Kill a Bird with Two Stones: Closing the Convergence Gaps in Non-Strongly Convex Optimization by Directly Accelerated SVRG with Double Compensation and Snapshots

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Abstract

Recently, some accelerated stochastic variance reduction algorithms such as Katyusha and ASVRG-ADMM achieve faster convergence than non-accelerated methods such as SVRG and SVRG-ADMM. However, there are still some gaps between the oracle complexities and their lower bounds. To fill in these gaps, this paper proposes a novel Directly Accelerated stochastic Variance reductIon algorithm with two Snapshots (DAVIS) for non-strongly convex (non-SC) unconstrained problems. Our theoretical results show that DAVIS achieves the optimal convergence rate $\mathcal{O}(1/(nS^2))$ and optimal gradient complexity $\mathcal{O}(n+\sqrt{nL/\epsilon})$, which is identical to its lower bound. To the best of our knowledge, this is the first directly accelerated algorithm that attains the lower bound and improves the convergence rate from $\mathcal{O}(1/S^2)$ to $\mathcal{O}(1/(nS^2))$. Moreover, we extend DAVIS and theoretical results to non-SC problems with an equality constraint, and prove that the proposed DAVIS-ADMM algorithm with double snapshots for each variable also attains the optimal convergence rate $\mathcal{O}(1/(nS))$ and optimal oracle complexity $\mathcal{O}\big(n + L/\epsilon\big)$ for such problems, and it is at least by a factor n/S faster than existing accelerated stochastic algorithms, where $n \gg S$ in general.

1. Introduction

Consider the following finite-sum composite convex minimization problem

$$\min_{x\in\mathbb{R}^d} \Big\{F(x) \triangleq \frac{1}{n}\sum\nolimits_{i=1}^{n} f_i(x) + h(x)\Big\}, \tag{1}$$

where each $f_i(\cdot)$ is convex, and $h(\cdot)$ is convex but possibly non-smooth. We define $f(x)=\frac{1}{n}\sum_{i=1}^{n}f_i(x)$. This problem arises frequently in machine learning, signal processing, statistics, and operations research (Bubeck, 2015), such as regularized empirical risk minimization. In many real-world applications, the number of component functions (i.e., n) is usually very large so that even first-order methods become computationally burdensome due to high per-iteration complexity $O(nd)$. Stochastic gradient descent (SGD) (Robbins & Monro, 1951) uses the gradient of only one (or a small batch of) randomly chosen f_i to estimate full gradient in each iteration, and enjoys a significantly lower cost $O(d)$.

In recent years, stochastic (or incremental) variance reduction methods have received extensive attention due to their low per-iteration cost and ability to handle large-scale problems. In particular, research on variance reduction methods (e.g., SAG (Roux et al., 2012), SDCA (Shalev-Shwartz & Zhang, 2013), SVRG (Johnson & Zhang, 2013), SAGA (Defazio et al., 2014), and their proximal variants, e.g., Prox-SVRG (Xiao & Zhang, 2014)), and stochastic variance reduced algorithms of the alternating direction method of multipliers (ADMM) (e.g., SAG-ADMM (Zhong & Kwok, 2014), SDCA-ADMM (Suzuki, 2014) and SVRG-ADMM (Zheng & Kwok, 2016)) have made exciting progress, e.g., linear convergence for strongly convex (SC) problems.

For solving the SC problem (1), the oracle complexity (i.e., the number of Incremental First-order Oracle calls and Proximal Oracle calls needed to find an ϵ-suboptimal solution) of the stochastic variance reduction methods mentioned above is $\mathcal{O}((n+\kappa)\log(1/\epsilon))$, while the complexity of accelerated deterministic methods including AGD (Nesterov, 1983) and APG (Beck & Teboulle, 2009) is $\mathcal{O}(n\sqrt{\kappa}\log(1/\epsilon))$, where κ is the condition number. Obviously, the complexities show that the variance reduction methods always converge faster than accelerated batch methods as long as $\kappa\leq\mathcal{O}(n^2)$. For non-strongly convex (non-SC) problems, they seem to yield slower convergence rates, e.g., $\mathcal{O}(1/S)$ for SVRG vs. $\mathcal{O}(1/S^2)$ for AGD and APG, where S is the length of the outer-loop or the number of iterations.

The momentum acceleration techniques for deterministic

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Table 1. Comparison of oracle complexities (i.e., the number of first-order oracle calls and proximal oracle calls (Lan, 2020; Xie et al., 2020)) and convergence rates of some stochastic methods for non-SC problems, where $S_0 := \lfloor \log_2(n) \rfloor + 1$. Note that we regard using reductions or proximal point variants as "Indirect" acceleration, such as Catalyst and Katyusha with reduction techniques.

Algorithms	SAGA (Defazio et al., 2014) SVRG (Johnson & Zhang, 2013)	Catalyst (Lin et al., 2015a)	Katyusha$^{\text{ns}}$ (Allen-Zhu, 2018)	Katyusha (Allen-Zhu, 2018)
Convergence rates	$\mathcal{O}\left(\frac{1}{S}\right)$	$\mathcal{O}\left(\frac{\log^4(nS)}{nS^2}\right)$	$\mathcal{O}\left(\frac{1}{S^2}\right)$	NA
Oracle complexities	$\mathcal{O}\left(\frac{n}{\epsilon}+\frac{L}{\epsilon}\right)$	$\mathcal{O}\left(\left(n+\sqrt{\frac{nL}{\epsilon}}\right)\log^2\left(\frac{1}{\epsilon}\right)\right)$	$\mathcal{O}\left(\frac{n}{\sqrt{\epsilon}}+\sqrt{\frac{nL}{\epsilon}}\right)$	$\mathcal{O}\left(n\log\left(\frac{1}{\epsilon}\right)+\sqrt{\frac{nL}{\epsilon}}\right)$
Direct	Yes	No	Yes	No
Algorithms	Varag (Lan et al., 2019)	VRADA (Song et al., 2020)	**DAVIS** **This paper**	Lower Bound (Woodworth & Srebro, 2016)
Convergence rates	$\mathcal{O}\left(\frac{1}{n(S-S_0+4)^2}\right)$	NA	$\mathcal{O}\left(\frac{1}{nS^2}\right)$	$\mathcal{O}\left(\frac{1}{nS^2}\right)$
Oracle complexities	$\mathcal{O}\left(n\log_2(n)+\sqrt{\frac{nL}{\epsilon}}\right)$	$\mathcal{O}\left(n\log_2\log_2(n)+\sqrt{\frac{nL}{\epsilon}}\right)$	$\mathcal{O}\left(n+\sqrt{\frac{nL}{\epsilon}}\right)$	$\mathcal{O}\left(n+\sqrt{\frac{nL}{\epsilon}}\right)$
Direct	Yes	Yes	Yes	–

