

1 Basic Knowledge

Def of ODE & ODEs: (1st order) ODE: $\frac{dy}{dt} = f(t, y)$ & ODEs: $\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y})$, $\mathbf{y} = (y_1, \dots, y_d)^T$, $\mathbf{f}(t, \mathbf{y}) = (f_1(t, \mathbf{y}), \dots, f_d(t, \mathbf{y}))^T$

Autonomous: $\frac{dy}{dt} = \mathbf{f}(\mathbf{y}) \Rightarrow$ autonomous ODE(s). $\parallel \Downarrow$ New Autonomous ODEs: $\frac{dy}{ds} = \mathbf{f}(y_{d+1}, \mathbf{y})$ and $\frac{dy_{d+1}}{ds} = 1$

· **Change to Autonomous:** For $\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y})$. Let $y_{d+1} = t$ and new independent variable s s.t. $\frac{dt}{ds} = 1 \uparrow$

Linearity: ODE: $\frac{dy}{dt} = f(t, y)$ is linearity if $f(t, y) = a(t)y + b(t)$ \parallel ODEs: If each ODE is linear, then the ODEs are linear.

Picard's Theorem: If $f(t, y)$ is continuous in $D := \{(t, y) : t_0 \leq t \leq T, |y - y_0| < K\}$ and $\exists L > 0$ (Lipschitz constant) s.t.

$\forall (t, u), (t, v) \in D \quad |f(t, u) - f(t, v)| \leq L|u - v|$ (ps: Can use MVT). And Assume that $M_f(T - t_0) \leq K$, $M_f := \max\{|f(t, u)| : (t, u) \in D\}$

\Rightarrow **Then**, \exists a unique continuously differentiable solution $y(t)$ to the IVP $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$ on $t \in [t_0, T]$.

Existence & Uniqueness Theorem: IVP $\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y})$, $\mathbf{y}(t_0) = \mathbf{y}_0$. If $f(t, y)$ and $\frac{\partial f}{\partial y_i}$ are continuous in a neighborhood of (t_0, \mathbf{y}_0) .

\Rightarrow **Then**, $\exists I := (t_0 - \delta, t_0 + \delta)$ s.t. \exists a unique continuously differentiable solution $\mathbf{y}(t)$ to the IVP on $t \in I$.

2 Acknowledge

Notation	Meaning	Notation	Meaning
$[a, b]$	Approximate function for $t \in [a, b]$	$t_0 = a \mid t_N = b$	Assume that $t_0 = a, t_N = b$
N	number of timesteps (i.e. Break up interval $[a, b]$ into N equal-length sub-intervals)	h	stepsize ($h = \frac{b-a}{N}$)
t_i	Define $N + 1$ points: t_0, t_1, \dots, t_N	t_m	$t_m = a + h \cdot m = t_0 + h \cdot m$
y_i	Approximation of y at point $t = t_i$ (Except y_0)	$y(t_i)$	Exact value of y at point $t = t_i$

3 Euler's Method and Taylor Series Method

Euler's Method Algorithm: Approximate ODE $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$ with number of steps N . (Similarly for ODEs)

\Rightarrow for $n = 0, 1, 2, \dots, N - 1$: $y_{n+1} = y_n + hf(t_n + nh, y_n) = y_n + hf(t_n, y_n)$ **end**

Lemma: If $v_{n+1} \leq Av_n + B$, then $v_n \leq A^n v_0 + \frac{A^n - 1}{A - 1}B$

Moreover, suppose $|y''| < M$ and $v_n = e_n := y_n - y(t_n)$, then $A = 1 + hL, B = h^2 M / 2$

Boundedness Theorem|Euler Method: For $\frac{dy}{dt} = f(t, y)$, $y(a) = y_0$:

\exists ¹ unique, ² twice differentiable, solution $y(t)$ on $[a, b]$, ³ y is continuous and ⁴ $|\frac{\partial f}{\partial y}| \leq L$.

