1 Definition of LP and General LP LPMS Note

1.1 Definition of LP

Linear Programming (LP): Let x_i: **Decision Variables** a_{ij}, b_i, c_i: **parameters** f: **Objective Function** **Constraints**: subject 后的限制条件

LP problem can be written as:

$$\begin{aligned} \text{maximize} \quad & f = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \\ & x_1 \ge 0, x_2 \ge 0, \cdots, x_n \ge 0 \end{aligned}$$

We can also write the LP problem in matrix form:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{aligned} \text{maximize} \quad & f = \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \le \mathbf{b} \quad \mathbf{x} \ge 0 \end{aligned}$$

At this time, we have the following definitions:

1. **Feasible Solution**: 可行解, 满足所有约束条件的解. (i.e. $A\mathbf{x} \le \mathbf{b}$)
2. **Optimal Solution**: 最优解 (可多个)
3. **Feasible Region**: 可行域, 所有可行解的集合.

Find Optimal Solution: **Graphical Method** or **Simplex Method**

1.2 General LP Problem

Slack Variables: 对每个不等式约束, 各引入 (Slack Variables) x_i $(i > n)$ 来将其转化为等式约束. (i.e. $A\mathbf{x} \le \mathbf{b} \Rightarrow \bar{A}\mathbf{x} = \mathbf{b}$)

LP problem can be written as: ps: $x_i \ge 0$ $(i > n)$.

$$\begin{aligned} \text{maximize} \quad & f = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{n+2} = b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + x_{n+m} = b_m \\ & x_1 \ge 0, x_2 \ge 0, \cdots, x_{n+m} \ge 0 \end{aligned}$$

We can also write the LP problem in matrix form:

$$\bar{A} = [\, A \;\; I_m \,], \mathbf{b} = \mathbf{b}, \bar{\mathbf{c}} = \begin{pmatrix} \mathbf{c} \\ \mathbf{0}_{m\times 1} \end{pmatrix}, \mathbf{x} = \begin{pmatrix} \mathbf{x} \\ \vdots \\ x_{n+m} \end{pmatrix}$$

$$\begin{aligned} \text{maximize} \quad & f = \bar{\mathbf{c}}^T\mathbf{x} \\ \text{subject to} \quad & \bar{A}\mathbf{x} = \mathbf{b} \quad \mathbf{x} \ge 0 \end{aligned}$$

At this time, we have the following definitions:

1. **Basic Solution**: Solution of $\bar{A}\mathbf{x} = \mathbf{b}$. → 得到: $\mathbf{x}_B$, $\mathbf{x}_\mathbf{N} = 0$(要求) (不一定 $\mathbf{x} \ge \mathbf{0}$, $\mathbf{x}_B$ 不一定 ≥ 0)
   ps: 只有把某一组 m 个线性无关列选为基，令其余变量固定为 0 后求得的那一个唯一向量才叫 Basic Solution
2. **Basic Feasible Solution (BFS)**: If a Basic Solution $\mathbf{x}_B, \mathbf{x}_\mathbf{N} = \mathbf{0}$ have: $\mathbf{x}_B \ge 0$, then $\mathbf{x}$ is a Basic Feasible Solution.
3. **Basic Variables**: $\mathbf{x}_B$ **Nonbasic Variables**: $\mathbf{x}_N$ **Basic Matrix**: B **Nonbasic Matrix**: N **Basic Index**: $\mathcal{B}$ **Nonbasic Index**: $\mathcal{N}$
   ps: 均为人为规定 $\mathcal{B}$ 和 $\mathcal{N}$ 后得到.

For the LP problem, we consider the following:

1. **Convex Set**: Set K is convex if, for any two points $\mathbf{x}, \mathbf{x}' \in K$, the line segment $\mathbf{x}_\theta = \theta\mathbf{x} + (1-\theta)\mathbf{x}'$ for $\theta \in [0,1]$ is also in K.
2. **Vertex of a Convex Set**: A vertex of a convex set K is a point $\mathbf{x} \in K$ such that it does not lie strictly within any line segment joining two points in K. (ps: there are $\frac{(m+n)!}{m!n!}$ vertices for LP problem)
3. **Theorem I**: If LP has a unique optimal solution, then it is a vertex. (optimal ↔ vertex)
4. **Theorem II**: If LP has a non-unique solution, ∃ optimal solution at vertex. (optimal ↔ vertex)

Remark: BFS and Vertices are NOT 1-1 (与它是否是唯一 optimal solution 无关!)

⋆ Happened if one of *basic variables* of *optimal solution* is zero. → Referred as **degenerate**.

