# Supplementary Material for "Bootstrapping Extreme Value Estimators"

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# A Appendix: Proofs

## A.1 Proofs for results in Section 1 and 2

**Proof of Lemma 1.3.** We first verify that given the observations $\{\tilde{X}_1, \cdots, \tilde{X}_n\}$, the conditional distribution of $\left\{\tilde{X}_j^*\right\}_{j=1}^n$ is the same as an i.i.d. sample drawn from $F_n$.

Obviously, conditional on $\tilde{X}_1, \cdots, \tilde{X}_n$, $\left\{X_j^*\right\}_{j=1}^n$ are i.i.d. Hence, we only need to check one distribution as follows:

$$\Pr(F_n^{\leftarrow}(1 - 1/Y_j^*) \le x | \tilde{X}_1, \cdots, \tilde{X}_n) = \Pr(Y_j^* \le 1/(1 - F_n(x)) | \tilde{X}_1, \cdots, \tilde{X}_n) = F_n(x).$$

Relation (1.5) follows directly from the calculation below:

$$\tilde{X}^*_{n-[ks],n} = F_n^{\leftarrow}\left(1 - \frac{1}{Y^*_{n-[ks],n}}\right) = \tilde{X}_{\lceil n\left(1 - \frac{1}{Y^*_{n-[ks],n}}\right)\rceil,n} = \tilde{X}_{n-[\frac{n}{Y^*_{n-[ks],n}}],n} = X_{n-[k\tilde{D}_n(s)],n}.$$

■

**Proof of Lemma 2.3.** Notice that $\left\{Y^*_{j,n}\right\}_{j=1}^n \overset{d}{=} \{1/U_{n-j+1,n}\}_{i=1}^n$ where $\{U_{j,n}\}_{i=1}^n$ are the order statistics from the standard uniform distribution. In addition, for all $s \in [0,1]$, $ks/n \le k/n \to 0$ as $n \to \infty$. Thus, we can use Inequality 1 on page 419, Shorack and Wellner (1986) to obtain the first half of the lemma.

The second half of the lemma follows from the fact that uniformly for all $s \in [0,1]$, as $n \to \infty$, $\tilde{D}_n(s) \le \tilde{D}_n(1) = \frac{n}{kY^*_{n-k,n}} \xrightarrow{P} 1$. ■

**Proof of Lemma 2.4.** The first equation in (2.4) follows from Lemma 2.4.10 in de Haan and Ferreira (2006) with $\gamma = -\xi$.

The second equation in (2.4) follows from the fact that

$$\sup_{0<s\le 1} \left|\frac{W_n^*(s)}{s^{1/2-\varepsilon}}\right| = O_P(1). \tag{A.1}$$

■

**Proof of Lemma 2.5.** Write

$$\left(\tilde{D}_n(s)\right)^{-\gamma-1} W_n\left(\tilde{D}_n(s)\right) - s^{-\gamma-1}W_n(s) = I_1\left(s\right) + I_2\left(s\right),$$

with

$$\begin{aligned} I_1\left(s\right) &:= \left(\tilde{D}_n(s)\right)^{-\gamma-1}\left(W_n\left(\tilde{D}_n(s)\right) - W_n\left(s\right)\right), \\ I_2\left(s\right) &:= W_n\left(s\right)\left(\left(\tilde{D}_n(s)\right)^{-\gamma-1} - s^{-\gamma-1}\right). \end{aligned}$$

For $I_1(s)$, we first apply (2.4) with $\xi = 1$ to obtain that uniformly for all $s \in [1/(k+1), 1]$, as $n \to \infty$,

$$\tilde{D}_n(s) - s = k^{-1/2}s^{1/2-\tau}O_P(1) \xrightarrow{P} 0,$$

for any $\tau > 0$. Hence, we can apply the modulus of continuity to the Brownian motion $W_n(t)$ to get that

$$\left|W_n(\tilde{D}_n(s)) - W_n(s)\right| \le (k^{-1/2}s^{1/2-\tau}O_P(1))^{1/2-\tau} = k^{-1/4+\tau/2}s^{1/4-\tau+\tau^2}O_p(1)$$

uniformly for all $s \in [1/(k+1), 1]$. As $n \to \infty$, since $\left(\tilde{D}_n(s)\right)^{-\gamma-1} \overset{P}{\asymp} s^{-\gamma-1}$ by Lemma 2.3, we get that

$$I_1(s) = k^{-1/4+\tau/2}s^{-\gamma-3/4-\tau+\tau^2}O_p(1),$$

uniformly for all $s \in [1/(k+1), 1]$. Notice that

$$\frac{k^{-1/4+\tau/2}s^{-\gamma-3/4-\tau+\tau^2}}{s^{-\gamma-1/2-\varepsilon}} = k^{-1/4+\tau/2}s^{-1/4-\tau+\tau^2+\varepsilon} \le \max(k^{-1/4+\tau/2}, Ck^{3\tau/2-\tau^2-\varepsilon}),$$

where the last inequality is due to the fact that $s^{-1/4-\tau+\tau^2+\varepsilon} \le \max(Ck^{1/4+\tau-\tau^2-\varepsilon}, 1)$ for all $s \in [1/(k+1), 1]$ and some constant $C > 0$. Therefore, by choosing $\tau < \min(1/2, 2\varepsilon/3)$, we get that as $n \to \infty$, $\frac{k^{-1/4+\tau/2}s^{-\gamma-3/4-\tau+\tau^2}}{s^{-\gamma-1/2-\varepsilon}} \to 0$ uniformly for all $s \in [1/(k+1), 1]$, which implies that $I_1(s) = s^{-\gamma-1/2-\varepsilon} o_P(1)$.

The term $I_2(s)$ is handled by applying (A.1) and the second equation in (2.4) with $\xi = -\gamma - 1$. We get that for any $\tau > 0$, as $n \to \infty$, uniformly for all $s \in [1/(k+1), 1]$,

$$I_2(s) = s^{1/2-\tau} O_P(1) \cdot k^{-1/2} s^{-\gamma-3/2-\tau} O_p(1) = k^{-1/2} s^{-\gamma-1-2\tau} O_p(1).$$

Similar to the way of handling $I_1(s)$, we have the following inequality

$$\frac{k^{-1/2} s^{-\gamma-1-2\tau}}{s^{-\gamma-1/2-\varepsilon}} = k^{-1/2} s^{-1/2-2\tau+\varepsilon} \leq \max(k^{-1/2}, Ck^{2\tau-\varepsilon}),$$

uniformly for all $s \in [1/(k+1), 1]$ and some constant $C > 0$. Therefore, by choosing $\tau < \varepsilon/2$, we get that as $n \to \infty$, uniformly for all $s \in [1/(k+1), 1]$, $I_2(s) = s^{-\gamma-1/2-\varepsilon} o_P(1)$. The lemma is proved by combining the two terms $I_1(s)$ and $I_2(s)$. ■

**Proof of Lemma 2.6.** Write

$$\begin{aligned}
&\left| \sqrt{k} A_0\left(\frac{n}{k}\right) \Psi_{\gamma,\rho}\left(\frac{1}{\tilde{D}_n(s)}\right) - \sqrt{k} A_0\left(\frac{n}{k}\right) \Psi_{\gamma,\rho}\left(\frac{1}{s}\right) \right| \\
=& \left| \sqrt{k} A_0\left(\frac{n}{k}\right) \right| \cdot \left| \Psi_{\gamma,\rho}\left(\frac{1}{\tilde{D}_n(s)}\right) - \Psi_{\gamma,\rho}\left(\frac{1}{s}\right) \right| \\
=& \left| \sqrt{k} A_0\left(\frac{n}{k}\right) \right| \cdot \left| \frac{1}{\tilde{D}_n(s)} - \frac{1}{s} \right| \cdot \left| \Psi'_{\gamma,\rho}\left(\theta \frac{1}{\tilde{D}_n(s)} + (1-\theta)\frac{1}{s}\right) \right| \\
=:& \, I_1 \cdot I_2 \cdot I_3
\end{aligned}$$

where $\theta := \theta_n(s) \in [0, 1]$.

