NODEA Note

Basic Knowledge

Def of ODE & ODEs: (1st order) ODE: $\frac{dy}{dt} = f(t,y)$ & ODEs: $\frac{dy}{dt} = \mathbf{f}(t,\mathbf{y})$, $\mathbf{y} = (y_1,...,y_d)^T$, $\mathbf{f}(t,\mathbf{y}) = (f_1(t,\mathbf{y}),...,f_d(t,\mathbf{y}))^T$ **Autonomous**: $\frac{dy}{dt} = \mathbf{f}(\mathbf{y}) \Rightarrow \text{ autonomous ODE}(\mathbf{s})$. $|| \Downarrow \text{ New Autonomous ODEs}: \frac{dy}{ds} = \mathbf{f}(y_{d+1},\mathbf{y}) \text{ and } \frac{dy_{d+1}}{ds} = 1$ • **Change to Autonomous**: For $\frac{dy}{dt} = \mathbf{f}(t,\mathbf{y})$. Let $y_{d+1} = t$ and *new* independent variable s s.t. $\frac{dt}{ds} = 1$

Linearity: ODE: $\frac{dy}{dt} = f(t, y)$ is linearity if f(t, y) = a(t)y + b(t) || ODEs: If each ODE is linear, then the ODEs are linear.

Picard's Theorem: If f(t,y) is continuous in $D := \{(t,y) : t_0 \le t \le T, |y-y_0| < K\}$ and $\exists L > 0$ (Lipschitz constant) s.t.

 $\forall (t,u), (t,v) \in D \quad |f(t,u)-f(t,v)| \leq L|u-v| \text{ (ps:Can use MVT)}. \text{ And Assume that } M_f(T-t_0) \leq K, M_f := \max\{|f(t,u)|: (t,u) \in D\}$ \Rightarrow **Then**, \exists a unique continuously differentiable solution y(t) to the IVP $\frac{dy}{dt} = f(t,y), y(t_0) = y_0$ on $t \in [t_0,T]$.

Existence & Uniqueness Theorem: IVP $\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y}), \mathbf{y}(t_0) = \mathbf{y}_0$. If f(t, y) and $\frac{\partial f}{\partial y_i}$ are continuous in a neighborhood of (t_0, \mathbf{y}_0) . ⇒ **Then**, $\exists I := (t_0 - \delta, t_0 + \delta)$ s.t. \exists a unique continuously differentiable solution $\mathbf{y}(t)$ to the IVP on $t \in I$.

Acknowledge

Notation	Meaning	Notation	Meaning
[<i>a</i> , <i>b</i>]	Approximate function for $t \in [a, b]$	$t_0 = a \mid t_N = b$	Assume that $t_0 = a$, $t_N = b$
N	number of timesteps (i.e. Break up interval [a, b] into N equal-length sub-intervals)	h	stepsize $(h = \frac{b-a}{N})$
t_i	Define $N+1$ points: $t_0, t_1,, t_N$	t_m	$t_m = a + h \cdot m = t_0 + h \cdot m$
y_i	Approximation of y at point $t = t_i$ (Except y_0)	$y(t_i)$	Exact value of y at point $t = t_i$

Euler's Method and Taylor Series Method

Euler's Method Algorithm: Approximate ODE $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$ with number of steps N. (Similarly for ODEs) \Rightarrow **for** n = 0, 1, 2, ..., N - 1: $y_{n+1} = y_n + hf(t_0 + nh, y_n) = y_n + hf(t_n, y_n)$ **end** (p. (ps: ↓ Can get

Boundedness Theorem: For $\frac{dy}{dt} = f(t,y)$, $y(a) = y_0$ and suppose there exists a unique, twice differentiable, solution y(t) on [a,b]. Suppose: y is continuous and $|\frac{\partial f}{\partial y}| \le L$. \Rightarrow the solution y_n given Euler's method satisfies: $e_n = |y_n - y(t_n)| \le Dh$, $D = e^{(b-a)L} \frac{M}{2L}$