2 Simplex Method

2.1 Some Definitions

For LP problem: max $f = \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} \le \mathbf{b}$ $\mathbf{x} \ge 0$

By adding slack variables, we can write it as: max $f = \bar{\mathbf{c}}^T\mathbf{x}$ s.t. $\bar{A}\mathbf{x} = \mathbf{b}$ $\mathbf{x} \ge 0$

1. **We can define Index**: $\mathcal{B}, \mathcal{N}$
2. **Basic Matrix**: B (整个 $\bar{A}$ 中在 index $\mathcal{B}$ 中的列) **Nonbasic Matrix**: N (整个 $\bar{A}$ 中在 index $\mathcal{N}$ 中的列) $\bar{A} \to [B \; N]$
3. **Basic Variables**: $\mathbf{x}_B$ (对应 B 列因有的变量) **Nonbasic Variables**: $\mathbf{x}_N$ (对应 N 列因有的变量) $\mathbf{x} \to \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix}$
4. **Matrix Cost**: $\mathbf{c}_B$ (f 中对应 **B** 变量的系数) **Nonbasic Cost**: $\mathbf{c}_N$ (f 中对应 **N** 变量的系数) $\bar{\mathbf{c}} \to \begin{pmatrix} \mathbf{c}_B \\ \mathbf{c}_N \end{pmatrix}$
5. **Other**: $\bar{A}\mathbf{x} = \mathbf{b} \Leftrightarrow B\mathbf{x}_B + N\mathbf{x}_N = \mathbf{b}$ || Let $\hat{\mathbf{b}} = B^{-1}\mathbf{b}$ $\hat{N} = B^{-1}N$ || Thus $\mathbf{x}_B = \hat{\mathbf{b}} - \hat{N}\mathbf{x}_N$
6. **Objective Value**: $f = \bar{\mathbf{c}}^T\mathbf{x} = \mathbf{c}_B^T\mathbf{x}_B + \mathbf{c}_N^T\mathbf{x}_N = \hat{f} + \hat{c}_N^T\mathbf{x}_N$ where $\hat{f} = \mathbf{c}_B^T\hat{b}$ $\hat{\mathbf{c}}_N = \mathbf{c}_N - \hat{N}\mathbf{c}_B = \mathbf{c}_N - N^TB^{-T}\mathbf{c}_B$

⋆ **Remark**: For *Basic Solution*, we let $\mathbf{x}_\mathbf{B} = \hat{\mathbf{b}}$ and $\mathbf{x}_\mathbf{N} = \mathbf{0}$. **Remark**: If basic Solution have $\hat{\mathbf{b}} \ge 0$, it is a **Basic Feasible Solution (BFS)**.

⋆ **Theorem**: If BFS have: $\hat{\mathbf{c}}_N \le 0$, it is an **Optimal Solution**

2.2 Simplex Algorithm

An Initial BFS: Try $\mathcal{B} = \{n+1, ..., n+m\}$ and $\mathcal{N} = \{1, ..., n\}$ $\Rightarrow$ $B = I, N = A, \mathbf{x}_B = \hat{\mathbf{b}} = \mathbf{b}, \mathbf{x}_N = \mathbf{0}, \mathbf{c}_B = \mathbf{0}, \mathbf{c}_N = \mathbf{c}$

Corollary: If $\hat{\mathbf{b}} = \mathbf{b} \geq 0$, it's a BFS.

Simplex Algorithm:

1. If **An Initial BFS** is BFS→2 || If we have an BFS→2 || Else: Go to **Phase I Problem**
2. If $\hat{\mathbf{c}}_N \leq 0$. ⇒ Optimal Solution || Else: Go to 3
3. Let $q \in$ (local index in N) ; $q' \in \mathcal{N}$ (global index) || $\hat{c}_i := \hat{\mathbf{c}}_N$ 中第 i 个分量 ($\hat{c}_i := \hat{\mathbf{c}}_N^T \mathbf{e}_i$) || such that:

   $\hat{c}_q = \hat{\mathbf{c}}_N$ 中最大的正的分量 ($c_q = \arg\max_{i=1,...,n\,;\,c_i>0} \hat{c}_i$) ; q' is the index corresponding to q in $\mathcal{N}$ (对应的在 $\mathcal{N}$ 的 index(global index))

   Thus, we find q, q' Go to 4
4. Let $p \in$ (local index in B) ; $p' \in \mathcal{B}$ (global index) || $\mathbf{a}_q := N$ 中的第 q 列($\mathbf{a}_q = N\mathbf{e}_q$) || Let $\widehat{\mathbf{a}_q} = B^{-1}\mathbf{a}_q$:

   If $\widehat{\mathbf{a}_q} \leq 0$ ⇒ LP is unbounded. || Else: LP is bounded→5
5. Let $p = \arg\min_{i=1,...,m\,;\,\hat{a}_{iq}>0} \frac{\widehat{b_i}}{\hat{a}_{iq}}$ p 是对应的 local index,not value, ⋆ 用 $\overline{\alpha} = \frac{\widehat{b_p}}{\hat{a}_{pq}}$ 代表值 $\widehat{b_i}$ 代表 $\hat{\mathbf{b}}$ 的第 i 个分量; $\hat{a}_{iq}$ 代表 $\widehat{\mathbf{a}_q}$ 的第 i 个分量

   Let p' be (global) index corresponding to p.

   Thus, we find p, p' Go to 6
6. Exchange p' and q' between $\mathcal{B}$ and $\mathcal{N}$ ⇒ Update $\mathcal{B}, \mathcal{N}, B, N, \hat{\mathbf{b}}, ...$ ps: New $B := B + (a_q - Be_p)e_p^T$
7. Calculate $\hat{\mathbf{c}}_N$. Go to 2

Remark: If we consider using $\mathbf{x} + \alpha\mathbf{d}$ where $\mathbf{d} \to [\mathbf{d}_B\ \mathbf{d}_N]^T$; $\mathbf{d}_N = \mathbf{e}_q$ $\mathbf{d}_B = -\hat{\mathbf{a}}_q$, then:

We have: $\begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{b}} \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -\hat{\mathbf{a}}_q \\ \mathbf{e}_q \end{pmatrix}$ and $f = \mathbf{c}_B^T\mathbf{x}_B + \mathbf{c}_N^T\mathbf{x}_N = \mathbf{c}_B^T(\hat{\mathbf{b}} - \alpha\hat{\mathbf{a}}_q) + \mathbf{c}_N^T(\alpha\mathbf{e}_q) = \hat{f} - \alpha(\mathbf{c}_N^T\mathbf{e}_q - \mathbf{c}_B^T B^{-1} N\mathbf{e}_q) = \hat{f} + \alpha\hat{\mathbf{c}}_N\mathbf{e}_q = \hat{f} + \alpha\hat{c}_q$

Phase I Problem: (If $\mathbf{b} \ngeq 0$ & No BFS)

1. Subtract **artificial variables** $x_{n+m+1}, ..., x_{n+m+m} \geq 0$ and change objective function f to:

$$
\begin{array}{llllllllll}
\max & f = & & & & & -x_{n+m+1} & \cdots & -x_{n+2m} & & \\
\text{s. t.} & a_{11}x_1 & + \cdots + & a_{1n}x_n & +x_{n+1} & & -x_{n+m+1} & & & = & b_1 \\
& \vdots & \vdots & & & \ddots & & \ddots & & \vdots & \\
& a_{m1}x_1 & + \cdots + & a_{mn}x_n & & +x_{n+m} & & & -x_{n+2m} & = & b_m \\
& x_1 \geq 0, & \ldots, & x_{n+2m} \geq 0 & & & & & & &
\end{array}
$$

2. Let $b_i := \mathbf{b}$ 的第 i 个分量. (i.e. $b_i = \mathbf{b}^T\mathbf{e}_i$)

   Case I: $b_i \geq 0$: $x_{n+i} = b_i$ is a Basic Variables. and $x_{n+m+i} = 0$ is a Nonbasic Variables.

   Case II: $b_i < 0$: $x_{n+m+i} = -b_i$ is a Basic Variables. and $x_{n+i} = 0$ is a Nonbasic Variables.

   Other: We put $x_1, ..., x_n$ as Nonbasic Variables.

   Thus we can find $\mathcal{B}, \mathcal{N}, B, N, \mathbf{x}_B = \hat{\mathbf{b}} = B^{-1}\mathbf{b} = |\mathbf{b}|, \mathbf{x}_N = \mathbf{0}, \mathbf{c}_B, \mathbf{c}_N$
3. Let **Size of Infeasibility**: is $\sum_{i=1}^{m} x_{n+m+i}$ $f = -\sum_{i=1}^{m} x_{n+m+i} \leq 0$
4. Consider this modified LP problem, by using Simplex Algorithm, we can find an optimal solution $\mathbf{x}$ and the value of f.

   ps: 使用刚刚我们选好的 $\mathcal{B}, \mathcal{N}$ 作为初始的 BFS.
5. Case I: If $f = 0$ (等价于 artificial variables 都是 Nonbasic Variables), $\mathbf{x}$ is a BFS for the original LP problem. (去除 artificial variables, 并回到原问题 (也叫 Phase II Problem))

   Case II: **If** $f < 0$: LP is infeasible (等价于存在 artificial variables 是 Basic Variables).

