HCV Note

1 Basic Knowledge

Useful Complex Number Properties: $|Re(z)|, |Im(z)| \le |z|$ $Re(z) = \frac{z+\bar{z}}{2}, Im(z) = \frac{z-\bar{z}}{2i}, |z|^2 = z\bar{z}$

Triangle (Reverse) Inequality: $|z_1 + z_2| \le |z_1| + |z_2|$ $|z_1| - |z_2| \le |z_1 - z_2|$ $(Re(zw) = 0 \Leftrightarrow \overline{zw} = -zw; Im(zw) = 0 \Leftrightarrow zw = \overline{zw})$

Argument: $\arg(z) := \{\theta : z = |z|e^{i\theta}\} = \{Arg(z) + 2\pi k : k \in \mathbb{Z}\}$ **Principle Value of Argument**: $Arg(z) \in (-\pi, \pi]$

· **Operations on Argument**: $arg(z_1 z_2) = arg(z_1) + arg(z_2)$ $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$ $arg(\bar{z}) = -arg(z)$

2 Holomorphic Functions

2.1 Open/Closed Set | Limit Point | limit of Sequence | Continuous of Function

Open/Closed/Punctured ε-disc: $D_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ $\overline{D}_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| \le \varepsilon\}$ $D'_\varepsilon(z_0) := \{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}$

Open/Closed Set in $\mathbb{C}$: $U \subset \mathbb{C}$ is **open** if $\forall z_0 \in U, \exists \varepsilon > 0, D_\varepsilon(z_0) \subseteq U$ U is **closed** if $\mathbb{C} \setminus U$ is open **Lemma**: $D_\varepsilon, D'_\varepsilon$ open, $\overline{D}_\varepsilon$ closed.

Limit Point of S: $z_0 \in \mathbb{C}$ is a limit point of S if: $\forall \varepsilon > 0, D'_\varepsilon(z_0) \cap S \ne \emptyset$ 聚点 **Bounded**: S is bounded if $\exists M > 0$ s.t. $|z| \le M, \forall z \in S$

Closed of Set S: $\overline{S}$:= 所有 S 的 limit point 和 S 的点. **Property**: Let $S \subseteq \mathbb{C}$, then S is closed $\Leftrightarrow$ $S = \overline{S}$.

Limit of sequence: Sequence $(z_n)_{n\in\mathbb{N}}$ has limit z if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \ge N \Rightarrow |z_n - z| < \varepsilon$. limit rules 依旧成立

1. **Lemma|Important**: $\lim z_n = z \Leftrightarrow \lim Re(z_n) = Re(z)$ and $\lim Im(z_n) = Im(z)$
2. **Cauchy**: Sequence $(z_n)_{n\in\mathbb{N}}$ is cauchy if: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall m, n \ge N \Rightarrow |z_m - z_n| < \varepsilon$ **Lemma**: Cauchy $\Leftrightarrow$ convergent.
3. **Lemma|Closed of Set**: $S \subseteq \mathbb{C}, z \in \mathbb{C}. \Rightarrow [\ z \in \overline{S} \Leftrightarrow \exists$ sequence $(z_n)_{n\in\mathbb{N}} \in S$ s.t. $\lim z_n = z\ ]$
4. **Bolzano-Weierstrass**: Every bounded sequence in $\mathbb{C}$ has a convergent subsequence.