optimization have been widely researched, e.g., the heavy-ball method (Polyak, 1964), Nesterov's accelerated gradient methods (Nesterov, 1983; 2013) and the optimized gradient method (Kim & Fessler, 2016). Recently, there has been a surge in interest in accelerating stochastic variance reduced methods such as (Frostig et al., 2015; Lin et al., 2015a; Mahdavi et al., 2013; Nitanda, 2014; Allen-Zhu, 2018; Murata & Suzuki, 2017; Hien et al., 2019). Lin et al. (2015a) presented an indirect acceleration (Catalyst) framework, which achieves the complexity of $\mathcal{O}((n+\sqrt{nL/\epsilon})\log^2(1/\epsilon))$ for non-SC problems, where L is a Lipschitz constant. Here, the methods via dummy regularization or reductions are regarded as indirect ones. As the direct acceleration of SVRG, Katyusha (Allen-Zhu, 2018) introduced the idea of negative momentum (i.e., Katyusha momentum). By combining Katyusha momentum with Nesterov's momentum, Katyusha achieves the complexity $\mathcal{O}((n+\sqrt{n\kappa})\log(1/\epsilon))$ for SC problems, which matches the complexity lower bound for minimizing convex finite-sum functions, proved by Lan & Zhou (2018b). Besides, several accelerated methods were proposed, e.g., APCG (Lin et al., 2015b), SDPC (Zhang & Xiao, 2015), Point-SAGA (Defazio, 2016) and RPDG (Lan & Zhou, 2018a). In particular, Allen-Zhu (2018) also proved that Katyusha directly (i.e., Katyusha$^{\text{ns}}$) attains the complexity $\mathcal{O}(n/\sqrt{\epsilon}+\sqrt{nL/\epsilon})$ for non-SC problems. Although by using reduction techniques, Katyusha obtains an improved complexity $\mathcal{O}(n\log(1/\epsilon)+\sqrt{nL/\epsilon})$, which is still worse than the optimal oracle bound in (Woodworth & Srebro, 2016), i.e., $\mathcal{O}(n+\sqrt{nL/\epsilon})$. More recently, Lan et al. (2019) proposed a directly accelerated (Varag) method, which obtains the complexity of $\mathcal{O}(n\log_2(n)+\sqrt{nL/\epsilon})$ for non-SC problems. However, similar to Varag, VRADA (Song et al., 2020) has an extra $\log_2$ factor compared with the complexity lower bound, $\Omega(n+\sqrt{nL/\epsilon})$. It is then natural to ask whether there exists a directly accelerated stochastic method that can attain the optimal oracle complexity.

This paper also considers the minimization problem (1) with a structured regularizer $h(Ax)$, such as graph-guided fused Lasso (Kim et al., 2009), where $A\in\mathbb{R}^{d_1\times d}$ is a given matrix. As the generalization of Problem (1), such problems can be formulated as the equality-constrained finite-sum problem,

$$\min_{x\in\mathbb{R}^d, w\in\mathbb{R}^{d_1}} \left\{F(x) := f(x)+h(w), \text{ s.t., } Ax = w\right\}, \quad (2)$$

where $A\in\mathbb{R}^{d_1\times d}$. In fact, the algorithm proposed in this paper and its convergence result can be extended to the more general problem (2) with the constraint $Ax+Bw=c$, where $A\in\mathbb{R}^{d_2\times d}$, $B\in\mathbb{R}^{d_2\times d_1}$, $c\in\mathbb{R}^{d_2}$. For the SC and equality-constrained problem (2), Suzuki (2014) and Zheng & Kwok (2016) proved that their variance reduction stochastic ADMM methods attain linear convergence for the special (i.e., the constraint in (2) is $Ax=w$) and general ADMM forms (i.e., the constraint in (2) becomes $Ax+Bw=c$), respectively. In SAG-ADMM and SVRG-ADMM, the convergence rate $\mathcal{O}(1/S)$ can be guaranteed for non-SC problems, which implies that there remains a gap in the convergence rates of between the stochastic ADMM and accelerated batch algorithms, i.e., $\mathcal{O}(1/S)$ vs. $\mathcal{O}(1/S^2)$.

For the equality-constrained composite convex problem (2), Xu et al. (2017) proposed a faster variant of SVRG-ADMM with an adaptive penalty parameter scheme. Liu et al. (2020) presented a momentum accelerated variant of SVRG-ADMM (called ASVRG-ADMM), and Li & Lin (2017) proposed an accelerated stochastic ADMM for solving a four-composite minimization problem. However, there also exist some similar gaps between the convergence rates, as well as the oracle complexities, of the existing methods and the optimal convergence rate.

Motivations: (**I**) For solving the non-SC problem (1) (e.g., ℓ_1-norm regularized problems), Katyusha (Allen-Zhu, 2018) and Varag (Lan et al., 2019) attain the oracle complexities of $\mathcal{O}(n/\sqrt{\epsilon}+\sqrt{nL/\epsilon})$ and $\mathcal{O}(n\log_2(n)+\sqrt{nL/\epsilon})$, respec-

Table 2. Comparison of convergence rates and oracle complexities of the stochastic ADMM methods for solving Problem (2), where those of ASVRG-ADMM are obtained with a boundedness assumption on the constraint sets of primal and dual variables (see Section 4.3 for details). Note that we can easily achieve the lower bounds for Problem (2) by using (Xie et al., 2020).

Algorithms	SAGA-ADMM (Zhong & Kwok, 2014)	SVRG-ADMM (Zheng & Kwok, 2016)	ASVRG-ADMM (Liu et al., 2020)	**DAVIS-ADMM** **This paper**	Lower Bound (Xie et al., 2020)
Convergence rates	$\mathcal{O}(\frac{1}{S})$	$\mathcal{O}(\frac{1}{S})$	$\mathcal{O}(\frac{1}{S^2})$	$\mathcal{O}(\frac{1}{nS})$	$\mathcal{O}(\frac{1}{nS})$
Oracle complexities	$\mathcal{O}\Big(\frac{n}{\epsilon}+\frac{L}{\epsilon}\Big)$	$\mathcal{O}\Big(\frac{n}{\epsilon}+\frac{L}{\epsilon}\Big)$	$\mathcal{O}\Big(\frac{n}{\sqrt{\epsilon}}+\sqrt{\frac{nL}{\epsilon}}\Big)$	$\mathcal{O}\Big(n+\frac{L}{\epsilon}\Big)$	$\mathcal{O}\Big(n+\frac{L}{\epsilon}\Big)$
Boundedness assumption	No	No	Yes	No	–

tively. However, the lower bound of the oracle complexity is $\mathcal{O}(n+\sqrt{nL/\epsilon})$ (Woodworth & Srebro, 2016). That is, there are some gaps between the convergence results in (Allen-Zhu, 2018; Lan et al., 2019) and the lower bound[1].

