

# 1 Definition of LP and General LP LPMS Note

## 1.1 Definition of LP

**Linear Programming (LP):** Let  $x_i$ : **Decision Variables**  $a_{ij}, b_i, c_i$ : **parameters**  $f$ : **Objective Function** **Constraints:** subject 后的限制条件

LP problem can be written as:

$$\begin{aligned} \text{maximize} \quad & f = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\ & x_1 \geq 0, x_2 \geq 0, \cdots, x_n \geq 0 \end{aligned}$$

We can also write the LP problem in matrix form:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{aligned} \text{maximize} \quad & f = \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \quad \mathbf{x} \geq 0 \end{aligned}$$

At this time, we have the following definitions:

1. **Feasible Solution:** 可行解, 满足所有约束条件的解. (i.e.  $A\mathbf{x} \leq \mathbf{b}$ )
2. **Optimal Solution:** 最优解 (可多个)
3. **Feasible Region:** 可行域, 所有可行解的集合.

**Find Optimal Solution: Graphical Method or Simplex Method**

## 1.2 General LP Problem

**Slack Variables:** 对每个不等式约束, 各引入 (Slack Variables)  $x_i$  ( $i > n$ ) 来将其转化为等式约束. (i.e.  $A\mathbf{x} \leq \mathbf{b} \Rightarrow \bar{A}\mathbf{x} = \mathbf{b}$ )

LP problem can be written as: ps:  $x_i \geq 0$  ( $i > n$ ).

$$\begin{aligned} \text{maximize} \quad & f = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{n+2} = b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + x_{n+m} = b_m \\ & x_1 \geq 0, x_2 \geq 0, \cdots, x_{n+m} \geq 0 \end{aligned}$$

We can also write the LP problem in matrix form:

$$\bar{A} = [A \quad I_m], \mathbf{b} = \mathbf{b}, \bar{\mathbf{c}} = \begin{pmatrix} \mathbf{c} \\ \mathbf{0}_{m \times 1} \end{pmatrix}, \mathbf{x} = \begin{pmatrix} \mathbf{x} \\ x_{n+m} \end{pmatrix}$$

$$\begin{aligned} \text{maximize} \quad & f = \bar{\mathbf{c}}^T \mathbf{x} \\ \text{subject to} \quad & \bar{A}\mathbf{x} = \mathbf{b} \quad \mathbf{x} \geq 0 \end{aligned}$$

At this time, we have the following definitions:

1. **Basic Solution:** Solution of  $\bar{A}\mathbf{x} = \mathbf{b}$ .  $\rightarrow$  得到:  $\mathbf{x}_B, \mathbf{x}_N = 0$  (要求) (不一定  $\mathbf{x} \geq 0$ ,  $\mathbf{x}_B$  不一定  $\geq 0$ )  
ps: 只有把某一组  $m$  个线性无关列选为基, 令其余变量固定为 0 后求得的那一个唯一向量才叫 Basic Solution
2. **Basic Feasible Solution (BFS):** If a Basic Solution  $\mathbf{x}_B, \mathbf{x}_N = 0$  have:  $\mathbf{x}_B \geq 0$ , then  $\mathbf{x}$  is a Basic Feasible Solution.
3. **Basic Variables:**  $\mathbf{x}_B$  **Nonbasic Variables:**  $\mathbf{x}_N$  **Basic Matrix:**  $B$  **Nonbasic Matrix:**  $N$  **Basic Index:**  $\mathcal{B}$  **Nonbasic Index:**  $\mathcal{N}$   
ps: 均为人为规定  $\mathcal{B}$  和  $\mathcal{N}$  后得到.

For the LP problem, we consider the following:

1. **Convex Set:** Set  $K$  is convex if, for any two points  $\mathbf{x}, \mathbf{x}' \in K$ , the line segment  $\mathbf{x}_\theta = \theta\mathbf{x} + (1 - \theta)\mathbf{x}'$  for  $\theta \in [0, 1]$  is also in  $K$ .
2. **Vertex of a Convex Set:** A vertex of a convex set  $K$  is a point  $\mathbf{x} \in K$  such that it does not lie strictly within any line segment joining two points in  $K$ . (ps: there are  $\frac{(m+n)!}{m!n!}$  vertices for LP problem)
3. **Theorem I:** If LP has a unique optimal solution, then it is a vertex. (optimal  $\leftrightarrow$  vertex)
4. **Theorem II:** If LP has a non-unique solution,  $\exists$  optimal solution at vertex. (optimal  $\leftrightarrow$  vertex)

**Remark:** BFS and Vertices are NOT 1-1 (与它是否是唯一 optimal solution 无关!)

★ Happened if one of *basic variables* of *optimal solution* is zero.  $\rightarrow$  Referred as **degenerate**.

