Learning from Prices: Amplification and Sentiments*

Ryan Chahrour† Gaetano Gaballo ‡
Boston College Banque de France

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Abstract

We provide a new theory of expectations-driven business cycles in an economy where all shocks are fundamental and consumers learn from the prices they observe. Learning from prices causes small changes in aggregate productivity to induce large shifts in aggregate beliefs, generating positive price-quantity comovement. Amplification may be strengthened when fundamental shocks become smaller so that, in the limit of arbitrarily small shocks, aggregate fluctuations appear to be driven by "sentiments." Providing noisy public information reinforces amplification rather than dampening it, while weakening price-quantity correlation. In extreme cases, quantities respond exclusively to noise while prices respond exclusively to fundamentals. Finally, we analyze equilibrium stability and find that equilibria exhibiting high amplification may be stable.

Keywords: animal spirits, expectational coordination, imperfect information.

JEL Classification: D82, D83, E3.

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†Boston College. Email: ryan.chahrour@bc.edu

‡Banque de France, Monetary Policy Division. Email: gaetano.gaballo@banque-france.fr. The views expressed in this paper do not necessarily reflect the opinion of the Banque de France or the Eurosystem.

1 Introduction

We propose a new mechanism – based on learning from prices – that delivers expectations-driven economic fluctuations without relying on any source of extrinsic noise. We show that when consumers learn from the prices of the goods they consume, higher prices can lead consumers to become unduly optimistic about their economic prospects. Initial optimism causes consumers to demand more goods, further increasing prices beyond their full-information level. The self-reinforcing nature of this feedback loop leads to equilibria in which small initial shocks to supply drive large changes in beliefs, inducing the positive price-quantity comovement typically associated with demand shocks. We show that such equilibria can occur in a standard economic environment and provide a rich set of testable implications for the study of business cycles.

The mechanism of this paper offers a new resolution to a long-standing challenge for macroeconomic theory: how to rationalize large fluctuations in economic outcomes with the rather small measured volatility of total factor productivity and other aggregate fundamentals that may drive those outcomes. Our analysis unifies two competing approaches to resolving this problem. First, it shares the insight of the recent noise shock and sentiment literature, which shows that fluctuations may be driven by expectational errors that are correlated across agents (Lorenzoni, 2009; Angeletos and La'O, 2013; Benhabib et al., 2015). Second, it shares the focus on amplification that characterizes the literature studying aggregate transmission mechanisms, which lead otherwise modest economic shocks to have larger aggregate consequences (Kiyotaki and Moore, 1997; Bernanke et al., 1999; Brunnermeier and Sannikov, 2014). In our environment, expectational errors originate with fundamental shocks and are amplified by agents' inference using endogenous price signals.

We begin our analysis with a static microfounded economy inspired by the large family metaphor introduced by Lucas (1980). The economy is divided into islands, each inhabited by three types of agents – producers, workers, and shoppers – belonging to the same economy-wide family. The utility of the family is perturbed by island-specific preference shocks that shift the relative importance of each island's contribution to family well-being, while producers are subject to a common (aggregate) productivity shock. We assume that each agent type observes the prices of the resources they use, which is sufficient for producers and workers to behave as if they had full information. Producers produce a local consumption good using local labor and a tradable endowment good. Workers supply labor according to the marginal disutility of labor, embedding the local preference shock in the equilibrium local wage. The choices of producers and workers combine to generate a price for the local good which reflects both exogenous shocks – one idiosyncratic, one aggregate – and the price of the endowment good, which is endogenous to the equilibrium actions of all agents in the economy.

For their part, shoppers do not observe the preference shock on their own island when they shop for the local consumption good. They are therefore uncertain about the marginal utility contributed to the household by consuming the local good and seek to infer this utility from the good's equilibrium price. A high local price could be a signal of high marginal utility of their local good or it could reflect a change in aggregate productivity, which plays the role of noise in the shopper's inference. If aggregate conditions do not move prices much, rational shoppers attribute price increases primarily to high marginal utility of their variety, so causing price increases to drive demand *up*. Nevertheless, since the average price level does reflect aggregate conditions, aggregate productivity shocks cause changes to the average beliefs of shoppers. The aggregate supply shock can thus coordinate an increase (or decrease) in demand across islands. In general equilibrium, the aggregate increase in demand raises the price of the endowment good, which in turn further pushes up the price of local goods, reinforcing shoppers' initial mistaken inference. In short, learning through prices leads to productivity-driven shifts in demand, while the feedback of aggregate conditions to local prices gives the potential for a strong amplification mechanism.

We next characterize equilibrium in the economy, describing cases with both unique and multiple equilibria. Under most circumstances, the informational feedbacks described above are *reinforcing*: the volatility of beliefs relative to fundamentals grows as fundamental volatility falls. Feedbacks are reinforcing because lowering the variance of aggregate shocks leads, *ceteris paribus*, to an increase in the precision of local price signals, which in turn causes agents to increase the inference weights they place on those signals. Higher weights lead to larger fluctuations in beliefs for similarly-sized realizations of the productivity shock.

When the local consumption price responds strongly enough to aggregate conditions, the feedback of actions into beliefs leads some equilibria to exhibit non-trivial aggregate fluctuations even in the limit of arbitrarily small aggregate productivity shocks. To an econometrician, the fluctuations emerging at the limit of no aggregate shocks would appear as shocks to "sentiments," but the origin of sentiment in this case is quite different from that described by Angeletos and La'O (2013) or Benhabib et al. (2015).[1] First, sentiments in our model emerge as a case of extreme sensitivity to fundamental shocks, rather than relying on extrinsic randomness. Second, our model demonstrates how the price system leads markets to endogenously coordinate on a particular shock that drives sentiments, rather than assuming coordination on the shock from the outset.

In addition to providing a novel foundation for sentiment shocks, our model exhibits several qualitative features that are attractive for business cycle analysis. In particular, we show that *all* the equilibria in our economy – not only limit cases with sentiment-like equilibria – feature positive price-quantity comovements in response to sufficiently small productivity shocks. When the aggregate shock in the price signal is small, the informational role of prices dominates their allocative role, so that agents react to prices more for what they mean than for the costs they impose: Higher prices lead to higher expected marginal utility, increasing quantity demanded by more than higher costs reduce it. When aggregate productivity shocks are small, the balance of these forces causes the aggregate demand schedule to become upward sloping, generating positive price-quantity comovements. Our static mechanism provides one alternative to the dynamic mechanism of Lorenzoni (2009), which also links productivity shocks with changes in expectations and demand-driven fluctuations, but relies on extrinsic shocks to information.

After establishing the key features of the baseline model of price-driven amplification,

[1]Indeed, we show that these limit equilibria have *exactly* the same stochastic properties as the equilibrium documented by Benhabib et al. (2015).

we go on to demonstrate several important implications of the model regarding the effects of noisy public information. We show that public information, which might be expected to prevent agents from making the large correlated errors associated with sentiments, actually facilitates the coordination failure that generates these expectations-driven fluctuations. In particular, adding an exogenous public signal exacerbates the informational feedback channel, making demand-driven fluctuations more likely rather than less. Moreover, whereas in the baseline model aggregate expectations are perfectly correlated with aggregate fundamentals, the additional public signal shifts the coordination of beliefs towards the noise in the signal. In this case, the economy appears to be driven by two types of shocks: one which reflects supply-side conditions, originating in the predicted component of productivity, and one which reflects demand conditions, originating in the combination of noise and unpredicted productivity shocks. Overall volatility in this case can rise or fall with an increase in the precision of public information, and the comovement of prices and quantities takes on intermediate values between minus one and one.

We discuss several extensions that demonstrate the robustness of the basic insight. First, we show how preferences, and concavity in the the disutility of labor in particular, can serve to increase the likelihood that strong informational amplification arises in equilibrium. Second, we show that a similar characterization of equilibria obtains when, instead of having a separate productivity shock, we introduce correlation in preference shocks. Indeed the aggregate consequences of the price-feedback mechanism arise in our economy whenever any fundamental shock has an aggregate component. Lastly, we show that while high prices do indeed spur total demand, the model need not imply the existence of a positive price-quantity relationship at the good-level.

One argument favoring recent models of sentiments, rather than traditional sunspot models of "animal spirits", is that the existence of sunspot equilibria typically relies on non-convexities in the payoff structure of private agents, which have proven problematic for the out-of-equilibrium convergence of agents' actions under higher-order belief and adaptive learning dynamics.[2] We therefore conclude by examining the stability properties of the equilibria we have emphasized. To do this, we consider two equilibrium selection techniques, higher-order belief stability and adaptive learnability, that have gained prominence in both microeconomics and other areas of macroeconomics. We show that the limiting sentiment equilibria do not survive either of these tests while the limiting sentiment-free equilibria do. Outside of the limit of vanishing common noise, however, equilibria with strong informational feedbacks, and positive comovement of price and quantity, can be stable.

In addition to the sunspot and sentiment literature (see Azariadis (1981); Cass and Shell (1983); Cooper and John (1988); Benhabib and Farmer (1994) among others), this paper belongs to a long literature that studies coordination games with incomplete information. Angeletos and Pavan (2007) characterize equilibria and the welfare consequences of exogenous signal structures. Amador and Weill (2010), Vives (2012), and Manzano and Vives (2011), among others, consider imperfect information models in which the endogeneity of signals plays an important role, including in generating multiple equilibria. Gaballo (2015) shows that information transmitted by prices can originate learnable dispersed-information equilibria in the limit of zero cross-sectional variance of fundamentals, where a non-learnable perfect-information equilibrium also exists. Recent work by Bergemann and Morris (2013) characterizes the full-set of incomplete information equilibria of similar coordination games, although it does not study their out-of-equilibrium properties. Related work by Bergemann et al. (2015) studies the exogenous information structures that gives rize to maximal aggregate volatility, and the extrema they find are typically achieved by the endogenous signal structures considered here. The studies of Hassan and Mertens (2011, 2014) have also shown that arbitrarily small deviations from rational expectations can generate non-trivial aggregate consequences, in a manner that resembles the multiplier effect that we demonstrate.

[2]See Guesnerie (2005) and Evans and Honkapohja (2001). A notable exception is Woodford (1990) who shows the existence of adaptively learneable sunspots; for a comprehensive discussion see also Evans and McGough (2011). Examples of stability under higher-order belief dynamics are found by Desgranges and Negroni (2003).

2 Amplification through Learning

2.1 A Microfounded Model

In this section, we develop a microfounded economy that endogenously generates the information structure we wish to study. In the microfounded economy, all shocks are fundamental and all signals are derived as endogenous outcomes of competitive markets.

Preferences and technology

To model heterogeneity of information, we employ the "family" metaphor first introduced by Lucas (1980) and more recently adopted by Amador and Weill (2010) and Angeletos and La'O (2010) among others. The economy is inhabited by a single household composed of a continuum of members. Each member can be a "shopper", a "producer" or a "worker". Members are evenly distributed across islands indexed by $i \in [0, 1]$. On each island i, a representative worker chooses how much labor type i to supply; a representative producer transforms labor type i into a local consumption good of variety i; and a representative shopper uses the household's budget resources to buy consumption goods of variety i, which are finally consumed by the household.

The utility function of the household is:

$$\int e^{\mu_i} \left(\log C_i - \phi N_i\right) di, \tag{1}$$

where C_i and N_i denote respectively consumption and labor of variety i, ϕ is a positive constant and e^{μ_i} is an island-specific preference shock with $\mu_i \sim N(0, \sigma_\mu)$ independently distributed across islands. The preference shock is meant to capture heterogeneity in the value of each island business (variety) in terms of overall utility of the household. The household is subject to the following budget constraint:

$$\int P_i C_i di = QZ + \int W_i N_i di + \int \Pi_i di, \tag{2}$$

where P_i is the price of the good of variety i, W_i is the nominal remuneration of labor hours type i, Π_i is the profit in island i, and Q is the price of the endowment which is available in a fixed supply of Z and trades freely across islands.

The tradable endowment good is combined with island-specific labor to produce the final good, C_i, according to the technology,

$$C_i = N_i^{\gamma} \left(e^{-\zeta} Z_{(i)}\right)^{1-\gamma}, \tag{3}$$

with $\gamma \in (0,1)$, where $Z_{(i)}$ denotes the quantity of the endowment good used in the production of the consumption good i and $e^{-\zeta}$ is an aggregate productivity shock distributed according to $\zeta \sim N(0, \sigma_\zeta)$. Note that the sign convention we employ implies that positive value for ζ corresponds to a negative productivity shock. Finally, market clearing requires

$$\int Z_{(i)} di = Z.$$

To study the role of prices in transmitting information, we assume that the only information about preference and productivity shocks that is available to household members is that provided by the (actual or shadow) prices of the resources they use. In practice, workers observe the marginal disutility of labor, producers observe the wage and price of the endowment, and shoppers observe the price of their local variety. Moreover, as in Angeletos and La'O (2013), we normalize to one the value of the Lagrange multiplier associated with the budget constraint of the household to serve as a numeraire. With these assumptions, we can write the maximization problem of the worker, the producer, and the shopper in island i, as follows:

$$\begin{aligned}
\text{worker} \quad &: \quad \max_{N_i} \; W_i N_i - e^{\mu_i} \phi N_i, && (4)\\
\text{producer} \quad &: \quad \max_{N_i, Z_{(i)}} \Pi_i \equiv \max_{N_i, Z_{(i)}} \; P_i C_i - W_i N_i - Q Z_{(i)}, && (5)\\
\text{shopper} \quad &: \quad \max_{C_i} \; E[e^{\mu_i} | P_i] \log C_i - P_i C_i. && (6)
\end{aligned}$$

For workers, knowledge of the marginal disutility of labor is sufficient for to post the optimal contingent labor supply schedule on the market. Similarly for producers, their knowledge of input prices allows them to correctly anticipate their selling price (as we will see formally in short). Only the problem of shoppers involves solving a non-trivial inference problem. The shopper does not know μ_i, which is an exogenous island-specific disturbance.