For $I_1$, we have that $\sqrt{k} A_0\left(\frac{n}{k}\right) = O(1)$ as $n \to \infty$.

According to Lemma 2.4 with $\xi = -1$, $I_2$ has the following expansion, as $n \to \infty$,

$$I_2 = \left| \left(\tilde{D}_n(s)\right)^{-1} - s^{-1} \right| = k^{-1/2} s^{-3/2-\varepsilon} O_P(1),$$

uniformly for all $s \in [1/(k+1), 1]$.

Finally, to handle $I_3$, notice that

$$\Psi'_{\gamma,\rho}(t) = \begin{cases} t^{\gamma+\rho-1} & \text{if } \rho < 0 \\ t^{\gamma-1}(\log t + 1/\gamma) & \text{if } \rho = 0 \neq \gamma \\ t^{-1}\log t & \text{if } \rho = 0 = \gamma \end{cases},$$

which implies for any $\delta' > 0$,

$$\left|\Psi'_{\gamma,\rho}(t)\right| \leq C \max(t^{\gamma+\rho-1+\delta'}, t^{\gamma+\rho-1-\delta'}), \tag{A.2}$$

for some $C = C(\delta') > 0$.

As $n \to \infty$, since $\tilde{D}_n(s) \overset{P}{\asymp} s$ according to Lemma 2.3, we get

$$I_3 = \left|\Psi'_{\gamma,\rho}\left(\theta(\tilde{D}_n(s))^{-1} + (1-\theta)\frac{1}{s}\right)\right| = s^{-\gamma-\rho+1-\delta'} O_P(1),$$

uniformly for all $s \in [1/(k+1), 1]$.

Combining the three factors $I_1, I_2$ and $I_3$, we conclude that as $n \to \infty$,

$$\begin{aligned} &\left|A_0\left(\frac{n}{k}\right)\Psi_{\gamma,\rho}\left(\frac{1}{\tilde{D}_n(s)}\right) - \sqrt{k}A_0\left(\frac{n}{k}\right)\Psi_{\gamma,\rho}\left(\frac{1}{s}\right)\right| \\ =&k^{-1/2}s^{-3/2-\varepsilon} \cdot s^{-\gamma-\rho+1-\delta'}O_P(1) = s^{-\gamma-1/2-\varepsilon}o_P(1), \end{aligned}$$

holds uniformly for all $s \in [1/(k+1), 1]$ by choosing $\delta' < 1/2$. ■

**Proof of Lemma 2.7.** The lemma follows directly from Lemma 2.3. ■

Before proving Theorem 2.1, we first prove the following proposition.

**Proposition A.1** *Under the same conditions as in Theorem 2.1, with the Brownian motions $\{W_n^*(s)\}$ given in Lemma 2.4, the function $b_0\left(\frac{n}{k}\right)$ and the Browninan motions $\{W_n(s)\}$ given* *in Proposition 1.1, for any $\varepsilon > 0$, as $n \to \infty$,*

$$\sqrt{k}\left(\frac{\tilde{X}^*_{n-[ks],n} - b_0\left(\frac{n}{k}\right)}{a_0\left(\frac{n}{k}\right)} - \frac{s^{-\gamma}-1}{\gamma}\right)$$
$$= s^{-\gamma-1}W_n^*(s) + s^{-\gamma-1}W_n(s) + \sqrt{k}A_0\left(\frac{n}{k}\right)\Psi_{\gamma,\rho}\left(\frac{1}{s}\right) + s^{-\gamma-1/2-\varepsilon}o_P(1), \tag{A.3}$$

*holds uniformly for all $s \in [1/(k+1), 1]$.*

**Proof of Proposition A.1.** The proposition follows directly from Lemmas 2.5 - 2.7 ■

**Proof of Theorem 2.1.** We first handle the case $\gamma \geq -1/2$. Note that in this case $b_0^*\left(\frac{n}{k}\right) = b_0\left(\frac{n}{k}\right)$. Hence Proposition A.1 shows that the (2.1) holds uniformly for $s \in [1/(k+1), 1]$. The proof of the extension to uniformity on $(0, 1]$ is very similar to the proof of Corollary 2.4.5 in de Haan and Ferreira (2006) and is thus omitted.

Next, we handle the case $\gamma < -1/2$. We first show that, for any $\varepsilon > 0$, as $n \to \infty$,

$$\sqrt{k}\left(\frac{b_0^*\left(\frac{n}{k}\right) - b_0\left(\frac{n}{k}\right)}{a_0\left(\frac{n}{k}\right)}\right) = s^{-\gamma-1/2-\varepsilon}o_P(1), \tag{A.4}$$

holds uniformly for all $s \in [1/(k+1), 1]$.

By taking $s = 1/(k+1)$ in (A.3), we get that as $n \to \infty$,

$$\sqrt{k}\left(\frac{\tilde{X}^*_{n,n} - b_0\left(\frac{n}{k}\right)}{a_0\left(\frac{n}{k}\right)} - \frac{(k+1)^{\gamma}-1}{\gamma}\right) = (k+1)^{\gamma+1}W_n^*\left(\frac{1}{k+1}\right) + (k+1)^{\gamma+1}W_n\left(\frac{1}{k+1}\right)$$
$$+ \sqrt{k}A_0\left(\frac{n}{k}\right)\Psi_{\gamma,\rho}(k+1) + (k+1)^{\gamma+1/2+\varepsilon}o_P(1),$$

which implies that

$$\sqrt{k}\left(\frac{b_0^*\left(\frac{n}{k}\right) - b_0\left(\frac{n}{k}\right)}{a_0\left(\frac{n}{k}\right)} - \frac{(k+1)^{\gamma}}{\gamma}\right) = (k+1)^{\gamma+1}W_n^*\left(\frac{1}{k+1}\right) + (k+1)^{\gamma+1}W_n\left(\frac{1}{k+1}\right)$$
$$+ \sqrt{k}A_0\left(\frac{n}{k}\right)\left(\Psi_{\gamma,\rho}(k+1) + \frac{1}{\gamma+\rho}1_{\rho<0}\right)$$
$$+ (k+1)^{\gamma+1/2+\varepsilon'}o_P(1).$$

Note that the relation holds for any $\varepsilon$, but we choose a specific $\varepsilon'$ such that $\gamma + 1/2 + \varepsilon' < 0$ and $\varepsilon' < \varepsilon$.