• **Lemma**: If $v_{n+1} \le Av_n + B$, then $v_n \le A^n v_0 + \frac{A^{n-1}}{A-1}B$ If $v_n = e_n := y_n - y(t_n)$, then A = 1 + hL, $B = h^2 M/2$ (suppose |y''| < M) **Order Notation (** \mathcal{O} **)**: we write $z(h) = \mathcal{O}(h^p)$ if $\exists \mathcal{C}, h_0 > 0$ s.t. $|z| \le \mathcal{C}h^p$, $0 < h < h_0$

Flow Map (Φ, Ψ) : $\Phi_{t_0,h}(y_0) = y(t_0 + h)$ Clearly, $\Phi(t_n + h) = y(t_n + h) = \Phi_h(y(t_n)) = y(t_{n+1})$.

 $\cdot \ \Psi_{t_n,h}(y_n) = y_{n+1} \text{:= Numerical method for ODE} \quad \text{Clearly, } \Psi(t_n+h) = y_{n+1} = \Psi_h(y_n)$

Taylor Series Method: Approximate ODE $\frac{dy}{dt} = f(t,y)$, $y(t_0) = y_0$ with *n-order Methods*: 用 Taylor Series 在 $t_0 + h$ 处展开保留到 n 阶

 $\Phi_{t,h}(y) = y + hf(t,y) + \frac{1}{2}h^2[f_t(t,y) + f_y(t,y)f(t,y)] + \frac{1}{6}y'''(t,y)h^3 + \cdots$ (For one variable y) $\cdot \text{ps: Taylor Series: } y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{h^2}{2}y''(t_0) + \cdots + \frac{h^{n-1}}{(n-1)!}y^{(n-1)}(t_0) + \frac{h^n}{n!}y^{(n)}(t^*), t^* \in [t,t+h]$ ps: $\frac{h^n}{n!}$

Convergence of One-Step Methods consider for autonomous y' = f(y)

Convergence | Consistent | Stable

Global Error: global error after n steps: $e_n := y_n - y(t_n)$ **Local Error**: For *one-step* method is: $le(y,h) = \Psi_h(y) - \Phi_h(y)$

Consistent: If $||le(y,h)|| \le Ch^{p+1} (\le \mathcal{O}(h^{p+1}))$, C > 0. \Rightarrow Consistent at order p. **Stable**: If $||\Psi_h(u) - \Psi_h(v)|| \le (1 + h\hat{L})||u - v||$

 $\textbf{Convergent} \text{: A method is convergent if:} \ \forall \ T, \lim_{h \to 0, \ h = T/N} \max_{n = 0, 1, \dots, N} ||e_n|| = 0 \\ \qquad \qquad \ \ \, \text{ } \ \ \, \text{Then the global error satisfies: } \max_{n = 0, 1, \dots, N} ||e_n|| = \mathcal{O}(h^p) \text{ } \text{ }_{\text{p-th order}}$ **Convergence of One-Step Method**: For y' = f(y), and a one-step method $\Psi_h(y)$ is 1 consistent at order p and 2 stable with \hat{L} $\uparrow \uparrow \uparrow$. (ps: $C = \frac{C}{\hat{I}}(e^{T\hat{L}} - 1)$)

More One-Step Methods | Runge-Kutta Methods | Collocation

Construction of More General one-step Method: For y' = f(y), $y(t_0) = y_0 \Rightarrow y(t+h) - y(t) = \int_t^{t+h} f(y(\tau)) d\tau$ Trapezoidal Method: $y_{n+1} = y_n + \frac{h}{2}(f(y_n) + f(y_{n+1}))$ Midpoint Method: $y_{n+1} = y_n + hf(\frac{y_n + y_{n+1}}{2})$

- One-Step Collocation Methods (By Lagrange Interpolating Polynomials):

