

1 Basic Knowledge

HCV Note

Useful Complex Number Properties: $|Re(z)|, |Im(z)| \leq |z|$ $Re(z) = \frac{z+\bar{z}}{2}, Im(z) = \frac{z-\bar{z}}{2i}, |z|^2 = z\bar{z}$

Triangle (Reverse) Inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$ $|z_1| - |z_2| \leq |z_1 - z_2|$ $(Re(zw) = 0 \Leftrightarrow \bar{z}\bar{w} = -zw; Im(zw) = 0 \Leftrightarrow zw = \bar{z}\bar{w})$

Argument: $arg(z) := \{\theta : z = |z|e^{i\theta}\} = \{Arg(z) + 2\pi k : k \in \mathbb{Z}\}$ **Principle Value of Argument:** $Arg(z) \in (-\pi, \pi]$

Operations on Argument: $arg(z_1 z_2) = arg(z_1) + arg(z_2)$ $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$ $arg(\bar{z}) = -arg(z)$

2 Holomorphic Functions

2.1 Open/Closed Set | Limit Point | limit of Sequence | Continuous of Function

Open/Closed/Punctured ε -disc: $D_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ $\bar{D}_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq \varepsilon\}$ $D'_\varepsilon(z_0) := \{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}$

Open/Closed Set in \mathbb{C} : $U \subseteq \mathbb{C}$ is **open** if $\forall z_0 \in U, \exists \varepsilon > 0, D_\varepsilon(z_0) \subseteq U$ U is **closed** if $\mathbb{C} \setminus U$ is open **Lemma:** $D_\varepsilon, D'_\varepsilon$ open, \bar{D}_ε closed.

Limit Point of S : $z_0 \in \mathbb{C}$ is a limit point of S if: $\forall \varepsilon > 0, D'_\varepsilon(z_0) \cap S \neq \emptyset$ **Bounded:** S is bounded if $\exists M > 0$ s.t. $|z| \leq M, \forall z \in S$

Closed of Set S : $\bar{S} :=$ 所有 S 的 limit point 和 S 的点. **Property:** Let $S \subseteq \mathbb{C}$, then S is closed $\Leftrightarrow S = \bar{S}$.

Limit of sequence: Sequence $(z_n)_{n \in \mathbb{N}}$ has limit z if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N \Rightarrow |z_n - z| < \varepsilon$. limit rules 依旧成立

1. **Lemma|Important:** $\lim z_n = z \Leftrightarrow \lim Re(z_n) = Re(z)$ and $\lim Im(z_n) = Im(z)$

2. **Cauchy:** Sequence $(z_n)_{n \in \mathbb{N}}$ is cauchy if: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N \Rightarrow |z_m - z_n| < \varepsilon$ **Lemma:** Cauchy \Leftrightarrow convergent.

3. **Lemma|Closed of Set:** $S \subseteq \mathbb{C}, z \in \mathbb{C} \Rightarrow [z \in \bar{S} \Leftrightarrow \exists \text{ sequence } (z_n)_{n \in \mathbb{N}} \in S \text{ s.t. } \lim z_n = z]$

4. **Bolzano-Weierstrass:** Every bounded sequence in \mathbb{C} has a convergent subsequence.

Complex Functions: $\forall f : \mathbb{C} \rightarrow \mathbb{C}$ we can write it as: $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ where $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$

Limit of Function: $a_0 \in \mathbb{C}$ is the limit of f at z_0 if: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |z - z_0| < \delta \Rightarrow |f(z) - a_0| < \varepsilon$ limit rules 依旧成立

Lemma|Important: $\lim_{z \rightarrow z_0} f(z) \Leftrightarrow \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = Re(a_0)$ and $\lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = Im(a_0)$

Useful Formula: $\lim_{z \rightarrow z_0} g(\bar{z}) = \lim_{z \rightarrow \bar{z}_0} g(z)$

continuous of Function: f is continuous at z_0 if: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$ continuous rules 依旧成立

1. **Lemma|Important:** f is continuous at $z_0 \Leftrightarrow u, v$ are continuous at (x_0, y_0)

2. **'Extreme Value Theorem':** f is continuous on a closed and bounded set $S \subseteq \mathbb{C}$, then $f(S)$ is closed and bounded.

3. **Lemma|continuous \Leftrightarrow open:** f is continuous $\Leftrightarrow \forall$ open set U , preimage $f^{-1}(U) := \{z \in \mathbb{C} | f(z) \in U\}$ is open.

2.2 Differentiable | Holomorphic Function | C-R Equation

Differentiable: Let $z_0 \in \mathbb{C}$ and $U \subseteq \mathbb{C}$ be neighborhood of z_0 , then $f : U \rightarrow \mathbb{C}$ is differentiable at z_0 if: $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

I. f is differentiable $\Rightarrow f$ is continuous. **II. Holomorphic \Leftrightarrow Differentiable + neighborhood** (除非是一个点时不成立, $|z|$) diff rules + chain rule 成立

Cauchy-Riemann Equations: If $z_0 = x_0 + iy_0, f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 \Rightarrow u_x = v_y, v_x = -u_y$ at (x_0, y_0) .

