HAlg Note

1 Basic Knowledge

Def of Group $(G,*)$: A set G with a operator $*$ is a group if: **Closure**: $\forall g,h\in G, g*h\in G$; **Associativity**: $\forall g,h,k\in G, (g*h)*k=g*(h*k)$; **Identity**: $\exists e\in G, \forall g\in G, e*g=g*e=g$; **Inverse**: $\forall g\in G, \exists g^{-1}\in G, g*g^{-1}=g^{-1}*g=e$. G,H groups, then $G\times H$ also.

Field (F): A set F is a field with two operators: (addition)$+: F\times F\to F; (\lambda,\mu)\to\lambda+\mu$ (multiplication)$\cdot : F\times F\to F; (\lambda,\mu)\to\lambda\mu$ if: $(F,+)$ and $(F\setminus\{0_F\},\cdot)$ are abelian groups with identity $0_F, 1_F$. and $\lambda(\mu+\nu)=\lambda\mu+\lambda\nu$ $e.g. Fields: \mathbb{R},\mathbb{C},\mathbb{Q},\mathbb{Z}/p\mathbb{Z}=\mathbb{F}_p$

F-Vector Space (V): A set V over a field F is a vector space if: V is an abelian group $V=(V,\dot{+})$ and $\forall\vec{v},\vec{w}\in V\ \lambda,\mu\in F$ e.g. $Poly: \mathbb{R}[x]_{<n}$
$\exists$ map $F\times V\to V: (\lambda,\vec{v})\to\lambda\vec{v}$ satisfies: **I**: $\lambda(\vec{v}\dot{+}\vec{w})=(\lambda\vec{v})\dot{+}(\lambda\vec{w})$ **II**: $(\lambda+\mu)\vec{v}=(\lambda\vec{v})\dot{+}(\mu\vec{v})$ **III**: $\lambda(\mu\vec{v})=(\lambda\mu)\vec{v}$ **IV**: $1_F\vec{v}=\vec{v}$

Vector Subspaces Criterion: $U\subseteq V$ is a subspace of V if: **I.** $\vec{0}\in U$ **II.** $\forall\vec{u},\vec{v}\in U, \forall\lambda\in F$: $\vec{u}+\vec{v}\in U$ and $\lambda\vec{u}\in U$ (or: $\lambda\vec{u}+\mu\vec{v}\in U$)

· **property**: If U,W are subspaces of V, then $U\cap W$ and $U+W$ are also subspaces of V. ps: $U+W:=\{\vec{u}+\vec{w}:\vec{u}\in U,\vec{w}\in W\}$

Complement-wise Operations: $\phi: V_1\times V_2\to V_1\oplus V_2$ by **I**:$(\vec{v_1},\vec{u_1})+(\vec{v_2},\vec{u_2}):=(\vec{v_1}+\vec{v_1},\vec{u_1}+\vec{u_2})$, $\lambda(\vec{v},\vec{u}):=(\lambda\vec{v},\lambda\vec{u})$ (ps:V_1,V_2 通过 ϕ 定义的 map 所形成的 vector space 记作 $V_1\oplus V_2$)

Projections: $pr_i: X_1\times\cdots\times X_n\to X_i$ by $(x_1,...,x_n)\mapsto x_i$ **Canonical Injections**: $in_i: X_i\to X_1\times\cdots\times X_n$ by $x\mapsto(0,...,0,x,0,...,0)$

2 Vector Spaces/Subspaces | Generating Set | Linear Independent | Basis

Generating (subspaces) $\langle T\rangle$: $\langle T\rangle:=\{\alpha_1\vec{v_1}+\cdots+\alpha_n\vec{v_n}:\alpha_i\in F,\vec{v_i}\in T, r\in\mathbb{N}\}$ $\langle\emptyset\rangle:=\{\vec{0}\}$ If T is subspace $\Rightarrow\langle T\rangle=T$.

