LPMS Note

1 General Linear Programming Problem

General LP Problem: **Decision Variables**: x_i **parameters**: a_{ij}, b_i, c_i **Objective Function**: f **Constraints**: subject to 后边的部分

LP problem can be written as:

$$\begin{aligned} \text{maximize} \quad & f = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \\ & x_1 \ge 0, x_2 \ge 0, \cdots, x_n \ge 0 \end{aligned}$$

We can also write the LP problem in matrix form:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{aligned} \text{maximize} \quad & f = \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \le \mathbf{b} \qquad \mathbf{x} \ge 0 \end{aligned}$$

Feasible Solution: If $\mathbf{x}$ satisfies all constraints (i.e. $A\mathbf{x} \le \mathbf{b}$), then $\mathbf{x}$ is a feasible solution. (可行解) **Optimal Sol**: (最优解) (可多个)

Find Optimal Solution: **Graphical Method**: 略. **Vertex Enumeration**: 所有顶点检查, 找出最优解. **Simplex Method**: 后面详述.

Slack Variables: For each inequality constraint, we introduce a slack variable x_i $(i > n)$ to convert it to an equation. (松弛变量)

LP problem can be written as: ps: $x_i \ge 0$ $(i > n)$.

$$\begin{aligned} \text{maximize} \quad & f = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{n+2} = b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + x_{n+m} = b_m \\ & x_1 \ge 0, x_2 \ge 0, \cdots, x_{n+m} \ge 0 \end{aligned}$$

We can also write the LP problem in matrix form:

$$\overline{A} = [\, A \;\; I_m \,], \mathbf{b} = \mathbf{b}, \overline{\mathbf{c}} = \begin{pmatrix} \mathbf{c} \\ \mathbf{0}_{m\times 1} \end{pmatrix}, \mathbf{x} = \begin{pmatrix} \mathbf{x} \\ \vdots \\ x_{n+m} \end{pmatrix}$$

$$\begin{aligned} \text{maximize} \quad & f = \overline{\mathbf{c}}^T\mathbf{x} \\ \text{subject to} \quad & \overline{A}\mathbf{x} = \mathbf{b} \qquad \mathbf{x} \ge 0 \end{aligned}$$

Solving: $\mathbf{x} = \mathbf{x}_b + \mathbf{v}$ where $\overline{A}\mathbf{v} = 0$ and $\mathbf{x}_b = [\mathbf{0}\ \mathbf{b}]^T$

Feasible Region: It's the set K of the solutions to $\overline{A}\mathbf{x} = \mathbf{b}$ **Convex Set**: The set K is convex if $\forall \mathbf{x}, \mathbf{x}' \in K, \forall \theta \in [0,1], \mathbf{x}_\theta = (1-\theta)\mathbf{x} + \theta\mathbf{x}' \in K$.

Vertex on Convex Set: A vertex of the convex set K is a point $\mathbf{x} \in K$ which doesn't lie strictly inside any line segment connecting two points in K.

· **Theorem**: If LP has a unique optimal solution is a vertex. **Theorem**: If LP has a non-unique solution, $\exists$ optimal solution at vertex

2 Simplex Method

Solve LP Problem: Assume $f = \overline{\mathbf{c}}^T\mathbf{x}$ with $\overline{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \ge 0$.

