Guided tour on Johansen's cointegration analysis

This guided tour contains mathematical formulas and/or Greek symbols and are therefore best viewed with Internet Explorer, because other web browsers may not display the "Symbol" fonts involved. For example, "β" should be displayed as the Greek letter "beta" rather than the Roman "b". If not, reload this guided tour in Internet Explorer, or make the latter your default web browser.

What is cointegration?

Cointegration is the phenomenon that each component $Y_{i,t}$, $i = 1,...,k$, of a vector time series process Y_t is a unit root process, possibly with drift, but certain linear combinations of the $Y_{i,t}$'s are stationary. Thus

$$Y_t = Y_{t-1} + \mu + V_t,$$

where V_t is a zero-mean k-variate stationary time series process and μ is a k-vector of drift parameters, but there exists a $k \times r$ matrix β with rank $r < k$ such that $\beta' Y_t$ is (trend) stationary.

In order to show that this is possible, let us assume that V_t can be wriiten as an infinite order vector moving average process:

$$V_t = C(L)e_t,$$

where e_t is i.i.d. k-variate white noise with unit variance matrix and $C(L)$ is a matrix-valued lag polynomial:

$$C(L) = C_0 + C_1 L + C_2 L^2 + C_3 L^3 + \text{...........},$$

with the C_j's $k \times k$ coefficient matrices and L the backshift lag operator (i.e., $Le_t = e_{t-1}$). Now write $C(L)$ as

$$C(L) = C(1) + [C(L) - C(1)] = C(1) + (1 - L)D(L),$$

where

$$D(L) = [C(L) - C(1)]/(1 - L).$$

This is always possible because $C(L) - C(1)$ is a zero matrix for $L = 1$, hence each element of this lag polynomal matrix has root 1 and thus these elements have a common factor $1 - L$. Next, denote

$$W_t = D(L)e_t.$$

Under some regularity conditions on the matrices C_j the W_t's are stationary.

We can now write

$$Y_t = Y_{t-1} + \mu + C(1)e_t + W_t - W_{t-1},$$

which by backward subtitution can be written as

$$Y_t = Y0 - W0 + \mu t + C(1)[e_t + ... + e_1] + W_t.$$

Apart from the term $Y0 - W0$ (which acts as a vector of intercepts) and the linear time trend μt, the nonstationarity of Y_t is due to the term $C(1)[e_t + ... + e_1]$. But if the matrix $C(1)$ is singular, with rank $k-r < k$, then there exist r linear combinations of the rows of $C(1)$ that are zero row vectors. The coefficients of these linear combinations form the $k \times r$ matrix β: $\beta' C(1) = O$. Thus

$$\beta' Y_t = \beta'(Y0 - W0) + \beta' \mu t + \beta' W_t.$$

For $t \to \infty$ the random vector $\beta' W_t$ becomes independent of $\beta'(Y0 - W0)$, so that for large t the latter may be treated as a vector of constant intercepts. Then $\beta' Y_t$ is trend stationary because $\beta' W_t$ is zero-mean stationary. The columns of the matrix β are called the *cointegrating vectors*, and the rank r of β is called the *cointegration rank*.

The Granger representation theorem

Clive Granger has shown that under some regularity conditions we can write a cointegrated process Y_t as a Vector Error Correction Model (VECM):

$$\Delta Y_t = \pi_0 + \pi_1 t + \Pi_1 \Delta Y_{t-1} + ... + \Pi_{p-1} \Delta Y_{t-p+1} + \alpha\beta' Y_{t-p} + U_t,$$

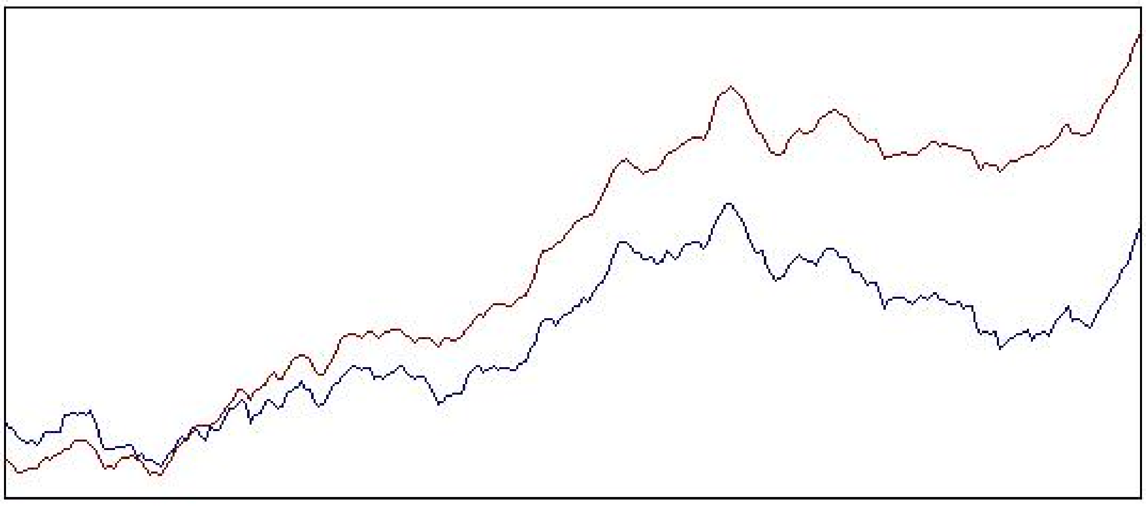
where Δ is the difference operator (i.e., $\Delta Y_t = Y_t - Y_{t-1}$), the U_t's are i.i.d. $(0,\Sigma)$, and

$$\pi_1 = \alpha\gamma_1$$

for some r-vector γ_1. The latter condition is called *cointegrating restrictions on the trend parameters*, and is necessary because otherwise Y_t would be vector unit root process with linear drift (which is rare in practice). Thus

$$\Delta Y_t = \pi_0 + \Pi_1 \Delta Y_{t-1} + ... + \Pi_{p-1} \Delta Y_{t-p+1} + \alpha[\gamma_1 t + \beta' Y_{t-p}] + U_t.$$

Since ΔY_t is stationary, we must have that $\gamma_1 t + \beta' Y_{t-p}$ is stationary, hence $\beta' Y_t$ is trend stationary. Consequently, the time series involved have drift, and veer apart, like in the following picture:



When do we need a time trend in the VECM?

Most nonstationary macroeconomic time series such as the log of GDP have drift: they display a trending pattern with nonstationary fluctuations around the deterministic time trend. However, a VECM without a time trend,

$$\Delta Y_t = \pi_0 + \Pi_1 \Delta Y_{t-1} + ... + \Pi_{p-1} \Delta Y_{t-p+1} + \alpha\beta' Y_{t-p} + U_t,$$

is also able to generate drift in Y_t because π_0 acts (in some way) as a vector of drift parameters. Thus drift in the Y_t process is no reason to include a time trend in the VECM.

You should include a time trend in the VECM if you suspect that:

- The components of the vector $\beta' Y_t$ (or some of them) are trend stationary,
  so that they veer apart, as in the above picture.
- Y_t is trend stationary rather than a multivariate unit root with drift process, and you want to test this hypothesis.
  Under the trend stationarity hypothesis the matrix $\alpha\beta'$ has full rank (k). One of Johansen's cointegration tests, the trace test, has this as alternative hypothesis.

Cointegrating restrictions on the intercept parameters of the VECM

Now suppose that in the case of a VECM without time trend,

$$\Delta Y_t = \pi_0 + \Pi_1 \Delta Y_{t-1} + ... + \Pi_{p-1} \Delta Y_{t-p+1} + \alpha\beta' Y_{t-p} + U_t,$$

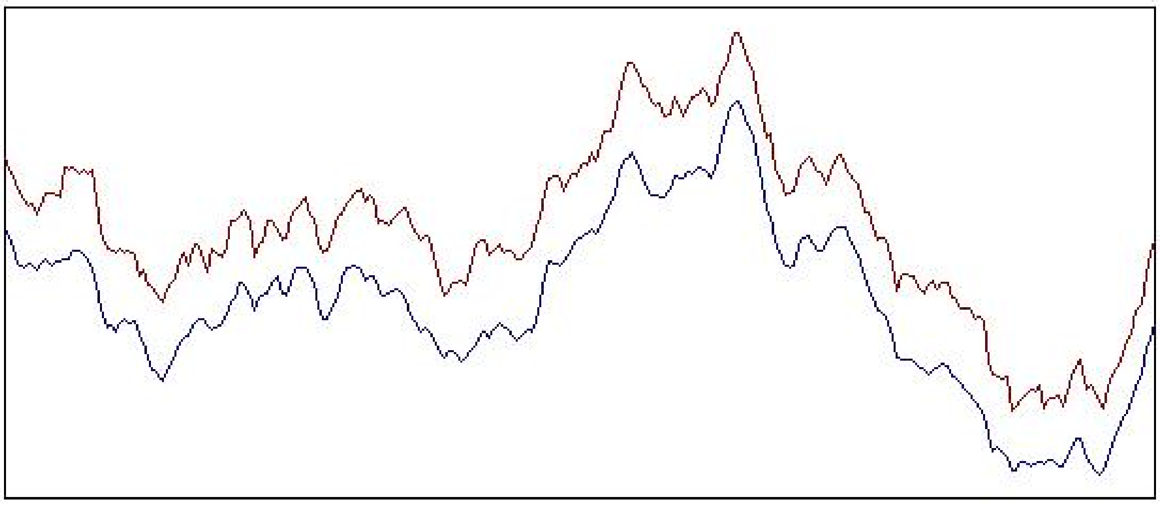
there exists a vector γ_0 such that

$$\pi_0 = \alpha\gamma_0.$$

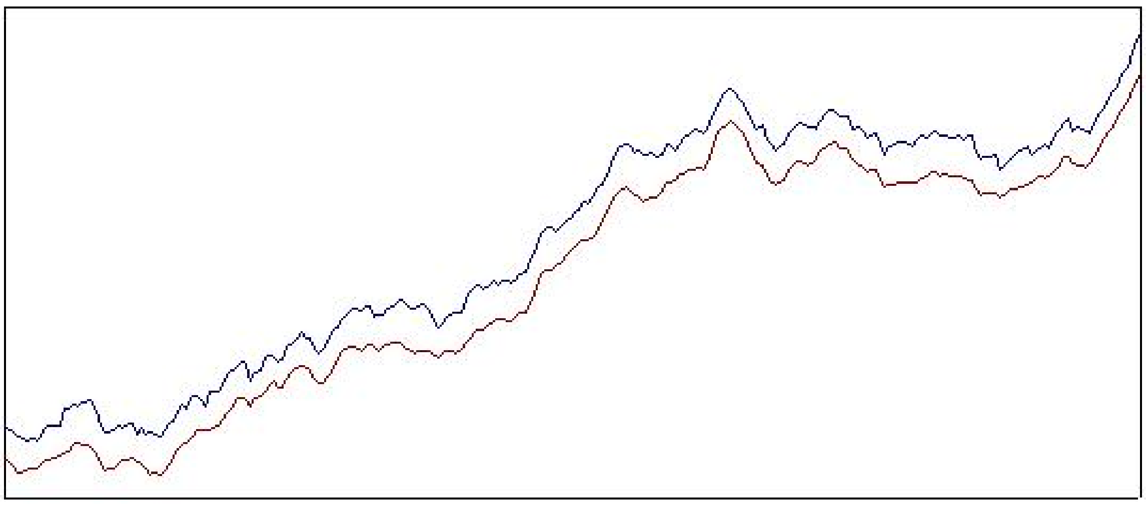
This condition is called *cointegrating restrictions on the intercept parameters*. Then the VECM can be written as

$$\Delta Y_t = \Pi_1 \Delta Y_{t-1} + ... + \Pi_{p-1} \Delta Y_{t-p+1} + \alpha[\gamma_0 + \beta' Y_{t-p}] + U_t.$$

In this case the components of $\gamma_0 + \beta' Y_t$ are **zero-mean stationary**, hence ΔY_t is zero-mean stationary, and therefore Y_t has no drift! Consequently, cointegrating restrictions on the intercept parameters are only appropriate if the time series run approximately parallel without drift, like in the following picture:



If the time series run parallel and upwards sloping, like



you should **not** impose these restrictions.

Johansen's cointegration tests

Soren Johansen's approach is to estimate the VECM by maximum likelihood, under various assumptions about the trend or intercept parameters and the number *r* of cointegrating vectors, and then conduct likelihood ratio tests. Assuming that the VECM errors U_t are independent $N_k[0,\Sigma]$ distributiod, and given the cointegrating restrictions on the trend or intercept parameters, the maximum likelihood $L_{max}(r)$ is a function of the cointegration rank *r*. Johansen proposes two types of tests for *r*:

- The lambda-max test.
  This test is based on the log-likelihood ratio $\ln[L_{max}(r)/L_{max}(r+1)]$, and is conducted sequentially for $r = 0,1,..,k-1$. The name comes from the fact that the test statistic involved is a maximum generalized eigenvalue. This test tests the null hypothesis that the cointegration rank is equal to *r* against the alternative that the cointegration rank is equal to $r+1$.

- The trace test.
  This test is based on the log-likelihood ratio $\ln[L_{max}(r)/L_{max}(k)]$, and is conducted sequentially for $r = k-1,...,1,0$. The name comes from the fact that the test statistic involved is the trace (= the sum of the diagonal elements) of a diagonal matrix of generalized eigenvalues. This test tests the null hypothesis that the cointegration rank is equal to *r* against the alternative that the cointegration rank is *k*. The latter implies that X_t is trend stationary.

Both tests have non-standard asymptotic null distributions. Moreover, given the cointegration rank *r* Johansen also derives likelihood ratio tests of the cointegrating restrictions on the intercept or trend parameters.

For further reading on cointegration, see my lecture notes on cointegration and the references to Johansen's papers at the end of this guided tour.

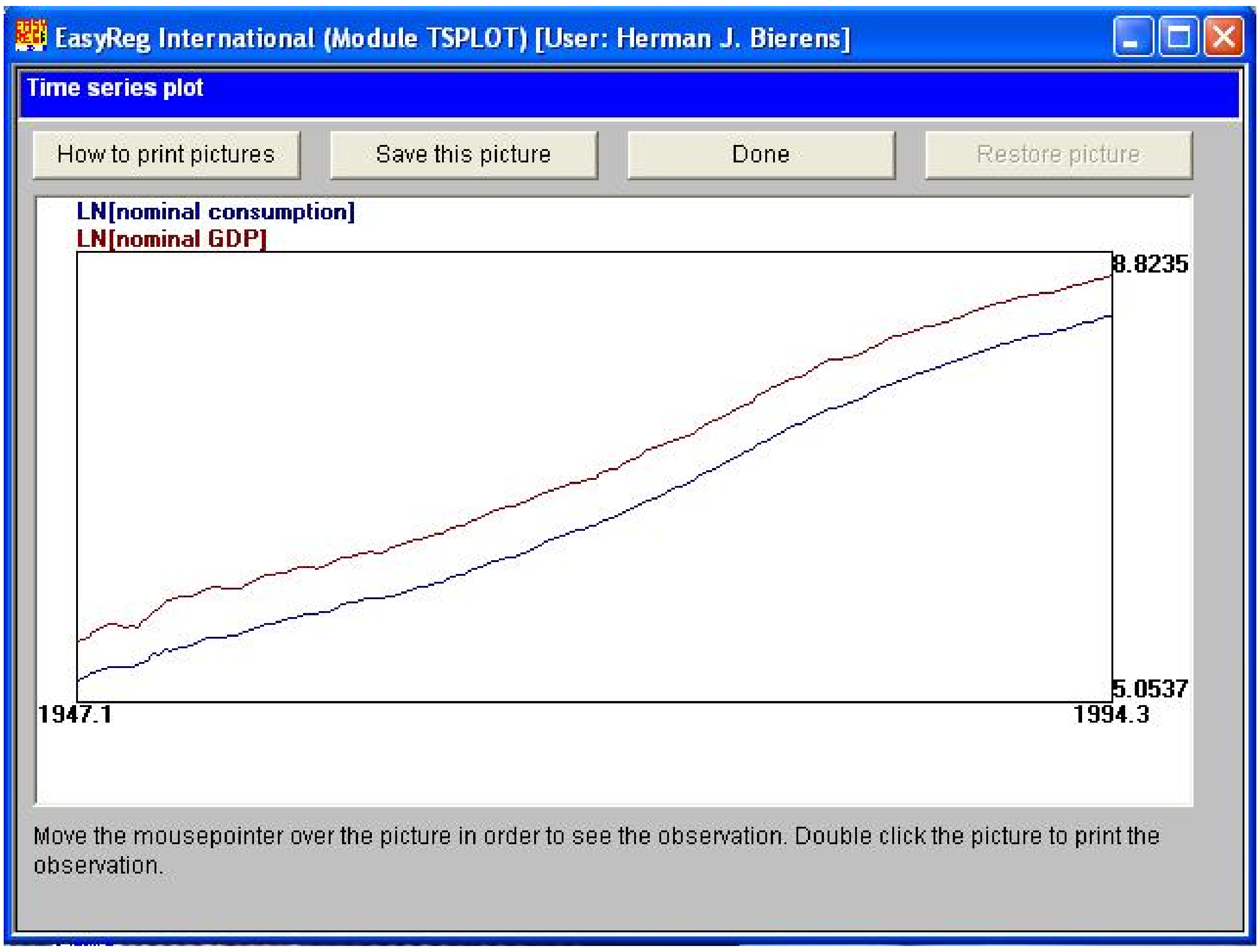
Cointegration analysis in practice

The data

The data are taken from the EasyReg data base, namely quarterly data on nominal consumption and nominal GDP for the US. The two time series are taken in logs (using the option Menu > Input > Transform variables).

Before you conduct cointegration analysis, you have to test whether the time series involved are unit root processes, using the option Menu > Data analysis > Unit root tests (root 1).

