

## 1 Basic Knowledge

**Def of ODE & ODEs:** (1st order) ODE:  $\frac{dy}{dt} = f(t, y)$  & ODEs:  $\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y})$ ,  $\mathbf{y} = (y_1, \dots, y_d)^T$ ,  $\mathbf{f}(t, \mathbf{y}) = (f_1(t, \mathbf{y}), \dots, f_d(t, \mathbf{y}))^T$

**Autonomous:**  $\frac{dy}{dt} = \mathbf{f}(\mathbf{y}) \Rightarrow$  autonomous ODE(s).  $\parallel \Downarrow$  New Autonomous ODEs:  $\frac{dy}{ds} = \mathbf{f}(y_{d+1}, \mathbf{y})$  and  $\frac{dy_{d+1}}{ds} = 1$

**Change to Autonomous:** For  $\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y})$ . Let  $y_{d+1} = t$  and new independent variable  $s$  s.t.  $\frac{dt}{ds} = 1 \uparrow$

**Linearity:** ODE:  $\frac{dy}{dt} = f(t, y)$  is linearity if  $f(t, y) = a(t)y + b(t)$   $\parallel$  ODEs: If each ODE is linear, then the ODEs are linear.

**Picard's Theorem:** If  $f(t, y)$  is continuous in  $D := \{(t, y) : t_0 \leq t \leq T, |y - y_0| < K\}$  and  $\exists L > 0$  (Lipschitz constant) s.t.  $\forall (t, u), (t, v) \in D \quad |f(t, u) - f(t, v)| \leq L|u - v|$  (ps: Can use MVT). And Assume that  $M_f(T - t_0) \leq K$ ,  $M_f := \max\{|f(t, u)| : (t, u) \in D\}$   
 $\Rightarrow$  **Then**,  $\exists$  a unique continuously differentiable solution  $y(t)$  to the IVP  $\frac{dy}{dt} = f(t, y)$ ,  $y(t_0) = y_0$  on  $t \in [t_0, T]$ .

**Existence & Uniqueness Theorem:** IVP  $\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y})$ ,  $\mathbf{y}(t_0) = \mathbf{y}_0$ . If  $f(t, y)$  and  $\frac{\partial f}{\partial y_i}$  are continuous in a neighborhood of  $(t_0, \mathbf{y}_0)$ .  
 $\Rightarrow$  **Then**,  $\exists I := (t_0 - \delta, t_0 + \delta)$  s.t.  $\exists$  a unique continuously differentiable solution  $\mathbf{y}(t)$  to the IVP on  $t \in I$ .

## 2 Acknowledge

Notation	Meaning	Notation	Meaning
$[a, b]$	Approximate function for $t \in [a, b]$	$t_0 = a \mid t_N = b$	Assume that $t_0 = a, t_N = b$
$N$	number of <b>timesteps</b> (i.e. Break up interval $[a, b]$ into $N$ equal-length sub-intervals)	$h$	<b>stepsize</b> ( $h = \frac{b-a}{N}$ )
$t_i$	Define $N + 1$ points: $t_0, t_1, \dots, t_N$	$t_m$	$t_m = a + h \cdot m = t_0 + h \cdot m$
$y_i$	Approximation of $y$ at point $t = t_i$ (Except $y_0$ )	$y(t_i)$	Exact value of $y$ at point $t = t_i$

## 3 Euler's Method and Taylor Series Method

**Euler's Method Algorithm:** Approx  $\frac{dy}{dt} = f(t, y)$ ,  $y(t_0) = y_0$  Euler Method:  $y_{n+1} = y_n + hf(t_0 + nh, y_n) = y_n + hf(t_n, y_n)$

**Lemma:** If  $v_{n+1} \leq Av_n + B \Rightarrow$  Then  $v_n \leq A^n v_0 + \frac{A^n - 1}{A - 1} B$  If  $|y''| < M$  and  $v_n = e_n := y_n - y(t_n)$ , then  $A = 1 + hL, B = h^2 M/2$

**Boundedness Theorem|Euler Method:** For  $\frac{dy}{dt} = f(t, y)$ ,  $y(a) = y_0$ :  
 $\exists$  <sup>1</sup> unique, <sup>2</sup> twice differentiable, solution  $y(t)$  on  $[a, b]$ , <sup>3</sup>  $y$  is continuous and <sup>4</sup>  $|\frac{\partial f}{\partial y}| \leq L$ .  
 $\Rightarrow$  the solution  $y_n$  given by Euler's method satisfies:  $e_n = |y_n - y(t_n)| \leq Dh$ ,  $D = e^{(b-a)L} \frac{M}{2L}$

**Order Notation ( $\mathcal{O}$ ):** we write  $z(h) = \mathcal{O}(h^p)$  if  $\exists C, h_0 > 0$  s.t.  $|z| \leq Ch^p, 0 < h < h_0$

