NODEA Note

1 Basic Knowledge

Def of ODE & ODEs: (1st order) ODE: $\frac{dy}{dt} = f(t, y)$ & ODEs: $\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y})$, $\mathbf{y} = (y_1, ..., y_d)^T, \mathbf{f}(t, \mathbf{y}) = (f_1(t, \mathbf{y}), ..., f_d(t, \mathbf{y}))^T$

Autonomous: $\frac{d\mathbf{y}}{dt} = \mathbf{f}(\mathbf{y}) \Rightarrow$ autonomous ODE(s). || $\Downarrow$ New Autonomous ODEs: $\frac{d\mathbf{y}}{ds} = \mathbf{f}(y_{d+1}, \mathbf{y})$ and $\frac{dy_{d+1}}{ds} = 1$

· **Change to Autonomous**: For $\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y})$. Let $y_{d+1} = t$ and *new* independent variable s s.t. $\frac{dt}{ds} = 1 \Uparrow$

Linearity: ODE: $\frac{dy}{dt} = f(t, y)$ is linearity if $f(t, y) = a(t)y + b(t)$ || ODEs: If each ODE is linear, then the ODEs are linear.

Picard's Theorem: If $f(t, y)$ is continuous in $D := \{(t, y) : t_0 \le t \le T, |y - y_0| < K\}$ and $\exists L > 0$ (Lipschitz constant) s.t.
$\forall (t, u), (t, v) \in D \quad |f(t, u) - f(t, v)| \le L|u - v|$ (ps:Can use MVT). And Assume that $M_f(T - t_0) \le K, M_f := \max\{|f(t, u)| : (t, u) \in D\}$
$\Rightarrow$ **Then**, $\exists$ a unique continuously differentiable solution $y(t)$ to the IVP $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$ on $t \in [t_0, T]$.

Existence & Uniqueness Theorem: IVP $\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y}), \mathbf{y}(t_0) = \mathbf{y}_0$. If $f(t, y)$ and $\frac{\partial f}{\partial y_i}$ are continuous in a neighborhood of $(t_0, \mathbf{y}_0)$.
$\Rightarrow$ **Then**, $\exists I := (t_0 - \delta, t_0 + \delta)$ s.t. $\exists$ a unique continuously differentiable solution $\mathbf{y}(t)$ to the IVP on $t \in I$.

2 Acknowledge

Notation	Meaning	Notation	Meaning
$[a, b]$	Approximate function for $t \in [a, b]$	$t_0 = a \mid t_N = b$	Assume that $t_0 = a, t_N = b$
N	number of **timesteps** (i.e. Break up interval $[a, b]$ into N equal-length sub-intervals)	h	**stepsize** $(h = \frac{b-a}{N})$
t_i	Define $N + 1$ points: $t_0, t_1, ..., t_N$	t_m	$t_m = a + h \cdot m = t_0 + h \cdot m$
y_i	Approximation of y at point $t = t_i$ (Except y_0)	$y(t_i)$	Exact value of y at point $t = t_i$

3 Euler's Method and Taylor Series Method

Euler's Method Algorithm: Approx $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$ Euler Method: $y_{n+1} = y_n + hf(t_0 + nh, y_n) = y_n + hf(t_n, y_n)$

By Taylor Series, for Euler Method, we have: $|le_n| \le |y''(\tau) \cdot \frac{h^2}{2}|$ where $\tau \in [t_n, t_{n+1}]$

Lemma: If $v_{n+1} \le Av_n + B \Rightarrow$ Then $v_n \le A^n v_0 + \frac{A^n - 1}{A - 1}B$ If $|y''| < M$ and $v_n = e_n := y_n - y(t_n)$, then $A = 1 + hL, B = h^2M/2$

Boundedness Theorem|Euler Method: For $\frac{dy}{dt} = f(t, y), y(a) = y_0$:
$\exists$ [1] *unique,* [2] *twice differentiable*, solution $y(t)$ on $[a, b]$, [3] y is continuous and [4] $|\frac{\partial f}{\partial y}| \le L$.
$\Rightarrow$ the solution y_n given by *Euler's method* satisfies: $e_n = |y_n - y(t_n)| \le Dh, D = e^{(b-a)L} \frac{M}{2L}$

Order Notation ($\mathcal{O}$): we write $z(h) = \mathcal{O}(h^p)$ if $\exists C, h_0 > 0$ s.t. $|z| \le Ch^p, 0 < h < h_0$

Flow Map (Φ, Ψ): Consider $\frac{dy}{dt} = f(t, y)$.

1. **Exact Flow Map (Φ)**: $\Phi_{t_n,h}(y_n) = y(t_n + h)$ 代表假设 $y(t_n) = y_n$ 的情况下，输入 y_n 在 $t_n + h$ 时刻的精确值; 当不写 t_n 角标时, 默认要算的前一个时间点已知/精确
2. **Numerical Flow Map (Ψ)**: $\Psi_{t_n,h}(y_n) = y_{n+1}$ 代表假设 $y(t_n) = y_n$ 的情况下，输入 y_n 在 $t_n + h$ 时刻的数值解; 当不写 t_n 角标时, 默认要算的前一个时间点已知/精确
   Remark: $\Phi_h(y(t_n)) = y(t_n + h)$ $\Psi_h(y(t_n)) = y_{n+1}$
   Find: Generally, use $\Phi_{t_0,h}(y_0) = y(t_0 + h)$ to find $y(t_0 + h)$; and $\Psi(y)$: Numerical method for ODE.