(**II**) Although by adding a SC proximal term into non-SC problems as in (Frostig et al., 2015; Lin et al., 2015a; Allen-Zhu, 2018), one can achieve faster convergence, this may hurt the performance of the algorithms in both theory and practice (Allen-Zhu & Yuan, 2016). The difficulty for the indirect methods is that it is really hard to choose the proximal parameter properly. **Can we design a simple algorithm for Problem (1) to close the gap in theory?**

(**III**) Moreover, for solving the non-SC structure-regularized problem (2), Table 2 shows that there is a big gap of convergence rates between prior works and the lower bound in (Xie et al., 2020). **Can we obtain the optimal convergence rate in both theory and practice?**

Our Main Contributions: To fill in the gaps, we propose a novel directly accelerated stochastic variance reduced gradient (DAVIS) method, which has two snapshots and new momentum accelerated rules with a new compensated stochastic gradient operator. We prove that DAVIS obtains an optimal convergence rate, $\mathcal{O}(1/(nS^2))$. Moreover, we prove that the oracle complexity of DAVIS is $\mathcal{O}(n+\sqrt{nL/\epsilon})$, which is identical to the lower bound in (Woodworth & Srebro, 2016). That is, our oracle complexity is by a factor $1/\sqrt{\epsilon}$ lower than Katyusha, and by a factor $\log_2(n)$ better than Varag, as shown in Table 1. To the best of our knowledge, this is the first directly accelerated method that attains the optimal complexity bound for the non-SC problem (1).

To answer the above-mentioned question, we also extend DAVIS to solve Problem (2) with a structured regularizer. For important emerging equality-constrained non-SC problems (e.g., graph-guided fused Lasso (Kim et al., 2009)), the best-known convergence rate of existing accelerated stochastic ADMM methods such as (Liu et al., 2020) is $\mathcal{O}(1/S^2)$ in the case with an assumption of boundedness on the constraint sets of primal and dual variables (see details in Section 4.3) or $\mathcal{O}(1/S)$ in the case without the assumption of boundedness. Our second main result is that we propose a directly accelerated stochastic ADMM (DAVIS-ADMM) algorithm, and prove that it attains the optimal rate of $\mathcal{O}(1/(nS))$ without the assumption of boundedness, as shown in Table 2. Moreover, DAVIS-ADMM attains the optimal gradient complexity $\mathcal{O}\big(n+L/\epsilon\big)$, which matches the lower bound in (Xie et al., 2020). To the best of our knowledge, this is the first method that obtains the optimal theoretical result for stochastic ADMMs.

[1] A work by Li (Li, 2021) appeared on arXiv, and can match the lower bound in (Woodworth & Srebro, 2016) for non-SC problems for a very wide range of ϵ.

2. Related Work

Throughout this paper, $\|\cdot\|$ denotes the Euclidean norm. We mostly focus on two types of problems (1) and (2), where each component function is L-smooth.

Assumption 1 (Smoothness)**.** *Each component function $f_i(\cdot)$ is L-smooth, i.e., for all $x, y \in \mathbb{R}^d$,*

$$\|\nabla f_i(x) - \nabla f_i(y)\| \le L\|x-y\|.$$

2.1. Stochastic Variance Reduction Methods for Unconstrained Optimization

In recent years, stochastic variance reduction methods such as (Roux et al., 2012; Shalev-Shwartz & Zhang, 2013; Johnson & Zhang, 2013; Defazio et al., 2014; Xiao & Zhang, 2014) have received extensive attention. Variance reduced gradient estimators such as SVRG (Johnson & Zhang, 2013) and SAGA (Defazio et al., 2014) use gradient information from previous iterates to construct a better estimate of the gradient at the current iterate x_k. In particular, the popular SVRG estimator (i.e., $\widetilde{\nabla} f_{i_k}(x_k)$) in (Johnson & Zhang, 2013; Zhang et al., 2013) uses the gradient at the snapshot $\widetilde{x}$ to progressively reduce the variance of the SGD estimator, $\nabla f_{i_k}(x_k)$, where i_k is chosen uniformly at random from $\{1, 2, \ldots, n\}$. Therefore, the SVRG estimator has been widely used in most stochastic variance reduced methods including Katyusha (Allen-Zhu, 2018) and Varag (Lan et al., 2019). More recently, there is a surge of interest in accelerating stochastic variance reduced methods such as Katyusha and Varag for solving unconstrained optimization problems.

By using negative momentum and both proximal descent and mirror descent steps, Katyusha (Allen-Zhu, 2018) can obtain improved convergence rates. In contrast, Varag (Lan et al., 2019) only requires the solution of one sub-problem instead of the two for Katyusha. More recently, VRADA (Song et al., 2020) can match the lower bound for non-SC problems up to a $\log_2\log_2(n)$ factor.

2.2. Stochastic Optimization for Structured Regularization Problems

To solve the equality-constrained problem (2), the alternating direction method of multipliers (ADMM) is an efficient optimization method. However, ADMM and its deterministic variants suffer from a high per iteration computational cost for large-scale problems. Thus, several stochastic ADMMs such as (Wang & Banerjee, 2012; Ouyang et al., 2013) have recently been proposed. The formulation (2) with the constraint $Ax+By=c$ is the general form of the ADMM (Boyd et al., 2011). With the constraint $Ax=w$, Problem (2) is a simpler optimization problem, e.g., generalized Lasso (Tibshirani & Taylor, 2011), which is called the special ADMM form. Together with the dual variable λ, the update steps of stochastic ADMMs are

$$w_k=\arg\min_{w}\Big\{h(w)+\phi_k(x_{k-1},w)\Big\},$$
$$x_k=\arg\min_{x}\Big\{x^T\widetilde{\nabla}f_{i_k}(x_{k-1})+\frac{\|x-x_{k-1}\|_Q^2}{2\eta_k}+\phi_k(x,w_k)\Big\},$$

and $\lambda_k=\lambda_{k-1}+Ax_k-w_k$, where $\eta_k>0$ is a parameter, $\phi_k(x,w)=\frac{\beta}{2}\|Ax-w+\lambda_{k-1}\|^2$, $\beta>0$ is a penalty parameter, and $\|x\|_Q^2=x^TQx$ with a given positive semi-definite matrix Q as in (Ouyang et al., 2013). Another possible solution is primal-dual hybrid gradient methods such as (Zhu & Chan, 2008; Goldstein et al., 2015). Note that we mainly focus on accelerating stochastic ADMM and refrain from discussing dual and primal-dual stochastic algorithms. Some researchers have adopted the variance reduced techniques mentioned above for ADMM, e.g., (Zhong & Kwok, 2014; Suzuki, 2014; Zheng & Kwok, 2016). More recently, some momentum accelerated stochastic ADMM algorithms (Li & Lin, 2017; Xu et al., 2017; Liu et al., 2020) have been proposed for solving Problem (2).