3 Degeneracy | Termination | Cycling | Sparsity

3.1 Degeneracy & Termination

Degeneracy: If $\hat{\mathbf{b}}$ has any zero component, then $\mathbf{x}$ is a **degenerate vertex.** (等价于 $\hat{\mathbf{b}} > 0$)

1. If a point is *degenerate*, then there may be multiple *BFS* at that point.
2. ⋆ If $\hat{b}_p = 0$, then $\overline{\alpha} = 0$, simplex algorithm may not terminate.

   ps: 对于 simplex algorithm 我们依靠 $\hat{\mathbf{a}}_q$ 中大于 0 的元素来决定下一个要交换的变量, 但是如果 $\hat{b}_p = 0$, 那么 $\overline{\alpha} = 0$, 可能会导致我们在这个点上循环

Termination of Simplex Algorithm in the absence of degeneracy: If no degenerate situation, then Simplex Algorithm will terminate in a finite number of steps.

Examples: 1.The worst case of Simplex Algorithm. 2. Cyclic case of Simplex Algorithm.

1. **Klee-Minty Problem**: $\max f = \sum_{j=1}^{n} 10^{j-1}x_j$; s.t. $x_i + 2\sum_{j=i+1}^{n} 10^{j-i}x_j \leq 100^{n-i}$ for $i = 1, ..., n$; $x_i \geq 0$ has 2^n vertices. **Worst Case**: $2^n - 1$ iterations.
2. **Hall-McKinnon Problem**: $\max f = x_1 - 5.5x_2 + 0.75x_3 - 5.75x_4$; s.t. $2.5x - 19.5x_2 - 3.5x_3 + 19.5x_4 + x_5 = 0$; $0.5x_1 - 3.5x_2 - 0.5x_3 + 3.5x_4 + x_6 = 0$; $x_i \geq 0$

3.2 Sparsity LP Problem

Implementation: 计算/编程中的计算化简　　Consider: $\hat{\mathbf{b}} = B^{-1}\mathbf{b}$ $\hat{\mathbf{c}}_N = \mathbf{c}_N - N^T B^{-T}\mathbf{c}_B$ $\hat{\mathbf{a}}_q = B^{-1}\mathbf{a}_q$

1. Solve: $B\hat{\mathbf{b}} = \mathbf{b}$ to get: $\hat{\mathbf{b}} = B^{-1}\mathbf{b}$
2. Solve: $B^T\pi = \mathbf{c}_B$ to get: $\pi = B^{-T}\mathbf{c}_B$
3. Solve: $\hat{\mathbf{c}}_N = \mathbf{c}_N - N^T\pi$ to get: $\hat{\mathbf{c}}_N = c_N - N^T B^{-T}\mathbf{c}_B$
4. Solve: $B^T\hat{\mathbf{a}}_q = \mathbf{a}_q$ to get: $\hat{\mathbf{a}}_q = B^{-T}\mathbf{a}_q$
5. Matrix B is **sparse** and **invertible.** And in Simplex Algorithm, the next matrix B obtained by $B := B + (a_q - Be_p)e_p^T$

Sparse LP Problem: For LP problem: $\max f = \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{b}$ $\mathbf{x} \geq 0$

It is **sparse** if the matrix A is sparse. (i.e. Most of the elements of A are zero)

Find Inverse of B: Decomposition B into LU where L is lower triangular and U is upper triangular.

Then we can solve $B\mathbf{x} = \mathbf{b}$ by solving ^{1} $L\mathbf{y} = \mathbf{b}$ and ^{2} $U\mathbf{x} = \mathbf{y}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} & l_{21}u_{14} + u_{24} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} & l_{31}u_{14} + l_{32}u_{24} + u_{34} \\ l_{41}u_{11} & l_{41}u_{12} + l_{42}u_{22} & l_{41}u_{13} + l_{42}u_{23} + l_{43}u_{33} & l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + u_{44} \end{bmatrix}$$