Then (A.4) is proved by checking the following relations: as $n \to \infty$, uniformly for all $s \in [1/(k+1), 1]$

$$
\begin{aligned}
&\sqrt{k} s^{\gamma+1/2+\varepsilon} \frac{(k+1)^{\gamma}}{\gamma} = o(1), \\
&(k+1)^{\gamma+1} s^{\gamma+1/2+\varepsilon} W\left(\frac{1}{k+1}\right) = o_P(1), \text{ with } W = W_n \text{ or } W = W_n^*, \\
&s^{\gamma+1/2+\varepsilon}\left(\Psi_{\gamma,\rho}(k+1) + \frac{1}{\gamma+\rho} 1_{\rho<0}\right) = o(1), \\
&s^{\gamma+1/2+\varepsilon}(k+1)^{\gamma+1/2+\varepsilon'} = O(1).
\end{aligned}
$$

We omit the details but remark that a key tool for checking these relations is

$$
s^{\gamma+1/2+\varepsilon} \leq \max(1, (k+1)^{-\gamma-1/2-\varepsilon}).
$$

Combining Proposition A.1 and the relation (A.4), we can replace the shift $b_0\left(\frac{n}{k}\right)$ in (A.3) by the new shift $b_0^*\left(\frac{n}{k}\right)$. Therefore, for $\gamma < -1/2$, the expansion (2.1) holds uniformly for $s \in [1/(k+1), 1]$.

Finally, for $\gamma < -1/2$, we need to extend the uniformity to $s \in (0, 1]$. Again the proof is very similar to the extension above, which follows the same lines as in the proof of Corollary 2.4.5 in de Haan and Ferreira (2006) and is thus omitted. ■

**Proof of Corollary 2.2.** The corollary follows directly by combining Proposition 1.1, Proposition A.1 and Lemma A.2 below. ■

**Lemma A.2** *Consider a probability space $(\Omega, \mathcal{F}, P)$. Suppose $\{\delta_n\}_{n=1}^{\infty}$ is a series of non-negative random variables defined in this probability space such that $\delta_n = o_P(1)$ as $n \to \infty$. Consider a series of $\sigma-$algebra $\{\mathcal{F}_n\}_{n=1}^{\infty}$. Then for any $\epsilon > 0$ $\Pr(\delta_n > \epsilon \mid \mathcal{F}_n) = o_P(1)$.*

**Proof of Lemma A.2.** We need to show that for any $\epsilon, \epsilon' > 0$,

$$\lim_{n\to\infty} \Pr(\Pr(\delta_n > \epsilon \,|\, \mathcal{F}_n) > \epsilon') = 0.$$

Since $\Pr(\delta_n > \epsilon \,|\, \mathcal{F}_n)$ is a random variable in $[0,1]$, by Markov inequality, we have that

$$0 \leq \Pr(\Pr(\delta_n > \epsilon \,|\, \mathcal{F}_n) > \epsilon') \leq \frac{1}{\epsilon'} \mathrm{E}\left(\Pr(\delta_n > \epsilon \,|\, \mathcal{F}_n)\right) = \frac{\Pr(\delta_n > \epsilon)}{\epsilon'}.$$

By taking $n \to \infty$, the Lemma follows. ■

## A.2 Proofs for results in Section 3

To prove Proposition 3.2, we are going to use the following two lemmas which extend Lemma 2.4.10 and Theorem 2.4.2 in de Haan and Ferreira (2006).

**Lemma A.3** *Let $Y_1, Y_2, \cdots$ be i.i.d. random variables following the standard Pareto distribution. Let $Y_{1,n} \leq Y_{2,n} \leq \cdots \leq Y_{n,n}$ be the order statistics. Suppose $k = O(n/(\log n)^4)$ as $n \to \infty$. Then, there exists a sequence of Brownian motions $\{W_n(x)\}_{x>0}$ such that for any $0 < \varepsilon < 1/2$,*

$$\sup_{1/(k\log k) \leq x \leq (\log k)^2} x^{\gamma+\frac{1}{2}+\varepsilon} \left| \sqrt{k}\left( \frac{\left(\frac{k}{n} Y_{n-[kx],n}\right)^{\gamma} - 1}{\gamma} - \frac{x^{-\gamma}-1}{\gamma} \right) - x^{-\gamma-1} W_n(x) \right| = o_P(1).$$

**Proof of Lemma A.3.** We split the region $x \in [1/(k\log k), (\log k)^2]$ into two parts $x \in R_1 := [1/(k\log k), 1/(k+1)]$ and $x \in R_2 := [1/(k+1), (\log k)^2]$.

The proof of the statement for $x \in R_2$ follows the same lines as that for Lemma 2.4.10 in de Haan and Ferreira (2006), with adaptations on line -6 of page 53 and on line 6 of page 54. In both cases the present conditions suffice. Details are thus omitted.

For $x \in R_1$, $Y_{n-[kx],n} = Y_{n,n}$. First we handle $\gamma \neq 0$. The statement is proved by verifying the following relations: as $n \to \infty$,

$$\sup_{1/(k\log k)\le x\le 1/(k+1)} x^{\gamma+\frac{1}{2}+\varepsilon}\left|\sqrt{k}\left(\frac{k}{n}Y_{n,n}\right)^{\gamma}\right| = o_P(1).$$
$$\sup_{1/(k\log k)\le x\le 1/(k+1)} x^{\gamma+\frac{1}{2}+\varepsilon}\left|\sqrt{k}x^{-\gamma}\right| = o_P(1).$$
$$\sup_{1/(k\log k)\le x\le 1/(k+1)} x^{\gamma+\frac{1}{2}+\varepsilon}\left|x^{-\gamma-1}W_n(x)\right| = o_P(1).$$

The first relation is verified by noting that as $n \to \infty$, $Y_{n,n}/n = O_P(1)$ and for sufficiently large $k$, uniformly for all $x \in R_1$,

$$x^{\gamma+1/2+\varepsilon} \le 2\max((\log k)^{-\gamma-1/2-\varepsilon}, 1)k^{-\gamma-1/2-\varepsilon}.$$

The second relation can be verified in a similar way but simpler. The third relation follows from the law of iterated logarithm.

The case $\gamma = 0$ can be handled in a similar way, with simpler calculation. ■

From now on, for simplicity, we use the notation $\cdot^{\pm\delta'} = \max\left(\cdot^{\delta'}, \cdot^{-\delta'}\right)$ and $\cdot^{\mp\delta'} = \min\left(\cdot^{\delta'}, \cdot^{-\delta'}\right)$.

**Proof of Proposition 3.2.** Under the conditions in the Proposition, we have that $k(\log k)^4/n \to 0$ as $n \to \infty$. We can apply inequality (2.3.19) in de Haan and Ferreira (2006) with replacing $tx$ and $t$ by $Y_{n-[kx],n}$ and $n/k$ respectively. Note the representation in Lemma 3.3. For any $\delta' > 0$ and sufficiently large $n$, for all $1/(k\log k) \le x \le (\log k)^2$,

$$\left|\frac{\frac{\tilde{X}_{n-[kx],n}-\tilde{b}_0\left(\frac{n}{k}\right)}{a_0\left(\frac{n}{k}\right)} - \frac{\left(\frac{k}{n}Y_{n-[kx],n}\right)^{\gamma}-1}{\gamma}}{A_0\left(\frac{n}{k}\right)} - \bar{\Psi}_{\gamma,\rho}\left(\frac{k}{n}Y_{n-[kx],n}\right)\right| \le \varepsilon'\left(\frac{k}{n}Y_{n-[kx],n}\right)^{\gamma+\rho\pm\delta'},$$

where $a_0$ and $A_0$ are the same functions as in Proposition 1.1, $\tilde{b}_0$ is the function given in this

Proposition, and

$$\bar{\Psi}_{\gamma,\rho}(x) = \begin{cases} \frac{x^{\gamma+\rho}}{\gamma+\rho}, & \gamma+\rho \neq 0, \rho < 0, \\ \log x, & \gamma+\rho = 0, \rho < 0, \\ \frac{1}{\gamma} x^{\gamma} \log x, & \rho = 0 \neq \gamma, \\ \frac{1}{2}(\log x)^2, & \rho = 0 = \gamma; \end{cases}$$

see (2.3.20) in de Haan and Ferreira (2006).

