NODEA Note

1 Basic Knowledge

Def of ODE & ODEs: (1st order) ODE: $\frac{dy}{dt} = f(t,y)$ & ODEs: $\frac{d\mathbf{y}}{dt} = \mathbf{f}(t,\mathbf{y})$, $\mathbf{y} = (y_1, ..., y_d)^T, \mathbf{f}(t,\mathbf{y}) = (f_1(t,\mathbf{y}), ..., f_d(t,\mathbf{y}))^T$

Autonomous: $\frac{d\mathbf{y}}{dt} = \mathbf{f}(\mathbf{y}) \Rightarrow$ autonomous ODE(s). || ⇓ New Autonomous ODEs: $\frac{d\mathbf{y}}{ds} = \mathbf{f}(y_{d+1}, \mathbf{y})$ and $\frac{dy_{d+1}}{ds} = 1$

· **Change to Autonomous**: For $\frac{d\mathbf{y}}{dt} = \mathbf{f}(t,\mathbf{y})$. Let $y_{d+1} = t$ and *new* independent variable s s.t. $\frac{dt}{ds} = 1$ ⇑

Linearity: ODE: $\frac{dy}{dt} = f(t,y)$ is linearity if $f(t,y) = a(t)y + b(t)$ || ODEs: If each ODE is linear, then the ODEs are linear.

Picard's Theorem: If $f(t,y)$ is continuous in $D := \{(t,y) : t_0 \le t \le T, |y - y_0| < K\}$ and $\exists L > 0$ (Lipschitz constant) s.t.

$\forall (t,u),(t,v) \in D \quad |f(t,u) - f(t,v)| \le L|u - v|$ (ps:Can use MVT). And Assume that $M_f(T - t_0) \le K, M_f := \max\{|f(t,u)| : (t,u) \in D\}$

⇒ **Then**, ∃ a unique continuously differentiable solution $y(t)$ to the IVP $\frac{dy}{dt} = f(t,y), y(t_0) = y_0$ on $t \in [t_0, T]$.

Existence & Uniqueness Theorem: IVP $\frac{d\mathbf{y}}{dt} = \mathbf{f}(t,\mathbf{y}), \mathbf{y}(t_0) = \mathbf{y}_0$. If $f(t,y)$ and $\frac{\partial f}{\partial y_i}$ are continuous in a neighborhood of $(t_0, \mathbf{y}_0)$.

⇒ **Then**, $\exists I := (t_0 - \delta, t_0 + \delta)$ s.t. ∃ a unique continuously differentiable solution $\mathbf{y}(t)$ to the IVP on $t \in I$.

2 Acknowledge

Notation	Meaning	Notation	Meaning
$[a,b]$	Approximate function for $t \in [a,b]$	$t_0 = a \mid t_N = b$	Assume that $t_0 = a, t_N = b$
N	number of **timesteps** (i.e. Break up interval $[a,b]$ into N equal-length sub-intervals)	h	**stepsize** ($h = \frac{b-a}{N}$)
t_i	Define $N+1$ points: $t_0, t_1, ..., t_N$	t_m	$t_m = a + h \cdot m = t_0 + h \cdot m$
y_i	Approximation of y at point $t = t_i$ (Except y_0)	$y(t_i)$	Exact value of y at point $t = t_i$

3 Euler's Method and Taylor Series Method

Euler's Method Algorithm: Approximate ODE $\frac{dy}{dt} = f(t,y), y(t_0) = y_0$ with number of steps N. (Similarly for ODEs)

⇒ **for** $n = 0, 1, 2, ..., N-1$: $y_{n+1} = y_n + hf(t_0 + nh, y_n) = y_n + hf(t_n, y_n)$ **end** (ps: ⇓ Can get $|y''| < M$)

Boundedness Theorem: For $\frac{dy}{dt} = f(t,y), y(a) = y_0$ and suppose there exists a unique, twice differentiable, solution $y(t)$ on $[a,b]$.