3. A Direct Optimal Stochastic Variance Reduction Algorithm

In this section, we propose a directly accelerated stochastic variance reduction gradient (DAVIS) algorithm for solving the non-SC problem (1). We first present a novel double snapshot acceleration framework for stochastic optimization, in which we need to compute the full gradient at the first snapshot and define a new update rule of the second snapshot in each outer loop. Moreover, we design a new stochastic variance reduction gradient estimator for each inner loop of our accelerated algorithm. Finally, we analyze the convergence properties of DAVIS, which show that it attains the optimal convergence rate $\mathcal{O}(1/(nS^2))$ and the optimal oracle complexity $\mathcal{O}(n+\sqrt{nL/\epsilon})$.

3.1. Main Ideas of DAVIS

As discussed above, there still exist some gaps between the oracle complexities of the directly accelerated algorithms (e.g., Katyusha and Varag) and the lower bound in (Woodworth & Srebro, 2016). Let x^* be an optimal solution of Problem (1) and $\widetilde{x}^0$ be a given starting vector, the convergence result of the directly accelerated version (i.e., Katyusha$^{\text{ns}}$) of Katyusha (Allen-Zhu, 2018) is $\mathcal{O}(\frac{F(\widetilde{x}^0)-F(x^*)}{S^2}+\frac{L\|x^*-\widetilde{x}^0\|^2}{nS^2})$. In order to achieve the optimal convergence rate $\mathcal{O}(1/(nS^2))$, we need to accelerate the rate of the first term from $\mathcal{O}(1/S^2)$ to $\mathcal{O}(1/(nS^2))$, and thus Algorithm 1 mainly includes a new extra snapshot and its update rule in each outer loop, one new stochastic gradient estimator and new momentum acceleration rules in each inner loop. Therefore, our accelerated algorithm attains the optimal convergence rate $\mathcal{O}(1/(nS^2))$ without restarting and without using any reduction techniques.

As most stochastic variance reduction methods including Katyusha (Allen-Zhu, 2018) and Varag (Lan et al., 2019), each epoch of our algorithms including Algorithm 1 consists of m inner-iterations, e.g., $m=2n$ as suggested in (Johnson & Zhang, 2013). In each outer loop of the proposed algorithm, we need to compute the full gradient at the first snapshot point, and design a new update rule for the second snapshot point. In each inner loop of our algorithm, we define a new compensated stochastic variance reduction gradient estimator, and then present a new momentum acceleration scheme. More details are given below.

3.2. New Scheme of Double Snapshots in Outer Loop

In the s-th outer loop of Algorithm 1, we design two snapshot points $\widetilde{x}^{s-1}$ and $\overline{x}^{s-1}$, both of which remain unchanged in all the inner loops inside the same outer loop. The first snapshot point $\widetilde{x}^{s-1}$ takes the same role as in most variance reduction methods such as SVRG and Katyusha. That is, we need to compute the full gradient of $f(\cdot)$ at $\widetilde{x}^{s-1}$ in each outer loop, which is used to gradually reduce the variance of the SVRG estimator. Moreover, we design a novel update rule for the second snapshot point $\overline{x}^{s-1}$ as follows:

$$\overline{x}^{s-1}=\theta_s\overline{z}^{s-1}+(1-\theta_s)\widetilde{x}^{s-1}, \tag{3}$$

where θ_s is a parameter (e.g., $\theta_s=\frac{2}{s+1}$), and the auxiliary variable z^{s-1} is obtained by solving the following problem:

$$\overline{z}^{s-1}=\operatorname*{argmin}_{z}\Big\{h(z)+\big\langle\nabla f(\widetilde{x}^{s-1}),z\big\rangle+\frac{\theta_s}{2m\eta}\|z-\widetilde{x}^{s-1}\|^2\Big\}.$$

Algorithm 1 DAVIS

Input: The number S of epochs, the number m of iterations per epoch.

Initialize: $\widetilde{x}^0$, $v_0^1 = z_0^1 = 0$, $\theta_1 = 1$, and η.

1: **for** $s = 1, 2, \ldots, S$ **do**
2: Compute the full gradient at the first snapshot point (i.e., $\widetilde{x}^{s-1}$), $\nabla f(\widetilde{x}^{s-1}) = \frac{1}{n}\sum_{i=1}^{n} \nabla f_i(\widetilde{x}^{s-1})$;
3: $\overline{z}^{s-1} = \mathrm{prox}_h^{\frac{\eta}{\theta_s}}(\widetilde{x}^{s-1} - \frac{m\eta}{\theta_s}\nabla f(\widetilde{x}^{s-1}))$;
4: $\overline{x}^{s-1} = \theta_s \overline{z}^{s-1} + (1-\theta_s)\widetilde{x}^{s-1}$; // *The second snapshot*
5: **for** $k = 1, 2, \ldots, m$ **do**
6: Update y_k^s via (4);
7: Pick i_k uniformly at random from $\{1, 2, \ldots, n\}$;
8: $\widetilde{\nabla}_{i_k}(y_k^s) = \nabla f_{i_k}(y_k^s) - \nabla f_{i_k}(\widetilde{x}^{s-1}) + \nabla f(\widetilde{x}^{s-1}) + \frac{m\theta_s}{\eta}(\overline{z}^{s-1} - \widetilde{x}^{s-1})$;
9: $z_k^s = \text{Prox-SGrad}(y_k^s)$;
10: $x_k^s = \frac{\theta_s}{m}(z_k^s - p_k^s) + y_k^s$;
11: **end for**
12: $\widetilde{x}^s = \frac{1}{m}\sum_{k=1}^{m} x_k^s$, $\theta_{s+1} = \frac{2}{s+2}$, $z_0^{s+1} = z_m^s$, $v_0^{s+1} = v_m^s$;
13: **end for**
14: Output: $\widetilde{x}^S$.

Here, η is a learning rate. Clearly, $\overline{z}^{s-1}$ is obtained by performing one deterministic gradient descent step from the snapshot point $\widetilde{x}^{s-1}$, which requires no gradient calculations. For the second snapshot point $\overline{x}^{s-1}$, we give its upper bound in Lemma 1 in Section 3.4.

3.3. New Stochastic Update Schemes in Inner Loop

In this subsection, we first define a new compensated s-tochastic variance reduction gradient estimator for the proposed algorithm, and then we design a new momentum acceleration update rule.

3.3.1. Compensated Stochastic Gradient Estimator

Before giving our new stochastic momentum acceleration scheme for our algorithm, we first define a new compensated gradient estimator.