 1. Lagrange Interpolating Polynomials: $\ell_i(x) = \prod_{j=1, j \neq i}^{s} \frac{x c_j}{c_i c_j} \in \mathbb{P}_{s-1}$ where $c_i \in F \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$
 - \Rightarrow **Polynomial Interpolation**: $\forall p(x) \in \mathbb{P}_{s}$ with $p(c_{i}) = g_{i} \in F \ \Rightarrow \ \exists ! \ p(x) = \sum_{i=1}^{s} g_{i}\ell_{i}(x)$ (Can be proved by Honour Algebra)
- 2. Quadrature Rule: If $g(t) \in \mathbb{P}_{p-1} \Rightarrow \text{Order } p \quad \Big| \quad \int_{t_0}^{t_0+h} g(t)dt = \int_0^1 g(t_0+hx)dx \approx h \sum_{i=1}^s b_i g(t_0+hc_i), b_i := \int_0^1 \ell_i(x)dx \quad \text{ps. } c_i \not \in \mathbb{P}_{p-1}$
- 3. Collocation Methods: For: $y(t_0) = y_0$, $y'(t_0 + c_i h) = f(y(t_0 + c_i h))$ ps: $c_i \not = (0.1]$ + (0.1] +

 $\Rightarrow F_i = f(y_n + h\sum_{j=1}^s a_{ij}F_j) \text{ and } y_{n+1} = y_n + h\sum_{i=1}^s b_iF_i \qquad \text{where } F_i := y'(t_0 + c_ih)$

 \cdot **Remark:** For choice of c_i : The optimal choice is attained by *Gauss-Legendre collocation methods*.

Runge-Kutta Methods: Let y' = f(t,y) **Stage Values**: $Y_i = y_n + h \sum_{j=1}^s a_{ij} f(Y_j)$ $i \in \{1,...,s\}$ $F_i = f(Y_i)$ 1. The RK method is the form: $y_{n+1} = y_n + h \sum_{i=1}^s b_i f(t_n + c_i, Y_i)$ for some values of b_i, a_{ij}, s, c_i for Autonomous: $c_i = \sum_{j=1}^s a_{ij}$

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- 2. Flow-map: $\Psi_h(y) = y + h \sum_{i=1}^{s} b_i f(t_n + c_i, Y_i(y, h))$ ps:weights: b_i ; internal coefficients: a_{ij}
- 3. We can using **Butcher Table** to represent the RK method (Appendix) **Explicit**: $a_{ij} = 0$ for $j \ge i$ (严格下三角行) **Implicit**: $\exists a_{ij} \ne 0$ for $j \ge i$ (Not Explicit)

4.3 Accuracy of RK Method | Order Condition

Some Notations: If $\mathbf{y} = f'(\mathbf{y})$ where $f(\mathbf{y}) : \mathbb{R}^d \to \mathbb{R}^d$. Def $f' = (\frac{\partial f_i}{\partial v_i}), 1 \le i \le d, 1 \le j \le d$ (Fight) $f'' = (\frac{\partial^2 f_i}{\partial v_i \partial v_i}), 1 \le i \le d$ $1 \le i \le d, 1 \le j, k \le d$

Def: $f''(\mathbf{a}, \mathbf{b}) = \sum_{j=1}^{d} \sum_{k=1}^{d} \frac{\partial^2 f_i}{\partial y_j \, \partial y_k} a_j b_k \quad | \ y' = f \quad y'' = \sum_{j=1}^{d} \frac{\partial f_i}{\partial y_j} f_j = f' f \quad y''' = \sum_{j=1}^{d} \sum_{k=1}^{d} \frac{\partial^2 f_i}{\partial y_j \, \partial y_k} y_j'(t) y_k'(t) + \sum_{j=1}^{d} \frac{\partial f_i}{\partial y_j} y_j''(t) = f''(f, f) + f' f' f$ $\cdot \Phi_h(y) = y + hf + \frac{h^2}{2} f' f + \frac{h^3}{6} [f''(f, f) + f' f' f] + \mathcal{O}(h^4)$ Order Condition: RK method: $y_{n+1} = y_n + h \sum_{i=1}^{s} b_i f(Y_i)$, Let $z(h) = \Phi_h(y)$