\Rightarrow the solution y_n given by Euler's method satisfies: $e_n = |y_n - y(t_n)| \leq Dh, D = e^{(b-a)L} \frac{M}{2L}$

Order Notation (\mathcal{O}): we write $z(h) = \mathcal{O}(h^p)$ if $\exists C, h_0 > 0$ s.t. $|z| \leq Ch^p, 0 < h < h_0$

Flow Map (Φ, Ψ): $\Phi_{t_0, h}(y_0) = y(t_0 + h)$ Clearly, $\Phi(t_n + h) = y(t_n + h) = \Phi_h(y(t_n)) = y(t_{n+1})$.

· $\Psi_{t_n, h}(y_n) = y_{n+1} :=$ Numerical method for ODE Clearly, $\Psi(t_n + h) = y_{n+1} = \Psi_h(y_n)$

Taylor Series Method: Approximate ODE $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$ with n -order Methods: 用 Taylor Series 在 $t_0 + h$ 处展开保留到 n 阶

· $\Phi_{t, h}(y) = y + hf(t, y) + \frac{1}{2}h^2[f_t(t, y) + f_y(t, y)f(t, y)] + \frac{1}{6}y'''(t, y)h^3 + \dots$ (For one variable y)

· ps: Taylor Series: $y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{h^2}{2}y''(t_0) + \dots + \frac{h^{n-1}}{(n-1)!}y^{(n-1)}(t_0) + \frac{h^n}{n!}y^{(n)}(t^*), t^* \in [t, t + h]$ ps: $y' = f, y'' = f_t + f_y f$

4 Convergence of One-Step Methods consider for autonomous $y' = f(y)$

4.1 Convergence | Consistent | Stable

Global Error: global error after n steps: $e_n := y_n - y(t_n)$ **Local Error:** For one-step method is: $le(y, h) = \Psi_h(y) - \Phi_h(y)$

Consistent: If $||le(y, h)|| \leq Ch^{p+1} (\leq \mathcal{O}(h^{p+1}))$, $C > 0 \Rightarrow$ Consistent at order p . **Stable:** If $||\Psi_h(u) - \Psi_h(v)|| \leq (1 + h\hat{L})||u - v||$

Convergent: A method is convergent if: $\forall T, \lim_{h \rightarrow 0} \max_{h=T/N, n=0, 1, \dots, N} ||e_n|| = 0$ \Downarrow Then the global error satisfies: $\max_{n=0, 1, \dots, N} ||e_n|| = \mathcal{O}(h^p)$ **p-th order**

Convergence of One-Step Method: For $y' = f(y)$, and a one-step method $\Psi_h(y)$ is ¹ consistent at order p and ² stable with $\hat{L} \uparrow$. (ps: $C = \frac{C}{L}(e^{T\hat{L}} - 1)$)

4.2 More One-Step Methods | Runge-Kutta Methods | Collocation

Construction of More General one-step Method: For $y' = f(y)$, $y(t_0) = y_0 \Rightarrow y(t + h) - y(t) = \int_t^{t+h} f(y(\tau))d\tau$

Trapezoidal Method: $y_{n+1} = y_n + \frac{h}{2}(f(y_n) + f(y_{n+1}))$ **Midpoint Method:** $y_{n+1} = y_n + hf(\frac{y_n + y_{n+1}}{2})$

One-Step Collocation Methods (By Lagrange Interpolating Polynomials):

1. **Lagrange Interpolating Polynomials:** $\ell_i(x) = \prod_{j=1, j \neq i}^s \frac{x - c_j}{c_i - c_j} \in \mathbb{P}_{s-1}$ where $c_i \in F \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$

\Rightarrow **Polynomial Interpolation:** $\forall p(x) \in \mathbb{P}_s$ with $p(c_i) = g_i \in F \Rightarrow \exists! p(x) = \sum_{i=1}^s g_i \ell_i(x)$ (Can be proved by Honour Algebra)

