

1 General LP

General LP Problem: Decision Variables: x_i parameters: a_{ij}, b_i, c_i Objective Function: f Constraints: subject to 后边的部分

LP problem can be written as:

$$\begin{array}{ll} \text{maximize} & f = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\ & x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \end{array}$$

Feasible Solution: If \mathbf{x} satisfies all constraints (i.e. $A\mathbf{x} \leq \mathbf{b}$), then \mathbf{x} is a feasible solution. (可行解) **Optimal Sol:** (最优解) (可多个)

Find Optimal Solution: Graphical Method: 略. **Vertex Enumeration:** 所有顶点检查, 找出最优解. **Simplex Method:** 后面详述.

Slack Variables: For each inequality constraint, we introduce a slack variable x_i ($i > n$) to convert it to an equation. (松弛变量)

LP problem can be written as: ps: $x_i \geq 0$ ($i > n$).

$$\begin{array}{ll} \text{maximize} & f = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{n+2} = b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + x_{n+m} = b_m \\ & x_1 \geq 0, x_2 \geq 0, \dots, x_{n+m} \geq 0 \end{array}$$

Feasible Region: It's the set K of the solutions to $\bar{A}\mathbf{x} = \mathbf{b}$ **Convex Set:** The set K is convex if $\forall \mathbf{x}, \mathbf{x}' \in K, \forall \theta \in [0, 1], \mathbf{x}_\theta = (1 - \theta)\mathbf{x} + \theta\mathbf{x}' \in K$.

Vertex on Convex Set: A vertex of the convex set K is a point $\mathbf{x} \in K$ which doesn't lie strictly inside any line segment connecting two points in K .

• **Theorem:** If LP has a unique optimal solution is a vertex.

Theorem: If LP has a non-unique solution, \exists optimal solution at vertex

We can also write the LP problem in matrix form:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{array}{ll} \text{maximize} & f = \mathbf{c}^T \mathbf{x} \\ \text{subject to} & A\mathbf{x} \leq \mathbf{b} \quad \mathbf{x} \geq 0 \end{array}$$

Optimal Sol: (最优解) (可多个)

We can also write the LP problem in matrix form:

$$\bar{A} = [A \ I_m], \mathbf{b} = \mathbf{b}, \bar{\mathbf{c}} = \begin{pmatrix} \mathbf{c} \\ \mathbf{0}_{m \times 1} \end{pmatrix}, \mathbf{x} = \begin{pmatrix} \mathbf{x} \\ \vdots \\ x_{n+m} \end{pmatrix}$$

$$\begin{array}{ll} \text{maximize} & f = \bar{\mathbf{c}}^T \mathbf{x} \\ \text{subject to} & \bar{\mathbf{A}}\mathbf{x} = \mathbf{b} \quad \mathbf{x} \geq 0 \end{array}$$

Solving: $\mathbf{x} = \mathbf{x}_b + \mathbf{v}$ where $\bar{A}\mathbf{v} = 0$ and $\mathbf{x}_b = [\mathbf{0} \ \mathbf{b}]^T$
t: The set K is convex if $\forall \mathbf{x}, \mathbf{x}' \in K, \forall \theta \in [0, 1], \mathbf{x}_\theta = (1 - \theta)\mathbf{x} + \theta\mathbf{x}' \in K$.

2 Simplex Method

2.1 Simplex Algorithm

Solve LP Problem: Assume $f = \bar{\mathbf{c}}^T \mathbf{x}$ with $\bar{\mathbf{A}}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$.

1. 修改 x_i /列顺序, A 中换顺序后得到可逆的 $B_{m \times m} \Rightarrow \bar{A}x = b \Leftrightarrow Bx_B + Nx_N = b \quad \bar{c} \rightarrow [c_B \ c_N]^T \quad \bar{A} \rightarrow [B \ N] \quad x \rightarrow [x_B \ x_N]^T \quad \text{Let } \hat{b} = B^{-1}b \quad \hat{N} = B^{-1}N$
2. **Solution:** $x_B = \hat{b} - \hat{N}x_N$ If $x_N = 0 \Rightarrow$ It's a basic solution. (But we need to check whether $x_B \geq 0$) Using \mathcal{B}, \mathcal{N} : Index set of independent/else.
3. **Basic Variables:** x_B **Nonbasic Variables:** x_N
4. **At Basic Solution:** $x_B = \hat{b}, x_N = 0 \Rightarrow$ If $\hat{b} \geq 0$. Corresponding x is: ¹ vertex of K ; ² **Basic Feasible Solution (BFS)**
5. **Basic Costs:** c_B^T **Nonbasic Costs:** c_N^T **Reduced Costs:** $\hat{c}_N = c_N - \hat{N}^T c_B = c_N - N^T B^{-T} c_B \quad \hat{f} = c_B^T \hat{b} \quad x_B, x_N \geq 0$
6. **Objective Value:** $f = \bar{c}^T x = c_B^T x_B + c_N^T x_N = \hat{f} + \hat{c}_N^T x_N$ If $\hat{c}_N \leq 0$, then $f \leq \hat{f} \Rightarrow$ Corresponding x is optimal.
7. If $\hat{c}_N \leq 0$ doesn't hold, we using **Simplex Algorithm**.