Meanwhile, the price of her good, P_i, depends on both local conditions and on the price of the aggregate tradable input, Q, which itself depends on the total demand for the endowment across all islands. Notice that the price Q is the only market link across islands: all other prices and quantities are island-specific.

Equilibria with learning from prices

The definition of equilibrium is formally given by the following.

Definition 1. *For a given realization of* $\{\mu_i\}_0^1$ *and* ζ*, a rational expectation equilibrium is a collection of prices* $\{\{P_i, W_i\}_0^1, Q\}$ *and quantities* $\{N_i, C_i, Z_{(i)}\}_0^1$ *such that agents' choices are optimal given the prices they observe, and markets clear.*

The first order conditions of the family members' problems are:

$$E[e^{\mu_i}|P_i] = C_i P_i, \tag{7}$$

$$W_i = e^{\mu_i}\phi, \tag{8}$$

$$Q = (1-\gamma)P_i N_i^{\gamma} Z_{(i)}^{-\gamma} e^{-(1-\gamma)\zeta}, \tag{9}$$

$$W_i = \gamma P_i N_i^{\gamma-1} Z_{(i)}^{1-\gamma} e^{-(1-\gamma)\zeta}, \tag{10}$$

Letting $x \equiv \log(X/\bar{X})$ for any level variable X, the full set of equilibrium conditions of the economy can be written in terms of log-deviations of each variable from their steady-state values $\bar{X}$. Combining the log-linear version of (9) and (10) we obtain the standard result,

$$p_i = \gamma w_i + (1-\gamma)(q+\zeta), \tag{11}$$

which states that the equilibrium price is a linear combination of the costs of factor inputs, corrected for productivity, with weights according to the share of that input in production.

From equation (8), it follows that $w_i = \mu_i$, i.e the wage is a direct measure of the island-specific preference shock. Combining the optimality condition for z in (9) with the production function in (3), it is possible to show that $q = p_i + c_i - z_{(i)}$. Then, using the log-linear version of shopper optimality in (7) and exploiting the market clearing condition in the market for the endowment, $\int z_{(i)} di = 0$, we have

$$q = \int E[\mu_i|p_i] di. \tag{12}$$

Equation (12) states that fluctuations in the price of the endowment are driven only by the correlated component of shoppers' expectations.

We can therefore rewrite the marginal cost expression in (11) as

$$p_i = \gamma\mu_i + (1-\gamma)\left(\int E[\mu_i|p_i]di + \zeta\right). \tag{13}$$

The signal structure implied by this final equation captures the endogenous feedback effect of inference *from prices* back *into* prices, and it is on this structure that we focus our subsequent analysis.

Before proceeding to an analytical characterization, it is helpful to spell out the economic intuition behind the inference problem being solved by shoppers. When shoppers see the equilibrium price of their good fluctuating, they cannot determine the extent to which the change they observe is due to island-specific rather than economy-wide factors. From equation (13), it is clear that an increase in price can be triggered by local factors – that is, by an increase in the local wage – in which case the higher price indicates an increase in the marginal value of the local variety. Nevertheless, the same increase in price could also be driven by global factors, that is either an increase in the price of endowment or a decrease in global productivity, which are not related to local conditions. The confusion of shoppers between these two sources of price fluctuations can sustain a situation where, everything else being equal, a small negative productivity shock moving final prices up is interpreted by shoppers on each island as a positive local preference shock, thereby triggering an increase in demand for the final goods, and thus also for the endowment, whose price feeds back and is reflected in the final goods price. Due to this loop, the very fact that all shoppers do their best to extract information from local prices amplifies the volatility of the price for the endowment, making shoppers' *equilibrium* inference worse.

The following proposition provides a characterization of equilibrium in terms of the profile of expectations, so that it will be easy to map the outcomes of the inference problem to the equilibria of the economy.

Characterization of the equilibrium. *An equilibrium is characterized by a profile of* *shoppers' expectation* $\{E[\mu_i|p_i]\}_{i=0}^{1}$ *so that, given (12), in each island* $i \in (0,1)$ *we have*

$$
\begin{aligned}
p_i &= \gamma\mu_i + (1-\gamma)(q+\zeta), &(14)\\
c_i &= E[\mu_i|p_i] - \gamma\mu_i - (1-\gamma)(q+\zeta), &(15)\\
w_i &= \mu_i &(16)\\
n_i &= E[\mu_i|p_i] - \mu_i, &(17)\\
z_{(i)} &= E[\mu_i|p_i] - q. &(18)
\end{aligned}
$$

A REE is one for which shoppers' expectations are rational expectations.

Proof. Derivations are provided in Appendix A.2. ∎

It is quick to check that, when shoppers have perfect information, the aggregate price and quantity move in opposite directions.[3] In particular, a positive productivity shock, by our convention a negative value for ζ, produces a typical supply-driven fluctuation: total production goes up and the average price level falls.

2.2 Inference with Endogenous Signals

This section presents a self-contained analysis of the signal extraction problem created by the information structure microfounded above. In it, we show how to solve the shoppers' inference problem, highlighting the strategic interaction engendered by the endogeneity of the price signal. In particular, we demonstrate that the informational feedback can generate amplification of fundamental shocks, in some cases strong enough to deliver non-trivial responses to vanishingly small shocks.

Best individual weight function

Given her price signal, p_i, which depends on the aggregate expectation, shopper i must infer μ_i, the marginal utility of her consumption type. The key feature of the resulting signal extraction problem is that the precision of the signal depends on the nature of average actions across the population and, in particular, on the average reaction of shoppers to their own price signals. This is a typical property of endogenous signals. A rational expectations

[3]Substitute $E[\mu_i|p_i]$ with μ_i, substitute (14) into (15) and take the integral on both sides to get $c = -p$.

equilibrium is therefore a situation in which the individual reaction to the signal is consistent with its actual precision, i.e. is an optimal response to the average reaction of others, which each individual takes as given.

We now characterize the equilibria of the economy. Since we assume that all stochastic elements are normal, the optimal forecasting strategy is linear. As a consequence the individual expectation is linear in p_i and can be written as

$$E[\mu_i|p_i] = a_i \left(\gamma\epsilon_i + (1-\gamma) \left(\int E[\mu_i|p_i]di + \zeta \right) \right), \tag{19}$$

where a_i is the coefficient, determined prior to the realization of shocks, which measures the strength of the reaction of shopper i's beliefs to the signal she receives. Since the signal is ex-ante identical for all shoppers, each uses a similar strategy, and we can recover the average expectation by integrating across the population:

$$\int E[\mu_i|p_i]di = a\,(1-\gamma) \left(\int E[\mu_i|p_i]di + \zeta \right), \tag{20}$$

with $a \equiv \int a_i di$ denoting the average weight applied on the signal.[4] Solving the expression above for the average expectation yields

$$\int E[\mu_i|p_i]di = \frac{a\,(1-\gamma)}{1-a\,(1-\gamma)}\zeta, \tag{21}$$

which is a non-linear function of the average weight. Importantly, this function features a singularity at the point $1/(1-\gamma)$. When $a < 1/(1-\gamma)$ the average expectation comoves with the common noise, whereas the opposite holds when $a > 1/(1-\gamma)$.

The variance of the aggregate expectation – equivalently, of the endowment price – is given by

$$\sigma_q^2(a) = \left(\frac{a\,(1-\gamma)}{1-a\,(1-\gamma)} \right)^2 \sigma^2, \tag{22}$$

where $\sigma_q^2 \equiv var(q)/\sigma_\epsilon^2$ and $\sigma^2 \equiv \sigma_\zeta^2/\sigma_\epsilon^2$ are the variances of the aggregate expectation and the variance of the aggregate shock respectively, once each is normalized by the variance of the idiosyncratic fundamental.

[4] Keep in mind that a_i and p_i do not covary as the former is a strategy fixed prior to the realization of uncertainty.

Substituting the average expectation in (21) into the price signal described in equation (13), we get an expression for the local price exclusively in terms of the idiosyncratic and aggregate shocks,

$$p_i = \gamma\epsilon_i + \frac{1-\gamma}{1-a\,(1-\gamma)}\zeta, \tag{23}$$

whose precision with regard to ϵ_i is given by

$$\tau(a) = \left(\frac{\gamma\,(1-a\,(1-\gamma))}{(1-\gamma)\sigma}\right)^2. \tag{24}$$

Since the average weight a determines the precision of the signal, rational expectations requires that the shopper's individually optimal signal weights is a function of the signal weights used by the rest of the population. We are now ready to compute such an optimal response.

Taking the average weight as given, it is straightforward to work out an expression for the optimal individual weight a_i such that $\mathrm{E}[p_i(\epsilon_i - a_i p_i)] = 0$, i.e. the covariance between the signal and forecast error is zero in expectation. This condition implies that the information is used optimally. The best individual weight is given by

$$a_i(a) = \frac{1}{\gamma}\left(\frac{\tau(a)}{1+\tau(a)}\right). \tag{25}$$

Given the linear-quadratic environment, we can interpret $a_i(a)$ in a game-theoretic fashion as an individual best reply to the profile of others' actions summarized by a sufficient statistic a. To be precise, every a_i is associated with one and only one contingent shopping strategy that describes the conditional expectation $E[\mu_i|p_i] = a_i p_i$ of shopper i, where p_i identifies a set of states of the world indistinguishable to the shopper i.

Equilibria

Given that agents face an informational structure with the same stochastic properties, a REE equilibrium has to be symmetric. This last requirement completes our notion of equilibrium which is formally stated below.

Definition 2. *A noisy rational expectations equilibrium (REE) is characterized by a profile of shoppers' expectations* $\{E[\mu_i|p_i]\}_{i=0}^{1}$ *such that* $E[\mu_i|p_i] = \hat{a}p_i$ *with* $a_i(\hat{a}) = \hat{a}$*, for each* $i \in (0,1)$.

Our game-theoretic interpretation of the optimal coefficient makes clear the equivalence between a rational expectations equilibrium and a Nash equilibrium: no one has any individual incentive to deviate when everybody else conforms to the equilibrium prescriptions.

An equilibrium of the model is given as a fixed-point of the individual best weight mapping given by equation (25). In practice, there are as many equilibria as intersections between $a_i(a)$ with the bisector. The fixed-point relation delivers a cubic equation, which may have one or three real roots. The following proposition characterizes these equilibrium points.

Proposition 1. *For $\gamma \geq 1/2$, there always exists a unique REE equilibrium for $\hat{a} = a_u \in (0, \gamma^{-1})$.*

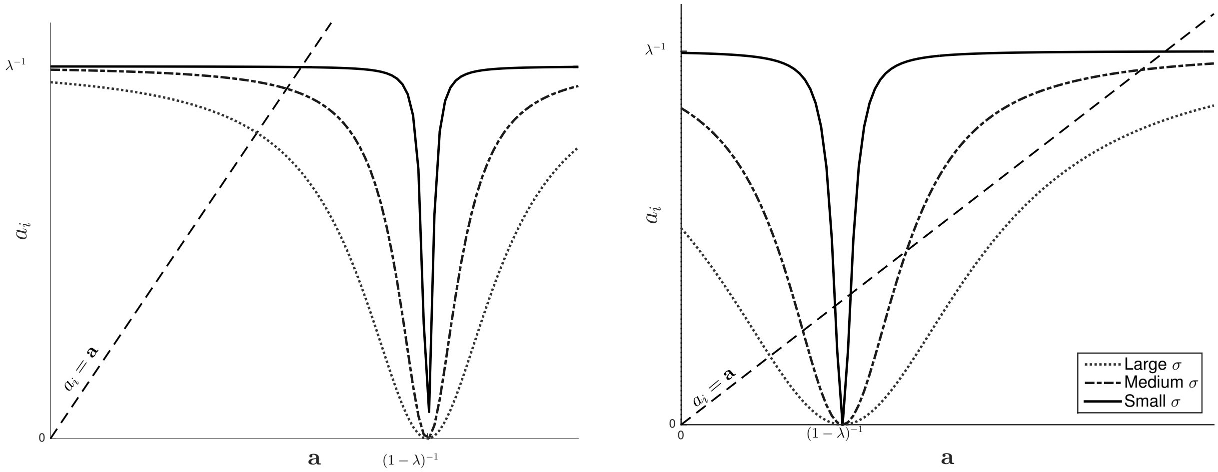
For $\gamma < 1/2$, there always exists a low REE equilibrium for $\hat{a} = a_- \in (0, (1-\gamma)^{-1})$. In addition, there exists a threshold $\bar{\sigma}^2$ such that, for any $\sigma^2 \in (0, \bar{\sigma}^2)$, a middle and a high REE equilibrium also exist for $\hat{a} = a_\circ$ and $\hat{a} = a_+$ respectively, both lying in the range $\left((1-\gamma)^{-1}, \gamma^{-1}\right)$.