Find Numerical Method|Taylor Series Method: Approx $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$ with *n-order Methods*

1. **Method**: 通过泰勒展开精确解, 取前 n 项作为近似解, 从而得到数值解.
2. **Taylor Series for Φ**: $\Phi_{t,h}(y) = y + hf(t, y) + \frac{1}{2}h^2[f_t(t, y) + f_y(t, y)f(t, y)] + \frac{1}{6}y'''(t, y)h^3 + \cdots$ (For one variable y) ps: $y' = f, y'' = f_t + f_y f$
3. **Taylor Series**: $y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{h^2}{2}y''(t_0) + \cdots + \frac{h^{n-1}}{(n-1)!}y^{(n-1)}(t_0) + \frac{h^n}{n!}y^{(n)}(t^*), t^* \in [t, t + h]$

4 Convergence of One-Step Methods consider for autonomous $y' = f(y)$

4.1 Convergence | Consistent | Stable

Global Error: global error after n steps: $e_n := y_n - y(t_n)$ **Local Error**: For *one-step* method is: $le(y, h) = \Psi_h(y) - \Phi_h(y)$

ps: More Exactly, $le_n = \Psi_h(y(t_n)) - \Phi_h(y(t_n)).$

Consistent: If $||le(y, h)|| \le Ch^{p+1} (\le \mathcal{O}(h^{p+1}))$, $C > 0$. $\Rightarrow$ Consistent at order p. **Stable**: If $||\Psi_h(u) - \Psi_h(v)|| \le (1 + h\hat{L})||u - v||$

Convergent: A method is convergent if:$\forall T$, $\lim_{h \to 0, h = T/N} \max_{n=0,1,...,N} ||e_n|| = 0$ $\Downarrow$ Then the global error satisfies: $\max_{n=0,1,...,N} ||e_n|| = \mathcal{O}(h^p)$ **p-th order**

Convergence of One-Step Method: For $y' = f(y)$, and a one-step method $\Psi_h(y)$ is [1] consistent at order p and [2] stable with $\hat{L} \Uparrow$. (ps:$C = \frac{C}{\hat{L}}(e^{T\hat{L}} - 1)$)

4.2 More One-Step Methods | Runge-Kutta Methods | Collocation

Construction of More General one-step Method: For $y' = f(y), y(t_0) = y_0 \Rightarrow y(t + h) - y(t) = \int_t^{t+h} f(y(\tau))d\tau$

Lagrange Interpolating Polynomials: For function $p(x)$. Consider points: $(c_1, g_1), ..., (c_s, g_s)$. where $p(c_i) = g_i$.

1. **Lagrange Interpolating Polynomials**: Let $\ell_i(x) = \prod_{j=1, j \ne i}^{s} \frac{x - c_j}{c_i - c_j} \in \mathbb{P}_{s-1}$
2. **Polynomial Interpolation**: $\exists! \, p(x) = \sum_{i=1}^{s} g_i \ell_i(x)$ (Can be proved by Honour Algebra)

Interpolatory Quadrature: 对于函数 $g(t) \in \mathbb{P}_{p-1}$, 我可以通过插值求积的方法来近似求解积分；以下展示 $[a, b]$ 上的插值求积。

1. Choose c_i points in $[a, b]$: $c_1, \ldots, c_s$. Let $g_i = g(c_i)$. By using c_i, g_i, we can get $\ell_i(x)$.
2. Define weights: $b_i := \int_a^b \ell_i(x)\, dx$. Then $\int_a^b g(t)\, dt \approx \sum_{i=1}^{s} b_i\, g(c_i)$.

One-Step Collocation Methods: 对于 $y' = f(y)$, $y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(y(t))dt$, 通过 Interpolatory Quadrature 来近似求解积分. 为了简化, 考虑 autonomous 的情况

1. Choose $c_1, ..., c_s$ in $[0,1]$, consider $t_i = t_n + c_i h$, then $t_i \in [t_n, t_{n+1}]$.
2. Let $F_i = f(y(t_i))$, then we can get $\ell_i(x)$ which pass through (c_i, F_i).
3. Let weights: $b_i = \int_0^1 \ell_i(x)dx$, and $a_{ij} = \int_0^{c_i} \ell_j(x)dx$. **Then** $\star\ y_{n+1} = y_n + h\sum_{i=1}^s b_i F_i.\ \star$
4. Moreover, we can get: $F_i = f(Y_i)$, where $Y_i = y_n + h\sum_{j=1}^s a_{ij}F_j$.
   ps: More Exactly, $Y_i = y_n + h\sum_{i=1}^s b_i f(t_n + c_i h, Y_i)$ and $y_{n+1} = y_n + h\sum_{i=1}^s b_i f(t_n + c_i h, Y_i)$
   Remark: For choice of c_i: The optimal choice is attained by *Gauss-Legendre collocation methods*.
   e.g. $s = 1$: $c_1 = \frac{1}{2}$; $s = 2$: $c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}, c_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}$; $s = 3$: $c_1 = \frac{1}{2} - \frac{\sqrt{15}}{10}, c_2 = \frac{1}{2}, c_3 = \frac{1}{2} + \frac{\sqrt{15}}{10}$

Runge-Kutta Methods: Let $y' = f(y)$ here we consider the autonomous case. The RK method has following form:

1. **Stage Values**: $Y_i = y_n + h\sum_{j=1}^s a_{ij} f(Y_j)$ $\ i \in \{1, ..., s\}$ $\qquad F_i = f(Y_i)$
2. **Update**: $y_{n+1} = y_n + h\sum_{i=1}^s b_i F_i = y_n + h\sum_{i=1}^s b_i f(Y_i)$ For Autonomous: $c_i = \sum_{j=1}^s a_{ij}$
   Remark: Flow-map: $\Psi_h(y) = y + h\sum_{i=1}^s b_i f(Y_i(y))$ ps:*weights*: b_i; *internal coefficients*: a_{ij}
   ps: We can using **Butcher Table** to represent the RK method (Appendix)
   Explicit: $a_{ij} = 0$ for $j \geq i$ (严格下三角行) **Implicit**: $\exists a_{ij} \neq 0$ for $j \geq i$ (Not Explicit)

4.3 Accuracy of RK Method | Order Condition

Some Notations: If $\mathbf{y} = f'(\mathbf{y})$ where $f(\mathbf{y}) : \mathbb{R}^d \to \mathbb{R}^d$. Def $f' = (\frac{\partial f_i}{\partial y_j})_{1 \leq i \leq d, 1 \leq j \leq d}$ (行向量) $\qquad f'' = (\frac{\partial^2 f_i}{\partial y_j \partial y_k})_{1 \leq i \leq d, 1 \leq j,k \leq d}$

· Def: $f''(\mathbf{a}, \mathbf{b}) = \sum_{j=1}^d \sum_{k=1}^d \frac{\partial^2 f_i}{\partial y_j \partial y_k} a_j b_k$ $\ |\ y' = f$ $\quad y'' = \sum_{j=1}^d \frac{\partial f_i}{\partial y_j} f_j = f'f$ $\quad y''' = \sum_{j=1}^d \sum_{k=1}^d \frac{\partial^2 f_i}{\partial y_j \partial y_k} y_j'(t) y_k'(t) + \sum_{j=1}^d \frac{\partial f_i}{\partial y_j} y_j''(t) = f''(f,f) + f'f'f$

· $\Phi_h(y) = y + hf + \frac{h^2}{2}f'f + \frac{h^3}{6}[f''(f,f) + f'f'f] + \mathcal{O}(h^4)$

Order Condition: RK method: $y_{n+1} = y_n + h\sum_{i=1}^s b_i f(Y_i)$, Let $z(h) = \Phi_h(y)$

$\Rightarrow$ If $z'(0) = y', z''(0) = y'', ..., z^{(n)} = y^{(n)} \Rightarrow$ **Convergent at order** n

· Order 1: $\sum_{i=1}^s b_i = 1$ $\qquad$ Order 2: (add) $\sum_{i=1}^s b_i c_i = \frac{1}{2}$ $\qquad$ Order 3: (add) $\sum_{i=1}^s b_i c_i^2 = \frac{1}{3}$ and $\sum_{i=1}^s \sum_{j=1}^s b_i a_{ij} c_j = \frac{1}{6}$

5 Stability of Runge-Kutta Methods **consider for autonomous** $y' = f(y)$

5.1 Basic Definition for Stability

Fixed Point-Exact: For ODEs $\frac{dy}{dt} = f(y)$, point y^* is fixed point if $f(y^*) = 0 \Leftrightarrow \Phi_t(y^*) = y^*$ **Set of Fixed Points**: $\mathcal{F} = \{y^* \in \mathbb{R}^d : f(y^*) = 0\}$

Fixed Point-Numerical: *One-step* method $\Psi_h(y)$, point y^* is fixed point if $y^* = \Psi_h(y^*)$ **Set of Fixed Points**: $\mathcal{F}_h = \{y^* \in \mathbb{R}^d : y^* = \Psi_h(y^*)\}$

Theorem: For Runge-Kutta method, $\mathcal{F} \subseteq \mathcal{F}_h$ $\qquad$ **Remark**: $\mathcal{F}_h \subseteq \mathcal{F}$ is NOT always true. $\quad$ If $\mathcal{F}_h = \mathcal{F}$, then the method is **regular**.

· the point in $\mathcal{F}_h \setminus \mathcal{F}$ is called **spurious fixed point**. $\qquad$ As $h \to \infty$, the *spurious* fixed points will tends to infinity.

· **Remark**: For Euler's Method, it's regular. (i.e. $\mathcal{F}_h = \mathcal{F}$)

Stability of Fixed Points: Fixed point y^*, the ODEs $\frac{dy}{dt} = f(y)$ with $y(0) = y_0$.

1. **Stable in the sense of Lyapunov**: Fixed point y^* is stable if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $||y_0 - y^*|| < \delta \Rightarrow ||y(t; y_0) - y^*|| < \varepsilon\ \forall t > 0$
2. **Asymptotically Stable**: Fixed point y^* is asymptotically stable if $\exists \delta > 0$ s.t. $||y_0 - y^*|| < \delta \Rightarrow \lim_{t \to \infty} ||y(t; y_0) - y^*|| = 0$
3. **Unstable**: Fixed point y^* is unstable if it's not stable. $\quad$ i.e. $\exists \epsilon > 0, \forall \delta > 0$ s.t. $||y_0 - y^*|| < \delta \Rightarrow ||y(t) - y^*|| \geq \varepsilon$ for some t.

5.2 Classification of Fixed Points

Linearization Theorem: Suppose $\frac{dy}{dt} = f(y)$, y^* is a fixed point. Let $J = f'(y^*)$ be the Jacobian matrix of f at y^*.

1. If $\forall$ eigenvalues of J in left complex half plane, then y^* is **asymptotically stable**.
2. If $\exists$ eigenvalues of J in right complex half plane, then y^* is **unstable**.

