HCV Note

1 Basic Knowledge

Useful Complex Number Properties: $|Re(z)|, |Im(z)| \le |z|$ $\quad Re(z) = \frac{z+\bar{z}}{2}, Im(z) = \frac{z-\bar{z}}{2i}, |z|^2 = z\bar{z}$

Triangle (Reverse) Inequality: $|z_1 + z_2| \le |z_1| + |z_2|$ $\quad |z_1| - |z_2| \le |z_1 - z_2|$ $\quad (Re(zw) = 0 \Leftrightarrow \overline{zw} = -zw; Im(zw) = 0 \Leftrightarrow zw = \overline{zw})$

Argument: $\arg(z) := \{\theta : z = |z|e^{i\theta}\} = \{Arg(z) + 2\pi k : k \in \mathbb{Z}\}$ **Principle Value of Argument**: $Arg(z) \in (-\pi, \pi]$

· **Operations on Argument**: $arg(z_1 z_2) = arg(z_1) + arg(z_2)$ $\quad arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$ $\quad arg(\bar{z}) = -arg(z)$

2 Holomorphic Functions

Open/Closed/Punctured ε-disc: $D_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ $\quad \overline{D}_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| \le \varepsilon\}$ $\quad D'_\varepsilon(z_0) := \{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}$

Open/Closed Set in $\mathbb{C}$: $U \subset \mathbb{C}$ is **open** if $\forall z_0 \in U, \exists \varepsilon > 0, D_\varepsilon(z_0) \subseteq U$ $\quad U$ is **closed** if $\mathbb{C} \setminus U$ is open **Lemma**: $D_\varepsilon, D'_\varepsilon$ open, $\overline{D}_\varepsilon$ closed.

Limit Point of S: $z_0 \in \mathbb{C}$ is a limit point of S if: $\forall \varepsilon > 0, D'_\varepsilon(z_0) \cap S \ne \emptyset$ 聚点 **Bounded**: S is bounded if $\exists M > 0$ s.t. $|z| \le M, \forall z \in S$

Closed of Set S: $\overline{S}$:= 所有 S 的 limit point 和 S 的点. **Property**: Let $S \subseteq \mathbb{C}$, then S is closed $\Leftrightarrow$ $S = \overline{S}$.

Limit of sequence: Sequence $(z_n)_{n\in\mathbb{N}}$ has limit z if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \ge N \Rightarrow |z_n - z| < \varepsilon$. limit rules 依旧成立

1. **Lemma|Important**: $\lim z_n = z \Leftrightarrow \lim Re(z_n) = Re(z)$ and $\lim Im(z_n) = Im(z)$
2. **Cauchy**: Sequence $(z_n)_{n\in\mathbb{N}}$ is cauchy if: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall m, n \ge N \Rightarrow |z_m - z_n| < \varepsilon$ **Lemma**: Cauchy $\Leftrightarrow$ convergent.
3. **Lemma|Closed of Set**: $S \subseteq \mathbb{C}, z \in \mathbb{C}. \Rightarrow [\ z \in \overline{S} \Leftrightarrow \exists$ sequence $(z_n)_{n\in\mathbb{N}} \in S$ s.t. $\lim z_n = z\ ]$
4. **Bolzano-Weierstrass**: Every bounded sequence in $\mathbb{C}$ has a convergent subsequence.

Complex Functions: $\forall f : \mathbb{C} \to \mathbb{C}$ we can write it as: $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$

Limit of Function: $a_0 \in \mathbb{C}$ is the limit of f at z_0 if: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |z - z_0| < \delta \Rightarrow |f(z) - a_0| < \varepsilon$ limit rules 依旧成立

· **Lemma|Important**: $\lim_{z\to z_0} f(z) \Leftrightarrow \lim_{(x,y)\to(x_0,y_0)} u(x, y) = Re(a_0)$ and $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = Im(a_0)$

continuous of Function: f is continuous at z_0 if: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$ continuous rules 依旧成立

1. **Lemma|Important**: f is continuous at $z_0 \Leftrightarrow u, v$ are continuous at (x_0, y_0)
2. **'Extreme Value Theorem'**: f is continuous on a closed and bounded set $S \subseteq \mathbb{C}$, then $f(S)$ is closed and bounded.
3. **Lemma|continuous⇔open**: f is continuous $\Leftrightarrow \forall$ open set U, preimage $f^{-1}(U) := \{z \in \mathbb{C} | f(z) \in U\}$ is open.