Operations that preserve convexity

- Composition – Vector composition

$$f(\mathbf{x}) = h(g(\mathbf{x})) = h(g_1(\mathbf{x}), \ldots, g_k(\mathbf{x})),$$

with $h : \mathbb{R}^k \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$. Again without loss of generality we can assume $n = 1$. As in the case $k = 1$, we start by assuming the functions are twice differentiable, with $\operatorname{dom} g = \mathbb{R}$ and $\operatorname{dom} h = \mathbb{R}^k$, in order to discover the composition rules. We have

$$f''(\mathbf{x}) = \mathbf{g}'(\mathbf{x})^T \nabla^2 h(\mathbf{g}(\mathbf{x}))\mathbf{g}'(\mathbf{x}) + \nabla h(\mathbf{g}(\mathbf{x}))^T \mathbf{g}''(\mathbf{x}).$$

f is convex if $\tilde{h}$ is **convex** and **nondecreasing** in each argument, and g_i is **convex**,
f is convex if $\tilde{h}$ is **convex** and **nonincreasing** in each argument, and g_i is **concave**.

Operations that preserve convexity

- Composition – Vector composition

Examples:

- Let $h(\mathbf{z}) = z_{[1]} + \cdots + z_{[r]}$, the sum of the r largest components of $\mathbf{z} \in \mathbb{R}^k$. Then h is convex and nondecreasing in each argument. Suppose $g_1, \ldots, g_k$ are convex functions on $\mathbb{R}^n$. Then the composition function $f = h \circ \mathbf{g}$, *i.e.*, the pointwise sum of the r largest g_i's, is convex.

- The function $h(\mathbf{z}) = \log(\sum_{i=1}^k e^{z_i})$ is convex and nondecreasing in each argument, so $\log(\sum_{i=1}^k e^{g_i(\mathbf{x})})$ is convex whenever g_i are.

- For $0 < p \leq 1$, the function $h(\mathbf{z}) = (\sum_{i=1}^k z_i^p)^{1/p}$ on $\mathbb{R}_+^k$ is concave, and its extension (which has the value $-\infty$ for $\mathbf{z} \not\succeq \mathbf{0}$ is nondecreasing in each component. So if g_i are concave and nonnegative, we conclude that $f(\mathbf{x}) = (\sum_{i=1}^k g_i(\mathbf{x})^p)^{1/p}$ is concave.

Operations that preserve convexity

- Composition – Vector composition

Examples:

- Suppose $p \geq 1$, and $g_1, \ldots, g_k$ are convex and nonnegative. Then the function $f(\mathbf{x}) = (\sum_{i=1}^k g_i(\mathbf{x})^p)^{1/p}$ is convex. To show this, we consider the function $h : \mathbb{R}^k \to \mathbb{R}$ defined as

$$h(\mathbf{z}) = \left(\sum_{i=1}^k \max\{z_i, 0\}^p \right)^{1/p},$$

with $\operatorname{dom} h = \mathbb{R}^k$, so $h = \tilde{h}$. This function is convex, and nondecreasing, so we conclude $h(\mathbf{g}(\mathbf{x}))$ is a convex function of $\mathbf{x}$. For $\mathbf{z} \succeq \mathbf{0}$, we have $h(\mathbf{z}) = (\sum_{i=1}^k z_i^p)^{1/p}$, so our conclusion is that $(\sum_{i=1}^k g_i(\mathbf{x})^p)^{1/p}$ is convex.

- The geometric mean $h(\mathbf{z}) = (\prod_{i=1}^k z_i)^{1/k}$ on $\mathbb{R}_+^k$ is concave and its extension is nondecreasing in each argument. It follows that if $g_1, \ldots, g_k$ are nonnegative concave functions, then so is their geometric mean, $(\prod_{i=1}^k g_i)^{1/k}$.

Operations that preserve convexity

- Minimization

If f is convex in $(\mathbf{x}, \mathbf{y})$, and C is a convex nonempty set, then the function

$$g(\mathbf{x}) = \inf_{\mathbf{y} \in C} f(\mathbf{x}, \mathbf{y}) \tag{1}$$

is convex in $\mathbf{x}$, provided $g(\mathbf{x}) > -\infty$ for some $\mathbf{x}$ (which implies $g(\mathbf{x}) > -\infty$ for all $\mathbf{x}$). The domain of g is the projection of $\operatorname{dom} f$ on its $\mathbf{x}$-coordinates, *i.e.*,

$$\operatorname{dom} g = \{\mathbf{x} \mid (\mathbf{x}, \mathbf{y}) \in \operatorname{dom} f \text{ for some } \mathbf{y} \in C\}.$$

Examples:

- The distance of a point $\mathbf{x}$ to a set $S \subseteq \mathbb{R}^n$, in the norm $\|\cdot\|$, is defined as

$$\operatorname{dist}(\mathbf{x}, S) = \inf_{\mathbf{y} \in S} \|\mathbf{x} - \mathbf{y}\|.$$

- Suppose h is convex. Then the function g defined below is convex:

$$g(\mathbf{x}) = \inf\{h(\mathbf{y}) \mid \mathbf{A}\mathbf{y} = \mathbf{x}\}.$$

Operations that preserve convexity

- Perspective of a function

If $f : \mathbb{R}^n \to \mathbb{R}$, then the *perspective* of f is the function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$g(\mathbf{x}, t) = tf(\mathbf{x}/t),$$

with domain

$$\mathrm{dom}\, g = \{(\mathbf{x}, t) \mid \mathbf{x}/t \in \mathrm{dom}\, f,\ t > 0\}.$$

The perspective operation preserves convexity: If f is a convex function, then so is its perspective function g.