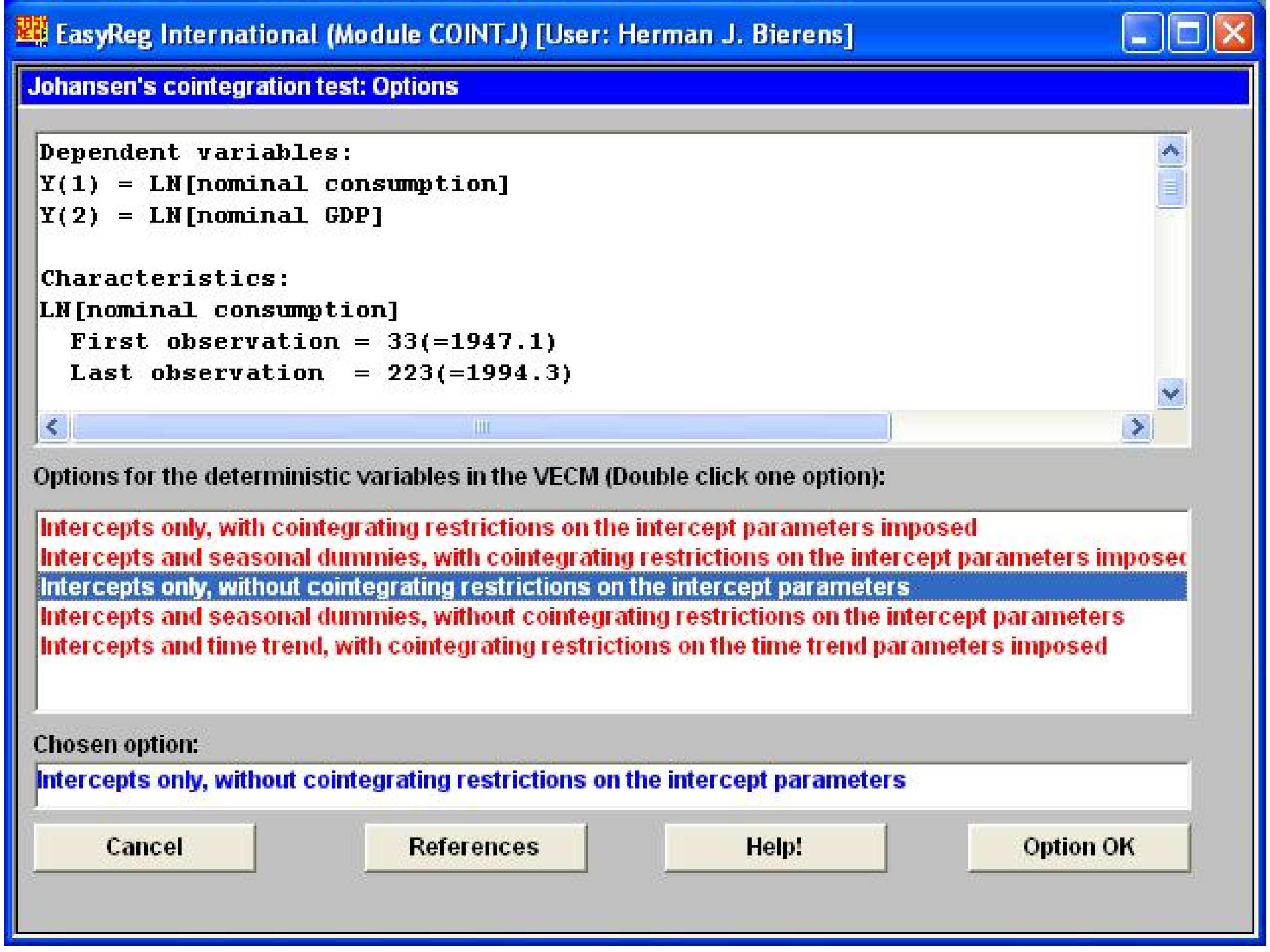
Also, always plot the data in order to determine whether there is drift, and whether you should include a time trend in the VECM. The data is plotted below:



Clearly, there is strong evidence of drift. The two series seem to run parallel at a distance, which suggests to use a VECM **without trend and without cointegrating restrictions imposed on the intercept parameters**.

Johansen's cointegration analysis in practice

Open Menu > Multiple equation models > Johansen's cointegration analysis, select the two time series, and select the option "Intercepts only, without cointegrating restrictions imposed":



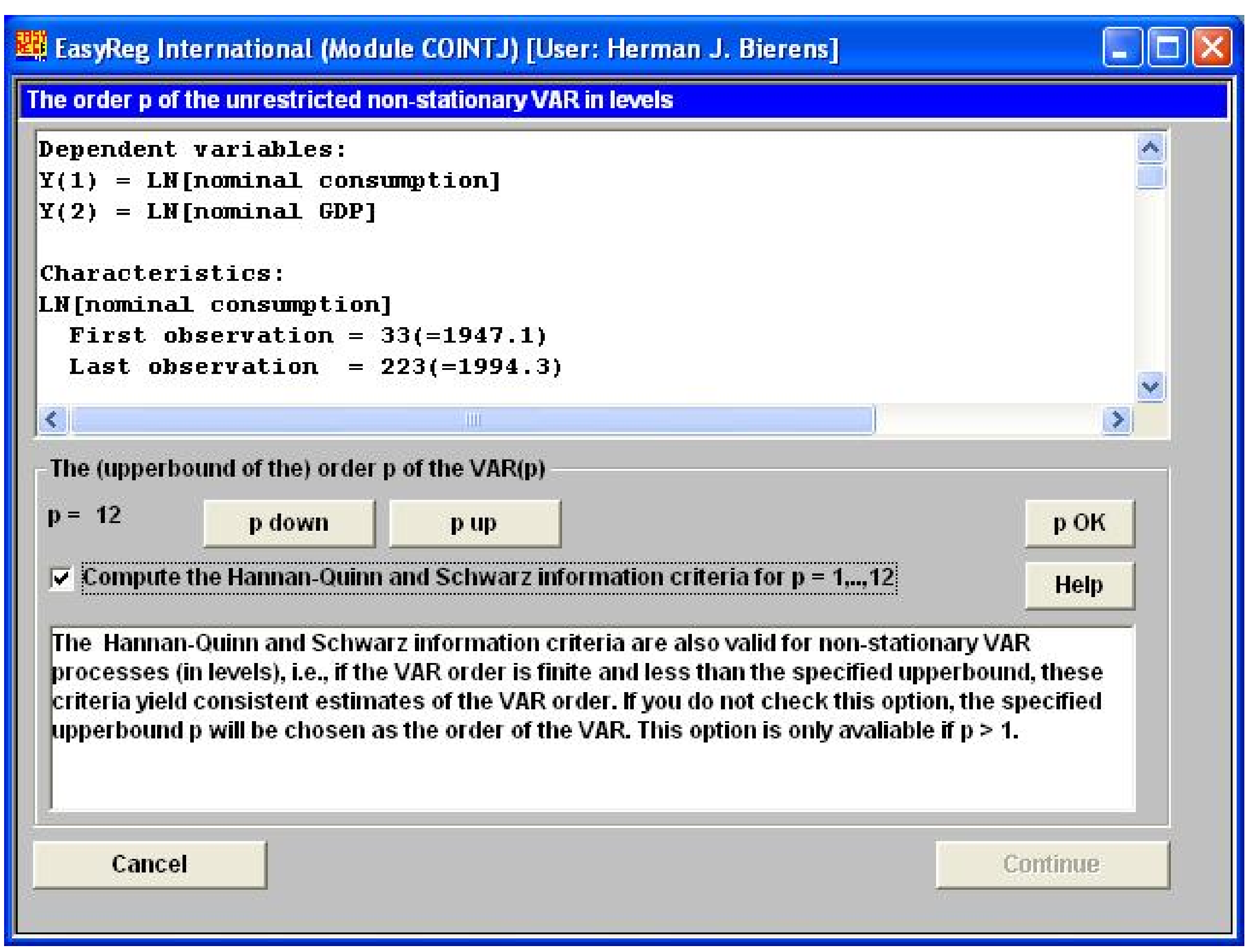
The two options to include seasonal dummy variables are only available if the time series are quarterly or monthly time series. For all other time series EasyReg will not show these options.

However, the time series involved are seasonally adjusted, so that there is no need for these options.

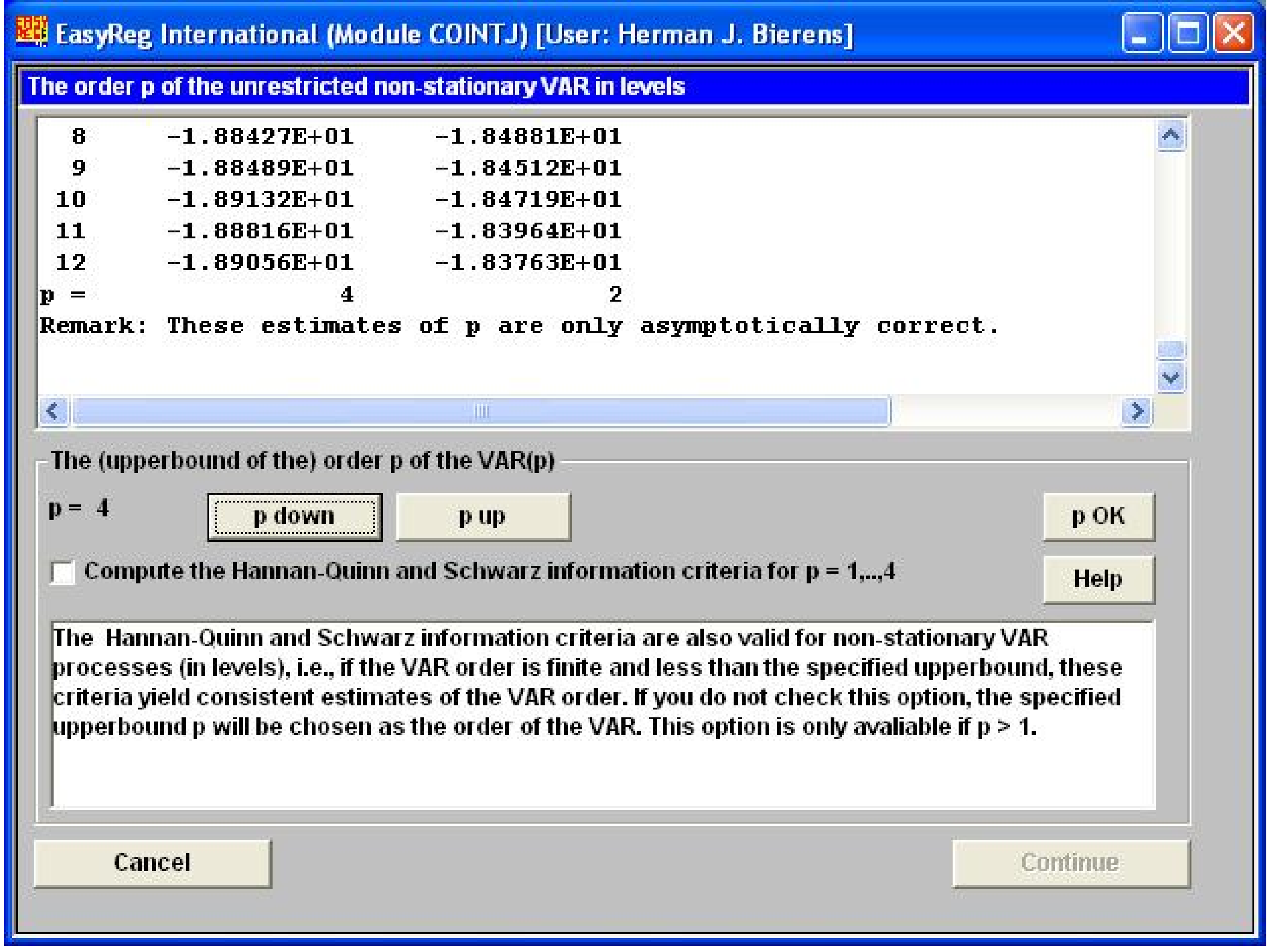
If you include a time trend in the VECM the cointegrating restrictions on the time trend parameters are automatically imposed. However, the validity of these restrictions will be tested.

If you click References then the references to Johansen's papers (see below) will be displayed in the text box, and written to the output file if there is cointegration.

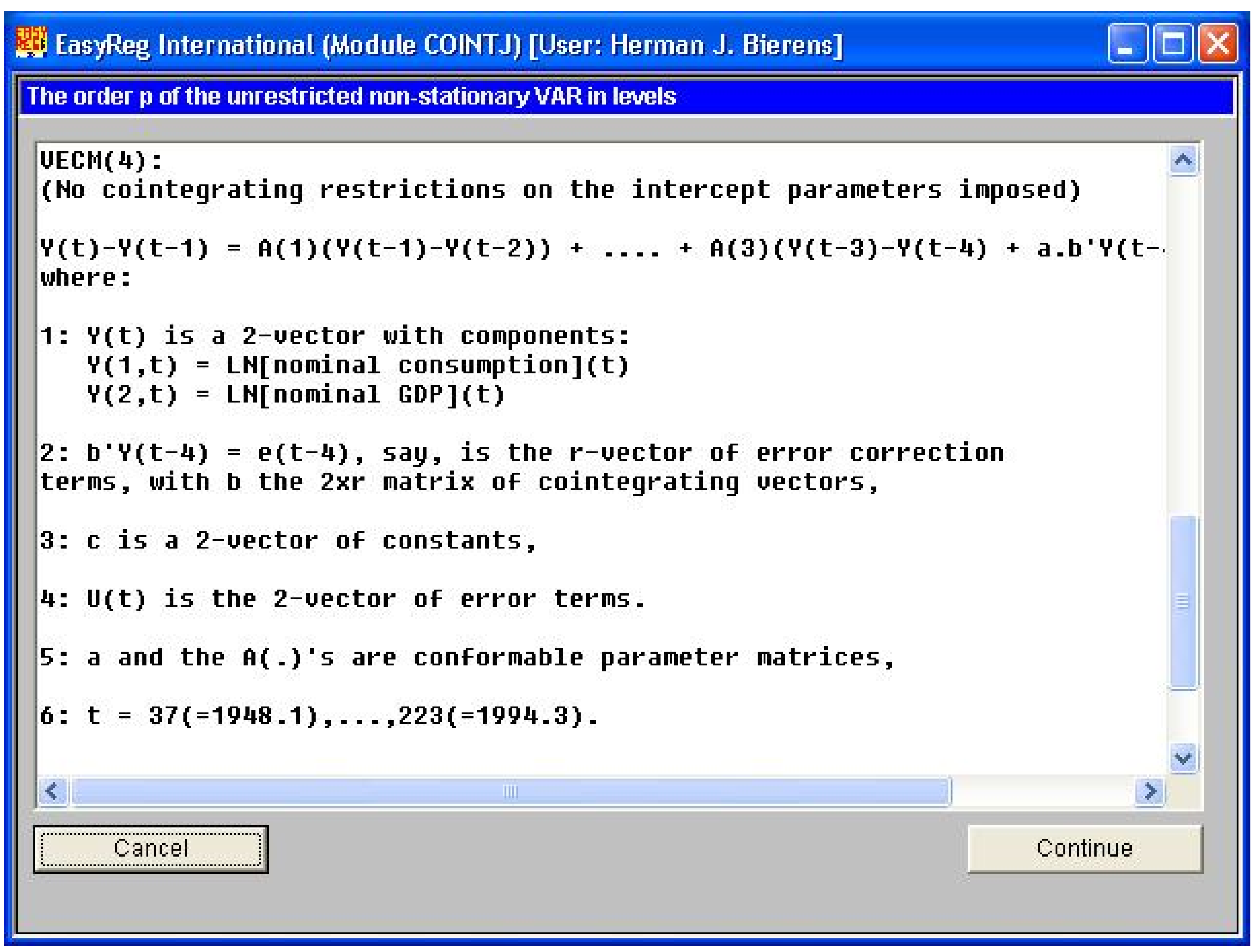
Click Option OK. The next window let you choose the order *p* of the VECM. EasyReg can do that automatically using the Hannan-Quinn and/or Schwarz information criteria. Since the Johansen approach is quite sensitive to the choice of *p*, I recommend to use this option. If so, the selected value of *p* is only an upper bound of the VECM order.