1. 修改 x_i/列顺序, A 中换顺序后得到可逆的 $B_{m\times m}$ $\Rightarrow$ $\overline{A}\mathbf{x} = \mathbf{b} \Leftrightarrow B\mathbf{x}_B + N\mathbf{x}_N = \mathbf{b}$ $\overline{\mathbf{c}} \to [\mathbf{c}_B\ \mathbf{c}_N]^T$ $\overline{A} \to [B\ N]$ $\mathbf{x} \to [\mathbf{x}_B\ \mathbf{x}_N]^T$ Let $\widehat{\mathbf{b}} = B^{-1}\mathbf{b}$ $\widehat{N} = B^{-1}N$
2. **Solution**: $\mathbf{x}_B = \widehat{\mathbf{b}} - \widehat{N}\mathbf{x}_N$ If $\mathbf{x}_N = 0 \Rightarrow$ It's a basic solution. (But we need to check whether $\mathbf{x}_B \ge 0$) Using $\mathcal{B}, \mathcal{N}$: Index set of independent/else.
3. **Basic Variables**: $\mathbf{x}_B$ **Nonbasic Variables**: $\mathbf{x}_N$
4. **At Basic Solution**: $\mathbf{x}_B = \widehat{\mathbf{b}}, \mathbf{x}_N = \mathbf{0}$ $\Rightarrow$ If $\widehat{\mathbf{b}} \ge 0$. Corresponding $\mathbf{x}$ is: [1] vertex of K; [2] **Basic Feasible Solution (BFS)**
5. **Basic Costs**: $\mathbf{c}_B^T$ **Nonbasic Costs**: $\mathbf{c}_N^T$ **Reduced Costs**: $\hat{\mathbf{c}}_N = \mathbf{c}_N - \widehat{N}^T\mathbf{c}_B = \mathbf{c}_N - N^TB^{-T}\mathbf{c}_B$ $\hat{f} = \mathbf{c}_B^T\widehat{\mathbf{b}}$ $\mathbf{x}_B, \mathbf{x}_N \ge 0$
6. **Objective Value**: $f = \overline{\mathbf{c}}^T\mathbf{x} = \mathbf{c}_B^T\mathbf{x}_B + \mathbf{c}_N^T\mathbf{x}_N = \hat{f} + \hat{\mathbf{c}}_N^T\mathbf{x}_N$ If $\hat{\mathbf{c}}_N \le 0$, then $f \le \hat{f}$ $\Rightarrow$ Corresponding $\mathbf{x}$ is optimal.
7. If $\hat{\mathbf{c}}_N \le 0$ doesn't hold, we using **Simplex Algorithm**.

Simplex Algorithm:

1. **Initial Basic Feasible Solution**: Try $\mathcal{B} = \{n+1, ..., n+m\}$ and $\mathcal{N} = \{1, ..., n\}$ $\Rightarrow$ $B = I, N = A$, $\mathbf{x}_B = \widehat{\mathbf{b}} = \mathbf{b}, \mathbf{x}_N = \mathbf{0}, \mathbf{c}_B = \mathbf{0}, \mathbf{c}_N = \mathbf{c}$
2. If $\mathbf{b} \ge 0$. $\Rightarrow$ Basis is feasible + cont. Else: solved by **Two-Phase Simplex Method.**
3. If $\hat{\mathbf{c}}_N \le 0$. $\Rightarrow$ Optimal Solution Else: cont.
4. Let $q' \in \mathcal{N}$ 对应 $\hat{\mathbf{c}}_N$ 中最大 positive 分量的 index 同理, 对应的最大正分量值为 $\hat{c}_q$ 对应的 variable 为 $x_{q'}$ 对应 N 中的 $[q']$ 列为 $\mathbf{a}_q$
5. Let $\widehat{\mathbf{a}_q} = B^{-1}\mathbf{a}_\mathbf{q}$. If $\widehat{\mathbf{a}_q} \le 0 \Rightarrow$ LP is unbounded. Else: LP is bounded + cont.
6. Let $p' \in \mathcal{B}$ be the index corresponding to $p = \arg\min_{i=1,...,m\,;\,\hat{a}_{iq}>0} \frac{\hat{b}_i}{\hat{a}_{iq}}$ p 是对应的 index,not value 用 $\overline{\alpha} = \frac{\hat{b}_p}{\hat{a}_{pq}}$ 代表值
   $\hat{b}_i, \hat{a}_{iq}$ 代表 $\widehat{\mathbf{b}}$ 的第 i 个分量, $\widehat{\mathbf{a}_q}$ 的第 i 个分量
   p' 代表能使 $\frac{\hat{b}_i}{\hat{a}_{iq}}$ 的值最小的 index, 前提条件是 $\hat{a}_{iq} > 0$ 对应的 variable 为 $x_{p'}$
7. Exchange p' and q' between $\mathcal{B}$ and $\mathcal{N}$ $\Rightarrow$ values of new $x_B = \widehat{\mathbf{b}} - \overline{\alpha}\,\hat{\mathbf{a}}_q$ Update $\mathcal{B}, \mathcal{N}, B, N, \widehat{\mathbf{b}}, ...$
8. Go to 3.