HAlg Note

1 Basic Knowledge

Def of Matrix: A mapping from $\{1, ..., n\} \times \{1, ..., m\}$ to a field F is called a $n \times m$ matrix over F.

· The set of all $n \times m$ matrices over F is denoted by $Mat(n \times m; F) := Maps(\{1, ..., n\} \times \{1, ..., m\}, F)$.

· If $n = m$, we sill speak of a **Square Matrix** and shorten the notation to $Mat(n; F)$.

Solution Sets of Inhomogeneous Systems of Linear Equations: Solution = 特解 (Particular Solution) + 通解 (Homogeneous solution)

Def of Group $(G, *)$: A set G with a operator $*$ is a group if: **Closure**: $\forall g, h \in G, g*h \in G$; **Associativity**: $\forall g, h, k \in G, (g*h)*k = g*(h*k)$;
Identity: $\exists e \in G, \forall g \in G, e * g = g * e = g$; **Inverse**: $\forall g \in G, \exists g^{-1} \in G, g * g^{-1} = g^{-1} * g = e$.

· **Properties of Group**: If G, H are groups, then $G \times H$ also.

Field (F): A set F is a field with two operators: (addition)$+ : F \times F \to F; (\lambda, \mu) \to \lambda + \mu$ (multiplication)$\cdot : F \times F \to F; (\lambda, \mu) \to \lambda\mu$ if:
$(F, +)$ and $(F \backslash \{0_F\}, \cdot)$ are abelian groups with identity $0_F, 1_F$. and $\lambda(\mu + \nu) = \lambda\mu + \lambda\nu$ $e.g. Fields : \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}/p\mathbb{Z}$

2 Vector Spaces

2.1 Vector Spaces | Product of Sets | Vector Subspaces | Power, Union, Intersection of Sets

F-Vector Space (V): A set V over a field F is a vector space if: V is an abelian group $V = (V, \dot{+})$ and $\forall \vec{v}, \vec{w} \in V$ $\lambda, \mu \in F$
a map $F \times V \to V : (\lambda, \vec{v}) \to \lambda\vec{v}$ satisfies: **I**: $\lambda(\vec{v} \dot{+} \vec{w}) = (\lambda\vec{v}) \dot{+} (\lambda\vec{w})$ **II**: $(\lambda + \mu)\vec{v} = (\lambda\vec{v}) \dot{+} (\mu\vec{v})$
III: $\lambda(\mu\vec{v}) = (\lambda\mu)\vec{v}$ **IV**: $1_F\vec{v} = \vec{v}$ ps:**I,II** are **Distributive Laws**; **III** is **Associative Law**. **Trivial Vector Space**: $V = \vec{0}$

1. **Properties of F-Vector Space**: **a.** $0_F\vec{v} = \vec{0}$ **b.** $(-1_F)\vec{v} = -\vec{v}$ **c.** $\lambda\vec{0} = \vec{0}$ **d.** If $\lambda\vec{v} = \vec{0}$, then $\lambda = 0$ or $\vec{v} = \vec{0}$ or both.
2. If V, W are F-vector spaces, then $V \times W$ is also.

Component: An individual entry x_i of an **n-tuple** $(x_1, ..., x_n)$ is called a component.

Projections (pr_i): $pr_i : X_1 \times X_2 \times \cdots \times X_n \to X_i$ with $(x_1, ..., x_n) \mapsto x_i$

Vector Subspace (U): $U \subseteq V$ is a subspace of V if: **I.** $\vec{0} \in U$ **II.** $\forall \vec{u}, \vec{v} \in U, \forall \lambda \in F$: $\vec{u} + \vec{v} \in U$ and $\lambda\vec{u} \in U$ (or: $\lambda\vec{u} + \mu\vec{v} \in U$)

1. If U_1, U_2 are subspaces of V. Then $U_1 \cap U_2$ and $U_1 + U_2$ are also. ps: $U_1 + U_2 := \{\vec{u} + \vec{v} : \vec{u} \in U_1, \vec{v} \in U_2\}$
2. **Vector Subspace Generated by T ($\langle T \rangle$)**: If T is a subset of a F-vector space V. $\Rightarrow$ $\langle T \rangle$ is the smallest subspace of V containing T.
   Also, we can get: $\langle T \rangle = span(T) := \{\sum_i c_i\vec{v_i} : if\ T = \{v_1, ..., v_i\}, c_i \in F\}$ $\forall \vec{v} \in \langle T \rangle, \langle T \cup \{\vec{v}\} \rangle = \langle T \rangle$
3. **Generating/Spanning Set**: V is a vector space. If $T \subseteq V$ and $\langle T \rangle = V$. $\Rightarrow$ T is a generating set of V.
4. **Finitely Generated**: $\exists$ T finite set, s.t. $V = \langle T \rangle$

Power of Set $\mathcal{P}(X)$: If X is a set, then $\mathcal{P}(X) := \{U : U \subseteq X\}$ (set of all subsets) ps: $\mathcal{U} \subseteq \mathcal{P}(X) \Rightarrow U$ is called a **system of subsets of** X.

1. **Empty System of subsets of X**: Empty System of subsets of X $:= \emptyset \in \mathcal{P}(X)$ (NOT $\{\emptyset\}$) $\star \bigcap \emptyset = X$ and $\bigcup \emptyset = \emptyset$ $\star$
2. **Def of Union**: For $\mathcal{U} \subseteq \mathcal{P}(X)$, $\bigcup_{U \in \mathcal{U}} U := \{x \in X : \exists U \in \mathcal{U}\ s.t.\ x \in U\}$
3. **Def of Intersection**: For $\mathcal{U} \subseteq \mathcal{P}(X)$, $\bigcap_{U \in \mathcal{U}} U := \{x \in X : \forall U \in \mathcal{U}, x \in U\}$

2.2 Linear Independence | Basis | Dimension

Linearly Independent: $L = \{\vec{v_1}, \vec{v_2}, ..., \vec{v_r}\}$ is linearly independent if: $\forall c_1, ..., c_r \in F, c_1\vec{v_1} + ... + c_r\vec{v_r} = \vec{0} \Rightarrow c_1 = ... = c_r = 0$.

· **Linearly Dependent**: L is linearly dependent if: $\exists \alpha_1, ..., \alpha_r$ not all zero s.t. $\alpha_1\vec{v_1} + ... + \alpha_r\vec{v_r} = \vec{0}$

· Empty set is linearly independent. Every nonzero one-element set is linearly independent.

2.3 Linear Maps | Rank-Nullity Theorem