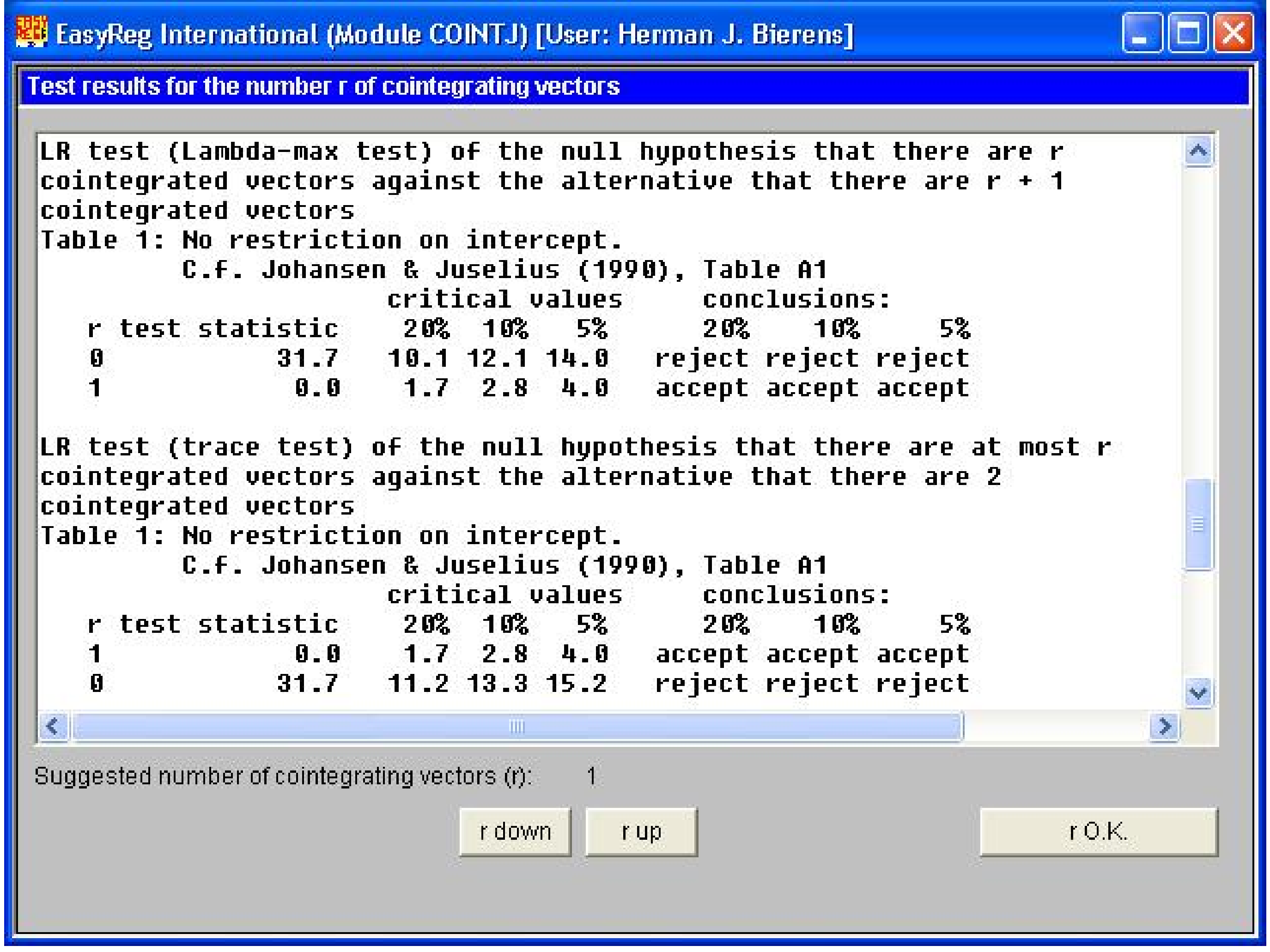
The Hannan-Quinn criterion suggests to choose $p = 4$, and the Schwarz criterion suggests to choose $p = 2$. I recommend to choose the maximum of the two: $p = 4$:



Click p OK:



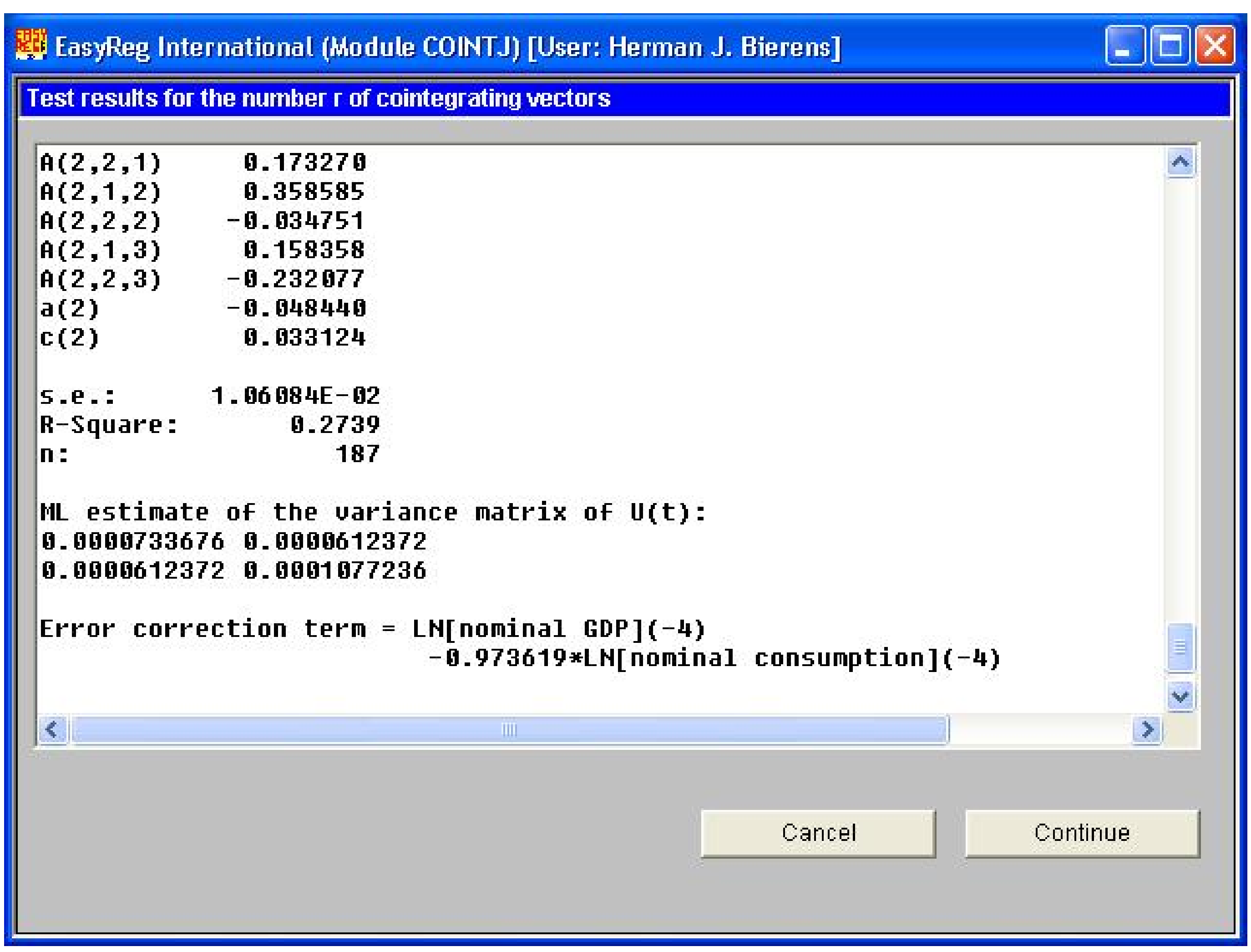
Click Continue:



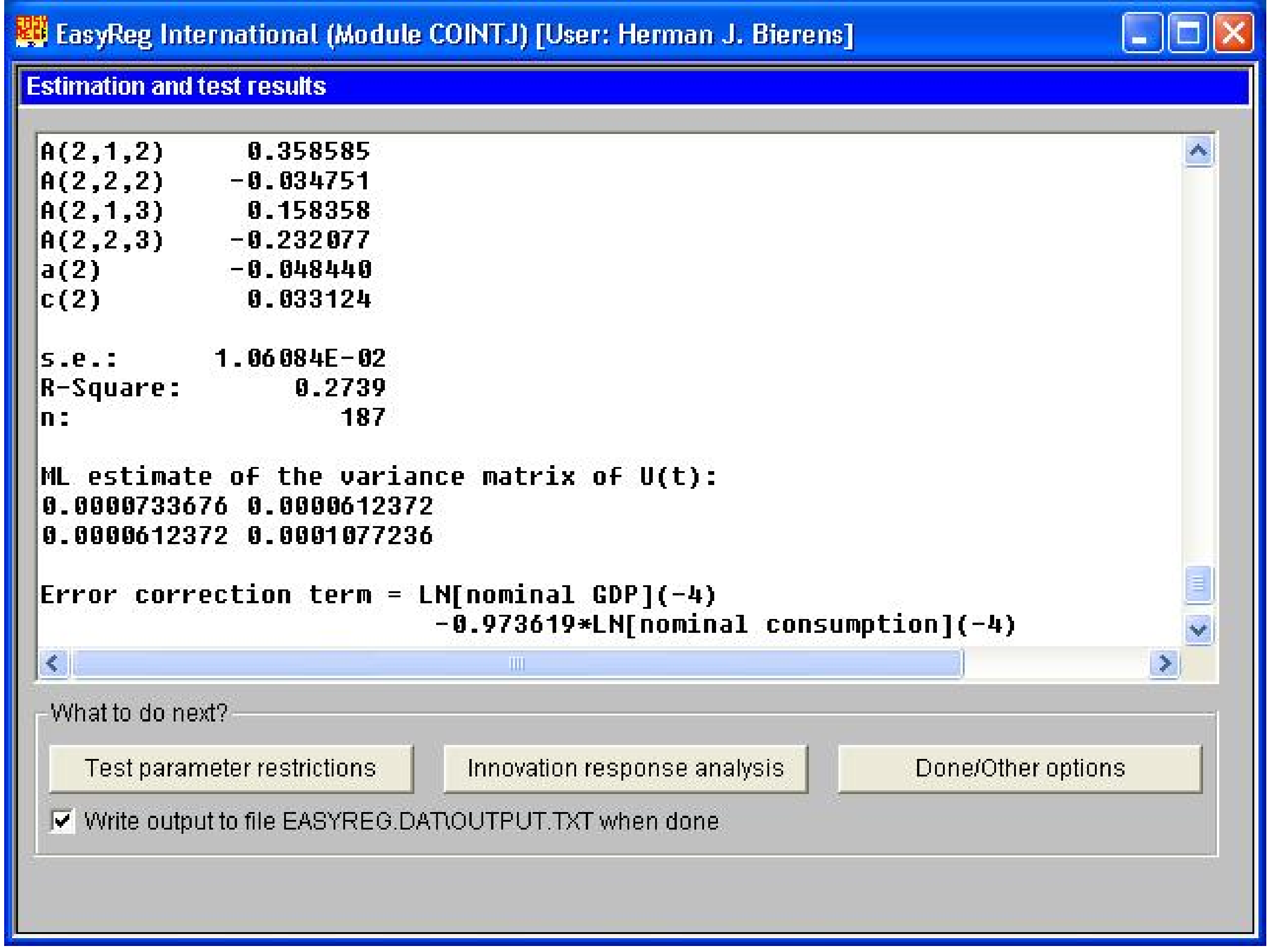
Here are the results for the lambda-max and trace tests. EasyReg will automatially analyze these results and make a recommendation about the cointegration order *r*. However, you should not blindly follow this suggestion, but draw your conclusion yourself. In this case however the suggested value *r* = 1 looks OK.

If you disagree with EasyReg, you can adjust the value of *r*.

Click Continue:



Click Continue again for further options:

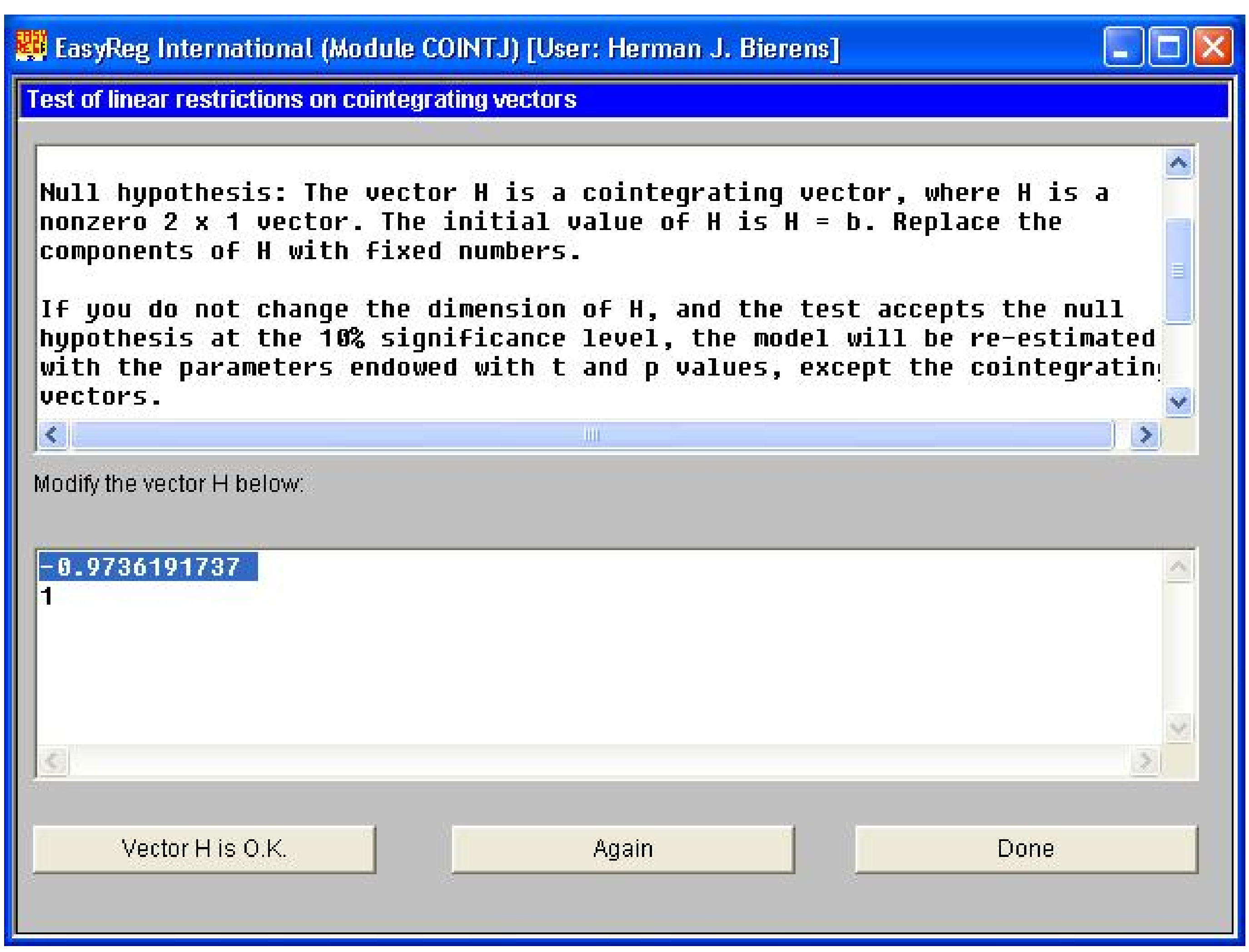


If you leave the box "write output to file OUTPUT.TXT" checked then all the estimation and test results will be appended to the output file when you click Done/Other options.

We have now two options. Upon completion of these options you will return to this window.

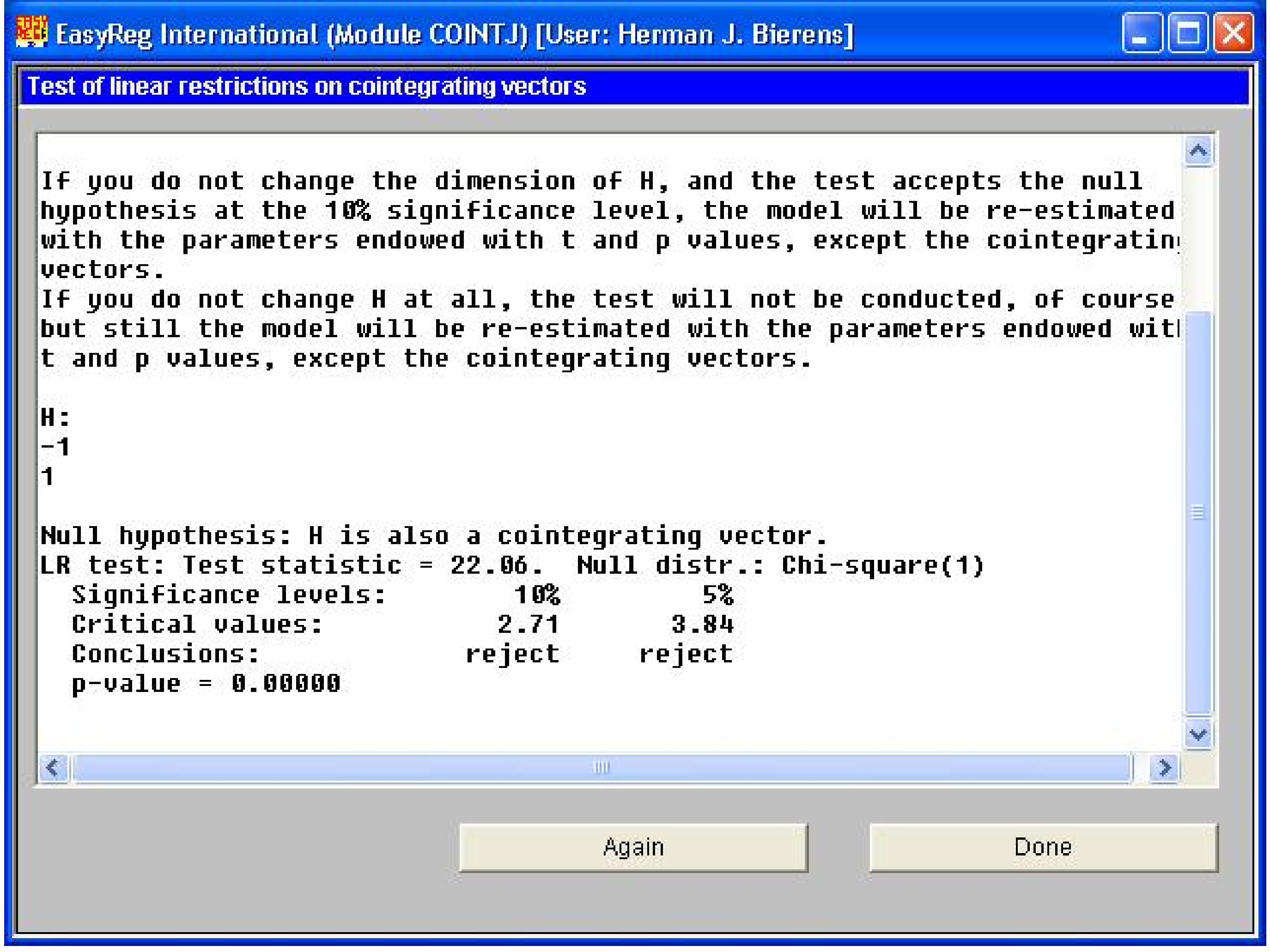
The first option will be discussed next. Thus click Test parameter restrictions:

Testing parameter restrictions on the cointegrating vectors



Suppose you want to test whether the cointegrating vector is β = (-1,1)' rather than (-0.9736191737,1)'. To do so, replace the value -0.9736191737 by -1, and click Vector H OK (H in EasyReg is the same as β above).

