

## 1 Basic Knowledge

**Def of Group**  $(G, *)$ : A set  $G$  with a operator  $*$  is a group if: **Closure**:  $\forall g, h \in G, g * h \in G$ ; **Associativity**:  $\forall g, h, k \in G, (g * h) * k = g * (h * k)$ ; **Identity**:  $\exists e \in G, \forall g \in G, e * g = g * e = g$ ; **Inverse**:  $\forall g \in G, \exists g^{-1} \in G, g * g^{-1} = g^{-1} * g = e$ .  $G, H$  groups, then  $G \times H$  also.

**Subgroup**:  $H \subseteq G$  is a subgroup if:  $\forall h_1, h_2 \in H$  **I**:  $H \neq \emptyset$ ; **II**:  $h_1 * h_2 \in H$ ; **III**:  $h_1^{-1} \in H$ .

**Field**  $(F)$ : A set  $F$  is a field with two operators: (addition)  $+: F \times F \rightarrow F; (\lambda, \mu) \rightarrow \lambda + \mu$  (multiplication)  $\cdot: F \times F \rightarrow F; (\lambda, \mu) \rightarrow \lambda \mu$  if:  
 $(F, +)$  and  $(F \setminus \{0_F\}, \cdot)$  are abelian groups with identity  $0_F, 1_F$ . and  $\lambda(\mu + \nu) = \lambda\mu + \lambda\nu$  e.g.  $Fields: \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$

**F-Vector Space**  $(V)$ : A set  $V$  over a field  $F$  is a vector space if:  $V$  is an abelian group  $V = (V, +)$  and  $\forall \vec{v}, \vec{w} \in V, \lambda, \mu \in F$  e.g.  $Poly: \mathbb{R}[x]_{<n}$   
 $\exists \text{ map } F \times V \rightarrow V: (\lambda, \vec{v}) \rightarrow \lambda\vec{v}$  satisfies: **I**:  $\lambda(\vec{v} + \vec{w}) = (\lambda\vec{v}) + (\lambda\vec{w})$  **II**:  $(\lambda + \mu)\vec{v} = (\lambda\vec{v}) + (\mu\vec{v})$  **III**:  $\lambda(\mu\vec{v}) = (\lambda\mu)\vec{v}$  **IV**:  $1_F\vec{v} = \vec{v}$

**Vector Subspaces Criterion**:  $U \subseteq V$  is a subspace of  $V$  if: **I**.  $\vec{0} \in U$  **II**.  $\forall \vec{u}, \vec{v} \in U, \forall \lambda \in F: \vec{u} + \vec{v} \in U$  and  $\lambda\vec{u} \in U$  (or:  $\lambda\vec{u} + \mu\vec{v} \in U$ )  
**property**: If  $U, W$  are subspaces of  $V$ , then  $U \cap W$  and  $U + W$  are also subspaces of  $V$ .  $ps: U + W := \{\vec{u} + \vec{w} : \vec{u} \in U, \vec{w} \in W\}$

**Complement-wise Operations**:  $\phi: V_1 \times V_2 \rightarrow V_1 \oplus V_2$  by  $I: (\vec{v}_1, \vec{u}_1) + (\vec{v}_2, \vec{u}_2) := (\vec{v}_1 + \vec{v}_2, \vec{u}_1 + \vec{u}_2), \lambda(\vec{v}, \vec{u}) := (\lambda\vec{v}, \lambda\vec{u})$  (ps:  $V_1, V_2$  通过  $\phi$  定义的 map 所形成的 vector space 记作  $V_1 \oplus V_2$ )

**Projections**:  $pr_i: X_1 \times \dots \times X_n \rightarrow X_i$  by  $(x_1, \dots, x_n) \mapsto x_i$  **Canonical Injections**:  $in_i: X_i \rightarrow X_1 \times \dots \times X_n$  by  $x \mapsto (0, \dots, 0, x, 0, \dots, 0)$

## 2 Vector Spaces/Subspaces | Generating Set | Linear Independent | Basis

**Generating (subspaces)**  $\langle T \rangle$ :  $\langle T \rangle := \{\alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n : \alpha_i \in F, \vec{v}_i \in T, r \in \mathbb{N}\}$   $\langle \emptyset \rangle := \{\vec{0}\}$  If  $T$  is subspace  $\Rightarrow \langle T \rangle = T$ .