Definition 1 (Compensated stochastic gradient estimator)**.** *We define a new compensated stochastic variance reduction gradient estimator for our DAVIS algorithm as follows:*

$$\widetilde{\nabla}_{i_k}(x) = \underbrace{\nabla f_{i_k}(x) - \nabla f_{i_k}(\widetilde{x}^{s-1}) + \nabla f(\widetilde{x}^{s-1})}_{\text{SVRG estimator}} + \underbrace{m\theta_s(\overline{z}^{s-1} - \widetilde{x}^{s-1})/\eta}_{\text{Compensated estimator}}.$$

It is clear that our stochastic variance reduction gradient estimator consists of two terms, i.e., the SVRG estimator independently proposed in (Johnson & Zhang, 2013; Zhang et al., 2013) and a new compensated estimator. Note that the new compensated term is introduced into the proposed gradient estimator $\widetilde{\nabla}_{i_k}(x)$, and plays a key role to offset the residual term in the upper bound of Lemma 1 (see the discussion in Section 3.4 for details).

3.3.2. Momentum Acceleration

We first define a new proximal stochastic gradient decent scheme for our algorithm.

Definition 2 (Prox-SGrad)**.** *The proximal stochastic gradient decent (*Prox-SGrad*) is defined as:*

$$\begin{aligned}&\text{Prox-SGrad}(x)\\ \triangleq\ &\operatorname*{argmin}_z \left\{h(z) + \langle\widetilde{\nabla}_{i_k}(x), z\rangle + \frac{m\theta_s}{2\eta}\|z - r_k^s\|^2\right\}\\ =\ &\mathrm{prox}_h^{\frac{\eta}{m\theta_s}}\big(r_k^s - \eta m\theta_s\widetilde{\nabla}_{i_k}(x)\big).\end{aligned}$$

where $r_k^s = p_k^s - 2(z^{s-1} - x^{s-1})$, and the key auxiliary variable p_k^s is defined below, and $\mathrm{prox}_h^{\frac{\eta}{m\theta_s}}(\cdot)$ is the standard proximal operator as in (Xiao & Zhang, 2014; Allen-Zhu, 2018) (see the Supplementary Material for some details and discussions about the coefficient of $\widetilde{\nabla}_{i_k}(x)$).

Next we design a new momentum acceleration scheme in each inner loop, and first give a new update rule for y_k^s at the k-th iteration of the s-th epoch:

$$y_k^s = \frac{\theta_s}{m}(z_k^s - p_k^s) + \frac{m - \theta_s - 1}{m}\overline{x}^{s-1} + \frac{1}{m}\widetilde{x}^{s-1}. \quad (4)$$

Here $p_k^s = 2(\overline{z}^{s-1} - \widetilde{x}^{s-1}) + \frac{3}{4}z_{k-1}^s + \frac{1}{4}p_{k-1}^s$ is designed to get a well-structured recursive form (i.e., $\|x^* - r_{k-1}^s\|^2 - \|x^* - r_k^s\|^2$) in our upper bound of one-iteration in Lemma 2 below. Moreover, we define the following momentum acceleration update rule for x_k^s:

$$x_k^s = \frac{\theta_s}{m}(z_k^s - p_k^s) + y_k^s, \quad (5)$$

where $z_k^s = \text{Prox-SGrad}(y_k^s)$. We give the following upper bound for our stochastic updates in one iteration (e.g., the k-th iteration) of Algorithm 1.

3.4. Optimal Convergence Guarantees

We analyze the convergence property of DAVIS. Theorem 1 below shows that DAVIS can improve the best-known convergence rate of some accelerated methods (e.g., Katyusha) from $\mathcal{O}(1/S^2)$ to $\mathcal{O}(1/(nS^2))$ for the non-SC problem (1). We also suppose that the distance between an initial point $\widetilde{x}^0$ and an optimal solution x^* can be bounded by a constant c, i.e., $\|x^* - \widetilde{x}^0\| \leq c$, which is a basic condition.

3.4.1. Core Lemmas

The proof of our main result relies on the one-iteration inequality in Lemma 3 below, which is a key lemma to obtain our theoretical result in Theorem 1. Lemma 3 consists of two key upper bounds in Lemmas 1 and 2. By using our double snapshot scheme in each outer loop of Algorithm 1, we can obtain the following result.

Lemma 1 (Upper bound of double snapshot update)**.** *Suppose that Assumption 1 holds. Let $\{\overline{x}^s\}$ be the sequence generated by our double snapshot scheme in Algorithm 1, for a given v_k^s, we have*

$$F(\overline{x}^{s-1}) - F(x^*) \leq (1-\theta_s)(F(\widetilde{x}^{s-1}) - F(x^*)) + \mathcal{R}^s + \frac{\theta_s^2}{2m\eta}\left(\left\|x^* - \widetilde{x}^{s-1}\right\|^2 - \left\|x^* - \overline{z}^{s-1}\right\|^2\right),$$

where $\mathcal{R}^s = \left(\frac{\theta_s^2}{2\eta} - \frac{\theta_s^2}{2m\eta}\right)\|\overline{z}^{s-1} - \widetilde{x}^{s-1}\|^2$.

Note that the upper bound in Lemma 1 has an additional term, which may be positive but we shall cancel it by using the upper bound of our stochastic update schemes in Lemma 2 below. Using our stochastic momentum accelerated scheme in each inner loop, we give the following result.

Lemma 2 (**Upper bound of one-iteration**)**.** *Suppose that Assumption 1 holds. Let $\{x_k^s, z_k^s\}$ be the sequence generated by Algorithm 1. Let $r_k^s = p_k^s + 2(\overline{z}^{s-1} - \overline{x}^{s-1})$, then*

$$\mathbb{E}[F(x_k^s) - F(x^*)] \leq \frac{\theta_s^2}{2\eta}\left(\left\|x^* - r_k^s\right\|^2 - \|x^* - r_{k+1}^s\|^2\right) + \mathcal{C}^s + \mathbb{E}\Big[\Big(1 - \frac{\theta_s + 1}{m}\Big)\big(F(\overline{x}^{s-1}) - F(x^*)\big)\Big] + \frac{1}{m}\mathbb{E}\big[F(\widetilde{x}^{s-1}) - F(x^*)\big],$$

where $\mathcal{C}^s = \left(\frac{\theta_s^2}{2m\eta} - \frac{\theta_s^2}{2\eta}\right)\|\overline{z}^{s-1} - \widetilde{x}^{s-1}\|^2$.

Remark 1. In Lemma 2, the term $\mathcal{C}_k^s$ is produced by our gradient estimator, and is used to compensate the additional term $\mathcal{R}^s$ in Lemma 1 (i.e., $\mathcal{R}^s + \mathcal{C}^s = 0$). As we expected, the designed stochastic descent step can be used to counteract the additional term, as shown in the detailed proof for Lemma 3 below (see the Supplementary Material for our proof sketch and their detailed proofs).