**Flow Map ( $\Phi, \Psi$ ):** Consider  $\frac{dy}{dt} = f(t, y)$ .

- Exact Flow Map ( $\Phi$ ):**  $\Phi_{t_n, h}(y_n) = y(t_n + h)$  代表假设  $y(t_n) = y_n$  的情况下, 输入  $y_n$  在  $t_n + h$  时刻的精确值; 当不写  $t_n$  角标时, 默认要算的前一个时间点已知/精确
- Numerical Flow Map ( $\Psi$ ):**  $\Psi_{t_n, h}(y_n) = y_{n+1}$  代表假设  $y(t_n) = y_n$  的情况下, 输入  $y_n$  在  $t_n + h$  时刻的数值解; 当不写  $t_n$  角标时, 默认要算的前一个时间点已知/精确

**Remark:**  $\Phi_h(y(t_n)) = y(t_n + h)$   $\Psi_h(y(t_n)) = y_{n+1}$

**Find:** Generally, use  $\Phi_{t_0, h}(y_0) = y(t_0 + h)$  to find  $y(t_0 + h)$ ; and  $\Psi(y)$ : Numerical method for ODE.

**Find Numerical Method|Taylor Series Method:** Approx  $\frac{dy}{dt} = f(t, y)$ ,  $y(t_0) = y_0$  with  $n$ -order Methods

- Method:** 通过泰勒展开精确解, 取前  $n$  项作为近似解, 从而得到数值解.
- Taylor Series for  $\Phi$ :**  $\Phi_{t, h}(y) = y + hf(t, y) + \frac{1}{2}h^2[f_t(t, y) + f_y(t, y)f(t, y)] + \frac{1}{6}y'''(t, y)h^3 + \dots$  (For one variable  $y$ ) ps:  $y' = f, y'' = f_t + f_y f$
- Taylor Series:**  $y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{h^2}{2}y''(t_0) + \dots + \frac{h^{n-1}}{(n-1)!}y^{(n-1)}(t_0) + \frac{h^n}{n!}y^{(n)}(t^*), t^* \in [t, t + h]$

## 4 Convergence of One-Step Methods consider for autonomous $y' = f(y)$

### 4.1 Convergence | Consistent | Stable

**Global Error:** global error after  $n$  steps:  $e_n := y_n - y(t_n)$  **Local Error:** For one-step method is:  $le(y, h) = \Psi_h(y) - \Phi_h(y)$

**Consistent:** If  $||le(y, h)|| \leq Ch^{p+1} (\leq \mathcal{O}(h^{p+1}))$ ,  $C > 0 \Rightarrow$  Consistent at order  $p$ . **Stable:** If  $||\Psi_h(u) - \Psi_h(v)|| \leq (1 + h\hat{L})||u - v||$

**Convergent:** A method is convergent if:  $\forall T, \lim_{h \rightarrow 0} \max_{h=T/N, n=0, 1, \dots, N} ||e_n|| = 0 \Downarrow$  Then the global error satisfies:  $\max_{n=0, 1, \dots, N} ||e_n|| = \mathcal{O}(h^p)$   $p$ -th order

**Convergence of One-Step Method:** For  $y' = f(y)$ , and a one-step method  $\Psi_h(y)$  is <sup>1</sup> consistent at order  $p$  and <sup>2</sup> stable with  $\hat{L} \uparrow$ . (ps:  $C = \frac{C}{L}(e^{T\hat{L}} - 1)$ )

### 4.2 More One-Step Methods | Runge-Kutta Methods | Collocation

**Construction of More General one-step Method:** For  $y' = f(y)$ ,  $y(t_0) = y_0 \Rightarrow y(t + h) - y(t) = \int_t^{t+h} f(y(\tau))d\tau$

**Lagrange Interpolating Polynomials:** For function  $p(x)$ . Consider points:  $(c_1, g_1), \dots, (c_s, g_s)$ . where  $p(c_i) = g_i$ .

- Lagrange Interpolating Polynomials:** Let  $\ell_i(x) = \prod_{j=1, j \neq i}^s \frac{x - c_j}{c_i - c_j} \in \mathbb{P}_{s-1}$
- Polynomial Interpolation:**  $\exists! p(x) = \sum_{i=1}^s g_i \ell_i(x)$  (Can be proved by Honour Algebra)

**Interpolatory Quadrature:** 对于函数  $g(t) \in \mathbb{P}_{p-1}$ , 我可以通过插值求积的方法来近似求解积分: 以下展示  $[a, b]$  上的插值求积。

- Choose  $c_i$  points in  $[a, b]$ :  $c_1, \dots, c_s$ . Let  $g_i = g(c_i)$ . By using  $c_i, g_i$ , we can get  $\ell_i(x)$ .
- Define weights:  $b_i := \int_a^b \ell_i(x) dx$ . Then  $\int_a^b g(t) dt \approx \sum_{i=1}^s b_i g(c_i)$ .

**One-Step Collocation Methods:** 对于  $y' = f(y)$ ,  $y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(y(t))dt$ , 通过 Interpolatory Quadrature 来近似求解积分. 为了简化, 考虑 autonomous 的情况

- Choose  $c_1, \dots, c_s$  in  $[0, 1]$ , consider  $t_i = t_n + c_i h$ , then  $t_i \in [t_n, t_{n+1}]$ .

