tag{5.60}$$

where **C** is a nonreducible matrix. Applying Theorem 5.1, on page 167, to the above equation would indicate that **C** is reducible. This is a contradiction since **C** is a nonreducible matrix.

Thus the k closed states correspond to the nonzero elements of **s** and the transient states of the chain correspond to the zero elements of **s**. This proves the theorem. ■

Example 5.10 Prove that the given transition matrix corresponds to a reducible Markov chain.

$$\mathbf{P} = \begin{bmatrix} 0.3 & 0.2 & 0.3 & 0.4 & 0.1 & 0.2 \\ 0 & 0.1 & 0.2 & 0 & 0 & 0 \\ 0 & 0.2 & 0.1 & 0 & 0 & 0 \\ 0.4 & 0.1 & 0.2 & 0.1 & 0.2 & 0.3 \\ 0.1 & 0.1 & 0.2 & 0.2 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0 & 0.3 & 0.4 & 0.1 \end{bmatrix}$$

We calculate the eigenvalues and eigenvectors for the transition matrix. The distribution vector associated with the eigenvalue $\lambda = 1$ is

$$\mathbf{s} = \begin{bmatrix} 0.25 & 0 & 0 & 0.25 & 0.25 & 0.25 \end{bmatrix}^t$$

Since we have zero elements, we conclude that we have a reducible Markov chain. The zero elements identify the transient states and the nonzero elements identify the closed states.

Closed States	Transient States
1, 4, 5, 6	2, 3

5.10 Identifying Reducible Composite Matrices

We can generalize Theorem 5.2 as follows. Let **P** be the transition matrix of a composite reducible Markov chain with u mutually exclusive closed states corresponding to the sets C_1, C_2, ..., C_u. The canonic form for the transition matrix of such a system will be

$$\mathbf{P} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_1 \\ \mathbf{0} & \mathbf{C}_1 & \cdots & \mathbf{0} & \mathbf{A}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{C}_u & \mathbf{A}_u \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{T} \end{bmatrix} \tag{5.61}$$

The eigenvalue $\lambda = 1$ corresponds to the eigenvectors $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_u$ such that

$$\mathbf{P}\,\mathbf{s}_1 = \mathbf{s}_1 \tag{5.62}$$
$$\mathbf{P}\,\mathbf{s}_2 = \mathbf{s}_2 \tag{5.63}$$
$$\vdots$$
$$\mathbf{P}\,\mathbf{s}_u = \mathbf{s}_u \tag{5.64}$$

The eigenvectors also satisfy the equations

$$\mathbf{C}_1\,\mathbf{s}_1 = \mathbf{s}_1 \tag{5.65}$$
$$\mathbf{C}_2\,\mathbf{s}_2 = \mathbf{s}_2 \tag{5.66}$$
$$\vdots$$
$$\mathbf{C}_u\,\mathbf{s}_u = \mathbf{s}_u \tag{5.67}$$

We can in fact write each eigenvector $\mathbf{s}_i$ in block form as

$$\mathbf{s}_1 = \begin{bmatrix} \mathbf{a}_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{bmatrix}^t \tag{5.68}$$
$$\mathbf{s}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{a}_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{bmatrix}^t \tag{5.69}$$
$$\vdots$$
$$\mathbf{s}_u = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{a}_u & \mathbf{0} \end{bmatrix}^t \tag{5.70}$$

where each $\mathbf{a}_i$ is a nonzero vector whose dimension matches $\mathbf{C}_i$ such that

$$\mathbf{C}_1\,\mathbf{a}_1 = \mathbf{a}_1 \tag{5.71}$$
$$\mathbf{C}_2\,\mathbf{a}_2 = \mathbf{a}_2 \tag{5.72}$$
$$\vdots$$
$$\mathbf{C}_u\,\mathbf{a}_u = \mathbf{a}_u \tag{5.73}$$

which means that $\mathbf{a}_i$ is a distribution (sum of its components is unity).

Vector $\mathbf{s}_i$ corresponds to the set of closed states C_i and the transient states of the chain correspond to the zero elements common to all the vectors $\mathbf{s}_i$.

Example 5.11 Assume a composite reducible transition matrix where the number of closed states is $u = 3$ such that the partitioned matrices are

$$\begin{aligned}
\mathbf{C}_1 &= \begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix} \\
\mathbf{C}_2 &= \begin{bmatrix} 0.5 & 0.1 \\ 0.5 & 0.9 \end{bmatrix} \\
\mathbf{C}_3 &= \begin{bmatrix} 0.2 & 0.3 \\ 0.8 & 0.7 \end{bmatrix} \\
\mathbf{A}_1 &= \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.1 \end{bmatrix} \\
\mathbf{A}_2 &= \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.1 \end{bmatrix} \\
\mathbf{A}_3 &= \begin{bmatrix} 0 & 0.2 \\ 0.1 & 0 \end{bmatrix} \\
\mathbf{T} &= \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.3 \end{bmatrix}
\end{aligned}$$

Determine the eigenvectors corresponding to the eigenvalue $\lambda = 1$ and identify the closed and transient states with the elements of those eigenvectors.