## 2 Simplex Method

### 2.1 Some Definitions

For LP problem:  $\max f = \mathbf{c}^T \mathbf{x}$  s.t.  $A\mathbf{x} \leq \mathbf{b} \quad \mathbf{x} \geq 0$

By adding slack variables, we can write it as:  $\max f = \bar{\mathbf{c}}^T \mathbf{x}$  s.t.  $\bar{A}\mathbf{x} = \mathbf{b} \quad \mathbf{x} \geq 0$

1. **We can define Index:**  $\mathcal{B}, \mathcal{N}$
2. **Basic Matrix:**  $B$  (整个  $\bar{A}$  中在 index  $\mathcal{B}$  中的列) **Nonbasic Matrix:**  $N$  (整个  $\bar{A}$  中在 index  $\mathcal{N}$  中的列)  $\bar{A} \rightarrow [B \quad N]$
3. **Basic Variables:**  $\mathbf{x}_B$  (对应  $B$  列因有的变量) **Nonbasic Variables:**  $\mathbf{x}_N$  (对应  $N$  列因有的变量)  $\mathbf{x} \rightarrow \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix}$
4. **Matrix Cost:**  $\mathbf{c}_B$  ( $f$  中对应  $B$  变量的系数) **Nonbasic Cost:**  $\mathbf{c}_N$  ( $f$  中对应  $N$  变量的系数)  $\bar{\mathbf{c}} \rightarrow \begin{pmatrix} \mathbf{c}_B \\ \mathbf{c}_N \end{pmatrix}$
5. **Other:**  $\bar{A}\mathbf{x} = \mathbf{b} \Leftrightarrow B\mathbf{x}_B + N\mathbf{x}_N = \mathbf{b}$  || Let  $\hat{\mathbf{b}} = B^{-1}\mathbf{b}$   $\hat{N} = B^{-1}N$  || Thus  $\mathbf{x}_B = \hat{\mathbf{b}} - \hat{N}\mathbf{x}_N$
6. **Objective Value:**  $f = \bar{\mathbf{c}}^T \mathbf{x} = \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N = \hat{f} + \hat{\mathbf{c}}_N^T \mathbf{x}_N$  where  $\hat{f} = \mathbf{c}_B^T \hat{\mathbf{b}}$   $\hat{\mathbf{c}}_N = \mathbf{c}_N - \hat{N}^T \mathbf{c}_B = \mathbf{c}_N - N^T B^{-T} \mathbf{c}_B$

★ **Remark:** For *Basic Solution*, we let  $\mathbf{x}_B = \hat{\mathbf{b}}$  and  $\mathbf{x}_N = 0$ . **Remark:** If basic Solution have  $\hat{\mathbf{b}} \geq 0$ , it is a **Basic Feasible Solution (BFS)**.

★ **Theorem:** If BFS have:  $\hat{\mathbf{c}}_N \leq 0$ , it is an **Optimal Solution**

## 2.2 Simplex Algorithm

**An Initial BFS:** Try  $B = \{n+1, \dots, n+m\}$  and  $\mathcal{N} = \{1, \dots, n\} \Rightarrow B = I, N = A, \mathbf{x}_B = \hat{\mathbf{b}} = \mathbf{b}, \mathbf{x}_N = \mathbf{0}, \mathbf{c}_B = \mathbf{0}, \mathbf{c}_N = \mathbf{c}$