If $z_0 = x_0 + iy_0, f = u + iv$ satisfies: $^1 u, v$ are continuously differentiable on a neighborhood of (x_0, y_0) and:

$^2 u, v$ satisfies Cauchy-Riemann Equations at (x_0, y_0) . $\Rightarrow f$ is differentiable at z_0 .

ps: 常见可导复数函数: $\exp(z), \sin z, \cos z, \log z, z^\alpha$, polynomial, $\sinh, \cosh, \Gamma(z), |z|^2$ (at 0), constant **ps:** 常见不可导复数函数: $\bar{z}, |z| \cdot \bar{z}, Re(z), Im(z), Arg(z)$

Harmonic Function: $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ is harmonic if: $\forall (x, y) \in \mathbb{R}^2 \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$ (Laplace Equation)

Lemma: If $f = u + iv$ is holomorphic on \mathbb{C} and u, v are twice continuously differentiable, $\Rightarrow u, v$ are harmonic.

Harmonic Conjugate: Let $u, v : U \rightarrow \mathbb{R}, U \subseteq \mathbb{R}^2$ be harmonic functions. u, v are harmonic conjugate if: $f = u + iv$ is holomorphic on U .

Properties of Polynomial: The domain of rational function and polynomial are always open. **Lemma:** If $P(z_0) = 0$ then $P(\bar{z}_0) = 0$

First-order Operator ∂ : $\partial := \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$ $\bar{\partial} := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$ $\parallel f = u + iv$ satisfies C-R Equations $\Leftrightarrow \bar{\partial} f = 0$

sin/cos Functions: $\sin z := \frac{e^{iz} - e^{-iz}}{2i}$ $\cos z := \frac{e^{iz} + e^{-iz}}{2}$ **Exponential Function:** $\exp(z) = e^x(\cos(y) + i \sin(y))$

1. $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$ $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$

2. $\sin(z + w) = \sin(z) \cos(w) + \cos(z) \sin(w)$ $\cos(z + w) = \cos(z) \cos(w) - \sin(z) \sin(w)$

3. $\sin^2 z + \cos^2 z = 1$ $\sin(z + \frac{\pi}{2}) = \cos(z)$ $\sin(z + 2k\pi) = \sin(z)$ $\cos(z + 2k\pi) = \cos(z)$ * $\sin z, \cos z$ NOT bounded.

Hyperbolic Functions: $\sinh z := \frac{\exp(z) - \exp(-z)}{2}$ $\cosh z := \frac{\exp(z) + \exp(-z)}{2}$ $\parallel \sinh(iz) = i \sin z$ $\cosh(iz) = \cos z$

Logarithm: Define multivalued function: $\log z := \{w \in \mathbb{C} : \exp w = z\}$ **Principal Branch:** $Log(z) := \ln |z| + i Arg(z)$

1. **I.** $\log(z) = \ln |z| + i \arg z = \{\ln |z| + i Arg(z) + i 2\pi k : k \in \mathbb{Z}\}$ **II.** $\log(zw) = \log(z) + \log(w)$ **III.** $\log(1/z) = -\log(z)$

2. **Branch of Logarithm:** $Log_\phi(z) := \ln |z| + i Arg_\phi(z)$ $Log_\phi(z)$ is holomorphic on D_ϕ

3. If $g : U \rightarrow \mathbb{C}$, then $Log_\phi(g(z))$ is holomorphic on $g^{-1}(D_\phi) \cap U$

4. $Log(z)$ not continuous on \mathbb{C} . $Log(z)$ not continuous on $Re(z) \leq 0, Im(z) = 0$.

Branch Cut|Cut Plane: Branch Cut $L_{z_0, \phi} := \{z \in \mathbb{C} : z = z_0 + re^{i\phi}, r \geq 0\}$ Cut Plane: $D_{z_0, \phi} := \mathbb{C} \setminus L_{z_0, \phi}$ $L_\phi = L_{0, \phi}; D_\phi = D_{0, \phi}$

Branch of Argument: $Arg_\phi(z) := z$ 的辐角, 但是角度限制在: $\phi < Arg_\phi(z) \leq \phi + 2\pi$. **ps:** $Arg_{-\pi}(z) = Arg(z)$

$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$
z^n	$n z^{n-1}$	$\exp(z)$	$\exp(z)$	$\sin(z)$	$\cos(z)$	$\cos(z)$	$-\sin(z)$	$\sinh(z)$	$\cosh(z)$	$\cosh(z)$	$\sinh(z)$	$\log_\phi z$	$\frac{1}{z} \quad z \in D_\phi$

Complex Powers: $z^\alpha := \{\exp(\alpha w) : w \in \log(z)\} = \{\exp[\alpha(\ln |z| + i Arg(z) + i 2k\pi)] : k \in \mathbb{Z}\}$

$\frac{d}{dz} z^\alpha = \alpha z^{\alpha-1} \quad z \in D_\phi$

I. If $\alpha \in \mathbb{Z}$, there is one value of z^α **II.** If $\alpha = \frac{p}{q}, \gcd(p, q) = 1, p, q \in \mathbb{Z}, q \neq 0$, there are exactly q values of z^α

III. If α is irrational or non-real, there are infinitely values z^α **IV.** $1^{1/q}, q \in \mathbb{Z}, q \neq 0$ is $\{1, w, \dots, w^{q-1}\}, w = \exp(i 2\pi/q)$

V. We prefer use $\exp(z)$ to denote single-valued function, and e^z to denote multi-valued function.

Principal Branch: $z^\alpha := \exp(\alpha Log(z))$

Operation: $z^\alpha z^\beta = z^{\alpha+\beta}$ (Using Principal Branch)