Operations that preserve convexity

- Perspective of a function

Examples:

- The perspective of the convex function $f(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle$ on $\mathbb{R}^n$ is

$$g(\mathbf{x}, t) = t \langle \mathbf{x}/t, \mathbf{x}/t \rangle = \frac{\langle \mathbf{x}, \mathbf{x} \rangle}{t},$$

  which is convex in $(\mathbf{x}, t)$ for $t > 0$.

- Consider the convex function $f(x) = -\log x$ on $\mathbb{R}_{++}$. Its perspective is

$$g(x, t) = -t \log(x/t) = t \log(t/x) = t \log t - t \log x,$$

  and is convex on $\mathbb{R}^2_{++}$. The function g is called the *relative entropy* of t and x.

The conjugate function

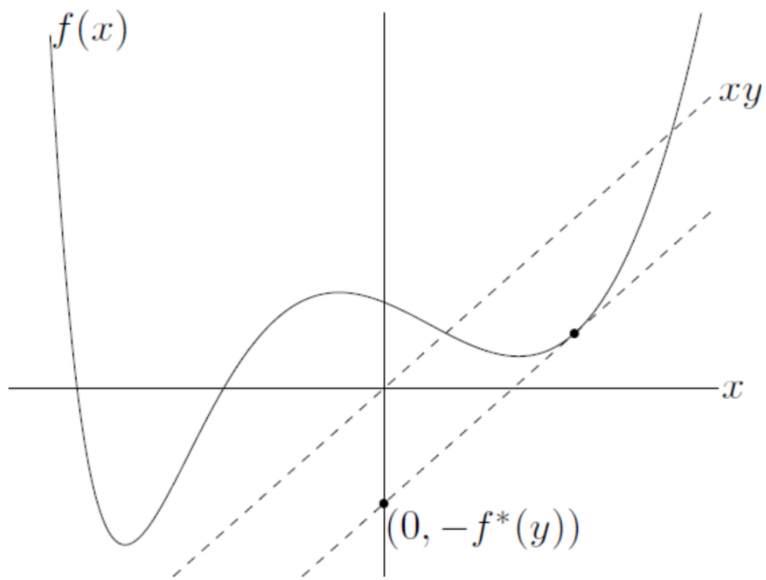
- Definition and examples

$\boxed{f^*(\mathbf{y}) \text{ is always convex!}}$

Let $f : \mathbb{R}^n \to \mathbb{R}$. The function

$$f^*(\mathbf{y}) = \sup_{\mathbf{x} \in \mathrm{dom}\, f} \left(\langle \mathbf{y}, \mathbf{x} \rangle - f(\mathbf{x}) \right), \qquad (1)$$

is called the *conjugate* of the function f. The domain of the conjugate function consists of $\mathbf{y} \in \mathbb{R}^n$ for which the supremum is finite, *i.e.*, for which the difference $\langle \mathbf{y}, \mathbf{x} \rangle - f(\mathbf{x})$ is bounded above on dom f.



The conjugate function

- Definition and examples

- *Affine function.* $f(x) = ax + b$. As a function of x, $yx - ax - b$ is bounded if and only if $y = a$. Therefore $\operatorname{dom} f^* = \{a\}$ and $f^*(a) = -b$.

- *Negative logarithm.* $f(x) = -\log x$, with $\operatorname{dom} f = \mathbb{R}_{++}$. The function $xy + \log x$ is unbounded above if $y \geq 0$ and reaches its maximum at $x = -1/y$ otherwise. Therefore, $\operatorname{dom} f^* = \{y \mid y < 0\} = -\mathbb{R}_{++}$ and $f^*(y) = -\log(-y) - 1$ for $y < 0$.

- *Exponential.* $f(x) = e^x$. $xy - e^x$ is unbounded if $y < 0$. For $y > 0$, $xy - e^x$ reaches its maximum at $x = \log y$, so we have $f^*(y) = y \log y - y$. For $y = 0$, $f^*(y) = \sup_x -e^x = 0$. So $\operatorname{dom} f^* = \mathbb{R}_+$ and $f^*(y) = y \log y - y$.

- *Negative entropy.* $f(x) = x \log x$, with $\operatorname{dom} f = \mathbb{R}_+$. The function $xy - x \log x$ is bounded above on $\mathbb{R}_+$ for all y, hence $\operatorname{dom} f^* = \mathbb{R}$. It attains its maximum at $x = e^{y-1}$, and substituting we find $f^*(y) = e^{y-1}$.

- *Inverse.* $f(x) = 1/x$ on $\mathbb{R}_{++}$. For $y > 0$, $yx - 1/x$ is unbounded above. For $y = 0$ this function has supremum 0; for $y < 0$ the supremum is attained at $x = (-y)^{-1/2}$. So $f^*(y) = -2(-y)^{1/2}$, with $\operatorname{dom} f^* = -\mathbb{R}_+$.