By Solving ■ → ■ → ■ → ■ → ■ → ■ → ■

4 Sensitive Analysis

4.1 RHS Sensitivity Consider $b_i \to b_i + \delta$

RHS Sensitivity: $\max f = \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0$ $\Rightarrow$ $\max f = \overline{\mathbf{c}}^T\mathbf{x}$ s.t. $\overline{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$

ps: Assume $\mathcal{B}, \mathcal{N}, B, N$ yield an *optimal solution*, with $\mathbf{x}_B = \hat{\mathbf{b}}, \mathbf{x}_N = \mathbf{0}$

Then, the new optimal values is: $\mathbf{x}_B = B^{-1}(\mathbf{b} + \delta\mathbf{e}_i) = \hat{\mathbf{b}} + \delta B^{-1}\mathbf{e}_i$ if $\mathbf{x}_B \geq 0$

With range of $\delta \in [\underline{\delta}, \overline{\delta}]$ $\underline{\delta} = \max\limits_{[B^{-1}]_{ji}>0} -\dfrac{\hat{b}_j}{[B^{-1}]_{ji}}$ $\overline{\delta} = \min\limits_{[B^{-1}]_{ji}<0} -\dfrac{\hat{b}_j}{[B^{-1}]_{ji}}$

Fair Prices: The objective function: $f = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}_B^T(\hat{\mathbf{b}} + \delta B^{-1}\mathbf{e}_i) = \hat{f} + \delta\pi_i$ where $\hat{f} = \mathbf{c}_B^T\hat{\mathbf{b}}$ $\pi_i = \mathbf{c}_B^T B^{-1}\mathbf{e}_i$ (fair price)

1. **If *price of unit amount*** $>$ π_i: Buying more of the resource is *unattractive*.
2. **If *price of unit amount*** $<$ π_i: Buying more of the resource is *attractive*.
3. **For Basic Slack**: If $n + i \in \mathcal{B}$ (slack variable x_{n+i} 是 basic slack) $\to$ $\pi_i = 0$ (Proof: $\pi_i = (B^{-T}\mathbf{c}_B)^T\mathbf{e}_i = \mathbf{c}_B^T B^{-1}\mathbf{e}_i = 0$)
4. **For Nonbasic Slack**: If $n + i \in \mathcal{N}$ (slack variable x_{n+i} 是 nonbasic slack) $\to$ $\pi_i = -\hat{c}_i$

(Proof: Assume $n + i \in \mathcal{N}$ 在 N 中的 (local) index 为 j. Since $\hat{c}_j = [\mathbf{c}_N]_j - [N]_j^T\pi$, $[\mathbf{c}_N]_j = 0$, $[N]_j^T = \mathbf{e}_i$; we have: $\pi_i = -\hat{c}_i$)

⋆ Range of δ is *lower/upper bound Sensitivity*

⋆ *Feasible Region* increases when $\overline{\delta}$ increases and $\underline{\delta}$ decreases.

4.2 Cost and Coefficient Sensitivity Consider $c_i \to c_i + \delta$ & $a_{ij} \to a_{ij} + \delta$

Cost Sensitivity: 即 $c_i \to c_i + \delta$. $\max f = \mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0$

1. 小范围的变化不会改变最优解. (If an LP cost coefficient is changed by a small amount, the optimal solution will not change.)
2. 如果 $c_i \to c_i + \delta$ 对应的 i 变量在 $\mathcal{B}$ 中 (basic variables), 那么 all reduced costs will change.
3. 如果 $c_i \to c_i + \delta$ 对应的 i 变量在 $\mathcal{N}$ 中 (nonbasic variables), 那么 only the reduced cost of *that* variables will change.

Coefficient Sensitivity: 即 $a_{ij} \to a_{ij} + \delta$. $\max f = \mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0$

1. 如果影响的变量在 $\mathcal{B}$ 中 (basic variables): B may become singular, the optimal solution may change, etc.
2. 如果影响的变量在 $\mathcal{N}$ 中 (nonbasic variables): One *reduced cost* will change, N will change.

5 Duality

Duality: For LP problem: $\max f = \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq 0$ **Primal** problem (P)

The **dual** problem is: $\min f = \mathbf{b}^T\mathbf{y}$ s.t. $A^T\mathbf{y} \geq \mathbf{c}$, $\mathbf{y} \geq 0$ **Dual** problem (D)

- It is *equivalence* to: $\max f = -\mathbf{b}^T\mathbf{y}$ s.t. $-A^T\mathbf{y} \leq -\mathbf{c}$, $\mathbf{y} \geq 0$

The **dual** of (D) is: $\min f = -\mathbf{c}^T\mathbf{z}$ s.t. $(-A^T)^T\mathbf{z} \geq (-\mathbf{b})$, (i.e. $-A\mathbf{z} \geq -\mathbf{b}$), $\mathbf{z} \geq 0$

- It is *equivalence* to: $\max f = \mathbf{c}^T\mathbf{z}$ s.t. $A\mathbf{z} \leq \mathbf{b}$, $\mathbf{z} \geq 0$, which is the **Primal** problem (P).