2. **Quadrature Rule:** If $g(t) \in \mathbb{P}_{p-1} \mid \int_{t_0}^{t_0+h} g(t)dt = \int_0^1 g(t_0 + hx)dx \approx h \sum_{i=1}^s b_i g(t_0 + hc_i), b_i := \int_0^1 \ell_i(x)dx$ ps: c_i 从 $[0, 1]$ 中取不同的

3. **Collocation Methods:** For: $y(t_0) = y_0, y'(t_0 + c_i h) = f(y(t_0 + c_i h))$ ps: c_i 从 $[0, 1]$ 中取不同的 Let: $a_{ij} := \int_0^{c_i} \ell_j(x)dx$ and $b_i := \int_0^1 \ell_i(x)dx$
 $\Rightarrow F_i = f(y_n + h \sum_{j=1}^s a_{ij} F_j)$ and $y_{n+1} = y_n + h \sum_{i=1}^s b_i F_i$ where $F_i := y'(t_0 + c_i h)$

· **Remark:** For choice of c_i : The optimal choice is attained by Gauss-Legendre collocation methods.

Runge-Kutta Methods: Let $y' = f(t, y)$ **Stage Values:** $Y_i = y_n + h \sum_{j=1}^s a_{ij} f(t_n + c_j h, Y_j)$ $i \in \{1, \dots, s\}$ $F_i = f(Y_i)$

1. The RK method is the form: $y_{n+1} = y_n + h \sum_{i=1}^s b_i f(t_n + c_i h, Y_i(y_n))$ for some values of b_i, a_{ij}, s, c_i for Autonomous:
 $c_i = \sum_{j=1}^s a_{ij}$

2. Flow-map: $\Psi_h(y) = y + h \sum_{i=1}^s b_i f(t_n + c_i h, Y_i(y))$ ps:weights: b_i ; internal coefficients: a_{ij}
3. We can using **Butcher Table** to represent the RK method (Appendix) **Explicit:** $a_{ij} = 0$ for $j \geq i$ (严格下三角行) **Implicit:** $\exists a_{ij} \neq 0$ for $j \geq i$ (Not Explicit)

4.3 Accuracy of RK Method | Order Condition

Some Notations: If $\mathbf{y} = f'(\mathbf{y})$ where $f(\mathbf{y}) : \mathbb{R}^d \rightarrow \mathbb{R}^d$. Def $f' = (\frac{\partial f_i}{\partial y_j})_{1 \leq i \leq d, 1 \leq j \leq d}$ (行向量) $f'' = (\frac{\partial^2 f_i}{\partial y_j \partial y_k})_{1 \leq i \leq d, 1 \leq j, k \leq d}$

$$\cdot \text{Def: } f''(\mathbf{a}, \mathbf{b}) = \sum_{j=1}^d \sum_{k=1}^d \frac{\partial^2 f_i}{\partial y_j \partial y_k} a_j b_k \quad | y' = f \quad y'' = \sum_{j=1}^d \frac{\partial f_i}{\partial y_j} f_j = f' f \quad y''' = \sum_{j=1}^d \sum_{k=1}^d \frac{\partial^2 f_i}{\partial y_j \partial y_k} y'_j(t) y'_k(t) + \sum_{j=1}^d \frac{\partial f_i}{\partial y_j} y''_j(t) = f''(f, f) + f' f' f$$
$$\cdot \Phi_h(y) = y + hf + \frac{h^2}{2}f'f + \frac{h^3}{6}[f''(f, f) + f'f'f] + \mathcal{O}(h^4)$$

Order Condition: RK method: $y_{n+1} = y_n + h \sum_{i=1}^s b_i f(Y_i)$, Let $z(h) = \Phi_h(y)$

$$\Rightarrow \text{If } z'(0) = y', z''(0) = y'', \dots, z^{(n)}(0) = y^{(n)} \Rightarrow \text{Convergent at order } n$$
$$\text{Order 1: } \sum_{i=1}^s b_i = 1 \quad \text{Order 2: (add) } \sum_{i=1}^s b_i c_i = \frac{1}{2} \quad \text{Order 3: (add) } \sum_{i=1}^s b_i c_i^2 = \frac{1}{3} \text{ and } \sum_{i=1}^s \sum_{j=1}^s b_i a_{ij} c_j = \frac{1}{6}$$