(Following is a special cases from HDE)

Generalized Eigenvectors: If λ is an repeated eigenvalue with eigenvalue ξ then:

Generalized Eigenvectors: η s.t. $(A - \lambda I)\eta = \xi$ $\qquad$ More generally: $(A - \lambda I)\eta_n = \eta_{n-1}$

Classification of Critical Points at y^* (Linear): $\quad r_1, r_2$ be sol of $det(J - \lambda I) = 0$. $\ ||\ \mathbb{C} : r = \lambda \pm i\mu (\mu > 0)$

If J constant, write sol: $\mathbf{x} = c_1 e^{r_1 t}\xi_1 + c_2 e^{r_2 t}\xi_2$ $\ ||\ $ $GM = 1$: $\mathbf{x} = c_1 e^{rt}\xi + c_2 e^{rt}(t\xi + \eta)$ $\qquad J = \begin{pmatrix} \partial_x F(\mathbf{x}_0) & \partial_y F(\mathbf{x}_0) \\ \partial_x G(\mathbf{x}_0) & \partial_y G(\mathbf{x}_0) \end{pmatrix}$ If $f(x,y) = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}$

$\mathbb{R}/\mathbb{C}$	Condition || Stability	Type || Name	Phase Plane Description	Other	
$\mathbb{R}$	$r_1 < r_2 < 0$ || asy.stab	N || NSk	向原点,ξ_2 直线,ξ_1 曲线,and ξ_1 周围 $y = \pm x^3$	$c_2 \neq 0, t \to \infty$:ξ_2 主导方向; $c_2 = 0, t \to \infty$:ξ_1 主导方向	PS:
	$r_1 > r_2 > 0$ || unstable	N || NSo	原点向外,ξ_2 直线,ξ_1 曲线,and ξ_1 周围 $y = \pm x^3$	$c_1 \neq 0, t \to \infty$:ξ_1 主导方向; $c_1 = 0, t \to \infty$:ξ_2 主导方向	**N** = Node
	$r_1 > 0 > r_2$ || unstable	SP || SP	$t \to \infty$,ξ_1 从原点向外,ξ_2 从外向原点 and: 像 $y = \pm\frac{1}{x}$, 同进同出	$t \to \pm\infty : \lvert\mathbf{x}\rvert \to \infty$; $t \to \infty : c_1, c_2 \neq 0, \lvert\mathbf{x}\rvert \to \infty, \xi_1$ 主导; $t \to \infty : c_2 = 0, \lvert\mathbf{x}\rvert \to \infty, \xi_1$ 主导; $t \to \infty : c_1 = 0, \lvert\mathbf{x}\rvert \to 0, \xi_2$ 主导	**PN** = Proper Node **IN** = Improper
	$r_1 = r_2 < 0$, GM=2 || asy.stab	PN || PN or Stable Star	直线 向原点	直线, u_1/u_2 is t independent	or: Degenerate Node
	$r_1 = r_2 > 0$, GM=2 || unstable	PN || PN or Unstable Star	直线 从原点向外	直线, u_1/u_2 is t independent	**SP** = Saddle Point
	$r_1 = r_2 < 0$, GM=1 || asy.stab	IN (AL:Type: SpP) || IN (Stable)	S 曲线, 向原点	$t \to \infty, \lvert\mathbf{x}\rvert \to 0, \xi$ 主导 ps: 旋转方向大体和 $\eta + c_2\xi$ 方向相同	**SpP** = spiral point
	$r_1 = r_2 > 0$, GM=1 || unstable	IN (AL:Type: SpP) || IN (Unstable)	S 曲线, 从原点向外	$t \to \infty, \lvert\mathbf{x}\rvert \to \infty, \xi$ 主导 ps: 旋转方向大体和 $\eta + c_2\xi$ 方向相同	or: Focus Point
$\mathbb{C}$	$\lambda \neq 0, \lambda > 0$ || unstable	SpP || Unstable Focus	向外椭圆 (elliptical) 螺旋	$t \to \infty, \lvert\mathbf{x}\rvert \to \infty$ ps: 考虑 $J = (a, b; c, d)$, 如果 bc>0, 顺时针，如果 bc<0, 逆时针	**C** = Center
	$\lambda \neq 0, \lambda < 0$ || asy.stab	SpP || Stable Focus	向内椭圆 (elliptical) 螺旋	$t \to \infty, \lvert\mathbf{x}\rvert \to 0$ ps: 考虑 $J = (a, b; c, d)$, 如果 bc>0, 顺时针, 如果 bc<0, 逆时针	**NSk** = Nodal Sink
	$\lambda = 0$ || stable (AL:Indeterminate)	C (AL:C or SpP) || C	椭圆 (elliptical) and 半长轴 ξ 实部方向	Bounded trajectory or $\exists$ Periodic Trajectories	**NSo** = Nodal Source

5.3 Stability of Fixed Points of Maps (Numerical)

Definition: For flow map Ψ from $\mathbb{R}^d \to \mathbb{R}^d$. Def $y^n(y_0) :=$ the n-th iterate of y_0 under Ψ. i.e. $y^n = y_n$; $y_n = \Psi(y_{n-1})$

Stability of Fixed Points of Maps: Fixed point y^*, the map Ψ with $y^* = \Psi(y^*)$.