Using Lemmas 1 and 2, we give the following upper bound of one iteration in Algorithm 1.

Lemma 3 (**Upper bound of one-epoch**)**.** *Suppose that Assumption 1 holds. Let $\{x_k^s\}$ be the sequence generated by Algorithm 1. Then we have*

$$\mathbb{E}[F(\widetilde{x}^s) - F(x^*)] \leq \left(1 - \theta_s + \frac{1}{m}\right)\left(F(\widetilde{x}^{s-1}) - F(x^*)\right) + \frac{\theta_s^2}{2m\eta}\left(\|x^* - \widetilde{x}^{s-1}\|^2 - \|x^* - \widetilde{x}^s\|^2\right).$$

3.4.2. Optimal Convergence Results

We can obtain the inequality of one-epoch by using Lemma 3, and telescope the inequality over all epochs to obtain the following result.

Theorem 1. *Suppose that each component function $f_i(\cdot)$ is L-smooth. Let $\widetilde{x}^s = \frac{1}{m}\sum_{k=1}^m x_k^s$ (i.e., the average point of the previous epoch), then the following result holds*

$$\mathbb{E}[F(\widetilde{x}^s) - F(x^*)] \leq \mathcal{O}\Big(\frac{F(\widetilde{x}^0) - F(x^*) + L\|x^* - \widetilde{x}^0\|^2}{mS^2}\Big).$$

Choosing $m = \Theta(n)$, Algorithm 1 achieves an ϵ-suboptimal solution using at most $\mathcal{O}(n + \sqrt{nL/\epsilon})$ iterations.

Remark 2. Theorem 1 shows that DAVIS achieves the optimal convergence rate $\mathcal{O}(1/(nS^2))$, while most existing directly accelerated methods including (Allen-Zhu, 2018; Zhou et al., 2018) attain the rate $\mathcal{O}(1/S^2)$. In particular, Algorithm 1 also attains the optimal oracle complexity $\mathcal{O}(n + \sqrt{nL/\epsilon})$, which matches the lower bound in (Woodworth & Srebro, 2016). In contrast, the complexities of Katyusha (Allen-Zhu, 2018), Varag (Lan et al., 2019) and VRADA (Song et al., 2020) are $\mathcal{O}(n/\sqrt{\epsilon} + \sqrt{nL/\epsilon})$, $\mathcal{O}(n\log_2(n) + \sqrt{nL/\epsilon})$ and $\mathcal{O}(n\log_2\log_2(n) + \sqrt{nL/\epsilon})$, respectively. In other words, DAVIS has both the optimal oracle complexity and optimal convergence rate for solving Problem (1). To the best of our knowledge, this is the first time that the optimal complexity bound is obtained through a directly accelerated algorithm for general convex finite-sum optimization in the literature.

3.5. Comparison with Existing Algorithms

There are some main differences between our DAVIS algorithm and existing accelerated stochastic algorithms such as Katyusha (Allen-Zhu, 2018), Varag (Lan et al., 2019), and VRADA (Song et al., 2020).

- One vs two snapshots: Both snapshots (i.e., $\widetilde{x}^s$ and $\overline{x}^s$) are introduced into our update rules in (4) and (5), while only one snapshot $\widetilde{x}^s$ is applied in most algorithms such as Katyusha, Varag and VRADA. Similarly, we use double snapshots for each variable in equality-constrained problems.
- SVRG estimator vs our compensated estimator: Most accelerated algorithms such as Katyusha, Varag and VRADA use the SVRG estimator proposed in (Johnson & Zhang, 2013; Zhang et al., 2013). In contrast, our gradient estimator is introduced to compensate the residual term in Lemma 1, and thus is one of main contributions of this paper.
- Negative momentum vs compensated momentum: Both Varag and Katyusha use the negative momentum proposed in (Allen-Zhu, 2018). By combining with our defined gradient estimator, we also design a new momentum accelerated rule in (5) to achieve an optimal convergence rate. Note that θ_s can be defined as: $\theta_s = \frac{2}{s+1}$ satisfying $\frac{1}{\theta_{s-1}^2} \geq \frac{1-\theta_s}{\theta_s^2}$. Different from Katyusha[ns], the coefficient of our momentum is θ_s/m, which can reduce the impact of the variance bound on the upper bound of one iteration in Lemma 2 from $O(1)$ to $O(1/m)$. Moreover, both Varag and DAVIS only require the solution of one subproblem in each iteration instead of

two subproblems in Katyusha.

4. An Optimal Stochastic ADMM Algorithm for Equality Constrained Optimization

In this section, we propose a novel directly accelerated stochastic variance reduction ADMM (DAVIS-ADMM) algorithm with double-snapshots to solve Problem (2), which improves the best-known convergence rate from $O(1/S^2)$ to $O(1/(nS))$ (see Algorithm 2 for the details of DAVIS-ADMM). We first let $Q_s=\gamma I-\frac{\eta\beta}{\theta_s}A^TA$ with $\gamma\geq\eta\beta\|A^TA\|_2+1$ to ensure that $Q_s\succeq I$, where $\|\cdot\|_2$ is the spectral norm, and $\|C\|_2$ is the spectral norm of a matrix C.

4.1. Double-Snapshot Scheme in DAVIS-ADMM

Existing stochastic variance reduction ADMMs such as SVRG-ADMM (Zheng & Kwok, 2016) and ASVRG-ADMM (Liu et al., 2020) use the three snapshots $\widetilde{x}^s$, $\widetilde{w}^s$ and $\widetilde{\lambda}^s$, while three additional snapshots $\overline{x}^s$, $\overline{w}^s$ and $\overline{\lambda}^s$ are also designed for our DAVIS-ADMM. All the snapshots in our DAVIS-ADMM are updated only in outer loop. More specifically, the update rules of the first three snapshots are defined as: $\widetilde{x}^s=\frac{1}{m}\sum_{k=1}^m x_k^s$, $\widetilde{w}^s=\frac{\theta_s}{m^2}\sum_{k=1}^m w_k^s+\left(1-\frac{2}{m}\right)\overline{w}^{s-1}$, and $\widetilde{\lambda}^s=\frac{1}{m}\sum_{k=1}^m\lambda_k^s$. And the update rules of the three new snapshots are given as follows:

$$\begin{aligned}\overline{x}^{s-1}&=\theta_s\overline{z}^{s-1}+(1-\theta_s)\widetilde{x}^{s-1},\\ \overline{w}^{s-1}&=\theta_s\overline{p}^{s-1}+(1-\theta_s)\widetilde{w}^{s-1},\\ \overline{\lambda}^{s-1}&=A\overline{z}^{s-1}-\overline{p}^{s-1}+\overline{\lambda}^{s-2}.\end{aligned}\tag{6}$$

Let $\phi^s(z,p)=\frac{1}{m}\|Az-p+\overline{\lambda}^{s-2}\|^2$, the auxiliary variables $\overline{z}^{s-1}=\arg\min_z\left\{\left\langle\nabla f(\widetilde{x}^{s-1}),z\right\rangle+\frac{\theta_s}{2m\eta}\|z-\widetilde{x}^{s-1}\|_{Q_s}^2+\phi^s(z,\overline{p}^{s-2})\right\}$, and $\overline{p}^{s-1}=\arg\min_p\left\{h(p)+\frac{\beta}{2}\phi^s(\overline{z}^{s-1},p)\right\}$.

