HCV Note

1 Basic Knowledge

Useful Complex Number Properties: $|Re(z)|, |Im(z)| \le |z|$ $\quad Re(z) = \frac{z+\bar{z}}{2}, Im(z) = \frac{z-\bar{z}}{2i}, |z|^2 = z\bar{z}$ $\quad$ In *circle*, $\bar{z} = |z|^2 z^{-1}$

Triangle (Reverse) Inequality: $|z_1 + z_2| \le |z_1| + |z_2|$ $\quad ||z_1| - |z_2|| \le |z_1 - z_2|$ $\quad (Re(zw) = 0 \Leftrightarrow \overline{zw} = -zw; Im(zw) = 0 \Leftrightarrow zw = \overline{zw})$

Argument: $\arg(z) := \{\theta : z = |z|e^{i\theta}\} = \{Arg(z) + 2\pi k : k \in \mathbb{Z}\}$ **Principle Value of Argument**: $Arg(z) \in (-\pi, \pi]$

· **Operations on Argument**: $arg(z_1 z_2) = arg(z_1) + arg(z_2)$ $\quad arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$ $\quad arg(\bar{z}) = -arg(z)$

2 Holomorphic Functions

2.1 Open/Closed Set | Limit Point | limit of Sequence | Continuous of Function

Open/Closed/Punctured ε-disc: $D_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ $\quad \overline{D}_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| \le \varepsilon\}$ $\quad D'_\varepsilon(z_0) := \{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}$

Open/Closed Set in $\mathbb{C}$: $U \subset \mathbb{C}$ is **open** if $\forall z_0 \in U, \exists \varepsilon > 0, D_\varepsilon(z_0) \subseteq U$ $\quad U$ is **closed** if $\mathbb{C} \setminus U$ is open $\quad$ **Lemma**: $D_\varepsilon, D'_\varepsilon$ open, $\overline{D}_\varepsilon$ closed.

Limit Point of S: $z_0 \in \mathbb{C}$ is a limit point of S if: $\forall \varepsilon > 0, D'_\varepsilon(z_0) \cap S \ne \emptyset$ 聚点 $\quad$ **Bounded**: S is bounded if $\exists M > 0$ s.t. $|z| \le M, \forall z \in S$

Closed of Set S: $\overline{S} :=$ 所有 S 的 limit point 和 S 的点. $\quad$ **Property**: Let $S \subseteq \mathbb{C}$, then S is closed $\Leftrightarrow S = \overline{S}$.

Limit of sequence: Sequence $(z_n)_{n\in\mathbb{N}}$ has limit z if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \ge N \Rightarrow |z_n - z| < \varepsilon$. limit rules 依旧成立

1. **Lemma|Important**: $\lim z_n = z \Leftrightarrow \lim Re(z_n) = Re(z)$ and $\lim Im(z_n) = Im(z)$
2. **Cauchy**: Sequence $(z_n)_{n\in\mathbb{N}}$ is cauchy if: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall m, n \ge N \Rightarrow |z_m - z_n| < \varepsilon$ $\quad$ **Lemma**: Cauchy $\Leftrightarrow$ convergent.
3. **Lemma|Closed of Set**: $S \subseteq \mathbb{C}, z \in \mathbb{C}. \Rightarrow [\ z \in \overline{S} \Leftrightarrow \exists$ sequence $(z_n)_{n\in\mathbb{N}} \in S$ s.t. $\lim z_n = z\ ]$
4. **Bolzano-Weierstrass**: Every bounded sequence in $\mathbb{C}$ has a convergent subsequence.

Complex Functions: $\forall f : \mathbb{C} \to \mathbb{C}$ we can write it as: $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$

Limit of Function: $a_0 \in \mathbb{C}$ is the limit of f at z_0 if: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |z - z_0| < \delta \Rightarrow |f(z) - a_0| < \varepsilon$ $\quad$ limit rules 依旧成立

· **Lemma|Important**: $\lim_{z\to z_0} f(z) \Leftrightarrow \lim_{(x,y)\to(x_0,y_0)} u(x, y) = Re(a_0)$ and $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = Im(a_0)$

· **Useful Formula**: $\lim_{z\to z_0} g(\,\bar{z}\,) = \lim_{z\to\overline{z_0}} g(z)$

continuous of Function: f is continuous at z_0 if: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$ $\quad$ continuous rules 依旧成立

1. **Lemma|Important**: f is continuous at $z_0 \Leftrightarrow u, v$ are continuous at (x_0, y_0)
2. **'Extreme Value Theorem'**: f is continuous on a closed and bounded set $S \subseteq \mathbb{C}$, then $f(S)$ is closed and bounded.
3. **Lemma|continuous⇔open**: f is continuous $\Leftrightarrow \forall$ open set U, preimage $f^{-1}(U) := \{z \in \mathbb{C} | f(z) \in U\}$ is open.