5 Stability of Runge-Kutta Methods

5.1 Basic Definition for Stability

Fixed Point-Exact: For ODEs $\frac{dy}{dt} = f(y)$, point y^* is fixed point if $f(y^*) = 0 \Leftrightarrow \Phi_t(y^*) = y^*$ **Set of Fixed Points:** $\mathcal{F} = \{y^* \in \mathbb{R}^d : f(y^*) = 0\}$

Fixed Point-Numerical: *One-step* method $\Psi_h(y)$, point y^* is fixed point if $y^* = \Psi_h(y^*)$ Set of Fixed Points: $\mathcal{F}_h = \{y^* \in \mathbb{R}^d : y^* = \Psi_h(y^*)\}$

Theorem: For Runge-Kutta method, $\mathcal{F} \subseteq \mathcal{F}_h$ **Remark:** $\mathcal{F}_h \subseteq \mathcal{F}$ is NOT always true. If $\mathcal{F}_h = \mathcal{F}$, then the method is **regular**.

the point in $\mathcal{F}_h \setminus \mathcal{F}$ is called **spurious fixed point**. As $h \rightarrow \infty$, the *spurious* fixed points will tends to infinity.

Stability of Fixed Points: Fixed point y^* , the ODEs $\frac{dy}{dt} = f(y)$ with $y(0) = y_0$.

1. **Stable in the sense of Lyapunov:** Fixed point y^* is stable if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\|y_0 - y^*\| < \delta \Rightarrow \|y(t; y_0) - y^*\| < \varepsilon \forall t > 0$
2. **Asymptotically Stable:** Fixed point y^* is asymptotically stable if $\exists \delta > 0$ s.t. $\|y_0 - y^*\| < \delta \Rightarrow \lim_{t \rightarrow \infty} \|y(t; y_0) - y^*\| = 0$
3. **Unstable:** Fixed point y^* is unstable if it's not stable. i.e. $\exists \varepsilon > 0, \forall \delta > 0$ s.t. $\|y_0 - y^*\| < \delta \Rightarrow \|y(t) - y^*\| \geq \varepsilon$ for some t .

5.2 Classification of Fixed Points

Linearization Theorem: Suppose $\frac{dy}{dt} = f(y)$, y^* is a fixed point. Let $J = f'(y^*)$ be the Jacobian matrix of f at y^* .

1. If \forall eigenvalues of J in left complex half plane, then y^* is **asymptotically stable**.
2. If \exists eigenvalues of J in right complex half plane, then y^* is **unstable**.

(Following is a special cases from HDE)

Generalized Eigenvectors: If λ is an repeated eigenvalue with eigenvalue ξ then:

Generalized Eigenvectors: η s.t. $(A - \lambda I)\eta = \xi$ More generally: $(A - \lambda I)\eta_n = \eta_{n-1}$

Classification of Critical Points at y^* (Linear): r_1, r_2 be sol of $\det(J - \lambda I) = 0$. || $\mathbb{C} : r = \lambda \pm i\mu (\mu > 0)$

If f constant, write sol: $\mathbf{x} = c_1 e^{r_1 t} \xi_1 + c_2 e^{r_2 t} \xi_2 \quad || \quad GM = 1: \mathbf{x} = c_1 e^{rt} \xi + c_2 e^{rt} (t\xi + \eta) \quad J = \begin{pmatrix} \partial_x F(x_0) & \partial_y F(x_0) \\ \partial_x G(x_0) & \partial_y G(x_0) \end{pmatrix} \text{ If } f(x, y) = \begin{pmatrix} F(x, y) \\ G(x, y) \end{pmatrix}$