The composite transition matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} & \mathbf{0} & \mathbf{A}_1 \\ \mathbf{0} & \mathbf{C}_2 & \mathbf{0} & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_3 & \mathbf{A}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{T} \end{bmatrix}$$

Let us find the eigenvalues and eigenvectors for $\mathbf{P}$. The eigenvalues are

$$\begin{aligned}
\lambda_1 &= -0.3 \\
\lambda_2 &= 1 \\
\lambda_3 &= 0.4 \\
\lambda_4 &= 1 \\
\lambda_5 &= -0.1 \\
\lambda_6 &= 1 \\
\lambda_7 &= 0.0268 \\
\lambda_8 &= 0.3732
\end{aligned}$$

The eigenvectors corresponding to unity eigenvalue (after normalization so their sums is unity) are

$$\begin{aligned}
\mathbf{s}_1 &= \begin{bmatrix} 0.4615 & 0.5385 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^t \\
\mathbf{s}_2 &= \begin{bmatrix} 0 & 0 & 0.1667 & 0.8333 & 0 & 0 & 0 & 0 \end{bmatrix}^t \\
\mathbf{s}_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0.2727 & 0.7273 & 0 & 0 \end{bmatrix}^t
\end{aligned}$$

The sets of closed and transient states are as follows.

Set	States
C_1	1, 2
C_2	3, 4
C_3	5, 6
T	7, 8

Problems

Reducible Markov Chains

For Problems 5.1–5.8: (a) Determine whether the given Markov matrices have absorbing or closed states; (b) express such matrices in the form given in (5.2) or (5.3); (c) identify the component matrices **C**, **A**, and **T**; and (d) identify the closed and transient states.

5.1

$$\mathbf{P} = \begin{bmatrix} 0.3 & 0 \\ 0.7 & 1 \end{bmatrix}$$

5.2

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0 & 0.2 \\ 0.3 & 1 & 0.3 \\ 0.2 & 0 & 0.5 \end{bmatrix}$$

5.3

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.3 & 0.5 & 0 \\ 0.2 & 0 & 1 \end{bmatrix}$$

5.4

$$\mathbf{P} = \begin{bmatrix} 0.7 & 0.5 & 0.1 \\ 0.3 & 0.5 & 0 \\ 0 & 0 & .9 \end{bmatrix}$$

5.5

$$\mathbf{P} = \begin{bmatrix} 0.2 & 0 & 0 \\ 0.3 & 1 & 0 \\ 0.5 & 0 & 1 \end{bmatrix}$$

5.6

$$\mathbf{P} = \begin{bmatrix} 0.1 & 0 & 0.5 & 0 \\ 0.2 & 1 & 0 & 0 \\ 0.3 & 0 & 0.5 & 0 \\ 0.4 & 0 & 0 & 1 \end{bmatrix}$$

5.7

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.2 \\ 0 & 0.2 & 1 & 0 \\ 0 & 0.3 & 0 & 0.8 \end{bmatrix}$$

5.8

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0.5 \\ 0 & 0.2 & 1 & 0 \\ 0 & 0.7 & 0 & 0.5 \end{bmatrix}$$

Composite Reducible Markov Chains

The transition matrices in Problems 5.9–5.12 represent composite reducible Markov chains. Identify the sets of closed and transient states, find the eigenvalues and eigenvectors for the matrices, and find the value of each matrix for large values of n, say when $n = 50$.

5.9

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0.3 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 1 & 0.6 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}$$

5.10

$$\mathbf{P} = \begin{bmatrix} 0.7 & 0 & 0 & 0 \\ 0.1 & 0.9 & 0.2 & 0 \\ 0.1 & 0.1 & 0.8 & 0 \\ 0.1 & 0 & 0 & 1 \end{bmatrix}$$

5.11

$$\mathbf{P} = \begin{bmatrix} 0.4 & 0 & 0 & 0 & 0 \\ 0.2 & 0.5 & 0.8 & 0 & 0 \\ 0.1 & 0.5 & 0.2 & 0 & 0 \\ 0.1 & 0 & 0 & 0.7 & 0.9 \\ 0.2 & 0 & 0 & 0.3 & 0.1 \end{bmatrix}$$

5.12

$$\mathbf{P} = \begin{bmatrix} 0.2 & 0 & 0 & 0.2 & 0 & 0.6 \\ 0 & 0.3 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.2 & 0.2 & 0.3 & 0 \\ 0 & 0.3 & 0 & 0.3 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0.7 & 0 \\ 0.8 & 0.2 & 0 & 0 & 0 & 0.4 \end{bmatrix}$$

Transient Analysis

5.13 Check whether the given transitions matrix is reducible or irreducible. Identify the closed and transient states and express the matrix in the form of (5.2) or (5.3) and identify the component matrices **C**, **A**, and **T**.