- Proposition**:  $\langle T \rangle$  is the smallest subspace containing  $T$ . (i.e.  $\langle T \rangle$  is the intersection of all subspaces containing  $T$ )
- Generating Set**:  $V$  is vector space,  $T \subseteq V$ .  $T$  is generating set of  $V$  if  $\langle T \rangle = V$ . **Finitely Generated**:  $\exists$  finite set  $T, \langle T \rangle = V$
- External Direct Sum**: 一个“代数结构”, 定义为 set 是  $V_1 \oplus \dots \oplus V_n := V_1 \times \dots \times V_n$  且有一组运算法则 component-wise operations
- Connect to Matrix**: Let  $E = \{\vec{v}_1, \dots, \vec{v}_n\}$ ,  $E$  is GS of  $V$ . Let  $A = [\vec{v}_1, \dots, \vec{v}_n] \Rightarrow \forall \vec{b} \in V, \exists \vec{x} = (x_1, \dots, x_n)^T$  s.t.  $A\vec{x} = \vec{b}$  (i.e. linear map:  $\phi: \vec{x} \mapsto A\vec{x}$  is surjective)

**Linearly Independent**:  $L = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$  is linearly independent if:  $\forall c_1, \dots, c_r \in F, c_1 \vec{v}_1 + \dots + c_r \vec{v}_r = \vec{0} \Rightarrow c_1 = \dots = c_r = 0$ .

**Connect to Matrix**: Let  $L = \{\vec{v}_1, \dots, \vec{v}_n\}$ ,  $L$  is LI of  $V$ . Let  $A = [\vec{v}_1, \dots, \vec{v}_n] \Rightarrow \forall \vec{x} \in F^n, A\vec{x} = 0$  (or  $\vec{0}$ )  $\Rightarrow \vec{x} = 0$  (or  $\vec{0}$ ) (i.e. linear map  $\phi: \vec{x} \mapsto A\vec{x}$  is injective)

**Basis & Dimension**: If  $V$  is finitely generated.  $\Rightarrow \exists$  subset  $B \subseteq V$  which is both LI and GS. ( $B$  is basis) **Dim**:  $\dim V := |B|$

**Connect to Matrix**: Let  $B = \{\vec{v}_1, \dots, \vec{v}_n\}$  is basis of  $V$ . Let  $A = [\vec{v}_1, \dots, \vec{v}_n] \Rightarrow \forall \vec{x} = (x_1, \dots, x_n)^T$  s.t.  $\phi: \vec{x} \mapsto A\vec{x}$  is 1-1 & onto (Bijection)

**Relation|GS,LI,Basis,dim**: Let  $V$  be vector space.  $L$  is linearly independent set,  $E$  is generating set,  $B$  is basis set.

- GS|LI**:  $|L| \leq |E|$  (can get: dim unique) **LI  $\rightarrow$  Basis**: If  $V$  finite generate  $\Rightarrow \forall L$  can extend to a basis. If  $L = \emptyset$ , prove  $\exists B$   $kerf \cap imf = \{0\}$
- Basis|max,min**:  $B \Leftrightarrow B$  is minimal GS ( $E \Leftrightarrow B$  is maximal LI ( $L$ )). **Uniqueness|Basis**: 每个元素都可以由 basis 唯一表示.
- Proper Subspaces**: If  $U \subset V$  is proper subspace, then  $\dim U < \dim V$ .  $\Rightarrow$  If  $U \subseteq V$  is subspace and  $\dim U = \dim V$ , then  $U = V$ .
- Dimension Theorem**: If  $U, W \subseteq V$  are subspaces of  $V$ , then  $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$

**Complementary**:  $U, W \subseteq V, U, W$  subspaces are complementary ( $V = U \oplus W$ ) if:  $\exists \phi: U \times W \rightarrow V$  by  $(\vec{u}, \vec{w}) \mapsto \vec{u} + \vec{w}$   
i.e.  $\forall \vec{v} \in V$ , we have unique  $\vec{u} \in U, \vec{w} \in W$  s.t.  $\vec{v} = \vec{u} + \vec{w}$ . ps: It's a linear map.