Proof. Given in Appendix A.1. ∎

Proposition 1 proves that, when the aggregate component receives relatively high weight in the signal, the model may exhibit a multiplicity. In particular, there are three equilibria whenever $\gamma < 1/2$ and the variance of the aggregate error is small enough; Otherwise a unique equilibrium exists. While an analytical characterization of the three equilibria is possible, the expressions themselves are rather complicated. Nevertheless, the relevant properties can be grasped from the reaction functions plotted in Figure 1 (see figure caption).

The slope of the $a_i(a)$ curve at the intersection with the bisector determines the nature of the strategic incentives underlying each equilibrium. Equilibria a_u and a_- are characterized by *substitutability in information* as $a_i'(\hat{a}) < 0$. In contrast, the equilibria $a_\circ$ and a_+ are characterized by *complementarity in information* as the optimal individual weight is marginally increasing in the average weight, i.e. $a_i'(\hat{a}) > 0$. [5] In fact, as soon as $a > (1-\gamma)^{-1}$, the higher the a the higher the precision of the signal, which further pushes up the optimal weight. The emergence of complementarity explains the upward sloping part of the best weight function, which is key for the existence of a multiplicity.

[5] See equation (65) in appendix A.1.



(a) $\gamma = 0.75$

(b) $\gamma = 0.25$

Figure 1: The figure illustrates four properties of $a_i(a)$ for given γ and σ: (i) $a_i(0) > 0$; (ii) $a_i'(a) < 0$ for $a \in (0, (1-\gamma)^{-1})$, and $a_i((1-\gamma)^{-1}) = 0$; (iii) $a_i'(a) > 0$ for for $a \in ((1-\gamma)^{-1}, \gamma^{-1})$ and $\lim_{a\to\infty} = \gamma^{-1}$; (iv) $\partial a_i(a)/\partial\sigma \geq 0$.

While complementarity is essential for the existence of multiple equilibria, it is neither necessary nor sufficient to imply a strong informational multiplier. To see this, define the informational multiplier, $\Gamma(\hat{a}) \equiv \sigma_q^2(\hat{a})/\sigma^2$, as the volatility of beliefs relative to the volatility of the shock ζ for some equilibrium point $\hat{a}$. We will say that the economy exhibits *amplifying* informational feedback whenever a fall in the volatility of the exogenous shock leads to an increase in $\Gamma(\hat{a})$, i.e. $\partial\Gamma(\hat{a})/\partial\sigma < 0$, and *dampening* feedback otherwise. The following proposition shows that the equilibria described in Proposition 1 can be classified by whether they exhibit amplifying or dampening information feedback.

Proposition 2. *The equilibria a_u, a_-, and $a_\circ$ all exhibit amplifying feedback, while the equilibrium a_+ exhibits dampening feedback.*

Proof. Given in Appendix A.1. ∎

The strategic nature of the interaction implied by the endogeneity of the information structure does not determine whether or not an equilibrium exhibits amplifying or dampening feedback. The nature amplifying or dampening of the informational feedback is decided by whether or not the equilibrium value of a gets closer to $(1-\gamma)^{-1}$ as σ shrinks. From a simple inspection of figure 1, we can easily see that $a_u, a_\circ$ and a_- features amplifying feedback, whereas only a_+ features dampening feedback. Nevertheless, the feedback effects in $a_\circ$ and a_- are distinct from that in a_u, as only the former approach infinite amplification as σ goes to zero. In the following section we study the source of this powerful amplification in more detail.

2.3 Low variance in productivity shocks

In this section we analyze the properties of the equilibria for sufficiently small values of σ^2. This case is particularly interesting because as the variance of the productivity shock decreases, everything else being equal, the volatility of the consumption price decreases and its informativeness increases. Hence, as aggregate volatility shrinks, the informational role of prices overcomes their allocative role and agents react to prices more for their informational content than for the costs they impose. This mechanism provides the key insight to understand the phenomena documented in this section.

Sentiment equilibria as limit case of strong amplifcation

Here, we show that learning from prices can generate such high amplification of fundamental shocks that it is able to sustain sizable aggregate fluctuations even at the limit $\sigma^2 \to 0$; we see this as a new characterization of sentiments-driven fluctuations, which have recently received growing attention in the literature. The intuition for this result is captured by Figures 2 and 3, which plot, for each equilibrium, the evolution of the signal precision and the variance of the average expectation, respectively, as a function of the volatility of productivity shocks. As σ shrinks, the unique and the high equilibrium, namely a_u and a_+, approach infinite precision and no aggregate volatility; on the contrary, the middle and the low equilibria $a_\circ$ and a_- converge to finite precision and sizable aggregate volatility.

The plots numerically demonstrate that, for the middle and low equilibrium, as σ goes to zero the informational feedback goes to infinity at a speed that makes the product of the two achieve a finite limit. The following proposition establishes the result formally.

Proposition 3. *In the limit* $\sigma^2 \to 0$

i. the unique equilibrium (for $\gamma \geq 1/2$*) and the high equilibrium (for* $\gamma < 1/2$*) converge to a point with no aggregate volatility:*

$$\lim_{\sigma^2 \to 0} a_{u,+} = \max\left(\frac{1}{\gamma}, \frac{1}{1-\gamma}\right) \qquad \lim_{\sigma^2 \to 0} \sigma_q^2(a_{u,+}) = 0. \tag{26}$$

ii. the low and middle equilibria (for $\gamma < 1/2$*) converge to the same point and exhibit non-trivial aggregate volatility:*

$$\lim_{\sigma^2 \to 0} a_{-,\circ} = (1-\gamma)^{-1} \qquad \lim_{\sigma^2 \to 0} \sigma_q^2(a_{\circ,-}) = \frac{\gamma(1-2\gamma)}{(1-\gamma)^2}. \tag{27}$$

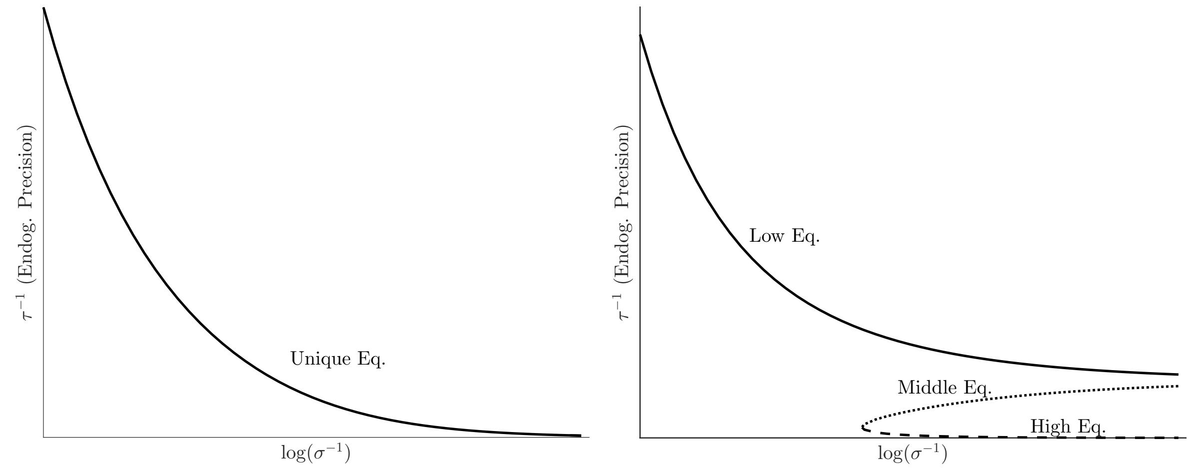
Proof. Given in Appendix A.1. ∎

In the limit of $\sigma \to 0$, the middle and the high equilibrium have the same stochastic properties of the sentiment equilibria uncovered in Benhabib et al. (2015). This means that, despite the infinitesimal size of the fundamental shock, its realization is able to coordinate fluctuations in agents' expectations of sizable measure.

The limiting result suggests that a strict dichotomy between fundamental and non-fundamental fluctuations is potentially misleading. Since endogenous signal structures can generate strong multiplier effects on small shocks, they deliver fluctuations that effectively span a continuum from purely fundamental-driven to purely sentiment-driven. Of course, this possibility does not preclude the existence of fluctuations that originate from truly payoff-irrelevant shocks, but the possibility of fundamental-based sentiments may appeal to those who find such fluctuations implausible.

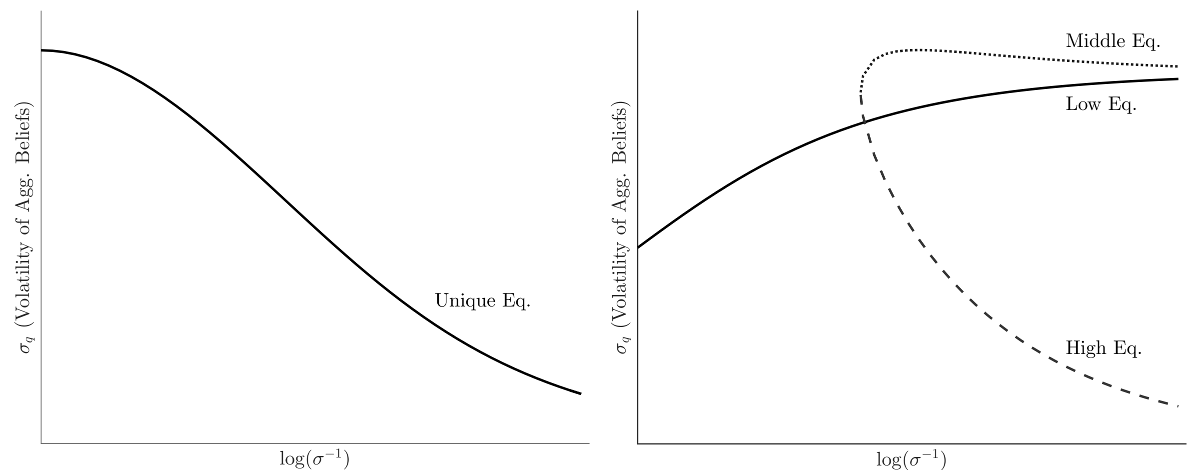
Moreover, in our economy, agents' coordination on small fundamental shocks as the drivers of beliefs arises endogenously arises through the competitive price system, rather than being assumed from the outset. The analysis of Benhabib et al. (2015), in contrast to ours, occurs at the limit point rather than approaching it, so it cannot explain the origins of coordination on a particular sentiment shock.

A special feature of our account of sentiments is that, as the economy approaches the limit, expectations and aggregate outcome are *perfectly* correlated with fundamentals, although the fluctuations in fundamentals themselves become progressively more difficult for



(a) $\gamma = 0.75$ (b) $\gamma = 0.25$

Figure 2: (Inverse) signal precision as a function of exogenous shock volatility.



(a) $\gamma = 0.75$ (b) $\gamma = 0.25$

Figure 3: Belief volatility approaching the limit.

the econometrician to measure. In Section 3, we show how the addition of a noisy public signal into the economy can generate expectations-driven fluctuations that either are imperfectly correlated or orthogonal to economic fundamentals.

A final implication of our basic analysis here is that the addition of a small amount of *aggregate* noise in the signal – in this case captured by the effect of productivity on the price signal – can sustain additional equilibria that do not arise under full information. A previous literature has demonstrated cases where adding *idiosyncratic* noise to signals can either eliminate (Morris and Shin, 1998) or generate (Gaballo, 2015) additional equilibria. But this is the first time it has been observed, to our knowledge, that adding aggregate noise can cause equilibria to proliferate.

Supply shocks generate demand-driven fluctuations

In this section, we show that the limiting sentiment-like equilibria share an important feature with all the equilibria of our economy: when aggregate shocks are not too large, final good prices, total output, the price of the endowment, and total employment positively comove. This happens in *all* equilibria because, as aggregate volatility falls, the informational value of the price signal rises, leading agents' beliefs about their local conditions to respond more strongly to it. Stronger aggregate effects on beliefs eventually lead the informational channel of prices to dominate, so that consumption increases in response to higher prices. Learning from prices thus provides a new mechanism for generating expectation-driven demand shocks in an economy hit only by fundamental shocks to productivity. In this respect, the fluctuations in our economy are similar to those studied by Lorenzoni (2009), although our mechanism is static and does not require the presence of exogenous shocks to information.