Suppose: y is continuous and $|\frac{\partial f}{\partial y}| \le L$. ⇒ the solution y_n given Euler's method satisfies: $e_n = |y_n - y(t_n)| \le Dh, D = e^{(b-a)L}\frac{M}{2L}$

· **Lemma**: If $v_{n+1} \le Av_n + B$, then $v_n \le A^n v_0 + \frac{A^n - 1}{A - 1}B$ If $v_n = e_n := y_n - y(t_n)$, then $A = 1 + hL, B = h^2 M/2$ (suppose $|y''| < M$)

Order Notation ($\mathcal{O}$): we write $z(h) = \mathcal{O}(h^p)$ if $\exists C, h_0 > 0$ s.t. $|z| \le Ch^p, 0 < h < h_0$

Flow Map (Φ, Ψ): $\Phi_{t_0,h}(y_0) = y(t_0 + h)$ Clearly, $\Phi(t_n + h) = y(t_n + h) = \Phi_h(y(t_n)) = y(t_{n+1})$.

· $\Psi_{t_n,h}(y_n) = y_{n+1}$:= Numerical method for ODE Clearly, $\Psi(t_n + h) = y_{n+1} = \Psi_h(y_n)$

Taylor Series Method: Approximate ODE $\frac{dy}{dt} = f(t,y), y(t_0) = y_0$ with *n-order Methods*: 用 Taylor Series 在 $t_0 + h$ 处展开保留到 n 阶

· $\Phi_{t,h}(y) = y + hf(t,y) + \frac{1}{2}h^2[f_t(t,y) + f_y(t,y)f(t,y)] + \frac{1}{6}y'''(t,y)h^3 + \cdots$ (For one variable y)

· ps: Taylor Series: $y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{h^2}{2}y''(t_0) + \cdots + \frac{h^{n-1}}{(n-1)!}y^{(n-1)}(t_0) + \frac{h^n}{n!}y^{(n)}(t^*), t^* \in [t, t+h]$ ps: $y' = f, y'' = f_t + f_y f$

4 Convergence of One-Step Methods consider for autonomous $y' = f(y)$

4.1 Convergence | Consistent | Stable

Global Error: global error after n steps: $e_n := y_n - y(t_n)$ **Local Error**: For *one-step* method is: $le(y,h) = \Psi_h(y) - \Phi_h(y)$

Consistent: If $||le(y,h)|| \le Ch^{p+1} (\le \mathcal{O}(h^{p+1}))$, $C > 0$. ⇒ Consistent at order p. **Stable**: If $||\Psi_h(u) - \Psi_h(v)|| \le (1 + h\hat{L})||u - v||$

Convergent: A method is convergent if:$\forall T$, $\lim_{h\to 0,\, h=T/N} \max_{n=0,1,...,N} ||e_n|| = 0$ ⇓ Then the global error satisfies: $\max_{n=0,1,...,N} ||e_n|| = \mathcal{O}(h^p)$ **p-th order**

Convergence of One-Step Method: For $y' = f(y)$, and a one-step method $\Psi_h(y)$ is [1] consistent at order p and [2] stable with $\hat{L}$ ⇑. (ps:$C = \frac{C}{\hat{L}}(e^{T\hat{L}} - 1)$)

4.2 More One-Step Methods | Runge-Kutta Methods | Collocation

Construction of More General one-step Method: For $y' = f(y), y(t_0) = y_0 \Rightarrow y(t+h) - y(t) = \int_t^{t+h} f(y(\tau))d\tau$

Trapezoidal Method: $y_{n+1} = y_n + \frac{h}{2}(f(y_n) + f(y_{n+1}))$ **Midpoint Method**: $y_{n+1} = y_n + hf(\frac{y_n + y_{n+1}}{2})$

One-Step Collocation Methods (By Lagrange Interpolating Polynomials):

1. **Lagrange Interpolating Polynomials**: $\ell_i(x) = \prod_{j=1,j\ne i}^{s} \frac{x - c_j}{c_i - c_j} \in \mathbb{P}_{s-1}$ where $c_i \in F \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$

   ⇒ **Polynomial Interpolation**: $\forall p(x) \in \mathbb{P}_s$ with $p(c_i) = g_i \in F \Rightarrow \exists!\, p(x) = \sum_{i=1}^{s} g_i \ell_i(x)$ (Can be proved by Honour Algebra)

2. **Quadrature Rule**: If $g(t) \in \mathbb{P}_{p-1} \Rightarrow$ Order p | $\int_{t_0}^{t_0+h} g(t)dt = \int_0^1 g(t_0 + hx)dx \approx h\sum_{i=1}^{s} b_i g(t_0 + hc_i), b_i := \int_0^1 \ell_i(x)dx$ ps: c_i 从 $[0,1]$ 中取不同的

3. **Collocation Methods**: For: $y(t_0) = y_0$, $y'(t_0 + c_i h) = f(y(t_0 + c_i h))$ ps: c_i 从 $[0,1]$ 中取不同的 Let: $a_{ij} := \int_0^{c_i} \ell_j(x)dx$ and $b_i := \int_0^1 \ell_i(x)dx$

   ⇒ $F_i = f(y_n + h\sum_{j=1}^{s} a_{ij}F_j)$ and $y_{n+1} = y_n + h\sum_{i=1}^{s} b_i F_i$ where $F_i := y'(t_0 + c_i h)$

· **Remark**: For choice of c_i: The optimal choice is attained by *Gauss-Legendre collocation methods*.