**Corollary:** If  $\hat{\mathbf{b}} = \mathbf{b} \geq 0$ , it's a BFS.

**Simplex Algorithm:**

1. If **An Initial BFS** is BFS  $\rightarrow 2$  || If we have an BFS  $\rightarrow 2$  || Else: Go to **Phase I Problem**
2. If  $\hat{\mathbf{c}}_N \leq 0 \Rightarrow$  Optimal Solution || Else: Go to 3
3. Let  $q \in (\text{local index in } N)$  ;  $q' \in \mathcal{N}$  (global index) ||  $\hat{c}_i := \hat{\mathbf{c}}_N$  中第  $i$  个分量 ( $\hat{c}_i := \hat{\mathbf{c}}_N^T \mathbf{e}_i$ ) || such that:  
 $\hat{c}_q = \hat{\mathbf{c}}_N$  中最大的正的分量 ( $c_q = \arg \max_{i=1, \dots, n; c_i > 0} \hat{c}_i$ ) ;  $q'$  is the index corresponding to  $q$  in  $\mathcal{N}$  (对应的在  $\mathcal{N}$  的 index(global index))  
 Thus, we find  $q, q'$  Go to 4
4. Let  $p \in (\text{local index in } B)$  ;  $p' \in \mathcal{B}$  (global index) ||  $\mathbf{a}_q := N$  中的第  $q$  列 ( $\mathbf{a}_q = N\mathbf{e}_q$ ) || Let  $\hat{\mathbf{a}}_q = B^{-1}\mathbf{a}_q$ :  
 If  $\hat{\mathbf{a}}_q \leq 0 \Rightarrow$  LP is unbounded. || Else: LP is bounded  $\rightarrow 5$
5. Let  $p = \arg \min_{i=1, \dots, m; \hat{a}_{iq} > 0} \frac{\hat{b}_i}{\hat{a}_{iq}}$   $p$  是对应的 local index, not value, \* 用  $\bar{\alpha} = \frac{\hat{b}_p}{\hat{a}_{pq}}$  代表值  $\hat{b}_i$  代表  $\hat{\mathbf{b}}$  的第  $i$  个分量;  $\hat{a}_{iq}$  代表  $\hat{\mathbf{a}}_q$  的第  $i$  个分量  
 Let  $p'$  be (global) index corresponding to  $p$ .  
 Thus, we find  $p, p'$  Go to 6
6. Exchange  $p'$  and  $q'$  between  $\mathcal{B}$  and  $\mathcal{N} \Rightarrow$  Update  $\mathcal{B}, \mathcal{N}, B, N, \hat{\mathbf{b}}, \dots$  ps: New  $B := B + (\alpha_q - B e_p) e_p^T$
7. Calculate  $\hat{\mathbf{c}}_N$ . Go to 2

**Remark:** If we consider using  $\mathbf{x} + \alpha \mathbf{d}$  where  $\mathbf{d} \rightarrow [\mathbf{d}_B \ \mathbf{d}_N]^T$ ;  $\mathbf{d}_N = \mathbf{e}_q \ \mathbf{d}_B = -\hat{\mathbf{a}}_q$ , then:

We have:  $\begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{b}} \\ \mathbf{0} \end{pmatrix} + \alpha \begin{pmatrix} -\hat{\mathbf{a}}_q \\ \mathbf{e}_q \end{pmatrix}$  and  $f = \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N = \mathbf{c}_B^T (\hat{\mathbf{b}} - \alpha \hat{\mathbf{a}}_q) + \mathbf{c}_N^T (\alpha \mathbf{e}_q) = \hat{f} - \alpha (\mathbf{c}_N^T \mathbf{e}_q - \mathbf{c}_B^T B^{-1} N \mathbf{e}_q) = \hat{f} + \alpha \hat{\mathbf{c}}_N \mathbf{e}_q = \hat{f} + \alpha \hat{c}_q$