2.2 Differentiable | Holomorphic Function | C-R Equation

Differentiable: Let $z_0 \in \mathbb{C}$ and $U \subseteq \mathbb{C}$ be neighborhood of z_0, then $f : U \to \mathbb{C}$ is differentiable at z_0 if: $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists.

· **I.** f is differentiable $\Rightarrow f$ is continuous. $\quad$ **II. Holomorphic** $\Leftrightarrow$ Differentiable + neighborhood (除非是一个点时不成立,|z|) $\quad$ diff rules + chain rule 成立

Cauchy-Riemann Equations: If $z_0 = x_0 + iy_0, f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 \Rightarrow u_x = v_y, v_x = -u_y$ **at** (x_0, y_0).

· If $z_0 = x_0 + iy_0, f = u + iv$ satisfies: $^1 u, v$ are *continuously differentiable* on a neighborhood of (x_0, y_0) and:
$^2 u, v$ satisfies Cauchy-Riemann Equations at (x_0, y_0). $\Rightarrow f$ is differentiable at z_0.

· ps: 常见可导复数函数: $\exp(z), \sin z, \cos z, \log z, z^\alpha$, polynomial, sinh, cosh, $\Gamma(z), |z|^2$ (at 0), constant $\quad$ ps: 常见不可导复数函数: $\bar{z}, |z| \cdot \bar{z}, Re(z), Im(z), \mathrm{Arg}(z)$

Harmonic Function: $h : \mathbb{R}^2 \to \mathbb{R}$ is harmonic if: $\forall (x, y) \in \mathbb{R}^2$ $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$ (**Laplace Equation**)

· **Lemma**: If $f = u + iv$ is holomorphic on $\mathbb{C}$ (and u, v are twice *continuously differentiable*) 可以不用, $\Rightarrow u, v$ are harmonic.

Harmonic Conjugate: Let $u, v : U \to \mathbb{R}, U \subseteq \mathbb{R}^2$ be harmonic functions. u, v are harmonic conjugate if: $f = u + iv$ is holomorphic on U.

Properties of Polynomial: The domain of rational function and polynomial are always open. $\quad$ **Lemma**: If $P(z_0) = 0$ then $P(\overline{z_0}) = 0$

First-order Operator ∂: $\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ $\quad \bar{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ $\quad || \quad f = u + iv$ satisfies C-R Equations $\Leftrightarrow \bar{\partial} f = 0$

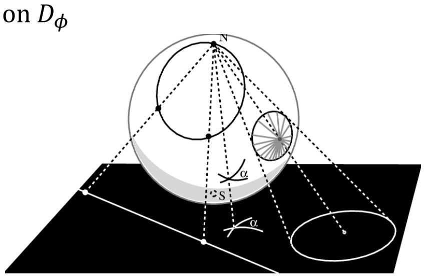
sin/cos Functions: $\sin z := \frac{e^{iz}-e^{-iz}}{2i}$ $\quad \cos z := \frac{e^{iz}+e^{-iz}}{2}$ $\quad$ **Exponential Function**: $\exp(z) = e^x(\cos(y) + i\sin(y))$

1. $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$ $\quad \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$
2. $\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$ $\quad \cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$
3. $\sin^2 z + \cos^2 z = 1$ $\quad \sin(z + \frac{\pi}{2}) = \cos(z)$ $\quad \sin(z + 2k\pi) = sin(z)$ $\quad \cos(z + 2k\pi) = \cos(z)$ $\quad \star \sin z, \cos z$ NOT bounded.

Hyperbolic Functions: $\sinh z := \frac{\exp(z)-\exp(-z)}{2}$ $\quad \cosh z := \frac{\exp(z)+\exp(-z)}{2}$ $\quad || \quad \sinh(iz) = i \sin z$ $\quad \cosh(iz) = \cos z$

Logarithm: Define *multivalued function*: $\log z := \{w \in \mathbb{C} : \exp w = z\}$ $\quad$ **Principal Branch**: $Log(z) := \ln|z| + iArg(z)$

1. **I.** $\log(z) = \ln|z| + i \arg z = \{\ln|z| + iArg(z) + i2\pi k : k \in \mathbb{Z}\}$ $\quad$ **II.** $\log(zw) = \log(z) + \log(w)$ $\quad$ **III.** $\log(1/z) = -\log(z)$
2. **Branch of Logarithm**: $Log_\phi(z) := \ln|z| + iArg_\phi(z)$ $\quad Log_\phi(z)$ is holomorphic on D_ϕ
3. If $g : U \to \mathbb{C}$, then $Log_\phi(g(z))$ is holomorphic on $g^{-1}(D_\phi) \cap U$
4. $Log(z)$ *not continuous* on $\mathbb{C}$. $\quad Log(z)$ not continuous on $Re(z) \le 0, Im(z) = 0$.
   Remark: $\log(x) + \log(x) \ne 2\log(x)$