1. **Proposition**: $\langle T\rangle$ is the smallest subspace containing T. (i.e. $\langle T\rangle$ is the intersection of all subspaces containing T)
2. **Generating Set**: V is vector space, $T\subseteq V$. T is generating set of V if $\langle T\rangle=V$. **Finitely Generated**: $\exists$ finite set T, $\langle T\rangle=V$
3. **External Direct Sum**: 一个" 代数结构", 定义为 set 是 $V_1\oplus\cdots\oplus V_n:=V_1\times\cdots\times V_n$ 且有一组运算法则 component-wise operations
4. **Connect to Matrix**: Let $E=\{\vec{v_1},...,\vec{v_n}\}$, E is GS of V. Let $A=[\vec{v_1},...,\vec{v_n}]\Rightarrow\forall\vec{b}\in V,\exists\vec{x}=(x_1,...,x_n)^T$ s.t. $A\vec{x}=\vec{b}$ (i.e. linear map:$\phi:\vec{x}\mapsto A\vec{x}$ is surjective)

Linearly Independent: $L=\{\vec{v_1},\vec{v_2},...,\vec{v_r}\}$ is linearly independent if: $\forall c_1,...,c_r\in F, c_1\vec{v_1}+\cdots+c_r\vec{v_r}=\vec{0}\Rightarrow c_1=\cdots=c_r=0$.

· **Connect to Matrix**: Let $L=\{\vec{v_1},...,\vec{v_n}\}$, L is LI of V. Let $A=[\vec{v_1},...,\vec{v_n}]\Rightarrow\forall\vec{x}\in F^n, A\vec{x}=0\ (or\ \vec{0})\Rightarrow\vec{x}=0(or\ \vec{0})$ (i.e. linear map ϕ:$\vec{x}\mapsto A\vec{x}$ is injective)

Basis & Dimension: If V is finitely generated. $\Rightarrow\exists$ subset $B\subseteq V$ which is both LI and GS. (B is basis) **Dim**: $\dim V:=|B|$

· **Connect to Matrix**: Let $B=\{\vec{v_1},...,\vec{v_n}\}$ is basis of V. Let $A=[\vec{v_1},...,\vec{v_n}]\Rightarrow\forall\vec{x}=(x_1,...,x_n)^T$ s.t. $\phi:\vec{x}\mapsto A\vec{x}$ is 1-1 & onto (Bijection)

Relation|GS,LI,Basis,dim: Let V be vector space. L is linearly independent set, E is generating set, B is basis set.

1. **GS|LI**: $|L|\le|E|$ (can get: dim unique) **LI→Basis**: If V finite generate $\Rightarrow\forall L$ can extend to a basis. If $L=\emptyset$, prove $\exists B$
2. **Basis|max,min**: $B\Leftrightarrow B$ is minimal GS $(E)\Leftrightarrow B$ is maximal LI (L). **Uniqueness|Basis**: 每个元素都可以由 basis 唯一表示.
3. **Proper Subspaces**: If $U\subset V$ is proper subspace, then $\dim U<\dim V$. $\Rightarrow$ If $U\subseteq V$ is subspace and $\dim U=\dim V$, then $U=V$.
4. **Dimension Theorem**: If $U,W\subseteq V$ are subspaces of V, then $\dim(U+W)=\dim U+\dim W-\dim(U\cap W)$

Complementary: $U,W\subseteq V$,U,V subspaces are complementary $(V=U\oplus W)$ if: $\exists\phi: U\times W\to V$ by $(\vec{u},\vec{w})\mapsto\vec{u}+\vec{w}$
i.e. $\forall\vec{v}\in V$, we have unique $\vec{u}\in U,\vec{w}\in W$ s.t. $\vec{v}=\vec{u}+\vec{w}$. ps: It's a linear map.