1. **Stable in the sense of Lyapunov**: y^* is stable if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $||y_0 - y^*|| < \delta \Rightarrow ||y^n(y_0) - y^*|| < \varepsilon \ \forall n \geq 0$
2. **Asymptotically Stable**: y^* is asymptotically stable if $\exists \delta > 0$ s.t. $||y_0 - y^*|| < \delta \Rightarrow \lim_{n\to\infty} ||y^n(y_0) - y^*|| = 0$
3. **Unstable**: y^* is unstable if it's not stable. i.e. $\exists \epsilon > 0, \forall \delta > 0$ s.t. $||y_0 - y^*|| < \delta \Rightarrow ||y^n(y_0) - y^*|| \geq \varepsilon$ for some n.

Spectral Radius: For matrix K, $\rho(K) = \max\{|\lambda| : \lambda \text{ is eigenvalue of } K\}$

Theorem|Spectral Radius: Let $z_n = ||K^n y_0||$, where $K \in \mathbb{R}^{d\times d}$ is the matrix. Then:

1. $\rho(K) < 1 \Leftrightarrow \lim_{n\to\infty} z_n = 0$
2. $\rho(K) > 1 \Leftrightarrow \lim_{n\to\infty} z_n = \infty$
3. If $\rho(K) = 1$ and *eigenvalues* of K are *semisimple* (i.e. No generalized eigenvector), then $\{z_n\}$ is bounded.

Theorem|Connect to Stability: For smooth (C^2) map Ψ, $y^* = \Psi(y^*)$. Let $K = \Psi'(y^*)$, for iteration $y_{n+1} = \Psi(y_n)$, we have:

1. $\rho(K) < 1 \Rightarrow y^*$ is *asymptotically stable*
2. $\rho(K) > 1 \Rightarrow y^*$ is *unstable*

5.4 Linear Stability of Numerical Methods

Special Case|Euler Method: For $\frac{dy}{dt} = By$, Using Euler method: $y_{n+1} = (I + hB)y_n$. where λ_i is eigenvalues of B. Assume $f(y) = \lambda y$

1. The origin is *stable* if $||I + h\lambda_i|| \leq 1 \ \forall i$
2. The origin is *asymptotically stable* if $||I + h\lambda_i|| < 1 \ \forall i$
3. The origin is *unstable* if $||I + hB|| > 1$

   ps: 即 $h\lambda_i$ 在复平面上以 $z = -1$ 为圆心, 半径为 1 的圆内 ← 称为 **Region of absolute stability**

Stability function R, P: Let P be polynomial function and R be rational function.

If RK is *explicit*, then $y_{n+1} = P(\mu)y_n$; If RK is *implicit*, then $y_{n+1} = R(\mu)y_n$ where $\mu = h\lambda$

Stability function $R(\mu)$|Special Case: For $\frac{dy}{dt} = \lambda y$ All RK methods can be written as: where: b^T, A are from *Butcher Table*. $\mathbf{1} = [1, ..., 1]^T$

I. $Y_i = y_n + \mu \sum_{j=1}^{s} a_{ij} Y_j \quad (Y = y_n \mathbf{1} + \mu A Y) \qquad y_{n+1} = y_n + \mu \sum_{j=1}^{s} b_j Y_j = y_n + \mu b^T Y$

II. $R(\mu) = 1 + \mu b^T (I - \mu A)^{-1} \mathbf{1}$ **III.** $y_{n+1} = R(\mu) y_n$ where $\mu = h\lambda$

Stability function $R(\mu)$|General: For $\frac{dy}{dt} = By$ where: b^T, A are from *Butcher Table*. Λ, U 是 B 的特征值分解 $U^{-1}BU = \Lambda$ 此时 z_n, y_n 是向量

I. Let $y_n = U z_n$ and $Y_i = U Z_i$:

Then $Z_i = z_n + h \sum_{j=1}^{s} a_{ij} \Lambda Z_j \quad (Z_j^{(i)} = z_n^{(i)} \mathbf{1} + \mu A Z_j^{(i)} \ \forall i) \qquad z_{n+1} = z_n + h \sum_{i=1}^{s} b_i \Lambda Z_i \quad (z_{n+1}^{(i)} = z_n^{(i)} + \mu \sum_{j=1}^{s} b_j Z_j^{(i)})$

II. $\frac{dz}{dt} = \Lambda z \quad \Rightarrow \quad \frac{dz^{(i)}}{dt} = \lambda_i z^{(i)} \quad \Rightarrow \quad z_{n+1}^{(i)} = R(\mu) z_n^{(i)}$ where $\mu = h\lambda_i$ (回到前一个)

Theorem: For $\frac{dy}{dt} = By$ with $\lambda_1, ..., \lambda_d$ be eigenvalues of B. The RK method is *stable*|*asy.stab* at *origin* iff:

The Same method also *stable*|*asy.stab* at *origin* for $\frac{dz}{dt} = \lambda_i z \ \forall i$

Corollary: For $\frac{dy}{dt} = By$ with B diagonalizable. An RK Method with *stability function* $R(\mu)$ is *stable*|*asy.stab*|*unstable* at *origin* iff: Assume $f(y) = \lambda_i y$

$|R(\mu)| \leq 1$ or $|R(\mu)| < 1$ or $|R(\mu)| > 1 \quad \forall \mu = h\lambda_i \ \forall i$ we can write $\sigma(B) = \{\lambda_1, ..., \lambda_d\}$ the set of eigenvalues of B

Remark: 这里的 $R(\mu)$ 是指 B 分解后的每一个特征值 λ_i 的 $R(\mu)$, 而不是 B 的 $R(\mu)$

5.5 Stability Region and A-stability

Stability Region: $\frac{dy}{dt} = By$. An RK method, the *stability region* is the set of μ where $\hat{R}(\mu) = |R(\mu)| < 1$. ($f(y) = \lambda y$, 如 y 是向量,$R(\mu)$ 按上面 corollary 的 remark 所说)