**Phase I Problem:** (If  $\mathbf{b} \not\geq 0$  & No BFS)

1. Subtract **artificial variables**  $x_{n+m+1}, \dots, x_{n+m+m} \geq 0$  and change objective function  $f$  to:  

$$\begin{array}{ccccccc} \max & f = & & & -x_{n+m+1} & \dots & -x_{n+2m} \\ \text{s. t.} & a_{11}x_1 & + & \dots & + & a_{1n}x_n & + x_{n+1} & -x_{n+m+1} & \dots & -x_{n+2m} & = & b_1 \\ & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ & a_{m1}x_1 & + & \dots & + & a_{mn}x_n & + x_{n+m} & & & -x_{n+2m} & = & b_m \\ & x_1 \geq 0, & \dots, & & & & & & & x_{n+2m} \geq 0 \end{array}$$
2. Let  $b_i := \mathbf{b}$  的第  $i$  个分量. (i.e.  $b_i = \mathbf{b}^T \mathbf{e}_i$ )  
 Case I:  $b_i \geq 0$ :  $x_{n+i} = b_i$  is a Basic Variables. and  $x_{n+m+i} = 0$  is a Nonbasic Variables.  
 Case II:  $b_i < 0$ :  $x_{n+m+i} = -b_i$  is a Basic Variables. and  $x_{n+i} = 0$  is a Nonbasic Variables.  
 Other: We put  $x_1, \dots, x_n$  as Nonbasic Variables.  
 Thus we can find  $\mathcal{B}, \mathcal{N}, B, N, \mathbf{x}_B = \hat{\mathbf{b}} = B^{-1}\mathbf{b} = |\mathbf{b}|, \mathbf{x}_N = \mathbf{0}, \mathbf{c}_B, \mathbf{c}_N$
3. Let **Size of Infeasibility**: is  $\sum_{i=1}^m x_{n+m+i} \quad f = -\sum_{i=1}^m x_{n+m+i} \leq 0$
4. Consider this modified LP problem, by using Simplex Algorithm, we can find an optimal solution  $\mathbf{x}$  and the value of  $f$ .  
 ps: 使用刚刚我们选好的  $\mathcal{B}, \mathcal{N}$  作为初始的 BFS.
5. Case I: If  $f = 0$  (等价于 **artificial variables** 都是 **Nonbasic Variables**),  $\mathbf{x}$  is a BFS for the original LP problem. (去除 artificial variables, 并回到原问题 (也叫 Phase II Problem))  
 Case II: If  $f < 0$ : LP is infeasible (等价于存在 **artificial variables** 是 **Basic Variables**).

## 3 Degeneracy | Termination | Cycling | Sparsity

### 3.1 Degeneracy & Termination

**Degeneracy:** If  $\hat{\mathbf{b}}$  has any zero component, then  $\mathbf{x}$  is a **degenerate vertex**. (等价于  $\hat{\mathbf{b}} > 0$ )

1. If a point is *degenerate*, then there may be multiple *BFS* at that point.

2. \* If  $\hat{b}_p = 0$ , then  $\bar{\alpha} = 0$ , simplex algorithm may not terminate.

ps: 对于 simplex algorithm 我们依靠  $\hat{\mathbf{a}}_q$  中大于 0 的元素来决定下一个要交换的变量, 但是如果  $\hat{b}_p = 0$ , 那么  $\bar{\alpha} = 0$ , 可能会导致我们在这个点上循环

**Termination of Simplex Algorithm in the absence of degeneracy:** If no degenerate situation, then Simplex Algorithm will terminate in a finite number of steps.

**Examples:** 1. The worst case of Simplex Algorithm. 2. Cyclic case of Simplex Algorithm.

1. **Klee-Minty Problem:**  $\max f = \sum_{j=1}^n 10^{j-1} x_j$  ; s.t.  $x_i + 2 \sum_{j=i+1}^n 10^{j-i} x_j \leq 100^{n-i}$  for  $i = 1, \dots, n$  ;  $x_i \geq 0$   
 has  $2^n$  vertices. **Worst Case:**  $2^n - 1$  iterations.

2. **Hall-McKinnon Problem:**  $\max f = x_1 - 5.5x_2 + 0.75x_3 - 5.75x_4$  ; s.t.  $2.5x_1 - 19.5x_2 - 3.5x_3 + 19.5x_4 + x_5 = 0; 0.5x_1 - 3.5x_2 - 0.5x_3 + 3.5x_4 + x_6 = 0; x_i \geq 0$

## 3.2 Sparsity LP Problem

**Implementation:** 计算/编程中的计算化简

$$\text{Consider: } \hat{\mathbf{b}} = B^{-1}\mathbf{b} \quad \hat{\mathbf{c}}_N = \mathbf{c}_N - N^T B^{-T} \mathbf{c}_B \quad \hat{\mathbf{a}}_q = B^{-1} \mathbf{a}_q$$

1. Solve:  $B\hat{\mathbf{b}} = \mathbf{b}$  to get:  $\hat{\mathbf{b}} = B^{-1}\mathbf{b}$
2. Solve:  $B^T \pi = \mathbf{c}_B$  to get:  $\pi = B^{-T} \mathbf{c}_B$
3. Solve:  $\hat{\mathbf{c}}_N = \mathbf{c}_N - N^T \pi$  to get:  $\hat{\mathbf{c}}_N = \mathbf{c}_N - N^T B^{-T} \mathbf{c}_B$
4. Solve:  $B^T \hat{\mathbf{a}}_q = \mathbf{a}_q$  to get:  $\hat{\mathbf{a}}_q = B^{-T} \mathbf{a}_q$
5. Matrix  $B$  is **sparse** and **invertible**. And in Simplex Algorithm, the next matrix  $B$  obtained by  $B := B + (a_q - B e_p) e_p^T$

**Sparse LP Problem:** For LP problem:  $\max f = \mathbf{c}^T \mathbf{x}$  s.t.  $A\mathbf{x} = \mathbf{b}$   $\mathbf{x} \geq 0$

It is **sparse** if the matrix  $A$  is sparse. (i.e. Most of the elements of  $A$  are zero)

**Find Inverse of  $B$ :** Decomposition  $B$  into  $LU$  where  $L$  is lower triangular and  $U$  is upper triangular.