Weak Duality Theorem: If $\mathbf{x}$ is feasible for (P) and $\mathbf{y}$ is feasible for (D), then $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$.

Corollary: If (P) is unbounded $\Rightarrow$ (D) is infeasible. || If (D) is unbounded $\Rightarrow$ (P) is infeasible.

Strong Duality Theorem: If $\mathbf{x}^*$ is optimal (basic) solution for (P), then:

$\mathbf{y}^* = \pi = B^{-T}\mathbf{c}_B$ is optimal solution for (D), and $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$.

Application: If LP problem: $\max f = \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} \le \mathbf{b}$, $\mathbf{x} \ge 0$ (P)

is changed to a **tightening** LP problem: $\max f = \mathbf{c}^T\mathbf{x}$ s.t. $\begin{bmatrix} A \\ \mathbf{a}^T \end{bmatrix}\mathbf{x} \le \begin{bmatrix} \mathbf{b} \\ b \end{bmatrix}$, $\mathbf{x} \ge 0$ (P')

Then, the **relaxation dual** problem is: $\min f = \mathbf{b}^T\mathbf{y} + by$ s.t. $A^T\mathbf{y} + \mathbf{a}y \ge \mathbf{c}$, $\mathbf{y} \ge 0$, $y \ge 0$ (D')

- It is *equivalence* to: $\max f = -\mathbf{b}^T\mathbf{y} - by$ s.t. $-A^T\mathbf{y} - \mathbf{a}y \le -\mathbf{c}$, $\mathbf{y} \ge 0$, $y \ge 0$

此时, 计算 (P') 的最优解可以通过计算 (D') 的最优解来得到. (通过 **dual simplex method** 计算或 equivalence to 一般的计算)

6 Example of format written in LP

Example| 构建 LP 问题模板:

Defining decision variables:

Let x_1 be the number of 1kg packets of Breakfast Blend made each day.

Let x_2 be the number of 1kg packets of Dinner Blend made each day.

Total income is: $f_I = 1.16x_1 + 1.42x_2$.

Total cost is: $f_C = 0.18x_1 + 0.36x_2 + 0.28x_1 + 0.16x_2 = 0.46x_1 + 0.52x_2$.

Thus, total profit is: $f_I - f_C = 1.16x_1 + 1.42x_2 - 0.46x_1 - 0.52x_2 = 0.7x_1 + 0.9x_2$.

The objective is to maximize total profit $f = 0.7x_1 + 0.9x_2$

Constraints:

1. The number of kilos of arabica used, $0.3x_1 + 0.6x_2$, must not exceed the supply of 1200kg.
2. The number of kilos of robusta used, $0.7x_1 + 0.4x_2$, must not exceed the supply of 1500kg.
3. The total number of kilos of coffee made, $x_1 + x_2$, must not exceed the capacity of 2400kg.

Thus, the LP Problem to be solved is:

$$\begin{aligned} \text{maximize} \quad & f = 0.7x_1 + 0.9x_2, \\ \text{subject to} \quad & 0.3x_1 + 0.6x_2 \le 1200, \\ & 0.7x_1 + 0.4x_2 \le 1500, \\ & x_1 + x_2 \le 2400, \\ & x_1, x_2 \ge 0. \end{aligned}$$

Introducing Slack Variables:

$$\begin{aligned} \text{maximize} \quad & f = 0.7x_1 + 0.9x_2, \\ \text{subject to} \quad & 0.3x_1 + 0.6x_2 + x_3 = 1200, \\ & 0.7x_1 + 0.4x_2 + x_4 = 1500, \\ & x_1 + x_2 + x_5 = 2400, \\ & x_1, x_2, x_3, x_4, x_5 \ge 0. \end{aligned}$$

Example|Iteration in Simplex Method|No Phase I,II:

The RHS $\mathbf{b} = \begin{bmatrix} 100 \\ 1 \end{bmatrix}$ is positive, so the "all-slack" basis $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$ yields a basic feasible solution.