Complex Functions: $\forall f : \mathbb{C} \to \mathbb{C}$ we can write it as: $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$

Limit of Function: $a_0 \in \mathbb{C}$ is the limit of f at z_0 if: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |z - z_0| < \delta \Rightarrow |f(z) - a_0| < \varepsilon$ limit rules 依旧成立

· **Lemma|Important**: $\lim_{z\to z_0} f(z) \Leftrightarrow \lim_{(x,y)\to(x_0,y_0)} u(x, y) = Re(a_0)$ and $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = Im(a_0)$

· **Useful Formula**: $\lim_{z\to z_0} g(\bar{z}) = \lim_{z\to\overline{z_0}} g(z)$

continuous of Function: f is continuous at z_0 if: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$ continuous rules 依旧成立

1. **Lemma|Important**: f is continuous at $z_0 \Leftrightarrow u, v$ are continuous at (x_0, y_0)
2. **'Extreme Value Theorem'**: f is continuous on a closed and bounded set $S \subseteq \mathbb{C}$, then $f(S)$ is closed and bounded.
3. **Lemma|continuous⇔open**: f is continuous $\Leftrightarrow \forall$ open set U, preimage $f^{-1}(U) := \{z \in \mathbb{C} | f(z) \in U\}$ is open.

2.2 Differentiable | Holomorphic Function | C-R Equation

Differentiable: Let $z_0 \in \mathbb{C}$ and $U \subseteq \mathbb{C}$ be neighborhood of z_0, then $f : U \to \mathbb{C}$ is differentiable at z_0 if: $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists.

· **I.** f is differentiable $\Rightarrow f$ is continuous. **II. Holomorphic** $\Leftrightarrow$ Differentiable + neighborhood (除非是一个点时不成立,|z|) diff rules + chain rule 成立

Cauchy-Riemann Equations: If $z_0 = x_0 + iy_0, f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 \Rightarrow u_x = v_y, v_x = -u_y$ **at** (x_0, y_0).

· If $z_0 = x_0 + iy_0, f = u + iv$ satisfies: [1] u, v are *continuously differentiable* on a neighborhood of (x_0, y_0) and:
[2] u, v satisfies Cauchy-Riemann Equations at (x_0, y_0). $\Rightarrow$ f is differentiable at z_0.

· ps: 常见可导复数函数: $\exp(z), \sin z, \cos z, \log z, z^\alpha$, polynomial, sinh, cosh, $\Gamma(z), |z|^2$ (at 0), constant ps: 常见不可导复数函数: $\bar{z}, |z| \cdot \bar{z}, Re(z), Im(z), \mathrm{Arg}(z)$

Harmonic Function: $h : \mathbb{R}^2 \to \mathbb{R}$ is harmonic if: $\forall (x, y) \in \mathbb{R}^2$ $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$ (**Laplace Equation**)

· **Lemma**: If $f = u + iv$ is holomorphic on $\mathbb{C}$ and u, v are twice *continuously differentiable*, $\Rightarrow$ u, v are harmonic.

Harmonic Conjugate: Let $u, v : U \to \mathbb{R}, U \subseteq \mathbb{R}^2$ be harmonic functions. u, v are harmonic conjugate if: $f = u + iv$ is holomorphic on U.

Properties of Polynomial: The domain of rational function and polynomial are always open. **Lemma**: If $P(z_0) = 0$ then $P(\overline{z_0}) = 0$

First-order Operator ∂: $\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ $\bar{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ || $f = u + iv$ satisfies C-R Equations $\Leftrightarrow \bar{\partial} f = 0$

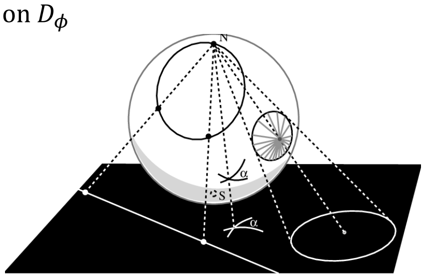
sin/cos Functions: $\sin z := \frac{e^{iz}-e^{-iz}}{2i}$ $\cos z := \frac{e^{iz}+e^{-iz}}{2}$ **Exponential Function**: $\exp(z) = e^x(\cos(y) + i\sin(y))$

1. $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$ $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$
2. $\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$ $\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$
3. $\sin^2 z + \cos^2 z = 1$ $\sin(z + \frac{\pi}{2}) = \cos(z)$ $\sin(z + 2k\pi) = sin(z)$ $\cos(z + 2k\pi) = \cos(z)$ ⋆$\sin z, \cos z$ NOT bounded.