Then we can solve  $B\mathbf{x} = \mathbf{b}$  by solving  $^1 L\mathbf{y} = \mathbf{b}$  and  $^2 U\mathbf{x} = \mathbf{y}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} & l_{21}u_{14} + u_{24} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} & l_{31}u_{14} + l_{32}u_{24} + u_{34} \\ l_{41}u_{11} & l_{41}u_{12} + l_{42}u_{22} & l_{41}u_{13} + l_{42}u_{23} + l_{43}u_{33} & l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + u_{44} \end{bmatrix}$$

By Solving  $\blacksquare \rightarrow \blacksquare \rightarrow \blacksquare \rightarrow \blacksquare \rightarrow \blacksquare \rightarrow \blacksquare \rightarrow \blacksquare$

## 4 Sensitive Analysis

### 4.1 RHS Sensitivity Consider $b_i \rightarrow b_i + \delta$

**RHS Sensitivity:**  $\max f = \mathbf{c}^T \mathbf{x}$  s.t.  $A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0 \Rightarrow \max f = \bar{\mathbf{c}}^T \mathbf{x}$  s.t.  $\bar{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$

ps: Assume  $\mathcal{B}, \mathcal{N}, B, N$  yield an optimal solution, with  $\mathbf{x}_B = \hat{\mathbf{b}}, \mathbf{x}_N = \mathbf{0}$

Then, the new optimal values is:  $\mathbf{x}_B = B^{-1}(\mathbf{b} + \delta \mathbf{e}_i) = \hat{\mathbf{b}} + \delta B^{-1} \mathbf{e}_i$  if  $\mathbf{x}_B \geq 0$

With range of  $\delta \in [\underline{\delta}, \bar{\delta}]$   $\underline{\delta} = \max_{[B^{-1}]_{ij} > 0} -\frac{\hat{b}_j}{[B^{-1}]_{ij}}$   $\bar{\delta} = \min_{[B^{-1}]_{ij} < 0} -\frac{\hat{b}_j}{[B^{-1}]_{ij}}$

**Fair Prices:** The objective function:  $f = \mathbf{c}_B^T \mathbf{x}_B = \mathbf{c}_B^T (\hat{\mathbf{b}} + \delta B^{-1} \mathbf{e}_i) = \hat{f} + \delta \pi_i$  where  $\hat{f} = \mathbf{c}_B^T \hat{\mathbf{b}}$   $\pi_i = \mathbf{c}_B^T B^{-1} \mathbf{e}_i$  (fair price)

1. If **price of unit amount**  $> \pi_i$ : Buying more of the resource is *unattractive*.
2. If **price of unit amount**  $< \pi_i$ : Buying more of the resource is *attractive*.
3. For **Basic Slack**: If  $n + i \in \mathcal{B}$  (slack variable  $x_{n+i}$  是 basic slack)  $\rightarrow \pi_i = 0$  (Proof:  $\pi_i = (B^{-T} \mathbf{c}_B)^T \mathbf{e}_i = \mathbf{c}_B^T B^{-1} \mathbf{e}_i = 0$ )
4. For **Nonbasic Slack**: If  $n + i \in \mathcal{N}$  (slack variable  $x_{n+i}$  是 nonbasic slack)  $\rightarrow \pi_i = -\hat{c}_i$

(Proof: Assume  $n + i \in \mathcal{N}$  in  $N$  中的 (local) index 为  $j$ . Since  $\hat{\mathbf{c}}_j = [\mathbf{c}_N]_j - [N]^T \pi, [\mathbf{c}_N]_j = 0, [N]^T_j = \mathbf{e}_i$ ; we have:  $\pi_i = -\hat{c}_i$ )

★ Range of  $\delta$  is *lower/upper bound Sensitivity*

★ *Feasible Region* increases when  $\bar{\delta}$  increases and  $\underline{\delta}$  decreases.