R/C	Condition Stability	Type Name	Phase Plane Description	Other	
R	$r_1 < r_2 < 0$ asystab	N NSk	向原点, ξ_2 直线, ξ_1 曲线, 和 ξ_1 周围 $y = \pm x^3$	$c_2 \neq 0, t \rightarrow \infty: \xi_2$ 主导方向; $c_2 = 0, t \rightarrow \infty: \xi_1$ 主导方向	PS:
	$r_1 > r_2 > 0$ unstable	N NSo	从原点向外, ξ_2 直线, ξ_1 曲线, 和 ξ_1 周围 $y = \pm x^3$	$c_1 \neq 0, t \rightarrow \infty: \xi_1$ 主导方向; $c_1 = 0, t \rightarrow \infty: \xi_2$ 主导方向	N = Node
	$r_1 > 0 > r_2$ unstable	SP SP	$t \rightarrow \infty, \xi_1$ 从原点向外, ξ_2 从外向原点 and: 像 $y = \pm \frac{1}{x}$, 同进同出	$t \rightarrow \pm \infty: x \rightarrow \infty; t \rightarrow \infty: c_1, c_2 \neq 0, x \rightarrow \infty: \xi_1$ 主导; $t \rightarrow \infty: c_2 = 0, x \rightarrow \infty: \xi_1$ 主导; $t \rightarrow \infty: c_1 = 0, x \rightarrow \infty: \xi_2$ 主导	PN = Proper Node IN = Improper
	$r_1 = r_2 < 0, GM=2$ asystab	PN PN or Stable Star	直线 向原点	直线, u_1/u_2 is t independent	or: Degenerate Node
	$r_1 = r_2 > 0, GM=2$ unstable	PN PN or Unstable Star	直线 从原点向外	直线, u_1/u_2 is t independent	SP = Saddle Point
	$r_1 = r_2 < 0, GM=1$ asystab	IN (AL.Type: SpP) IN (Stable)	S 曲线, 向原点	$t \rightarrow \infty, x \rightarrow 0, \xi_2$ 主导 ps: 旋转方向大体和 $\eta + c_2 \xi_2$ 方向相同	SpP = spiral point
	$r_1 = r_2 > 0, GM=1$ unstable	IN (AL.Type: SpP) IN (Unstable)	S 曲线, 从原点向外	$t \rightarrow \infty, x \rightarrow 0, \xi_2$ 主导 ps: 旋转方向大体和 $\eta + c_2 \xi_2$ 方向相同	or: Focus Point
C	$\lambda \neq 0, \lambda > 0$ unstable	SpP Unstable Focus	向外椭圆 (elliptical) 螺旋	$t \rightarrow \infty, x \rightarrow 0$ ps: 考虑 $J = (a, b, c, d)$, 如果 $bc < 0$, 顺时针, 如果 $bc > 0$, 逆时针	C = Center
	$\lambda \neq 0, \lambda < 0$ asystab	SpP Stable Focus	向内椭圆 (elliptical) 螺旋	$t \rightarrow -\infty, x \rightarrow 0$ ps: 考虑 $J = (a, b, c, d)$, 如果 $bc < 0$, 顺时针, 如果 $bc > 0$, 逆时针	NSK = Nodal Sink
	$\lambda = 0$ stable (AL:Indeterminate)	C (AL.C SpP) C	椭圆 (elliptical) 和 半长轴 ξ 实部方向	Bounded trajectory or \exists Periodic Trajectories	NSo = Nodal Source

5.3 Stability of Fixed Points of Maps (Numerical)

Definition: For flow map Ψ from $\mathbb{R}^d \rightarrow \mathbb{R}^d$. Def $y^n(y_0) :=$ the n -th iterate of y_0 under Ψ . i.e. $y^n = y_n$; $y_n = \Psi(y_{n-1})$

Stability of Fixed Points of Maps: Fixed point y^* , the map Ψ with $y^* = \Psi(y^*)$.