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0.25 & 0 \\ 0 & 0 & 1 & 0.25 & 0 \\ 0 & 0.5 & 0 & 0.25 & 0 \\ 0.5 & 0 & 0 & 0.25 & 0.5 \end{bmatrix}$$

Find the value of $\mathbf{P}^{10}$ using (5.8) and (5.9) on page 158 and verify your results using repeated multiplications.

5.14 Assume the transition matrix $\mathbf{P}$ has the structure given in (5.1) or (5.2). Prove that $\mathbf{P}^n$ also possesses the same structure as the original matrix and prove also that the component matrices $\mathbf{C}$, $\mathbf{A}$, and $\mathbf{T}$ have the same properties as the original component matrices.

5.15 Find an expression for the transition matrix using (5.13) at time $n = 4$ and $n = 20$ for the reducible Markov chain characterized by the transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.7 & 0.9 & 0.3 & 0.3 & 0.2 \\ 0.3 & 0.1 & 0.2 & 0.1 & 0.3 \\ 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & 0.2 & 0.2 & 0.3 \\ 0 & 0 & 0.1 & 0.2 & 0.2 \end{bmatrix}$$

Find the value of $\mathbf{P}^{10}$ using (5.8) and verify your results using repeated multiplications.

Reducible Markov Chains at Steady-State

5.16 In Section 5.7, it was asserted that the transition matrix for a reducible Markov chain will have the form of (5.22) where all the columns of the matrix are identical. Prove that assertion knowing that

(a) all the columns of $\mathbf{C}_1$ are all identical
(b) matrix $\mathbf{C}_1$ is column stochastic
(c) matrix $\mathbf{Y}^\infty$ is column stochastic
(d) the columns of $\mathbf{Y}^\infty$ are identical to the columns of $\mathbf{C}_1$.

5.17 Find the steady-state transition matrix and distribution vector for the reducible Markov chain characterized by the matrix

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.2 & 0.5 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 & 0.2 & 0.3 \\ 0 & 0 & 0.2 & 0.2 & 0.1 \\ 0 & 0 & 0.2 & 0.2 & 0.1 \\ 0 & 0 & 0 & 0.3 & 0.4 \end{bmatrix}$$

5.18 Find the steady-state transition matrix and distribution vector for the reducible Markov chain characterized by the matrix

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.2 & 0.5 & 0.1 \\ 0.1 & 0.8 & 0.1 & 0.2 \\ 0 & 0 & 0.4 & 0.2 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

5.19 Find the steady-state transition matrix and distribution vector for the reducible Markov chain characterized by the matrix

$$\mathbf{P} = \begin{bmatrix} 0.1 & 0.2 & 0.5 & 0.1 & 0.1 \\ 0.5 & 0.7 & 0.1 & 0.2 & 0.3 \\ 0.4 & 0.1 & 0.4 & 0.2 & 0.1 \\ 0 & 0 & 0 & 0.2 & 0.1 \\ 0 & 0 & 0 & 0.3 & 0.4 \end{bmatrix}$$

5.20 Find the steady-state transition matrix and distribution vector for the reducible Markov chain characterized by the matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0.2 & 0.2 & 0.1 & 0.1 \\ 0 & 0.3 & 0.1 & 0.2 & 0.3 \\ 0 & 0.1 & 0.4 & 0.2 & 0.1 \\ 0 & 0.3 & 0.1 & 0.2 & 0.1 \\ 0 & 0.1 & 0.2 & 0.3 & 0.4 \end{bmatrix}$$

5.21 Find the steady-state transition matrix and distribution vector for the reducible Markov chain characterized by the matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0.2 & 0.1 & 0.1 \\ 0 & 1 & 0.1 & 0.2 & 0.3 \\ 0 & 0 & 0.4 & 0.2 & 0.1 \\ 0 & 0 & 0.1 & 0.2 & 0.1 \\ 0 & 0 & 0.2 & 0.3 & 0.4 \end{bmatrix}$$

Note that this matrix has two absorbing states.

