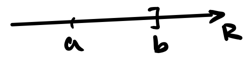
1 区间的长度与推广.

(0:00~6:00)

1) 

区间的长度 $\lambda((a,b]) = b-a$.

能否推广到任意集合中?

$$\lambda: \mathcal{P}(\mathbb{R}) \longrightarrow R_+ \cup \{+\infty\}$$

测度.

2) 我们希望这一测度有什么性质?

i 区间的测度是区间的长度:

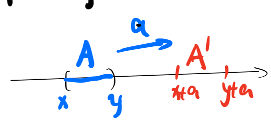
$$\lambda((a,b]) = b-a$$

ii λ是$\mathbb{R}$的幂集上的函数:

$$\lambda: \mathcal{P}(\mathbb{R}) \to R_+ \cup \{+\infty\}.$$

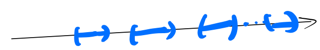
iii 平移不变性:

平移: $A+u = \{y+u : y \in A\}$



$$\forall A \subseteq \mathbb{R},\ \forall u \in \mathbb{R},\ \lambda(A+u) = \lambda(A)$$

iv 可列可加性:



$$A = \bigcup_{\sigma} A_\sigma.\quad A_\sigma \cap A_k = \emptyset$$

$$\lambda(A) = \sum_{k=1}^{\infty} \lambda(A_k)$$

这样的一个测度函数存在吗? 不存在.

证明: 思路: 找到一个不可测集 $A \subseteq \mathbb{R}$

1) $A = \bigsqcup_i A_i$, $\lambda(A_i) = \lambda(A_j)$. $i=j$ 平移.

且 $A \subseteq B$.

claim: if $A \subseteq B$, $\lambda(A) \le \lambda(B)$

$$\lambda(A) = \sum_i \lambda(A_i) \le \lambda(B)$$

$$\Rightarrow \lambda(A) = 0.$$

2) 存在区间 $E \subseteq A$, $\lambda(E) > 0$.

$$\lambda(A) \geq \lambda(E)$$

矛盾. A不可测!

Claim: if $A \subseteq B \subseteq \mathbb{R}$. 则 $\lambda(A) \le \lambda(B)$

$B = A \cup B \backslash A \quad (A \cap (B \backslash A) = \emptyset)$

$\lambda(B) = \lambda(A) + \lambda(B \backslash A)$

① $\lambda(B \backslash A) = +\infty \Rightarrow \lambda(B) = +\infty \quad \lambda(A) \le \lambda(B)$

② $\lambda(B \backslash A)$ 有界. $\quad \lambda(A) \le \lambda(B)$

$\lambda([0,1]) = \lambda([0,1] \cup \phi) = \lambda([0,1]) + \lambda(\phi)$

$\Rightarrow \lambda(\phi) = 0.$ $\square$.

定义关系 ~: $x \sim y$, $x, y \in \mathbb{R}$ if $x - y \in \mathbb{Q}$.

易证 ~ 是等价关系. $\begin{cases} x - x = 0 \in \mathbb{Q} \\ x - y = q. \quad y - x = -q \\ x - y = q_1, \ y - z = q_2, \quad x - z = q_1 + q_2 \in \mathbb{Q} \end{cases}$

等价类: $[X] = \{y \sim X : \forall y \in \mathbb{R}\}$

商集: $\Lambda = \mathbb{R}/\sim$. 所有等价类构成的集合.

选出代表元构成集合 $\Omega \subseteq \mathbb{R}$. 令 $\Omega \subseteq [0,1)$

令 $A = \bigcup\limits_{\substack{q \in \mathbb{Q} \\ -1 < q < 1}} (\Omega + q)$

Claim: $p, q \in \mathbb{Q}, p \ne q$. 则. $(\Omega + p) \cap (\Omega + q) = \phi$

证明: 假设 $(\Omega + p) \cap (\Omega + q) \ne \phi$

$\exists x = \alpha + p \quad \alpha \in \Omega$

$= \beta + q \quad \beta \in \Omega$

$\Rightarrow \alpha - \beta = p - q \in \mathbb{Q}$.

$\Rightarrow \alpha \sim \beta$.

$\Rightarrow p = q$. $\square$

Claim: $A \subset (-1, 2)$

$\forall x \in A$. $\quad \Omega \subseteq (0,1) \quad q \in (-1,1)$

$a = x + q \in (-1, 2)$ $\square$

$$\lambda\Big(\bigsqcup_{\substack{q\in\mathbb{Q}\\ -1<q<1}}(\Omega+q)\Big)=\sum_{\substack{q\in\mathbb{Q}\\ -1<q<1}}\lambda(\Omega+q)=\sum_{\substack{q\in\mathbb{Q}\\ -1<q<1}}\lambda(\Omega)\le 3.$$

$$\lambda(A)=0.$$

Claim: $[0,1)\subseteq A$.

$\forall x\in[0,1)$.

$\exists\,\alpha\in[x]\cap\Omega$.

$\alpha\in[0,1)$ $\alpha-x=q\in\mathbb{Q}$, $q\in(-1,1)$

因此, $x\in\Omega-q$ $\square$.

$\lambda(A)\ge\lambda([0,1))=1.$

$\square$.

总结:

1) 推广"长度"这一概念.

i $\lambda(I)=|I|$

ii $\lambda:\mathcal{P}(\mathbb{R})\longrightarrow\mathbb{R}_+\cup\{+\infty\}$

iii $\lambda(A+q)=\lambda(A)$ $\forall A\subseteq\mathbb{R},\forall q\in\mathbb{R}.$

iiii $\lambda(\bigsqcup_i A_i)=\sum_i\lambda(A_i)$

2) 证明了满足全部四条性质的λ不存在.

接受一条性质不成立 $\Rightarrow$ ii

不可测集.

$\Longrightarrow$ 最大的可测集集合是什么?