- Let  $F_i = f(y(t_i))$ , then we can get  $\ell_i(x)$  which pass through  $(c_i, F_i)$ .
- Let weights:  $b_i = \int_0^1 \ell_i(x)dx$ , and  $a_{ij} = \int_0^{c_i} \ell_j(x)dx$ . **Then**  $\star y_{n+1} = y_n + h \sum_{i=1}^s b_i F_i$ .  $\star$
- Moreover, we can get:  $F_i = f(Y_i)$ , where  $Y_i = y_n + h \sum_{j=1}^s a_{ij} F_j$ .

**Remark:** For choice of  $c_i$ : The optimal choice is attained by Gauss-Legendre collocation methods.

e.g.  $s = 1: c_1 = \frac{1}{2}; \quad s = 2: c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}, c_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}; \quad s = 3: c_1 = \frac{1}{2} - \frac{\sqrt{15}}{10}, c_2 = \frac{1}{2}, c_3 = \frac{1}{2} + \frac{\sqrt{15}}{10}$

**Runge-Kutta Methods:** Let  $y' = f(y)$  here we consider the autonomous case. The RK method has following form:

- Stage Values:**  $Y_i = y_n + h \sum_{j=1}^s a_{ij} f(Y_j) \quad i \in \{1, \dots, s\} \quad F_i = f(Y_i)$
- Update:**  $y_{n+1} = y_n + h \sum_{i=1}^s b_i F_i = y_n + h \sum_{i=1}^s b_i f(Y_i)$  For Autonomous:  $c_i = \sum_{j=1}^s a_{ij}$

**Remark:** Flow-map:  $\Psi_h(y) = y + h \sum_{i=1}^s b_i f(Y_i(y))$  ps: weights:  $b_i$ ; internal coefficients:  $a_{ij}$

ps: We can using **Butcher Table** to represent the RK method (Appendix)

**Explicit:**  $a_{ij} = 0$  for  $j \geq i$  (严格下三角行) **Implicit:**  $\exists a_{ij} \neq 0$  for  $j \geq i$  (Not Explicit)

### 4.3 Accuracy of RK Method | Order Condition

**Some Notations:** If  $y = f'(y)$  where  $f(y) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ . Def  $f' = (\frac{\partial f_i}{\partial y_j})_{1 \leq i \leq d, 1 \leq j \leq d}$  (行向量)  $f'' = (\frac{\partial^2 f_i}{\partial y_j \partial y_k})_{1 \leq i \leq d, 1 \leq j, k \leq d}$

· Def:  $f''(a, b) = \sum_{j=1}^d \sum_{k=1}^d \frac{\partial^2 f_i}{\partial y_j \partial y_k} a_j b_k \quad | y' = f \quad y'' = \sum_{j=1}^d \frac{\partial f_i}{\partial y_j} f_j = f' f \quad y''' = \sum_{j=1}^d \sum_{k=1}^d \frac{\partial^2 f_i}{\partial y_j \partial y_k} y'_j(t) y'_k(t) + \sum_{j=1}^d \frac{\partial f_i}{\partial y_j} y''_j(t) = f''(f, f) + f' f' f$

·  $\Phi_h(y) = y + hf + \frac{h^2}{2} f' f + \frac{h^3}{6} [f''(f, f) + f' f' f] + \mathcal{O}(h^4)$

**Order Condition:** RK method:  $y_{n+1} = y_n + h \sum_{i=1}^s b_i f(Y_i)$ , Let  $z(h) = \Phi_h(y)$

$\Rightarrow$  If  $z'(0) = y', z''(0) = y'', \dots, z^{(n)}(0) = y^{(n)} \Rightarrow$  **Convergent at order  $n$**

· Order 1:  $\sum_{i=1}^s b_i = 1$  Order 2: (add)  $\sum_{i=1}^s b_i c_i = \frac{1}{2}$  Order 3: (add)  $\sum_{i=1}^s b_i c_i^2 = \frac{1}{3}$  and  $\sum_{i=1}^s \sum_{j=1}^s b_i a_{ij} c_j = \frac{1}{6}$

## 5 Stability of Runge-Kutta Methods consider for autonomous $y' = f(y)$

### 5.1 Basic Definition for Stability

**Fixed Point-Exact:** For ODEs  $\frac{dy}{dt} = f(y)$ , point  $y^*$  is fixed point if  $f(y^*) = 0 \Leftrightarrow \Phi_t(y^*) = y^*$  **Set of Fixed Points:**  $\mathcal{F} = \{y^* \in \mathbb{R}^d : f(y^*) = 0\}$

**Fixed Point-Numerical:** One-step method  $\Psi_h(y)$ , point  $y^*$  is fixed point if  $y^* = \Psi_h(y^*)$  **Set of Fixed Points:**  $\mathcal{F}_h = \{y^* \in \mathbb{R}^d : y^* = \Psi_h(y^*)\}$

**Theorem:** For Runge-Kutta method,  $\mathcal{F} \subseteq \mathcal{F}_h$  **Remark:**  $\mathcal{F}_h \subseteq \mathcal{F}$  is NOT always true. If  $\mathcal{F}_h = \mathcal{F}$ , then the method is **regular**.