Runge-Kutta Methods: Let $y' = f(t,y)$ **Stage Values**: $Y_i = y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j)$ $i \in \{1, ..., s\}$ $F_i = f(Y_i)$

1. The RK method is the form: $y_{n+1} = y_n + h\sum_{i=1}^{s} b_i f(t_n + c_i, Y_i)$ for some values of b_i, a_{ij}, s, c_i for Autonomous: $c_i = \sum_{j=1}^{s} a_{ij}$
2. Flow-map: $\Psi_h(y) = y + h\sum_{i=1}^{s} b_i f(t_n + c_i, Y_i(y,h))$ ps:*weights*: b_i; *internal coefficients*: a_{ij}
3. We can using **Butcher Table** to represent the RK method (Appendix) **Explicit**: $a_{ij} = 0$ for $j \ge i$ (严格下三角行) **Implicit**: $\exists a_{ij} \ne 0$ for $j \ge i$ (Not Explicit)

4.3 Accuracy of RK Method | Order Condition

Some Notations: If $\mathbf{y} = f'(\mathbf{y})$ where $f(\mathbf{y}) : \mathbb{R}^d \to \mathbb{R}^d$. Def $f' = (\frac{\partial f_i}{\partial y_j})_{1 \le i \le d, 1 \le j \le d}$ (行向量) $f'' = (\frac{\partial^2 f_i}{\partial y_j \partial y_k})_{1 \le i \le d, 1 \le j, k \le d}$

· Def: $f''(\mathbf{a}, \mathbf{b}) = \sum_{j=1}^{d} \sum_{k=1}^{d} \frac{\partial^2 f_i}{\partial y_j \partial y_k} a_j b_k$ | $y' = f$ $y'' = \sum_{j=1}^{d} \frac{\partial f_i}{\partial y_j} f_j = f'f$ $y''' = \sum_{j=1}^{d} \sum_{k=1}^{d} \frac{\partial^2 f_i}{\partial y_j \partial y_k} y_j'(t) y_k'(t) + \sum_{j=1}^{d} \frac{\partial f_i}{\partial y_j} y_j''(t) = f''(f, f) + f'f'f$

· $\Phi_h(y) = y + hf + \frac{h^2}{2} f'f + \frac{h^3}{6}[f''(f, f) + f'f'f] + \mathcal{O}(h^4)$

Order Condition: RK method: $y_{n+1} = y_n + h \sum_{i=1}^{s} b_i f(Y_i)$, Let $z(h) = \Phi_h(y)$ $\Rightarrow$ If $z'(0) = y', z''(0) = y'', ..., z^{(n)} = y^{(n)}$ $\Rightarrow$ **Convergent at order n**

· Order 1: $\sum_{i=1}^{s} b_i = 1$ Order 2: (add) $\sum_{i=1}^{s} b_i c_i = \frac{1}{2}$ Order 3: (add) $\sum_{i=1}^{s} b_i c_i^2 = \frac{1}{3}$ and $\sum_{i=1}^{s} \sum_{j=1}^{s} b_i a_{ij} c_j = \frac{1}{6}$

5 Appendix

5.1 Useful Series | Common RK Methods

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ $\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ $(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ $\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$

$\frac{1}{1+x} = 1 - x + x^2 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$ $\ln(x) = (x-1) - \frac{(x-1)^2}{2} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$, $x > 0$

Common Runge-Kutta Methods (Butcher Table):

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}$$

Example

$$\begin{array}{c|c} 0 & \\ \hline & 1 \end{array}$$

RK1 (Euler's Method)

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array}$$

RK2 (Heun's Method)

$$\begin{array}{c|ccc} 0 & & & \\ 1/2 & 1/2 & & \\ 1 & -1 & 2 & \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

RK3

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

RK4 (Classical/Famous)