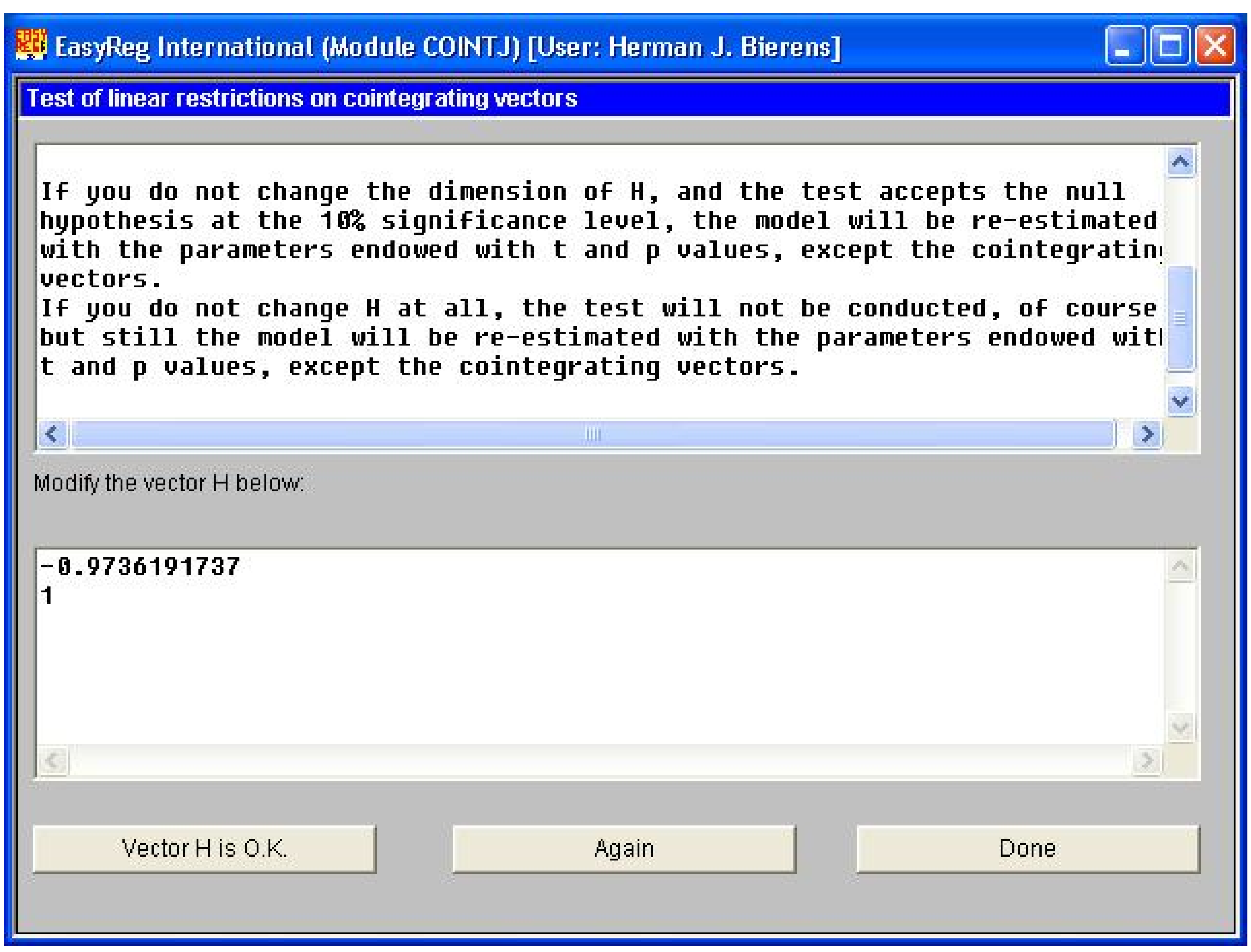
In general you have to replace all estimated entries (the numbers between -1 and 1) by fixed numbers, except for the 1-s themselves because they are due to normalisation. If the matrix H (alias β) has two or more columns, you may erase one or more columns but of course you should leave at least one column.



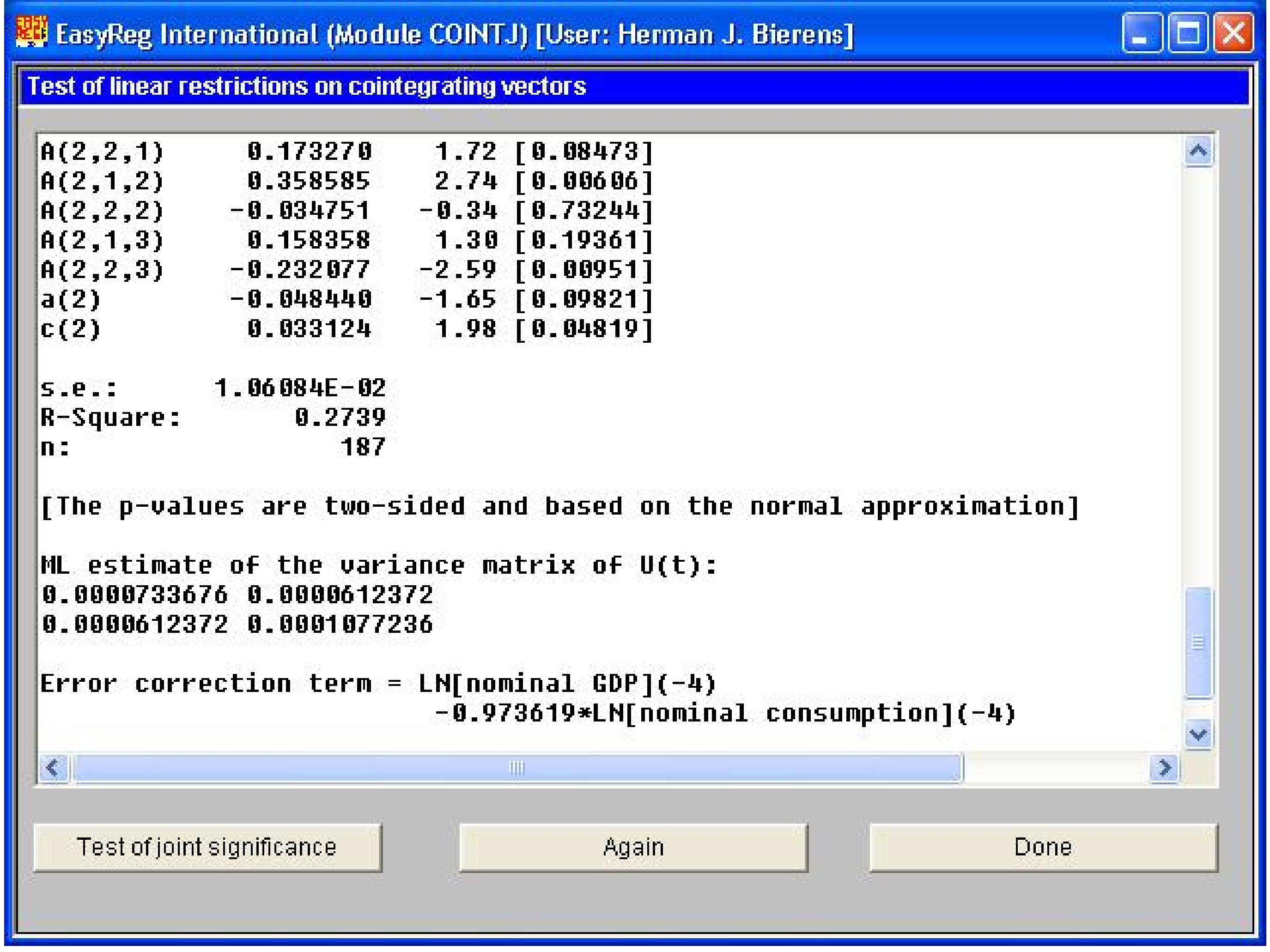
The hypothesis is firmly rejected at both significance levels.

One way of double-checking this hypothesis is to form the linear combinations involved (using Menu > Input > Transform variables) and test whether they are unit root processes. In this case I have made a new variable, LN[nominal GDP]-LN[nominal consumption], and conducted various unit root and trend stationarity tests. It appears that this variable is a unit root process, so that (-1,1)' is **not** a cointegrating vector.

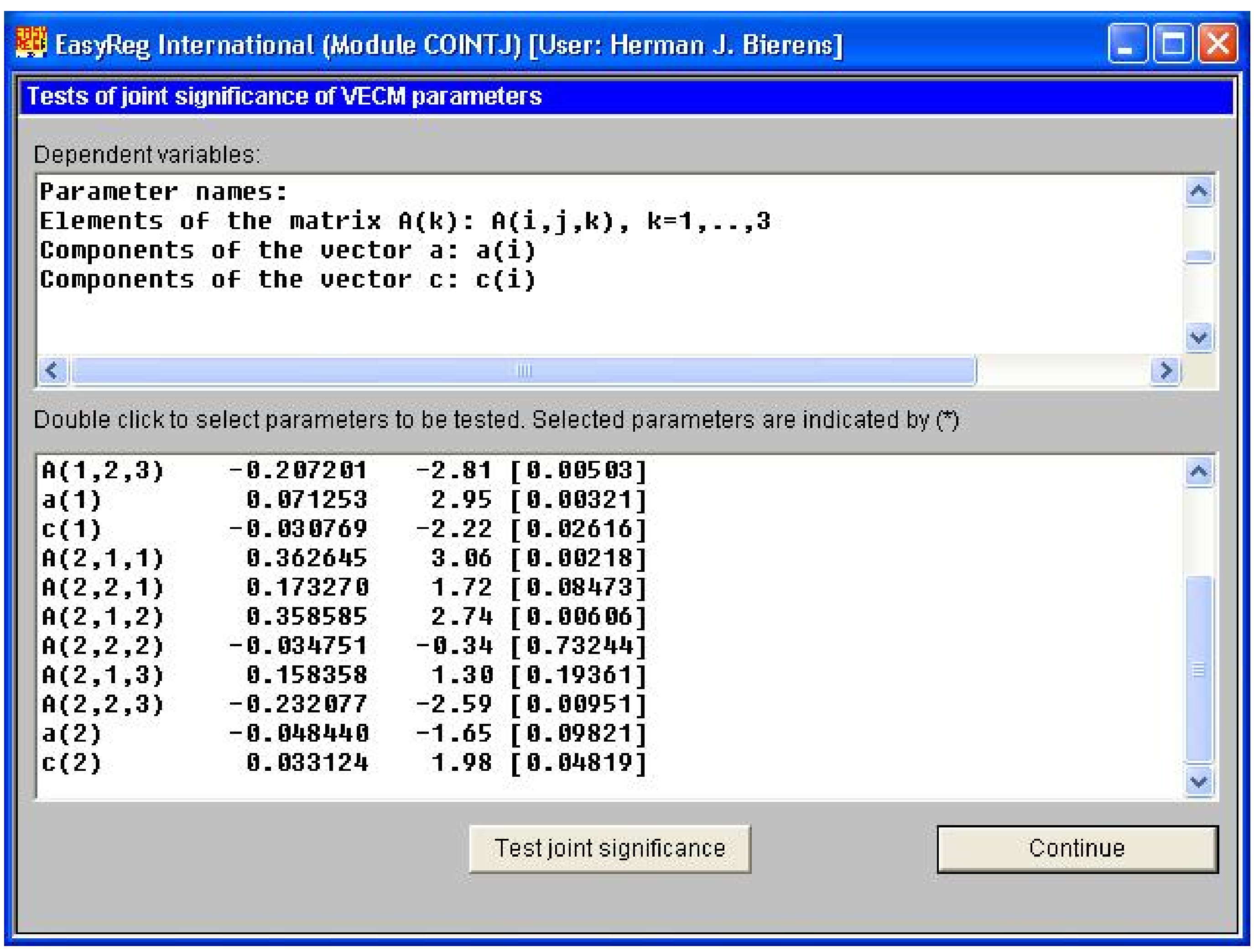
Click Again:



Now do not change H, and click "Vector H is OK". Then the VECM will be re-estimated, and the parameter estimates will now be endowed with t and p values, except the parameters in the estimated cointegrating vector b. Since the latter estimates are super-consistent, we may treat them as the true values:

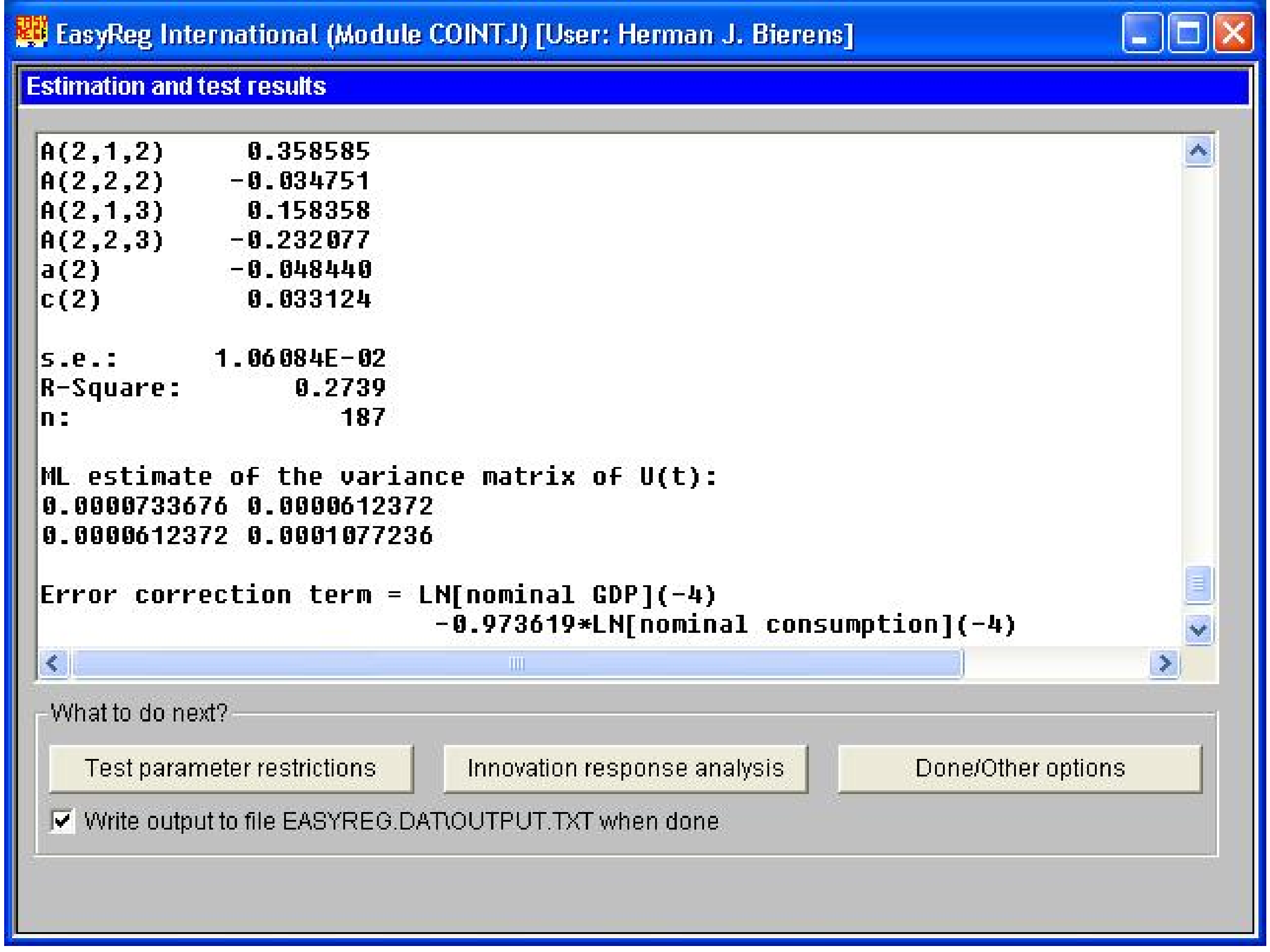


You can now test the joint significance of the VECM parameters. To show that, click "Test of joint significance":



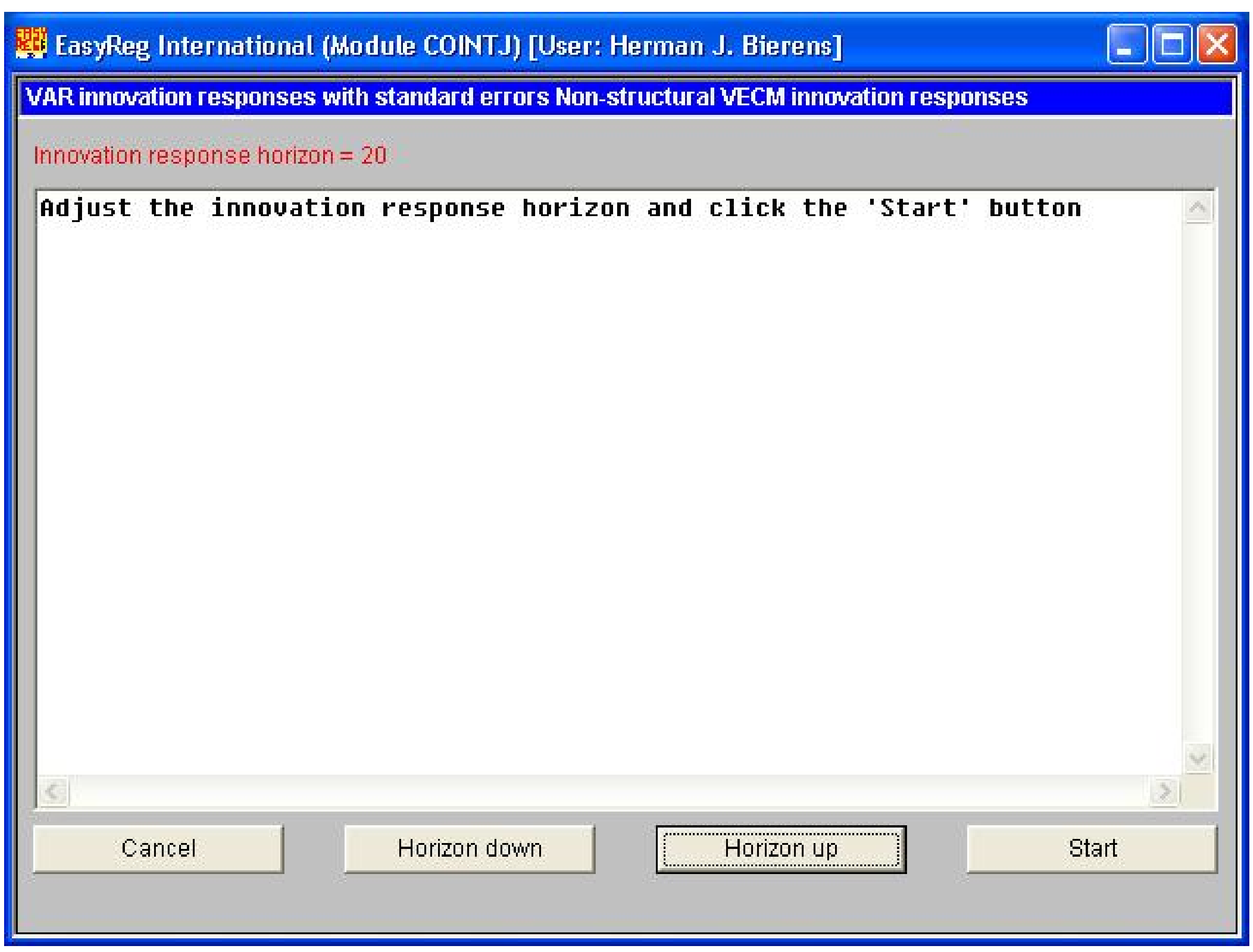
Double-click the parameters you want to test, and click "Test joint significance". The procedure is the same as for VAR models, and will therefore not be further explained.