Branch Cut|Cut Plane: *Branch Cut* $L_{z_0,\phi} := \{z \in \mathbb{C} : z = z_0 + re^{i\phi}, r \ge 0\}$

· *Cut Plane*: $D_{z_0,\phi} := \mathbb{C} \setminus L_{z_0,\phi}$ $\quad L_\phi = L_{0,\phi}; D_\phi = D_{0,\phi}$

· If $Log_\phi(z)$ is holomorphic on D_ϕ, then $Log_\phi(z - a)$ is holomorphic on $D_{a,\phi}$

Branch of Argument: $Arg_\phi(z) := z$ 的辐角, 但是角度限制在: $\phi < Arg_\phi(z) \le \phi + 2\pi$. $\quad$ ps: $Arg_{-\pi}(z) = Arg(z)$

$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$
z^n	nz^{n-1}	$\exp(z)$	$\exp(z)$	$\sin(z)$	$\cos(z)$	$\cos(z)$	$-\sin(z)$	$\sinh(z)$	$\cosh(z)$	$\cosh(z)$	$\sinh(z)$	$Log_\phi z$	$\frac{1}{z}$ $_{z\in D_\phi}$

Complex Powers: $z^\alpha := \{\exp(\alpha w) : w \in \log(z)\} = \{\exp[\alpha(\ln|z| + iArg(z) + i2k\pi)] : k \in \mathbb{Z}\}$ $\frac{d}{dz}z^\alpha = \alpha z^{\alpha-1}$ $_{z\in D_\phi}$

I. If $\alpha \in \mathbb{Z}$, there is one value of z^α **II.** If $\alpha = \frac{p}{q}, \gcd(p,q) = 1, p, q \in \mathbb{Z}, q \neq 0$, there are exactly q values of z^α

III. If α is *irrational* or *non-real*, there are infinitely values z^α **IV.** $1^{1/q}, q \in \mathbb{Z}, q \neq 0$ is $\{1, w, ..., w^{q-1}\}, w = \exp(i2\pi/q)$

V. We prefer use $\exp(z)$ to denote single-valued function, and e^z to denote multi-valued function.

Principal Branch: $z^\alpha := \exp(\alpha Log(z))$ **Operation**: $z^\alpha z^\beta = z^{\alpha+\beta}$ (Using Principal Branch) **NB**: $(z_1z_2)^\alpha \neq z_1^\alpha z_2^\alpha$; $(z^\alpha)^\beta \neq z^{\alpha\beta}$

3 Conformal Maps and Mobius Transformations

Conformal: Let U be open set and $f : U \to \mathbb{C}$. Then f is conformal iff: f *preserves angles*. i.e. 任意两条曲线/直线之间的角度在 f 作用下不变.

Important Theorem: If $f : U \to \mathbb{C}$ is *holomorphic*, then $\forall z_0 \in U, f'(z_0) \neq 0$, f *preserves angles*.

i.e. $\forall$ curves C_1, C_2 in U. If C_1, C_2 intersecting at a point $z_0 \in U$. C_1, C_2 在 z_0 切线的夹角与 $f(C_1), f(C_2)$ 在 $f(z_0)$ 切线的夹角一样.

Extended Complex Plane: $\tilde{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and define that $a + \infty = \infty, b \cdot \infty = \infty, \frac{b}{0} = \infty, \frac{b}{\infty} = 0$.

Riemann Sphere: Consider $(X, Y, Z) \in \mathbb{R}^3$: $^1z = X + iY \in \mathbb{C}$ is the point $(X, Y, 0)$ and $^2Z = 0$ is the complex plane.

1. Define the Riemann Sphere: $S^2 := \{(X, Y, Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1\}$ and consider the **North Pole** is point $N := (0,0,1)$
2. Define $\phi : \mathbb{C} \to S^2$ by N 点与 $z = (X, Y, 0)$ 点连线与 S^2 的交点为 $\phi(z)$ Thus $\lim_{|z|\to\infty} \phi(z) = N$ $\phi(\infty) := N$
3. Calculation shows that: $\phi(z) = \phi(x + iy) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$ $\psi(X, Y, Z) = \begin{cases} \frac{X+iY}{1-Z}, (X, Y, Z) \neq N \\ \infty \quad, (X, Y, Z) = N \end{cases}$

   Remark: $\phi : \tilde{\mathbb{C}} \to S^2$ is bijection and it's inverse $\psi : S^2 \to \tilde{\mathbb{C}}$ is the **stereographic projection**
4. Stereographic projection $\psi(X, Y, Z)$ maps a circle to either a circle or a straight line. (见上图)