The consequences of endogenous information for business cycle comovements are most intuitively seen by analyzing the aggregate demand and aggregate supply schedules in our economy. Given (3), (7) and (17), we can express aggregate demand and supply as

$$AD \quad : \quad c = q - p, \tag{28}$$

$$AS \quad : \quad c = \gamma q - (1-\gamma)\zeta. \tag{29}$$

When the endowment price q has no effect on shoppers' beliefs, this relationship implies a standard downward sloping aggregate demand relation. However, once we account for the equilibrium feedback of prices into shoppers' inference, the aggregate demand and the aggregate supply relations become

$$AD \quad : \quad p = \frac{1}{a-1}c \tag{30}$$

$$AS \quad : \quad p = \frac{1}{\gamma a}c + \frac{1-\gamma}{\gamma a}\zeta \tag{31}$$

where a denotes the average weight put on the price by shoppers who form their expectations according to $E[\mu_i|p_i] = ap_i$, so that $q = \int E[\mu_i|p_i]di = ap$.

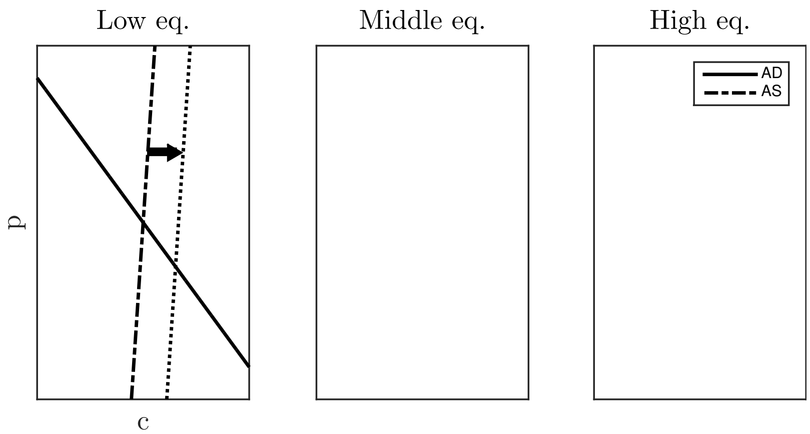
Crucially, the relation in (30) implies that aggregate demand is upward sloping for any a larger than unity. That is, price and quantity will move together, despite the nature of the shock hitting the economy! As the relative variance σ decreases, this will be true *for all equilibria* in the economy. Even the equilibrium a_h, which displays no fluctuations in the limit $\sigma \to 0$, exhibits comovements in prices and quantities away from the limit, *as if* the economy is hit by a common preference shock. In fact, the equilibrium condition $a > 1$ always entails a situation in which the informative effect of prices is more important than their allocative effect, that is, movements in expected marginal utility of a good more than compensate a change in its price. In the model driven by aggregate productivity shocks, the consequences for aggregate demand have immediate implications for the comovement of price and quantity in the economy.

Proposition 4. *For σ^2 sufficiently small, all equilibria exhibit comovement of aggregate output, employment, the price level, and the price of the endowment.*

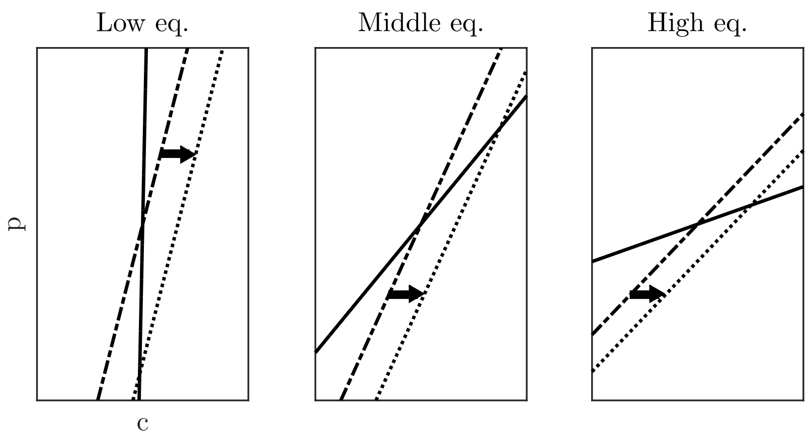
Proof. The results follows from continuity of the best response function, and the observation that all limit equilibria entail $\hat{a} > 1$. ∎

As theorem shows, with sufficiently informative prices, the economy will always demonstrate positive comovement between prices and quantities, even though the corresponding full-information will exhibit covariance of the opposite sign. The learning-through-prices mechanism can thus easily reverse the typical assumptions made regarding the identification of demand and supply shocks, and it does not require being at the limit or a particular equilibrium selection to do so.

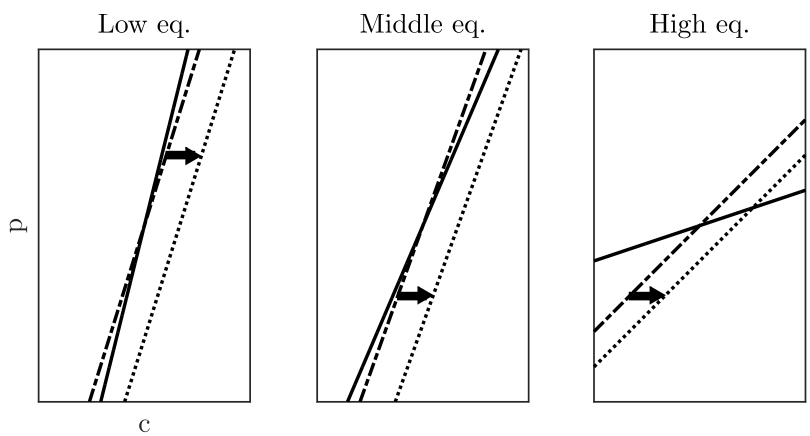
Figure 4 plots aggregate supply and demand relations for different values of the relative volatility σ, in a case where a multiplicity is possible ($\gamma = 0.25$). As σ shrinks, the slope of aggregate demand in the low equilibrium first switches signs and then, as σ approaches the limit, it becomes nearly parallel with the aggregate supply curve of the economy. In



(a) Large σ



(b) Medium σ



(c) Small σ

Figure 4: Aggregate supply and demand in the microfounded model.

particular, in the limit this situation corresponds to

$$AD \quad : \quad p = \frac{\gamma}{1-\gamma}c \tag{32}$$

$$AS \quad : \quad p = c + \frac{1-\gamma}{\zeta} \tag{33}$$

when considering a_u and a_+ for which $\lim_{\sigma \to 0} a_i(a_{u,+}) = \gamma^{-1}$, and

$$AD \quad : \quad p = \frac{1-\gamma}{\gamma}c \tag{34}$$

$$AS \quad : \quad p = \frac{1-\gamma}{\gamma}c + \frac{(1-\gamma)^2}{\gamma}\zeta \tag{35}$$

when considering $a_\circ$ and a_- for which $\lim_{\sigma \to 0} a_i(a_{\circ,-}) = (1-\gamma)^{-1}$.

The figure provides an easy intuition for the extremely large informational multiplier implied by "sentiment-like" equilibria, as even small shifts in aggregate supply imply large changes in the equilibrium quantity of the intermediate input. Notice, as well, that even away from the limit, all equilibria also demonstrate upward-sloping aggregate demand. Therefore, the same information mechanism which delivers sentiments as a special case, more generally offers the potential to be an important amplification mechanism for non-trivial aggregate shocks, which show up in expectation-driven fluctuations.

3 Noisy public information

Given that agents generally observe some indicators of aggregate conditions, a natural question is whether the results above generalize to a situation in which agents observe some signal regarding the fundamental shock that, in the baseline model, drives beliefs in the economy. In this section, we therefore consider expanding the information set of agents to include a second signal of the form

$$g = \zeta + \vartheta,$$

where $\vartheta \sim N(0, \sigma_\vartheta^2)$ is a common aggregate noise term in agents' signal regarding the fundamental shock. In this case, agents form expectations with weights on both the price signal and the public signal. In particular, the public signal will be useful to refine the information of the price signal, allowing agents to partial out a portion of the aggregate productivity shock that blurs the inference of shoppers.

Signal extraction with public information

Let us consider the following linear forecasting rule

$$E[\mu_i|p_i, g] = a_i p_i - b_i a_i (1-\gamma) g, \tag{36}$$

where a_i is the same as before and b_i represents the weight that shopper i puts on the public signal, which we re-scale for convenience by two deterministic factors a_i and $(1-\gamma)$. Therefore, we can rewrite shoppers' expectation as

$$E[\mu_i|\tilde{p}_i] = a_i \tilde{p}_i, \tag{37}$$

where

$$\tilde{p}_i \equiv \gamma\mu_i + (1-\gamma)\left(\int E[\mu_i|\tilde{p}_i]di + \zeta - b_i g\right), \tag{38}$$

represents a new signal embodying the best information available to the individual shopper, i.e. the signal with the highest precision given g.[6] In particular, the highest precision of the new signal $\tilde{p}_i$ is obtained when b_i is set to minimize the variance of the correlated component, which depends on ζ and ϑ, taking the average weight b as fixed. To recover the best weight function $b_i(b)$, we need to work out the average expectation in analogy to the previous section to obtain

$$\tilde{p}_i = \gamma\mu_i + (1-\gamma)\left(\frac{a(1-\gamma)}{1-a(1-\gamma)}(\zeta - bg) + \zeta - b_i g\right), \tag{39}$$

where $a = \int a_i di$ and $b = \int b_i di$, are the average weights. Therefore $b_i(b)$ is the value of b_i that minimizes the varaince of the term multiplied by $(1-\gamma)$ in (39). It is easy to observe that

$$\hat{b} = \frac{\sigma_\vartheta^{-2}}{\sigma_\zeta^{-2} + \sigma_\vartheta^{-2}} \tag{40}$$

minimizes $E[(\zeta - \hat{b}g)^2|g]$, and in particular $E[(\zeta - \hat{b}g)g] = 0$. Therefore, we can conclude that $b_i(\hat{b}) = \hat{b}$ is such optimal value.

[6] By the Frisch-Waugh theorem, the projection of μ_i on $\{p_i, g\}$ is equivalent to the projection of μ_i on $\tilde{p}_i \equiv p_i - E[p_i|g]$, where $E[p_i|g] = b_i(1-\gamma)g$.

Proposition 5. *Suppose that agents' information consists of* $\{p_i, g\}$*. Then, the equilibrium of the economy is characterized by* $E[\mu_i|p_i, g] = a_i\tilde{p}_i$ *where*

$$\tilde{p}_i = \gamma\mu_i + (1-\gamma)(q + \tilde{\zeta}) \tag{41}$$

where $\tilde{\zeta}$ *is given by*

$$\tilde{\zeta} \equiv (1-b)\zeta - b\vartheta, \tag{42}$$

with $b = \sigma_\vartheta^{-2}/(\sigma_\zeta^{-2} + \sigma_\vartheta^{-2})$ *and variance*

$$\tilde{\sigma}_\zeta^2 \equiv (\sigma_\zeta^{-2} + \sigma_\vartheta^{-2})^{-1}. \tag{43}$$

The equilibrium values $\{a_u, a_-, a_\circ, a_+\}$ *and the conditions for their existence are isomorphic to the ones in the baseline economy once* $\tilde{\sigma}_\zeta^2$ *takes the place of* σ_ζ^2*.*

Proof. Given in appendix A.1. ∎

Equation (43) has several important implications. First, the one and two-signal models clearly coincide as $\sigma_\vartheta^2 \to 0$, and the aggregate public signal become uninformative. Moreover, to the extent that the aggregate signal is informative, its effect corresponds to a decrease in the exogenous shock, ζ, affecting in agents' price signals, thereby pushing the economy towards a situation of multiple equilibria. Within the low equilibrium, this implies an *increase* in the variance of the average expectation of shoppers.

This result makes clear that the addition of the aggregate signal does not eliminate the potential for the endogenous private signal of agents to deliver correlated errors. To the contrary, whenever the economy without the public signal exhibits reinforcing informational feedback, increasing the precision of the public signal strengthens the feedback generated by agents' endogenous signals and pushes the economy closer to the limit point in which there are multiple equilibria, including one which exhibits large aggregate fluctuations driven by beliefs.

Economic implications

While the signal extraction problem of the model with an aggregate signal is isomorphic to the baseline economy, several important differences emerge with respect to the qualitative business cycle implications for consumption and prices. To see these, notice that in equilibrium the average endowment price q is now a function of both the average price in the economy and of the public signal,

$$q = a(p - (1-\gamma)bg).$$

Substituting into the aggregate demand and supply expression in equation (29) and (28) yields

$$AD \;:\; c = (a-1)p - a(1-\gamma)bg \tag{44}$$

$$AS \;:\; c = \gamma\left(ap - a(1-\gamma)bg\right) - (1-\gamma)\zeta. \tag{45}$$

Notice that *both* aggregate demand and aggregate supply are implicitly shifted by the productivity shock ζ and on the noise in the aggregate signal ϑ, via their appearances in the public signal g. Moreover, since the coefficients on these two shocks are different, they will have potentially different implications for aggregate supply and demand.