Click: "Continue". Then you will jump back to the following window:



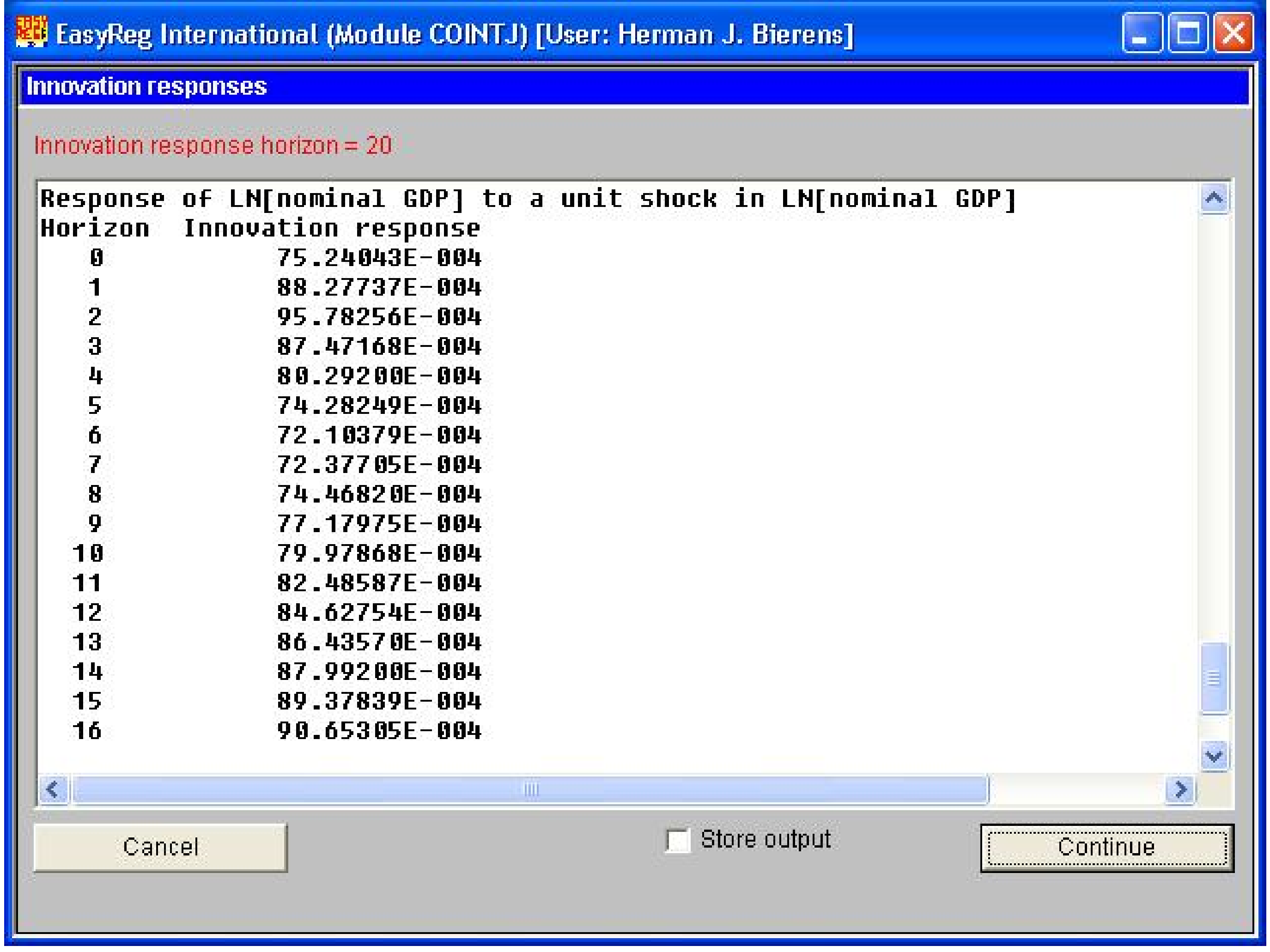
Innovation response analysis

Click "Innovation response analysis". Then you can conduct (non-structural) innovation response analysis, similar to the stationary VAR case:



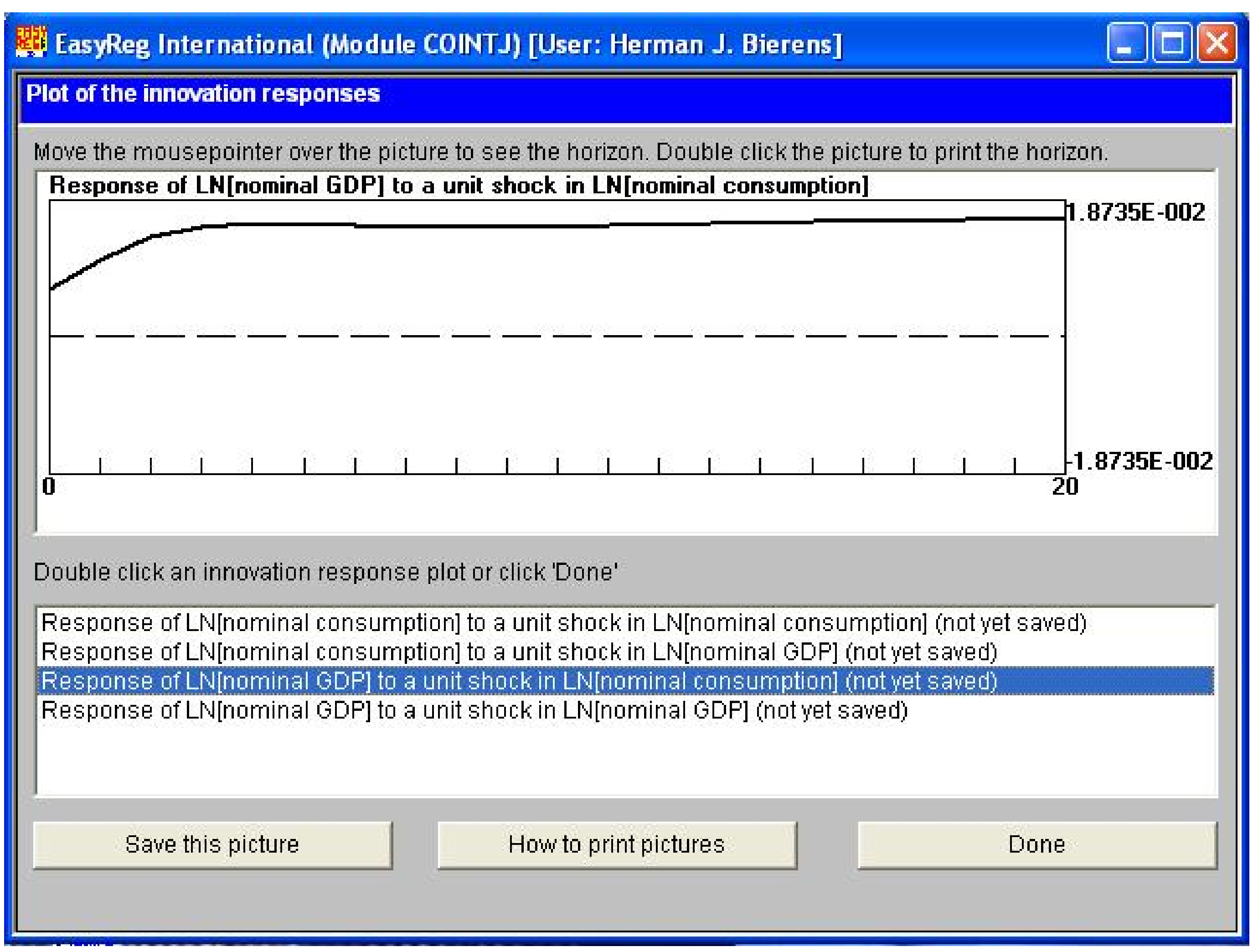
I have chosen an innovation response horizon of 20 periods, which corresponds to 5 years.

Click Start. Then the innovation responses are computed:



The numerical values of the innovation responses are displayed in this window. Usually you will only be interested in the innovation response plots, so that then there is no need to write the numerical innovation responses to the output file. Therefore, you may leave the box "store output" unchecked.

Click Continue:



Since some of the maximum likelihood parameter estimates have nonstandard asymptotic distributions, the innovation response plots are not endowed with standard error bands.

Another diffence with standard innovation response analysis is that in general the innovation responses do not taper off to zero. Thus, innovation shocks may induce permanent level shifts, as is apparent from the innovation response plot above.

The output

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Johansen's cointegration analysis:

Dependent variables:
Y(1)  = LN[nominal consumption]
Y(2)  = LN[nominal GDP]

Characteristics:
LN[nominal consumption]
  First observation = 33(=1947.1)
  Last observation  = 223(=1994.3)
  Number of usable observations: 191
  Minimum value: 5.0536948E+000
  Maximum value: 8.4462341E+000
  Sample mean:   6.6648432E+000
```

```
LN[nominal GDP]
  First observation = 33(=1947.1)
  Last observation  = 223(=1994.3)
  Number of usable observations: 191
  Minimum value: 5.4236276E+000
  Maximum value: 8.8234566E+000
  Sample mean:   7.0953749E+000


Information criteria:
  p     Hannan-Quinn          Schwarz
  1     -1.85644E+01     -1.85034E+01
  2     -1.89468E+01     -1.88447E+01
  3     -1.89676E+01     -1.88242E+01
  4     -1.89808E+01     -1.87958E+01
  5     -1.89483E+01     -1.87214E+01
  6     -1.89272E+01     -1.86580E+01
  7     -1.88742E+01     -1.85625E+01
  8     -1.88427E+01     -1.84881E+01
  9     -1.88489E+01     -1.84512E+01
 10     -1.89132E+01     -1.84719E+01
 11     -1.88816E+01     -1.83964E+01
 12     -1.89056E+01     -1.83763E+01
p =                4                2
Remark: These estimates of p are only asymptotically correct.

Chosen VAR(p) order: p = 4


VECM(4):
(No cointegrating restrictions on the intercept parameters imposed)

Y(t)-Y(t-1) = A(1)(Y(t-1)-Y(t-2)) + .... + A(3)(Y(t-3)-Y(t-4) + a.b'Y(t-4) + c  + U(t),
where:

1: Y(t) is a 2-vector with components:
   Y(1,t) = LN[nominal consumption](t)
   Y(2,t) = LN[nominal GDP](t)

2: b'Y(t-4) = e(t-4), say, is the r-vector of error correction
terms, with b the 2xr matrix of cointegrating vectors,

3: c is a 2-vector of constants,

4: U(t) is the 2-vector of error terms.

5: a and the A(.)'s are conformable parameter matrices,

6: t = 37(=1948.1),...,223(=1994.3).


Matrix Skk:
0.9132700666 0.8989685819
0.8989685819 0.8854573767

Matrix Sko[Soo^-1]Sok:
0.0222742845 0.0232120848
0.0232120848 0.0241894866

Generalized Eigenvalues of Sko[Soo^-1]Sok w.r.t. Skk:
0.1557922928 0.0000325597

Corresponding Eigenvectors:
-0.9736191737 1
1             -0.9595275184


LR test (Lambda-max test) of the null hypothesis that there are r
cointegrated vectors against the alternative that there are r + 1
cointegrated vectors
Table 1: No restriction on intercept.
         C.f. Johansen & Juselius (1990), Table A1
                      critical values     conclusions:
   r test statistic    20%  10%   5%      20%    10%     5%
   0            31.7    10.1 12.1 14.0   reject reject reject
   1             0.0     1.7  2.8  4.0   accept accept accept

LR test (trace test) of the null hypothesis that there are at most r
cointegrated vectors against the alternative that there are 2
cointegrated vectors
Table 1: No restriction on intercept.
         C.f. Johansen & Juselius (1990), Table A1
                      critical values     conclusions:
   r test statistic    20%  10%   5%      20%    10%     5%
   1             0.0     1.7  2.8  4.0   accept accept accept
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   0             31.7   11.2 13.3 15.2   reject reject reject

LR test (Lambda-max test) of the null hypothesis that there are r
cointegrated vectors against the alternative that there are r + 1
cointegrated vectors
Table 2:  Restrictions on intercept, but not imposed.
         C.f. Johansen & Juselius (1990), Table A2
                      critical values     conclusions:
   r test statistic    20%  10%   5%      20%    10%     5%
   0             31.7   10.7 12.8 14.6   reject reject reject
   1              0.0    4.9  6.7  8.1   accept accept accept

LR test (trace test) of the null hypothesis that there are at most r
cointegrated vectors against the alternative that there are 2
cointegrated vectors
Table 2:  Restrictions on intercept, but not imposed.
         C.f. Johansen & Juselius (1990), Table A2
                      critical values     conclusions:
   r test statistic    20%  10%   5%      20%    10%     5%
   1              0.0    4.9  6.7  8.1   accept accept accept
   0             31.7   13.0 15.6 17.8   reject reject reject

Conclusion: r =1


Standardized cointegrating vector:
-0.9736192  LN[nominal consumption]
         1  LN[nominal GDP]

Remark: This result will be used to test restrictions on the cointegrating vector.

VECM(4):
(No cointegrating restrictions on the intercept parameters imposed)

Y(t)-Y(t-1) = A(1)(Y(t-1)-Y(t-2)) + .... + A(3)(Y(t-3)-Y(t-4) + a.b'Y(t-4) + c  + U(t),
where:

1: Y(t) is a 2-vector with components:
   Y(1,t) = LN[nominal consumption](t)
   Y(2,t) = LN[nominal GDP](t)

2: b'Y(t-4) = e(t-4), say, is the 1-vector of error correction
terms, with b the 2x1 matrix of cointegrating vectors: b =
-0.9736191737
1

3: c is a 2-vector of constants,

4: U(t) is the 2-vector of error terms.

5: a and the A(.)'s are conformable parameter matrices,

6: t = 37(=1948.1),...,223(=1994.3).


ML estimation results for the VECM:
Parameter names:
Elements of the matrix A(k): A(i,j,k), k=1,..,3
Components of the vector a: a(i)
Components of the vector c: c(i)


Equation 1: DIF1[LN[nominal consumption]]
Parameter ML estimate
A(1,1,1)    -0.262023
A(1,2,1)     0.288099
A(1,1,2)     0.258230
A(1,2,2)     0.045008
A(1,1,3)     0.173523
A(1,2,3)    -0.207201
a(1)         0.071253
c(1)        -0.030769

s.e.:      8.75480E-03
R-Square:       0.2766
n:                 187

Equation 2: DIF1[LN[nominal GDP]]
Parameter ML estimate
A(2,1,1)     0.362645
A(2,2,1)     0.173270
A(2,1,2)     0.358585
A(2,2,2)    -0.034751
A(2,1,3)     0.158358
A(2,2,3)    -0.232077
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a(2)         -0.048440
c(2)          0.033124

s.e.:      1.06084E-02
R-Square:        0.2739
n:                  187

ML estimate of the variance matrix of U(t):
0.0000733676 0.0000612372
0.0000612372 0.0001077236

Error correction term = LN[nominal GDP](-4)
                        -0.973619*LN[nominal consumption](-4)



Variables:
LN[nominal consumption]
LN[nominal GDP]

Null hypothesis: The vector H is a cointegrating vector, where H is a
nonzero 2 x 1 vector. The initial value of H is H = b. Replace the
components of H with fixed numbers.

If you do not change the dimension of H, and the test accepts the null
hypothesis at the 10% significance level, the model will be re-estimated
with the parameters endowed with t and p values, except the cointegrating
vectors.
If you do not change H at all, the test will not be conducted, of course,
but still the model will be re-estimated with the parameters endowed with
t and p values, except the cointegrating vectors.

H:
-1
1

Null hypothesis: H is also a cointegrating vector.
LR test: Test statistic = 22.06.  Null distr.: Chi-square(1)
  Significance levels:        10%         5%
  Critical values:           2.71       3.84
  Conclusions:             reject     reject
  p-value = 0.00000
Variables:
LN[nominal consumption]
LN[nominal GDP]

Null hypothesis: The vector H is a cointegrating vector, where H is a
nonzero 2 x 1 vector. The initial value of H is H = b. Replace the
components of H with fixed numbers.

If you do not change the dimension of H, and the test accepts the null
hypothesis at the 1% significance level, the model will be re-estimated
with the parameters endowed with t and p values, except the cointegrating
vectors.
If you do not change H at all, the test will not be conducted, of course,
but still the model will be re-estimated with the parameters endowed with
t and p values, except the cointegrating vectors.

H:
-0.9736191737
1

Null hypothesis: H is also a cointegrating vector.

However, since you have not changed H, testing the null hypothesis makes no sense!

Re-estimation of the VECM
VECM(4):
(No cointegrating restrictions on the intercept parameters imposed)

Y(t)-Y(t-1) = A(1)(Y(t-1)-Y(t-2)) + .... + A(3)(Y(t-3)-Y(t-4) + a.b'Y(t-4) + c  + U(t),
where:

1: Y(t) is a 2-vector with components:
   Y(1,t) = LN[nominal consumption](t)
   Y(2,t) = LN[nominal GDP](t)

2: b'Y(t-4) = e(t-4), say, is the 1-vector of error correction
terms, with b the 2x1 matrix of cointegrating vectors: b =
-0.9736191737
1

3: c is a 2-vector of constants,

4: U(t) is the 2-vector of error terms.
```

```
5: a and the A(.)'s are conformable parameter matrices,

6: t = 37(=1948.1),...,223(=1994.3).