· the point in  $\mathcal{F}_h \setminus \mathcal{F}$  is called **spurious fixed point**. As  $h \rightarrow \infty$ , the *spurious* fixed points will tends to infinity.

**Stability of Fixed Points:** Fixed point  $y^*$ , the ODEs  $\frac{dy}{dt} = f(y)$  with  $y(0) = y_0$ .

- Stable in the sense of Lyapunov:** Fixed point  $y^*$  is stable if  $\forall \varepsilon > 0, \exists \delta > 0$  s.t.  $\|y_0 - y^*\| < \delta \Rightarrow \|y(t; y_0) - y^*\| < \varepsilon \forall t > 0$
- Asymptotically Stable:** Fixed point  $y^*$  is asymptotically stable if  $\exists \delta > 0$  s.t.  $\|y_0 - y^*\| < \delta \Rightarrow \lim_{t \rightarrow \infty} \|y(t; y_0) - y^*\| = 0$
- Unstable:** Fixed point  $y^*$  is unstable if it's not stable. i.e.  $\exists \varepsilon > 0, \forall \delta > 0$  s.t.  $\|y_0 - y^*\| < \delta \Rightarrow \|y(t) - y^*\| \geq \varepsilon$  for some  $t$ .

### 5.2 Classification of Fixed Points

**Linearization Theorem:** Suppose  $\frac{dy}{dt} = f(y)$ ,  $y^*$  is a fixed point. Let  $J = f'(y^*)$  be the Jacobian matrix of  $f$  at  $y^*$ .

- If  $\forall$  eigenvalues of  $J$  in left complex half plane, then  $y^*$  is **asymptotically stable**.
- If  $\exists$  eigenvalues of  $J$  in right complex half plane, then  $y^*$  is **unstable**.

(Following is a special cases from HDE)

**Generalized Eigenvectors:** If  $\lambda$  is an repeated eigenvalue with eigenvalue  $\xi$  then:

Generalized Eigenvectors:  $\eta$  s.t.  $(A - \lambda I)\eta = \xi$  More generally:  $(A - \lambda I)\eta_n = \eta_{n-1}$

**Classification of Critical Points at  $y^*$  (Linear):**  $r_1, r_2$  be sol of  $\det(J - \lambda I) = 0$ .  $\parallel \mathbb{C} : r = \lambda \pm i\mu (\mu > 0)$

If  $J$  constant, write sol:  $\mathbf{x} = c_1 e^{r_1 t} \xi_1 + c_2 e^{r_2 t} \xi_2 \parallel GM = 1: \mathbf{x} = c_1 e^{rt} \xi + c_2 e^{rt} (t\xi + \eta) \quad J = \begin{pmatrix} \partial_x F(x_0) & \partial_y F(x_0) \\ \partial_x G(x_0) & \partial_y G(x_0) \end{pmatrix} \text{ If } f(x, y) = \begin{pmatrix} F(x, y) \\ G(x, y) \end{pmatrix}$

R/C	Condition    Stability	Type    Name	Phase Plane Description	Other	
R	$r_1 < r_2 < 0$    asy.stab	N    NSK	向原点, 直线 $\xi_1$ 曲线, and $\xi_1$ 周围 $y = \pm x^3$	$c_2 \neq 0, t \rightarrow \infty: \xi_2$ 主导方向; $c_2 = 0, t \rightarrow \infty: \xi_1$ 主导方向	PS: N = Node PN = Proper Node IN = Improper or: Degenerate Node SP = Saddle Point SpP = spiral point or: Focus Point C = Center NSK = Nodal Sink NSO = Nodal Source
	$r_1 > r_2 > 0$    unstable	N    NSO	原点向外, $\xi_2$ 直线, $\xi_1$ 曲线, and $\xi_1$ 周围 $y = \pm x^3$	$c_1 \neq 0, t \rightarrow \infty: \xi_1$ 主导方向; $c_1 = 0, t \rightarrow \infty: \xi_2$ 主导方向	
	$r_1 > 0 > r_2$    unstable	SP    SP	$t \rightarrow \infty: \xi_1$ 从原点向外, $\xi_2$ 从外向原点 and: 像 $y = \pm \frac{1}{x}$ , 同进同出	$t \rightarrow \pm \infty:  x  \rightarrow \infty; \quad t \rightarrow \infty: c_1, c_2 \neq 0,  x  \rightarrow \infty: \xi_1$ 主导; $t \rightarrow \infty: c_2 = 0,  x  \rightarrow \infty: \xi_1$ 主导; $t \rightarrow \infty: c_1 = 0,  x  \rightarrow 0: \xi_2$ 主导	
	$r_1 = r_2 < 0, GM=2$    asy.stab	PN    PN or Stable Star	直线 向原点	直线, $u_1/u_2$ is $t$ independent	
	$r_1 = r_2 > 0, GM=2$    unstable	PN    PN or Unstable Star	直线 从原点向外	直线, $u_1/u_2$ is $t$ independent	
	$r_1 = r_2 < 0, GM=1$    asy.stab	IN (AL-Type: SpP)    IN (Stable)	S 曲线, 向原点	$t \rightarrow \infty,  x  \rightarrow 0, \xi$ 主导 ps: 旋转方向大体和 $\eta + c_2 \xi$ 方向相同	
	$r_1 = r_2 > 0, GM=1$    unstable	IN (AL-Type: SpP)    IN (Unstable)	S 曲线, 从原点向外	$t \rightarrow \infty,  x  \rightarrow \infty, \xi$ 主导 ps: 旋转方向大体和 $\eta + c_2 \xi$ 方向相同	
C	$\lambda \neq 0, \lambda > 0$    unstable	SpP    Unstable Focus	向外椭圆 (elliptical) 螺旋	$t \rightarrow \infty,  x  \rightarrow \infty$ ps: 考虑 $J = (a, b; c, d)$ , 如果 $bc > 0$ , 顺时针, 如果 $bc < 0$ , 逆时针	C = Center NSK = Nodal Sink NSO = Nodal Source
	$\lambda \neq 0, \lambda < 0$    asy.stab	SpP    Stable Focus	向内椭圆 (elliptical) 螺旋	$t \rightarrow \infty,  x  \rightarrow 0$ ps: 考虑 $J = (a, b; c, d)$ , 如果 $bc > 0$ , 顺时针, 如果 $bc < 0$ , 逆时针	
	$\lambda = 0$    stable (AL: Indeterminate)	C (AL: C or SpP)    C	椭圆 (elliptical) 和 半长轴 $\xi$ 实部方向	Bounded trajectory or $\exists$ Periodic Trajectories	