Hyperbolic Functions: $\sinh z := \frac{\exp(z)-\exp(-z)}{2}$ $\cosh z := \frac{\exp(z)+\exp(-z)}{2}$ || $\sinh(iz) = i\sin z$ $\cosh(iz) = \cos z$

Logarithm: Define *multivalued function*: $\log z := \{w \in \mathbb{C} : \exp w = z\}$ **Principal Branch**: $Log(z) := \ln|z| + iArg(z)$

1. **I.** $\log(z) = \ln|z| + i\arg z = \{\ln|z| + iArg(z) + i2\pi k : k \in \mathbb{Z}\}$ **II.** $\log(zw) = \log(z) + \log(w)$ **III.** $\log(1/z) = -\log(z)$
2. **Branch of Logarithm**: $Log_\phi(z) := \ln|z| + iArg_\phi(z)$ $Log_\phi(z)$ is holomorphic on D_ϕ
3. If $g : U \to \mathbb{C}$, then $Log_\phi(g(z))$ is holomorphic on $g^{-1}(D_\phi) \cap U$
4. $Log(z)$ *not continuous* on $\mathbb{C}$. $Log(z)$ not continuous on $Re(z) \le 0, Im(z) = 0$.
   Remark: $\log(x) + \log(x) \ne 2\log(x)$



Branch Cut|Cut Plane: *Branch Cut* $L_{z_0,\phi} := \{z \in \mathbb{C} : z = z_0 + re^{i\phi}, r \ge 0\}$

· *Cut Plane*: $D_{z_0,\phi} := \mathbb{C} \setminus L_{z_0,\phi}$ $L_\phi = L_{0,\phi}; D_\phi = D_{0,\phi}$

· If $Log_\phi(z)$ is holomorphic on D_ϕ, then $Log_\phi(z - a)$ is holomorphic on $D_{a,\phi}$

Branch of Argument: $Arg_\phi(z) := z$ 的辐角, 但是角度限制在: $\phi < Arg_\phi(z) \le \phi + 2\pi$. ps: $Arg_{-\pi}(z) = Arg(z)$

$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$
z^n	nz^{n-1}	$\exp(z)$	$\exp(z)$	$\sin(z)$	$\cos(z)$	$\cos(z)$	$-\sin(z)$	$\sinh(z)$	$\cosh(z)$	$\cosh(z)$	$\sinh(z)$	$Log_\phi z$	$\frac{1}{z}\ {}_{z\in D_\phi}$

Complex Powers: $z^\alpha := \{\exp(\alpha w) : w \in \log(z)\} = \{\exp[\alpha(\ln|z| + iArg(z) + i2k\pi)] : k \in \mathbb{Z}\}$ $\frac{d}{dz}z^\alpha = \alpha z^{\alpha-1}\ {}_{z\in D_\phi}$

I. If $\alpha \in \mathbb{Z}$, there is one value of z^α **II.** If $\alpha = \frac{p}{q}$, $\gcd(p,q) = 1, p, q \in \mathbb{Z}, q \neq 0$, there are exactly q values of z^α

III. If α is *irrational* or *non-real*, there are infinitely values z^α **IV.** $1^{1/q}, q \in \mathbb{Z}, q \neq 0$ is $\{1, w, ..., w^{q-1}\}, w = \exp(i2\pi/q)$

V. We prefer use $\exp(z)$ to denote single-valued function, and e^z to denote multi-valued function.

Principal Branch: $z^\alpha := \exp(\alpha Log(z))$ **Operation**: $z^\alpha z^\beta = z^{\alpha+\beta}$ (Using Principal Branch) **NB**: $(z_1z_2)^\alpha \neq z_1^\alpha z_2^\alpha$; $(z^\alpha)^\beta \neq z^{\alpha\beta}$

3 Conformal Maps and Mobius Transformations

3.1 Conformal Maps & Definition of Mobius Transformations

Conformal: Let U be open set and $f : U \to \mathbb{C}$. Then f is conformal iff: f *preserves angles*. i.e. 任意两条曲线/直线之间的角度在 f 作用下不变.