1. **Stable in the sense of Lyapunov:** y^* is stable if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\|y_0 - y^*\| < \delta \Rightarrow \|y^n(y_0) - y^*\| < \varepsilon \quad \forall n \geq 0$
2. **Asymptotically Stable:** y^* is asymptotically stable if $\exists \delta > 0$ s.t. $\|y_0 - y^*\| < \delta \Rightarrow \lim_{n \rightarrow \infty} \|y^n(y_0) - y^*\| = 0$
3. **Unstable:** y^* is unstable if it's not stable. i.e. $\exists \varepsilon > 0, \forall \delta > 0$ s.t. $\|y_0 - y^*\| < \delta \Rightarrow \|y^n(y_0) - y^*\| \geq \varepsilon$ for some n .

Spectral Radius: For matrix K , $\rho(K) = \max\{|\lambda| : \lambda \text{ is eigenvalue of } K\}$

Theorem|Spectral Radius: Let $z_n = \|K^n y_0\|$, where $K \in \mathbb{R}^{d \times d}$ is the matrix. Then:

1. $\rho(K) < 1 \Leftrightarrow \lim_{n \rightarrow \infty} z_n = 0$
2. $\rho(K) > 1 \Leftrightarrow \lim_{n \rightarrow \infty} z_n = \infty$
3. If $\rho(K) = 1$ and *eigenvalues* of K are *semisimple* (i.e. No generalized eigenvector), then $\{z_n\}$ is bounded.

Theorem|Connect to Stability: For smooth (C^2) map Ψ , $y^* = \Psi(y^*)$. Let $K = \Psi'(y^*)$, for iteration $y_{n+1} = \Psi(y_n)$, we have:

1. $\rho(K) < 1 \Rightarrow y^*$ is asymptotically stable
2. $\rho(K) > 1 \Rightarrow y^*$ is unstable

5.4 Linear Stability of Numerical Methods

Special Case|Euler Method: For $\frac{dy}{dt} = By$, Using Euler method: $y_{n+1} = (I + hB)y_n$. where λ_i is eigenvalues of B . Assume $f(y) = \lambda y$

1. The origin is *stable* if $||I + h\lambda_i|| \leq 1 \ \forall i$
2. The origin is *asymptotically stable* if $||I + h\lambda_i|| < 1 \ \forall i$
3. The origin is *unstable* if $||I + hB|| > 1$

ps: 即 $h\lambda_i$ 在复平面上以 $z = -1$ 为圆心, 半径为 1 的圆内 \leftarrow 称为 **Region of absolute stability**

Stability function R, P : Let P be polynomial function and R be rational function.

If RK is *explicit*, then $y_{n+1} = P(\mu)y_n$; If RK is *implicit*, then $y_{n+1} = R(\mu)y_n$ where $\mu = h\lambda$

Stability function $R(\mu)$ |Special Case: For $\frac{dy}{dt} = \lambda y$ All RK methods can be written as: where: b^T, A are from *Butcher Table*. $\mathbf{1} = [1, ..., 1]^T$

I. $Y_i = y_n + \mu \sum_{j=1}^s a_{ij} Y_j$ ($Y = y_n \mathbf{1} + \mu AY$) $y_{n+1} = y_n + \mu \sum_{b=1}^s b_i Y_j = y_n + \mu b^T Y$

II. $R(\mu) = 1 + \mu b^T (I - \mu A)^{-1} \mathbf{1}$ **III.** $y_{n+1} = R(\mu)y_n$ where $\mu = h\lambda$

Stability function $R(\mu)$ |General: For $\frac{dy}{dt} = By$ where: b^T, A are from *Butcher Table*. Λ, U is B 的特征值分解 $U^{-1}BU = \Lambda$ 此时 z_n, y_n 是向量