Mobius Transformation: A Mobius Transformation is a function form: $f(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$; $ad \neq bc$

1. **Remark**: $g(z) = \frac{f(z)}{\sqrt{ad-bc}}$ satisfies $ad - bc = 1$ | If a, b, c, d defined a mobius transformation, then $\lambda a, \lambda b, \lambda c, \lambda d$ also.
2. For Complex Matrix: $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(M) = ad - bc = 1$. We define $f_M = \frac{az+b}{cz+d}$ **I.**$f_{M_1M_2} = f_{M_1}f_{M_2}$ **II.**$f_{M^{-1}} = f_M^{-1}$
3. Extended $f(z)$ from $\mathbb{C}$ to $\tilde{\mathbb{C}}$ by: $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$
4. **Translation**: $f(z) = z + b \Leftrightarrow \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ **Rotation**: $f(z) = az, a = e^{i\theta}$ $(|a| = 1) \Leftrightarrow \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & -e^{i\theta/2} \end{pmatrix}$ **Dilation**: $f(z) = rz, r > 0 \Leftrightarrow \begin{pmatrix} \sqrt{r} & 0 \\ 0 & 1/\sqrt{r} \end{pmatrix}$

   Inversion: $f(z) = 1/z \Leftrightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ f **fixes the point at infinity**: If $f(\infty) = \infty$ ps: 除了 inversion 其他都是 fix the point at infinity.
5. **Theorem**: $f(z) = \frac{az+b}{cz+d}$ be a Mobius Transformation. $\Rightarrow$ ^{1}If $f(\infty) = \infty$: f is a composition of finite *Translation, Rotation, Dilation* $\Rightarrow c = 0, f(z) = \frac{a}{d}z + \frac{b}{d}$

   ^{2} If $f(\infty) < \infty$: f is composition of finite *Translation, Rotation, Dilation* and only one *inversion*. $\Rightarrow f(z) = \frac{(bc-ad)/c^2}{z+d/c} + \frac{a}{c}$

Properties of Mobius Transformation: *Important*: ⋆ Möbius transformations map circlines to circlines. ⋆

1. For mobius transformation $f(z) = \frac{az+b}{cz+d}$, if: $\exists z_1, z_2, z_3 \in \mathbb{C}$ distinct points. $f(z_1) = z_1, f(z_2) = z_2, f(z_3) = z_3 \Rightarrow f$ is identity.
2. If $z_1, z_2, z_3 \in \tilde{\mathbb{C}}$ distinct points. $\exists!$ mobius transformation $f(z)$ s.t. $f(z_1) = 1, f(z_2) = 0, f(z_3) = \infty$
3. If $(z_1, z_2, z_3), (w_1, w_2, w_3) \in \tilde{\mathbb{C}}$ distinct points. Then $\exists!$ mobius transformation $f(z)$ s.t. $f(z_i) = w_i$, $\forall i \in \{1,2,3\}$

   ps:Method to construct 2: If $z_i < \infty, f(z) = \frac{z_1-z_3}{z_1-z_2} \cdot \frac{z-z_2}{z-z_3}$ If $z_i = \infty$, $f(z) = \frac{z-z_2}{z-z_3}$ $,z_1 = \infty$ $f(z) = \frac{z_1-z_3}{z-z_3}$ $,z_2 = \infty$; $f(z) = \frac{z-z_2}{z_1-z_2}$ $,z_3 = \infty$

   ps:Method to construct 3: For 3: Let $f := h^{-1} \circ g$ where $g(z_i), h(w_i) = \{1, 0, \infty\}$ like part 2.

Geometric Meaning by using Mobius Transformation|Exponential|Complex Powers:

1. **Rotation**: $f(z) = e^{-i\theta}z$ is a rotation by θ (anticlockwise) about the origin. Specially, $f(z) = iz$ is a rotation by $\frac{\pi}{2}$
2. **Extend**: $f(z) = \exp(\alpha z)$ 原来的图像进行拉长, 以及旋转 (如果带 θ 带 i 时) e.g. $\{z : 0 < Im(z) < 1\}$ 可以被拉长到 $\{z : 0 < Im(z)\}$
3. **Angle Extend**: $f(z) = z^\alpha$ 原来的图像辐角范围收缩或放大
4. **Circlines**: **I.** 单位圆到实轴, $f(z) = \frac{z-i}{z+i}$ **II.** 实轴到单位圆, $f(z) = i\frac{1+z}{1-z}$

   III. 单位圆到虚轴, $f(z) = \frac{z-1}{z+1}$ **IV.** 虚轴到单位圆, $f(z) = \frac{1+iz}{1-iz}$

Cross-Ratio: cross-ratio $[z_1, z_2, z_3, z_4] := f(z_1)$ where f is mobius transformation s.t. $f(z_2) = 1, f(z_3) = 0, f(z_4) = \infty$