The conjugate function

- Definition and examples

Example 1 (Strictly convex quadratic function)**.** *Consider* $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x}$, *with* $\mathbf{Q} \in \mathbb{S}^n_{++}$. *The function* $\langle \mathbf{y}, \mathbf{x} \rangle - \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x}$ *is bounded above as a function of* $\mathbf{x}$ *for all* $\mathbf{y}$. *It attains its maximum at* $\mathbf{x} = \mathbf{Q}^{-1}\mathbf{y}$, *so*

$$f^*(\mathbf{y}) = \frac{1}{2}\mathbf{y}^T\mathbf{Q}^{-1}\mathbf{y}.$$

Example 2 (Indicator function)**.** *Let* I_S *be the indicator function of a (not necessarily convex) set* $S \subseteq \mathbb{R}^n$, i.e., $I_S(\mathbf{x}) = 0$ *on* $\operatorname{dom} I_S = S$. *Its conjugate is*

$$I_S^*(\mathbf{y}) = \sup_{\mathbf{x} \in S} \langle \mathbf{y}, \mathbf{x} \rangle,$$

which is the support function of the set S.

The conjugate function

- Definition and examples

Example 3 (Log-determinant). *We consider* $f(\mathbf{X}) = \log\det \mathbf{X}^{-1}$ *on* $\mathbb{S}^n_{++}$.

$$f^*(\mathbf{Y}) = \sup_{\mathbf{X}\succ\mathbf{0}} (\operatorname{tr}(\mathbf{Y}\mathbf{X}) + \log\det \mathbf{X}).$$

We first show that $\operatorname{tr}(\mathbf{Y}\mathbf{X}) + \log\det\mathbf{X}$ *is unbounded above unless* $\mathbf{Y} \prec \mathbf{0}$. *If* $\mathbf{Y} \not\prec \mathbf{0}$, *then* $\mathbf{Y}$ *has an eigenvector* $\mathbf{v}$, *with* $\|\mathbf{v}\|_2 = 1$, *and eigenvalue* $\lambda \geq 0$. *Taking* $\mathbf{X} = \mathbf{I} + t\mathbf{v}\mathbf{v}^T$ *we find that*

$$\begin{aligned} \operatorname{tr}(\mathbf{Y}\mathbf{X}) + \log\det\mathbf{X} &= \operatorname{tr}(\mathbf{Y}) + t\lambda + \log\det(\mathbf{I} + t\mathbf{v}\mathbf{v}^T) \\ &= \operatorname{tr}(\mathbf{Y}) + t\lambda + \log(1+t) \longrightarrow +\infty. \end{aligned}$$

Now consider the case $\mathbf{Y} \prec \mathbf{0}$. *The optimum* $\mathbf{X}$ *satisfies:*

$$\nabla_{\mathbf{X}}(\operatorname{tr}(\mathbf{Y}\mathbf{X}) + \log\det\mathbf{X}) = \mathbf{Y} + \mathbf{X}^{-1} = \mathbf{0},$$

which yields $\mathbf{X} = -\mathbf{Y}^{-1}$. *Therefore we have*

$$f^*(\mathbf{Y}) = \log\det(-\mathbf{Y})^{-1} - n, \quad \operatorname{dom} f^* = -\mathbb{S}^n_{++}.$$

The conjugate function

- Definition and examples

Example 4 (Log-sum-exp function). *We consider* $f(\mathbf{x}) = \log(\sum_{i=1}^n e^{x_i})$. *By setting the gradient of* $\langle \mathbf{y}, \mathbf{x} \rangle - f(\mathbf{x})$ *w.r.t.* $\mathbf{x}$ *equal to zero, we obtain the condition*

$$y_i = \frac{e^{x_i}}{\sum_{j=1}^n e^{x_j}}, \quad i = 1, \ldots, n.$$

These equations are solvable for $\mathbf{x}$ *if and only if* $\mathbf{y} \succ \mathbf{0}$ *and* $\langle \mathbf{1}, \mathbf{y} \rangle = 1$. *By substituting the expression for* y_i *into* $\langle \mathbf{y}, \mathbf{x} \rangle - f(\mathbf{x})$ *we obtain* $f^*(\mathbf{y}) = \sum_{i=1}^n y_i \log y_i$. *This expression for* f^* *is still correct if some components of* $\mathbf{y}$ *are zero, as long as* $\mathbf{y} \succeq \mathbf{0}$ *and* $\langle \mathbf{1}, \mathbf{y} \rangle = 1$, *and we interpret* $0 \log 0$ *as* 0.

In fact $\operatorname{dom} f^* = \{\mathbf{y} | \langle \mathbf{1}, \mathbf{y} \rangle = 1, \mathbf{y} \succeq \mathbf{0}\}$. *To show this, suppose that a component of* $\mathbf{y}$ *is negative, say,* $y_k < 0$. *Then we can show that* $\langle \mathbf{y}, \mathbf{x} \rangle - f(\mathbf{x})$ *is unbounded above by choosing* $x_k = -t$, *and* $x_i = 0$, $i \neq k$, *and letting* t *go to infinity.*

If $\mathbf{y} \succeq \mathbf{0}$ *but* $\langle \mathbf{1}, \mathbf{y} \rangle \neq 1$, *we choose* $\mathbf{x} = t\mathbf{1}$, *so that*

$$\langle \mathbf{y}, \mathbf{x} \rangle - f(\mathbf{x}) = t(\langle \mathbf{1}, \mathbf{y} \rangle - 1) - \log n \longrightarrow \infty.$$

The conjugate function

- Definition and examples

Example 5 (Norm). *Let* $\|\cdot\|$ *be a norm on* $\mathbb{R}^n$, *with dual norm* $\|\cdot\|_*$. *The conjugate of* $f(\mathbf{x}) = \|\mathbf{x}\|$ *is*

$$f^*(\mathbf{y}) = I_{\mathcal{B}}(\mathbf{y}),$$

where $\mathcal{B} = \{\mathbf{y} | \|\mathbf{y}\|_* \leq 1\}$ *is the unit ball of the dual norm.*

Example 6 (Norm squared). *Now consider the function* $f(\mathbf{x}) = (1/2)\|\mathbf{x}\|^2$, *where* $\|\cdot\|$ *is a norm, with dual norm* $\|\cdot\|_*$. *Its conjugate is* $f^*(\mathbf{y}) = (1/2)\|\mathbf{y}\|_*^2$.