I. Let $y_n = Uz_n$ and $Y_i = UZ_i$:

Then $Z_i = z_n + h \sum_{j=1}^s a_{ij} \Lambda Z_j$ ($z_j^{(i)} = z_n^{(i)} \mathbf{1} + \mu A Z_j^{(i)} \ \forall i$) $z_{n+1} = z_n + h \sum_{i=1}^s b_i \Lambda Z_i$ ($z_{n+1}^{(i)} = z_n^{(i)} + \mu \sum_{j=1}^s b_j Z_j^{(i)}$)

II. $\frac{dz}{dt} = \Lambda z \Rightarrow \frac{dz^{(i)}}{dt} = \lambda_i z^{(i)} \Rightarrow z_{n+1}^{(i)} = R(\mu) z_n^{(i)}$ where $\mu = h\lambda_i$ (回到前一个)

Theorem: For $\frac{dy}{dt} = By$ with $\lambda_1, ..., \lambda_d$ be eigenvalues of B . The RK method is *stable|asy.stab* at *origin* iff:

The Same method also *stable|asy.stab* at *origin* for $\frac{dz}{dt} = \lambda_i z \ \forall i$

Corollary: For $\frac{dy}{dt} = By$ with B diagonalizable. An RK Method with *stability function $R(\mu)$* is *stable|asy.stab|unstable* at *origin* iff: Assume $f(y) = \lambda_i y$

$|R(\mu)| \leq 1$ or $|R(\mu)| < 1$ or $|R(\mu)| > 1 \ \forall \mu = h\lambda_i \ \forall i$ we can write $\sigma(B) = \{\lambda_1, ..., \lambda_d\}$ the set of eigenvalues of B

Remark: 这里的 $R(\mu)$ 是指 B 分解后的每一个特征值 λ_i 的 $R(\mu)$, 而不是 B 的 $R(\mu)$

5.5 Stability Region and A-stability

Stability Region: $\frac{dy}{dt} = By$. An RK method, the *stability region* is the set of μ where $\hat{R}(\mu) = |R(\mu)| < 1$. ($f(y) = \lambda y$, 如 y 是向量, $R(\mu)$ 按上面 corollary 的 remark 所说)

1. Euler's Method: $\hat{R}(\mu) = |1 + \mu| \Rightarrow \mu \in \{z \in \mathbb{C} : |1 + z| < 1\}$ (-1 处半径为 1 的圆)
2. Trapezoidal Rule: $\hat{R}(\mu) = \left| \frac{1+\mu/2}{1-\mu/2} \right| \Rightarrow \mu \in \{z \in \mathbb{C} : |1 + z/2| < |1 - z/2|\}$ (left complex half-plane, A-stable)
3. Implicit Euler: $\hat{R}(\mu) = |1 - \mu|^{-1} \Rightarrow \mu \in \{z \in \mathbb{C} : |1 - z| > 1\}$ (-1 处半径为 1 的圆外侧)
4. RK4: $\hat{R}(\mu) = \left| 1 + \mu + \frac{\mu^2}{2} + \frac{\mu^3}{6} + \frac{\mu^4}{24} \right| \Rightarrow$ Using $R(\mu) = e^{i\theta}$ to find the region.

A-Stable: An RK method is *A-stable* if its *stability region* contains the entire *left complex half-plane*. (i.e. $\Re(z) < 0$)