1. **Formulas**: $[z_1, z_2, z_3, z_4] = \frac{z_1-z_3}{z_1-z_4}\frac{z_2-z_4}{z_2-z_3}$ $[\infty, z_2, z_3, z_4] = \frac{z_2-z_4}{z_2-z_3}$ $[z_1, \infty, z_3, z_4] = \frac{z_1-z_3}{z_1-z_4}$ $[z_1, z_2, \infty, z_4] = \frac{z_2-z_4}{z_1-z_4}$ $[z_1, z_2, z_3, \infty] = \frac{z_1-z_3}{z_2-z_3}$
2. **Theorem**: If f is a mobius transformation, $[f(z_1), f(z_2), f(z_3), f(z_4)] = [z_1, z_2, z_3, z_4]$ **z_i's in this "small section" are distinct.**

4 Complex Integration

4.1 Line Integral

Integrable: $f : [a, b] \to \mathbb{C}$ as $f(t) = u(t) + iv(t)$ is integrable if: u, v are both integrable on $[a, b]$ and for $f(t)$:

1. **Def**: $\int_a^b f(t)dt := \int_a^b u(t)dt + i\int_a^b v(t)dt$
2. **Property I.** $\alpha f + \beta g$ is integrable and $\int_a^b (\alpha f + \beta g)dt = \alpha\int_a^b f(t)dt + \beta\int_a^b g(t)dt$
3. **Property II.** If f is *continuous* and $\frac{dF}{dt} = f(t)$ for $F : [a, b] \to \mathbb{C}$ is differentiable. $\Rightarrow \int_a^b f(t)dt = F(b) - F(a)$

4. **Property III.** If f is *continuous* $\Rightarrow \left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$.

Parameters Curves: A parametrized curve connecting z_0 to z_1 is a *continuous* function $\gamma : [t_0, t_1] \to \mathbb{C}$ s.t. $\gamma(t_0) = z_0, \gamma(t_1) = z_1$

If $z_0 = x_0 + iy_0, z_1 = x_1 + iy_1$, then $\gamma(t) = x(t) + iy(t)$ *continuous* functions. s.t. $x(t_0) = x_0, x(t_1) = x_1, y(t_0) = y_0, y(t_1) = y_1$

Regular: γ is regular if $\gamma'(t) \neq 0$ for all $t \in [t_0, t_1]$ **Remark**: Curve $\gamma([t_0, t_1]) = \Gamma$ is *closed and bdd*.

Integral Along Curve: Let $\gamma : [t_0, t_1] \to \mathbb{C}$ be a *regular* curve s.t. $\gamma([t_0, t_1]) = \Gamma$ and $f : \Gamma \to \mathbb{C}$ is *continuous*.

1. ⋆ **Def**: $\int_\Gamma f(z)dz := \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)dt$ ⋆
2. **Circle at zero**: *Circle Centred at 0 with radius R*: $\gamma : [0,1] \to \mathbb{C}$ by $\gamma(t) = R\exp(2\pi i t)$
3. **Constant Function**: If $f(z) = c$; $\gamma : [a, b] \to \mathbb{C}$. Then $\int_\Gamma f(z)dz = \int_b^a c \cdot \gamma'(z)dz = c \cdot (\gamma(b) - \gamma(a))$

Arclength of Curve: Let $\gamma : [t_0, t_1] \to \mathbb{C}$ be a *regular* curve. $\gamma(t) = x(t) + iy(t)$ Then arclength $\ell(\Gamma) := \int_{t_0}^{t_1} |\gamma'(t)|dt = \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2}dt$

Lemma: If Γ is an arc of a circle of radius r traced though angle θ, then $\ell(\Gamma) = r\theta$ (扇形弧长)

Properties of Integral Along Curve: Let Γ be a *regular* curve and $f, g : \Gamma \to \mathbb{C}$ be *continuous*, and $\alpha, \beta \in \mathbb{C}$

1. **M-L Lemma**: $|\int_\Gamma f(z)dz| \leq \max_{z\in\Gamma} |f(z)|\ell(\Gamma)$
2. **Lemma**: $\int_\Gamma (\alpha f + \beta g)dz = \alpha \int_\Gamma f(z)dz + \beta \int_\Gamma g(z)dz$ $\int_{-\Gamma} f(z)dz = -\int_\Gamma f(z)dz$ Here: $\tilde{\gamma}(t) := \gamma(b-t)$ have $\tilde{\gamma}([a, b]) = -\Gamma$
3. **Change of Variables**: If $^1\gamma : [a, b] \to \Gamma$, and $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \Gamma$ are *two parametrizations* of Γ;
   $^2\ \exists \lambda : [\tilde{a}, \tilde{b}] \to [a, b]$ s.t. $\lambda'(t) > 0$ and $\tilde{\gamma}(t) = \gamma(\lambda(t))$ (防止曲线回头) $\Rightarrow \int_a^b f(\gamma(t))\gamma'(t)dt = \int_{\tilde{a}}^{\tilde{b}} f(\tilde{\gamma}(t))\tilde{\gamma}'(t)dt$.
   (特别的, 如果 Γ 是 closed,f 在 Γ 上的积分与哪里选择起/终点无关)