### 5.3 Stability of Fixed Points of Maps (Numerical)

**Definition:** For flow map  $\Psi$  from  $\mathbb{R}^d \rightarrow \mathbb{R}^d$ . Def  $y^n(y_0) :=$  the  $n$ -th iterate of  $y_0$  under  $\Psi$ . i.e.  $y^n = y_n; y_n = \Psi(y_{n-1})$

**Stability of Fixed Points of Maps:** Fixed point  $y^*$ , the map  $\Psi$  with  $y^* = \Psi(y^*)$ .

- Stable in the sense of Lyapunov:**  $y^*$  is stable if  $\forall \varepsilon > 0, \exists \delta > 0$  s.t.  $\|y_0 - y^*\| < \delta \Rightarrow \|y^n(y_0) - y^*\| < \varepsilon \forall n \geq 0$
- Asymptotically Stable:**  $y^*$  is asymptotically stable if  $\exists \delta > 0$  s.t.  $\|y_0 - y^*\| < \delta \Rightarrow \lim_{n \rightarrow \infty} \|y^n(y_0) - y^*\| = 0$
- Unstable:**  $y^*$  is unstable if it's not stable. i.e.  $\exists \varepsilon > 0, \forall \delta > 0$  s.t.  $\|y_0 - y^*\| < \delta \Rightarrow \|y^n(y_0) - y^*\| \geq \varepsilon$  for some  $n$ .

**Spectral Radius:** For matrix  $K$ ,  $\rho(K) = \max\{|\lambda| : \lambda \text{ is eigenvalue of } K\}$

**Theorem|Spectral Radius:** Let  $z_n = \|K^n y_0\|$ , where  $K \in \mathbb{R}^{d \times d}$  is the matrix. Then:

1.  $\rho(K) < 1 \Leftrightarrow \lim_{n \rightarrow \infty} z_n = 0$
2.  $\rho(K) > 1 \Leftrightarrow \lim_{n \rightarrow \infty} z_n = \infty$
3. If  $\rho(K) = 1$  and *eigenvalues* of  $K$  are *semisimple* (i.e. No generalized eigenvector), then  $\{z_n\}$  is bounded.

**Theorem|Connect to Stability:** For smooth ( $C^2$ ) map  $\Psi$ ,  $y^* = \Psi(y^*)$ . Let  $K = \Psi'(y^*)$ , for iteration  $y_{n+1} = \Psi(y_n)$ , we have:

1.  $\rho(K) < 1 \Rightarrow y^*$  is *asymptotically stable*
2.  $\rho(K) > 1 \Rightarrow y^*$  is *unstable*

## 5.4 Linear Stability of Numerical Methods

**Special Case|Euler Method:** For  $\frac{dy}{dt} = By$ , Using Euler method:  $y_{n+1} = (I + hB)y_n$ . where  $\lambda_i$  is eigenvalues of  $B$ . Assume  $f(y) = \lambda y$

1. The origin is *stable* if  $\|I + h\lambda_i\| \leq 1 \forall i$
2. The origin is *asymptotically stable* if  $\|I + h\lambda_i\| < 1 \forall i$
3. The origin is *unstable* if  $\|I + h\lambda_i\| > 1$

ps: 即  $h\lambda_i$  在复平面上以  $z = -1$  为圆心, 半径为 1 的圆内  $\leftarrow$  称为 **Region of absolute stability**

