

## 1 Basic Knowledge

**Def of ODE & ODEs:** (1st order) ODE:  $\frac{dy}{dt} = f(t, y)$  & ODEs:  $\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y})$ ,  $\mathbf{y} = (y_1, \dots, y_d)^T$ ,  $\mathbf{f}(t, \mathbf{y}) = (f_1(t, \mathbf{y}), \dots, f_d(t, \mathbf{y}))^T$

**Autonomous:**  $\frac{dy}{dt} = \mathbf{f}(\mathbf{y}) \Rightarrow$  autonomous ODE(s).  $\parallel \Downarrow$  New Autonomous ODEs:  $\frac{dy}{ds} = \mathbf{f}(y_{d+1}, \mathbf{y})$  and  $\frac{dy_{d+1}}{ds} = 1$

· **Change to Autonomous:** For  $\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y})$ . Let  $y_{d+1} = t$  and new independent variable  $s$  s.t.  $\frac{dt}{ds} = 1 \uparrow$

**Linearity:** ODE:  $\frac{dy}{dt} = f(t, y)$  is linearity if  $f(t, y) = a(t)y + b(t)$   $\parallel$  ODEs: If each ODE is linear, then the ODEs are linear.

**Picard's Theorem:** If  $f(t, y)$  is continuous in  $D := \{(t, y) : t_0 \leq t \leq T, |y - y_0| < K\}$  and  $\exists L > 0$  (Lipschitz constant) s.t.

$\forall (t, u), (t, v) \in D \quad |f(t, u) - f(t, v)| \leq L|u - v|$  (ps: Can use MVT). And Assume that  $M_f(T - t_0) \leq K$ ,  $M_f := \max\{|f(t, u)| : (t, u) \in D\}$

$\Rightarrow$  **Then**,  $\exists$  a unique continuously differentiable solution  $y(t)$  to the IVP  $\frac{dy}{dt} = f(t, y)$ ,  $y(t_0) = y_0$  on  $t \in [t_0, T]$ .

**Existence & Uniqueness Theorem:** IVP  $\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y})$ ,  $\mathbf{y}(t_0) = \mathbf{y}_0$ . If  $f(t, y)$  and  $\frac{\partial f}{\partial y_i}$  are continuous in a neighborhood of  $(t_0, \mathbf{y}_0)$ .

$\Rightarrow$  **Then**,  $\exists I := (t_0 - \delta, t_0 + \delta)$  s.t.  $\exists$  a unique continuously differentiable solution  $\mathbf{y}(t)$  to the IVP on  $t \in I$ .

## 2 Acknowledge

Notation	Meaning	Notation	Meaning
$[a, b]$	Approximate function for $t \in [a, b]$	$t_0 = a \mid t_N = b$	Assume that $t_0 = a, t_N = b$
$N$	number of <b>timesteps</b> (i.e. Break up interval $[a, b]$ into $N$ equal-length sub-intervals)	$h$	<b>stepsize</b> ( $h = \frac{b-a}{N}$ )
$t_i$	Define $N + 1$ points: $t_0, t_1, \dots, t_N$	$t_m$	$t_m = a + h \cdot m = t_0 + h \cdot m$
$y_i$	Approximation of $y$ at point $t = t_i$ (Except $y_0$ )	$y(t_i)$	Exact value of $y$ at point $t = t_i$

## 3 Euler's Method and Taylor Series Method

**Euler's Method Algorithm:** Approximate ODE  $\frac{dy}{dt} = f(t, y)$ ,  $y(t_0) = y_0$  with number of steps  $N$ . (Similarly for ODEs)

$\Rightarrow$  for  $n = 0, 1, 2, \dots, N - 1$ :  $y_{n+1} = y_n + hf(t_0 + nh, y_n) = y_n + hf(t_n, y_n)$  **end** (ps:  $\Downarrow$  Can get  $|y''| < M$ )

**Boundedness Theorem:** For  $\frac{dy}{dt} = f(t, y)$ ,  $y(a) = y_0$  and suppose there exists a unique, twice differentiable, solution  $y(t)$  on  $[a, b]$ .

Suppose:  $y$  is continuous and  $|\frac{\partial f}{\partial y}| \leq L$ .  $\Rightarrow$  the solution  $y_n$  given Euler's method satisfies:  $e_n = |y_n - y(t_n)| \leq Dh$ ,  $D = e^{(b-a)L} \frac{M}{2L}$

· **Lemma:** If  $v_{n+1} \leq Av_n + B$ , then  $v_n \leq A^n v_0 + \frac{A^n - 1}{A - 1} B$  If  $v_n = e_n := y_n - y(t_n)$ , then  $A = 1 + hL$ ,  $B = h^2 M / 2$  (suppose  $|y''| < M$ )

**Order Notation ( $\mathcal{O}$ ):** we write  $z = \mathcal{O}(h^p)$  if  $\exists C, h_0 > 0$  s.t.  $|z| \leq Ch^p$ ,  $0 < h < h_0$

**Flow Map ( $\Phi$ ):**  $\Phi_h(y)$  is a flow function if:  $\Phi_{t_0, h}(y) = y(t_0 + h; t_0, y_0)$  Approx:  $\Psi_h(y) := \hat{\Phi}_h(y)$  where  $\Psi(y_n) = y_{n+1}$

**Taylor Series Method:** Approximate ODE  $\frac{dy}{dt} = f(t, y)$ ,  $y(t_0) = y_0$  with  $n$ -order Methods: 用 Taylor Series 在  $t_0 + h$  处展开保留到  $n$  阶

· ps: Taylor Series:  $y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{h^2}{2}y''(t_0) + \dots + \frac{h^{n-1}}{(n-1)!}y^{(n-1)}(t_0) + \frac{h^n}{n!}y^{(n)}(t^*)$ ,  $t^* \in [t, t + h]$   $y' = f$ ,  $y'' = f_t + f_y f$

## 4 Convergence of One-Step Methods consider for autonomous $y' = f(y)$

**Global Error:** global error after  $n$  steps:  $e_n := y_n - y(t_n)$  **Local Error:** For one-step method is:  $le(y, h) = \Psi_h(y) - \Phi_h(y)$

**Consistent:** If  $||le(y, h)|| < Ch^{p+1} (< \mathcal{O}(h^{p+1}))$ ,  $C > 0$ .  $\Rightarrow$  Consistent at order  $p$ . **Stable:** If  $||\Psi_h(u) - \Psi_h(v)|| \leq (1 + h\hat{L})||u - v||$

**Convergent:** A method is convergent if:  $\forall T, \lim_{h \rightarrow 0, h=T/N} \max_{n=0,1,\dots,N} ||e_n|| = 0$   $\Downarrow$  Then the global error satisfies:  $\max_{n=0,1,\dots,N} ||e_n|| = \mathcal{O}(h^p)$   $p$ -th order

**Convergence of One-Step Method:** For  $y' = f(y)$ , and a one-step method  $\Psi_h(y)$  is <sup>1</sup> consistent at order  $p$  and <sup>2</sup> stable with  $\hat{L}$   $\uparrow$ . (ps:  $C = \frac{C}{\hat{L}}(e^{T\hat{L}} - 1)$ )

**Polynomial Interpolation:**