 $\Rightarrow \text{If } z'(0) = y', z''(0) = y'', ..., z^{(n)} = y^{(n)} \Rightarrow \text{Convergent at order } n$ $\cdot \text{ Order 1: } \sum_{i=1}^{s} b_i = 1 \qquad \text{Order 2: (add) } \sum_{i=1}^{s} b_i c_i = \frac{1}{2} \qquad \text{Order 3: (add) } \sum_{i=1}^{s} b_i c_i^2 = \frac{1}{3} \text{ and } \sum_{i=1}^{s} \sum_{j=1}^{s} b_i a_{ij} c_j = \frac{1}{6}$

Stability of Runge-Kutta Methods consider for autonomous y' = f(y)

5.1 Basic Definition for Stability

Fixed Point-Exact: For ODEs $\frac{dy}{dt} = f(y)$, point y^* is fixed point if $f(y^*) = 0 \Leftrightarrow \Phi_t(y^*) = y^*$ Set of Fixed Points: $\mathcal{F} = \{y^* \in \mathbb{R}^d : f(y^*) = 0\}$ Fixed Point-Numerical: One-step method $\Psi_h(y)$, point y^* is fixed point if $y^* = \Psi_h(y^*)$ Set of Fixed Points: $\mathcal{F}_h = \{y^* \in \mathbb{R}^d : y^* = \Psi_h(y^*)\}$

Remark: $\mathcal{F}_h \subseteq \mathcal{F}$ is NOT always true. **Theorem**: For Runge-Kutta method, $\mathcal{F} \subseteq \mathcal{F}_h$

· the point in $\mathcal{F}_h \setminus \mathcal{F}$ is called **spurious fixed point**. As $h \to \infty$, the *spurious* fixed points will tends to infinity.

Stability of Fixed Points: Fixed point y^* , the ODEs $\frac{dy}{dt} = f(y)$ with $y(0) = y_0$.

- 1. **Stable in the sense of Lyapunov**: Fixed point y^* is stable if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $||y_0 y^*|| < \delta \Rightarrow ||y(t; y_0) y^*|| < \varepsilon \ \forall t > 0$
- 2. **Asymptotically Stable**: Fixed point y^* is asymptotically stable if $\exists \delta > 0$ s.t. $||y_0 y^*|| < \delta \Rightarrow \lim_{t \to \infty} ||y(t; y_0) y^*|| = 0$
- 3. **Unstable**: Fixed point y^* is unstable if it's not stable. i.e. $\exists \epsilon > 0, \forall \delta > 0$ s.t. $||y_0 y^*|| < \delta \Rightarrow ||y(t) y^*|| \ge \varepsilon$ for some t.

Classification of Fixed Points 5.2

Linearization Theorem: Suppose $\frac{dy}{dt} = f(y)$, y^* is a fixed point. Let $J = f'(y^*)$ be the Jacobian matrix of f at y^* . 1. If \forall eigenvalues of J in left complex half plane, then y^* is **asymptotically stable**.

- 2. If \exists eigenvalues of J in right complex half plane, then y^* is **unstable**.