3 Linear Mapping | Rank-Nullity| Matrices | Change of Basis

ps: 默认 V,W **F-Vector Spaces.**

Linear Mapping/Homomorphism(Hom): $f: V\to W$ is *linear map* if: $\forall\vec{v}_1,\vec{v}_2\in V,\forall\lambda\in F$. $f(\vec{v}_1+\vec{v}_2)=f(\vec{v}_1)+f(\vec{v}_2)$ and $f(\lambda\vec{v}_1)=\lambda f(\vec{v}_1)$

· **Isomorphism**: = LM & Bij. **Endomorphism(End)**: = LM & $V=W$. **Automorphism(Aut)**: = LM & $V=W$ **Monomorphism**: = LM & 1-1. **Epimorphism**: = LM & onto.

· **Kernel**: $\ker f:=\{\vec{v}\in V: f(\vec{v})=\vec{0}\}$(It's subspace) **Image**: $Imf:=\{f(\vec{v}):\vec{v}\in V\}$(It's subspace) **Rank**:= $\dim(Imf)$ **Nullity**:= $\dim(\ker f)$ **Fixed Point** X^f: $X^f:=\{x\in X: f(x)=x\}$

Property of Linear Map: Let $f,g\in Hom$: **a**.$f(\vec{0})=\vec{0}$ **b.** f is 1-1 iff $\ker f=\{\vec{0}\}$ **c.** $f\circ g$ is linear map.

1. **Determined**: f is determined by $f(\vec{b_i}), \vec{b_i}\in\mathcal{B}_{basis}$ ($\star$ **i.e.** $f(\sum_i\lambda_i\vec{v_i}):=\sum_i\lambda_i f(\vec{v_i})$)
2. **Classification of Vector Spaces**: $\dim V=n\Leftrightarrow f: F^n\xrightarrow{\sim}V$ by $f(\lambda_1,...,\lambda_n)\mapsto\sum_{i=1}^n\lambda_i\vec{v_i}$ is isomorphism.
3. **Left/Right Inverse**: f is 1-1 $\Rightarrow\exists$ left inverse g s.t. $g\circ f=id$ 考虑 direct sum f is onto $\Rightarrow\exists$ right inverse g s.t. $f\circ g=id$
4. $^\ominus$**More of Left/Right Inverse**: $f\circ g=id\Rightarrow g$ is 1-1 and f is onto. 使用 kernel=0 来证明

Rank-Nullity Theorem: For linear map $f: V\to W$, $\dim V=\dim(\ker f)+\dim(Imf)$ Following are properties:

1. **Injection**: f is 1-1 $\Rightarrow\dim V\le\dim W$ **Surjection**: f is onto $\Rightarrow\dim V\ge\dim W$ Moreover, $\dim W=\dim imf$ iff f is onto.
2. **Same Dimension**: f is isomorphism $\Rightarrow\dim V=\dim W$ **Matrix**: $\forall M$, column rank $c(M)$ = row rank $r(M)$.
3. **Relation**: If V,W finite generate, and $\dim V=\dim W$, Then: f is isomorphism $\Leftrightarrow f$ is 1-1 $\Leftrightarrow f$ is onto.

Matrix: For $A_{n\times m}, B_{m\times p}, AB_{n\times p}:=(AB)_{ij}=\sum_{k=1}^m a_{ik}b_{kj}$ **Transpose**: $A^T_{m\times n}:=(A^T)_{ij}=a_{ji}$

Invertible Matrices: A is invertible if $\exists B,C$ s.t. $BA=I$ and $AC=I$ || $\exists B, BA=I\Leftrightarrow\exists C, AC=I\Leftrightarrow\exists A^{-1}$ ${}_{\mathcal{B}}[f^{-1}]_{\mathcal{A}}={}_{\mathcal{A}}[f]_{\mathcal{B}}^{-1}$

Representing matrix of linear map ${}_{\mathcal{B}}[f]_{\mathcal{A}}$: $f: V\to W$ be linear map, $\mathcal{A}=\{\vec{v_1},...,\vec{v_n}\}$ is basis of V, $\mathcal{B}=\{\vec{w_1},...,\vec{w_m}\}$ is basis of W.