With the Brownian motions $\{W_n(x)\}$ defined in Lemma A.3, we can write

$$\begin{aligned} &\left| \sqrt{k} \left( \frac{\tilde{X}_{n-[kx],n} - \bar{b}_0\left(\frac{n}{k}\right)}{a_0\left(\frac{n}{k}\right)} - \frac{x^{-\gamma}-1}{\gamma} \right) - x^{-\gamma-1} W_n(x) \right| \\ \leq &\left| \sqrt{k} \left( \frac{\left(\frac{k}{n} Y_{n-[kx],n}\right)^{\gamma} - 1}{\gamma} - \frac{x^{-\gamma}-1}{\gamma} \right) - x^{-\gamma-1} W_n(x) \right| \\ &+ \left| \sqrt{k} A_0\left(\frac{n}{k}\right) \bar{\Psi}_{\gamma,\rho}\left(\frac{k}{n} Y_{n-[kx],n}\right) \right| \\ &+ \sqrt{k} \left| A_0\left(\frac{n}{k}\right) \right| \varepsilon' \left(\frac{k}{n} Y_{n-[kx],n}\right)^{\gamma+\rho\pm\delta'} \\ =: &I_1 + I_2 + I_3. \end{aligned}$$

We show that $x^{\gamma+1/2+\varepsilon} I_j = o_p(1)$ uniformly on $1/(k \log k) \leq x \leq (\log k)^2$ for $j = 1, 2, 3$ and for any $0 < \varepsilon < 1/2$.

For $I_1$, it follows from Lemma A.3.

For $I_2$, note that for any $\delta' > 0$ there exists $C > 0$ such that $\bar{\Psi}_{\gamma,\rho}(t) \leq C t^{\gamma+\rho\pm\delta'}$ for all $t > 0$. We split the region $x \in [1/(k \log k), (\log k)^2]$ into two regions $x \in R_1 := [1/(k \log k), 1/(k+1)]$ and $x \in R_2 := [1/(k+1), (\log k)^2]$.

Uniformly for all $x \in R_1$, we have that $Y_{n-[kx],n} = Y_{n,n}$ and $Y_{n,n}/n = O_P(1)$ as $n \to \infty$, which implies that $\bar{\Psi}_{\gamma,\rho}\left(\frac{k}{n} Y_{n-[kx],n}\right) = O_P(1) k^{\gamma+\rho+\delta'}$. Together with the facts that, for sufficiently large $k$, uniformly for all $x \in R_1$,

$$x^{\gamma+1/2+\varepsilon} \leq 2 \max((\log k)^{-\gamma-1/2-\varepsilon}, 1) k^{-\gamma-1/2-\varepsilon},$$

and $\sqrt{k}A_0\left(\frac{n}{k}\right) \to 0$ as $n \to \infty$, we can verify that $x^{\gamma+1/2+\varepsilon}I_2 = o_p(1)$ holds uniformly for $x \in R_1$, by taking $\delta' < \varepsilon$.

Next, since $k(\log k)^2/n \to 0$ as $n \to \infty$, we can apply Lemma 2.3 with treating $k(\log k)^2$ as $k$ therein, to obtain that uniformly for all $s \in [1/(k(\log k)^2 + 1), 1]$,

$$\frac{n}{k(\log k)^2 Y^*_{n-[k(\log k)^2 s],n}} \overset{P}{\asymp} s.$$

By replacing $x = (\log k)^2 s$, we get that as $n \to \infty$, uniformly for all $x \in [1/(k+1), (\log k)^2]$, $\frac{n}{kY^*_{n-[kx],n}} \overset{P}{\asymp} x$, which leads to

$$\left|\bar{\Psi}\left(\frac{k}{n}Y_{n-[kx],n}\right)\right| \leq C\left(\frac{k}{n}Y_{n-[kx],n}\right)^{\gamma+\rho\pm\delta'} = x^{-\gamma-\rho\pm\delta'}O_p(1).$$

Therefore, by choosing $\delta' < 1/2$, as $n \to \infty$, uniformly for all $x \in R_2$

$$x^{\gamma+1/2+\varepsilon}I_2 = \left|\sqrt{k}A_0\left(\frac{n}{k}\right)\right| \cdot x^{1/2-\rho+\varepsilon\pm\delta'}O_p(1) = \left|\sqrt{k}A_0\left(\frac{n}{k}\right)\right| \cdot (\log k)^{1-2\rho+\varepsilon+2\delta'}O_p(1) = o_P(1).$$

The last step follows from the condition of the Proposition with taking $\tau = 1 - 2\rho + \varepsilon + 2\delta'$. Combining the two regions yields the proof for $I_2$.

Finally, $I_3$ is handled in a similar way as $I_2$ by splitting the region into $R_1$ and $R_2$. The proof for $x \in R_1$ is simpler and thus omitted. For $x \in R_2$, write (using Lemma 2.3),

$$I_3 = \sqrt{k}\left|A_0\left(\frac{n}{k}\right)\right| x^{-\gamma-\rho\pm\delta'}O_p(1).$$

Therefore, as $n \to \infty$, according to the condition of the Proposition,

$$x^{\gamma+1/2+\varepsilon}I_3 = \sqrt{k}\left|A_0\left(\frac{n}{k}\right)\right| x^{1/2-\rho+\varepsilon\pm\delta'}O_p(1) = o_P(1).$$

uniformly for all $x \in R_2$. The last step follows the same argument as that for $I_2$ with $x \in R_2$.

Combining the three terms, we proved the Proposition. ■

**Proof of Lemma 3.3.** We first verify that given the observations $\{\tilde{X}_1, \cdots, \tilde{X}_n\}$, the conditional distribution of $\{X_i^*\}_{i=1}^k$ is the same as an i.i.d. sample of block maxima obtained from observations drawn from $F_n$, with block size $m$.

Obviously, conditional on $\tilde{X}_1, \cdots, \tilde{X}_n$, $\{X_i^*\}_{i=1}^k$ are i.i.d. Hence, we only need to check one distribution as follows:

$$\begin{aligned}\Pr(F_n^{\leftarrow}(\Phi((mZ_i^*)) \le x | \tilde{X}_1, \cdots, \tilde{X}_n) =& \Pr(\Phi(mZ_i^*) \le F_n(x) | \tilde{X}_1, \cdots, \tilde{X}_n) \\ =& \exp\left(-\frac{m}{\Phi^{-1}(F_n(x))}\right) = (F_n(x))^m.\end{aligned}$$

Relation (3.4) follows directly from the calculation below:

$$X^*_{\lceil ks \rceil, k} = F_n^{\leftarrow}(\Phi(mZ^*_{\lceil ks \rceil, k})) = \tilde{X}_{\lceil n\Phi(mZ^*_{\lceil ks \rceil, k}) \rceil, n} = \tilde{X}_{n-[n-n\Phi(mZ^*_{\lceil ks \rceil, k})], n} = \tilde{X}_{n-[kD_n(s)], n}.$$

■

**Proof of Lemma 3.4.** Since $x - \frac{x^2}{2} < 1 - e^{-x} < x$ for $x > 0$, we get the expansion

$$1 - \Phi(x) = \frac{1}{x}\left(1 - \frac{\theta(x)}{2x}\right),$$

with $0 < \theta(x) < 1$. Consequently,

$$D_n(s) = \frac{1}{Z^*_{\lceil ks \rceil, k}}\left(1 - \frac{\Theta_n(s)}{2mZ^*_{\lceil ks \rceil, k}}\right), \tag{A.5}$$

where $0 < \Theta_n(s) < 1$.