Solving for the full equilibrium of the economy, yields the following expression for price and consumption,

$$p = (1-\gamma)\left(\frac{1-\hat{a}(1-\gamma)\hat{b}}{1-\hat{a}(1-\gamma)}\zeta - \frac{\hat{a}(1-\gamma)\hat{b}}{1-\hat{a}(1-\gamma)}\vartheta\right) \tag{46}$$

$$c = -(1-\gamma)\left(\frac{1-\hat{a}+\hat{a}\gamma\hat{b}}{1-\hat{a}(1-\gamma)}\zeta + \frac{\hat{a}\gamma\hat{b}}{1-\hat{a}(1-\gamma)}\vartheta\right) \tag{47}$$

Comparing (46) and (47), it is clear that the coefficients on the fundamental ζ will take opposite signs whenever $\hat{b}$ is sufficiently close to one. That is, productivity shocks can induce negative comovements when the public signal is sufficiently precise. This result contrasts with the case of the previous section without a public signal, when the same productivity shock induced perfect positive comovement. On the other hand, it is immediate to see that noise shocks in the public signal will *always* induce positive comovement in the economy, regardless of the precision of the information. Unconditional comovement in the economy will therefore depend on the contribution of each shock to overall correlations and, in general, will fall somewhere strictly between zero and one.

In Figure 5, we document how the correlation between average consumption price and total output (on the y-axis) may be affected by the variance of the noise in the noisy public signal (on the x-axis), in case $\gamma = 0.75$ in the left panel and $\gamma = 0.25$ in the right panel. In particular, we fix value of σ for which multiple equilibria exist with $\gamma = 0.25$. For large variance of noise in the aggregate signal, price-quantity correlation can be either 1 or -1. Perfect negative correlation arises only for the low equilibrium when σ is sufficiently large; for smaller values of σ all equilibria exhibit positive price-quantity correlation as explained in the previous section.

As the variance of the noise in the public signal shrinks, in the unqiue and high equilibrium supply-driven fluctuations crowd-out demand-driven ones, whereas the contrary happens in the low equilibrium. This occurs because the low equilibrium converges to a sentiment-like equilibrium where expectations become very sensitive to shocks, whereas high and unique equilibrium converge to no fluctuations in expectations. At that limit, in the high and unique equilibrium, productivity shocks hit without having any information impact, which explains the negative correlation. The low equilibrium instead converges to a limit correlation that is now different from 1 due to the presence of sizable productivity shocks, whose information role is reduced by the presence of public information. The middle equilibrium achieves the same correlation value as the low equilibrium since both equilibria converge to the same limit outcome.

Therefore, Figure 5 shows that the addition of public signal – and therefore of an alternative source of aggregating noise – is one possible approach to breaking the perfect correlation, either positive or negative, between p and c that is implied by the baseline model. Instead, as the aggregate signal becomes sufficiently precise, the economy behaves as if it is hit by *both* supply and demand side disturbances, leading to imperfect but potentially positive correlation between the price and quantity.

Figures 6 and 7 show the decomposition of the variance of aggregate consumption and the average price in the same cases $\gamma = 0.75$ and $\gamma = 0.25$, respectively, illustrated in figure 5. Indeed, increasing the precision of the public signal generally increases the role of common

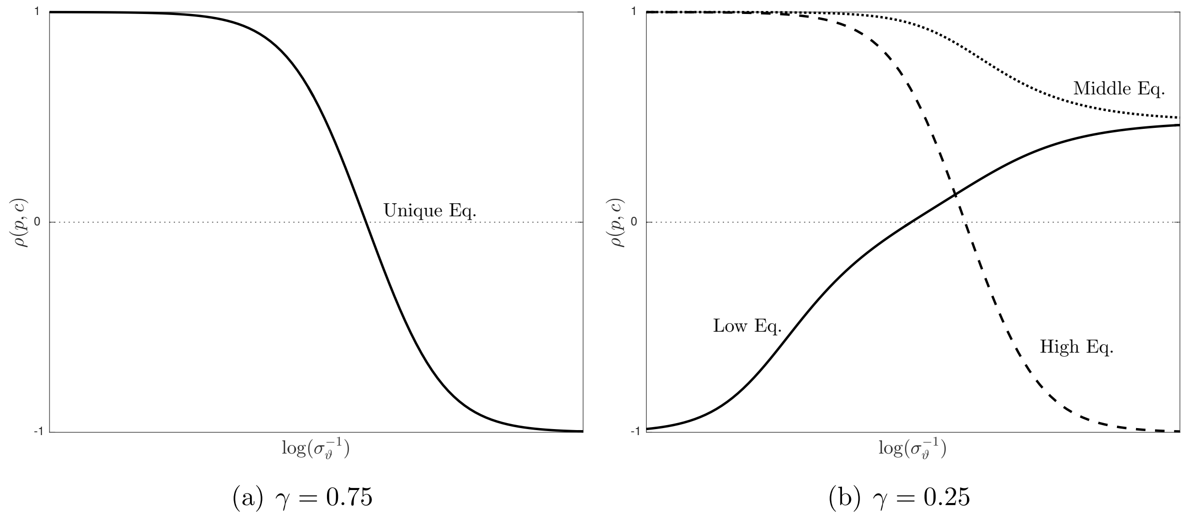
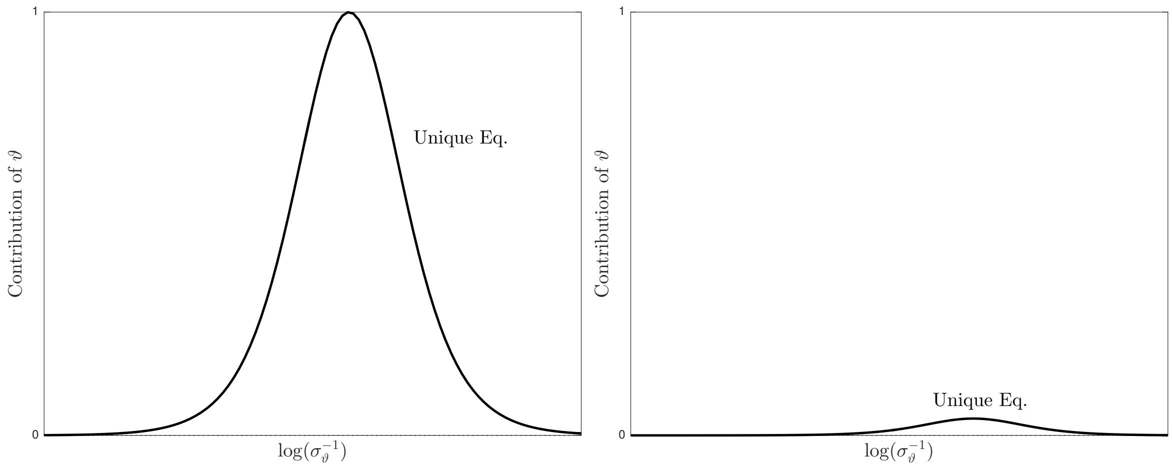


Figure 5: Correlation between average price and quantity as a function of the variance of noise in the public signal.

noise in the low and middle equilibrium, although some non-monotonicity does arise. On the other hand, the figure for the high and the unique equilibrium is striking. It turns out that almost all the variance in those equilibria is driven by productivity shocks when the variance of such noise is either sufficiently small or sufficiently high. Nevertheless, in the unique equilibrium case, there exists an intermediate value of σ_ϑ for which consumption is driven entirely by common noise, and prices entirely by productivity shocks. In a mirror-like situation, for the high equilibrium there exists an intermediate value of σ_ϑ for which consumption is driven entirely by productivity shocks, and prices entirely by common noise.

In particular, looking at (46) we can easily see that prices do not depend on fundamental shocks when $a = ((1-\gamma)b)^{-1}$, where recall $b < 1$. Such a value can only be achieved by $a_\circ$ and a_+ for a single (different) value of σ, provided $b > \gamma/(1-\gamma)$, which ensures $(1-\gamma)b \in ((1-\gamma)^{-1}, \gamma^{-1})$. On the other hand, consumption does not depend on fundamental shocks when $a = 1/(1-\gamma b)$. Such a value can only be achieved by a_u for a single value of σ, provided $b > (1-\gamma)/\gamma$, which can only be for $\gamma > 1/2$ given that $b < 1$.

These cases demonstrates the possibility that, far from the limit of infinitesimal variance, our rich structure can generate equilibria where price fluctuations and consumption fluctuations are driven by shocks of a different nature. This result does not rely on the multiplicity



(a) Consumption (b) Price level

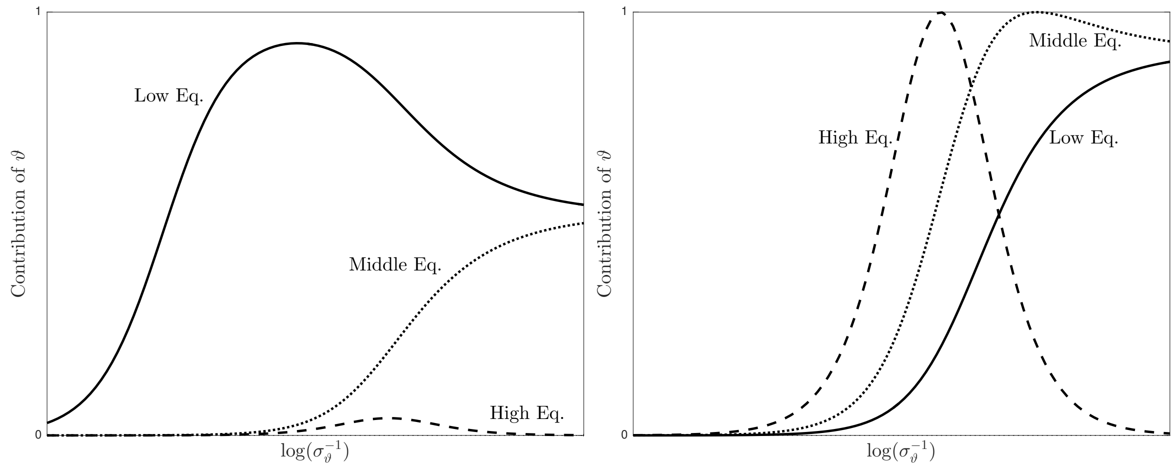
Figure 6: Variance decomposition when $\lambda = 0.75$: aggregate noise v.s. fundamentals.

of equilibria.

Summing up, the introduction of public information opens the way for aggregate beliefs to become disconnected from aggregate productivity shock and simultaneous roles for "supply" and "demand" fluctuations in the economy, despite the presence of a single aggregate shock. In the limit of arbitrarily small noise in the public signal, the model delivers fluctuations in aggregate beliefs that – although coordinated by noise in the public signal – could not be so-attributed by the econometrician who observes only imperceptibly small eventual revisions to the public data. Finally, we have showed that, even when far from any limit, there exist equilibria where fluctuations in prices and quantities can be explained by shocks of different nature.

4 Extensions and Discussion

This section presents several extensions to the basic setup, showing that the insights of the main mechanism are robust to various modeling details. In the first we show that convexity in the disutility of labor, (i) expands the range of γ for which strong informational multipliers – and equilibrium multiplicity - may arise, and (ii) induces wages to comove positively along with prices and quantities for sufficiently small values of σ. We then show



(a) Consumption (b) Price level

Figure 7: Variance decomposition when $\lambda = 0.25$: aggregate noise v.s. fundamentals.

that allowing for correlated shocks to the good-specific taste shocks μ_i does not materially affect the conclusions of the baseline model with productivity shocks. Finally, we allow for the disaggregation of goods at the island level to demonstrate that existence of upward-sloping aggregate demand in our model does not require the existence of Giffen-type goods at the micro level.

4.1 Convexity in labor disutility

Here, we show how our framework easily extends in the case of convex disutility in labor. Consider the utility function of the household is now

$$\int e^{\mu_i} \left(\log C_i - \phi N_i^{1+\alpha}\right) di \tag{48}$$

where $\alpha > 0$ accounts for the inverse of the Frisch labor elasticity. In this case the local wage will not be a direct measure of the island-specific preference shock, but it will be also a function of the quantity traded on the labor market. In Appendix A.2 we report the detailed derivation. Here we just state how the parameter α affects the characterization of the equilibrium.

Characterization of the equilibrium, extended case. *An equilibrium is charac-*

terized by a profile of shoppers' expectation $\{E[\mu_i|p_i]\}_{i=0}^{1}$ *so that, given (12), in each island* $i \in (0,1)$ *we have*

$$\begin{aligned} p_i &= \frac{\gamma}{1+\alpha}\mu_i + \frac{\alpha\gamma}{1+\alpha}E[\mu_i|p_i] + (1-\gamma)(q+\zeta), && (49)\\ c_i &= \frac{1+\alpha(1-\gamma)}{1+\alpha}E[\mu_i|p_i] - \frac{\gamma}{1+\alpha}\mu_i - (1-\gamma)(q+\zeta), && (50)\\ w_i &= \mu_i + \alpha n_i && (51)\\ n_i &= \frac{1}{1+\alpha}E[\mu_i|p_i] - \mu_i, && (52)\\ z_{(i)} &= E[\mu_i|p_i] - q. && (53) \end{aligned}$$

A REE is one for which shoppers' expectations are rational expectations.