Important Theorem: If $f : U \to \mathbb{C}$ is *holomorphic*, then $\forall z_0 \in U, f'(z_0) \neq 0$, f *preserves angles*.

i.e. $\forall$ curves C_1, C_2 in U. If C_1, C_2 intersecting at a point $z_0 \in U$. C_1, C_2 在 z_0 切线的夹角与 $f(C_1), f(C_2)$ 在 $f(z_0)$ 切线的夹角一样.

Extended Complex Plane: $\widetilde{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and define that $a + \infty = \infty, b \cdot \infty = \infty, \frac{b}{0} = \infty, \frac{b}{\infty} = 0$.

Riemann Sphere: Consider $(X,Y,Z) \in \mathbb{R}^3$: ${}^1 z = X + iY \in \mathbb{C}$ is the point $(X,Y,0)$ and ${}^2 Z = 0$ is the complex plane.

1. Define the Riemann Sphere: $S^2 := \{(X,Y,Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1\}$ and consider the **North Pole** is point $N := (0,0,1)$
2. Define $\phi : \mathbb{C} \to S^2$ by N 点与 $z = (X,Y,0)$ 点连线与 S^2 的交点为 $\phi(z)$ Thus $\lim_{|z|\to\infty} \phi(z) = N$ $\phi(\infty) := N$
3. Calculation shows that: $\phi(z) = \phi(x+iy) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$ $\psi(X,Y,Z) = \begin{cases} \frac{X+iY}{1-Z}, (X,Y,Z) \neq N \\ \infty \quad , (X,Y,Z) = N \end{cases}$

   Remark: $\phi : \widetilde{\mathbb{C}} \to S^2$ is bijection and it's inverse $\psi : S^2 \to \widetilde{\mathbb{C}}$ is the **stereographic projection**
4. Stereographic projection $\psi(X,Y,Z)$ maps a circle to either a circle or a straight line. (见上图)

Mobius Transformation: A Mobius Transformation is a function form: $f(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$; $ad \neq bc$

1. **Remark**: $g(z) = \frac{f(z)}{\sqrt{ad-bc}}$ satisfies $ad - bc = 1$ | If a, b, c, d defined a mobius transformation, then $\lambda a, \lambda b, \lambda c, \lambda d$ also.
2. For Complex Matrix: $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(M) = ad - bc = 1$. We define $f_M = \frac{az+b}{cz+d}$ **I.** $f_{M_1M_2} = f_{M_1}f_{M_2}$ **II.** $f_{M^{-1}} = f_M^{-1}$
3. Extended $f(z)$ from $\mathbb{C}$ to $\widetilde{\mathbb{C}}$ by: $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$
4. **Translation**: $f(z) = z + b \Leftrightarrow \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ **Rotation**: $f(z) = az, a = e^{i\theta}$ $(|a| = 1) \Leftrightarrow \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & -e^{i\theta/2} \end{pmatrix}$ **Dilation**: $f(z) = rz, r > 0 \Leftrightarrow \begin{pmatrix} \sqrt{r} & 0 \\ 0 & 1/\sqrt{r} \end{pmatrix}$

   Inversion: $f(z) = 1/z \Leftrightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ f **fixes the point at infinity**: If $f(\infty) = \infty$ ps: 除了 inversion 其他都是 fix the point at infinity.
5. **Theorem**: $f(z) = \frac{az+b}{cz+d}$ be a Mobius Transformation. $\Rightarrow$ ^{1}If $f(\infty) = \infty$: f is a composition of finite *Translation, Rotation, Dilation* $\Rightarrow c = 0, f(z) = \frac{a}{d}z + \frac{b}{d}$