5.22 Consider the state transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1-p & 0 & q \\ p & 0 & 1-q \end{bmatrix}$$

(a) Can this matrix represent a reducible Markov chain?
(b) Find the distribution vector at equilibrium.
(c) What values of p and q give $s_0 = s_1 = s_2$?

5.23 Consider a discrete-time Markov chain in which the transition probabilities are given by

$$p_{ij} = q^{|i-j|} p$$

For a 3×3 case, what are the values of p and q to make this a reducible Markov chain? What are the values of p and q to make this an irreducible Markov chain and find the steady-state distribution vector.

5.24 Consider the coin-tossing Example 5.3 on page 156. Derive the equilibrium distribution vector and comment on it for the cases $p < q$, $p = q$, and $p > q$.

5.25 Rewrite (5.10) on page 158 to take into account the fact that some of the eigenvalues of $\mathbf{C}$ might be repeated using the results of Section 3.14 on page 103.

Identification of Reducible Markov Chains

Use the results of Section 5.9 to verify that the transition matrices in the following problems correspond to reducible Markov chains and identify the closed and transient states. Rearrange each matrix to the standard form as in (5.2) or (5.3).

5.26

$$\mathbf{P} = \begin{bmatrix} 0.3 & 0.1 & 0.4 \\ 0 & 0.1 & 0 \\ 0.7 & 0.8 & 0.6 \end{bmatrix}$$

5.27

$$\mathbf{P} = \begin{bmatrix} 0.4 & 0.6 & 0 \\ 0.4 & 0.3 & 0 \\ 0.2 & 0.1 & 1 \end{bmatrix}$$

5.28

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.3 & 0 & 0 \\ 0.1 & 0.1 & 0 & 0 \\ 0.3 & 0.4 & 0.2 & 0.9 \\ 0.1 & 0.2 & 0.8 & 0.1 \end{bmatrix}$$

5.29

$$\mathbf{P} = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0.3 & 0.1 & 0.5 & 0.3 \\ 0.4 & 0.3 & 0.4 & 0.2 \\ 0.2 & 0.6 & 0.1 & 0.5 \end{bmatrix}$$

5.30

$$\mathbf{P} = \begin{bmatrix} 0.2 & 0.3 & 0 & 0.3 & 0 \\ 0.2 & 0.4 & 0 & 0 & 0 \\ 0.3 & 0 & 0.5 & 0.1 & 0.2 \\ 0.2 & 0 & 0 & 0.4 & 0 \\ 0.1 & 0.3 & 0.5 & 0.2 & 0.8 \end{bmatrix}$$

5.31

$$\mathbf{P} = \begin{bmatrix} 0.2 & 0.1 & 0.3 & 0.1 & 0.6 \\ 0 & 0.3 & 0 & 0.2 & 0 \\ 0.2 & 0.2 & 0.4 & 0.1 & 0.3 \\ 0 & 0.1 & 0 & 0.4 & 0 \\ 0.6 & 0.3 & 0.3 & 0.2 & 0.1 \end{bmatrix}$$

5.32

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0.4 & 0 & 0.5 & 0 \\ 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0.8 & 0 & 0.3 \\ 0.2 & 0.2 & 0 & 0.5 & 0 \\ 0 & 0.2 & 0.2 & 0 & 0.7 \end{bmatrix}$$

5.33

$$\mathbf{P} = \begin{bmatrix} 0.1 & 0 & 0.2 & 0 & 0 & 0.3 \\ 0.2 & 0.5 & 0.1 & 0 & 0.4 & 0.1 \\ 0.1 & 0 & 0.3 & 0 & 0 & 0.3 \\ 0.2 & 0 & 0.1 & 1 & 0 & 0 \\ 0.3 & 0.5 & 0.2 & 0 & 0.6 & 0.1 \\ 0.1 & 0 & 0.1 & 0 & 0 & 0.2 \end{bmatrix}$$

5.34

$$\mathbf{P} = \begin{bmatrix} 0.1 & 0.5 & 0 & 0.1 & 0 & 0 & 0.6 \\ 0.7 & 0.3 & 0 & 0.2 & 0 & 0 & 0.3 \\ 0 & 0 & 0.5 & 0.3 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0.3 & 0 \\ 0 & 0 & 0.5 & 0 & 0.9 & 0.2 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0.4 & 0 \\ 0.2 & 0.2 & 0 & 0 & 0 & 0.1 & 0.1 \end{bmatrix}$$

References

1. R.A. Horn and C.R. Johnson, *Matrix Analysis, Cambridge University Press, Cambridge, 1985.*