Contour: A curve Γ is *contour* if it's *finite union of regular curves* $\Gamma_1, \Gamma_2, ..., \Gamma_n$. Each Γ_i is **regular component** of Γ

Contour Integral: If $f : \Gamma \to \mathbb{C}$ is *continuous* and Γ is a *contour*. Then $\int_\Gamma f(z)dz := \sum_{i=1}^n \int_{\Gamma_i} f(z)dz$

4.2 Independent of Path

Domain: $D \subseteq \mathbb{C}$ is a *domain* if it's *open* and *connected*. (i.e. 任意两点都存在 contour(Γ) 将其连接, 并都在 D 里面)

Lemma: Let $D \subseteq \mathbb{C}$ be a domain. If $u : D \to \mathbb{C}$ is *differentiable*, with $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$. $\Rightarrow u$ is *constant* on D. ⇓ Clearly, F is *holomorphic*

Antiderivative: Let D be a domain. For $f : D \to \mathbb{C}$ be *continuous* and $F : D \to \mathbb{C}$ s.t. $F'(z) = f(z)$ for all $z \in D$. Then F is an *antiderivative* of f.

Fundamental Theorem of Calculus: D domain; $f : D \to \mathbb{C}$ *continuous*; $F : D \to \mathbb{C}$ *antiderivative* of f. Contour Γ in D connecting z_0 to z_1.

Then $\int_\Gamma f(z)dz = F(z_1) - F(z_0)$

1. D domain, if $f : D \to \mathbb{C}$ is *holomorphic* and $f'(z) = 0, \forall z \in D$. $\Rightarrow f$ is *constant* on D.
2. **Path-Independence Lemma**: D domain, f *continuous* on D. Then:
   f has *antiderivative* on D $\Leftrightarrow \int_\Gamma f(z)dz = 0\ \forall$ *closed contours* Γ in D $\Leftrightarrow \int_\Gamma f(z)dz$ is *path-independent*.

4.3 Cauchy's Theorem

Simple: A *contour* Γ is *simple* if it doesn't intersect itself except at the endpoints. **Loop**: A *contour* Γ is a *loop* if it's *simple* and $\Gamma(t_0) = \Gamma(t_1)$

Jordan Curve Theorem: $\forall\ \Gamma$ be *Loop* **Interior** $Int(\Gamma)$: Γ 的内部,bounded. **Exterior** $Ext(\Gamma)$: Γ 的外部,unbounded. **Boundary** Γ 的边界, Γ itself.

And $Int(\Gamma)$ is bounded *domain* $Ext(\Gamma)$ is unbounded *domain*. **Remark**: $Int(\Gamma)$ is *open* and $Ext(\Gamma)$ is *open* also.

· **Common Loop**: $C_r(z_0)$ is a circle of radius r centered at z_0 Corresponding $\gamma(t) = z_0 + r\exp(2\pi i t)\ t \in [0,1]$

· **Positive-Oriented**: If Γ is a *loop*, then Γ is *positive-oriented* if: 按方向走时, 内部在左边 (as we move along the curve in the direction of parametrization, the interior is on the left-hand side.)

Remark: Unless otherwise stated, all loops shall be *positively-oriented*.

Simply-Connected: A domain D is *simply-connected* if: $\forall$ *loop* Γ in D, $Int(\Gamma) \subseteq D$ (即没有洞的 domain/open set)

Cauchy Integral Theorem: If Γ is *Loop*, f is holomorphic in $Int(\Gamma) \cup \Gamma$ (Inside and on Γ), then $\int_\Gamma f(z)dz = 0$

Corollary: If D is *simply-connected* domain and $f : D \to \mathbb{C}$ is *holomorphic* on D. Then $f(z)$ has *antiderivative* on D. ⋆

即: 在没有洞的 open set 上如果都是 holomorphic, 那么都有 antiderivative.

Remark: 如果 loop Γ 上和以内没有穿过任何非 holomorphic 点, 那么 $f(z)$ 的积分值不变.