   ^{2} If $f(\infty) < \infty$: f is composition of finite *Translation, Rotation, Dilation* and only one *inversion*. $\Rightarrow f(z) = \frac{(bc-ad)/c^2}{z+d/c} + \frac{a}{c}$

Properties of Mobius Transformation: *Important*: ⋆ Möbius transformations map circlines to circlines. ⋆

1. For mobius transformation $f(z) = \frac{az+b}{cz+d}$, if: $\exists z_1, z_2, z_3 \in \mathbb{C}$ distinct points. $f(z_1) = z_1, f(z_2) = z_2, f(z_3) = z_3 \Rightarrow f$ is identity.
2. If $z_1, z_2, z_3 \in \widetilde{\mathbb{C}}$ distinct points. $\exists!$ mobius transformation $f(z)$ s.t. $f(z_1) = 1, f(z_2) = 0, f(z_3) = \infty$
3. If $(z_1, z_2, z_3), (w_1, w_2, w_3) \in \widetilde{\mathbb{C}}$ distinct points. Then $\exists!$ mobius transformation $f(z)$ s.t. $f(z_i) = w_i$, $\forall i \in \{1,2,3\}$

   ps:Method to construct 2: If $z_i < \infty, f(z) = \frac{z_1-z_3}{z_1-z_2} \cdot \frac{z-z_2}{z-z_3}$ If $z_i = \infty$, $f(z) = \frac{z-z_2}{z-z_3}$ $,z_1 = \infty$ $f(z) = \frac{z_1-z_3}{z-z_3}$ $,z_2 = \infty$; $f(z) = \frac{z-z_2}{z_1-z_2}$ $,z_3 = \infty$

   ps:Method to construct 3: For 3: Let $f := h^{-1} \circ g$ where $g(z_i), h(w_i) = \{1, 0, \infty\}$ like part 2.

Geometric Meaning by using Mobius Transformation|Exponential|Complex Powers:

1. **Rotation**: $f(z) = e^{-i\theta}z$ is a rotation by θ (anticlockwise) about the origin. Specially, $f(z) = iz$ is a rotation by $\frac{\pi}{2}$
2. **Extend**: $f(z) = \exp(\alpha z)$ 原来的图像进行拉长, 以及旋转 (如果带 θ 带 i 时) e.g. $\{z : 0 < Im(z) < 1\}$ 可以被拉长到 $\{z : 0 < Im(z)\}$
3. **Angle Extend**: $f(z) = z^\alpha$ 原来的图像辐角范围收缩或放大
4. **Circlines**: **I.** 单位圆到实轴, $f(z) = \frac{z-i}{z+i}$ **II.** 实轴到单位圆, $f(z) = i\frac{1+z}{1-z}$

   III. 单位圆到虚轴, $f(z) = \frac{z-1}{z+1}$ **IV.** 虚轴到单位圆, $f(z) = \frac{1+iz}{1-iz}$

Cross-Ratio: cross-ratio $[z_1, z_2, z_3, z_4] := f(z_1)$ where f is mobius transformation s.t. $f(z_2) = 1, f(z_3) = 0, f(z_4) = \infty$

1. **Formulas**: $[z_1,z_2,z_3,z_4] = \frac{z_1-z_3}{z_1-z_4}\frac{z_2-z_4}{z_2-z_3}$ $[\infty,z_2,z_3,z_4] = \frac{z_2-z_4}{z_2-z_3}$ $[z_1,\infty,z_3,z_4] = \frac{z_1-z_3}{z_1-z_4}$ $[z_1,z_2,\infty,z_4] = \frac{z_2-z_4}{z_1-z_4}$ $[z_1,z_2,z_3,\infty] = \frac{z_1-z_3}{z_2-z_3}$
2. **Theorem**: If f is a mobius transformation, $[f(z_1), f(z_2), f(z_3), f(z_4)] = [z_1, z_2, z_3, z_4]$ **z_i's in this "small section" are distinct.**