We first prove the second half of the lemma: as $n \to \infty$, uniformly for all $s \in [1/(k+1), k/(k+1)]$,

$$\frac{1}{Z^*_{\lceil ks \rceil, k}} \overset{P}{\asymp} -\log s. \tag{A.6}$$

To prove (A.6), we first apply Inequality 1 on p.419 in Shorack and Wellner (1986) to the quantile process of the i.i.d. uniform distributed random variables $\left\{\exp\left(-\frac{1}{Z^*_{\lceil ks \rceil, k}}\right)\right\}_{s \in (0,1)}$ to obtain the following inequalities. For any given $\delta > 0$, there exists sufficiently large positive values $M, M' > 1$, such that with probability higher than $1-\delta$, we have

$$\frac{s}{M} \le \exp\left(-\frac{1}{Z^*_{\lceil ks\rceil,k}}\right) \le Ms, \qquad \text{for} \quad \frac{1}{k+1} \le s \le \frac{1}{M}, \tag{A.7}$$

$$\frac{1-s}{M'} \le 1-\exp\left(-\frac{1}{Z^*_{\lceil ks\rceil,k}}\right) \le (1-s)M_1, \qquad \text{for} \;\; 1-\frac{1}{M'} \le s \le 1-\frac{1}{k+1}. \tag{A.8}$$

If $\frac{1}{M}+\frac{1}{M'} < \frac{1}{2}$, we consider three regions $s \in [1/(k+1), 1/(2M)]$, $s \in [1/(2M), 1-1/(2M')]$ and $s \in [1-1/(2M'), k/(k+1)]$. If $\frac{1}{M}+\frac{1}{M'} \ge \frac{1}{2}$, the middle region is covered by the other two.

For $s \in [1/(k+1), 1/(2M)]$, from (A.7) we get that

$$1+\frac{\log M}{-\log s} \ge \frac{1}{(-\log s)Z^*_{\lceil ks\rceil,k}} \ge 1-\frac{\log M}{-\log s}. \tag{A.9}$$

Since $0 < \frac{\log M}{-\log s} \le \frac{\log M}{\log M+\log 2} < 1$, (A.6) is proved for this region.

Similarly, for $s \in [1-1/(2M'), k/(k+1)]$, from (A.8), we get that

$$\frac{-\log\left(1-\frac{1-s}{M'}\right)}{-\log s} \le \frac{1}{(-\log s)Z^*_{\lceil ks\rceil,k}} \le \frac{-\log(1-(1-s)M')}{-\log s}. \tag{A.10}$$

Since $s \ge 1-1/(2M')$, and $1-(1-s)M' \ge 1/2$ , there exist proper constants $c_1, c_2 > 1$ such that

$$\begin{aligned} &c_1(1-s) \ge -\log s \ge 1-s, \\ &-\log(1-(1-s)M') \le c_2(1-s)M' \quad \text{and} \quad -\log(1-(1-s)/M') \ge (1-s)/M'. \end{aligned}$$

Applying these bounds to (A.10), the term $\frac{1}{(-\log s)Z^*_{\lceil ks\rceil,k}}$ can be further bounded by

$$\frac{1}{c_1 M'} \le \frac{1}{(-\log s)Z^*_{\lceil ks\rceil,k}} \le c_2 M',$$

which proves (A.6) for this region.

Finally, in the middle region $s \in [1/(2M), 1-1/(2M')]$, we apply Theorem 3 on p.95 in Shorack and Wellner (1986): given any small $\eta > 0$, for sufficiently large $k$, uniformly for all $s \in [1/(2M), 1-1/(2M')]$,

$$s - \eta < \exp\left(-\frac{1}{Z^*_{\lceil ks \rceil, k}}\right) < s + \eta, \quad \text{a.s.}$$

Therefore,

$$\frac{-\log(s-\eta)}{-\log s} > \frac{1}{(-\log s) Z^*_{\lceil ks \rceil, k}} > \frac{-\log(s+\eta)}{-\log s}.$$

By choosing $\eta < \min\left(\frac{1}{2M}, \frac{1}{2M'}\right)$, we prove (A.6) for the middle region.

Combining the three region, we obtain (A.6). Together $m/\log k \to \infty$ as $n \to \infty$, we get that $\frac{\Theta_n(s)}{2mZ^*_{\lceil ks \rceil, k}} = o_P(1)$, holds uniformly for all $s \in [1/(k+1), k/(k+1)]$, which completes the proof of the first half of the lemma. ■

**Proof of Lemma 3.5.** Recall that $k = O(n^l)$ for some $0 < l < 2/3$ and $m = n/k$. By choosing $\nu > 0$ such that $2 - 1/l < \nu < 1/2$, we get that as $n \to \infty$, $\frac{k^{1-\nu}}{m} = \frac{k^{2-\nu}}{n} = O(1)$.

We first show a version of (3.6) as follows: there exists a sequence of Brownian bridges $B_1^*, B_2^*, \ldots$ such that as $k \to \infty$, uniformly for all $s \in [1/(k+1), k/(k+1)]$,

$$\begin{aligned} &\sqrt{k}\left(\frac{(D_n(s))^{\xi} - 1}{\xi} - \frac{(-\log s)^{\xi} - 1}{\xi}\right) \\ &= -\frac{B_k^*(s)}{s(-\log s)^{1-\xi}} + k^{-1/2+\nu} s^{\nu-1} \frac{(1-s)^{\nu}}{(-\log s)^{1-\xi}} O_P(1), \end{aligned} \tag{A.11}$$

for any $\xi \in \mathbb{R}$.

By checking condition (6.1.2) in Csörgő and Horváth (1993, p. 369) for $Y_i^* := \frac{\left(Z_i^*\right)^{-\xi} - 1}{-\xi}$, we can apply Theorem 6.2.1 therein to get that for any $0 < \nu < 1/2$, there exists a sequence of Brownian bridges $B_k^*$ such that as $k \to \infty$,

$$k^{1/2-\nu}\left(s\left(1-s\right)\right)^{-\nu}\left|\sqrt{k}s(-\log s)^{1-\xi}\left(\frac{\left(Z^*_{\lceil ks\rceil,k}\right)^{-\xi}-1}{-\xi}-\frac{(-\log s)^{\xi}-1}{-\xi}\right)-B_k^*\left(s\right)\right|=O_P(1),$$

where $O_P(1)$ is uniform for $s \in [1/(k+1), k/(k+1)]$.

Hence, to prove (A.11), it is sufficient to show that as $k \to \infty$, for all $\xi \in \mathbb{R}$,

$$k^{1-\nu}s^{1-\nu}\left(1-s\right)^{-\nu}\left(-\log s\right)^{1-\xi}\left|\frac{\left(D_n\left(s\right)\right)^{\xi}-1}{\xi}-\frac{\left(Z^*_{\lceil ks\rceil,k}\right)^{-\xi}-1}{\xi}\right|=O_P(1),$$

where $O_P(1)$ is uniform for $s \in [1/(k+1), k/(k+1)]$.

Recall the expansion of the $\Phi$ function used in (A.5). Consequently, since $mZ^*_{\lceil ks\rceil,k} \to \infty$ as $n \to \infty$, we have

$$\left|\frac{\left(D_n\left(s\right)\right)^{\xi}-1}{\xi}-\frac{\left(Z^*_{\lceil ks\rceil,k}\right)^{-\xi}-1}{\xi}\right|\leq\left(Z^*_{\lceil ks\rceil,k}\right)^{-\xi}\frac{\Theta_n(s)}{2mZ^*_{\lceil ks\rceil,k}}=\frac{(-\log s)^{1+\xi}}{m}O_p(1),$$

where in the last step we use the result in Lemma 3.4.

