

## 1 General LP

**General LP Problem:** Decision Variables:  $x_i$  parameters:  $a_{ij}, b_i, c_i$  Objective Function:  $f$  Constraints: subject to 后边的部分

LP problem can be written as:

$$\begin{aligned} \text{maximize} \quad & f = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\ & x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \end{aligned}$$

**Feasible Solution:** If  $\mathbf{x}$  satisfies all constraints (i.e.  $A\mathbf{x} \leq \mathbf{b}$ ), then  $\mathbf{x}$  is a feasible solution. (可行解) **Optimal Sol:** (最优解) (可多个)

**Find Optimal Solution: Graphical Method:** 略. **Vertex Enumeration:** 所有顶点检查, 找出最优解. **Simplex Method:** 后面详述.

**Slack Variables:** For each inequality constraint, we introduce a slack variable  $x_i$  ( $i > n$ ) to convert it to an equation. (松弛变量)

LP problem can be written as: ps:  $x_i \geq 0$  ( $i > n$ ).

$$\begin{aligned} \text{maximize} \quad & f = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{n+2} = b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + x_{n+m} = b_m \\ & x_1 \geq 0, x_2 \geq 0, \dots, x_{n+m} \geq 0 \end{aligned}$$

**Feasible Region:** It's the set  $K$  of the solutions to  $\bar{A}\mathbf{x} = \mathbf{b}$  **Convex Set:** The set  $K$  is convex if  $\forall \mathbf{x}, \mathbf{x}' \in K, \forall \theta \in [0, 1], \mathbf{x}_\theta = (1-\theta)\mathbf{x} + \theta\mathbf{x}' \in K$ .

**Vertex on Convex Set:** A vertex of the convex set  $K$  is a point  $\mathbf{x} \in K$  which doesn't lie strictly inside any line segment connecting two points in  $K$ .

**Theorem:** If LP has a unique optimal solution is a vertex.

**Theorem:** If LP has a non-unique solution,  $\exists$  optimal solution at vertex

We can also write the LP problem in matrix form:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{aligned} \text{maximize} \quad & f = \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \quad \mathbf{x} \geq 0 \end{aligned}$$

We can also write the LP problem in matrix form:

$$\bar{A} = [A \ I_m], \mathbf{b} = \mathbf{b}, \bar{\mathbf{c}} = \begin{pmatrix} \mathbf{c} \\ \mathbf{0}_{m \times 1} \end{pmatrix}, \mathbf{x} = \begin{pmatrix} \mathbf{x} \\ x_{n+m} \end{pmatrix}$$

$$\begin{aligned} \text{maximize} \quad & f = \bar{\mathbf{c}}^T \mathbf{x} \\ \text{subject to} \quad & \bar{A}\mathbf{x} = \mathbf{b} \quad \mathbf{x} \geq 0 \end{aligned}$$

**Solving:**  $\mathbf{x} = \mathbf{x}_b + \mathbf{v}$  where  $\bar{A}\mathbf{v} = 0$  and  $\mathbf{x}_b = [\mathbf{0} \ \mathbf{b}]^T$

## 2 Simplex Method

### 2.1 Simplex Algorithm

**Solve LP Problem:** Assume  $f = \bar{\mathbf{c}}^T \mathbf{x}$  with  $\bar{A}\mathbf{x} = \mathbf{b}$  and  $\mathbf{x} \geq 0$ .

- 修改  $x_i$ /列顺序,  $A$  中换顺序后得到可逆的  $B_{m \times m} \Rightarrow \bar{A}\mathbf{x} = \mathbf{b} \Leftrightarrow B\mathbf{x}_B + N\mathbf{x}_N = \mathbf{b} \quad \bar{\mathbf{c}} \rightarrow [\mathbf{c}_B \ \mathbf{c}_N]^T \quad \bar{A} \rightarrow [B \ N] \quad \mathbf{x} \rightarrow [\mathbf{x}_B \ \mathbf{x}_N]^T \quad \text{Let } \hat{\mathbf{b}} = B^{-1}\mathbf{b} \quad \hat{N} = B^{-1}N$
- Solution:**  $\mathbf{x}_B = \hat{\mathbf{b}} - \hat{N}\mathbf{x}_N$  If  $\mathbf{x}_N = 0 \Rightarrow$  It's a basic solution. (But we need to check whether  $\mathbf{x}_B \geq 0$ ) Using  $\mathcal{B}, \mathcal{N}$ : Index set of independent/else.
- Basic Variables:**  $\mathbf{x}_B$  **Nonbasic Variables:**  $\mathbf{x}_N$
- At Basic Solution:**  $\mathbf{x}_B = \hat{\mathbf{b}}, \mathbf{x}_N = \mathbf{0} \Rightarrow$  If  $\hat{\mathbf{b}} \geq 0$ . Corresponding  $\mathbf{x}$  is: <sup>1</sup> vertex of  $K$ ; <sup>2</sup> **Basic Feasible Solution (BFS)**
- Basic Costs:**  $\mathbf{c}_B^T$  **Nonbasic Costs:**  $\mathbf{c}_N^T$  **Reduced Costs:**  $\hat{\mathbf{c}}_N = \mathbf{c}_N - \hat{N}^T \mathbf{c}_B = \mathbf{c}_N - N^T B^{-T} \mathbf{c}_B \quad \hat{f} = \mathbf{c}_B^T \hat{\mathbf{b}} \quad \mathbf{x}_B, \mathbf{x}_N \geq 0$
- Objective Value:**  $f = \bar{\mathbf{c}}^T \mathbf{x} = \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N = \hat{f} + \hat{\mathbf{c}}_N^T \mathbf{x}_N$  If  $\hat{\mathbf{c}}_N \leq 0$ , then  $f \leq \hat{f} \Rightarrow$  Corresponding  $\mathbf{x}$  is optimal.
- If  $\hat{\mathbf{c}}_N \leq 0$  doesn't hold, we using **Simplex Algorithm**.

**Simplex Algorithm:**

- Initial Basic Feasible Solution:** Try  $\mathcal{B} = \{n+1, \dots, n+m\}$  and  $\mathcal{N} = \{1, \dots, n\} \Rightarrow B = I, N = A, \mathbf{x}_B = \hat{\mathbf{b}} = \mathbf{b}, \mathbf{x}_N = \mathbf{0}, \mathbf{c}_B = \mathbf{0}, \mathbf{c}_N = \mathbf{c}$
- If  $\mathbf{b} \geq 0 \Rightarrow$  Basis is feasible + cont. Else: Basis  $B$  is not feasible. Go to "Cases If  $\mathbf{b} \not\geq 0$ "
- If  $\hat{\mathbf{c}}_N \leq 0 \Rightarrow$  Optimal Solution Else: cont.
- Let  $q' \in \mathcal{N}$  对应  $\hat{\mathbf{c}}_N$  中最大 positive 分量的 index 同理, 对应的最大正分量值为  $\hat{c}_q$  对应的 variable 为  $x_{q'}$  对应  $N$  中的  $[q]$  列为  $\mathbf{a}_q$
- Let  $\hat{\mathbf{a}}_q = B^{-1}\mathbf{a}_q$ . If  $\hat{\mathbf{a}}_q \leq 0 \Rightarrow$  LP is unbounded. Else: LP is bounded + cont.
- Let  $p' \in \mathcal{B}$  be the index corresponding to  $p = \arg \min_{i=1, \dots, m} \frac{\hat{b}_i}{\hat{a}_{iq}}; \hat{a}_{iq} > 0$   $p$  是对应的 index, not value 用  $\bar{\alpha} = \frac{\hat{b}_p}{\hat{a}_{pq}}$  代表值  
 $\hat{b}_i, \hat{a}_{iq}$  代表  $\hat{\mathbf{b}}$  的第  $i$  个分量,  $\hat{\mathbf{a}}_q$  的第  $i$  个分量  
 $p'$  代表能使  $\frac{\hat{b}_i}{\hat{a}_{iq}}$  的值最小的 index, 前提条件是  $\hat{a}_{iq} > 0$  对应的 variable 为  $x_{p'}$
- Exchange  $p'$  and  $q'$  between  $\mathcal{B}$  and  $\mathcal{N} \Rightarrow$  Update  $\mathcal{B}, \mathcal{N}, B, N, \hat{\mathbf{b}}, \dots$  ps: New  $B := B + (a_q - B e_p) e_p^T$
- Go to 3.

**Case: If  $\mathbf{b} \not\geq 0$ : Phase I Problem**

- Subtract **artificial variables**  $x_{n+m+1}, \dots, x_{n+m+m} \geq 0$  and change objective function  $f$  to:

$$\begin{aligned} \max \quad & f = \\ \text{s. t.} \quad & a_{11}x_1 + \cdots + a_{1n}x_n + x_{n+1} - x_{n+m+1} - \cdots - x_{n+2m} = b_1 \\ & \vdots \quad \vdots \quad \ddots \quad \vdots \\ & a_{m1}x_1 + \cdots + a_{mn}x_n + x_{n+m} - x_{n+2m} = b_m \\ & x_1 \geq 0, \dots, x_{n+2m} \geq 0 \end{aligned}$$

- Let Basic Variables:  $\mathbf{x}_B$  = 第  $i$  个元素是:  $x_{n+i} = b_i$  (if  $b_i \geq 0$ ) 是:  $x_{n+m+i} = -b_i$  (if  $b_i < 0$ ) \*  
 Let Nonbasic Variables:  $\mathbf{x}_N$  = 第  $i$  个元素是:  $x_{n+m+i} = 0$  (if  $b_i \geq 0$ ) 是:  $x_{n+i} = b_i$  (if  $b_i < 0$ )  
 Let Basic Matrix  $B$  = 第  $i$  列是:  $\mathbf{e}_i$  (if  $b_i \geq 0$ ) 否则:  $-\mathbf{e}_i$  其他列是  $N$  的对应列      Let  $\hat{\mathbf{b}} = B^{-1}\mathbf{b} = |\mathbf{b}| \geq 0$   
 Let  $\mathbf{c}_B = \mathbf{x}_B$  对应的  $f$  中的系数       $\mathbf{c}_N$  = 同理       $\hat{\mathbf{c}}_N = \mathbf{c}_N - N^T B^{-T} \mathbf{c}_B$
  - Size of Infeasibility:** is  $\sum_{i=1}^m x_{n+m+i} \quad f = -\sum_{i=1}^m x_{n+m+i} \leq 0$
  - If  $f = 0$ ,  $\mathbf{x}$  is a BFS.  $\Rightarrow$  Go to **Phase II Problem** (恢复到之前的  $f$  并删除 artificial variables)  
 Else: Go to **Simplex Algorithm** 得到辅助问题 Phase I 的最优解.      If  $f < 0$ : LP is infeasible.      If  $f = 0$ : LP is feasible.  $\Rightarrow$  **Phase II**
- Case: If  $\mathbf{b} \geq 0$ : Phase II Problem**

## 2.2 More thing about Simplex Algorithm

**Degeneracy:** If  $\hat{\mathbf{b}}$  has any zero component, then  $\mathbf{x}$  is a **degenerate vertex**.      如果  $\hat{\mathbf{b}}$  的第  $p$  个分量  $\hat{b}_p = 0$ , 那么单纯形法可能会陷入循环.

**Theorem:** If no degenerate situation, then Simplex Algorithm will terminate in a finite number of steps.

**Examples:** 1. The worst case of Simplex Algorithm.      2. Cyclic case of Simplex Algorithm.

- Klee-Minty Problem:**  $\max f = \sum_{j=1}^n 10^{j-1} x_j \quad ; \quad \text{s.t. } x_i + 2 \sum_{j=i+1}^n 10^{j-i} x_j \leq 100^{n-i} \text{ for } i = 1, \dots, n \quad ; \quad x_i \geq 0$   
 has  $2^n$  vertices.      **Worst Case:**  $2^n - 1$  iterations.
- Hall-McKinnon Problem:**  $\max f = x_1 - 5.5x_2 + 0.75x_3 - 5.75x_4 \quad ; \quad \text{s.t. } 2.5x_1 - 19.5x_2 - 3.5x_3 + 19.5x_4 + x_5 = 0; 0.5x_1 - 3.5x_2 - 0.5x_3 + 3.5x_4 + x_6 = 0; x_i \geq 0$

## 2.3 Sparsity LP Problem

**Implementation:** 计算/编程中的计算化简      Consider:  $\hat{\mathbf{b}} = B^{-1}\mathbf{b} \quad \hat{\mathbf{c}}_N = \mathbf{c}_N - N^T B^{-T} \mathbf{c}_B \quad \hat{\mathbf{a}}_q = B^{-1} \mathbf{a}_q$

- Solve:  $B\hat{\mathbf{b}} = \mathbf{b}$  to get:  $\hat{\mathbf{b}} = B^{-1}\mathbf{b}$
- Solve:  $B^T \pi = \mathbf{c}_B$  to get:  $\pi = B^{-T} \mathbf{c}_B$
- Solve:  $\hat{\mathbf{c}}_N = \mathbf{c}_N - N^T \pi$  to get:  $\hat{\mathbf{c}}_N = \mathbf{c}_N - N^T B^{-T} \mathbf{c}_B$
- Solve:  $B^T \hat{\mathbf{a}}_q = \mathbf{a}_q$  to get:  $\hat{\mathbf{a}}_q = B^{-T} \mathbf{a}_q$
- Matrix  $B$  is **sparse** and **invertible**. And in Simplex Algorithm, the next matrix  $B$  obtained by  $B := B + (\mathbf{a}_q - B\mathbf{e}_p)\mathbf{e}_p^T$

**Sparse LP Problem:** For LP problem:  $\max f = \mathbf{c}^T \mathbf{x} \quad \text{s.t. } A\mathbf{x} = \mathbf{b} \quad \mathbf{x} \geq 0$

It is **sparse** if the matrix  $A$  is sparse. (i.e. Most of the elements of  $A$  are zero)

**Find Inverse of  $B$ :** Use **Gaussian Elimination** to decomposition  $B$  into  $LU$  where  $L$  is lower triangular and  $U$  is upper triangular.

Then we can solve  $B\mathbf{x} = \mathbf{b}$  by solving  $^1 L\mathbf{y} = \mathbf{b}$  and  $^2 U\mathbf{x} = \mathbf{y}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} & l_{21}u_{14} + u_{24} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} & l_{31}u_{14} + l_{32}u_{24} + u_{34} \\ l_{41}u_{11} & l_{41}u_{12} + l_{42}u_{22} & l_{41}u_{13} + l_{42}u_{23} + l_{43}u_{33} & l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + u_{44} \end{bmatrix}$$

By Solving  $\text{red} \rightarrow \text{orange} \rightarrow \text{green} \rightarrow \text{blue} \rightarrow \text{black} \rightarrow \text{red}$

## 3 Sensitive Analysis

**RHS Sensitivity:** Consider  $b_i \rightarrow b_i + \delta \quad | \quad \max f = \mathbf{c}^T \mathbf{x}$  subject to  $A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0 \Rightarrow$  (Add Slack variables)  $\max f = \bar{\mathbf{c}}^T \mathbf{x}$  subject to  $\bar{A}\mathbf{x} = \bar{\mathbf{b}}, \mathbf{x} \geq 0$

Assume  $\mathcal{B}, \mathcal{N}, B, N$  yield an optimal solution, with  $\mathbf{x}_B = \hat{\mathbf{b}}, \mathbf{x}_N = \mathbf{0}$

Then, the new optimal values is:  $\mathbf{x}_B = B^{-1}(\mathbf{b} + \delta \mathbf{e}_i) = \hat{\mathbf{b}} + \delta B^{-1} \mathbf{e}_i$  if  $\mathbf{x}_B \geq 0$

With range of  $\delta \in [\underline{\delta}, \bar{\delta}] \quad \underline{\delta} = \max_{[B^{-1}]_{ij} > 0} -\frac{\hat{b}_j}{[B^{-1}]_{ij}} \quad \bar{\delta} = \min_{[B^{-1}]_{ij} < 0} -\frac{\hat{b}_j}{[B^{-1}]_{ij}} \quad \parallel \text{nonbasic slack: } \pi_i = -\bar{c}_i, \bar{c}_i \text{ 指 reduced cost } \bar{\mathbf{c}}_N \text{ 的第 } i \text{ 个分量; basic slack: } \pi_i = 0$

**Fair Prices:** The objective function will change by  $\delta$  as:  $f = \hat{f} + \delta \pi_i$  where  $\hat{f} = \mathbf{c}_B^T \hat{\mathbf{b}} \quad \pi_i = \mathbf{c}_B^T B^{-1} \mathbf{e}_i$  (fair price)

If price of unit amount  $< \pi_i$ , buying more of the resource is attractive.      If  $>$ , is unattractive.

ps: 如果  $b_i \rightarrow b_i + \delta$  对应的  $\mathcal{B}$  中的第  $i$  个变量是 basic slack, 那么  $\pi_i = 0$ . (simply no fair price price for more of the resource)

ps: 如果  $b_i \rightarrow b_i + \delta$  对应的  $\mathcal{B}$  中的第  $i$  个变量不是 slack, (Same as  $x_{n+i}$  is nonbasic slack), 那么  $\pi_i = -\bar{c}_i$ .

\* Range of  $\delta$  is lower/upper bound Sensitivity,      \* Feasible Region increases when  $\bar{\delta}$  increases and  $\underline{\delta}$  decreases.

**Cost Sensitivity:** 即  $c_i \rightarrow c_i + \delta$ .  $\max f = \mathbf{c}^T \mathbf{x}$  subject to  $A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0$

- 小范围的变化不会改变最优解. (If an LP cost coefficient is changed by a small amount, the optimal solution will not change.)
- 如果  $c_i \rightarrow c_i + \delta$  对应的  $i$  变量在  $\mathcal{B}$  中 (basic variables), 那么 all reduced costs will change.
- 如果  $c_i \rightarrow c_i + \delta$  对应的  $i$  变量在  $\mathcal{N}$  中 (nonbasic variables), 那么 only the reduced cost of that variables will change.

**Coefficient Sensitivity:** 即  $a_{ij} \rightarrow a_{ij} + \delta$ .  $\max f = \mathbf{c}^T \mathbf{x}$  subject to  $A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0$

- 如果影响的变量在  $\mathcal{B}$  中 (basic variables):  $B$  may become singular, the optimal solution may change, etc.
- 如果影响的变量在  $\mathcal{N}$  中 (nonbasic variables): One reduced cost will change,  $N$  will change.

## 4 Example of format written in LP

**Example|** 构建 LP 问题模板:

**Defining decision variables:**

Let  $x_1$  be the number of 1kg packets of Breakfast Blend made each day.

Let  $x_2$  be the number of 1kg packets of Dinner Blend made each day.

**Total income is:**  $f_I = 1.16x_1 + 1.42x_2$ .

**Total cost is:**  $f_C = 0.18x_1 + 0.36x_2 + 0.28x_1 + 0.16x_2 = 0.46x_1 + 0.52x_2$ .

**Thus, total profit is:**  $f_I - f_C = 1.16x_1 + 1.42x_2 - 0.46x_1 - 0.52x_2 = 0.7x_1 + 0.9x_2$ .

**The objective is to maximize total profit**  $f = 0.7x_1 + 0.9x_2$

**Constraints:**

1. The number of kilos of arabica used,  $0.3x_1 + 0.6x_2$ , must not exceed the supply of 1200kg.
2. The number of kilos of robusta used,  $0.7x_1 + 0.4x_2$ , must not exceed the supply of 1500kg.
3. The total number of kilos of coffee made,  $x_1 + x_2$ , must not exceed the capacity of 2400kg.

**Thus, the LP Problem to be solved is:**

$$\begin{aligned} &\text{maximize} && f = 0.7x_1 + 0.9x_2, \\ &\text{subject to} && 0.3x_1 + 0.6x_2 \leq 1200, \\ &&& 0.7x_1 + 0.4x_2 \leq 1500, \\ &&& x_1 + x_2 \leq 2400, \\ &&& x_1, x_2 \geq 0. \end{aligned}$$

**Introducing Slack Variables:**

$$\begin{aligned} &\text{maximize} && f = 0.7x_1 + 0.9x_2, \\ &\text{subject to} && 0.3x_1 + 0.6x_2 + x_3 = 1200, \\ &&& 0.7x_1 + 0.4x_2 + x_4 = 1500, \\ &&& x_1 + x_2 + x_5 = 2400, \\ &&& x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$