6 Linear Multistep Methods

7 Appendix

7.1 Common Numerical Method | Order Condition

Method	Formula	Order	Stability
Euler's Method	$y_{n+1} = y_n + h f(t_n, y_n)$	1	$ 1 + h\lambda < 1$
Backward Euler	$y_{n+1} = y_n + h f(t_{n+1}, y_{n+1})$	1	$\left \frac{1}{1-h\lambda} \right < 1$ (A-stable)
Trapezoidal Rule	$y_{n+1} = y_n + \frac{h}{2} \left[f(t_n, y_n) + f(t_{n+1}, y_{n+1}) \right]$	2	A-stable; $R(z) = \frac{1+z/2}{1-z/2}$
Explicit Midpoint	$y_{n+1} = y_n + h f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n, y_n)\right)$	2	$\left 1 + h\lambda + \frac{(h\lambda)^2}{2} \right < 1$
Heun's Method	$y_{n+1} = y_n + \frac{h}{2} \left[f(t_n, y_n) + f\left(t_{n+1}, y_n + h f(t_n, y_n)\right) \right]$	2	$\left 1 + h\lambda + \frac{(h\lambda)^2}{2} \right < 1$
Theta Method	$y_{n+1} = y_n + h \left[(1-\theta) f(t_n, y_n) + \theta f(t_{n+1}, y_{n+1}) \right]$	1 (or 2 if $\theta = \frac{1}{2}$)	$R(z) = \frac{1+(1-\theta)z}{1-\theta z}$
RK4 Method	见 Butcher Table	4	$R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}$
2-Stage Gauss-Legendre	$\frac{1}{2} - \frac{\sqrt{3}}{6}$ $\frac{1}{4}$ $\frac{1}{4} - \frac{\sqrt{3}}{6}$	4	A-stable
	$\frac{1}{2} + \frac{\sqrt{3}}{6}$ $\frac{1}{4} + \frac{\sqrt{3}}{6}$ $\frac{1}{4}$		
		1/2 1/2	

RK Order Condition

1. **order 1:** $\sum_{i=1}^s b_i = 1$
2. **order 2:** $\sum_{i=1}^s b_i c_i = \frac{1}{2}$
3. **order 3:** $\sum_{i=1}^s b_i c_i^2 = \frac{1}{3}$ and $\sum_{i=1}^s \sum_{j=1}^s b_i a_{ij} c_j = \frac{1}{6}$
4. **order 4:** $\sum_{i=1}^s b_i c_i^3 = \frac{1}{4}$, $\sum_{i=1}^s \sum_{j=1}^s b_i a_{ij} c_j^2 = \frac{1}{8}$, $\sum_{i=1}^s \sum_{j=1}^s b_i a_{ij} c_j^2 = \frac{1}{12}$, $\sum_{i=1}^s \sum_{j=1}^s \sum_{k=1}^s b_i a_{ij} a_{jk} c_k = \frac{1}{24}$

7.2 Useful Series | Common RK Methods

Common Runge-Kutta Methods (Butcher Table):

c_1	a_{11}	\cdots	a_{1s}	0	0	0	0
\vdots	\vdots	\ddots	\vdots	1	1	$1/2$	$1/2$
c_s	a_{s1}	\cdots	a_{ss}	$1/2$	$1/2$	1	1
	b_1	\cdots	b_s			$1/6$	$2/3$
						$1/6$	$1/6$
Example				RK1		RK2 (Heun's Method)	
				(Euler's Method)		RK3	
						RK4 (Classical/Famous)	

Useful Series:

$f(x)$	Taylor	Series	R	$f(x)$	Taylor	Series	R
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + \dots$	1	$\frac{1}{(1-x)^2}$	$\sum_{n=1}^{\infty} nx^{n-1}$	$1 + 2x + 3x^2 + 4x^3 + \dots$	1
$\frac{2}{(1-x)^3}$	$\sum_{n=2}^{\infty} n(n-1)x^{n-2}$	$2 + 6x + 12x^2 + 20x^3 + \dots$	1	e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	∞
$\ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$	1	$-\ln(1-x)$	$\sum_{n=1}^{\infty} \frac{x^n}{n}$	$x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$	1
$\sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$	∞	$\cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$	∞
$\arctan x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$	1	$\sinh x$	$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$	∞
$\cosh x$	$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$	∞	$(1+x)^k$	$\sum_{n=0}^{\infty} \binom{k}{n} x^n$	$1 + kx + \frac{k(k-1)x^2}{2!} + \dots$	1
$\ln x$	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$	$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots$	$1, 0 < x < 2$	$\frac{1}{1+x}$	$\sum_{n=0}^{\infty} (-1)^n x^n$	$1 - x + x^2 - x^3 + \dots$	1