### 4.2 Cost and Coefficient Sensitivity Consider $c_i \rightarrow c_i + \delta$ & $a_{ij} \rightarrow a_{ij} + \delta$

**Cost Sensitivity:** 即  $c_i \rightarrow c_i + \delta$ .  $\max f = \mathbf{c}^T \mathbf{x}$  subject to  $A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0$

1. 小范围的变化不会改变最优解. (If an LP cost coefficient is changed by a small amount, the optimal solution will not change.)
2. 如果  $c_i \rightarrow c_i + \delta$  对应的  $i$  变量在  $\mathcal{B}$  中 (basic variables), 那么 all reduced costs will change.
3. 如果  $c_i \rightarrow c_i + \delta$  对应的  $i$  变量在  $\mathcal{N}$  中 (nonbasic variables), 那么 only the reduced cost of *that* variables will change.

**Coefficient Sensitivity:** 即  $a_{ij} \rightarrow a_{ij} + \delta$ .  $\max f = \mathbf{c}^T \mathbf{x}$  subject to  $A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0$

1. 如果影响的变量在  $\mathcal{B}$  中 (basic variables):  $B$  may become singular, the optimal solution may change, etc.
2. 如果影响的变量在  $\mathcal{N}$  中 (nonbasic variables): One *reduced cost* will change,  $N$  will change.

## 5 Duality

**Duality:** For LP problem:  $\max f = \mathbf{c}^T \mathbf{x}$  s.t.  $A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0$  **Primal problem** (P)

The **dual** problem is:  $\min f = \mathbf{b}^T \mathbf{y}$  s.t.  $A^T \mathbf{y} \geq \mathbf{c}, \mathbf{y} \geq 0$  **Dual problem** (D)

- It is *equivalence* to:  $\max f = -\mathbf{b}^T \mathbf{y}$  s.t.  $-A^T \mathbf{y} \leq -\mathbf{c}, \mathbf{y} \geq 0$

The **dual** of (D) is:  $\min f = -\mathbf{c}^T \mathbf{z}$  s.t.  $(-A^T)^T \mathbf{z} \geq (-\mathbf{b}),$  (i.e.  $-A\mathbf{z} \geq -\mathbf{b}), \mathbf{z} \geq 0$

- It is *equivalence* to:  $\max f = \mathbf{c}^T \mathbf{z}$  s.t.  $A\mathbf{z} \leq \mathbf{b}, \mathbf{z} \geq 0$ , which is the **Primal** problem (P).

**Weak Duality Theorem:** If  $\mathbf{x}$  is feasible for (P) and  $\mathbf{y}$  is feasible for (D), then  $\mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y}$ .

**Corollary:** If (P) is unbounded  $\Rightarrow$  (D) is infeasible. || If (D) is unbounded  $\Rightarrow$  (P) is infeasible.

**Strong Duality Theorem:** If  $\mathbf{x}^*$  is optimal (basic) solution for (P), then:

$\mathbf{y}^* = \pi = B^{-T} \mathbf{c}_B$  is optimal solution for (D), and  $\mathbf{c}^T \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^*$ .

**Application:** If LP problem:  $\max f = \mathbf{c}^T \mathbf{x} \quad \text{s.t. } A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq 0 \quad (P)$

is changed to a **tightening** LP problem:  $\max f = \mathbf{c}^T \mathbf{x} \quad \text{s.t. } \begin{bmatrix} A \\ \mathbf{a}^T \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} \mathbf{b} \\ b \end{bmatrix}, \quad \mathbf{x} \geq 0 \quad (P')$

Then, the **relaxation dual** problem is:  $\min f = \mathbf{b}^T \mathbf{y} + by \quad \text{s.t. } A^T \mathbf{y} + \mathbf{a}y \geq \mathbf{c}, \quad \mathbf{y} \geq 0, \quad y \geq 0 \quad (D')$

- It is *equivalence* to:  $\max f = -\mathbf{b}^T \mathbf{y} - by \quad \text{s.t. } -A^T \mathbf{y} - \mathbf{a}y \leq -\mathbf{c}, \quad \mathbf{y} \geq 0, \quad y \geq 0$

此时, 计算  $(P')$  的最优解可以通过计算  $(D')$  的最优解来得到. (通过 **dual simplex method** 计算或 equivalence to 一般的计算)

## 6 Example of format written in LP

**Example| 构建 LP 问题模板:**

**Defining decision variables:**

Let  $x_1$  be the number of 1kg packets of Breakfast Blend made each day.

Let  $x_2$  be the number of 1kg packets of Dinner Blend made each day.