1. ${}_{\mathcal{B}}[f]_{\mathcal{A}}:=A$ (matrix) where $f(\vec{v}_{i\in\mathcal{A}})=\sum_{j\in\mathcal{B}}A_{ji}\vec{w}_j$ $\exists M_{\mathcal{B}}^{\mathcal{A}}: Hom_F(V,W)\xrightarrow{\sim}Mat(n\times m;F)$
2. If $\vec{v}\in V$, then ${}_{\mathcal{A}}[\vec{v}]:=\mathbf{b}$ (vector) where $\vec{v}=\sum_{i\in\mathcal{A}}\mathbf{b}_i\vec{v}_i$
3. **Theorems**: $[f\circ g]=[f]\circ[g]$ ${}_{\mathcal{C}}[f\circ g]_{\mathcal{A}}={}_{\mathcal{C}}[f]_{\mathcal{B}}\circ_{\mathcal{B}}[g]_{\mathcal{A}}$ ${}_{\mathcal{B}}[f(\vec{v})]={}_{\mathcal{B}}[f]_{\mathcal{A}}\circ_{\mathcal{A}}[\vec{v}]$ ${}_{\mathcal{A}}[f]_{\mathcal{A}}=I\Leftrightarrow f=id$
4. **Change of Basis**: Define *Change of Basis Matrix*:$={}_{\mathcal{A}}[id_V]_{\mathcal{B}}$ ${}_{\mathcal{B}'}[f]_{\mathcal{A}'}={}_{\mathcal{B}'}[id_W]_{\mathcal{B}}\circ_{\mathcal{B}}[f]_{\mathcal{A}}\circ_{\mathcal{A}}[id_V]_{\mathcal{A}'}$ ${}_{\mathcal{A}'}[f]_{\mathcal{A}'}={}_{\mathcal{A}}[id_V]^{-1}_{\mathcal{A}'}\circ_{\mathcal{A}}[f]_{\mathcal{A}}\circ_{\mathcal{A}}[id_V]_{\mathcal{A}'}$

Elementary Matrix: $I+\lambda E_{ij}$ (cannot $I-E_{ii}$) 就是初等矩阵, 左乘代表 j 行乘 λ 倍加到第 i 行, 右乘代表 j 列乘 λ 倍加到第 i 列 $\Rightarrow$ Invertible!

1. 交换 i,j 列/行: $P_{ij}=diag(1,...,1,-1,1,...,1)(I+E_{ij})(I-E_{ji})(I+E_{ij})$ where -1 in jth place.
2. **Row Echelon Form|Smith Normal Form**: $\tilde{A}$: REF 通过左乘初等矩阵可以实现 $\tilde{\tilde{A}}$: $S(n,m,r)$ 通过 $\tilde{A}$ 右乘初等矩阵可以实现

Smith Normal Form: $\forall A,\exists$ invertible P,Q s.t. $PAQ=S(n,m,r):=n\times m$ 的矩阵, 对角线前 r 个是 1, 后面 0. **Lemma**: $r=r(A)=c(A)$

Similar Matrices: If $N=T^{-1}MT$, then M,N are similar. *Special Case*: If $N={}_{\mathcal{B}}[f]_{\mathcal{B}}, M={}_{\mathcal{A}}[f]_{\mathcal{A}}$, then $N=T^{-1}MT$. where $T={}_{\mathcal{A}}[id_V]_{\mathcal{B}}$

Trance: $tr(A):=\sum_i a_{ii}$ and $tr(f):=tr({}_{\mathcal{A}}[f]_{\mathcal{A}})$ | $tr(AB)=tr(BA)$ $tr(\lambda A+\mu B)=\lambda tr(A)+\mu tr(B)$ $tr(N)=tr(M)$ if M,N similar.