**Stability function  $R, P$ :** Let  $P$  be polynomial function and  $R$  be rational function.

If RK is *explicit*, then  $y_{n+1} = P(\mu)y_n$ ; If RK is *implicit*, then  $y_{n+1} = R(\mu)y_n$  where  $\mu = h\lambda$

**Stability function  $R(\mu)$ |Special Case:** For  $\frac{dy}{dt} = \lambda y$  All RK methods can be written as: where:  $b^T, A$  are from *Butcher Table*.  $\mathbf{1} = [1, \dots, 1]^T$

$$\text{I. } Y_i = y_n + \mu \sum_{j=1}^s a_{ij} Y_j \quad (Y = y_n \mathbf{1} + \mu AY) \quad y_{n+1} = y_n + \mu \sum_{b=1}^s b_b Y_b = y_n + \mu b^T Y$$

$$\text{II. } R(\mu) = 1 + \mu b^T (I - \mu A)^{-1} \mathbf{1} \quad \text{III. } y_{n+1} = R(\mu) y_n \quad \text{where } \mu = h\lambda$$

**Stability function  $R(\mu)$ |General:** For  $\frac{dy}{dt} = By$  where:  $b^T, A$  are from *Butcher Table*.  $\Lambda, U$  is  $B$  的特征值分解  $U^{-1}BU = \Lambda$  此时  $z_n, y_n$  是向量

I. Let  $y_n = Uz_n$  and  $Y_i = UZ_i$ :

$$\text{Then } Z_i = z_n + h \sum_{j=1}^s a_{ij} \Lambda Z_j \quad (z_j^{(i)} = z_n^{(i)} \mathbf{1} + \mu A Z_j^{(i)} \quad \forall i) \quad z_{n+1} = z_n + h \sum_{i=1}^s b_i \Lambda Z_i \quad (z_{n+1}^{(i)} = z_n^{(i)} + \mu \sum_{j=1}^s b_j Z_j^{(i)})$$

$$\text{II. } \frac{dz}{dt} = \Lambda z \Rightarrow \frac{dz^{(i)}}{dt} = \lambda_i z^{(i)} \Rightarrow z_{n+1}^{(i)} = R(\mu) z_n^{(i)} \quad \text{where } \mu = h\lambda_i \quad (\text{回到前一个})$$

**Theorem:** For  $\frac{dy}{dt} = By$  with  $\lambda_1, \dots, \lambda_d$  be eigenvalues of  $B$ . The RK method is *stable|asy.stab* at *origin* iff:

The Same method also *stable|asy.stab* at *origin* for  $\frac{dz}{dt} = \lambda_i z \forall i$

**Corollary:** For  $\frac{dy}{dt} = By$  with  $B$  diagonalizable. An RK Method with *stability function*  $R(\mu)$  is *stable|asy.stab|unstable* at *origin* iff: Assume  $f(y) = \lambda_1 y$

$|R(\mu)| \leq 1$  or  $|R(\mu)| < 1$  or  $|R(\mu)| > 1 \quad \forall \mu = h\lambda_i \quad \forall i$  we can write  $\sigma(B) = \{\lambda_1, \dots, \lambda_d\}$  the set of eigenvalues of  $B$

**Remark:** 这里的  $R(\mu)$  是指  $B$  分解后的每一个特征值  $\lambda_i$  的  $R(\mu)$ , 而不是  $B$  的  $R(\mu)$

## 5.5 Stability Region and A-stability

**Stability Region:**  $\frac{dy}{dt} = By$ . An RK method, the *stability region* is the set of  $\mu$  where  $\hat{R}(\mu) = |R(\mu)| < 1$ . ( $f(y) = \lambda y$ , 如  $y$  是向量,  $R(\mu)$  按上面 corollary 的 remark 所说)

1. Euler's Method:  $\hat{R}(\mu) = |1 + \mu| \Rightarrow \mu \in \{z \in \mathbb{C} : |1 + z| < 1\}$  (-1 处半径为 1 的圆)
2. Trapezoidal Rule:  $\hat{R}(\mu) = \left| \frac{1 + \mu/2}{1 - \mu/2} \right| \Rightarrow \mu \in \{z \in \mathbb{C} : |1 + z/2| < |1 - z/2|\}$  (left complex half-plane, A-stable)
3. Implicit Euler:  $\hat{R}(\mu) = |1 - \mu|^{-1} \Rightarrow \mu \in \{z \in \mathbb{C} : |1 - z| > 1\}$  (-1 处半径为 1 的圆外侧)
4. RK4:  $\hat{R}(\mu) = \left| 1 + \mu + \frac{\mu^2}{2} + \frac{\mu^3}{6} + \frac{\mu^4}{24} \right| \Rightarrow$  Using  $R(\mu) = e^{i\theta}$  to find the region.