**Total income is:**  $f_I = 1.16x_1 + 1.42x_2$ .

**Total cost is:**  $f_C = 0.18x_1 + 0.36x_2 + 0.28x_1 + 0.16x_2 = 0.46x_1 + 0.52x_2$ .

**Thus, total profit is:**  $f_I - f_C = 1.16x_1 + 1.42x_2 - 0.46x_1 - 0.52x_2 = 0.7x_1 + 0.9x_2$ .

**The objective is to maximize total profit**  $f = 0.7x_1 + 0.9x_2$

**Constraints:**

1. The number of kilos of arabica used,  $0.3x_1 + 0.6x_2$ , must not exceed the supply of 1200kg.
2. The number of kilos of robusta used,  $0.7x_1 + 0.4x_2$ , must not exceed the supply of 1500kg.
3. The total number of kilos of coffee made,  $x_1 + x_2$ , must not exceed the capacity of 2400kg.

**Thus, the LP Problem to be solved is:**

$$\begin{aligned} &\text{maximize} && f = 0.7x_1 + 0.9x_2, \\ &\text{subject to} && 0.3x_1 + 0.6x_2 \leq 1200, \\ &&& 0.7x_1 + 0.4x_2 \leq 1500, \\ &&& x_1 + x_2 \leq 2400, \\ &&& x_1, x_2 \geq 0. \end{aligned}$$

**Introducing Slack Variables:**

$$\begin{aligned} &\text{maximize} && f = 0.7x_1 + 0.9x_2, \\ &\text{subject to} && 0.3x_1 + 0.6x_2 + x_3 = 1200, \\ &&& 0.7x_1 + 0.4x_2 + x_4 = 1500, \\ &&& x_1 + x_2 + x_5 = 2400, \\ &&& x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

**Example|Iteration in Simplex Method|No Phase I,II:**

The RHS  $\mathbf{b} = \begin{bmatrix} 1200 \\ 1 \end{bmatrix}$  is positive, so the “all-slack” basis  $\mathcal{B} = \{3, 4\}$  and  $\mathcal{N} = \{1, 2\}$  yields a basic feasible solution.

**Iteration 1:**

- For  $\mathcal{B} = \{3, 4\}$  and  $\mathcal{N} = \{1, 2\}$ ,  $B = I$ ,  $N = \begin{bmatrix} 1 & 20 \\ 0 & 1 \end{bmatrix}$ ,  $\mathbf{c}_B = \mathbf{0}$  and  $\mathbf{c}_N = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$ , giving  $\hat{\mathbf{b}} = B^{-1}\mathbf{b} = \begin{bmatrix} 1200 \\ 1 \end{bmatrix}$ , and  $\hat{\mathbf{c}}_N = \mathbf{c}_N - N^T B^{-T} \mathbf{c}_B = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$ .
- Choose  $q = 2$  since  $\hat{c}_q = 10$ , and hence  $q' = 2$
- Thus  $\mathbf{a}_q = \begin{bmatrix} 20 \\ 0 \end{bmatrix}$ . Form  $\hat{\mathbf{a}}_q = B^{-1}\mathbf{a}_q = \begin{bmatrix} 20 \\ 1 \end{bmatrix}$ . Since  $\hat{\mathbf{a}}_q$  is positive, the LP is bounded.
- Choose  $p = \argmin \left\{ \frac{100}{(i \frac{20}{1})'} (i \frac{1}{1} 2) \right\} = 2$  and hence  $p' = 4$

**Iteration 2:** ... **Iteration 4:** 第一行一样, + ... Since  $\hat{\mathbf{c}}_N \leq 0$ , the optimal solution is  $\mathbf{x} = \dots$ , where values of  $x_i, \dots = 0$  since  $\mathcal{N} = \dots$

## 7 Additional Proofs

**If BFS has  $\hat{\mathbf{c}}_N \leq 0$ , it is an Optimal Solution | Show that  $f = \hat{f} + \hat{\mathbf{c}}_N^T \mathbf{x}_N$**

Proof: As  $\mathbf{x}_B = \hat{\mathbf{b}} - \hat{N}\mathbf{x}_N$ , we have  $f = \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N = \mathbf{c}_B^T (\hat{\mathbf{b}} - \hat{N}\mathbf{x}_N) + \mathbf{c}_N^T \mathbf{x}_N = \dots = \hat{f} + \hat{\mathbf{c}}_N^T \mathbf{x}_N$ . Since  $\hat{\mathbf{c}}_N \leq 0$ , we have  $f \leq \hat{f}$ , so it is an Optimal Solution.