1. Euler's Method: $\hat{R}(\mu) = |1 + \mu| \quad \Rightarrow \quad \mu \in \{z \in \mathbb{C} : |1 + z| < 1\}$ (-1 处半径为 1 的圆)
2. Trapezoidal Rule: $\hat{R}(\mu) = \left|\frac{1+\mu/2}{1-\mu/2}\right| \quad \Rightarrow \quad \mu \in \{z \in \mathbb{C} : |1 + z/2| < |1 - z/2|\}$ (left complex half-plane, A-stable)
3. Implicit Euler: $\hat{R}(\mu) = |1 - \mu|^{-1} \quad \Rightarrow \quad \mu \in \{z \in \mathbb{C} : |1 - z| > 1\}$ (-1 处半径为 1 的圆外侧)
4. RK4: $\hat{R}(\mu) = \left|1 + \mu + \frac{\mu^2}{2} + \frac{\mu^3}{6} + \frac{\mu^4}{24}\right| \quad \Rightarrow$ Using $R(\mu) = e^{i\theta}$ to find the region.

A-Stable: An RK method is *A-stable* if its *stability region* contains the entire *left complex half-plane*. (i.e. $\Re(z) < 0$)

6 Linear Multistep Methods consider for autonomous $y' = f(y)$

Assume $\frac{dy}{dt} = f(y)$ with $y(t_0) = y_0$. Let y_n' denote $f(y_n)$; Let $y'(t_n)$ denote $f(y(t_n))$

6.1 Derivation of LMM | Algebra Operators

Linear Multistep Methods (LMM): For k-step LMM: $\alpha_k y_{n+k} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f(y_{n+j})$ where ${}^1\alpha_k \neq 0$, ${}^2\ \alpha_0 \neq 0$ or $\beta_0 \neq 0$

· ps: Usually, coefficients are *normalized* to have $\alpha_k = 1$ or $\sum_{j=0}^{k} \beta_j = 1$. **Implicit**: If $\beta_k \neq 0$ **Explicit**: If $\beta_k = 0$

AB Schemes Construction|Using Interpolation: Adams-Bashforth schemes can be constructed by: Consider k points (t_{n+j}, y_{n+j}') for $j = 0, ..., k-1$.

1. Let $\prod_k^f(t)$ be the *Lagrange polynomial* which passes through (t_{n+j}, y_{n+j}').
2. The AB scheme is: $y_{n+k} = y_{n+k-1} + \int_{t_{n+k-1}}^{t_{n+k}} \prod_k^f(t) dt$

   Remark: Adams-Moulton schemes 同理: 考虑 $k + 1$ points (t_{n+j}, y_{n+j}') for $j = 0, ..., k$.

   Then, we can found $\widehat{\prod}_k^f(t)$, and $\quad y_{n+k} = y_{n+k-1} + \int_{t_{n+k-1}}^{t_{n+k}} \widehat{\prod}_k^f(t) dt$

Algebra Operators: Algebra Operators is a function which maps a function to another function.

1. **shift operator**: $E_h g(t) = g(t+h)$ **forward difference operator**: $\Delta_h g(t) = g(t+h) - g(t)$
2. **Identity Operator**: $1g(t) = g(t)$ **Differentiation operator**: $Dg(t) = g'(t)$
3. **backward difference operator**: $\nabla_h g(t) = g(t) - g(t-h)$

Properties of Algebra Operators:

<table>
<tr><td>$\Delta_h = E_h - 1$</td><td>$E_h = e^{hD}$</td><td>$e^{hD} = 1 + \Delta_h$</td><td>$D = \frac{1}{h}\ln[1+\Delta_h]$</td><td>$g(t) = e^{(t-t_n)D} g(t_n)$</td><td>$g(t_{n+1}) = e^{hD} g(t_n)$</td></tr>
<tr><td>$E_h^{-1} = e^{-hD}$</td><td colspan="2">$D = -\frac{1}{h}\ln[E_h^{-1}] = -\frac{1}{h}\ln[1-\nabla_h]$</td><td>$1 - E_h^{-1} = \nabla_h$</td><td colspan="2">$D = \frac{1}{h}[\nabla_h + \frac{1}{2}\nabla_h^2 + \frac{1}{3}\nabla_h^3 + \cdots]$</td></tr>
<tr><td colspan="3">$e^{hD} g(t) = g(t+h) = g(t) + hDg(t) + \frac{h^2}{2} D^2 g(t) + \cdots$</td><td colspan="3">$g(t) = \left[1 + \frac{t-t_n}{1!\cdot h}\Delta_h + \frac{(t-t_n)(t-t_n-h)}{2!\cdot h^2}\Delta_h^2 + \frac{(t-t_n)(t-t_n-h)(t-t_n-2h)}{3!\cdot h^3}\Delta_h^3 + \cdots\right] g(t_n)$</td></tr>
</table>

BDF Method: For $y' = f(t, y(t))$. Since $Dy(t) = y'(t)$ and $D = \frac{1}{h}[\nabla_h + \frac{1}{2}\nabla_h^2 + \frac{1}{3}\nabla_h^3 + \cdots]$.
we can get the BDF method by $\frac{1}{h}[\nabla_h + \frac{1}{2}\nabla_h^2 + \frac{1}{3}\nabla_h^3 + \cdots] y(t) = f(t, y(t))$. 选择 D 的前几项作为估计.