ML estimation results for the VECM:
Parameter names:
Elements of the matrix A(k): A(i,j,k), k=1,..,3
Components of the vector a: a(i)
Components of the vector c: c(i)


Equation 1: DIF1[LN[nominal consumption]]
Parameter ML estimate t-value [p-value]
A(1,1,1)     -0.262023    -2.68 [0.00730]
A(1,2,1)      0.288099     3.47 [0.00051]
A(1,1,2)      0.258230     2.39 [0.01663]
A(1,2,2)      0.045008     0.54 [0.59159]
A(1,1,3)      0.173523     1.73 [0.08433]
A(1,2,3)     -0.207201    -2.81 [0.00503]
a(1)          0.071253     2.95 [0.00321]
c(1)         -0.030769    -2.22 [0.02616]

s.e.:       8.75480E-03
R-Square:        0.2766
n:                  187

Equation 2: DIF1[LN[nominal GDP]]
Parameter ML estimate t-value [p-value]
A(2,1,1)      0.362645     3.06 [0.00218]
A(2,2,1)      0.173270     1.72 [0.08473]
A(2,1,2)      0.358585     2.74 [0.00606]
A(2,2,2)     -0.034751    -0.34 [0.73244]
A(2,1,3)      0.158358     1.30 [0.19361]
A(2,2,3)     -0.232077    -2.59 [0.00951]
a(2)         -0.048440    -1.65 [0.09821]
c(2)          0.033124     1.98 [0.04819]

s.e.:       1.06084E-02
R-Square:        0.2739
n:                  187

[The p-values are two-sided and based on the normal approximation]

ML estimate of the variance matrix of U(t):
0.0000733676 0.0000612372
0.0000612372 0.0001077236

Error correction term = LN[nominal GDP](-4)
                        -0.973619*LN[nominal consumption](-4)
```

Time-varying cointegration

The Bierens-Martins test

Bierens and Martins (2010) propose a test of the null hypothesis that the cointegrating vector (or matrix of cointegrating vectors) β is constant (which is the standard assumption), against the alternative hypothesis that β is a function of time: β_t. Thus, the Time-Varying Vector Error Correction Model (TV-VECM) becomes

$$\Delta Y_t = \pi_0 + \pi_1 t + \Pi_1 \Delta Y_{t}-1 + ... + \Pi_{p-1} \Delta Y_{t-p}+1 + \alpha\beta_t' Y_{t-p} + U_t,$$

where all the other parameters are the same as before.

The Bierens-Martins test is implemented by approximating β_t via a series expansion in terms of Chebyshev time polynomials:

$$\beta_t = \xi_0 + \xi_1 P_{1,T}(t) + ... + \xi_m P_{m,T}(t),$$

where $P_{i,T}(t)$ is a Chebyshev time polynomial of order i, i.e.,

$$\begin{aligned} P_{0,T}(t) &= 1,\ P_{i,T}(t) = \sqrt{2}\cos\left(i\pi\left(t-0.5\right)/T\right), \\ t &= 1,2,...,T,\ i = 1,2,3,... \end{aligned}$$

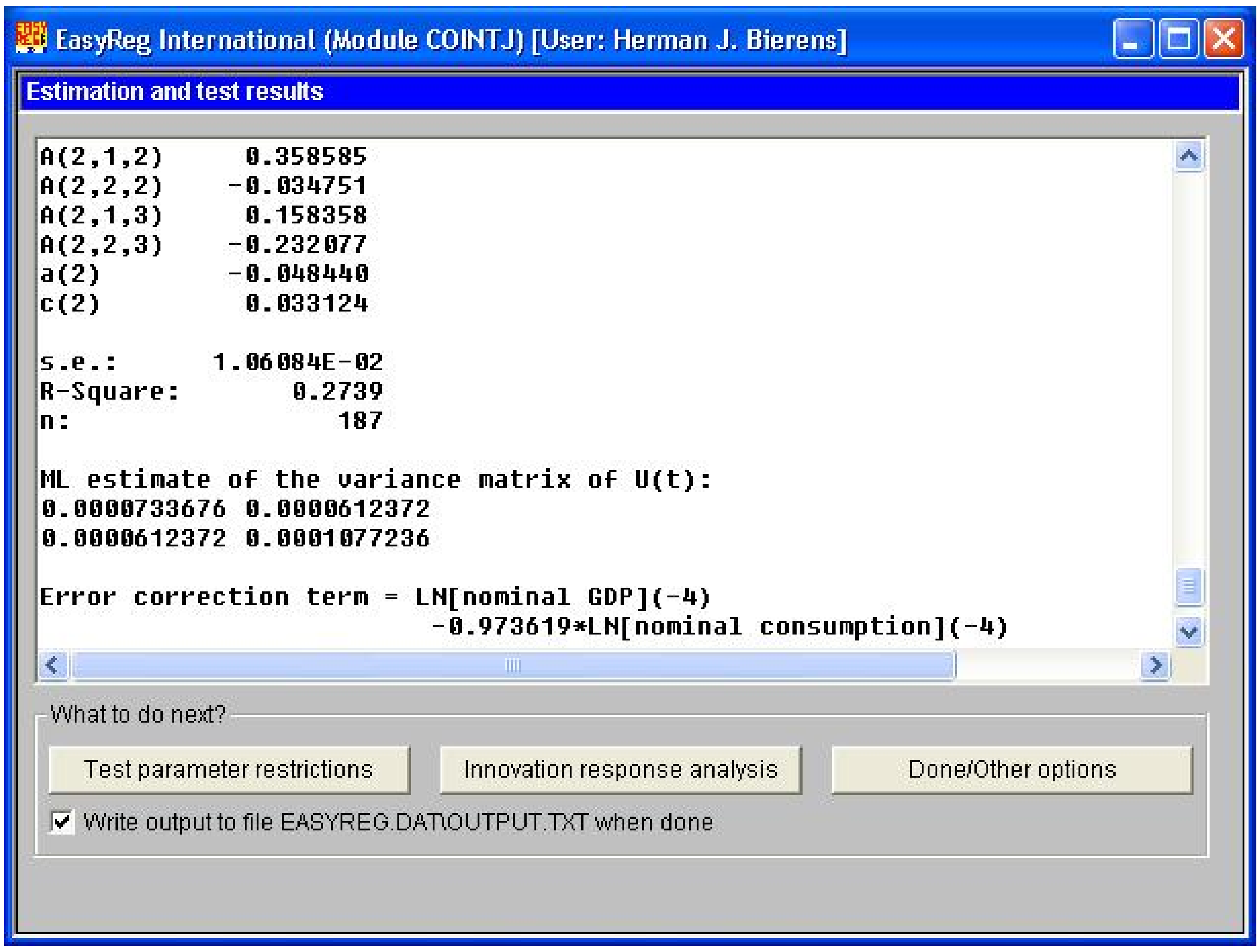
The null hypothesis of time-invariant cointegration corresponds to the hypothesis that

$$H_0: \xi_i = O \text{ for } i = 1,2,...,m.$$

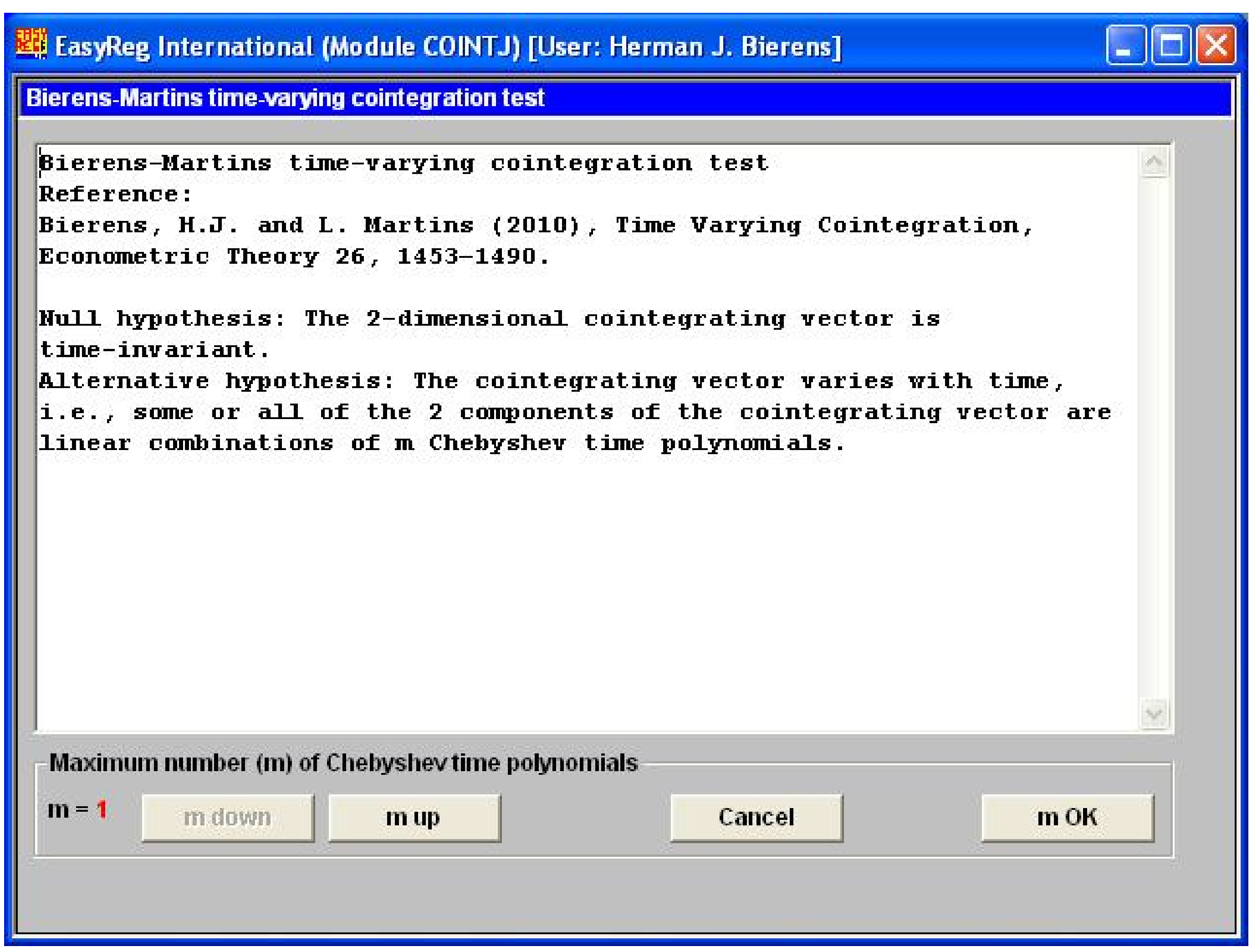
Under this null hypothesis the test statistic involved has a χ^2 null distribution.

The Bierens-Martins test in EasyReg

The Bierens-Martins test is part of the Johansen's cointegration module, as follows. Once you return to the window

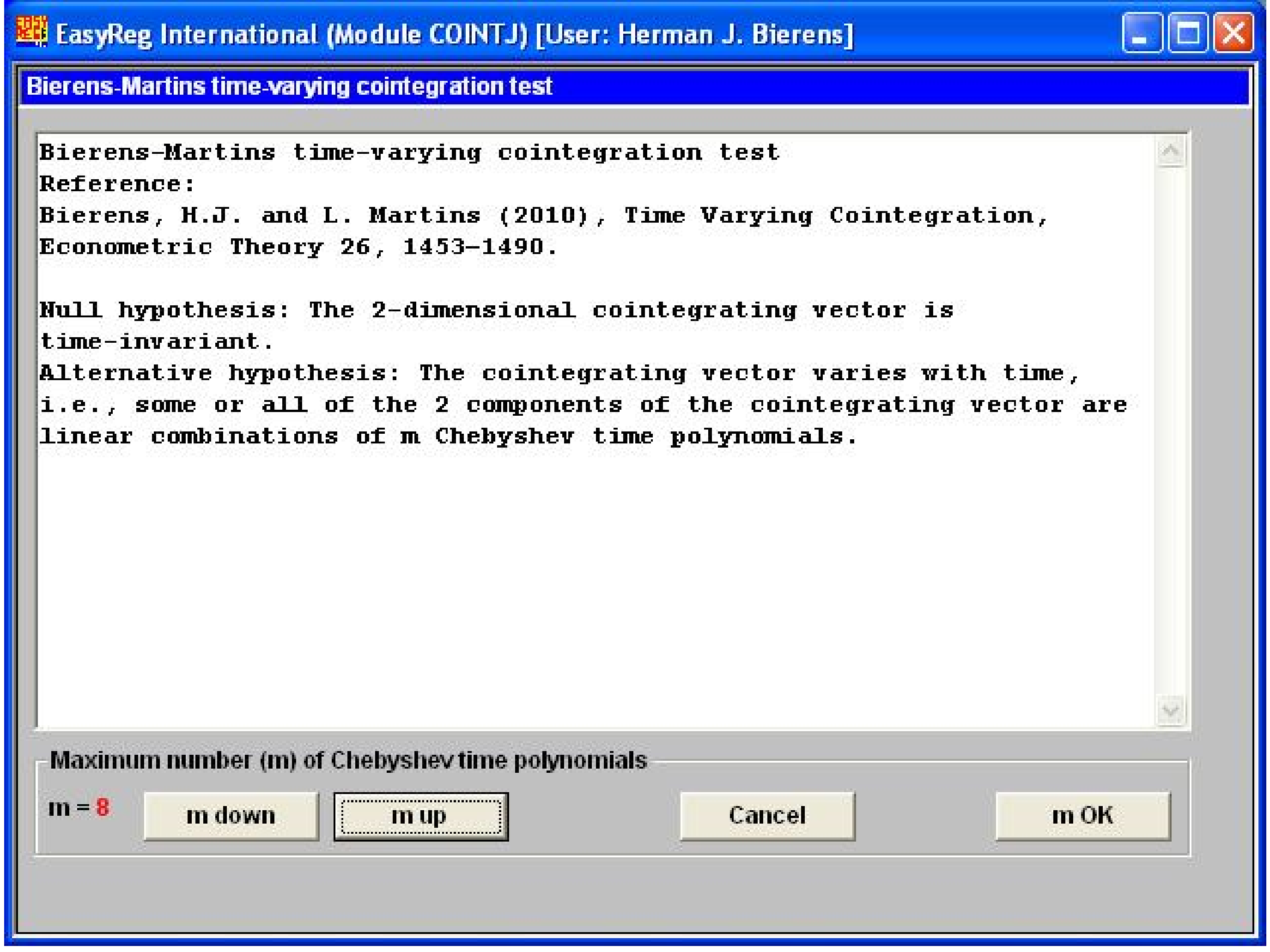


click "Done/Other options". Then the following window appears:



However, this option is only available if the number of cointegrating vectors *r*, i.e., the number of columns of β, is at least 1, and less than the number of rows of β.

For the time series involved the standard Johansen approach indicates $r = 1$, which is adopted for the Bierens-Martins test as well.



Select a maximum order of the Chebyshev time polynomials. In this example I have chosen 8. Click "m OK". Then the Bierens-Martins test is conducted in eight-fold, for *m* = 1,2,...,8, and the results are written to the output file OUTPUT.TXT.

```
Bierens-Martins time-varying cointegration test for m = 1
Test statistic = 17.65
Asymptotic null distribution: Chi-square(2)
p-value = 0.00015
Significance levels:        10%         5%
Critical values:           4.61       5.99
Conclusions:             reject     reject

Bierens-Martins time-varying cointegration test for m = 2
Test statistic = 18.38
Asymptotic null distribution: Chi-square(4)
p-value = 0.00104
Significance levels:        10%         5%
Critical values:           7.78       9.49
Conclusions:             reject     reject

Bierens-Martins time-varying cointegration test for m = 3
Test statistic = 21.53
Asymptotic null distribution: Chi-square(6)
p-value = 0.00147
Significance levels:        10%         5%
Critical values:          10.64      12.59
Conclusions:             reject     reject

Bierens-Martins time-varying cointegration test for m = 4
Test statistic = 38.34
Asymptotic null distribution: Chi-square(8)
p-value = 0.00001
Significance levels:        10%         5%
Critical values:          13.36      15.51
Conclusions:             reject     reject

Bierens-Martins time-varying cointegration test for m = 5
Test statistic = 43.25
Asymptotic null distribution: Chi-square(10)
p-value = 0.00000
Significance levels:        10%         5%
Critical values:          15.99      18.31
Conclusions:             reject     reject

Bierens-Martins time-varying cointegration test for m = 6
Test statistic = 49.09
Asymptotic null distribution: Chi-square(12)
p-value = 0.00000
Significance levels:        10%         5%
Critical values:          18.55      21.03
Conclusions:             reject     reject

Bierens-Martins time-varying cointegration test for m = 7
Test statistic = 50.69
Asymptotic null distribution: Chi-square(14)
p-value = 0.00000
Significance levels:        10%         5%
```

```
Critical values:          21.06      23.68
Conclusions:             reject     reject