**Why is the LP unbounded if  $\hat{\mathbf{a}}_q \leq 0$ ? | What is  $\hat{\mathbf{a}}_q$ ?**

Proof: At  $\mathbf{x}$ , the most positive reduced cost is  $\hat{c}_q$ . Now we construct a direction vector  $\mathbf{d}$ , such that  $\mathbf{d} \Leftrightarrow \begin{pmatrix} \mathbf{d}_B \\ \mathbf{d}_N \end{pmatrix}$  where  $\mathbf{d}_B = -B^{-1}N\mathbf{e}_q = -\hat{\mathbf{a}}_q$  and  $\mathbf{d}_N = \mathbf{e}_q$ . Then, we have  $\mathbf{x} + \alpha\mathbf{d}$  can be

partitioned as  $\begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{b}} \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -\hat{\mathbf{a}}_q \\ \mathbf{e}_q \end{pmatrix}$  We can see that the objective function is  $f = \hat{f} + \alpha\hat{c}_q$  increasing strictly if  $\hat{c}_q > 0$ . If  $\hat{\mathbf{a}}_q \leq 0$ , then  $-\hat{\mathbf{a}}_q \geq 0$ , so  $\mathbf{x} + \alpha\mathbf{d}$  is feasible for all  $\alpha \geq 0$ . Thus, the LP is unbounded.

**(Cont.) What is  $p = \arg \min_{i=1, \dots, m; \hat{a}_{iq} > 0} \frac{\hat{b}_i}{\hat{a}_{iq}}$ ?**

Proof: (Cont.) As  $\begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{b}} \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -\hat{\mathbf{a}}_q \\ \mathbf{e}_q \end{pmatrix}$ , we can see that, for the component of  $\hat{\mathbf{a}}_q$ , if  $\hat{a}_{iq} > 0$ , then  $\hat{b}_i - \alpha\hat{a}_{iq} \geq 0$ , objective function is strictly increasing; if  $\hat{a}_{iq} < 0$ , then  $\hat{b}_i - \alpha\hat{a}_{iq} \leq 0$ , objective function is strictly decreasing; if  $\hat{a}_{iq} = 0$ , then  $\hat{b}_i - \alpha\hat{a}_{iq} = \hat{b}_i$ , objective function is constant. Thus, we can see that, the most negative component of  $\hat{\mathbf{b}}$  is  $\hat{b}_p$  and the most positive component of  $\hat{\mathbf{a}}_q$  is  $\hat{a}_{pq}$ , so we have  $p = \arg \min_{i=1, \dots, m; \hat{a}_{iq} > 0} \frac{\hat{b}_i}{\hat{a}_{iq}}$ .

**Why in the Simplex Algorithm, we exchange  $p'$  and  $q'$  can get a new BFS?**

Proof: 1. It's clear that new index  $\mathcal{B}$  and  $\mathcal{N}$  form a partition of  $\{1, 2, \dots, n+m\}$ . 2. Only  $x_{q'}$  is increased from 0 and  $q'$  leaves  $\mathcal{N}$ , and  $x_{p'}$  is decreased to 0. So that they can be consider as new nonbasic variables and basic variables. 3. The new basis B is nonsingular (invertible): New Basic matrix is  $B_{New} = B + (\mathbf{a}_q - B\mathbf{e}_p)\mathbf{e}_p^T = BE$  where  $E = I + (\hat{\mathbf{a}}_q - B\mathbf{e}_p)\mathbf{e}_p^T$ ;  $B\hat{\mathbf{a}}_q = \mathbf{a}_q$  Thus  $B_{New}$  is nonsingular if and only if  $E$  is nonsingular. By Sherman-Morrison Formula,  $E$  is nonsingular iff  $\hat{a}_{pq}$  is nonzero. But we have  $\hat{a}_{pq} > 0$  so that  $B_{New}$  is nonsingular.

**Termination of Simplex Algorithm in the absence of degeneracy**

Proof: If no degenerate situation,  $\hat{\mathbf{b}} > 0$ . Thus  $\hat{\alpha} = \frac{\hat{b}_p}{\hat{a}_{pq}} > 0$ , and hence the objective function is strictly increasing. (By  $\bar{\alpha}\hat{c}_q > 0$ ) So the algorithm cannot return to a basic feasible solution previously visited. Thus the algorithm will terminate in a finite number of steps. (As it's bounded)