Iteration 1:

- For $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$, $B = I$, $N = \begin{bmatrix} 1 & 20 \\ 0 & 1 \end{bmatrix}$, $\mathbf{c}_B = \mathbf{0}$ and $\mathbf{c}_N = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$, giving $\hat{\mathbf{b}} = B^{-1}\mathbf{b} = \begin{bmatrix} 100 \\ 1 \end{bmatrix}$, and $\hat{\mathbf{c}}_N = \mathbf{c}_N - N^T B^{-T}\mathbf{c}_B = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$.
- Choose $q = 2$ since $\hat{c}_q = 10$, and hence $q' = 2$
- Thus $\mathbf{a}_q = \begin{bmatrix} 20 \\ 0 \end{bmatrix}$. Form $\hat{\mathbf{a}}_q = B^{-1}\mathbf{a}_q = \begin{bmatrix} 20 \\ 1 \end{bmatrix}$. Since $\hat{\mathbf{a}}_q$ is positive, the LP is bounded.
- Choose $p = \operatorname{argmin}\left\{ \underset{(i=1)}{\frac{100}{20}}, \underset{(i=2)}{\frac{1}{1}} \right\} = 2$ and hence $p' = 4$

Iteration 2: $\cdots$ **Iteration 4**: 第一行一样, + $\cdots$ Since $\hat{\mathbf{c}}_N \le 0$, the optimal solution is $\mathbf{x} = \cdots$, where values of $x_i, \ldots = 0$ since $\mathcal{N} = \ldots$

Example|Iteration in Simplex Method|Phase I, II:

The original LP Problem is:

maximize $f = -3x_1 - 5x_2$

subject to $-2x_1 - 3x_2 + x_3 = -6$

$x_1, x_2, x_3 \ge 0$

Phase I Problem is:

minimize $f = -x_4$

subject to $-2x_1 - 3x_2 + x_3 + x_4 = -6$

$x_1, x_2, x_3, x_4 \ge 0$

We start from the basic feasible solution given by $\mathcal{B} = \{4\}$ and $\mathcal{N} = \{1, 2, 3\}$. Hence $B = [-1]$, $N = [-2 \;\; -3 \;\; 1]$, ... (开始普通的 Simplex Method) 算出 optimal solution/或者如果什么时候 artificial variable 都是 nonbasic variables

去除 artificial variable, 变成 Phase II Problem (即变成原来的 LP Problem), 我们在 Phase I 找到的 Basic feasible solution 是 Phase II Problem 的一个 Basic feasible solution.

7 Additional Proofs

If BFS has $\hat{\mathbf{c}}_N \le 0$, it is an Optimal Solution | Show that $f = \hat{f} + \hat{\mathbf{c}}_N^T\mathbf{x}_N$

Proof: As $\mathbf{x}_B = \hat{b} - \hat{N}\mathbf{x}_N$, we have $f = \mathbf{c}_B^T\mathbf{x}_B + \mathbf{c}_N^T\mathbf{x}_N = \mathbf{c}_B^T(\hat{b} - \hat{N}\mathbf{x}_N) + \mathbf{c}_N^T\mathbf{x}_N = \cdots = \hat{f} + \hat{\mathbf{c}}_N^T\mathbf{x}_N$. Since $\hat{\mathbf{c}}_N \le 0$, we have $f \le \hat{f}$, so it is an Optimal Solution.

Why is the LP unbounded if $\hat{\mathbf{a}}_q \le 0$? | What is $\hat{\mathbf{a}}_q$?

Proof: At $\mathbf{x}$, the most positive reduced cost is $\hat{c}_q$. Now we construct a direction vector $\mathbf{d}$, such that $\mathbf{d} \Leftrightarrow \begin{pmatrix} \mathbf{d}_B \\ \mathbf{d}_N \end{pmatrix}$ where $\mathbf{d}_B = -B^{-1}N\mathbf{e}_q = -\hat{\mathbf{a}}_q$ and $\mathbf{d}_N = \mathbf{e}_q$. Then, we have $\mathbf{x} + \alpha\mathbf{d}$ can be partitioned as $\begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{b}} \\ 0 \end{pmatrix} + \alpha\begin{pmatrix} -\hat{\mathbf{a}}_q \\ \mathbf{e}_q \end{pmatrix}$ We can see that the objective function is $f = \hat{f} + \alpha\hat{c}_q$ increasing strictly if $\hat{c}_q > 0$. If $\hat{\mathbf{a}}_q \le 0$, then $-\hat{\mathbf{a}}_q \ge 0$, so $\mathbf{x} + \alpha\mathbf{d}$ is feasible for all $\alpha \ge 0$. Thus, the LP is unbounded.

(Cont.) What is $p = \arg\min_{i=1,\ldots,m\,;\,\hat{a}_{iq}>0} \frac{\hat{b}_i}{\hat{a}_{iq}}$?