Hence (A.11) is proved by checking that as $k \to \infty$,

$$k^{1-\nu}s^{1-\nu}\left(1-s\right)^{-\nu}\left(-\log s\right)^{1-\xi}\cdot\frac{(-\log s)^{1+\xi}}{m}=O(1),$$

holds uniformly for $s \in [1/(k+1), k/(k+1)]$, which follows from $\frac{k^{1-\nu}}{m}=O(1)$ and the facts that $s^{1-\nu}(1-s)^{-\nu}(\log s)^2$ is bounded for $s \in (0,1)$.

Then, to prove (3.6), we only need to check that for any $0<\lambda<1/2$, as $n \to \infty$,

$$\frac{k^{-1/2+\nu}s^{\nu-1}\frac{(1-s)^{\nu}}{(-\log s)^{1-\xi}}}{s^{-1/2-\lambda}\frac{(1-s)^{1/2-\lambda}}{(-\log s)^{1-\xi}}}=k^{-1/2+\nu}\cdot\left(s(1-s)\right)^{\nu+\lambda-1/2}$$

converges to zero, uniformly for $s \in [1/(k+1), k/(k+1)]$. Notice that for $s \in$

$[1/(k+1), k/(k+1)]$, $1/(4k) \le s(1-s) \le 1/4$, it implies that

$$(s(1-s))^{\nu+\lambda-1/2} \le \max((4k)^{-\nu-\lambda+1/2}, 4^{-\nu-\lambda+1/2}).$$

Here the maximum of the two depends on $\nu+\lambda > 1/2$ or $\nu+\lambda < 1/2$. In either case, as $n \to \infty$,

$$k^{-1/2+\nu} \cdot (s(1-s))^{\nu+\lambda-1/2} \le 4^{-\nu-\lambda+1/2} \max(k^{-\lambda}, k^{-1/2+\nu}) \to 0,$$

which completes the proof of (3.6).

Next we prove (3.7) from (A.11) by using the law of iterated logarithm for the Brownian bridge $B_k^*$: for any given $\tau > 0$, uniformly for all $s \in [1/(k+1), k/(k+1)]$

$$B_k^*(s) = (s(1-s))^{1/2} \left(\max\left(-\log s, -\log(1-s)\right)\right)^{\tau} O_P(1) = (s(1-s))^{1/2} (\log k)^{\tau} O_P(1).$$

Hence, uniformly for all $s \in [1/(k+1), k/(k+1)]$,

$$\begin{aligned}\frac{(D_n(s))^{\xi} - (-\log s)^{\xi}}{\xi} &= k^{-1/2} s^{-1/2} (1-s)^{1/2} \frac{(\log k)^{\tau}}{(-\log s)^{1-\xi}} O_P(1) \\ &\quad + k^{\nu-1} \frac{s^{\nu-1}(1-s)^{\nu}}{(-\log s)^{1-\xi}} O_P(1) \\ &= k^{-1/2} s^{-1/2} (1-s)^{1/2} \frac{(\log k)^{\tau}}{(-\log s)^{1-\xi}} O_P(1).\end{aligned}$$

■

**Proof of Lemma 3.6.** We will consider the two regions $1/(k+1) \le s \le 1/2$ and $1/2 \le s \le k/(k+1)$ separately.

Firstly, for all $1/(k+1) \le s \le 1/2$, we apply (3.7) with $\xi = -1$ and get that uniformly for any $0 < \tau < 2$ and $1/(k+1) \le s \le 1/2$, as $n \to \infty$,

$$\left|(D_n(s))^{-1} - (-\log s)^{-1}\right| = k^{-1/2} s^{-1/2} (1-s)^{1/2} (-\log s)^{-2} (\log k)^{\tau} O_p(1) \xrightarrow{P} 0,$$

Hence, we can apply the modulus of continuity to the Brownian motion $tW_n(1/t)$ to get that for any $\tau > 0$, as $n \to \infty$, uniformly for all $1/(k+1) \le s \le 1/2$,

$$\begin{aligned}
&\left|(D_n(s))^{-1} W_n(D_n(s)) - (-\log s)^{-1} W_n(-\log s)\right| \\
=&\left|(D_n(s))^{-1} - (-\log s)^{-1}\right|^{1/2-\tau} O_P(1) \\
=&\left(k^{-1/2} s^{-1/2} (1-s)^{1/2} (-\log s)^{-2} (\log k)^{\tau}\right)^{1/2-\tau} O_p(1) \\
=&k^{-1/4+\tau/2} s^{-1/4+\tau/2} (-\log s)^{-1+2\tau} (\log k)^{\tau(1/2-\tau)} O_p(1). \qquad \text{(A.12)}
\end{aligned}$$

Next, for all $1/(k+1) \le s \le 1/2$, write

$$(D_n(s))^{-\gamma-1} W_n(D_n(s)) - (-\log s)^{-\gamma-1} W_n(-\log s) = I_1(s) + I_2(s),$$

with

$$\begin{aligned}
I_1(s) &:= (D_n(s))^{-\gamma} \left((D_n(s))^{-1} W_n(D_n(s)) - (-\log s)^{-1} W_n(-\log s)\right), \\
I_2(s) &:= (-\log s)^{-1} W_n(-\log s) \left((D_n(s))^{-\gamma} - (-\log s)^{-\gamma}\right).
\end{aligned}$$

By combining (A.12) with Lemma 3.4, we get that as $n \to \infty$,

$$I_1(s) = k^{-1/4+\tau/2} s^{-1/4+\tau/2} (-\log s)^{-\gamma-1+2\tau} (\log k)^{\tau(1/2-\tau)} O_p(1),$$

holds uniformly for $1/(k+1) \le s \le 1/2$.

Then we handle $I_2(s)$ by applying (3.7) with $\xi = -\gamma$ to obtain uniformly for all $1/(k+1) \le s \le 1/2$, as $n \to \infty$,

$$\left|\frac{(D_n(s))^{-\gamma} - (-\log s)^{-\gamma}}{\gamma}\right| = k^{-1/2} s^{-1/2} (-\log s)^{-1-\gamma} (\log k)^{\tau} O_p(1).$$

Together with the fact that

$$(-\log s)^{-1} W_n(-\log s) = (-\log s)^{-1/2+\tau} O_p(1) = (-\log s)^{-1/2} (\log k)^{\tau} O_p(1)$$

for all $1/(k+1) \le s \le 1/2$, we get that

$$I_2(s) = k^{-1/2} s^{-1/2} (-\log s)^{-3/2-\gamma} (\log k)^{2\tau} O_p(1).$$

Finally, to prove the lemma for $1/(k+1) \le s \le 1/2$, we need to verify that as $n \to \infty$, $I_i(s)/\left(s^{-1/2-\lambda} \frac{(1-s)^{1/2-\lambda}}{(-\log s)^{1+\gamma}}\right) \xrightarrow{P} 0$ holds uniformly for all $1/(k+1) \le s \le 1/2$, for both $i = 1, 2$. We only check for $I_1(s)$, while $I_2(s)$ can be handled in a similar way.

$$\frac{I_1(s)}{s^{-1/2-\lambda} \frac{(1-s)^{1/2-\lambda}}{(-\log s)^{1+\gamma}}} = k^{-1/4+\tau/2} (\log k)^{\tau(1/2-\tau)} s^{1/4+\tau/2+\lambda} (1-s)^{\lambda-1/2} (-\log s)^{2\tau} O_P(1)$$

