Problem 1:

① Classical Brownian Motion

$$P_t = P_{t-1} + r_t$$

Expected value:

$$E(P_t) = E(P_{t-1}) + E(r_t)$$

since $r_t \sim N(0, \sigma^2)$

$$E(P_t) = E(P_{t-1})$$

$$\Rightarrow E(P_t) = P_{t-1}$$

Standard Deviation:

$$std(P_t) = std(P_{t-1}) + std(r_t)$$

since $r_t \sim N(0, \sigma^2)$

$$std(P_t) = \sigma$$

② Arithmetic Return System

$$P_t = P_{(t-1)}(1 + r_t)$$

Expected value:

$$E(P_t) = E(P_{(t-1)}(1 + r_t))$$

$$= E(P_{(t-1)}) + E(P_{(t-1)} \cdot r_t)$$

$$= E(P_{(t-1)}) + E(P_{(t-1)}) \cdot E(r_t)$$

Since $r_t \sim N(0, \sigma^2)$, $E(r_t) = 0$

$$E(P_t) = E(P_{(t-1)})$$

$$= P_{t-1}$$

Standard Deviation:

$$E[P_t^2] = E[P_{t-1}^2] + E[P_{t-1}^2] \cdot E[r_t^2]$$

since $r_t \sim N(0, \sigma^2)$, $E[r_t^2) = \sigma^2$

$$E[P_t^2] = E[P_{t-1}^2] \cdot (1 + \sigma^2)$$

$$= P_{t-1}^2 \cdot (1 + \sigma^2)$$

$$E[P_t]^2 = P_{t-1}^2$$

$$Var(P_t) = E[P_t^2] - E[P_t]^2$$

$$= P_{t-1}^2 \left((1 + \sigma^2) - 1\right)$$

$$std(P_t) = P_{t-1} \cdot \sigma$$

③

Log return

$$P_t = P_{t-1} e^{r_t}$$

Expected Value:

since $r_t \sim N(0, \sigma^2)$, $E[e^{r_t}] = e^{\mu + \frac{\sigma^2}{2}}$

$\mu = 0$, $E[e^{r_t}] = e^{\frac{\sigma^2}{2}}$

$$E[P_t] = E[P_{t-1} \cdot e^{r_t}]$$

$$= E[P_{t-1}] \cdot E[e^{r_t}]$$

$$= P_{t-1} \cdot e^{\frac{\sigma^2}{2}}$$

Standard deviation:

Since $r_t \sim N(0, \sigma^2)$,

P_t follows a log-normal distrition, which

If $X \sim \text{Log-Normal}(\mu, \sigma^2)$, then

$$SD[X] = \sqrt{(e^{\sigma^2} - 1) e^{2\mu + \sigma^2}}$$

In GBM for a single period, $\mu = \log(P_{t-1})$,

σ remain the same, therefore,

$$std(P_t) = P_{(t-1)} \cdot \sqrt{e^{\sigma^2} - 1}$$

Problem 1 simulation:

Set Pt-1 to be 100 and sigma to be 0.2 and run simulation:

Expected value for Classical Brownian Motion: 100.00

Expected value for Arithmetic Return System: 100.00

Expected value for Log Return or Geometric Brownian Motion: 100.02

Expected standard deviation for Classical Brownian Motion: 0.02

Expected standard deviation for Arithmetic Return System: 2.00

Expected standard deviation for Log Return or Geometric Brownian Motion: 2.00

Simulated price using Classical Brownian Motion: 100.00

Simulated price using Arithmetic Return System: 100.02

Simulated price using Log Return or Geometric Brownian Motion: 100.04

Simulated standard deviation using Classical Brownian Motion: 0.02

Simulated standard deviation using Arithmetic Return System: 1.99

Simulated standard deviation using Log Return or Geometric Brownian Motion: 1.99

Conclusion:

- The simulated prices for all models are generally the same as the result derived from analytical expression. While arithmetic and log return models show slight increases from the initial, with **Log Return** showing the highest simulated price of $100.04. This is consistent with the expected values, where the compounding effect in the log return model slightly pushes up the price.
- The simulated standard deviations for all models also very close to the result derived. The **Classical Brownian Motion** (0.02) accurately reflect the input volatility of returns. In contrast, **Arithmetic Return System** and **Log Return** show simulated

standard deviations (1.99) that are much closer to each other and suggest a high level of price volatility over time. These values indicate the models' sensitivity to the volatility of returns, with the geometric compounding effect slightly moderating the volatility in the log return model compared to the arithmetic model. Also, the 10000 random draws of return may not enough, if we conduct more random draws the differences between them would be smaller.

- Overall, the results from expectation and simulation are almost the same.

Problem 2:

Here are the arithmetic returns for all prices using function “return_calculate”:

```
           Date      AAPL      ABBV       ABT       ACN      ADBE  ...      VRTX        VZ       WFC       WMT       XOM       ZTS
0    2022-09-02 -0.013611 -0.015674 -0.022320 -0.016344 -0.006450  ... -0.031564 -0.012198 -0.007323 -0.011005  0.018323 -0.015244
1    2022-09-06 -0.008215  0.009613  0.002049 -0.002147  0.000435  ... -0.006618 -0.004843 -0.011526 -0.004962 -0.006695 -0.000892
2    2022-09-07  0.009254  0.008140  0.019375  0.011642  0.031007  ...  0.018841 -0.000486  0.022388  0.025691 -0.008531  0.022698
3    2022-09-08 -0.009618  0.013049  0.021872  0.004185  0.010297  ...  0.015996  0.005842  0.031706  0.005083  0.008179 -0.011908
4    2022-09-09  0.018840  0.006405  0.013927  0.008994  0.029064  ...  0.010796  0.022265  0.002211  0.003005  0.016753  0.036721
..          ...       ...       ...       ...       ...       ...  ...       ...       ...       ...       ...       ...       ...
260  2023-09-18  0.016913  0.011964 -0.007273  0.010719  0.006674  ...  0.008885 -0.007695  0.011176 -0.007410  0.008055 -0.003329
261  2023-09-19  0.006181 -0.002923 -0.003267  0.004590  0.017411  ...  0.017784 -0.003877 -0.004835 -0.000245 -0.002635  0.012970
262  2023-09-20 -0.019992  0.000782 -0.003874 -0.001670 -0.010910  ... -0.016608  0.005988 -0.008098  0.003244 -0.007926 -0.002748
263  2023-09-21 -0.008889 -0.001172 -0.013462 -0.015151 -0.040875  ...  0.005592 -0.009226 -0.012829 -0.012141 -0.014089 -0.026725
264  2023-09-22  0.004945 -0.004497 -0.007884  0.013109 -0.001907  ... -0.013407 -0.000300 -0.025756  0.002656  0.001568  0.000283

[265 rows x 102 columns]
```

In order to remove the mean from a series so that the mean of the series becomes 0, I subtract the mean of the series from each element in the series.

After remove the mean, here is the current mean of “META” return:

```
Mean of META de_mean series: 1.25685625429263e-18
```

As shown above, the mean is now effectively zero.

Then, I start calculating VaR under different circumstances.

1. Using a normal distribution.
    1. This method assumes that returns are normally distributed. The VaR is calculated using the mean and standard deviation of the returns.
2. Using a normal distribution with an Exponentially Weighted variance (λ = 0. 94)
    1. This method uses exponentially weighted moving average (EWMA) for variance estimation.
3. Using a MLE fitted T distribution.
    1. This method involves fitting a T-distribution to the returns using Maximum Likelihood Estimation (MLE) and then calculating VaR from the fitted distribution.
4. Using a fitted AR(1) model.
    1. An AR(1) model predicts future returns based on past returns. The residuals from this model can be used to calculate VaR.
5. Using a Historic Simulation.
    1. This method uses the empirical distribution of past returns.

Here are the results after calculation:

```
VaR using Normal Distribution: 5.4287%
VaR using Normal Distribution with EWMA: 3.0137%
VaR using MLE Fitted T Distribution: 0.1424%
VaR using AR(1) Model: 5.4229%
VaR using Historic Simulation: 3.9484%
```

Conclusions by comparing the 5 values:

1. **Normal Distribution (5.4287%)**: This approach assumes normality of returns, which is a significant limitation given that financial market returns often exhibit fat tails and skewness not captured by a normal distribution. The relatively high VaR suggests a conservative estimate of risk under the normality assumption.

2. **Normal Distribution with EWMA (3.0137%)**: Incorporating exponentially weighted moving averages to account for the varying volatility over time provides a more responsive estimate of VaR. The lower VaR compared to the standard normal distribution method reflects the dynamic adjustment to recent market conditions, potentially offering a more accurate reflection of current market risks.

3. **MLE Fitted T Distribution (0.1424%)**: This method fit the distribution to the data, potentially capturing the fat tails and skewness of returns more accurately than the normal distribution. The significantly lower VaR suggests that this method might underestimate risk or that the fitted distribution captures events more precisely, leading to a lower probability of extreme losses under normal market conditions.

4. **Fitted AR(1) Model (5.4229%)**: The AR model considers the serial correlation in returns. The VaR is similar to that of the normal distribution method, indicating that while the AR(1) model incorporates time-series characteristics, it might not fully address the distributional properties of returns such as skewness and kurtosis.

5. **Historic Simulation (3.9484%)**: This non-parametric approach directly uses historical returns to estimate VaR, avoiding assumptions about the returns' distribution. The result is somewhere in the middle range of the methods tested, reflecting actual historical outcomes without assuming a specific distribution. This method is intuitive and straightforward but may not fully account for changes in market conditions or structural breaks in the data.

Problem 3:

Methods: In order to calculate the VaR of each portfolio as well as total VaR, first I calculate daily returns using the “return_calculate” function in problem 2. Then I adjust the DataFrame to have dates as the index. With the adjusted DataFrame, calculate the exponentially weighted covariance matrix. After that, to calculate portfolio variances, I iterate through each portfolio, and sum the product of holdings, weights, and the covariance between each pair of stocks within the portfolio. Then, Convert the variances into VaR figures for each portfolio using a specified confidence level.

Here are the final results of VaR of each portfolio as well as total VaR using exponentially weighted covariance:

```
Portfolio Value at Risk (VaR):
A: $46.6555
B: $44.8284
C: $46.9466

Total VaR of the Portfolio
$131.3449
```

Conclusion:

- **Portfolio VaRs**: The VaRs for portfolios A, B, and C were 46.66, 44.83, and 46.95, respectively. These values represent the maximum expected loss at a 95% confidence level, meaning there is a 5% chance that the portfolio could lose more than this value over the specified period.
- **Total VaR**: The combined VaR for all three portfolios was calculated to be 131.34. The total VaR provides a consolidated view of the risk across all holdings, offering insight into the overall risk exposure when considering the portfolios collectively.

I choose to use historical simulation to calculate VaR again. The reason I choose historical simulation is that this method doesn't assume normal distribution of returns but instead uses the actual distribution of historical returns to estimate the potential loss.

Here are the methods how I use historical simulation: use the daily prices to calculate daily returns for each stock. Then, aggregate the daily returns for each portfolio based on the holdings. After that, apply historical simulation to calculate VaR for each portfolio. Finally, Aggregate all holdings to calculate the total portfolio value, then calculate VaR for the total holdings.

Here are the final results of VaR of each portfolio as well as total VaR using historical simulation:

```
Portfolio Value at Risk (VaR):
A: $57.5341
B: $65.4816
C: $70.0502

Total VaR of the Portfolio:
$189.2008
```

Conclusion:

Historical simulation uses the actual distribution of returns, capturing extreme market movements and tail risks that might not be well-represented by a parametric model like EWC. EWC gives more weight to recent observations, which can make the VaR more sensitive to recent market volatility. If there were significant market downturns in the historical period used for simulation, this might result in higher VaR estimates, as observed my results.