Theorem: Let $z_0 \in \mathbb{C}$, Γ be *Loop*. Then $\int_\Gamma \frac{1}{z-z_0} = \begin{cases} 2\pi i & if\ z_0 \in \mathrm{Int}(\Gamma) \\ 0 & \text{otherwise} \end{cases}$

Deformation Theorem: Let Γ_1, Γ_2 be *loops*, and f is *holomorphic* on $(Int(\Gamma_1) \setminus Int(\Gamma_2)) \cup (Int(\Gamma_2) \setminus Int(\Gamma_1))$, Γ_1 , Γ_2. Then $\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$

即: 两个 *loop* Γ_1 和 Γ_2 及它们围成的区域中 (除公共区域) 上, 函数 $f(z)$ 全纯, 那么它们的路径积分相等 ps: 可以是内外 loop, 也可以是交叉的 loop

4.4 Cauchy's Integral Formula

Cauchy's Integral Formula: Γ *Loop*, $f(z)$ *holomorphic* inside and on Γ, $z_0 \in Int(\Gamma)$, $\Rightarrow f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz$

ps: We always use it to calculate: $\int_\Gamma \frac{f(z)}{z-z_0}dz$ if $f(z)$ is holomorphic on and inside Γ (*loop*), and $z_0 \in Int(\Gamma)$. $\Rightarrow \int_\Gamma \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$

Theorem: D be *domain*, Γ be *contour* in D, $g : D \to \mathbb{C}$ *continuous* on Γ, Then:

Function Defined as: $G : D \setminus \Gamma \to \mathbb{C}$ by $G(z) = \int_\Gamma \frac{g(w)}{w-z}dw$ is *holomorphic* on $D \setminus \Gamma$ and $G'(z) = \int_\Gamma \frac{g(w)}{(w-z)^2}dw$

Moreover, function $H : D \setminus \Gamma \to \mathbb{C}$ by $H(z) = \int_\Gamma \frac{g(w)}{(w-z)^n}dw$ is *holomorphic* on $D \setminus \Gamma$ and $H'(z) = n\int_\Gamma \frac{g(w)}{(w-z)^{n+1}}dw$

⋆ **Corollary**: If D is *domain* and f is *holomorphic* on D, then f is *infinitely differentiable* on D, and all of its derivatives are *holomorphic* on D.

Generalized Cauchy's Integral Formula: Γ *Loop*, $f(z)$ *holomorphic* inside and on Γ, $z \in Int(\Gamma)$, $n \in \mathbb{N}$, $\Rightarrow f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)^{n+1}}dw$

ps: We always use it to calculate: $\int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}dz$ if $f(z)$ is holomorphic on and inside Γ (*loop*), and $z_0 \in Int(\Gamma)$. $\Rightarrow \int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(z_0)$

Morera Theorem: Let D is *domain*, if $f : D \to \mathbb{C}$ is *continuous* and $\int_\Gamma f(z)dz = 0$ for all *loop* Γ in D. $\Rightarrow f$ is *holomorphic* on D.

4.5 Liouville's Theorem, FTA and Maximum Modulus Principle

Useful Formula: If ^1D domain; $^2\exists R > 0, z_0 \in \mathbb{C}$ s.t. $\overline{D}_R(z_0) \subseteq D$; ^3f is holomorphic on D

1. Then $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + R\exp(it))dt$.
2. If $|f(z)| < M, \forall z \in D$. Then $|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}$.
3. If $\max_{z\in\overline{D}_R(z_0)} |f(z)| = |f(z_0)|$. Then f is *constant* on $\overline{D}_R(z_0)$.

Criteria Constant Function: If $f : \mathbb{C}(or\ D) \to \mathbb{C}$ is *holomorphic* and *bounded* on: D domain

1. **Liouville's Theorem**: $|f(z)| < M$ *bounded* on $\forall z \in \mathbb{C}$, $\Rightarrow$ $f(z)$ is *constant*.
2. **Maximum Modulus Principle**: $|f(z)|$ *bounded* on $\forall z \in D$, and $|f(z)|$ has *maximum* at $z_0 \in D$. $\Rightarrow$ $f(z)$ is *constant*.
   Remark I: 意思是对于 $f(z)$ holomorphic 且在 domain 上 bounded, 如果 $|f(z)|$ 在 domain 上有最大值 (非边界), 那么 $f(z)$ 是 constant.
   Remark II : ⋆ If function f is *holomorphic* on a *bounded domain* D and *continuous* up to the boundary of D.
   $\Rightarrow$ f has *maximum* modulus on the boundary of D. 若 f 在 D 内全纯, 且在 ∂D 上连续, 则 f 在 $D \cup \partial D$ 最大值一定在边界上. 特别地, 若 f 不是常数, 则最大值只能在边界上取到.
3. **Maximum/Minimum Principle for Harmonic Functions**: If D domain, $\phi : D \to \mathbb{R}$ is *harmonic*, and ϕ is *bounded above/below* on D by M, with $\phi(z_0) = M$ for some $z_0 \in D$. $\Rightarrow$ ϕ is *constant* on D.
   Remark: 对于调和函数 $\phi : D \to \mathbb{R}$, 如果 f 不是常数, 那么最大值只能在边界上取到.

Fundamental Theorem of Algebra: If $P : \mathbb{C} \to \mathbb{C}$ is a non-constant *polynomial*. $\Rightarrow$ P has a at least one *root* in $\mathbb{C}$.

5 infinity Series

5.1 Basic Properties, Convergence Test, Series of Functions and M-Test

Partial Sum: A Series $\sum_{n=0}^{\infty} z_n$ is *convergent* if *partial sums* $S_n = \sum_{k=0}^{n} z_k$ is *convergent*. **Remark**: $\sum z_n$ is *convergent* $\Rightarrow \lim z_n = 0$.

Comparison Test: If $|z_n| \leq M_n$ for all $n \in \mathbb{N}$ and $\sum M_n$ is *convergent*. $\Rightarrow$ $\sum z_n$ is *convergent*.

Lemma|'Geometric Series': For $c \in \mathbb{C}, \sum_{n=0}^{\infty} c^n$ is *convergent* $\Leftrightarrow |c| < 1$. **Remark**: $\sum_{n=0}^{\infty} c^n = \frac{1}{1-c}$

Ratio Test: For $\sum z_n$, let $L = \lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|$. If $L < 1$, then $\sum z_n$ is *convergent*. If $L > 1$, then $\sum z_n$ is *divergent*. If $L = 1$, conclude nothing.

Converge Pointwise: Seq $f_n : S \to \mathbb{C}$ *pointwise convergent* to $f : S \to \mathbb{C}$ if $\forall \varepsilon > 0, \forall z \in S, \exists N_{\varepsilon,z} \in \mathbb{N}$ s.t. $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N$

Uniform Convergence: Seq $f_n : S \to \mathbb{C}$ *uniformly convergent* to $f : S \to \mathbb{C}$ if $\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ s.t. $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N$ and $\forall z \in S$

1. **Lemma|Continuous**: If $f_n : S \to \mathbb{C}$ is *uniformly convergent* and *continuous* to $f : S \to \mathbb{C}$, then f is *continuous* on S.
2. **Lemma|Integral**: If $f_n : S \to \mathbb{C}$ is *uniformly convergent* and *continuous* to $f : S \to \mathbb{C}$, then $\int_\Gamma f_n(z)dz$ *convergent* to $\int_\Gamma f(z)dz$.
3. **Lemma|Integral**: If $f_n : S \to \mathbb{C}$ is *continuous* s.t. $\sum_{n=0}^{\infty} f_n(z)$ is *uniformly convergent* on S, then $\int_\Gamma \sum_{n=0}^{\infty} f_n(z)dz = \sum_{n=0}^{\infty}\int_\Gamma f_n(z)dz$.
4. **Lemma|Holomorphic**: If D is *simply-connected* domain, $f_n : D \to \mathbb{C}$ is *holomorphic* and *uniformly convergent* to f. $\Rightarrow f$ *holomorphic* on D.

Weierstrass M-Test: For $f_n : S \to \mathbb{C}$, if $\exists M_n \geq 0, n_0 \in \mathbb{N}$ s.t. $|f_n(z)| \leq M_n$ for $\forall z \in S, n \geq n_0$.
If $\sum_{n=0}^{\infty} M_n$ is *convergent*. $\Rightarrow$ $\sum_{n=0}^{\infty} f_n(z)$ is *uniformly convergent* on S.

6 Appendix

6.1 Convergence Test for Real Series

Divergence Test: If $\lim a_n \neq 0 \Rightarrow \sum a_n$ diverges. (If $\sum a_n$ convergent $\Rightarrow \lim a_n = 0$.) **p-Test**: $\sum \frac{1}{n^p}$ convergent iff $p > 1$

Comparison Test: If $0 < a_n < b_n, \sum b_n$ convergent $\Rightarrow \sum a_n$ also ; $\sum a_n$ divergent $\Rightarrow \sum b_n$ also.

Integral Test: Let $f : [1, \infty) \to \mathbb{R}$ is 非负递减, $a_n = f(n)$. Then $\sum a_n$ converges iff $\int_1^{\infty} f(x)dx < \infty$.

Absolutely Convergence: $\sum a_n$ convergent absolutely iff $\sum |a_n|$ convergent. **If convergent abs $\Rightarrow$ convergent.**

Alternating Series Test: If a_n decreasing, $a_n \geq 0, \lim a_n = 0$. Then $\sum(-1)^{n-1}a_n$ convergent.

Cauchy's Condensation Test: If $a_n \geq 0, a_n$ decreasing, $\Rightarrow$ [$\sum a_n convergent \Leftrightarrow \sum 2^n a_{2^n}\ also$]