Bierens-Martins time-varying cointegration test for m = 8
Test statistic = 53.76
Asymptotic null distribution: Chi-square(16)
p-value = 0.00001
Significance levels:        10%         5%
Critical values:          23.54       26.3
Conclusions:             reject     reject
```

Obviously, the null hypothesis of time-invariant cointegration is strongly rejected. Thus,

$$\beta_t = \xi_0 + \xi_1 P_{1,T}(t) + \ldots + \xi_m P_{m,T}(t),$$

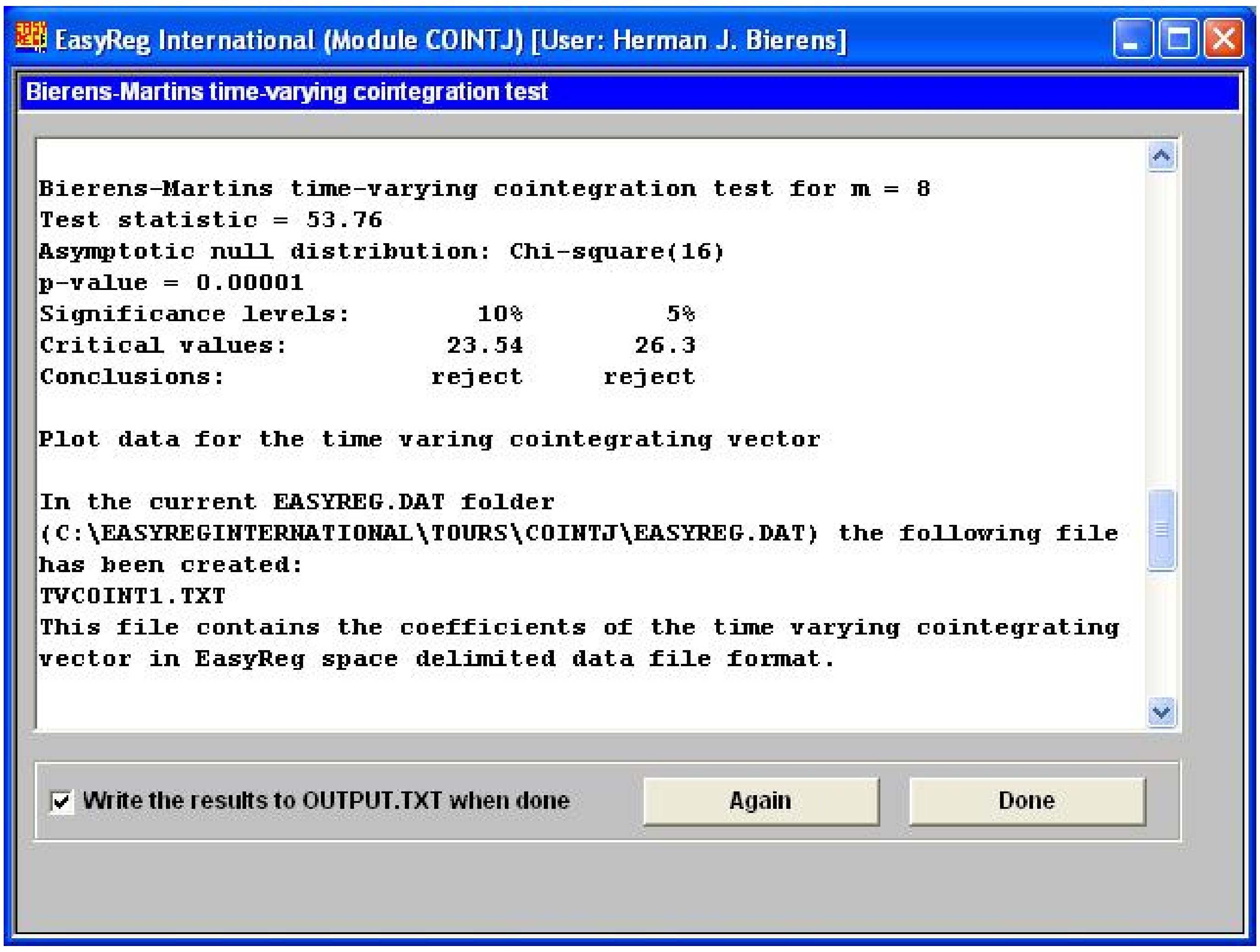
where some or all of the ξ_i's for i = 1,2,...,m are non-zero.

Plotting the time-varying cointegrating vectors

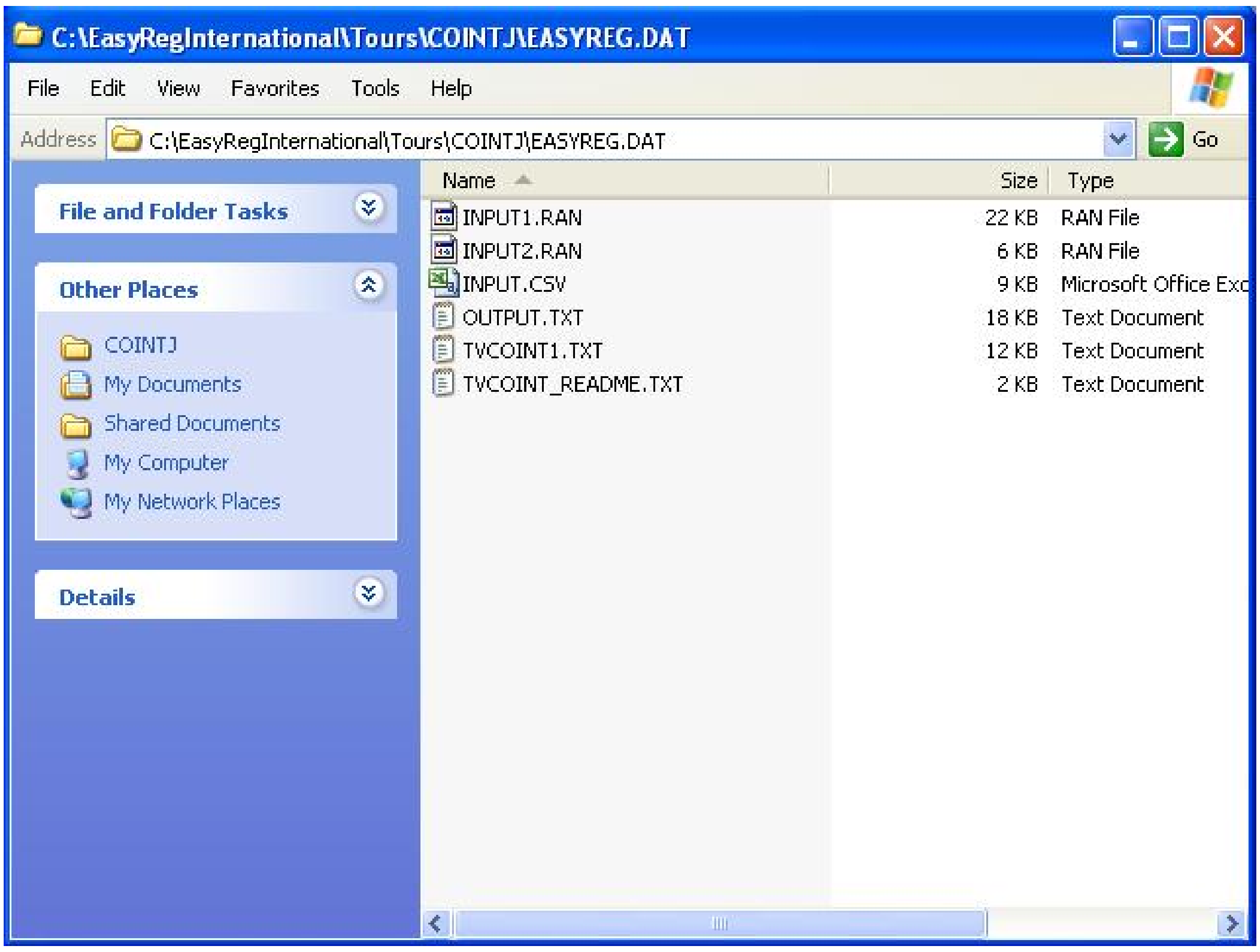
If the null hypothesis for m = 8 is rejected, EasyReg writes the components of each column of

$$\beta_t = \xi_0 + \xi_1 P_{1,T}(t) + \ldots + \xi_8 P_{8,T}(t)$$

to EasyReg input files in space delimited text format, in the current sub-folder EASYREG.DAT.

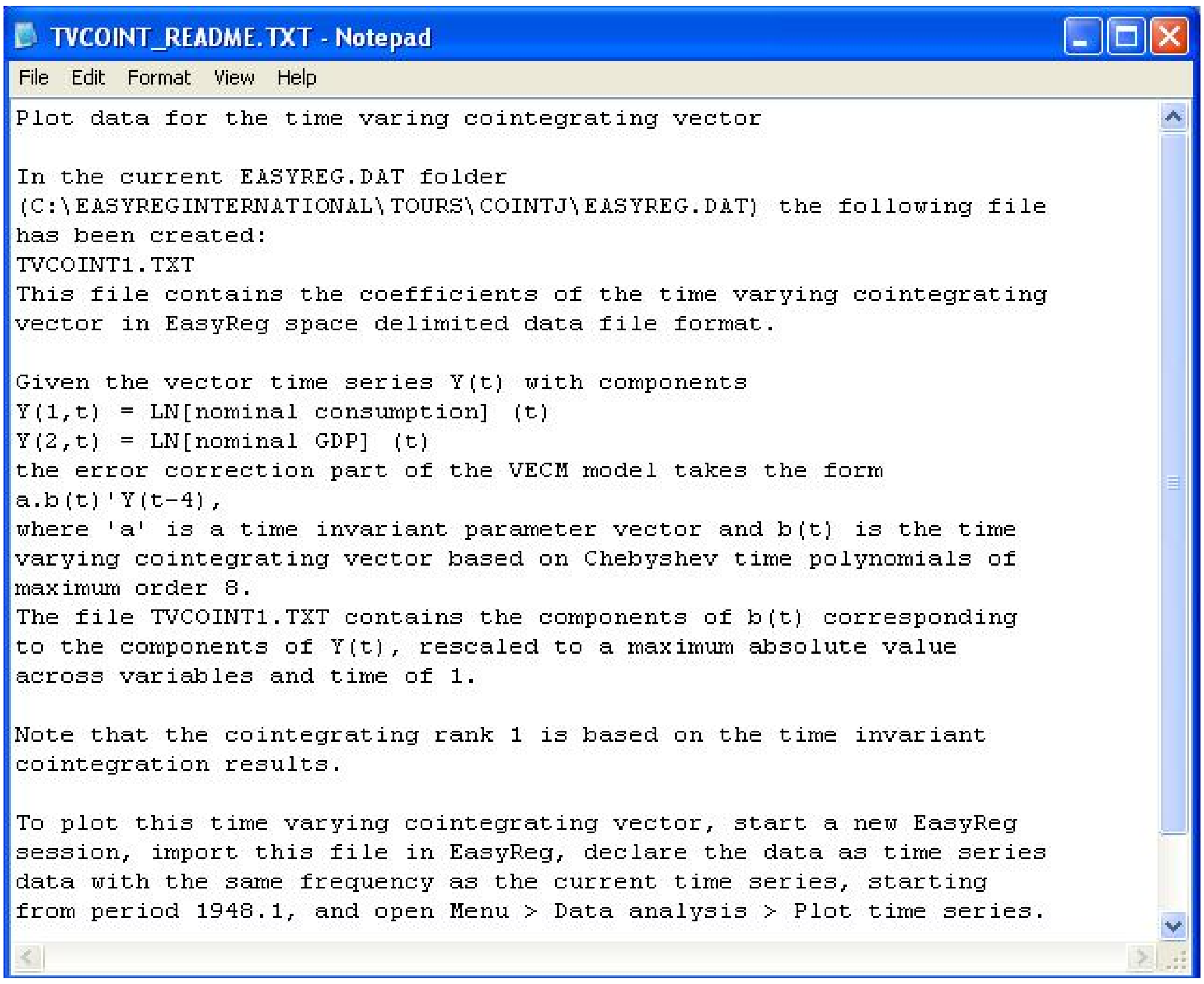


Since β_t has only one column, only one file TVCOINT1.TXT is created.



If β_t would have had 2 columns, for example, a second file TVCOINT2.TXT would have be created as well, containing the the components of the second column of β_t .

The file TVCOINT_README.TXT contains instructions how to plot the components of β_t .



TVCOINT_README.TXT - Notepad

File Edit Format View Help

```
Plot data for the time varing cointegrating vector

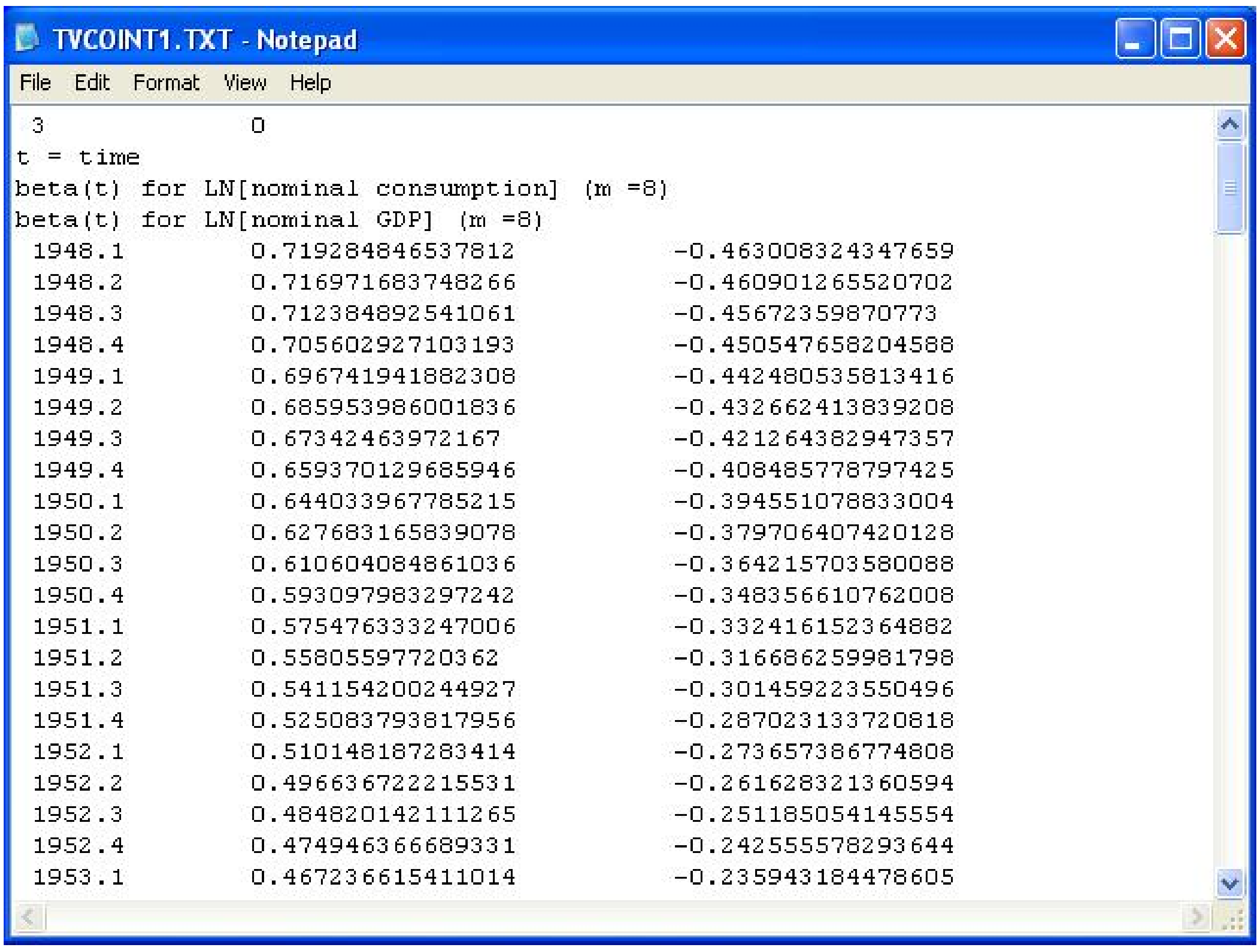
In the current EASYREG.DAT folder
(C:\EASYREGINTERNATIONAL\TOURS\COINTJ\EASYREG.DAT) the following file
has been created:
TVCOINT1.TXT
This file contains the coefficients of the time varying cointegrating
vector in EasyReg space delimited data file format.

Given the vector time series Y(t) with components
Y(1,t) = LN[nominal consumption] (t)
Y(2,t) = LN[nominal GDP] (t)
the error correction part of the VECM model takes the form
a.b(t)'Y(t-4),
where 'a' is a time invariant parameter vector and b(t) is the time
varying cointegrating vector based on Chebyshev time polynomials of
maximum order 8.
The file TVCOINT1.TXT contains the components of b(t) corresponding
to the components of Y(t), rescaled to a maximum absolute value
across variables and time of 1.

Note that the cointegrating rank 1 is based on the time invariant
cointegration results.

To plot this time varying cointegrating vector, start a new EasyReg
session, import this file in EasyReg, declare the data as time series
data with the same frequency as the current time series, starting
from period 1948.1, and open Menu > Data analysis > Plot time series.
```

As said before, the file TVCOINT1.TXT is an EasyReg input file in space delimited text format:

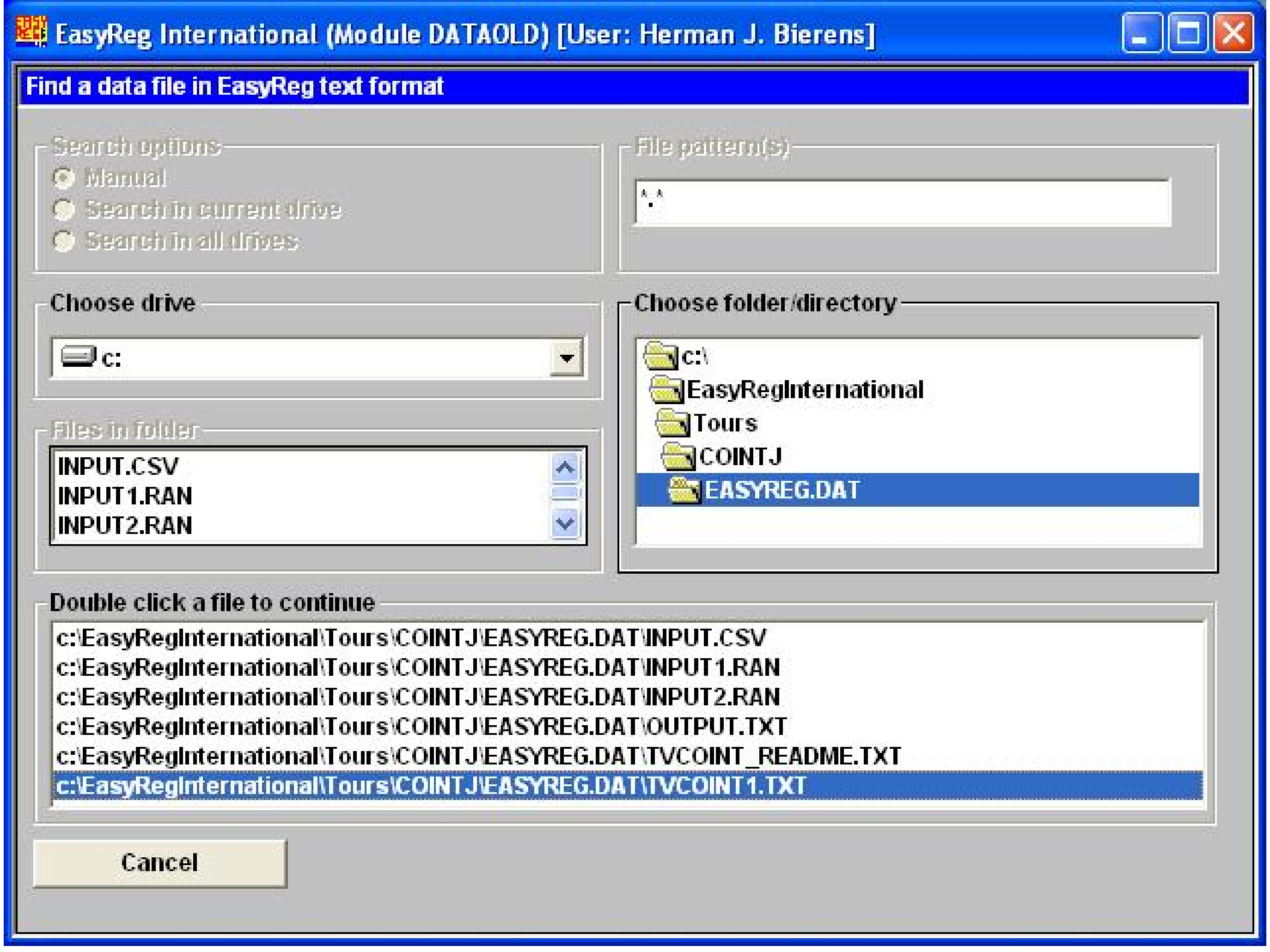


TVCOINT1.TXT - Notepad

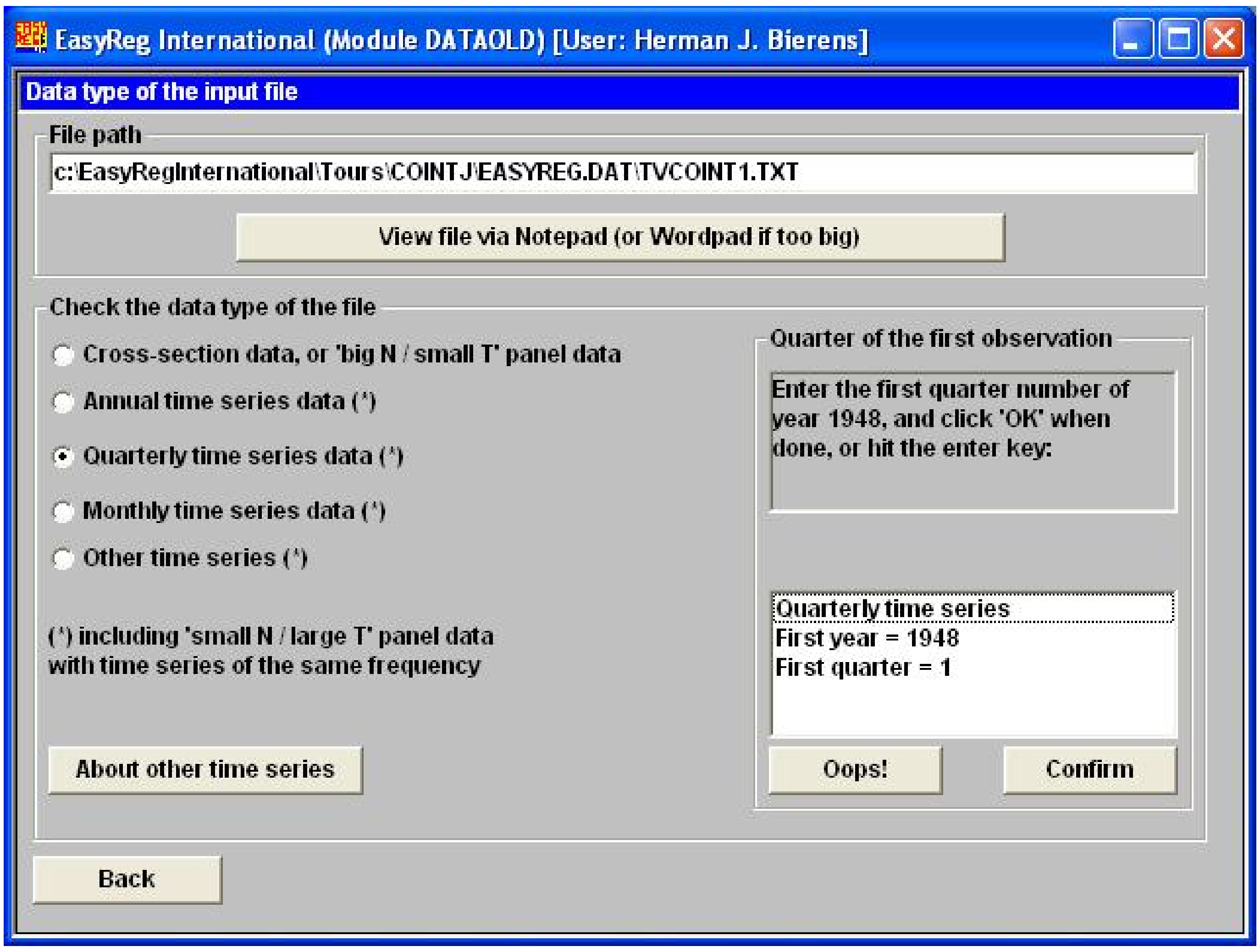
File Edit Format View Help

```
 3            0
t = time
beta(t) for LN[nominal consumption] (m =8)
beta(t) for LN[nominal GDP] (m =8)
 1948.1       0.719284846537812        -0.463008324347659
 1948.2       0.716971683748266        -0.460901265520702
 1948.3       0.712384892541061        -0.45672359870773
 1948.4       0.705602927103193        -0.450547658204588
 1949.1       0.696741941882308        -0.442480535813416
 1949.2       0.685953986001836        -0.432662413839208
 1949.3       0.67342463972167         -0.421264382947357
 1949.4       0.659370129685946        -0.408485778797425
 1950.1       0.644033967785215        -0.394551078833004
 1950.2       0.627683165839078        -0.379706407420128
 1950.3       0.610604084861036        -0.364215703580088
 1950.4       0.593097983297242        -0.348356610762008
 1951.1       0.575476333247006        -0.332416152364882
 1951.2       0.55805597720362         -0.316686259981798
 1951.3       0.541154200244927        -0.301459223550496
 1951.4       0.525083793817956        -0.287023133720818
 1952.1       0.510148187283414        -0.273657386774808
 1952.2       0.496636722215531        -0.261628321360594
 1952.3       0.484820142111265        -0.251185054145554
 1952.4       0.474946366689331        -0.242555578293644
 1953.1       0.467236615411014        -0.235943184478605
```

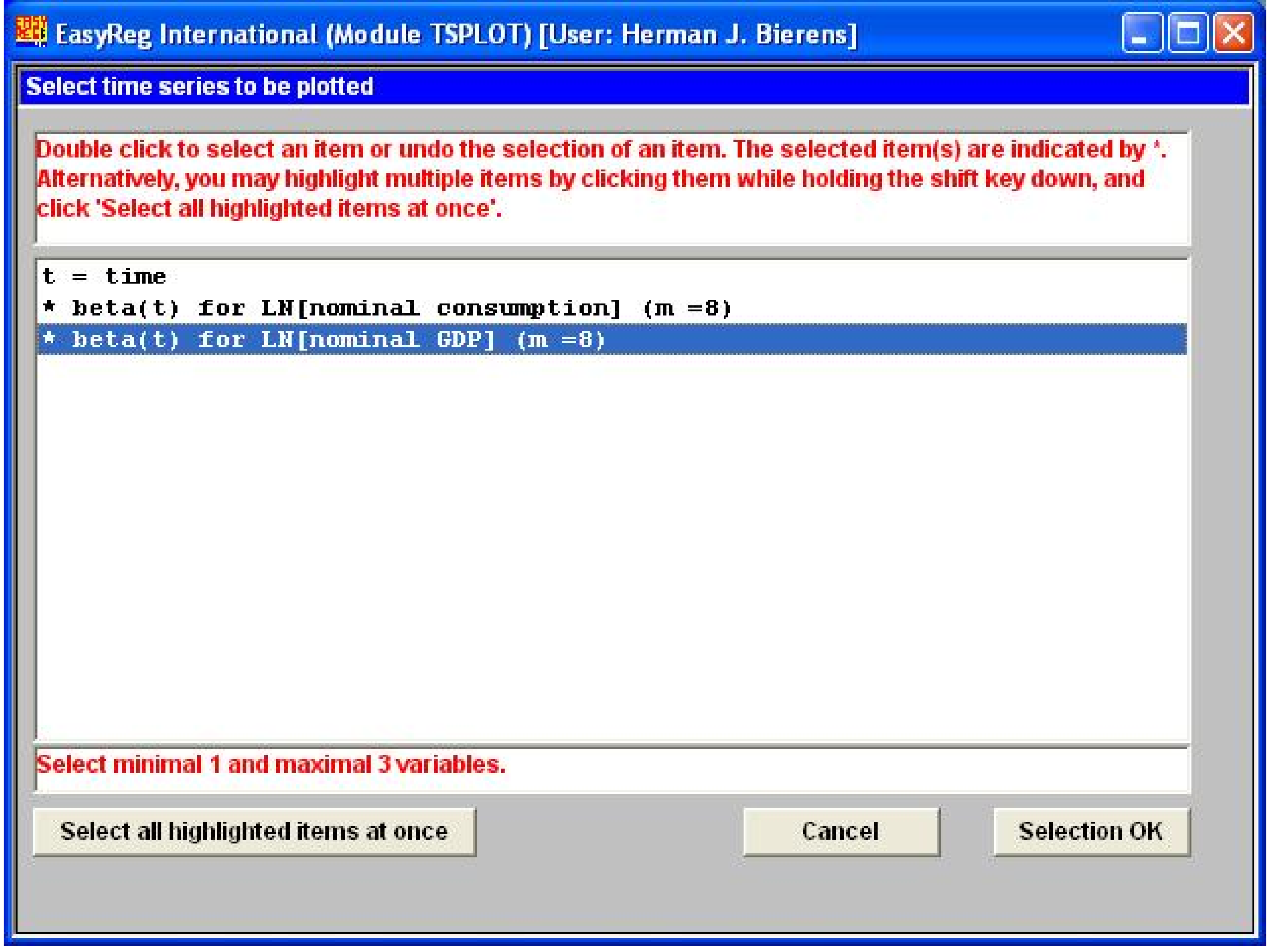
The next step is to import this file in EasyReg, as quarterly time series. See the guided tour involved (off-line or online) on how to do that. Only the two key steps involved are displayed here. First, surf to the current sub-folder EASYREG.DAT and double-click on TVCOINT1.TXT:



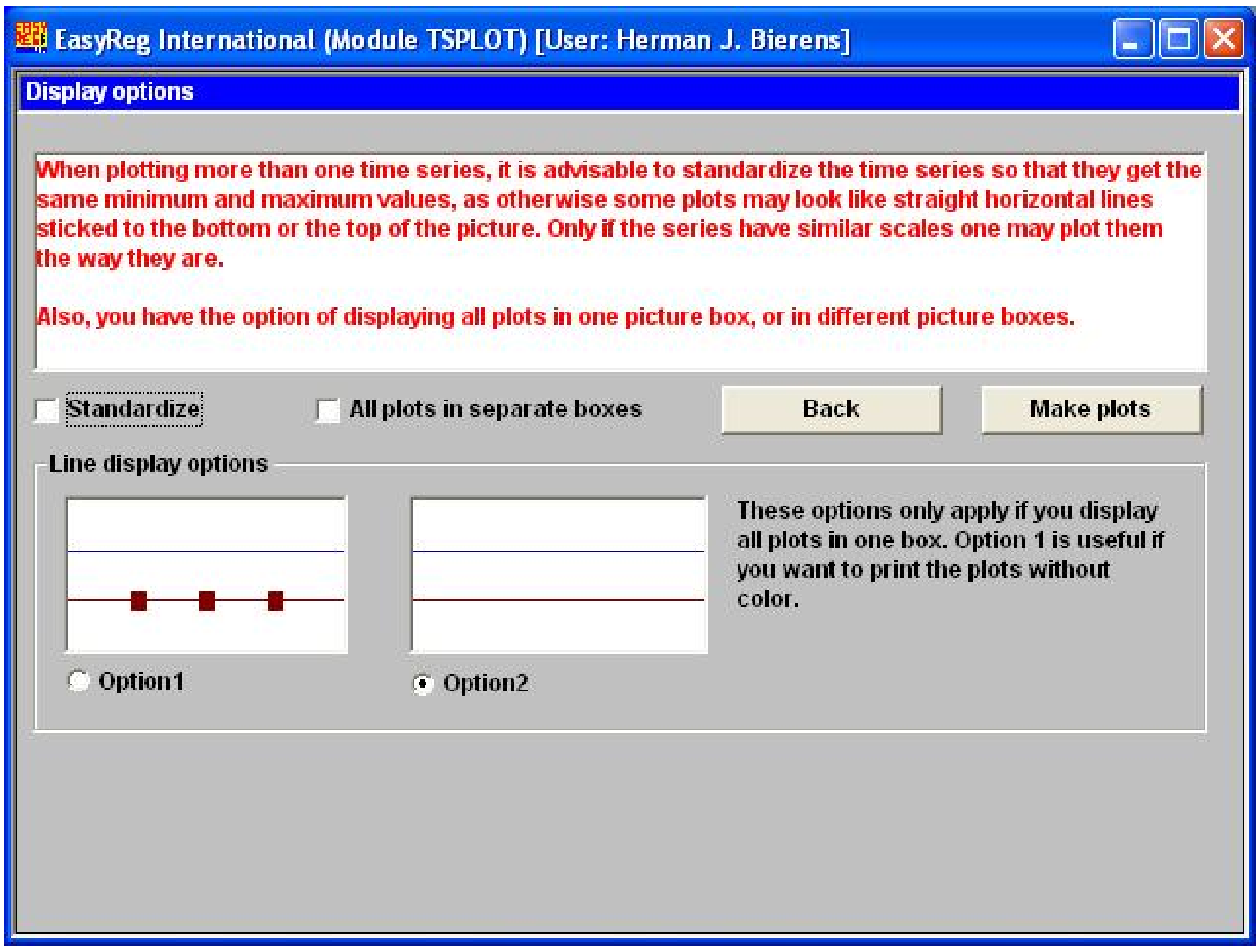
Declare the data as quarterly time series, starting from year 1948, quarter 1:



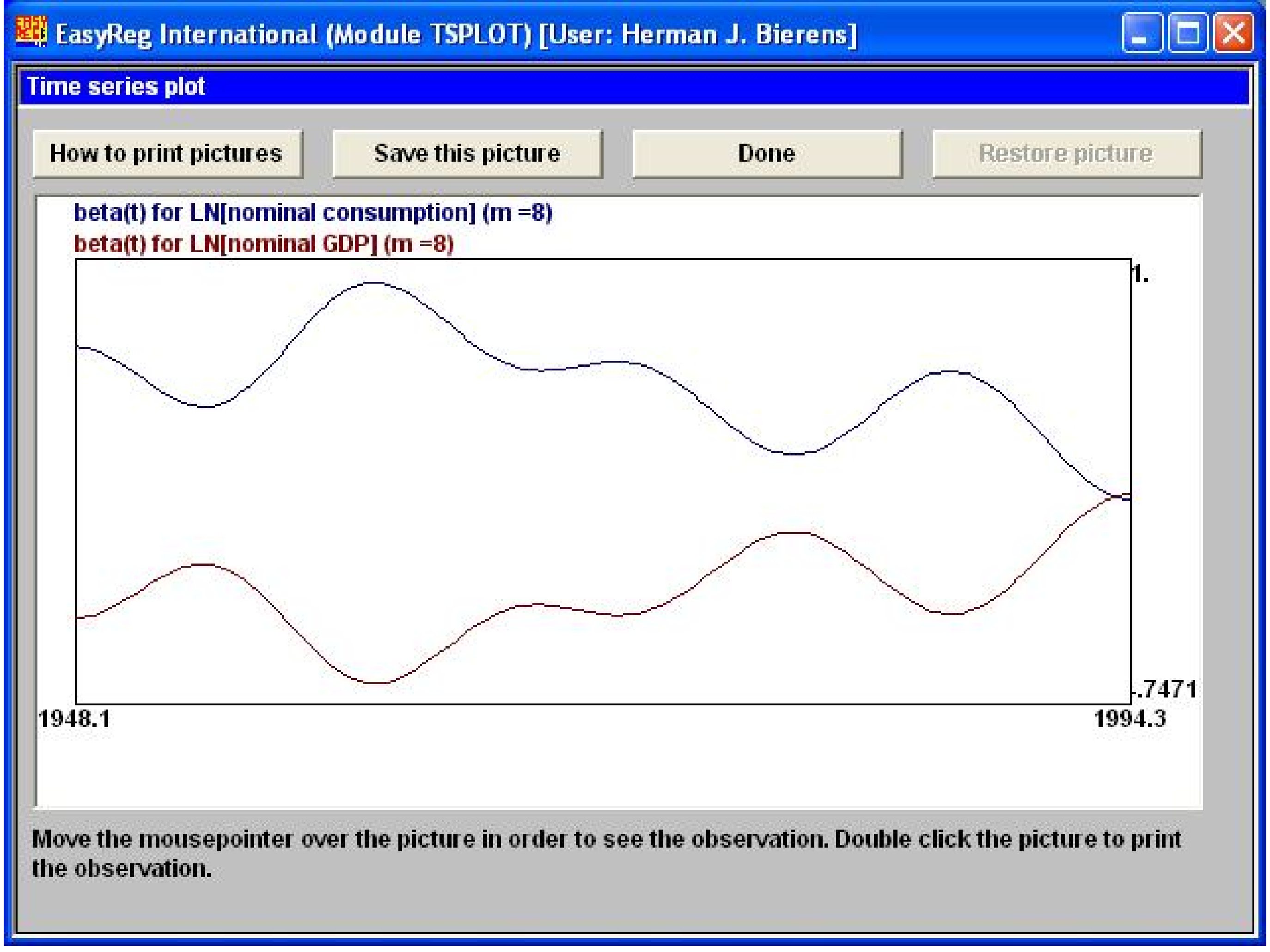
Once you have imported this data file in EasyReg, open Menu > Data analysis > Plot time series, and select the components of β_t .



Because the time series are already standardized between -1 and 1, there is no need to standardize them once more.



Finally, here are the plot results:



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This is the end of the guided tour on Johansen's cointegration analysis