Since the function $s^a(1-s)^b(-\log s)^c$ is a continuous function on $1/(k+1) \le s \le 1/2$, it is maximized either at a fixed $s_0 = s_0(a,b,c) \in (0, 1/2]$ or at the boundary $1/(k+1)$, for any fixed $a, b, c \in \mathbb{R}$. Therefore as $n \to \infty$, by choosing $\tau < 1/2$,

$$\begin{aligned} & k^{-1/4+\tau/2} (\log k)^{\tau(1/2-\tau)} s^{1/4+\tau/2+\lambda} (1-s)^{\lambda-1/2} (-\log s)^{2\tau} \\ \le & \max(k^{-1/4+\tau/2} (\log k)^{\tau(1/2-\tau)} \cdot C, k^{-1/2-\lambda} (\log k)^{\tau(5/2-\tau)}) \to 0, \end{aligned}$$

uniformly for all $1/(k+1) \le s \le 1/2$, which implies that $I_1(s)/\left(s^{-1/2-\lambda} \frac{(1-s)^{1/2-\lambda}}{(-\log s)^{1+\gamma}}\right) \xrightarrow{P} 0$. A similar argument leads to the same conclusion for $I_2(s)$.

Second, we handle the region $1/2 \le s \le k/(k+1)$ in a similar way. We only sketch the steps and intermediate results here while the details are omitted.

Write

$$(D_n(s))^{-\gamma-1} W_n(D_n(s)) - (-\log s)^{-\gamma+1} W_n(-\log s) = I_3(s) + I_4(s),$$

with

$$I_3\left(s\right) := \left(D_n\left(s\right)\right)^{-\gamma-1}\left(W_n\left(D_n\left(s\right)\right) - W_n\left(-\log s\right)\right),$$
$$I_4\left(s\right) := W_n\left(-\log s\right)\left(\left(D_n\left(s\right)\right)^{-\gamma-1} - \left(-\log s\right)^{-\gamma-1}\right).$$

By applying (3.7) with $\xi = -1$, we obtain that uniformly for all $1/2 \leq s \leq k/(k+1)$ as $n \to \infty$,

$$\left|D_n\left(s\right) - \left(-\log s\right)\right| = k^{-1/2}(1-s)^{1/2}\left(\log k\right)^{\tau} O_p(1) \xrightarrow{P} 0.$$

Then, we can apply the modulus of continuity to the Brownian motion $W_n(t)$ to obtain that for any $\tau > 0$, as $n \to \infty$, uniformly for all $1/2 \leq s \leq k/(k+1)$,

$$\begin{aligned}|W_n(D_n\left(s\right)) - W_n(-\log s)| &= \left(k^{-1/2}(1-s)^{1/2}\left(\log k\right)^{\tau}\right)^{1/2-\tau} O_p(1)\\ &= k^{-1/4+\tau/2}(1-s)^{1/4-\tau/2}(\log k)^{\tau(1/2-\tau)}O_P(1).\end{aligned}$$

Together with Lemma 3.4, we get that as $n \to \infty$,

$$|I_3(s)| = k^{-1/4+\tau/2}(1-s)^{1/4-\tau/2}(\log k)^{\tau(1/2-\tau)}(-\log s)^{-\gamma-1}O_P(1)$$

holds uniformly for all $1/2 \leq s \leq k/(k+1)$.

The term $I_4(s)$ is handled by applying the uniform bound as $|W_n\left(-\log s\right)| \leq (-\log s)^{1/2}\left|\log|\log s|\right|^{\tau} O_p(1) = (-\log s)^{1/2}(\log k)^{\tau}O_p(1)$ for all $1/2 \leq s \leq k/(k+1)$ and any $\tau > 0$. Together with applying (3.7) with $\xi = -\gamma - 1$, we get that uniformly for $1/2 \leq s \leq k/(k+1)$, as $n \to \infty$,

$$|I_4(s)| = k^{-1/2}(1-s)^{1/2}(-\log s)^{-3/2-\gamma}(\log k)^{2\tau}O_p(1).$$

Again, to prove the lemma for $1/2 \leq s \leq k/(k+1)$, we need to verify that as $n \to \infty$, $I_i(s)/\left(s^{-1/2-\lambda}\frac{(1-s)^{1/2-\lambda}}{(-\log s)^{1+\gamma}}\right) \xrightarrow{P} 0$ holds uniformly, for both $i = 3, 4$. This is achieved by taking any $\tau < 1/2$ and using a similar argument as that for $I_1(s)$. $\blacksquare$

**Proof of Lemma 3.7.** The lemma is proved by directly applying Lemma 3.4 with further checking that as $n \to \infty$,

$$\frac{(-\log s)^{-\gamma-1/2-\varepsilon}}{s^{-1/2-\lambda}\frac{(1-s)^{1/2-\lambda}}{(-\log s)^{1+\gamma}}} = (-\log s)^{1/2-\varepsilon} s^{1/2+\lambda}(1-s)^{\lambda-1/2} = O(1),$$

holds for all $s \in [1/(k+1), k/(k+1)]$ and $\varepsilon < \lambda$. $\blacksquare$

## A.3 Proofs for results in Section 4

**Proof of Theorem 4.3.** Since $\sqrt{k}A(n/k) \to 0$, as $n \to \infty$, we get that $\tilde{b}_n(\gamma, \rho) \to 0$. Proposition 4.1 implies that as $n \to \infty$,

$$\sup_{x\in\mathbb{R}} \left| \Pr\left(\sqrt{k}(\hat{\gamma}_{POT} - \gamma) \leq x\right) - \Pr(\tilde{L}(W, \gamma) \leq x) \right| \to 0,$$

where $W$ is a standard Brownian motion. Hence, it is left to show that

$$\sup_{x\in\mathbb{R}} \left| \Pr\left(\sqrt{k}(\hat{\gamma}^*_{POT} - \hat{\gamma}_{POT}) \leq x \mid \tilde{X}_1, \ldots, \tilde{X}_n\right) - \Pr(\tilde{L}(W, \gamma) \leq x) \right| \xrightarrow{P} 0.$$

Notice that $\tilde{L}(W, \gamma)$ follows a Gaussian distribution. The uniform convergence is thus implied by point wise convergence and Lemma A.4 below. Hence, we only need to show that for a given $x \in \mathbb{R}$, as $n \to \infty$,

$$\Pr\left(\sqrt{k}(\hat{\gamma}^*_{POT} - \hat{\gamma}_{POT}) \leq x \mid \tilde{X}_1, \ldots, \tilde{X}_n\right) \xrightarrow{P} \Pr(\tilde{L}(W, \gamma) \leq x)$$

Recall the relation (4.1): as $n \to \infty$,

$$\sqrt{k}(\hat{\gamma}^*_{POT} - \hat{\gamma}_{POT}) = \tilde{L}(W^*_n, \gamma) + \delta_n,$$

where $\delta_n = o_P(1)$. Following Lemma A.2, we get that as $n \to \infty$,

$$\Pr(|\delta_n| > \epsilon \mid \tilde{X}_1, \ldots, \tilde{X}_n) = o_P(1).$$

Together with the fact that the processes $\{W_n^*\}$ are independent of the observations, the point wise convergence follows. ■

**Lemma A.4** *Assume that $g_n : \mathbb{R} \to \mathbb{R}$ is a series of monotone and non-decreasing random functions with $g_n(-\infty) = 0$ and $g_n(+\infty) = 1$. In addition, as $n \to \infty$,*

$$g_n(x) \xrightarrow{P} g(x), \quad \textit{for all } x \in \mathbb{R}.$$

*where, $g(x)$ is a continuous deterministic function on $\mathbb{R}$ with $g(-\infty) = 0$ and $g(+\infty) = 1$. Then as $n \to \infty$, $\sup_{x\in\mathbb{R}} |g_n(x) - g(x)| \xrightarrow{P} 0$.*

**Proof of Lemma A.4.** For any given $\varepsilon > 0$, we choose and integer $m > 2/\varepsilon$. Define $x_i = g^{\leftarrow}\left(\frac{i}{m}\right)$, for $1 \le i \le m-1$, where $\cdot^{\leftarrow}$ indicates the left-continuous inverse function. In addition, define, $x_0 = -\infty$, $x_m = +\infty$, such that we have $x_0 \le x_1 \le \ldots \le x_m$. Then, due to the point wise convergence as $x_1, \ldots, x_{m-1}$, there exists $n_0 = n(m)$ such that for all $n > n_0$ and

$$\Pr\left(\bigcap_{i=1}^{m-1} \left\{|g_n(x_i) - g(x_i)| < \frac{\varepsilon}{2}\right\}\right) > 1 - \varepsilon.$$

Note that for any $x \in \mathbb{R}$, there exists some $j \in \{0, 1, \ldots, m-1\}$ such that $x \in [x_j, x_{j+1}]$. Then on the set $\bigcap_{i=1}^{m-1} \left\{|g_n(x_i) - g(x_i)| < \frac{\varepsilon}{2}\right\}$, we have that

$$\begin{aligned}
|g_n(x) - g(x)| &\le \max(g_n(x_{j+1}) - g(x_j), g(x_{j+1}) - g_n(x_j)) \\
&\le \frac{1}{m} + \max(|g_n(x_j) - g(x_j)|, |g_n(x_{j+1}) - g(x_{j+1})|) < \varepsilon.
\end{aligned}$$

In other words, for all $n > n_0$, $\Pr\left(\sup_{x\in\mathbb{R}} |g_n(x) - g(x)| < \varepsilon\right) > 1 - \varepsilon$. ■

**Proof of Proposition 4.5.** We first show that the probability weighted moments based on the bootstrapped sample, $\beta_q^*$ have the following asymptotic expansions: as $n \to \infty$, for $q = 0, 1, 2$

$$\begin{aligned}
&\sqrt{k}\left(\frac{(q+1)\beta_q^* - \tilde{b}_0(m)}{a_0(m)} - \frac{(q+1)^\gamma \Gamma(1-\gamma) - 1}{\gamma}\right) \\
=&L_q\left(\frac{B_k^*(s)}{s(-\log s)^{1+\gamma}}\right) + L_q\left(\frac{W_n(-\log s)}{(-\log s)^{1+\gamma}}\right) + o_P(1), \qquad \text{(A.13)}
\end{aligned}$$

where the functions $\tilde{b}_0(m)$ and $a_0(m)$, the Brownian bridges $\{B_k^*\}$ and the Brownian motions $\{W_n\}$ are the same as in Theorem 3.1, and the operators $L_q$ are the same as in Proposition 4.4.

The proof of (A.13) follows the same lines as in the proof of Theorem 2.2 in Ferreira and de Haan (2015) while changing the shift and scale to $\tilde{b}_0(m)$ and $a_0(m)$. In addition, we need to check the following integrals at the two corners close to 0 and 1 (analogous to (10) and (11) in Ferreira and de Haan (2015)): as $n \to \infty$,

$$\sqrt{k}(q+1)\int_0^{1/(k+1)}\left(\frac{X^*_{\lceil ks\rceil,k} - \tilde{b}_0(m)}{a_0(m)} - \frac{(-\log s)^{-\gamma} - 1}{\gamma}\right)s^q ds = o_P(1), \qquad \text{(A.14)}$$

$$\sqrt{k}(q+1)\int_{k/(k+1)}^{1}\left(\frac{X^*_{\lceil ks\rceil,k} - \tilde{b}_0(m)}{a_0(m)} - \frac{(-\log s)^{-\gamma} - 1}{\gamma}\right)ds = o_P(1). \qquad \text{(A.15)}$$

First, we show (A.14). For $q = 0, 1, 2$ and $\gamma < 1/2$, as $n \to \infty$, we have that

$$\sqrt{k}(q+1)\int_0^{1/(k+1)}\frac{(-\log s)^{-\gamma} - 1}{\gamma}s^q ds = o_P(1).$$

Thus, we only need to show that as $n \to \infty$,

$$\sqrt{k}(q+1)\int_0^{1/(k+1)}\frac{X^*_{\lceil ks\rceil,k} - \tilde{b}_0(m)}{a_0(m)}s^q ds = o_P(1)$$

Notice that for all $s \in (0, 1/(k+1)]$, $X^*_{\lceil ks\rceil,k} = X^*_{1,k} = \tilde{X}_{n-[kD_n(1/k)],n}$. Write $\tilde{X}_i = U(Y_i)$, where $Y_1, Y_2, \ldots, Y_n$ are i.i.d. Pareto distributed random variables. Then we get that

$\tilde{X}_{n-[kD_n(1/k)],n} = U(Y_{n-[kD_n(1/k)],n})$. Lemma 3.4 implies that $D_n(1/k) \overset{P}{\asymp} \log k$ as $n \to \infty$. Thus $Y_{n-[kD_n(1/k)],n} \overset{P}{\to} \infty$ as $n \to \infty$. Hence, we can apply inequality (2.3.19) in de Haan and Ferreira (2006) with $tx = Y_{n-[kD_n(1/k)],n}$ an $t = m = n/k$ to obtain the upper (and lower bound) of $\frac{X^*_{\lceil ks \rceil,k} - \tilde{b}_0(m)}{a_0(m)}$ as

$$\frac{\left(\frac{k}{n} Y_{n-[kD_n(1/k)],n}\right)^\gamma - 1}{\gamma} + A_0\left(\frac{n}{k}\right) \bar{\Psi}_{\gamma,\rho}\left(\frac{k}{n} Y_{n-[kD_n(1/k)],n}\right)$$
$$\pm\, \varepsilon \left| A_0\left(\frac{n}{k}\right)\right| \max\left(\left(\frac{k}{n} Y_{n-[kD_n(1/k)],n}\right)^{\gamma+\rho+\delta}, \left(\frac{k}{n} Y_{n-[kD_n(1/k)],n}\right)^{\gamma+\rho-\delta}\right),$$

where $\bar{\Psi}$ function is given in the proof of Proposition 3.2. We can verify that both upper and lower bounds are at the level $o(\sqrt{k})$ as $n \to \infty$ by noting that $Y_{n-[kD_n(1/k)],n} \overset{P}{\asymp} \frac{n}{k \log k}$. Consequently, (A.14) is proved.

Next, we can verify (A.15) in a similar way by noting that for all $s \in [k/(k+1), 1]$, $X^*_{\lceil ks \rceil,k} = X^*_{k,k} = \tilde{X}_{n-[kD_n(1-1/k)],n}$, $D_n(1-1/k) \overset{P}{\asymp} 1/k$ as $n \to \infty$, and $\gamma < 1/2$.

With verifying (A.14) and (A.15), we conclude that (A.13) holds. The rest of the proof regarding the asymptotic expansion of the bootstrap PWM estimator follows the same lines as in the proof of Theorem 2.3 in Ferreira and de Haan (2015). ∎

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