6.2 Order of Accuracy|Consistency

First/Second Characteristic Polynomials: For k-step LMM: $\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f(y_{n+j})$, we define:
First Poly: $\rho(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta^j$ **Second Poly**: $\sigma(\zeta) = \sum_{j=0}^{k} \beta_j \zeta^j$

Linear Case: For scalar, linear, test equation $y' = \lambda y$, we have $\rho(\zeta) - h\lambda\sigma(\zeta) = 0$.
"General Solution": $y_n = C_1 \zeta_1^n + \ldots + C_k \zeta_k^n$ where $\zeta_1, \ldots, \zeta_k$ are roots of $\rho(\zeta) - h\lambda\sigma(\zeta) = 0$.

Residual: $r_n := \sum_{j=0}^{k} \alpha_j y(t_{n+j}) - h \sum_{j=0}^{k} \beta_j y'(t_{n+j})$ Residual accumulated(累积) in the $n+k-1$-th step.

1. **Taylor Series Expansion**|$y(t_{n+j})$: $y(t_{n+j}) = y(t_n) + jhy'(t_n) + \frac{j^2h^2}{2} y''(t_n) + \cdots = \sum_{i=0}^{\infty} \frac{(jh)^i}{i!} y^{(i)}(t_n)$
2. **Taylor Series Expansion**|$y'(t_{n+j})$: $y'(t_{n+j}) = y'(t_n) + jhy''(t_n) + \frac{j^2h^2}{2} y'''(t_n) + \cdots = \sum_{i=0}^{\infty} \frac{(jh)^i}{i!} y^{(i+1)}(t_n)$

Consistency: An LMM is *consistent* if $r_n = \mathcal{O}(h^{p+1})$ for all sufficiently smooth f. with p be the order of the method.

1. **Test I**: LMM is *consistent* with order p if: $\sum_{j=0}^{k} \alpha_j = 0$ and $\sum_{j=0}^{k} j^i \alpha_j = i \sum_{j=0}^{k} j^{i-1} \beta_j$ for $i = 1, \ldots, p$
2. **Test II**: LMM is *consistent* with order p if: $\rho(e^z) - z\sigma(e^z) = \mathcal{O}(z^{p+1})$.
3. **Test III**: LMM is *consistent* with order p if: $\frac{\rho(z)}{\log(z)} - \sigma(z) = \mathcal{O}((z-1)^p)$.
   Remark: Test I shows that: $\rho(1) = 0 \Rightarrow 1$ is always a root of $\rho(\zeta) = 0$.
   Special Thing: If it's consistent $\Rightarrow$ $\rho'(1) = \sigma(1)$

6.3 Convergence of LMM

Starting Procedure: A LLM is incomplete without a starting procedure. (i.e. 需要初始值 $y_1, \ldots, y_{k-1}$)

Root Condition: A LMM satisfies the *root condition* if: [1] all roots of $\rho(\zeta) = 0$ have modulus $|\zeta| \leq 1$.
[2] only one root of $\rho(\zeta) = 0$ has modulus $|\zeta| = 1$.

Convergence Theorem: A k-step LMM with starting procedure satisfying $\lim_{h\to 0} y_j = y(t_0 + jh)$ for $j = 1, \ldots, k-1$. (i.e. 初始值 y_j 收敛到精确值 $y(t_0 + jh)$)
The LMM is convergent $\Leftrightarrow$ LMM is consistent with $p \geq 1$ and satisfies the root condition.
Remark: If starting procedure is p-th order accurate (i.e. $y_j = y(t_0 + jh) + \mathcal{O}(h^p)$) $\Rightarrow$ The LMM is convergent (with order p) i.e. $\max_{0\leq n\leq N} |y_n - y(t_n)| \leq Ch^p$

Order of Convergence: The *maximum* order p of a k-step LLM *satisfying the root condition* is:
$p = k$ (Explicit Method); $p = k+1$ (Implicit Method|odd k); $p = k+2$ (Implicit Method|even k).

6.4 Stability

Stability Region: For a test problem $y' = \lambda y$, let $z = h\lambda$, then k-step LMM have, we consider the equation: $\rho(\zeta) - z\sigma(\zeta) = 0$.
The *stability region* is $\mathcal{S} = \{z \in \mathbb{C} : \rho(\zeta) - z\sigma(\zeta) = 0$ has all roots ζ with $|\zeta| < 1\}$
The *boundary of stability region* is $\partial\mathcal{S} = \left\{z \in \mathbb{C} : z = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}, \theta \in [-\pi, \pi]\right\}$

A-Stable|Unconditionally Stable: A LMM is *A-stable* if its *stability region* contains the entire *left complex half-plane*. (i.e. $\Re(z) < 0$)

Theorem: An A-stable LMM has order $p \leq 2$.

7 Appendix

7.1 Common Numerical Method | Order Condition

One-step Methods:

Method	Formula	Order	Stability
Euler's Method	$y_{n+1} = y_n + h\,f(t_n, y_n)$	1	$\lvert 1 + h\lambda \rvert < 1$
Backward Euler	$y_{n+1} = y_n + h\,f(t_{n+1}, y_{n+1})$	1	$\left\lvert \frac{1}{1-h\lambda} \right\rvert < 1$ (A-stable)
Trapezoidal Rule	$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]$	2	A-stable; $R(z) = \frac{1+z/2}{1-z/2}$
Midpoint Method	$y_{n+1} = y_n + h\,f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}f(t_n, y_n)\right)$	2	$\left\lvert 1 + h\lambda + \frac{(h\lambda)^2}{2} \right\rvert < 1$
Heun's Method	$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f\left(t_{n+1}, y_n + h\,f(t_n, y_n)\right)\right]$	2	$\left\lvert 1 + h\lambda + \frac{(h\lambda)^2}{2} \right\rvert < 1$
Theta Method	$y_{n+1} = y_n + h\left[(1-\theta)f(t_n, y_n) + \theta f(t_{n+1}, y_{n+1})\right]$	1 (or 2 if $\theta = \frac{1}{2}$)	$R(z) = \frac{1+(1-\theta)z}{1-\theta z}$
RK4 Method	见 Butcher Table	4	$R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}$
2-Stage Gauss-Legendre	$\begin{array}{c\|cc} \frac{1}{2} - \frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & 1/2 & 1/2 \end{array}$	4	A-stable

Multi-step Methods:

Name	Formula	Step	Accuracy
Leapfrog Method	$y_{n+2} = y_n + 2h\,f(t_{n+1}, y_{n+1})$	2	
Adams-Bashforth Method 1	$y_{n+1} = y_n + h\,f(t_n, y_n)$	1	
Adams-Bashforth Method 2	$y_{n+2} = y_{n+1} + \frac{h}{2}\left[3f(t_{n+1}, y_{n+1}) - f(t_n, y_n)\right]$	2	
Adams-Bashforth Method 3	$y_{n+3} = y_{n+2} + \frac{h}{12}\left[23f(t_{n+2}, y_{n+2}) - 16f(t_{n+1}, y_{n+1}) + 5f(t_n, y_n)\right]$	3	
Backward Differentiation Formula 2	$y_{n+2} = \frac{4}{3}y_{n+1} - \frac{1}{3}y_n + \frac{2h}{3}f(t_{n+2}, y_{n+2})$	2	
Backward Differentiation Formula 3	$y_{n+3} = \frac{18}{11}y_{n+2} - \frac{9}{11}y_{n+1} + \frac{2}{11}y_n + \frac{6h}{11}f(t_{n+3}, y_{n+3})$	3	
Class of Adams-Moulton Methods: $\alpha_k = 1, \alpha_{k-1} = -1, \alpha_j = 0, \forall j < k-1$ | Class of Backward Differentiation Formula (BDF): $\beta_j = 0, \forall j < k$			

RK Order Condition

1. **order 1**: $\sum_{i=1}^{s} b_i = 1$
2. **order 2**: $\sum_{i=1}^{s} b_i c_i = \frac{1}{2}$
3. **order 3**: $\sum_{i=1}^{s} b_i c_i^2 = \frac{1}{3}$ and $\sum_{i=1}^{s}\sum_{j=1}^{s} b_i a_{ij} c_j = \frac{1}{6}$
4. **order 4**: $\sum_{i=1}^{s} b_i c_i^3 = \frac{1}{4}$, $\sum_{i=1}^{s}\sum_{j=1}^{s} b_i a_{ij} c_j^2 = \frac{1}{8}$, $\sum_{i=1}^{s}\sum_{j=1}^{s} b_i\, a_{ij} c_j^2 = \frac{1}{12}$, $\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{s} b_i a_{ij} a_{jk} c_k = \frac{1}{24}$

7.2 Useful Series | Common RK Methods

Common Runge-Kutta Methods (Butcher Table):

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}$$

Example

$$\begin{array}{c|c} 0 & \\ \hline & 1 \end{array}$$

RK1 (Euler's Method)

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array}$$

RK2 (Heun's Method)

$$\begin{array}{c|ccc} 0 & & & \\ 1/2 & 1/2 & & \\ 1 & -1 & 2 & \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

RK3

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

RK4 (Classical/Famous)

Useful Series:

$f(x)$	Taylor	Series	R	$f(x)$	Taylor	Series	R
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + \ldots$	1	$\frac{1}{(1-x)^2}$	$\sum_{n=1}^{\infty} nx^{n-1}$	$1 + 2x + 3x^2 + 4x^3 + \ldots$	1
$\frac{2}{(1-x)^3}$	$\sum_{n=2}^{\infty} n(n-1)x^{n-2}$	$2 + 6x + 12x^2 + 20x^3 + \ldots$	1	e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$	∞
$\ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$	1	$-\ln(1-x)$	$\sum_{n=1}^{\infty} \frac{x^n}{n}$	$x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$	1
$\sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$	∞	$\cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$	∞
$\arctan x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$	1	$\sinh x$	$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$	∞
$\cosh x$	$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$	∞	$(1+x)^k$	$\sum_{n=0}^{\infty} \binom{k}{n} x^n$	$1 + kx + \frac{k(k-1)x^2}{2!} + \ldots$	1
$\ln x$	$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{(x-1)^n}{n}$	$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \ldots$	$1, 0 < x < 2$	$\frac{1}{1+x}$	$\sum_{n=0}^{\infty} (-1)^n x^n$	$1 - x + x^2 - x^3 + \ldots$	1

If $R(z) - e^z = \mathcal{O}(z^{p+1})$, then we can *assume* the order of the method is p.

Inverse of 2×2 **Matrix**: For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have: $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Explain in one sentence what it means to say that Euler's Method is a first order method:

On a sufficiently smooth problem, with stepsize h the local error behaves like $\mathcal{O}(h^2)$.

Perform a calculation to explain why one typically uses a log-log plot to determine the order p **of a numerical method**:

If the global error satisfies $E(h) \approx \mathcal{O}(h^p)$, then taking the logarithm of both sides gives: $\log(E(h)) \approx p\log(h)$. So, if we plot $\log(E(h))$ vs. $\log(h)$, the slope of the line will be p, indicating the order (p) of the method.