4.2. Stochastic Update Rules in Inner Loop

Like Algorithm 1, DAVIS-ADMM uses the same momentum accelerated rules in (4) and (5) for y_k^s and x_k^s. Unlike existing accelerated algorithms, the weight in DAVIS-ADMM is θ_s/m. Then we can remove the constraint (i.e., $\theta_s\leq1-\frac{L\eta}{1-L\eta}$) for θ_s in (Liu et al., 2020). That is, the initial value is set to $\theta_1=1$ for our DAVIS-ADMM, while that in (Liu et al., 2020) requires to satisfy the condition $\theta_1\leq1-\frac{L\eta}{1-L\eta}$, which is a reason that DAVIS-ADMM improves the rate from $O(1/S)$ to $O(1/nS)$, as shown in Table 2. The update rule of θ_s is $\theta_s=(\sqrt{\theta_{s-1}^4+4\theta_{s-1}^2}-\theta_{s-1}^2)/2$. Let $\phi_k^s(z,w)=\frac{\beta}{2}\|Az-w+\lambda_{k-1}^s\|^2$, the mini-batch compensated gradient estimator $\widehat{\nabla}_{I_k}(x)$ for equality-constrained problems is defined as follows.

Definition 3.

$$\widehat{\nabla}_{I_k}(x)=g_{I_k}(x)+\frac{m\theta_s}{\eta}Q_s(\overline{z}^{s-1}-\widetilde{x}^{s-1}),$$

where $g_{I_k}(x)=\nabla f_{I_k}(x)-\nabla f_{I_k}(\widetilde{x}^{s-1})+\nabla f(\widetilde{x}^{s-1})$, *and* $I_k\subset\{1,2,\ldots,n\}$ *is a randomly chosen mini-batch of size* b. *Compared with the gradient estimator in Definition 1, an additional matrix* Q_s *is introduced into the estimator.*

Algorithm 2 DAVIS-ADMM

Input: S and m.
Initialize: $\widetilde{x}^0$, $\widetilde{w}^0$, $\widetilde{\lambda}^{s-1}$, $\theta_1=1$, and η.
for $s=1,2,\ldots,S$ **do**
  Compute the full gradient at the snapshot $\widetilde{x}^{s-1}$, $\nabla f(\widetilde{x}^{s-1})=\frac{1}{n}\sum_{i=1}^n\nabla f_i(\widetilde{x}^{s-1})$;
  Update the snapshots $\overline{x}^{s-1}$, $\overline{w}^{s-1}$ and $\overline{\lambda}^{s-1}$ via (6);
  for $k=1,2,\ldots,m$ **do**
    Pick i_k uniformly at random from $\{1,2,\ldots,n\}$;
    $\widehat{\nabla}_{I_k}(x)=g_{I_k}(x)+\frac{m\theta_s}{\eta}Q_s(\overline{z}^{s-1}-\widetilde{x}^{s-1})$;
    $w_k^s=\arg\min_w\left\{h(w)+\frac{\beta}{2}\|Az_{k-1}^s-w+\lambda_{k-1}^s\|^2\right\}$;
    $y_k^s=\frac{\theta_s}{m}w_k^s+\frac{m-\theta_s-1}{m}\overline{w}^{s-1}+\frac{1}{m}\widetilde{w}^{s-1}$;
    $z_k^s=\arg\min_z\left\{\langle\widehat{\nabla}_{I_k}(y_k^s),z\rangle+\phi_k^s(z,w_k^s)\right.$
      $\left.+\frac{m\theta_s}{2\eta}\|z-p_k^s-(\overline{z}^{s-1}-\widetilde{x}^{s-1})\|_{Q_s}^2\right\}$;
    $x_k^s=\frac{\theta_s}{m}z_k^s+(1-\frac{\theta_s}{m})\overline{x}^{s-1}$, $\lambda_k^s=Az_k^s-w_k^s+\lambda_{k-1}^s$;
  end for
  $\widetilde{\lambda}^s=\frac{1}{m}\sum_{k=1}^m\lambda_k^s$, $\theta_s=\frac{\sqrt{\theta_{s-1}^4+4\theta_{s-1}^2}-\theta_{s-1}^2}{2}$,
  $\widetilde{x}^s=\frac{1}{m}\sum_{k=1}^m x_k^s$, $\widetilde{w}^s=\frac{\theta_s}{m^2}\sum_{k=1}^m w_k^s+\left(1-\frac{2}{m}\right)\overline{w}^{s-1}$.
end for
Output: $\widetilde{x}^S$, $\widetilde{y}^S$.

4.3. Optimal Convergence Rate for Problem (2)

We analyze the convergence property of DAVIS-ADMM. Let x^* be an optimal solution of Problem (2), w^* and λ^* be the corresponding solutions, we give the convergence criterion (Zheng & Kwok, 2016) for our analysis.

Definition 4 (Convergence criterion). *Given a constant* $\delta\geq0$, *a nonnegative convergence criterion is defined as:* $\phi(\widetilde{x}^S,\widetilde{w}^S)=P(\widetilde{x}^S,\widetilde{w}^S)+\delta\|A\widetilde{x}^S-\widetilde{w}^S\|\geq0$, *where* $P(x,w)=f(x)-f(x^*)-\nabla f(x^*)^T(x-x^*)+h(w)-h(w^*)-\widehat{\nabla}h(w^*)^T(w-w^*)$, *and* $\widehat{\nabla}h(w)$ *is the (sub)gradient of* $h(\cdot)$ *at* w.

Previous work such as (He & Yuan, 2012; Azadi & Sra, 2014) requires the assumption of boundedness on the constraint sets of primal and dual variables (i.e., suppose $x\in\mathcal{X}$ and $\lambda\in\Lambda$, where $\mathcal{X}$ and Λ are compact convex sets with diameters $D_{\mathcal{X}}=\sup_{x_1,x_2\in\mathcal{X}}\|x_1-x_2\|$ and $D_\Lambda=\sup_{\lambda_1,\lambda_2\in\Lambda}\|\lambda_1-\lambda_2\|$, respectively) when proving the convergence of ADMMs, and ASVRG-ADMM obtains the rate $\mathcal{O}(1/S^2)$ with the boundedness assumption. However, we can remove such assumption, which is in fact a strong assumption, and provide the following convergence result.