Example 7 (Support function). $f(\mathbf{x}) = S_{\mathcal{C}}(\mathbf{x}) = \sup_{\mathbf{y}\in\mathcal{C}} \langle \mathbf{x}, \mathbf{y} \rangle$, *where* $\mathcal{C}$ *is a close convex set. Then its conjugate is the indicator function of* $\mathcal{C}$: $f^*(\mathbf{y}) = I_{\mathcal{C}}(\mathbf{y})$.

The conjugate function

- Basic properties

Fenchel's inequality:

$$f(\mathbf{x}) + f^*(\mathbf{y}) \geq \langle \mathbf{x}, \mathbf{y} \rangle, \quad \forall \mathbf{x}, \mathbf{y}.$$

For example, if $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2$ then $f^*(\mathbf{y}) = \frac{1}{2}\|\mathbf{y}\|_*^2$ and thus

$$\frac{1}{2}\|\mathbf{x}\|^2 + \frac{1}{2}\|\mathbf{y}\|_*^2 \geq \langle \mathbf{x}, \mathbf{y} \rangle,$$

which is generalization of $\frac{1}{2}\|\mathbf{x}\|_2^2 + \frac{1}{2}\|\mathbf{y}\|_2^2 \geq \langle \mathbf{x}, \mathbf{y} \rangle$.

When $f(\mathbf{x}) = (1/2)\mathbf{x}^T\mathbf{Q}\mathbf{x}$, where $\mathbf{Q} \in \mathbb{S}_{++}^n$, we obtain the inequality

$$\langle \mathbf{x}, \mathbf{y} \rangle \leq (1/2)\mathbf{x}^T\mathbf{Q}\mathbf{x} + (1/2)\mathbf{y}^T\mathbf{Q}^{-1}\mathbf{y}.$$

The conjugate function

- Basic properties

Conjugate of the conjugate

Theorem: If f is proper, convex and closed, then $f^{**} = f$.

Proof. It is easy to prove that $f^{**} \le f$, i.e. $\mathrm{epi} f^{**} \subseteq \mathrm{epi} f$. Suppose $\exists (\mathbf{x}_0^T, \gamma)^T \notin \mathrm{epi} f$, where $\gamma \ge f^{**}(\mathbf{x}_0)$. Since f is proper, convex and closed, its epigraph is closed and does not include a vertical line. Thus there exists a hyperplane parameterized as $(\mathbf{w}^T, \zeta)^T$ $(\zeta \neq 0)$ and μ such that it strictly separates $\mathrm{epi} f$ and the point $(\mathbf{x}_0^T, \gamma)^T$:

$$(\mathbf{w}^T, \zeta)(\mathbf{x}^T, t)^T < \mu < (\mathbf{w}^T, \zeta)(\mathbf{x}_0^T, \gamma)^T, \quad \forall \mathbf{x} \in \mathrm{dom}\, f, t \ge f(\mathbf{x}).$$

Since t can be arbitrarily large, ζ must be negative. W.l.o.g, we assume $\zeta = -1$. Then taking $t = f(\mathbf{x})$,

$$\mathbf{w}^T \mathbf{x} - f(\mathbf{x}) < \mu < \mathbf{w}^T \mathbf{x}_0 - \gamma \le \mathbf{w}^T \mathbf{x}_0 - f^{**}(\mathbf{x}_0), \quad \forall \mathbf{x} \in \mathrm{dom}\, f.$$

Thus $f^*(\mathbf{w}) < \mathbf{w}^T \mathbf{x}_0 - f^{**}(\mathbf{x}_0)$, contradicting Fenchel's inequality.

The conjugate function

- Basic properties

Useful for finding a convex surrogate of a nonconvex function!

Conjugate of the conjugate

Theorem: For any f, f^{**} is the largest convex function not exceeding f, i.e., *convex envelop* of f.

Proof. We first have $f^{***}(\mathbf{y}) = f^*(\mathbf{y})$ because $f^*(\mathbf{y})$ is a propoer, convex and closed function. And it is easy to prove that if $f_1(\mathbf{x}) \leq f_2(\mathbf{x})$, then $f_1^*(\mathbf{y}) \geq f_2^*(\mathbf{y})$.

Suppose g is proper, convex and closed, and $f^{**}(\mathbf{x}) \leq g(\mathbf{x}) \leq f(\mathbf{x})$. Then by the above results,

$$f^*(\mathbf{y}) \leq g^*(\mathbf{y}) \leq f^{***}(\mathbf{y}) = f^*(\mathbf{y}).$$

Thus $f^*(\mathbf{y}) = g^*(\mathbf{y})$ and hence $f^{**} = g^{**} = g$.

The conjugate function

- Basic properties

Conjugate of the conjugate

Examples. 1. The convex envelope of ℓ_0-pseudo-norm on unit ℓ_1 ball is the ℓ_1-norm.