**A-Stable:** An RK method is *A-stable* if its *stability region* contains the entire *left complex half-plane*. (i.e.  $\Re(z) < 0$ )

## 6 Linear Multistep Methods consider for autonomous $y' = f(y)$

Assume  $\frac{dy}{dt} = f(y)$  with  $y(t_0) = y_0$ . Let  $y_n$  denote  $f(y_n)$ .

### 6.1 Derivation of LMM | Algebra Operators

**Linear Multistep Methods (LMM):** For  $k$ -step LMM:  $\alpha_k y_{n+k} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h \sum_{j=0}^k \beta_j f(y_{n+j})$  where  $^1 \alpha_k \neq 0, ^2 \alpha_0 \neq 0$  or  $\beta_0 \neq 0$

· ps: Usually, coefficients are *normalized* to have  $\alpha_k = 1$  or  $\sum_{j=0}^k \beta_j = 1$ . **Implicit:** If  $\beta_k \neq 0$  **Explicit:** If  $\beta_k = 0$

**AB Schemes Construction|Using Interpolation:** Adams-Bashforth schemes can be constructed by: Consider  $k$  points  $(t_{n+j}, y'_{n+j})$  for  $j = 0, \dots, k-1$ .

1. Let  $\prod_k^f(t)$  be the *Lagrange polynomial* which passes through  $(t_{n+j}, y'_{n+j})$ .

2. The AB scheme is:  $y_{n+k} = y_{n+k-1} + \int_{t_{n+k-1}}^{t_{n+k}} \prod_k^f(t) dt$

**Remark:** Adams-Moulton schemes 同理: 考虑  $k+1$  points  $(t_{n+j}, y'_{n+j})$  for  $j = 0, \dots, k$ .

Then, we can found  $\hat{\prod}_k^f(t)$ , and  $y_{n+k} = y_{n+k-1} + \int_{t_{n+k-1}}^{t_{n+k}} \hat{\prod}_k^f(t) dt$

**Algebra Operators:** Algebra Operators is a function which maps a function to another function.

1. **shift operator:**  $E_h g(t) = g(t+h)$  **forward difference operator:**  $\Delta_h g(t) = g(t+h) - g(t)$
2. **Identity Operator:**  $1g(t) = g(t)$  **Differentiation operator:**  $Dg(t) = g'(t)$
3. **backward difference operator:**  $\nabla_h g(t) = g(t) - g(t-h)$

**Properties of Algebra Operators:**

$\Delta_h = E_h - 1$	$E_h = e^{hD}$	$e^{hD} = 1 + \Delta_h$	$D = \frac{1}{h} \ln[1 + \Delta_h]$	$g(t) = e^{(t-t_n)D} g(t_n)$	$g(t_{n+1}) = e^{hD} g(t_n)$
$E_h^{-1} = e^{-hD}$	$D = -\frac{1}{h} \ln[E_h^{-1}] = -\frac{1}{h} \ln[1 - \nabla_h]$	$1 - E_h^{-1} = \nabla_h$	$D = \frac{1}{h} [\nabla_h + \frac{1}{2} \nabla_h^2 + \frac{1}{3} \nabla_h^3 + \dots]$		
$e^{hD} g(t) = g(t+h) = g(t) + hDg(t) + \frac{h^2}{2} D^2 g(t) + \dots$			$g(t) =$	$1 + \frac{t-t_n}{1! \cdot h} \Delta_h + \frac{(t-t_n)(t-t_n-h)}{2! \cdot h^2} \Delta_h^2 + \frac{(t-t_n)(t-t_n-h)(t-t_n-2h)}{3! \cdot h^3} \Delta_h^3 + \dots$	$g(t_n)$

**BDF Method:** For  $y' = f(t, y(t))$ . Since  $Dy(t) = y'(t)$  and  $D = \frac{1}{h} [\nabla_h + \frac{1}{2} \nabla_h^2 + \frac{1}{3} \nabla_h^3 + \dots]$ .

we can get the BDF method by  $\frac{1}{h} [\nabla_h + \frac{1}{2} \nabla_h^2 + \frac{1}{3} \nabla_h^3 + \dots] y(t) = f(t, y(t))$ . 选择  $D$  的前几项作为估计.

## 6.2 Order of Accuracy

7 Appendix

7.1 Common Numerical Method | Order Condition

One-step Methods:

Method	Formula	Order	Stability		
Euler’s Method	$y_{n+1} = y_n + h f(t_n, y_n)$	1	$ 1 + h\lambda  < 1$		
Backward Euler	$y_{n+1} = y_n + h f(t_{n+1}, y_{n+1})$	1	$\left \frac{1}{1-h\lambda}\right  < 1$ (A-stable)		
Trapezoidal Rule	$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]$	2	A-stable; $R(z) = \frac{1+z/2}{1-z/2}$		
Midpoint Method	$y_{n+1} = y_n + h f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n, y_n)\right)$	2	$\left 1 + h\lambda + \frac{(h\lambda)^2}{2}\right  < 1$		
Heun’s Method	$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f\left(t_{n+1}, y_n + h f(t_n, y_n)\right)\right]$	2	$\left 1 + h\lambda + \frac{(h\lambda)^2}{2}\right  < 1$		
Theta Method	$y_{n+1} = y_n + h\left[(1 - \theta)f(t_n, y_n) + \theta f(t_{n+1}, y_{n+1})\right]$	1 (or 2 if $\theta = \frac{1}{2}$ )	$R(z) = \frac{1+(1-\theta)z}{1-\theta z}$		
RK4 Method	见 Butcher Table	4	$R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}$		
2-Stage Gauss-Legendre	$\frac{1}{2} - \frac{\sqrt{3}}{6}$	$\frac{1}{4}$	$\frac{1}{4} - \frac{\sqrt{3}}{6}$	4	A-stable
	$\frac{1}{2} + \frac{\sqrt{3}}{6}$	$\frac{1}{4} + \frac{\sqrt{3}}{6}$	$\frac{1}{4}$		
		1/2	1/2		