## 3 Linear Mapping | Rank-Nullity| Matrices | Change of Basis

ps: 默认  $V, W$   $F$ -Vector Spaces.

**Linear Mapping/Homomorphism(Hom)**:  $f: V \rightarrow W$  is linear map if:  $\forall \vec{v}_1, \vec{v}_2 \in V, \forall \lambda \in F. f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$  and  $f(\lambda \vec{v}_1) = \lambda f(\vec{v}_1)$

**Isomorphism**: = LM & Bij. **Endomorphism(End)**: = LM &  $V = W$ . **Automorphism(Aut)**: = LM &  $V = W$  **Monomorphism**: = LM & 1-1. **Epimorphism**: = LM & onto.

**Kernel**:  $\ker f := \{\vec{v} \in V : f(\vec{v}) = \vec{0}\}$  (It's subspace) **Image**:  $Imf := \{f(\vec{v}) : \vec{v} \in V\}$  (It's subspace) **Rank**:  $\dim(Imf)$  **Nullity**:  $\dim(\ker f)$  **Fixed Point**  $X^f: X^f := \{x \in X : f(x) = x\}$

**Property of Linear Map**: Let  $f, g \in Hom$ : **a**.  $f(\vec{0}) = \vec{0}$  **b**.  $f$  is 1-1 iff  $\ker f = \{\vec{0}\}$  **c**.  $f \circ g$  is linear map.

- Determined**:  $f$  is determined by  $f(\vec{b}_i), \vec{b}_i \in \mathcal{B}_{basis}$  (\* i.e.  $f(\sum_i \lambda_i \vec{v}_i) := \sum_i \lambda_i f(\vec{v}_i)$ )
- Classification of Vector Spaces**:  $\dim V = n \Leftrightarrow f: F^n \xrightarrow{\sim} V$  by  $f(\lambda_1, \dots, \lambda_n) \mapsto \sum_{i=1}^n \lambda_i \vec{v}_i$  is isomorphism.
- Left/Right Inverse**:  $f$  is 1-1  $\Rightarrow \exists$  left inverse  $g$  s.t.  $g \circ f = id$  考虑 direct sum  $f$  is onto  $\Rightarrow \exists$  right inverse  $g$  s.t.  $f \circ g = id$
- More of Left/Right Inverse**:  $f \circ g = id \Rightarrow g$  is 1-1 and  $f$  is onto. 使用 kernel=0 来证明

**Rank-Nullity Theorem**: For linear map  $f: V \rightarrow W, \dim V = \dim(\ker f) + \dim(Imf)$  Following are properties:

- Injection**:  $f$  is 1-1  $\Rightarrow \dim V \leq \dim W$  **Surjection**:  $f$  is onto  $\Rightarrow \dim V \geq \dim W$  Moreover,  $\dim W = \dim imf$  iff  $f$  is onto.
- Same Dimension**:  $f$  is isomorphism  $\Rightarrow \dim V = \dim W$  **Matrix**:  $\forall M$ , column rank  $c(M) = \text{row rank } r(M)$ .
- Relation**: If  $V, W$  finite generate, and  $\dim V = \dim W$ , Then:  $f$  is isomorphism  $\Leftrightarrow f$  is 1-1  $\Leftrightarrow f$  is onto.

**Matrix**: For  $A_{n \times m}, B_{m \times p}, AB_{n \times p} := (AB)_{ij} = \sum_{k=1}^m a_{ik} b_{kj}$  **Transpose**:  $A_{m \times n}^T := (A^T)_{ij} = a_{ji}$

**Invertible Matrices**:  $A$  is invertible if  $\exists B, C$  s.t.  $BA = I$  and  $AC = I$  ||  $\exists B, BA = I \Rightarrow \exists C, AC = I \Leftrightarrow \exists A^{-1}$   $_{\mathcal{B}}[f^{-1}]_{\mathcal{A}} =_{\mathcal{A}}[f]_{\mathcal{B}}^{-1}$

**Representing matrix of linear map**  $_{\mathcal{B}}[f]_{\mathcal{A}}: f: V \rightarrow W$  be linear map,  $\mathcal{A} = \{\vec{v}_1, \dots, \vec{v}_n\}$  is basis of  $V, \mathcal{B} = \{\vec{w}_1, \dots, \vec{w}_m\}$  is basis of  $W$ .