Proof. We first compute $f^*(\mathbf{y}) = \sup_{\|\mathbf{x}\|\le 1}(\langle \mathbf{y}, \mathbf{x}\rangle - \|\mathbf{x}\|_0)$. When $\|\mathbf{x}\| \le 1$,

$$f^*(\mathbf{y}) \le \|\mathbf{y}\|_\infty \|\mathbf{x}\|_1 - \|\mathbf{x}\|_0 \le \|\mathbf{y}\|_\infty - \|\mathbf{x}\|_0.$$

$$f^*(\mathbf{y}) = \begin{cases} \|\mathbf{y}\|_\infty - 1, & \text{if } \|\mathbf{y}\|_\infty \ge 1, \\ 0, & \text{if } \|\mathbf{y}\|_\infty < 1 \end{cases} = \max(\|\mathbf{y}\|_\infty - 1, 0).$$

Next, we compute $f^{**}(\mathbf{x}) = \sup_{\mathbf{y}}(\langle \mathbf{y}, \mathbf{x}\rangle - f^*(\mathbf{y}))$:

$$f^{**}(\mathbf{x}) \le \|\mathbf{y}\|_\infty \|\mathbf{x}\|_1 - \max(\|\mathbf{y}\|_\infty - 1, 0).$$

$$f^{**}(\mathbf{x}) = \begin{cases} \|\mathbf{x}\|_1, & \text{if } \|\mathbf{x}\|_1 \le 1, \\ +\infty, & \text{if } \|\mathbf{x}\|_1 > 1. \end{cases}$$

The conjugate function

- Basic properties

Conjugate of the conjugate

Examples. 2. The convex envelope of rank on unit 2-norm ball is the nuclear-norm.

Proof. 1) Compute rank*. Let $q = \min(m, n)$. By von Neumann's trace thm,

$$\operatorname{rank}^*(\mathbf{Y}) = \sup_{\|\mathbf{X}\|_2 \le 1} \{\langle \mathbf{X}, \mathbf{Y}\rangle - \operatorname{rank}(\mathbf{X})\} \le \sup_{\|\mathbf{X}\|_2 \le 1} \left\{ \sum_{i=1}^{q} \sigma_i(\mathbf{X})\sigma_i(\mathbf{Y}) - \operatorname{rank}(\mathbf{X}) \right\}.$$

The equality can hold when $\mathbf{X}$ and $\mathbf{Y}$ have the same singular subspaces. Suppose $\operatorname{rank}(\mathbf{X}) = r$, $0 \le r \le q$. Then

$$\operatorname{rank}^*(\mathbf{Y}) = \sup_{\|\mathbf{X}\|_2 \le 1} \left\{ \sum_{i=1}^{r} \sigma_i(\mathbf{Y}) - r \right\} = \sup_{\|\mathbf{X}\|_2 \le 1} \left\{ \sum_{i=1}^{r} (\sigma_i(\mathbf{Y}) - 1) \right\}.$$

So r should be chosen such that $\sigma_r(\mathbf{Y}) > 1$ and $\sigma_{r+1}(\mathbf{Y}) \le 1$, and

$$\operatorname{rank}^*(\mathbf{Y}) = \sum_{i=1}^{r} (\sigma_i(\mathbf{Y}) - 1).$$

The conjugate function

- Basic properties

2) Compute rank**. By von Neumann's trace thm. again,

$$\mathrm{rank}^{**}(\mathbf{X}) = \sup_{\mathbf{Y}}\{\langle \mathbf{X}, \mathbf{Y}\rangle - \mathrm{rank}^*(\mathbf{Y})\} \leq \sup_{\mathbf{Y}} \left\{ \sum_{i=1}^{q} \sigma_i(\mathbf{X})\sigma_i(\mathbf{Y}) - \sum_{i=1}^{r} (\sigma_i(\mathbf{Y}) - 1) \right\},$$

where r is such that $\sigma_r(\mathbf{Y}) > 1$ and $\sigma_{r+1}(\mathbf{Y}) \leq 1$. The equality can also hold.

If $\|\mathbf{X}\|_2 > 1$, we can choose $\sigma_1(\mathbf{Y})$ large enough so that $\mathrm{rank}^{**}(\mathbf{X}) \to +\infty$, because the coefficient of $\sigma_1(\mathbf{Y})$ is a positive value $\sigma_1(\mathbf{X}) - 1$. If $\|\mathbf{X}\|_2 \leq 1$,

$$\begin{aligned}
&\sum_{i=1}^{q} \sigma_i(\mathbf{X})\sigma_i(\mathbf{Y}) - \sum_{i=1}^{r} (\sigma_i(\mathbf{Y}) - 1) \\
=&\sum_{i=1}^{q} \sigma_i(\mathbf{X})\sigma_i(\mathbf{Y}) - \sum_{i=1}^{r} (\sigma_i(\mathbf{Y}) - 1) - \sum_{i=1}^{q} \sigma_i(\mathbf{X}) + \sum_{i=1}^{q} \sigma_i(\mathbf{X}) \\
=&\sum_{i=1}^{r} (\sigma_i(\mathbf{X}) - 1)(\sigma_i(\mathbf{Y}) - 1) + \sum_{i=r+1}^{q} \sigma_i(\mathbf{X})(\sigma_i(\mathbf{Y}) - 1) + \sum_{i=1}^{q} \sigma_i(\mathbf{X}) \\
\leq&\sum_{i=1}^{q} \sigma_i(\mathbf{X}) = \|\mathbf{X}\|_*.
\end{aligned}$$

The equality can hold. Thus $\mathrm{rank}^{**}(\mathbf{X}) = \|\mathbf{X}\|_*$ over the set $\{\mathbf{X} \mid \|\mathbf{X}\|_2 \leq 1\}$.