Theorem 2. *Suppose Assumption 1 holds. Let the constant* $c_1=2\|A^TA\|_2\|x^*-\widetilde{x}^0\|^2+2\|\lambda^*-\widetilde{\lambda}^0\|^2+8\delta^2+10\|\lambda^*\|^2$

and choose $m = \Theta(n)$*, then*

$$\begin{aligned}&\mathbb{E}\big[\phi(\widetilde{x}^S, \widetilde{w}^S)\big]\\ &\leq \mathcal{O}\Big(\frac{2\phi(\widetilde{x}^0, \widetilde{w}^0) + \|x^* - \widetilde{x}^0\|^2_{Q_1}/\eta}{n(S+1)} + \frac{c_1\beta}{n(S+1)}\Big).\end{aligned}$$

Remark 3. Theorem 2 shows that without the boundedness assumption, the convergence rate of DAVIS-ADMM is $\mathcal{O}(1/(nS))$, while the best-known convergence result as in (Liu et al., 2020) with a strong boundedness assumption is $\mathcal{O}(1/S^2)$, as shown in Table 2. That is, DAVIS-ADMM improves the best-known convergence rate from $\mathcal{O}(1/S^2)$ to $\mathcal{O}(1/(nS))$ without the boundedness assumption, which matches the lower bound in (Xie et al., 2020). The upper bound in Theorem 2 only relies on the constant c_1, while the theoretical result of ASVRG-ADMM (Liu et al., 2020) requires that $\mathcal{X}$ and Λ are bounded with the diameters $D_{\mathcal{X}}$ and D_Λ. Moreover, our DAVIS-ADMM algorithm and convergence result can be extended to the deterministic setting. When the mini-batch size is $b = n$, and $m = 1$, DAVIS-ADMM degenerates to its deterministic version and the convergence rate of our deterministic DAVIS-ADMM algorithm becomes $\mathcal{O}(1/S)$, which is consistent with the optimal convergence rate of accelerated deterministic methods such as (Ouyang & Xu, 2020).

5. Experimental Results

In this section, we evaluate the performance of our algorithms for solving the non-SC problems (1) and (2). Detailed experimental setup is given in the Supplementary Material.

5.1. ℓ_1-Norm Regularized Logistic Regression

We first apply Algorithm 1 to solve the ℓ_1-norm regularized logistic regression problem, and compare it with the state-of-the-art accelerated methods such as SVRG++ (Allen-Zhu & Yuan, 2016), Katyusha and Varag (Lan et al., 2019). The experimental results of all the accelerated methods on Adult and Covtype are shown in Figure 1, where the regularization parameter is 10^{-5} (see the Supplementary Material for more results). We observe that our DAVIS consistently outperforms other accelerated methods, which empirically verifies our theoretical result.

5.2. Graph-Guided Fused Lasso

We also evaluate the performance of DAVIS-ADMM for solving the graph-guided fused Lasso problem: $\min_x\Big\{\frac{1}{n}\sum_{i=1}^{n} f_i(x) + \lambda\|Ax\|_1\Big\}$, where $f_i(\cdot)$ is the logistic loss function, $\lambda \geq 0$ is the regularization parameter, $A{=}[G; I]$, and G is the sparsity pattern of the graph obtained by sparse inverse covariance selection as in (Banerjee et al., 2008). Figure 2 shows the experimental results of SVRG-ADMM (Zheng & Kwok, 2016), ASVRG-ADMM (Liu et al., 2020) and DAVIS-ADMM. All the results show that the accelerated methods (i.e., ASVRG-ADMM and DAVIS-ADMM) outperform the non-accelerated stochastic ADMM method, i.e., SVRG-ADMM. In particular, DAVIS-ADMM converges much faster than the other methods, which is consistent with our convergence guarantee.

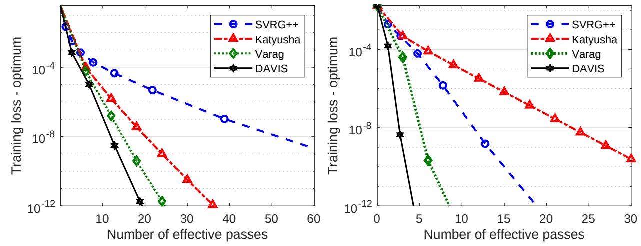

Figure 1. Comparison of all the methods for solving ℓ_1-norm regularized logistic regression problems on Adult and Covtype.

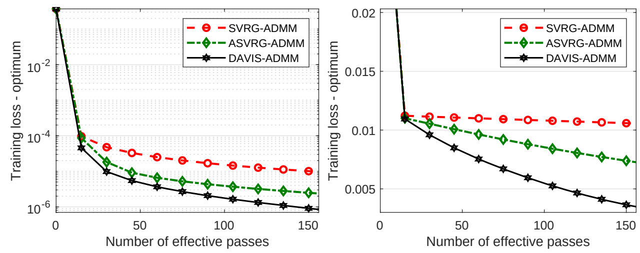

Figure 2. Comparison of all the methods for solving graph-guided fused Lasso problems on Adult and Covtype, where the regularization parameter is $\lambda = 10^{-5}$.

6. Conclusions

In this paper, we proposed two efficient directly optimal stochastic variance reduction algorithms for unconstrained and equality-constrained finite-sum problems, respectively. The proposed algorithms have simple update rules, and thus their per-iterations have similar computational costs as existing accelerated methods (e.g., Katyusha and ASVRG-ADMM). In particular, we theoretically analyzed the convergence properties of our algorithms, and our theoretical results show that our algorithms obtain the optimal convergence rates and optimal oracle complexities for both non-SC unconstrained and equality-constrained problems, respectively. They are identical to the lower bounds provided in (Woodworth & Srebro, 2016; Xie et al., 2020). That is, our algorithms are a factor n faster than both existing accelerated algorithms (e.g., Katyusha), i.e., $\mathcal{O}(1/nS^2)$ vs. $\mathcal{O}(1/S^2)$, and a factor $\frac{n}{S}$ faster than accelerated stochastic ADMM algorithms, i.e., $\mathcal{O}(1/nS)$ vs. $\mathcal{O}(1/S^2)$.

Our directly accelerated algorithms can be easily extended to solve more complex problems, e.g., (Wang et al., 2017). As our future work, our method can allow the applications of non-uniform sampling and non-Euclidean Bregman distance for solving more different types of optimization problems.

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