- $_{\mathcal{B}}[f]_{\mathcal{A}} := A$  (matrix) where  $f(\vec{v}_{i \in \mathcal{A}}) = \sum_{j \in \mathcal{B}} A_{ji} \vec{w}_j$   $\exists M_{\mathcal{B}}^{\mathcal{A}}: Hom_F(V, W) \xrightarrow{\sim} Mat(n \times m; F)$
- If  $\vec{v} \in V$ , then  $_{\mathcal{A}}[\vec{v}] := \mathbf{b}$  (vector) where  $\vec{v} = \sum_{i \in \mathcal{A}} \mathbf{b}_i \vec{v}_i$
- Theorems**:  $[f \circ g] = [f] \circ [g]$   $_{\mathcal{C}}[f \circ g]_{\mathcal{A}} =_{\mathcal{C}}[f]_{\mathcal{B}} \circ_{\mathcal{B}}[g]_{\mathcal{A}}$   $_{\mathcal{B}}[f(\vec{v})] =_{\mathcal{B}}[f]_{\mathcal{A}} \circ_{\mathcal{A}}[\vec{v}]$   $_{\mathcal{A}}[f]_{\mathcal{A}} = I \Leftrightarrow f = id$
- Change of Basis**: Define Change of Basis Matrix:  $_{\mathcal{A}}[id_V]_{\mathcal{B}} =_{\mathcal{B}'}[f]_{\mathcal{A}'} =_{\mathcal{B}'}[id_W]_{\mathcal{B}} \circ_{\mathcal{B}}[f]_{\mathcal{A}} \circ_{\mathcal{A}}[id_V]_{\mathcal{A}'}$   $_{\mathcal{A}'}[f]_{\mathcal{A}'} =_{\mathcal{A}}[id_V]_{\mathcal{A}'}^{-1} \circ_{\mathcal{A}}[f]_{\mathcal{A}} \circ_{\mathcal{A}}[id_V]_{\mathcal{A}'}$

**Elementary Matrix**:  $I + \lambda E_{ij}$  (cannot  $I - E_{ii}$ ) 就是初等矩阵, 左乘代表  $j$  行乘  $\lambda$  倍加到第  $i$  行, 右乘代表  $j$  列乘  $\lambda$  倍加到第  $i$  列  $\Rightarrow$  Invertible!

- 交换  $i, j$  列/行:  $P_{ij} = diag(1, \dots, 1, -1, 1, \dots, 1)(I + E_{ij})(I - E_{ji})(I + E_{ij})$  where  $-1$  in  $j$ th place.
- Row Echelon Form|Smith Normal Form**:  $\tilde{A}: REF$  通过左乘初等矩阵可以实现  $\tilde{A}: S(n, m, r)$  通过  $\tilde{A}$  右乘初等矩阵可以实现

**Smith Normal Form:**  $\forall A, \exists$  invertible  $P, Q$  s.t.  $PAQ = S(n, m, r) := n \times m$  的矩阵, 对角线前  $r$  个是 1, 后面 0. **Lemma:**  $r = r(A) = c(A)$   
 · Every linear map  $f : V \rightarrow W$  can be representing by  ${}_B[f]_A = S(n, m, r)$  for some basis  $\mathcal{A}, \mathcal{B}$  of  $V, W$ .

**Similar Matrices:**  $N = T^{-1}MT \Leftrightarrow M, N$  are similar. *Special Case:* If  $N = {}_B[f]_B, M = {}_A[f]_A$ , then  $N = T^{-1}MT$ . where  $T = {}_A[id_V]_B$

1. If  $A \sim B$  iff  $A$  is similar to  $B$ , then  $\sim$  is an equivalence relation.  ${}_{\mathcal{A}'}[f]_{\mathcal{A}'} \sim_{\mathcal{A}} [f]_{\mathcal{A}}$

2. If  $\mathcal{B} = \{p(\vec{v}_1), \dots, p(\vec{v}_n)\}$  and  $\mathcal{A} = \{\vec{v}_1, \dots, \vec{v}_n\}$  where  $p : V \xrightarrow{\sim} V$ . Then  ${}_{\mathcal{A}}[id_V]_{\mathcal{B}} = {}_{\mathcal{A}}[p]_{\mathcal{A}}$