Simplex Algorithm:

1. **Initial Basic Feasible Solution:** Try $\mathcal{B} = \{n+1, \dots, n+m\}$ and $\mathcal{N} = \{1, \dots, n\} \Rightarrow B = I, N = A, \mathbf{x}_B = \widehat{\mathbf{b}} = \mathbf{b}, \mathbf{x}_N = \mathbf{0}, \mathbf{c}_B = \mathbf{0}, \mathbf{c}_N = \mathbf{c}$
2. If $\mathbf{b} \geq \mathbf{0} \Rightarrow$ Basis is feasible + cont. Else: Basis B is not feasible. Go to “Cases If $\mathbf{b} \not\geq \mathbf{0}$ ”
3. If $\widehat{\mathbf{c}}_N \leq \mathbf{0} \Rightarrow$ Optimal Solution Else: cont.
4. Let $q' \in \mathcal{N}$ 对应 $\widehat{\mathbf{c}}_N$ 中最大 positive 分量的 index 同理, 对应的最大正分量值为 \widehat{c}_q 对应的 variable 为 $x_{q'}$ 对应 N 中的 $[q']$ 列为 \mathbf{a}_q
5. Let $\widehat{\mathbf{a}}_q = B^{-1}\mathbf{a}_q$. If $\widehat{\mathbf{a}}_q \leq \mathbf{0} \Rightarrow$ LP is unbounded. Else: LP is bounded + cont.
6. Let $p' \in \mathcal{B}$ be the index corresponding to $p = \arg \min_{i=1, \dots, m; \widehat{a}_{iq} > 0} \frac{\widehat{b}_i}{\widehat{a}_{iq}}$ p 是对应的 index, not value 用 $\bar{\alpha} = \frac{\widehat{b}_p}{\widehat{a}_{pq}}$ 代表值
 $\widehat{b}_i, \widehat{a}_{iq}$ 代表 $\widehat{\mathbf{b}}$ 的第 i 个分量, $\widehat{\mathbf{a}}_q$ 的第 i 个分量
 p' 代表能使 $\frac{\widehat{b}_i}{\widehat{a}_{iq}}$ 的值最小的 index, 前提条件是 $\widehat{a}_{iq} > 0$ 对应的 variable 为 $x_{p'}$
7. Exchange p' and q' between \mathcal{B} and $\mathcal{N} \Rightarrow$ Update $\mathcal{B}, \mathcal{N}, B, N, \widehat{\mathbf{b}}, \dots$ ps: New $B := B + (a_q - B e_p) e_p^T$
8. Go to 3.

Case: If $b \neq 0$: Phase I Problem

1. Subtract **artificial variables** $x_{n+m+1}, \dots, x_{n+m+m} \geq 0$ and change objective function f to:

$$\begin{array}{ll} \max & f = \\ \text{s. t.} & a_{11}x_1 + \dots + a_{1n}x_n + x_{n+1} - x_{n+m+1} - \dots - x_{n+2m} = b_1 \\ & \vdots \\ & a_{m1}x_1 + \dots + a_{mn}x_n + x_{n+m} - x_{n+2m} = b_m \\ & x_1 \geq 0, \dots, x_{n+2m} \geq 0 \end{array}$$