(Following is a special cases from HDE)

Classification of Critical Points at y^* (Linear): r_1, r_2 be sol of $det(J - \lambda I) = 0$. $|| \mathbb{C} : r = \lambda \pm i\mu(\mu > 0)$ If J constant, write sol: $\mathbf{x} = c_1 e^{r_1 t} \xi_1 + c_2 e^{r_2 t} \xi_2$ || GM = 1: $\mathbf{x} = c_1 e^{rt} \xi + c_2 e^{rt} (t \xi + \eta)$ $\int_{J} = \begin{pmatrix} \partial_x F(\mathbf{x}_0) & \partial_y F(\mathbf{x}_0) \\ \partial_x G(\mathbf{x}_0) & \partial_y G(\mathbf{x}_0) \end{pmatrix} \text{If } f(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} F(\mathbf{x}, \mathbf{y}) \\ G(\mathbf{x}, \mathbf{y}) \end{pmatrix}$

R/C	Condition Stability	Type Name	Phase Plane Description	Other	
	$r_1 < r_2 < 0$ asy.stab	N NSk	向原点, ξ_2 直线, ξ_1 曲线,and ξ_1 周围 $y = \pm x^3$	$c_2 \neq 0, t \rightarrow \infty$: ξ_2 主导方向; $c_2 = 0, t \rightarrow \infty$: ξ_1 主导方向	PS:
	$r_1 > r_2 > 0$ unstable	N NSo	原点向外, ξ_2 直线, ξ_1 曲线,and ξ_1 周围 $y = \pm x^3$	$c_1 \neq 0, t \rightarrow \infty$: ξ_1 主导方向; $c_1 = 0, t \rightarrow \infty$: ξ_2 主导方向	N = Node
R		CD II CD	$t \rightarrow \infty$, ξ_1 从原点向外, ξ_2 从外向原点	$t \to \pm \infty : \mathbf{x} \to \infty; t \to \infty : c_1, c_2 \neq 0, \mathbf{x} \to \infty, \xi_1 \pm \theta;$	PN = Proper Node
	$r_1 > 0 > r_2$ unstable	SP SP	and: 像 $y = \pm \frac{1}{r}$, 同进同出	$t \to \infty : c_2 = 0, \mathbf{x} \to \infty, \xi_1 \pm \theta; t \to \infty : c_1 = 0, \mathbf{x} \to 0, \xi_2 \pm \theta$	IN = Improper
	$r_1 = r_2 < 0$, GM=2 asy.stab	PN PN or Stable Star	直线 向原点	直线, u_1/u_2 is t independent	or: Degenerate Node
	$r_1 = r_2 > 0$, GM=2 unstable	PN PN or Unstable Star	直线 从原点向外	直线, u_1/u_2 is t independent	SP = Saddle Point
	$r_1 = r_2 < 0$, GM=1 asy.stab	0 , GM=1 asystab IN (AL:Type: SpP) IN (Stable) S 曲线,向原点 $t \rightarrow \infty$, $ \mathbf{x} \rightarrow 0$, ξ 主导 ps: 旋转方向大体和 $\eta + c_2 \xi$ 方向相同		$t \to \infty$, $ \mathbf{x} \to 0$, ξ 主导 ps: 旋转方向大体和 $\eta + c_2 \xi$ 方向相同	SpP = spiral point
	$r_1 = r_2 > 0$, GM=1 unstable	= r_2 > 0, GM=1 unstable IN (AL:Type: SpP) IN (Unstable) S 曲线, 从原点向外 $t \rightarrow \infty$, x → ∞, x → ∞		$t \to \infty$, $ \mathbf{x} \to \infty$, ξ 主导 ps: 旋转方向大体和 $\eta + c_2 \xi$ 方向相同	or: Focus Point
	$\lambda \neq 0, \lambda > 0$ unstable	SpP Unstable Focus	向外椭圆 (elliptical) 螺旋	$t \to \infty$, $ \mathbf{x} \to \infty$ ps: 考虑 $J = (a, b; c, d)$, 如果 bc>0, 顺时针,如果 bc<0, 逆时针	C = Center
C	λ ≠ 0, λ < 0 asy.stab	SpP Stable Focus	向内椭圆 (elliptical) 螺旋	$t \to \infty$, $ \mathbf{x} \to 0$ ps: 考虑 $J = (a, b; c, d)$, 如果 bc>0, 顺时针, 如果 bc<0, 逆时针	NSk = Nodal Sink
	$\lambda = 0 \mid \mid \text{stable (AL:Indeterminate)}$	C (AL:C or SpP) C	椭圆 (elliptical) and 半长轴 ξ 实部方向	Bounded trajectory or ∃ Periodic Trajectories	NSo = Nodal Source

Stability of Fixed Points of Maps (Numerical)

Definition: For flow map Ψ from $\mathbb{R}^d \to \mathbb{R}^d$. Def $y^n(y_0) :=$ the n-th iterate of y_0 under Ψ . $(y^n = y_n)$ e.g. $y^2(y_0) = \Psi(\Psi(y_0)) = y_2$ **Stability of Fixed Points of Maps**: Fixed point y^* , the map Ψ with $y^* = \Psi(y^*)$.

- 1. **Stable in the sense of Lyapunov**: y^* is stable if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $||y_0 y^*|| < \delta \Rightarrow ||y^n(y_0) y^*|| < \varepsilon \ \forall n \ge 0$
- 2. **Asymptotically Stable**: y^* is asymptotically stable if $\exists \delta > 0$ s.t. $||y_0 y^*|| < \delta \Rightarrow \lim_{n \to \infty} ||y^n(y_0) y^*|| = 0$
- 3. **Unstable**: y^* is unstable if it's not stable. i.e. $\exists \epsilon > 0, \forall \delta > 0$ s.t. $||y_0 y^*|| < \delta \Rightarrow ||y^n(y_0) y^*|| \ge \varepsilon$ for some n.

Appendix

6.1 Useful Series | Common RK Methods

$$e^{x} = 1 + x + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^{n}}{n!} \quad \sin(x) = x - \frac{x^{3}}{3!} + \frac{x^{5}}{5!} + \dots = \sum_{n=0}^{\infty} (-1)^{n} \frac{x^{2n+1}}{(2n+1)!} \quad \cos(x) = 1 - \frac{x^{2}}{2!} + \frac{x^{4}}{4!} + \dots = \sum_{n=0}^{\infty} (-1)^{n} \frac{x^{2n}}{(2n)!} \\ \ln(1+x) = x - \frac{x^{2}}{2} + \frac{x^{3}}{3} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n}}{n} \quad \arctan(x) = x - \frac{x^{3}}{3} + \frac{x^{5}}{5} + \dots = \sum_{n=0}^{\infty} (-1)^{n} \frac{x^{2n+1}}{2n+1} \quad \sinh(x) = x + \frac{x^{3}}{3!} + \frac{x^{5}}{5!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \\ \cosh(x) = 1 + \frac{x^{2}}{2!} + \frac{x^{4}}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad (1+x)^{k} = 1 + kx + \frac{k(k-1)x^{2}}{2!} + \dots = \sum_{n=0}^{\infty} {k \choose n} x^{n} \frac{1}{1-x} = 1 + x + x^{2} + \dots = \sum_{n=0}^{\infty} x^{n} \\ \frac{1}{1+x} = 1 - x + x^{2} + \dots = \sum_{n=0}^{\infty} (-1)^{n} x^{n} \quad \ln(x) = (x-1) - \frac{(x-1)^{2}}{2} - \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^{n}}{n}, \ x > 0 \\ \textbf{Common Runge-Kutta Methods (Butcher Table)}:$$

c_1 a_{11} \cdots a_{1s} \vdots \vdots \ddots \vdots c_s a_{s1} \cdots a_{ss}	0 1	0 1 1	0 1/2 1/2 1 -1 2		0 1/2 1/2	1/2	1/2	1			
$b_1 \cdots b_s$ Example	RK1 (Euler's Method)	1/2 1/2 RK2 (Heun's Method)		1/6 RI	2/3 3	1/6	· <u>1</u>	1/6 RK4 (Cla	1/3	1/3 Eamou	1/6