3. If  $V$  is a vector space over  $F$ ,  $[A, B]$  are similar matrices.  $\Leftrightarrow A = {}_{\mathcal{A}}[f]_{\mathcal{A}}, B = {}_{\mathcal{B}}[f]_{\mathcal{B}}$  for some basis  $\mathcal{A}, \mathcal{B}; f : V \rightarrow V$

4. Set of *Endomorphism* is in a bijection correspondence with the equivalence class of matrices under  $\sim$ . 一个自同态 **End** 就对应一个相似矩阵的等价类

**Trace:**  $tr(A) := \sum_i a_{ii}$  and  $tr(f) := tr({}_{\mathcal{A}}[f]_{\mathcal{A}}) \mid tr(AB) = tr(BA) \quad tr(\lambda A + \mu B) = \lambda tr(A) + \mu tr(B) \quad tr(N) = tr(M)$  if  $M, N$  similar.

## 4 Rings | Polynomials | Ideals | Subrings

### 4.1 Rings | Polynomial Rings

**Ring**  $(R, +, \cdot)$ : A set  $R$  with two operators  $+, \cdot$  is a ring if:

e.g.  $Mat(n, F); R[X]; \mathbb{Z}/m\mathbb{Z}; \mathbb{Z}$

1.  $(R, +)$  is an *abelian group* with identity  $0_R$ .

**Commutative Ring:** add:  $\forall a, b \in R, ab = ba$ .

2.  $(R, \cdot)$  is a **monoid** with identity  $1_R$ . i.e. **Associativity:**  $\forall a, b, c \in R, (a \cdot b) \cdot c = a \cdot (b \cdot c)$ . **Identity:**  $\forall a \in R, 1_R \cdot a = a \cdot 1_R = a$ .

3. **Distributive:**  $\forall a, b, c \in R: a \cdot (b + c) = a \cdot b + a \cdot c$  and  $(b + c) \cdot a = b \cdot a + c \cdot a$ .

ps: 默认 monoid 是 closure 的

· If  $R$  is ring  $\Rightarrow [1_R = 0_R \Leftrightarrow R = \{0\}]$  i.e. For any non-zero ring,  $1_R \neq 0_R$

**Field:** Commutative ring + multiplicative inverse = Field.

**Properties of Ring:**  $\forall a, b \in R$ .

**I.**  $0 \cdot a = a \cdot 0 = 0$

**II.**  $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$

**III.**  $(-a) \cdot (-b) = a \cdot b$

**Unit:**  $a \in R$  is unit if it's *Invertible*. i.e.  $\exists a^{-1} \in R$  s.t.  $aa^{-1} = a^{-1}a = 1_R$

**Group of Unit**  $(R^\times, \cdot) := \{a \in R : a \text{ is unit}\}$

**Zero-divisors:**  $a \in R$  is *zero-divisor* if  $\exists b \in R, b \neq 0$  s.t.  $ab = 0$  or  $ba = 0$

*Field has no zero-divisors.*

e.g.  $\mathbb{Z}^\times = \{-1, 1\}$ ;  $1_R$  is a unit.

**Integral Domain:** A *commutative ring*  $R$  is an integral domain if it has no zero-divisors.

e.g.  $\mathbb{Z}/p\mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}, \dots$

**Properties of Integral Domain:**  $\forall a, b \in R$ .

**I.**  $ab = 0 \Rightarrow a = 0$  or  $b = 0$

**II.**  $a, b \neq 0 \Rightarrow ab \neq 0$

**III.**  $ac = bc, a \neq 0 \Rightarrow b = c$

· *Field is Integral Domain*

**Every finite integral domain is a field**

$\mathbb{Z}/p\mathbb{Z}$  is field iff  $p$  is prime.

e.g. (integral domain)  $\mathbb{Z}; \mathbb{Z}/p\mathbb{Z}$

**Polynomial Ring**  $R[X]: R[X] := \{a_n X^n + \dots + a_1 X + a_0 : a_i \in R, n \in \mathbb{N}\}$  where  $X$  is **indeterminate**  $\Leftarrow X \notin R$  and  $\forall x \in R, Xa = aX$

1. **Degree:**  $\deg(P) := \max\{n \in \mathbb{N} : a_n \neq 0\}$

**Leading Coefficient:**  $a_n$

**Monic:**  $a_n = 1$

ps: Polynomial NOT a function

2. **Lemma:**  ${}^1 R$  integral domain/no zero-divisors  $\Rightarrow R[X]$  also.