Proof. Derivations are provided in Appendix A.2. ∎

The characterization of the equilibrium remains qualitatively unaffected for all the variables except for the wage, which now positively comoves with unemployment. Hence, the local price is now affected by the individual expectation of the representative local shopper as the equilibrium quantities of labor depend on shoppers' demand. One can easily show that the analysis above applies also to this case, once the price signal is conveniently transformed.

In fact, notice that the information transmitted by the local price can be equivalently represented by

$$\hat{p}_i = \frac{1+\alpha}{1+\alpha(1-\gamma)}\left(p_i - \frac{\alpha\gamma}{1+\alpha}E[\mu_i|p_i]\right) = \hat{\gamma}\mu_i + (1-\hat{\gamma})(q+\zeta),$$

where $\hat{\gamma} = \gamma/(1+\alpha(1-\gamma))$. In practice, we subtracted the individual expectation – which is obviously an information available to the shopper – and we re-scaled the signal to obtain a form that encompasses (13).

Hence, the analysis of section 2.2 follows through substituting the original price signal p_i with the equivalent one $\hat{p}_i$. As a consequence, now multiple equilibria exist with $\hat{\gamma} < 1/2$ which could well obtain even with $\gamma > 1/2$ for a sufficiently high α. Moreover, in this case, we obtain demand-driven fluctuations in which wages also positively comove with the average consumption price, the price of the endowment, total output and total employment.

4.2 Correlation in island-specific shocks

It turns out that assuming that the aggregate term in the signal given by (13) is uncorrelated with agents' objectives is not essential for generating any of the qualitative features of the economy we have described. To prove this, consider a version of the model in which preference shocks are correlated, that is $\mu_i = \mu + \epsilon_i$ where $\mu \sim N\left(0, \sigma_\mu^2\right)$, and there are no productivity shocks. Notice that, previously, productivity shocks were acting as noise in the signal, since shoppers were only interested in the forecast of μ_i. Now, the aggregate term μ represents a common objective in the signal extraction problem of shoppers.

According to (13) the price signal is now expressed as

$$p_i = \gamma(\mu + \epsilon_i) + (1-\gamma)\int E[\mu + \epsilon_i | p_i] di, \tag{54}$$

which no longer embeds a productivity shock. Nonetheless, correlated fundamentals generate confusion between the idiosyncratic and common component of the signal. As before, the individual expectation of a shopper type i is formed according to the linear rule $E[\mu+\epsilon_i|p_i] = a_i p_i$. Hence, the signal embeds the average expectation, which again leads the precision of the signal to depend on the average weight a. Following the analysis of the earlier section, the realization of the price signal can be rewritten as

$$p_i = \gamma\epsilon_i + \frac{\gamma}{1 - a(1-\gamma)}\mu. \tag{55}$$

where a represents the average weight placed on the signal by other shoppers. The variance of the average expectation, or equivalently of the price of the endowment, is given by

$$\sigma_q^2(a) = \left(\frac{\gamma a}{1 - a(1-\gamma)}\right)^2 \sigma^2, \tag{56}$$

which is slightly different from (22). As before, agents take the average weight as given, when setting their optimal individual weight a_i such that $\mathrm{E}[p_i(\mu + \epsilon_i - a_i p_i)] = 0$, which implies the best individual weight

$$a_i(a) = \frac{1}{\gamma}\left(\frac{(1 - a(1-\gamma))^2 + (1 - a(1-\gamma))\,\sigma^2}{(1 - a(1-\gamma))^2 + \sigma^2}\right). \tag{57}$$

While the best response function in equation (57) is slightly different than that of equation (25) for the case with productivity shocks, we can prove that the characterization of the limit equilibria is identical.

Proposition 6. *In the limit $\sigma_\mu^2 \to 0$, the equilibria of the economy converge to the same points as the baseline economy:*

$$\lim_{\sigma_\mu^2 \to 0} a_e^\mu = \lim_{\sigma^2 \to 0} a_e \qquad \lim_{\sigma_\mu^2 \to 0} \sigma^2(a_e^\mu) = \lim_{\sigma^2 \to 0} \sigma^2(a_e) \qquad \textit{for } e \in \{u, -, \circ, +\} \tag{58}$$

Proof. Given in Appendix A.1. ∎

More generally, it is possible to show that propositions 1 through 3 follow identically, and their proofs in parallel with only the obvious algebraic substitutions. Most importantly, the best response function (57) converges in the limit to (25), the best-response function of the model with productivity shocks.

4.3 A Theory of Giffen goods?

One possible objection to the realism of our mechanism is the implication that the consumption of island-specific good C_i is rising in its price, i.e. that local consumption goods appear to be Giffen goods. Such behavior at the good level is not an essential aspect of our story. The most natural way to avoid this complication is to presume that, *within* islands, quantity-choosing firms produce a continuum of goods indexed by (i, j) which are then aggregated at the island-level good by a standard Dixit-Stiglitz aggregator, $C_i = \left(\int C_{i,j}^{1-\frac{1}{\theta}}\right)^{\frac{1}{1-\frac{1}{\theta}}} dj$ with $\theta > 1$.

Suppose now that each (i, j) producer is hit with an idiosyncratic, mean-zero productivity shock, $\upsilon_{i,j}$. In this case, the price of good $c_{i,j}$ in logs turns out to be

$$p_{i,j} = \upsilon_{i,j} + \gamma\mu_i + (1 - \gamma)(q + \zeta).$$

In this case, demand for the good $c_{i,j}$ is governed by the standard formula

$$c_{i,j} = -\theta(p_{i,j} - p_i) + c_i,$$

which reflects a substitution effect governed by the standard elasticity parameter at *at the good level:* an econometrician studying good-level prices would find no evidence that the typical good is Giffen. Nevertheless, the total price level on island i,

$$p_i = \int p_{i,j} dj = \gamma \mu_i + (1-\gamma)(q+\zeta)$$

is both identical to its value in the baseline economy, and reflects the optimal (even) weighting of the signals $p_{i,j}$ that shoppers use in equilibrium to infer their local demand shock: the subsequent analysis of the island-level and aggregate economy is not affected.

5 Equilibria Selection

This section exploits the characterization of the individual best response function in the game implied by dispersed information to examine the stability of equilibria under two popular out-of-equilibrium beliefs dynamics: rationalizability and adaptive learning. We show that equilibria with reinforcing feedback may be stable, although sentiment-like equilibria are generally excluded by these tests.

5.1 Higher-Order Belief Dynamics

In our case a "rational"expectation is characterized by a mapping $a_i(a) : \Re \rightarrow \Re$ which associates with any value of the average weight a an individual best weight $a_i(a)$. A REE is an equilibrium weight $\hat{a}$ such that $a_i(\hat{a}) = \hat{a}$ for each $i \in (0,1)$, and $\hat{a}$ reflects the precision of the endogenous signal at the equilibrium. But how can people achieve common knowledge that others will conform to the equilibrium prescription so that $\hat{a}$ is actually the correct weight?

This is an old question on the epistemic foundations of Nash equilibrium with an important tradition in decision theory. A widely accepted concept is that of the rationalizable set (Bernheim, 1984; Pearce, 1984), defined as the profile set surviving to iterated deletion of never best replies. This criterion exploits implications from common knowledge of rationality in the model.

Guesnerie (1992) introduces the rationalizability argument to macroeconomics in the context of complete information competitive economies. Here we adapt Guesnerie's original setup in a dispersed information model focusing on the best-expectation coordination game entailed by the maps $\{a_i(a)\}_{i\in(0,1)}$. In contrast to the original Guesnerie setting, here agents agree on the unconditional expectation of their idiosyncratic fundamental which is exogenous to the average behavior but are uncertain about the precision of the information they are looking at. Nothing in the model guarantees that agents use the same conditional distribution to forecast their idiosyncratic fundamental. In the following we will check whether the assumption of common knowledge is sufficient to restrict the agents' strategic space to the REE prescriptions.

Initially we will take a local point of view. Suppose it is common knowledge that the individual weights on the signal lie in a neighborhood $\mathcal{F}(\hat{a})$ of $\hat{a}$, is this a sufficient condition for convergence in higher-order beliefs to $\hat{a}$? The process of iterated deletion of never-best replies works as follows. Let τ index the iterative round of deletion. If $a_{i,0} \in \mathcal{F}(\hat{a})$ for each i then $a_0 \in \mathcal{F}(\hat{a})$. Nevertheless, the latter implies that second order beliefs are justified in which $a_{i,1} = a_i(a_0)$ for each i, so that $a_{i,1} \in a_i(\mathcal{F}(\hat{a}))$. As a consequence $a_1 \in a_i(\mathcal{F}(\hat{a}))$. One can iterate the argument showing that $a_{i,\tau} \in a_i^{\tau}(\mathcal{F}(\hat{a}))$. Hence we have the following.

Definition 3. *A REE $\hat{a}$ is a locally unique rationalizable outcome if and only if there exists a neighborhood $\mathcal{F}(\hat{a})$ of $\hat{a}$ such that $\lim_{\tau\to\infty} a_i^{\tau}(\mathcal{F}(\hat{a})) = \hat{a}$.*

When a REE is a locally unique rationalizable outcome we can conclude that the equilibrium is stable to a sufficiently small higher-order beliefs perturbation (or is eductively stable in Guesnerie's language). In other words, the equilibrium is robust to beliefs that others could locally deviate from it, as agents conclude that no rational conjecture can sustain such a deviation.

A global qualification of the higher-order belief stability criterion obtains when the best response function entails a contraction for each point of the domain of a, that is when $\lim_{\tau\to\infty} a_i^{\tau}(\Re) = \hat{a}$. When an equilibrium is the globally unique rationalizable outcome, then this is the only profile of strategies that rational agents will play. In this sense the theory provides a complete out-of-equilibrium belief dynamics converging to the unique equilibrium. On the other hand, notice that uniqueness of a REE is not sufficient to guarantee stability to even arbitrarily small perturbation of higher-order beliefs.

Belief convergence requires that $a_i(a)$ entails a contracting map. For a locally rationalizable REE a necessary and sufficient condition is $|a_i'(\hat{a})| < 1$. The proposition below states the result.

Proposition 7. *The low and unique equilibrium are locally unique rationalizable equilibrium provided σ is large enough. Whenever the middle and the high equilibria exist, the latter is always a locally unique rationalizable equilibrium, whereas the former is never. In the limit of $\sigma \to 0$ the middle and the low equilibria are never stable under higher-order beliefs dynamics, whereas the unique equilibrium is.*

Proof. Given in Appendix A.1 ∎

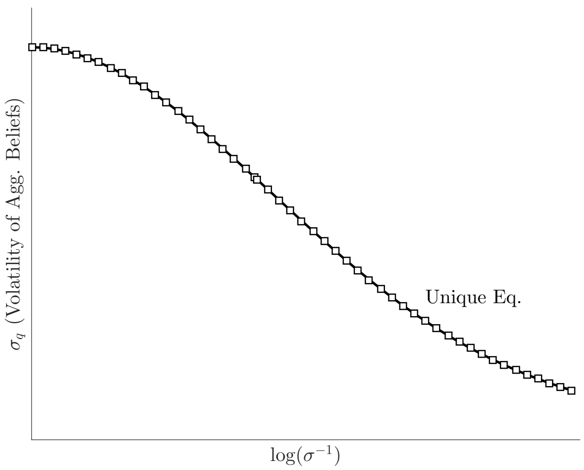
One can easily show that $a_\circ$ is never a locally unique rationalizable outcome from qualitative properties associated with the equilibria. First, $a_i(1-\gamma)^{-1} = 0$ lies below the forty-five degrees line. Second, for $a > (1-\gamma)^{-1}$ the best weight function is always monotonically increasing. These two observations taken jointly require $a_i'(a_\circ) > 1$, and thus are sufficient to claim that whenever the middle equilibrium $a_\circ$ exists, then it is not a locally unique rationalizable outcome.

A second result is that whenever a_+ exists distinct from $a_\circ$, it is always a locally unique rationalizable outcome since the first derivative at this equilibrium has to be bounded in $(0,1)$ to meet the 45 degree line. In the knife-edge case that $a_\circ = a_+$ the fix point map is tangent to the bisector, meaning $a_i'(a_+) = 1$, which does not satisfy the condition for rationalizability.

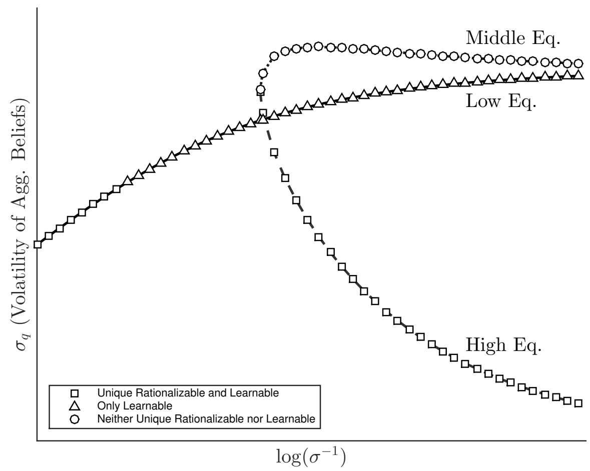
To establish the convergence properties of a_-, one needs to check that there is a threshold $\underline{\sigma}$ such that for any $\sigma \in (0, \underline{\sigma})$ this equilibrium is not locally rationalizable, whereas it is otherwise.

To give an intuition notice that in the case of the two limit equilibrium outcomes we have $\lim_{\sigma\to 0} a_- = \lim_{\sigma\to 0} a_\circ = (1-\gamma)^{-1}$ for which the derivative obtains as $\lim_{\sigma\to 0} a_i' = \pm\infty$. On the other hand $a_i'(a_-)$ increases in σ with 0 as an upper bound so that there exists a

$\underline{\sigma}$ such that for any $\sigma > \underline{\sigma}$ the low a_- is always rationalizable. Therefore there could be a multiplicity (two) of rationalizable REE for intermediate values of $\sigma \in (\underline{\sigma}, \bar{\sigma})$. To understand if this is the case we perform the following numerical analysis.



(a) $\gamma = 0.75$



(b) $\gamma = 0.25$

Figure 8: Volatilities and equilibrium stability approaching the limit.

The first panel of Figure 8 illustrates the size of expectation volatility generated by the unique-equilibrium economy as a function of the inverse of σ, and shows the equilibrium is always rationalizable. The second panel plots beliefs volatility and stability properties for the three equilibria (whenever they exist) of the model with $\gamma = 0.25$. For σ high enough only the low equilibrium exists. The volatility generated at that equilibrium is monotonically decreasing in σ. The low equilibrium is a locally unique rationalizable outcome provided σ is sufficiently large. With sufficiently low σ, the middle and the high equilibrium exist too. The latter is always a locally unique rationalizable outcome, whereas the former is never. The same picture shows that the unique equilibrium is always unique locally rationalizable outcome.

Notice that in the example illustrated in figure 8 there is no region in which multiple locally unique rationalizable outcomes exist. Moreover, there exists a region in which the low equilibrium is the only equilibrium, but it is not a locally unique rationalizable outcome. Finally, only for sufficiently small σ can a globally unique rationalizable outcome arise,

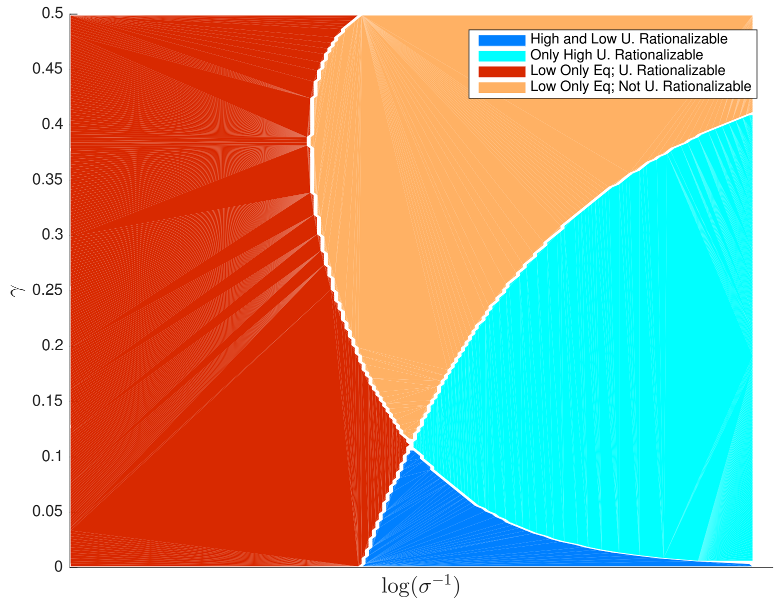


Figure 9: Stability properties for different signal weights.

originating in the low equilibrium.

In Figure 9, we show through numerical investigation that for sufficiently low values of γ there is a region in which two equilibria, the high and the low, emerge as locally unique rationalizable outcomes. Nevertheless this would not arise in the limit of infinite precision where only the high equilibrium remains a locally unique rationalizable equilibrium.

5.2 Adaptive Learning

To address the question of learnability of the rational expectation equilibria we have analyzed, we now suppose that agents behave like econometricians, rather than game-theorists. That is, agents individually set their weights consistently with data generated by possibly out-of-equilibrium replications of the signal extraction problem, without internalizing the effect of the ongoing process of learning in the economy. In practice, at time t they set a weight $a_{i,t}$ which is estimated from the sample distribution of signals collected from past repetition of the signal extraction problem. If agents are close to a locally adaptively stable equilibrium, this implies that once estimates are close enough to the equilibrium values they will almost surely converge to the equilibrium.

It has been shown that the asymptotic behavior of statistical learning algorithms can be studied by stochastic approximation techniques (for details refer to Marcet and Sargent, 1989a,b and Evans and Honkapohja, 2001). To see how this works in our context, consider the case that agents learn about the optimal weight according to an optimal adaptive learning scheme:

$$a_{i,t} = a_{i,t-1} + \gamma_t \; S_{i,t-1}^{-1} \; p_{i,t} \left(\mu_{i,t} - a_{i,t-1} p_{i,t} \right) \tag{59}$$

$$S_{i,t} = S_{i,t-1} + \gamma_{t+1} \left(p_{i,t}^2 - S_{i,t-1} \right), \tag{60}$$

where γ_t is a decreasing gain with $\sum \gamma_t = \infty$ and $\sum \gamma_t^2 = 0$, and matrix $S_{i,t}$ is the estimated variance of the signal rewritten with a convenient time index. The following formally defines adaptive stability.

Definition 4. *A REE $\hat{a}$ is a locally learnable equilibrium if and only if there exists a neighborhood $F(\hat{a})$ of $\hat{a}$ such that, given an initial estimate $a_{i,0} \in F(\hat{a})$, it is $\lim_{t\to\infty} a_{i,t} \overset{a.s}{=} \hat{a}$.*

Adaptive learning describes out-of-equilibrium dynamics, which can explain how agents can (or fail to) converge to a REE. The collective use of statistical techniques, although it does not account for the fully fledged effect of collective learning, may converge to a situation in which agents' estimates about the precision of the signal are correct. Such a convergence point is necessarily a REE.

The global qualification of the learnability criterion is obtained when convergence occurs almost surely irrespective of any initial condition, that is $\lim_{t\to\infty} a_{i,t} \overset{a.s}{=} \hat{a}$ for any $a_{i,0} \in \Re$. Notice that, in contrast to the rationalizability criterion, there could exist a unique globally learnable equilibrium despite the existence of multiple rational expectation equilibria. This is because the stochasticity of the learning process will always displace estimates temporarily away from equilibrium values. Nevertheless, if there only exists one REE, then if it is learnable it has to be globally learnable.

To check local learnability of the REE, suppose we are already close to the rest point of the system. That is, consider the case $\int \lim_{t\to\infty} a_{i,t} \; di = \hat{a}$ where $\hat{a}$ is one among the equilibrium points $\{a_-, a_\circ, a_+\}$ and so

$$\lim_{t\to\infty} S_{i,t} = \sigma_s^2(\hat{a}) = \gamma^2\sigma_\mu^2 + \frac{(1-\gamma)^2}{(1-\hat{a}(1-\gamma))^2}\sigma_\zeta^2. \tag{61}$$

According to stochastic approximation theory, we can write the associated ODE governing the stability around the equilibria as

$$\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \int \lim_{t\to\infty} \mathrm{E}\left[S_{i,t-1}^{-1} p_{i,t}\left(\mu_{i,t} - a_{i,t-1}p_{i,t}\right)\right] di = \\
&= \sigma_s^2(\hat{a})^{-1} \int \mathrm{E}\left[p_{i,t}\left(\mu_{i,t} - a_{i,t-1}p_{i,t}\right)\right] di = \\
&= \sigma_s^2(\hat{a})^{-1}\left(\gamma\sigma_\mu^2 - a_{i,t-1}\left(\gamma^2\sigma_\mu^2 + \frac{(1-\gamma)^2}{(1-a_{t-1}(1-\gamma))^2}\sigma_\zeta^2\right)\right) = \\
&= a_i(a) - a.
\end{aligned}$$

For asymptotic local stability to hold, the eigenvalues of the Jacobian of $a_i(a)$ calculated at the equilibrium have to lie inside the unit circle. The relevant condition for stability is therefore $a_i'(a) < 1$. The result is stated by the following proposition.

Proposition 8. *The unique equilibrium is always learnable. Whenever the middle and high equilibria exist, the latter is locally learnable, whereas the former it is not. The low equilibrium is always locally learnable, except at the limit $\sigma \to 0$, and it is globally learnable provided σ is large enough.*

Proof. Given in Appendix A.1 ∎

In practice, we have proved that $a_i(a)$ corresponds, at least locally around the equilibrium, to the dynamic map called the "projected T-map" in the adaptive learning literature. The T-map is a correspondence between the parameters used to calibrate the individual forecasting rule and the ones that would be optimal given observed data. It is an useful tool to recover information on the local out-of-equilibrium dynamics when expectations are formed recursively as information is gathered through time.

Referring to figure 2, the slope of the curves at the intersection of the bisector features the stable or unstable nature of the equilibrium. In particular, notice that the middle equilibrium defines two distinct basins of attraction for the learnable equilibria. As σ decreases the basin of attraction of the high equilibrium grows from below. This means that estimates are more and more likely to converge to the high equilibrium, the sentiment-free one, as σ gets smaller. At the limit $\sigma \to 0$, the low equilibrium is no longer learnable from above, meaning that for any estimate a larger than a_{-}, no matter how close to a_{-}, is fated to trigger convergence to the high equilibrium. This would suggest that although sufficiently negative shocks to the estimates can lead to a persistent deviation in the lower basin of attraction of the low "sentiment" equilibrium, long run-convergence can only obtain at the high sentiment-free equilibrium.

Our learnability results contrast with the original stability analysis in Benhabib et al. (2015). In their approach, agents treat the signal as exogenous, conjecturing a common precision and then updating dynamically. In our characterization instead, agents' learning incorporates the endogenous relationship between the signal precision and the average action and this endogeneity generates a coordination issue not contemplated by Benhabib et al. (2015). The differences in our results suggest that apparently differences in the microfoundations of sentiment-like fluctuations can lead to very different conclusions about their stability, and thus deserve close attention in this literature.

6 Conclusion

Endogenous structures of asymmetric information can deliver strong multipliers on common disturbances and thus offer a potential foundation for expectation-driven economic fluctuations. Here we have demonstrated that a single analysis can address such fluctuations whether they originate in common fundamentals or a mixture of fundamentals and common noise. Given the great amplification power of this mechanism, "sentiment" equilibria may, paradoxically, originate from economic fundamentals themselves and need not originate with shocks disconnected from the physical economic environment. Instead, expectation-driven fluctuations can be initiated by small changes in fundamentals that, under full-information, would warrant far smaller reactions.

The mechanism behind this result is a strong feedback loop that arises when agents observe, and draw inference from, endogenous variables. We microfounded such endogenous signals as competitive prices. We have therefore naturally embedded the mechanism described above in a microfounded economy, in which the essential features are local (idiosyncratic) demand-side shocks and final good firms learning from prices that reflect a combination of aggregate and local conditions. Our approach provides foundations for the correlated signal structures that are essential for "sentiment" fluctuations to arise.

In this economy, final goods firms observing higher input prices partially attribute those prices to favorable local demand conditions. When the effect of prices on inference is strong enough, observation of a higher price leads local firms to demand more, rather than fewer, inputs. Through this process, the informational effects of prices can lead to upward-sloping aggregate demand, reversing the typical comovements associated with supply shocks.

A Appendix

A.1 Proofs of Propositions

Proof of Proposition 1. To prove uniqueness for $\gamma \geq 1/2$, observe that the function $a_i(a)$ is continuous, bounded above by γ^{-1}, and monotonically decreasing in the range $(-\infty, (1-\gamma)^{-1})$. From $\gamma \geq 1/2$, we have $(1-\gamma)^{-1} > \gamma^{-1}$. Thus $a_i(a)$ intersects the 45 degree line a single time.

To prove the existence of a_-, notice that $\lim_{a\to-\infty} a_i = \gamma^{-1}$ and $a_i((1-\gamma)^{-1}) = 0$. By continuity, an equilibrium $a_- \in (0, (1-\gamma)^{-1})$ must always exist. Moreover a_- must be monotonically decreasing in σ^2 as a_i is monotonically decreasing in σ^2.

We now assess the conditions under which additional equilibria may also exist. Because $\lim_{a\to\infty} a_i = \gamma^{-1}$, the existence of a second equilibria (crossing the 45 degree line in figure 1) implies the existence of a third. Thus, we need to to check whether or not the difference $a_i(a) - a$ is positive anywhere in the range $a > (1-\gamma)^{-1}$. Such a difference is positive if and only if

$$\Phi(\sigma) \equiv \gamma (1 - a(1-\gamma))^2 (1-\gamma a) - a(1-\gamma)^2 \sigma^2 > 0 \tag{62}$$

which requires $a < \gamma^{-1}$ as a necessary condition. Therefore, if two other equilibria exist they must lie in $((1-\gamma)^{-1}, \gamma^{-1})$. Fixing $a \in ((1-\gamma)^{-1}, \gamma^{-1})$, $\lim_{\sigma\to 0} \Phi(\sigma)$ is clearly positive, implying that there always exists a threshold $\bar{\sigma}$ such that two equilibria $a_+, a_\circ \in ((1-\gamma)^{-1}, \gamma^{-1})$ exist with $a_+ \geq a_\circ$ for $\sigma^2 \in (0, \bar{\sigma}^2)$. ∎

Proof of Proposition 2. Notice that $\partial\Gamma/\partial a > 0$ if and only if $\gamma < \min\{(1-\gamma)^{-1}, \gamma^{-1}\}$. The left-hand side of the fixed point expression in (25) is downward sloping in a and falling in σ, implying the fixed-point intersection a_u and a_- must increases as σ falls. Similarly, $a_\circ$ falls and a_+ grows as σ falls, implying the amplifying feedback for the former and dampening feedback for the latter. ∎

Proof of Proposition 3. To prove the limiting statement for $\gamma \geq 1/2$, consider any point $a_\delta = \frac{1-\delta}{1-\gamma}$ such that $\delta > 0$. Then, we have

$$a_i(a_\delta) = \frac{\gamma\delta^2}{\gamma^2\delta^2 + \sigma^2(1-\gamma)^2.} \tag{63}$$

Since $\lim_{\sigma^2\to 0} a_i(a_\delta) = \frac{1}{\gamma}$ for any δ, the unique equilibrium must converge to the same point. That the variance of this equilibrium approaches zero follows from equation (21).

To prove the limiting statement for $\gamma < 1/2$, recall the monotonicity of $a_i(a)$ on the range $(0, (1-\gamma)^{-1}$. Following the logic of proposition 1, for any point a_δ in that range, $\lim_{\sigma^2\to 0} a_i(a_\delta) = \gamma^{-1}$, while $a_i((1-\gamma)^{-1}) = 0$. Thus, the intersection defining a_- must approach $(1-\gamma)^{-1}$. An analogous argument for the point just to the right of $(1-\gamma)^{-1}$

establishes that a_- converges to the same value. Finally, the bounded monotonic behavior of $a_i(a)$ establishes that for the high equilibrium $\lim_{\sigma^2 \to 0} a_+ = \gamma^{-1}$.

That the output variance of the high equilibrium in the limit $\sigma \to 0$ is zero follows from equation (22). The limiting variance of the other limit equilibrium equilibria can be established by noticing that (25) implies

$$\frac{\sigma^2}{(1-a(1-\gamma))^2} = \frac{\gamma(1-a\gamma)}{(1-\gamma)} \tag{64}$$

which, substituted in to (22), gives (27) for $a \to (1-\gamma)^{-1}$. ∎

Proof of Proposition 6. We can prove that a sentiment-free equilibrium with no aggregate variance exists for $a = \gamma^{-1}$ by simple substitution in (57). The limiting variance of the other limit equilibrium at the singularity $a \to (1-\gamma)^{-1}$ can be established by noticing that (57) implies

$$\frac{\sigma^2}{(1-a(1-\gamma))^2} = \frac{1-a\gamma}{a\gamma} + \frac{1-a(1-\gamma)}{a\gamma}\frac{\sigma^2}{(1-a(1-\gamma))^2},$$

which gives

$$\frac{\sigma^2}{(1-a(1-\gamma))^2} = -\frac{1-a\gamma}{1-a}.$$

Substituted into (56), this gives (27) for $a \to (1-\gamma)^{-1}$. ∎

Proposition 7. The derivative of $a_i(a)$ with respect to a is given by:

$$a_i'(a) = -\frac{2\gamma\,(1-\gamma)^3\,(1-(1-\gamma)\,a)\,\sigma^2}{\left((1-\gamma)^2\,\sigma^2 + (1-(1-\gamma)a)^2\,\gamma^2\right)^2}, \tag{65}$$

which is positive whenever $a > 1/(1-\gamma)^{-1}$. Then necessarily $a_i'(a_\circ) > 1$ and $a_i'(a_+) \in (0,1)$. Concerning the stability of a_- notice that $\lim_{\sigma \to \infty} a_i'(a_-) = 0$ and

$$\lim_{\sigma^2 \to 0, a \to (1-\gamma)_\pm^{-1}} a_i'(a) = \pm\infty$$

given that, at the limit $\sigma \to 0$, σ^2 and $(1-a\,(1-\gamma))^2$ go to zero at the same speed. On the other hand, concerning the stability of a_u, which exists when $1-\gamma < \gamma$, notice that $\lim_{\sigma \to \infty} a_i'(a_u) = 0$ and

$$\lim_{\sigma^2 \to 0, a \to \gamma^{-1}} a_i'(a) = \lim_{\sigma^2 \to 0, a \to \gamma^{-1}} -\frac{2\gamma\,(1-\gamma)^3\,(1-(1-\gamma)\,a)\,\sigma^2}{\left((1-\gamma)^2\,\sigma^2 + (1-(1-\gamma)a)^2\,\gamma^2\right)^2} = 0_-,$$

which proves that the unique equilibrium a_u is locally unique rationalizable at the limit. ∎

Proposition 8. The derivative $a_i'\,(a)$ at the four equilibria has been already studied. We know that $a_i'\,(a_u) < 0$, $a_i'\,(a_+) \in (0,1)\,,\, a_i'\,(a_-) < 0$ and $a_i'\,(a_\circ) > 1$. Nevertheless at the limit $\sigma \to 0$ where $a_+ = a_\circ$ coincide there is no neighborhood to qualify a_+ a locally learnable REE. ∎

A.2 Derivations of the model

Proof. In this section we provide all the details about the solution of the microfounded model, extending its formulation to include a convexity in the disutility of labor. For the baseline analysis in the text, we assume the convexity parameter $\alpha = 0$.

The utility function of the household is

$$\int e^{\mu_i} \left(\log C_i - \phi N_i^{1+\alpha}\right) di \tag{66}$$

where $\alpha > 0$ accounts for the inverse of the Firsh labor elasticity. Therefore, the maximization problem of the three representative members of the family are written now as

$$\begin{aligned}
\text{producer} &: \max_{N_i, Z_{(i)}} \Pi_i = \max_{N_i, z_{(i)}} \{P_i C_i - W_i N_i - Q Z_{(i)}\} \\
\text{worker} &: \max_{N_i} \{-e^{\mu_i} \phi N_i^{1+\alpha} + \Lambda W_i N_i\} \\
\text{shopper} &: \max_{C_i} \{E[e^{\mu_i} | P_i] \log C_i - E[\Lambda | P_i] P_i C_i\}
\end{aligned}$$

subjet to

$$\int P_i C_i di = \int W_i N_i di + QZ + \int \Pi_i di$$

where Λ denotes the marginal utility of relaxing the constraint, i.e. the Lagrangian associated to the budget constraints in the maximization problems.

The steady state of the economy is given by:

$$\begin{aligned}
\frac{1}{C} &= \Lambda P \\
(1+\alpha)\phi N^{\alpha} &= \Lambda W \\
Q &= (1-\gamma) P N^{\gamma} \bar{Z}^{-\gamma} \\
W &= \gamma P N^{(\gamma-1)} \bar{Z}^{1-\gamma} \\
C &= N^{\gamma} \bar{Z}^{1-\gamma}.
\end{aligned}$$

Notice that we have 6 variables C, P, N, W, Q, Λ and only 5 restrictions, i.e. the price level P is not determined. Hence, we need to express the price level in terms of a numeraire, which we conveniently choose as $\Lambda = 1$.

The corresponding system of equilibrium conditions expressed in log-linear term is:

$$\begin{aligned}
\mu_i + \alpha n_i &= w_i \\
E[\mu_i | p_i] - c_i &= p_i \\
w_i &= p_i + (\gamma - 1) n_i + (1-\gamma) z_{(i)} - (1-\gamma)\zeta \\
q &= p_i + \gamma n_i - \gamma z_{(i)} - (1-\gamma)\zeta \\
c_i &= \gamma n_i + (1-\gamma) z_{(i)} - (1-\gamma)\zeta
\end{aligned}$$

plus the market clearing condition for the endowment $\int z_{(i)} di = 0$.

Aggregate variables. Averaging the two sides of the fist relation we have $w = \alpha n$. We have then

$$\begin{aligned} \alpha n &= p + (\gamma - 1)n - (1-\gamma)\zeta \\ q &= p + \gamma n - (1-\gamma)\zeta \\ c &= \gamma n - (1-\gamma)\zeta \\ \int E[\mu_i|p_i] - c &= p \end{aligned}$$

which constitutes a linear system in four unknown p, q, n, c that can be expressed as function of two states $\int E[\mu_i|p_i]di, \zeta$. This system can be written as

$$\begin{bmatrix} p \\ q \\ n \\ c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1-\gamma+\alpha & 0 \\ 1 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma^{-1} \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ n \\ c \end{bmatrix} + \begin{bmatrix} 0 & 1-\gamma \\ 0 & \gamma - 1 \\ 0 & (1-\gamma)\,\gamma^{-1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \int E[\mu_i|p_i]di \\ \zeta \end{bmatrix}$$

whose solution is

$$\begin{aligned} p &= \frac{1+\alpha-\gamma}{1+\alpha} \int E[\mu_i|p_i]di + (1-\gamma)\zeta \\ q &= \int E[\mu_i|p_i]di \\ n &= \frac{1}{1+\alpha} \int E[\mu_i|p_i]di \\ c &= \frac{\gamma}{1+\alpha} \int E[\mu_i|p_i]di - (1-\gamma)\zeta \end{aligned}$$

Island-specific variables. The relevant system of equations is

$$\begin{aligned} E[\mu_i|p_i] - c_i &= p_i \\ c_i &= \gamma n_i + (1-\gamma) z_{(i)} - (1-\gamma)\zeta \\ w_i &= p_i + (\gamma-1)n_i + (1-\gamma)z_{(i)} - (1-\gamma)\zeta \\ q &= p_i + \gamma n_i - \gamma z_{(i)} - (1-\gamma)\zeta \end{aligned}$$

which constitutes a linear system in four unknown $p_i, c_i, n_i, z_{(i)}$ that can be expressed as function of four states $\mu_i, \zeta, q, E[\mu_i|p_i]$. This system can be written as

$$\begin{bmatrix} p_i \\ c_i \\ n_i \\ z_{(i)} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & \gamma & 1-\gamma \\ \frac{1}{1+\alpha-\gamma} & 0 & 0 & \frac{1-\gamma}{1+\alpha-\gamma} \\ \gamma^{-1} & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_i \\ c_i \\ n_i \\ z_{(i)} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \gamma - 1 & 0 & 0 \\ -\frac{1}{1+\alpha-\gamma} & -\frac{1-\gamma}{1+\alpha-\gamma} & 0 & 0 \\ 0 & (\gamma-1)\gamma^{-1} & -\gamma^{-1} & 0 \end{bmatrix} \begin{bmatrix} \mu_i \\ \zeta \\ q \\ E[\mu_i|p_i] \end{bmatrix}$$

where we already used $w_i = \mu_i + \alpha\mu_i$. The solution of the system is

$$\begin{aligned}
c_i &= -\frac{\gamma}{1+\alpha}\mu_i + \frac{1+\alpha(1-\gamma)}{1+\alpha}E[\mu_i|p_i] - (1-\gamma)(q+\zeta) \\
p_i &= \frac{\gamma}{1+\alpha}\mu_i + \frac{\alpha\gamma}{1+\alpha}E[\mu_i|p_i] + (1-\gamma)(q+\zeta) \\
n_i &= \frac{1}{1+\alpha}\left(E[\mu_i|p_i] - \mu_i\right) \\
z_{(i)} &= -q + E[\mu_i|p_i].
\end{aligned}$$

which is consistent with the expression for their relative aggregate variables.

In the case $\alpha \neq 0$ notice that the price signal can be equivalently written as

$$\tilde{p}_i = \frac{1+\alpha}{1+\alpha(1-\gamma)}\left(p_i - \frac{\alpha\gamma}{1+\alpha}E[\mu_i|p_i]\right) = \tilde{\gamma}\mu_i + (1-\tilde{\gamma})(q+\zeta)$$

where $\tilde{\gamma} = \gamma/(1+\alpha(1-\gamma))$. Notice that now limit sentiment equilibria exist with $\tilde{\gamma} < 1/2$ which could well obtain even with $\gamma > 1/2$ for a sufficiently high α. ∎

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