Multi-step Methods:

Name	Formula	Step	Accuracy
Leapfrog Method	$y_{n+2} = y_n + 2h f(t_{n+1}, y_{n+1})$	2	
Adams-Bashforth Method 1	$y_{n+1} = y_n + h f(t_n, y_n)$	1	
Adams-Bashforth Method 2	$y_{n+2} = y_{n+1} + \frac{h}{2} [3f(t_{n+1}, y_{n+1}) - f(t_n, y_n)]$	2	
Adams-Bashforth Method 3	$y_{n+3} = y_{n+2} + \frac{h}{12} [23f(t_{n+2}, y_{n+2}) - 16f(t_{n+1}, y_{n+1}) + 5f(t_n, y_n)]$	3	
Backward Differentiation Formula 2	$y_{n+2} = \frac{4}{3}y_{n+1} - \frac{1}{3}y_n + \frac{2h}{3}f(t_{n+2}, y_{n+2})$	2	
Backward Differentiation Formula 3	$y_{n+3} = \frac{18}{11}y_{n+2} - \frac{9}{11}y_{n+1} + \frac{2}{11}y_n + \frac{6h}{11}f(t_{n+3}, y_{n+3})$	3	
Class of Adams-Moulton Methods: $\alpha_k = 1, \alpha_{k-1} = -1, \alpha_j = 0, \forall j < k - 1$   Class of Backward Differentiation Formula (BDF): $\beta_j = 0, \forall j < k$			

RK Order Condition

1. order 1:  $\sum_{i=1}^s b_i = 1$
2. order 2:  $\sum_{i=1}^s b_i c_i = \frac{1}{2}$
3. order 3:  $\sum_{i=1}^s b_i c_i^2 = \frac{1}{3}$  and  $\sum_{i=1}^s \sum_{j=1}^s b_i a_{ij} c_j = \frac{1}{6}$
4. order 4:  $\sum_{i=1}^s b_i c_i^3 = \frac{1}{4}$ ,  $\sum_{i=1}^s \sum_{j=1}^s b_i a_{ij} c_j^2 = \frac{1}{8}$ ,  $\sum_{i=1}^s \sum_{j=1}^s b_i a_{ij} c_j^2 = \frac{1}{12}$ ,  $\sum_{i=1}^s \sum_{j=1}^s \sum_{k=1}^s b_i a_{ij} a_{jk} c_k = \frac{1}{24}$

7.2 Useful Series | Common RK Methods

Common Runge-Kutta Methods (Butcher Table):

$c_1$	$a_{11}$	$\cdots$	$a_{1s}$	0	0	0	0
$\vdots$	$\vdots$	$\ddots$	$\vdots$	1	1	1/2	1/2
$c_s$	$a_{s1}$	$\cdots$	$a_{ss}$		1/2	1	1
	$b_1$	$\cdots$	$b_s$			1/6	2/3
Example	RK1 (Euler's Method)			RK2 (Heun's Method)			RK3
							RK4 (Classical/Famous)

Useful Series:

$f(x)$	Taylor	Series	$R$	$f(x)$	Taylor	Series	$R$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + \dots$	1	$\frac{1}{(1-x)^2}$	$\sum_{n=1}^{\infty} nx^{n-1}$	$1 + 2x + 3x^2 + 4x^3 + \dots$	1
$\frac{2}{(1-x)^3}$	$\sum_{n=2}^{\infty} n(n-1)x^{n-2}$	$2 + 6x + 12x^2 + 20x^3 + \dots$	1	$e^x$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	$\infty$
$\ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$	1	$-\ln(1-x)$	$\sum_{n=1}^{\infty} \frac{x^n}{n}$	$x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$	1
$\sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$	$\infty$	$\cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$	$\infty$
$\arctan x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$	1	$\sinh x$	$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$	$\infty$
$\cosh x$	$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$	$\infty$	$(1+x)^k$	$\sum_{n=0}^{\infty} \binom{k}{n} x^n$	$1 + kx + \frac{k(k-1)x^2}{2!} + \dots$	1
$\ln x$	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$	$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots$	$1, 0 < x < 2$	$\frac{1}{1+x}$	$\sum_{n=0}^{\infty} (-1)^n x^n$	$1 - x + x^2 - x^3 + \dots$	1