${}^2 R$  integral domain or no zero-divisor  $\Rightarrow \deg(PQ) = \deg(P) + \deg(Q)$

3. **Division and Remainder:** If  $R$  is *integral domain* and  $P, Q \in R[X], Q$  **monic**  $\exists! A, B \in R[X]$  s.t.  $P = AQ + B$  and  $\deg(B) < \deg(Q)$

4. **Function | Factorize:** If  $R$  is *commutative ring*  $\Rightarrow {}^1 R[X] \rightarrow \text{Maps}(R, R)$  (可以视作函数)  ${}^2 \lambda \in R$  is root of  $P \Leftrightarrow (X - \lambda) \mid P(X)$

5. **Roots:** If  $R$  is *Integral domain*:  $P$  has at most  $\deg(P)$  roots.

**Algebraically Closed:**  $R = F$  field is *algebraically closed* if every non-constant polynomial has a root in  $F$ .

e.g.  $\mathbb{C}$

· **Decomposes:** If  $F$  field is *algebraically closed*  $\Rightarrow P$  decomposes into:  $P(X) = a(X - \lambda_1) \cdots (X - \lambda_n), a \in F^\times$  i.e.  $a \neq 0$

### 4.2 Homomorphism | Ideals | Subrings

**Ideal**  $(I \trianglelefteq R)$ : A subset  $I \subseteq R$  (ring) is an ideal if: **I.**  $I \neq \emptyset$  **II.**  $\forall a, b \in I, a - b \in I$  **III.**  $\forall i \in I, \forall r \in R, ri, ir \in I$

1. **Generated Ideal**  ${}_R\langle T \rangle: T \subseteq R$  (ring), where  $R$  is *commutative ring*. We define  ${}_R\langle T \rangle := \{\sum_{i=1}^n r_i t_i : n \in \mathbb{N}, r_i \in R, t_i \in T\}$

· **Lemma:**  ${}_R\langle T \rangle$  is the smallest ideal containing  $T$ .

**Principal Ideal:**  ${}_R\langle a \rangle$  **Proper Ideal:**  $I \neq R$

ps:  $R$  一定是 commutative ring

2. If  $I, J$  are *ideals* of  $R$ . Then  $I + J; I \cap J$  are also ideals.

**Subring Test:**  $R' \subseteq R$  (ring) is a subring if: **I.**  $1_R \in R'$  **II.**  $\forall a, b \in R', a - b \in R'$  **III.**  $\forall a, b \in R', ab \in R'$

· If  $f : R \rightarrow S$  is *ring homomorphism*, and  $R'$  is subring of  $R$ .  $\Rightarrow f(R')$  is subring of  $S$ .

**Ring Homomorphism:**  $R, S$  are rings,  $f : R \rightarrow S$  is *ring homomorphism* if: **I.**  $f(a + b) = f(a) + f(b)$  **II.**  $f(ab) = f(a)f(b)$   $f(1_R) = 1_S$  is NOT need

1. **Second Def:**  $f$  is ring homomorphism if:  $f : (R, +) \rightarrow (S, +)$  is group homomorphism and  $f(xy) = f(x)f(y)$ .

2. **I.**  $f(0_R) = 0_S$  **II.**  $f(-a) = -f(a)$  **III.**  $f(a^n) = (f(a))^n$  **IV.**  $f(x - y) = f(x) - f(y)$  **V.**  $f(mx) = mf(x)$

3. **Kernel:**  $\ker f := \{a \in R : f(a) = 0_S\}$  is an *ideal* **Image:**  $\text{Im} f := \{f(a) : a \in R\}$  is a *subring*. **1-1:**  $f$  is 1-1  $\Leftrightarrow \ker f = \{0_R\}$

## 5 Inner Product Spaces | Orthogonal Complement / Proj | Adjoints and Self-Adjoint

## 6 Jordan Normal Form | Spectral Theorem