The conjugate function

- Basic properties

Useful for finding the gradient for the dual problem!

Differentiable functions

The conjugate of a differentiable function f is also called the *Legendre transform* of f.

Suppose f is convex and differentiable, with $\boxed{\mathbf{dom}\, f = \mathbb{R}^n.}$ Any maximizer $\mathbf{x}^*$ of $\langle \mathbf{y}, \mathbf{x} \rangle - f(\mathbf{x})$ satisfies $\mathbf{y} = \nabla f(\mathbf{x}^*)$, and conversely, if $\mathbf{x}^*$ satisfies $\mathbf{y} = \nabla f(\mathbf{x}^*)$, then $\mathbf{x}^*$ maximizes $\langle \mathbf{y}, \mathbf{x} \rangle - f(\mathbf{x})$. Therefore, if $\mathbf{y} = \nabla f(\mathbf{x}^*)$, we have

$$f^*(\mathbf{y}) = \mathbf{x}^{*T} \nabla f(\mathbf{x}^*) - f(\mathbf{x}^*).$$

This allows us to determine $f^*(\mathbf{y})$ for any $\mathbf{y}$ for which we can solve the gradient equation $\mathbf{y} = \nabla f(\mathbf{z})$ for $\mathbf{z}$.

We can express this another way. Let $\mathbf{z} \in \mathbb{R}^n$ be arbitrary and define $\mathbf{y} = \nabla f(\mathbf{z})$. Then we have

$$f^*(\mathbf{y}) = \mathbf{z}^T \nabla f(\mathbf{z}) - f(\mathbf{z}).$$

If f is convex and closed, then

$$\mathbf{y} \in \partial f(\mathbf{x}) \Leftrightarrow \mathbf{x} \in \partial f^*(\mathbf{y}) \Leftrightarrow f(\mathbf{x}) + f^*(\mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle.$$

The conjugate function

- Basic properties

Scaling and composition with affine transformation
For $a > 0$ and $b \in \mathbb{R}$, the conjugate of $g(x) = af(x) + b$ is $g^*(y) = af^*(y/a) - b$.
Suppose $\mathbf{A} \in \mathbb{R}^{n\times n}$ is nonsingular and $\mathbf{b} \in \mathbb{R}^n$. Then the conjugate of $g(\mathbf{x}) = f(\mathbf{A}\mathbf{x} + \mathbf{b})$ is

$$g^*(\mathbf{y}) = f^*(\mathbf{A}^{-T}\mathbf{y}) - \mathbf{b}^T\mathbf{A}^{-T}\mathbf{y},$$

with $\mathrm{dom}\, g^* = \mathbf{A}^T \mathrm{dom}\, f^*$.

The conjugate function

- Basic properties

Separable functions

If $f(\mathbf{u}, \mathbf{v}) = f_1(\mathbf{u}) + f_2(\mathbf{v})$, where f_1 and f_2 are convex functions with conjugates f_1^* and f_2^*, respectively, then

$$f^*(\mathbf{w}, \mathbf{z}) = f_1^*(\mathbf{w}) + f_2^*(\mathbf{z}).$$

In other words, the conjugate of a *separable* convex function is the sum of the conjugates.

Envelope function and Proximal mapping

Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a closed proper function. For a scalar $c > 0$, define the corresponding *envelope function* $\mathrm{Env}_c f$ and the *proximal mapping* $\mathrm{Prox}_c f$ by

Involution: $(f \boxtimes g)(\mathbf{x}) = \inf_{\mathbf{y}+\mathbf{z}=\mathbf{x}} f(\mathbf{y}) + g(\mathbf{z})$.

$$\begin{aligned} \mathrm{Env}_c f(\mathbf{x}) &= \inf_{\mathbf{w}} \left\{ f(\mathbf{w}) + \frac{1}{2c}\|\mathbf{w}-\mathbf{x}\|^2 \right\}, \\ \mathrm{Prox}_c f(\mathbf{x}) &= \operatorname*{argmin}_{\mathbf{w}} \left\{ f(\mathbf{w}) + \frac{1}{2c}\|\mathbf{w}-\mathbf{x}\|^2 \right\}. \end{aligned} \tag{1}$$

Extremely useful for updating iterates!

Remark:

1. The envelope function $\mathrm{Env}_c f$ is an underestimate of the function f, i.e., $\mathrm{Env}_c f(\mathbf{x}) \le f(\mathbf{x})$, $\forall \mathbf{x} \in \mathbb{R}^n$. Furthermore, $\mathrm{Env}_c f$ is a real-valued continuous function, whereas f itself may only be extended real-valued and lower semi-continuous.

2. It is obvious that $\mathbf{u} = \mathrm{Prox}_c f(\mathbf{x}) \Leftrightarrow \mathbf{x} - \mathbf{u} \in \partial f(\mathbf{u})$.

Envelope function and Proximal mapping

Examples: $f(\mathbf{x}) = \chi_C(\mathbf{x})$

Projection onto halfspace: $\mathcal{C} = \{\mathbf{x}|\mathbf{a}^T\mathbf{x} \leq b\}$ ($\mathbf{a} \neq \mathbf{0}$):

$$P_{\mathcal{C}}(\mathbf{x}) = \begin{cases} \mathbf{x} + \frac{b-\mathbf{a}^T\mathbf{x}}{\|\mathbf{a}\|^2}\mathbf{a}, & \text{if } \mathbf{a}^T\mathbf{x} > b, \\ \mathbf{x}, & \text{if } \mathbf{a}^T\mathbf{x} \leq b. \end{cases}$$

Projection onto hyperbox $\mathcal{C} = \{\mathbf{x}|\mathbf{l} \leq \mathbf{x} \leq \mathbf{u}\}$:

$$(P_{\mathcal{C}}(\mathbf{x}))_i = \begin{cases} l_i, & \text{if } x_i \leq l_i, \\ x_i, & \text{if } l_i \leq \mathbf{x}_i \leq u_i, \\ u_i, & \text{if } x_i \geq u_i. \end{cases}$$

Projection onto nonnegative orthant: $\mathcal{C} = \mathbb{R}^n_+$:

$$P_{\mathcal{C}}(\mathbf{x}) = \max(\mathbf{x}, \mathbf{0}).$$

Envelope function and Proximal mapping

Projection onto positive semi-definite cone: $\mathcal{C} = \mathbb{S}^n_+$:

$$P_{\mathcal{C}}(\mathbf{X}) = \sum_{i=1}^{n} \max(\lambda_i, 0)\mathbf{q}_i\mathbf{q}_i^T,$$

where $\sum_{i=1}^{n} \lambda_i \mathbf{q}_i \mathbf{q}_i^T$ is the eigenvalue decomposition of $\mathbf{X}$.
Projection onto unit Euclidean ball: $\mathcal{C} = \{\mathbf{x} | \|\mathbf{x}\| \leq 1\}$:

$$P_{\mathcal{C}}(\mathbf{x}) = \begin{cases} \frac{1}{\|\mathbf{x}\|}\mathbf{x}, & \text{if } \|\mathbf{x}\| > 1, \\ \mathbf{x}, & \text{if } \|\mathbf{x}\| \leq 1. \end{cases}$$

Projection onto unit matrix 2-norm ball: $\mathcal{C} = \{\mathbf{X} | \|\mathbf{X}\|_2 \leq 1\}$:

$$P_{\mathcal{C}}(\mathbf{X}) = \begin{cases} \mathbf{U}\,\mathrm{Diag}(\min(\boldsymbol{\sigma}, 1))\mathbf{V}^T, & \text{if } \|\mathbf{X}\|_2 > 1, \\ \mathbf{X}, & \text{if } \|\mathbf{X}\|_2 \leq 1. \end{cases}$$

where $\mathbf{U}\,\mathrm{Diag}(\boldsymbol{\sigma})\mathbf{V}^T$ is the SVD of $\mathbf{X}$.

Envelope function and Proximal mapping

Examples: $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^t\mathbf{x} + \mathbf{c}$, $\|\mathbf{x}\|_1$, $\sum_{i=1}^{n} \log x_i$, $f(\mathbf{X}) = \|\mathbf{X}\|_*$

Theorem 1. *For each $\tau > 0$ and $\mathbf{Y} \in \mathbb{R}^{n_1 \times n_2}$,*

$$\mathcal{D}_\tau(\mathbf{Y}) = \underset{\mathbf{X}}{\operatorname{argmin}} \left\{ \tau\|\mathbf{X}\|_* + \frac{1}{2}\|\mathbf{X} - \mathbf{Y}\|_F^2 \right\}, \tag{1}$$

where $\mathcal{D}_\tau(\mathbf{Y})$ is the singular value thresholding operator defined as

$$\mathcal{D}_\tau(\mathbf{Y}) = \mathbf{U}\,\mathrm{diag}(\{(\sigma_i - \tau)_+\})\mathbf{V}^T, \tag{2}$$

in which $\mathbf{U}\,\mathrm{diag}(\{\sigma_i\})\mathbf{V}^T$ is the SVD of $\mathbf{Y}$.

Envelope function and Proximal mapping

Separable functions

If $f(\mathbf{u},\mathbf{v}) = f_1(\mathbf{u}) + f_2(\mathbf{v})$, where f_1 and f_2 are closed proper functions, then

$$\mathrm{Prox}_c f(\mathbf{u},\mathbf{v}) = \mathrm{Prox}_c f_1(\mathbf{u}) + \mathrm{Prox}_c f_2(\mathbf{v}).$$

Scaling and composition with orthogonal transformation

For $a > 0$ and $b \in \mathbb{R}$, the conjugate of $g(x) = f(ax + b)$ is

$$\mathrm{Prox}_c g(x) = a^{-1}(\mathrm{Prox}_{ca^2} f(ax + b) - b).$$

Suppose $g(\mathbf{x}) = f(\mathbf{A}\mathbf{x}+\mathbf{b})$, where $\mathbf{A} \in \mathbb{R}^{n\times m}$ satisfies $\mathbf{A}\mathbf{A}^T = \lambda^{-1}\mathbf{I}$ $(\lambda > 0)$ and $\mathbf{b} \in \mathbb{R}^n$. Then

$$\mathrm{Prox}_c g(\mathbf{x}) = (\mathbf{I} - \lambda\mathbf{A}^T\mathbf{A})\mathbf{x} + \lambda\mathbf{A}^T(\mathrm{Prox}_{c\lambda^{-1}} f(\mathbf{A}\mathbf{x} + \mathbf{b}) - \mathbf{b}).$$

Envelope function and Proximal mapping

Proof. $\mathbf{w} = \operatorname{Prox}_c g(\mathbf{x})$ is the solution of the optimization problem:

$$\min_{\mathbf{w},\mathbf{z}} f(\mathbf{z}) + \frac{1}{2c}\|\mathbf{w} - \mathbf{x}\|^2, \quad s.t. \quad \mathbf{A}\mathbf{w} + \mathbf{b} = \mathbf{z}.$$

Eliminating $\mathbf{w}$ gives: $\mathbf{w} = \mathbf{x} + \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}(\mathbf{z} - \mathbf{b} - \mathbf{A}\mathbf{x}) = (\mathbf{I} - \lambda\mathbf{A}^T\mathbf{A})\mathbf{x} + \lambda\mathbf{A}^T(\mathbf{z} - \mathbf{b})$. The optimal $\mathbf{z}$ is the minimizer of

$$f(\mathbf{z}) + \frac{\lambda^2}{2c}\|\mathbf{A}^T(\mathbf{z} - \mathbf{b} - \mathbf{A}\mathbf{x})\|^2 = f(\mathbf{z}) + \frac{\lambda}{2c}\|\mathbf{z} - \mathbf{b} - \mathbf{A}\mathbf{x}\|^2,$$

which is $\mathbf{z} = \operatorname{Prox}_{c\lambda^{-1}} f(\mathbf{A}\mathbf{x} + \mathbf{b})$.

Envelope function and Proximal mapping

Theorem 2. *For any function f, $\mathrm{Prox}_c\, f(\mathbf{x})$ is a monotonic function of $\mathbf{x}$ in the sense that:*

$$\langle \mathbf{x}_1 - \mathbf{x}_2, \mathbf{y}_1 - \mathbf{y}_2 \rangle \geq 0, \quad \forall \mathbf{y}_i \in Prox_c f(\mathbf{x}_i), i = 1, 2. \tag{1}$$

Proposition 1. *Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a closed proper convex function and $c > 0$. The envelope function $Env_c f$ is convex and smooth and its gradient is given by*

$$\nabla Env_c f(\mathbf{x}) = \frac{1}{c}(\mathbf{x} - Prox_c f(\mathbf{x})). \tag{2}$$

The envelope function $\mathrm{Env}_c f$ is smooth, regardless of whether f is smooth.

Envelope function and Proximal mapping

Proposition 2. *Let* $f : \mathbb{R}^n \to (-\infty, +\infty]$ *be a closed proper convex function and* $c > 0$*. The proximal mapping* $\operatorname{Prox}_c f$ *is single-valued and is continuous:* $\operatorname{Prox}_c f(x) \to \operatorname{Prox}_{c^*} f(\mathbf{x}^*)$ *whenever* $(\mathbf{x}, c) \to (\mathbf{x}^*, c^*)$*, with* $c^* > 0$*.*

$\boxed{\textbf{Theorem 3 (Moreau Decomposition). } \mathbf{x} = \operatorname{Prox}_c f(\mathbf{x}) + c \operatorname{Prox}_{c^{-1}} f^*(c^{-1}\mathbf{x}).}$

Proof. Let $\mathbf{u} = \operatorname{Prox}_c f(\mathbf{x})$. Then

$$\begin{aligned}\mathbf{u} = \operatorname{Prox}_c f(\mathbf{x}) &\iff -c^{-1}(\mathbf{u} - \mathbf{x}) \in \partial f(\mathbf{u}) \\ &\iff \mathbf{u} \in \partial f^*(-c^{-1}(\mathbf{u} - \mathbf{x}))\end{aligned}$$

Let $\mathbf{z} = -c^{-1}(\mathbf{u} - \mathbf{x})$. Then $\mathbf{u} = \mathbf{x} - c\mathbf{z}$ and thus

$$\begin{aligned}&\iff \mathbf{x} - c\mathbf{z} \in \partial f^*(\mathbf{z}) \\ &\iff \mathbf{0} \in \partial f^*(\mathbf{z}) + c(\mathbf{z} - c^{-1}\mathbf{x}) \\ &\iff \mathbf{z} = \operatorname{Prox}_{c^{-1}} f^*(c^{-1}\mathbf{x}).\end{aligned}$$

So $\mathbf{x} = \mathbf{u} + c\mathbf{z} = \operatorname{Prox}_c f(\mathbf{x}) + c \operatorname{Prox}_{c^{-1}} f^*(c^{-1}\mathbf{x})$.

Envelope function and Proximal mapping

Example: proximal mapping of a norm.
We know that if $f(\mathbf{x}) = \|\mathbf{x}\|$, then $f^*(\mathbf{y}) = I_{\mathcal{B}}(\mathbf{y})$, where $\mathcal{B}$ is the unit ball of the dual norm $\|\cdot\|_*$. Then by Moreay decomposition:

$$\begin{aligned}
\mathrm{Prox}_c f(\mathbf{x}) &= \mathbf{x} - c\mathrm{Prox}_{c^{-1}} f^*(\mathbf{x}/c) \\
&= \mathbf{x} - cP_{\mathcal{B}}(\mathbf{x}/c) \\
&= \mathbf{x} - P_{c\mathcal{B}}(\mathbf{x}).
\end{aligned}$$

Examples: $f(\mathbf{x}) = \|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, $f(\mathbf{X}) = \|\mathbf{X}\|_2$, $\|\mathbf{X}\|_*$.