2. Let Basic Variables: \mathbf{x}_B = 第 i 个元素是: $x_{n+i} = b_i$ (if $b_i \geq 0$) 是: $x_{n+m+i} = -b_i$ (if $b_i < 0$) *
- Let Nonbasic Variables: \mathbf{x}_N = 第 i 个元素是: $x_{n+m+i} = 0$ (if $b_i \geq 0$) 是: $x_{n+i} = b_i$ (if $b_i < 0$)
- Let Basic Matrix B = 第 i 列是: \mathbf{e}_i (if $b_i \geq 0$) 否则: $-\mathbf{e}_i$ 其他列是 N 的对应列 Let $\hat{\mathbf{b}} = B^{-1}\mathbf{b} = |\mathbf{b}| \geq 0$
- Let $\mathbf{c}_B = \mathbf{x}_B$ 对应的 f 中的系数 \mathbf{c}_N = 同理 $\hat{\mathbf{c}}_N = \mathbf{c}_N - N^T B^{-T} \mathbf{c}_B$
3. Size of Infeasibility: is $\sum_{i=1}^m x_{n+m+i} \quad f = -\sum_{i=1}^m x_{n+m+i} \leq 0$
4. If $f = 0$, \mathbf{x} is a BFS. \Rightarrow Go to **Phase II Problem** (恢复到之前的 f 并删除 artificial variables)
- Else: Go to **Simplex Algorithm** 得到辅助问题 Phase I 的最优解. If $f < 0$: LP is infeasible. If $f = 0$: LP is feasible. \Rightarrow **Phase II**
- Case: If $\mathbf{b} \geq 0$: Phase II Problem**

2.2 More thing about Simplex Algorithm

Degeneracy: If $\hat{\mathbf{b}}$ has any zero component, then \mathbf{x} is a **degenerate vertex**. 如果 $\hat{\mathbf{b}}$ 的第 p 个分量 $\hat{b}_p = 0$, 那么单纯形法可能会陷入循环.

Theorem: If no degenerate situation, then Simplex Algorithm will terminate in a finite number of steps.

Examples: 1. The worst case of Simplex Algorithm. 2. Cyclic case of Simplex Algorithm.

- Klee-Minty Problem:** $\max f = \sum_{j=1}^n 10^{j-1} x_j$; s.t. $x_i + 2 \sum_{j=i+1}^n 10^{j-i} x_j \leq 100^{n-i}$ for $i = 1, \dots, n$; $x_i \geq 0$
has 2^n vertices. **Worst Case:** $2^n - 1$ iterations.
- Hall-McKinnon Problem:** $\max f = x_1 - 5.5x_2 + 0.75x_3 - 5.75x_4$; s.t. $2.5x_1 - 19.5x_2 - 3.5x_3 + 19.5x_4 + x_5 = 0$; $0.5x_1 - 3.5x_2 - 0.5x_3 + 3.5x_4 + x_6 = 0$; $x_i \geq 0$

2.3 Sparsity LP Problem

Implementation: 计算/编程中的计算化简 Consider: $\hat{\mathbf{b}} = B^{-1}\mathbf{b}$ $\hat{\mathbf{c}}_N = \mathbf{c}_N - N^T B^{-T} \mathbf{c}_B$ $\hat{\mathbf{a}}_q = B^{-1} \mathbf{a}_q$

- Solve: $B\hat{\mathbf{b}} = \mathbf{b}$ to get: $\hat{\mathbf{b}} = B^{-1}\mathbf{b}$
- Solve: $B^T \pi = \mathbf{c}_B$ to get: $\pi = B^{-T} \mathbf{c}_B$
- Solve: $\hat{\mathbf{c}}_N = \mathbf{c}_N - N^T \pi$ to get: $\hat{\mathbf{c}}_N = \mathbf{c}_N - N^T B^{-T} \mathbf{c}_B$
- Solve: $B^T \hat{\mathbf{a}}_q = \mathbf{a}_q$ to get: $\hat{\mathbf{a}}_q = B^{-T} \mathbf{a}_q$
- Matrix B is **sparse** and **invertible**. And in Simplex Algorithm, the next matrix B obtained by $B := B + (\mathbf{a}_q - B\mathbf{e}_p)\mathbf{e}_p^T$

Sparse LP Problem: For LP problem: $\max f = \mathbf{c}^T \mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{b}$ $\mathbf{x} \geq 0$

It is **sparse** if the matrix A is sparse. (i.e. Most of the elements of A are zero)

Find Inverse of B : Use **Gaussian Elimination** to decomposition B into LU where L is lower triangular and U is upper triangular.

Then we can solve $B\mathbf{x} = \mathbf{b}$ by solving $^1 L\mathbf{y} = \mathbf{b}$ and $^2 U\mathbf{x} = \mathbf{y}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} & l_{21}u_{14} + u_{24} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} & l_{31}u_{14} + l_{32}u_{24} + u_{34} \\ l_{41}u_{11} & l_{41}u_{12} + l_{42}u_{22} & l_{41}u_{13} + l_{42}u_{23} + l_{43}u_{33} & l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + u_{44} \end{bmatrix}$$

By Solving ■ \rightarrow ■ \rightarrow ■ \rightarrow ■ \rightarrow ■ \rightarrow ■