Proof: (Cont.) As $\begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{b}} \\ 0 \end{pmatrix} + \alpha\begin{pmatrix} -\hat{\mathbf{a}}_q \\ \mathbf{e}_q \end{pmatrix}$, we can see that, for the component of $\hat{\mathbf{a}}_q$, if $\hat{a}_{iq} > 0$, then $\hat{b}_i - \alpha\hat{a}_{iq} \ge 0$, objective function is strictly increasing; if $\hat{a}_{iq} < 0$, then $\hat{b}_i - \alpha\hat{a}_{iq} \le 0$, objective function is strictly decreasing; if $\hat{a}_{iq} = 0$, then $\hat{b}_i - \alpha\hat{a}_{iq} = \hat{b}_i$, objective function is constant. Thus, we can see that, the most negative component of $\hat{\mathbf{b}}$ is $\hat{b}_p$ and the most positive component of $\hat{\mathbf{a}}_q$ is $\hat{a}_{pq}$, so we have $p = \arg\min_{i=1,\ldots,m\,;\,\hat{a}_{iq}>0} \frac{\hat{b}_i}{\hat{a}_{iq}}$.

Why $B_{new} = B + (\mathbf{a}_q - B\mathbf{e}_p)\mathbf{e}_p^T$

Since the new $\mathcal{B}$ is the old $\mathcal{B}$ with entry p replaced by the q th index from $\mathcal{N}$, the new basis matrix is the old basis matrix with column p replaced by the vector $\mathbf{a}_q$, which is column q of N. Moreover, $[B + (\mathbf{a}_q - B\mathbf{e}_p)\mathbf{e}_p^T]\mathbf{e}_j = \{B\mathbf{e}_j, \text{if } j \ne p \;;\; \mathbf{a}_q, \text{if } j = p$

Why in the Simplex Algorithm, we exchange p' and q' can get a new BFS?

Proof: 1. It's clear that new index $\mathcal{B}$ and $\mathcal{N}$ form a partition of $\{1, 2, \ldots, n+m\}$. 2. Only $x_{q'}$ is increased from 0 and q' leaves $\mathcal{N}$, and $x_{p'}$ is decreased to 0. So that they can be consider as new nonbasic variables and basic variables. 3. The new basis B is nonsingular (invertible): New Basic matrix is $B_{New} = B + (\mathbf{a}_q - B\mathbf{e}_p)e_p^T = BE$ where $E = I + (\hat{\mathbf{a}}_q - \mathbf{e}_p)e_p^T$; $B\hat{\mathbf{a}}_q = \mathbf{a}_q$ Thus B_{New} is nonsingular if and only if E is nonsingular. By Sherman-Morrison Formula, E is nonsingular iff $\hat{a}_{pq}$ is nonzero. But we have $\hat{a}_{pq} > 0$ so that B_{New} is nonsingular.

(ps: Sherman-Morrison Formula: If $A = I + uv^T$, then $A^{-1} = I - \frac{1}{1+v^T u} uv^T$)

Termination of Simplex Algorithm in the absence of degeneracy

Proof: If no degenerate situation, $\hat{\mathbf{b}} > 0$. Thus $\hat{\alpha} = \frac{\hat{b}_p}{\hat{a}_{pq}} > 0$, and hence the objective function is strictly increasing. (By $\overline{\alpha}\hat{c}_q > 0$) So the algorithm cannot return to a basic feasible solution previously visited. Thus the algorithm will terminate in a finite number of steps. (As it's bounded)

Proof of Week Duality Theorem

Proof(Inequality): $\mathbf{c}^T\mathbf{x} = (\mathbf{c}^T\mathbf{x})^T = \mathbf{x}^T\mathbf{c} \leq \mathbf{x}^T(A^T\mathbf{y}) = (A\mathbf{x})^T\mathbf{y} \leq \mathbf{b}^T\mathbf{y}$

Proof(Corollary): If (P) is unbounded, but (D) is feasible: Let $\mathbf{y}$ be feasible for (D), then $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$, $\mathbf{b}^T\mathbf{y}$ is an upper bound for (P), but (P) is unbounded, which is a contradiction. So (D) is infeasible. If (D) is unbounded, but (P) is feasible: Let $\mathbf{x}$ be feasible for (P), then $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$, $\mathbf{c}^T\mathbf{x}$ is an lower bound for (D), but (D) is unbounded, which is a contradiction. So (P) is infeasible.

Proof of Strong Duality Theorem

Proof: