

1 Basic Knowledge

Useful Complex Number Properties: $|Re(z)|, |Im(z)| \leq |z|$ $Re(z) = \frac{z+\bar{z}}{2}, Im(z) = \frac{z-\bar{z}}{2i}, |z|^2 = z\bar{z}$

Useful Tools: $Re(z) = 0 \Leftrightarrow z = -\bar{z}$ $Im(z) = 0 \Leftrightarrow z = \bar{z}$ In circle, $\bar{z} = |z|^2 z^{-1}$

Triangle (Reverse) Inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$ $||z_1| - |z_2|| \leq |z_1 - z_2|$ $\ominus (Re(zw) = 0 \Leftrightarrow \overline{zw} = -zw; Im(zw) = 0 \Leftrightarrow zw = \overline{zw})$

Argument: $arg(z) := \{\theta : z = |z|e^{i\theta}\} = \{Arg(z) + 2\pi k : k \in \mathbb{Z}\}$ **Principle Value of Argument:** $Arg(z) \in (-\pi, \pi]$

Operations on Argument: $arg(z_1 z_2) = arg(z_1) + arg(z_2)$ $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$ $arg(\bar{z}) = -arg(z)$

2 Holomorphic Functions

2.1 Open/Closed Set | Limit Point | limit of Sequence | Continuous of Function

Open/Closed/Punctured ε -disc: $D_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ $\bar{D}_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq \varepsilon\}$ $D'_\varepsilon(z_0) := \{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}$

Open/Closed Set in \mathbb{C} : $U \subset \mathbb{C}$ is **open** if $\forall z_0 \in U, \exists \varepsilon > 0, D_\varepsilon(z_0) \subseteq U$ U is **closed** if $\mathbb{C} \setminus U$ is open **Lemma:** $D_\varepsilon, D'_\varepsilon$ open, \bar{D}_ε closed.

Limit Point of S : $z_0 \in \mathbb{C}$ is a limit point of S if: $\forall \varepsilon > 0, D'_\varepsilon(z_0) \cap S \neq \emptyset$ **Bounded:** S is bounded if $\exists M > 0$ s.t. $|z| \leq M, \forall z \in S$

Closed of Set S : $\bar{S} :=$ 所有 S 的 limit point 和 S 的点. **Property:** Let $S \subseteq \mathbb{C}$, then S is closed $\Leftrightarrow S = \bar{S}$.

Limit of sequence: Sequence $(z_n)_{n \in \mathbb{N}}$ has limit z if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N \Rightarrow |z_n - z| < \varepsilon$. limit rules 依旧成立

1. **Lemma|Important:** $\lim z_n = z \Leftrightarrow \lim Re(z_n) = Re(z)$ and $\lim Im(z_n) = Im(z)$

2. **Cauchy:** Sequence $(z_n)_{n \in \mathbb{N}}$ is cauchy if: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N \Rightarrow |z_m - z_n| < \varepsilon$ **Lemma:** Cauchy \Leftrightarrow convergent.

3. **Lemma|Closed of Set:** $S \subseteq \mathbb{C}, z \in \mathbb{C} \Rightarrow [z \in \bar{S} \Leftrightarrow \exists \text{ sequence } (z_n)_{n \in \mathbb{N}} \in S \text{ s.t. } \lim z_n = z]$

4. **Bolzano-Weierstrass:** Every bounded sequence in \mathbb{C} has a convergent subsequence.

Complex Functions: $\forall f : \mathbb{C} \rightarrow \mathbb{C}$ we can write it as: $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ where $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$

Limit of Function: $a_0 \in \mathbb{C}$ is the limit of f at z_0 if: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |z - z_0| < \delta \Rightarrow |f(z) - a_0| < \varepsilon$ limit rules 依旧成立

Lemma|Important: $\lim_{z \rightarrow z_0} f(z) \Leftrightarrow \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = Re(a_0)$ and $\lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = Im(a_0)$

Useful Formula: $\lim_{z \rightarrow z_0} g(\bar{z}) = \lim_{z \rightarrow \bar{z}_0} g(z)$ $\lim_{z \rightarrow z_0} \overline{f(z)} = \overline{\lim_{z \rightarrow z_0} f(z)}$

continuous of Function: f is continuous at z_0 if: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$ continuous rules 依旧成立

1. **Lemma|Important:** f is continuous at $z_0 \Leftrightarrow u, v$ are continuous at (x_0, y_0)

2. **'Extreme Value Theorem':** f is continuous on a closed and bounded set $S \subseteq \mathbb{C}$, then $f(S)$ is closed and bounded.

3. **Lemma|continuous \Leftrightarrow open:** f is continuous $\Leftrightarrow \forall$ open set U , preimage $f^{-1}(U) := \{z \in \mathbb{C} | f(z) \in U\}$ is open.

2.2 Differentiable | Holomorphic Function | C-R Equation

Differentiable: Let $z_0 \in \mathbb{C}$ and $U \subseteq \mathbb{C}$ be neighborhood of z_0 , then $f : U \rightarrow \mathbb{C}$ is differentiable at z_0 if: $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

I. f is differentiable $\Rightarrow f$ is continuous. **II. Holomorphic \Leftrightarrow Differentiable + neighborhood** (除非是一个点时不成立, $|z|$) diff rules + chain rule 成立

Cauchy-Riemann Equations: If $z_0 = x_0 + iy_0, f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 \Rightarrow u_x = v_y, v_x = -u_y$ at (x_0, y_0) .

If $z_0 = x_0 + iy_0, f = u + iv$ satisfies: $^1 u, v$ are continuously differentiable on a neighborhood of (x_0, y_0) and:

$^2 u, v$ satisfies Cauchy-Riemann Equations at (x_0, y_0) . $\Rightarrow f$ is differentiable at z_0 .

ps: 常见可导复数函数: $\exp(z), \sin z, \cos z, \log z, z^\alpha$, polynomial, $\sinh, \cosh, \Gamma(z), |z|^2$ (at 0), constant **ps:** 常见不可导复数函数: $\bar{z}, |z| \cdot \bar{z}, Re(z), Im(z), Arg(z)$

Harmonic Function: $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ is harmonic if: $\forall (x, y) \in \mathbb{R}^2 \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$ (Laplace Equation)

Lemma: If $f = u + iv$ is holomorphic on \mathbb{C} (and u, v are twice continuously differentiable) 可以不用, $\Rightarrow u, v$ are harmonic. $\ominus (u, v \text{ harmonic} + CR \Leftrightarrow f \text{ holomorphic})$

Harmonic Conjugate: Let $u, v : U \rightarrow \mathbb{R}, U \subseteq \mathbb{R}^2$ be harmonic functions. u, v are harmonic conjugate if: $f = u + iv$ is holomorphic on U .

Properties of Polynomial: The domain of rational function and polynomial are always open. **Lemma:** If $P(z_0) = 0$ then $P(\bar{z}_0) = 0$

First-order Operator ∂ : $\partial := \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$ $\bar{\partial} := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$ $|| f = u + iv$ satisfies C-R Equations $\Leftrightarrow \bar{\partial} f = 0$

sin/cos Functions: $\sin z := \frac{e^{iz} - e^{-iz}}{2i}$ $\cos z := \frac{e^{iz} + e^{-iz}}{2}$ **Exponential Function:** $\exp(z) = e^x(\cos(y) + i \sin(y))$

1. $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$ $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$

2. $\sin(z + w) = \sin(z) \cos(w) + \cos(z) \sin(w)$ $\cos(z + w) = \cos(z) \cos(w) - \sin(z) \sin(w)$

3. $\sin^2 z + \cos^2 z = 1$ $\sin(z + \frac{\pi}{2}) = \cos(z)$ $\sin(z + 2k\pi) = \sin(z)$ $\cos(z + 2k\pi) = \cos(z)$ * $\sin z, \cos z$ NOT bounded.

Hyperbolic Functions: $\sinh z := \frac{\exp(z) - \exp(-z)}{2}$ $\cosh z := \frac{\exp(z) + \exp(-z)}{2}$ $|| \sinh(iz) = i \sin z$ $\cosh(iz) = \cos z$

Logarithm: Define multivalued function: $\log z := \{w \in \mathbb{C} : \exp w = z\}$ **Principal Branch:** $Log(z) := \ln |z| + i Arg(z)$

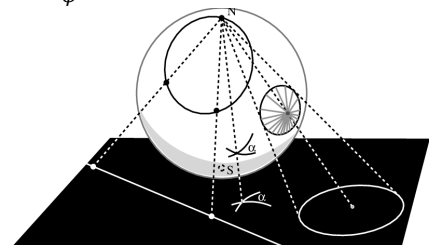
1. **I.** $\log(z) = \ln |z| + i \arg z = \{\ln |z| + i Arg(z) + i 2\pi k : k \in \mathbb{Z}\}$ **II.** $\log(zw) = \log(z) + \log(w)$ **III.** $\log(1/z) = -\log(z)$

2. **Branch of Logarithm:** $Log_\phi(z) := \ln |z| + i Arg_\phi(z)$ $Log_\phi(z)$ is holomorphic on D_ϕ

3. If $g : U \rightarrow \mathbb{C}$, then $Log_\phi(g(z))$ is holomorphic on $g^{-1}(D_\phi) \cap U$

4. $Log(z)$ not continuous on \mathbb{C} . $Log(z)$ not continuous on $Re(z) \leq 0, Im(z) = 0$.

Remark: $\log(x) + \log(x) \neq 2 \log(x)$



Branch Cut|Cut Plane: $Branch\ Cut\ L_{z_0, \phi} := \{z \in \mathbb{C} : z = z_0 + re^{i\phi}, r \geq 0\}$

· **Cut Plane:** $D_{z_0, \phi} := \mathbb{C} \setminus L_{z_0, \phi} \quad L_{\phi} = L_{0, \phi}; D_{\phi} = D_{0, \phi}$

· If $Log_{\phi}(z)$ is holomorphic on D_{ϕ} , then $Log_{\phi}(z - a)$ is holomorphic on $D_{a, \phi}$

Branch of Argument: $Arg_{\phi}(z) := z$ 的辐角, 但是角度限制在: $\phi < Arg_{\phi}(z) \leq \phi + 2\pi$. ps: $Arg_{-\pi}(z) = Arg(z)$

$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$
z^n	nz^{n-1}	$\exp(z)$	$\exp(z)$	$\sin(z)$	$\cos(z)$	$\cos(z)$	$-\sin(z)$	$\sinh(z)$	$\cosh(z)$	$\cosh(z)$	$\sinh(z)$	$Log_{\phi} z$	$\frac{1}{z} \quad z \in D_{\phi}$

Complex Powers: $z^{\alpha} := \{\exp(\alpha w) : w \in \log(z)\} = \{\exp[\alpha(\ln|z| + iArg(z) + i2k\pi)] : k \in \mathbb{Z}\}$ $\frac{d}{dz} z^{\alpha} = \alpha z^{\alpha-1} \quad z \in D_{\phi}$

- I. If $\alpha \in \mathbb{Z}$, there is one value of z^{α} II. If $\alpha = \frac{p}{q}$, $\gcd(p, q) = 1, p, q \in \mathbb{Z}, q \neq 0$, there are exactly q values of z^{α}
- III. If α is irrational or non-real, there are infinitely values of z^{α} IV. $1^{1/q}, q \in \mathbb{Z}, q \neq 0$ is $\{1, w, \dots, w^{q-1}\}, w = \exp(i2\pi/q)$
- V. We prefer use $\exp(z)$ to denote single-valued function, and e^z to denote multi-valued function.

Principal Branch: $z^{\alpha} := \exp(\alpha Log(z))$ **Operation:** $z^{\alpha} z^{\beta} = z^{\alpha+\beta}$ (Using Principal Branch) **NB:** $(z_1 z_2)^{\alpha} \neq z_1^{\alpha} z_2^{\alpha}; (z^{\alpha})^{\beta} \neq z^{\alpha\beta}$

3 Conformal Maps and Mobius Transformations

Conformal: Let U be open set and $f : U \rightarrow \mathbb{C}$. Then f is conformal iff: f preserves angles. i.e. 任意两条曲线/直线之间的角度在 f 作用下不变.

Important Theorem: If $f : U \rightarrow \mathbb{C}$ is holomorphic, then $\forall z_0 \in U, f'(z_0) \neq 0, f$ preserves angles.

i.e. \forall curves C_1, C_2 in U . If C_1, C_2 intersecting at a point $z_0 \in U$. C_1, C_2 在 z_0 切线的夹角与 $f(C_1), f(C_2)$ 在 $f(z_0)$ 切线的夹角一样.

Extended Complex Plane: $\tilde{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and define that $a + \infty = \infty, b \cdot \infty = \infty, \frac{b}{\infty} = 0, \frac{\infty}{\infty} = 0$.

Riemann Sphere: Consider $(X, Y, Z) \in \mathbb{R}^3: {}^1z = X + iY \in \mathbb{C}$ is the point $(X, Y, 0)$ and ${}^2Z = 0$ is the complex plane.

1. Define the Riemann Sphere: $S^2 := \{(X, Y, Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1\}$ and consider the **North Pole** is point $N := (0, 0, 1)$

2. Define $\phi : \mathbb{C} \rightarrow S^2$ by N 点与 $z = (X, Y, 0)$ 点连线与 S^2 的交点为 $\phi(z)$ Thus $\lim_{|z| \rightarrow \infty} \phi(z) = N \quad \phi(\infty) := N$

3. Calculation shows that: $\phi(z) = \phi(x + iy) = \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right)$ $\psi(X, Y, Z) = \begin{cases} \frac{X+iY}{1-Z}, & (X, Y, Z) \neq N \\ \infty, & (X, Y, Z) = N \end{cases}$

Remark: $\phi : \tilde{\mathbb{C}} \rightarrow S^2$ is bijection and it's inverse $\psi : S^2 \rightarrow \tilde{\mathbb{C}}$ is the **stereographic projection**

4. Stereographic projection $\psi(X, Y, Z)$ maps a circle to either a circle or a straight line. (见上图)

Mobius Transformation: A Mobius Transformation is a function form: $f(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}; ad \neq bc$

1. **Remark:** $g(z) = \frac{\frac{1}{\sqrt{ad-bc}}(az+b)}{\frac{1}{\sqrt{ad-bc}}(cz+d)}$ satisfies $ad - bc = 1$ | If a, b, c, d defined a mobius transformation, then $\lambda a, \lambda b, \lambda c, \lambda d$ also.

2. **Useful Property:** For $f(z) = \frac{az+b}{cz+d}, M = \frac{1}{ad-bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $f^{-1}(z) = \frac{dz-b}{-cz+a}$ | Why $ad - bc \neq 0: f'(z) \neq 0 \forall z \in \mathbb{C}$ iff $ad - bc \neq 0$

3. For Complex Matrix: $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(M) = ad - bc = 1$. We define $f_M = \frac{az+b}{cz+d}$ I. $f_{M_1 M_2} = f_{M_1} f_{M_2}$
II. $f_{M^{-1}} = f_M^{-1}$

4. Extended $f(z)$ from \mathbb{C} to $\tilde{\mathbb{C}}$ by: $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$

5. **Translation:** $f(z) = z + b \Leftrightarrow \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ **Rotation:** $f(z) = az, a = e^{i\theta} (|a| = 1) \Leftrightarrow \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & -e^{i\theta/2} \end{pmatrix}$ **Dilation:** $f(z) = rz, r > 0 \Leftrightarrow \begin{pmatrix} \sqrt{r} & 0 \\ 0 & 1/\sqrt{r} \end{pmatrix}$

Inversion: $f(z) = 1/z \Leftrightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ f fixes the point at infinity: If $f(\infty) = \infty$ ps: 除了 inversion 其他都是 fix the point at infinity.

6. **Theorem:** $f(z) = \frac{az+b}{cz+d}$ be a Mobius Transformation. \Rightarrow 1 If $f(\infty) = \infty: f$ is a composition of finite Translation, Rotation, Dilation $\Rightarrow c = 0, f(z) = \frac{a}{d}z + \frac{b}{d}$

2 If $f(\infty) < \infty: f$ is composition of finite Translation, Rotation, Dilation and only one inversion. $\Rightarrow f(z) = \frac{(bc-ad)/c^2}{z+d/c} + \frac{a}{c}$

Properties of Mobius Transformation: Important: ★ Möbius transformations map circlines to circlines. ★

1. For mobius transformation $f(z) = \frac{az+b}{cz+d}$, if: $\exists z_1, z_2, z_3 \in \mathbb{C}$ distinct points. $f(z_1) = z_1, f(z_2) = z_2, f(z_3) = z_3 \Rightarrow f$ is identity.

2. If $z_1, z_2, z_3 \in \tilde{\mathbb{C}}$ distinct points. $\exists!$ mobius transformation $f(z)$ s.t. $f(z_1) = 1, f(z_2) = 0, f(z_3) = \infty$

3. If $(z_1, z_2, z_3), (w_1, w_2, w_3) \in \tilde{\mathbb{C}}$ distinct points. Then $\exists!$ mobius transformation $f(z)$ s.t. $f(z_i) = w_i, \forall i \in \{1, 2, 3\}$

ps:Method to construct 2: If $z_i < \infty, f(z) = \frac{z_1 - z_3}{z_1 - z_2} \cdot \frac{z - z_2}{z - z_3}$ If $z_i = \infty, f(z) = \frac{z - z_2}{z - z_3}, z_1 = \infty \Rightarrow f(z) = \frac{z_1 - z_3}{z_1 - z_2}, z_2 = \infty; f(z) = \frac{z - z_2}{z_1 - z_2}, z_3 = \infty$

ps:Method to construct 3: For 3: Let $f := h^{-1} \circ g$ where $g(z_i), h(w_i) = \{1, 0, \infty\}$ like part 2.

Geometric Meaning by using Mobius Transformation|Exponential|Complex Powers:

1. **Rotation:** $f(z) = e^{-i\theta} z$ is a rotation by θ (anticlockwise) about the origin. Specially, $f(z) = iz$ is a rotation by $\frac{\pi}{2}$

2. **Extend:** $f(z) = \exp(\alpha z)$ 原来的图像进行拉长, 以及旋转 (如果带 θ 带 i 时) e.g. $\{z : 0 < Im(z) < 1\}$ 可以被拉长到 $\{z : 0 < Im(z)\}$

3. **Angle Extend:** $f(z) = z^{\alpha}$ 原来的图像辐角范围收缩或放大

4. **Circlines:** I. 实轴到单位圆, $f(z) = \frac{z-i}{z+i}$ II. 单位圆到实轴, $f(z) = i \frac{1+z}{1-z}$ III. 虚轴到单位圆, $f(z) = \frac{z-1}{z+1}$ IV. 单位圆到虚轴, $f(z) = \frac{1+z}{1-z}$

Cross-Ratio: cross-ratio $[z_1, z_2, z_3, z_4] := f(z_1)$ where f is mobius transformation s.t. $f(z_2) = 1, f(z_3) = 0, f(z_4) = \infty$

1. **Formulas:** $[z_1, z_2, z_3, z_4] = \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3}$ $[\infty, z_2, z_3, z_4] = \frac{z_2 - z_4}{z_2 - z_3}$ $[z_1, \infty, z_3, z_4] = \frac{z_1 - z_3}{z_1 - z_4}$ $[z_1, z_2, \infty, z_4] = \frac{z_2 - z_4}{z_1 - z_4}$ $[z_1, z_2, z_3, \infty] = \frac{z_1 - z_3}{z_2 - z_3}$

2. **Theorem:** If f is a mobius transformation, $[f(z_1), f(z_2), f(z_3), f(z_4)] = [z_1, z_2, z_3, z_4]$ z_i 's in this "small section" are distinct.

3. **Application:** \exists mobius transformation f s.t. $f(z_1) = w_1, f(z_2) = w_2, f(z_3) = w_3, f(z_4) = w_4 \Leftrightarrow [w_1, w_2, w_3, w_4] = [z_1, z_2, z_3, z_4]$

ps: \uparrow 原因: $(\Rightarrow), [z_1, z_2, z_3, z_4] = [f(z_1), f(z_2), f(z_3), f(z_4)] = [w_1, w_2, w_3, w_4]; (\Leftarrow), [w_1, w_2, w_3, w_4] = [z_1, z_2, z_3, z_4] \Rightarrow \exists f: w_2 \rightarrow 1, w_3 \rightarrow 0, w_4 \rightarrow \infty; g: z_2 \rightarrow 1, z_3 \rightarrow 0, z_4 \rightarrow \infty; f(w_1) = g(z_1) = g^{-1} \circ f$

4 Complex Integration

4.1 Line Integral

Integrable: $f : [a, b] \rightarrow \mathbb{C}$ as $f(t) = u(t) + iv(t)$ is integrable if: u, v are both integrable on $[a, b]$ and for $f(t)$:

1. **Def:** $\int_a^b f(t)dt := \int_a^b u(t)dt + i \int_a^b v(t)dt$
2. **Property I.** $\alpha f + \beta g$ is integrable and $\int_a^b (\alpha f + \beta g)dt = \alpha \int_a^b f(t)dt + \beta \int_a^b g(t)dt$
3. **Property II.** If f is continuous and $\frac{dF}{dt} = f(t)$ for $F : [a, b] \rightarrow \mathbb{C}$ is differentiable. $\Rightarrow \int_a^b f(t)dt = F(b) - F(a)$
4. **Property III.** If f is continuous $\Rightarrow \left| \int_a^b f(t)dt \right| \leq \int_a^b |f(t)|dt$.

Parameters Curves: A parametrized curve connecting z_0 to z_1 is a continuous function $\gamma : [t_0, t_1] \rightarrow \mathbb{C}$ s.t. $\gamma(t_0) = z_0, \gamma(t_1) = z_1$

If $z_0 = x_0 + iy_0, z_1 = x_1 + iy_1$, then $\gamma(t) = x(t) + iy(t)$ continuous functions. s.t. $x(t_0) = x_0, x(t_1) = x_1, y(t_0) = y_0, y(t_1) = y_1$

Regular: γ is regular if $\gamma'(t) \neq 0$ for all $t \in [t_0, t_1]$ **Remark:** Curve $\gamma([t_0, t_1]) = \Gamma$ is closed and bdd.

Integral Along Curve: Let $\gamma : [t_0, t_1] \rightarrow \mathbb{C}$ be a regular curve s.t. $\gamma([t_0, t_1]) = \Gamma$ and $f : \Gamma \rightarrow \mathbb{C}$ is continuous.

1. **★ Def:** $\int_{\Gamma} f(z)dz := \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)dt$ ★
2. **Circle at zero:** Circle Centred at 0 with radius R : $\gamma : [0, 1] \rightarrow \mathbb{C}$ by $\gamma(t) = R \exp(2\pi it)$
3. **Constant Function:** If $f(z) = c$; $\gamma : [a, b] \rightarrow \mathbb{C}$. Then $\int_{\Gamma} f(z)dz = \int_a^b c \cdot \gamma'(z)dz = c \cdot (\gamma(b) - \gamma(a))$

Arclength of Curve: Let $\gamma : [t_0, t_1] \rightarrow \mathbb{C}$ be a regular curve. $\gamma(t) = x(t) + iy(t)$ Then arclength $\ell(\Gamma) := \int_{t_0}^{t_1} |\gamma'(t)|dt = \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2}dt$

Lemma: If Γ is an arc of a circle of radius r traced through angle θ , then $\ell(\Gamma) = r\theta$ (扇形弧长)

Properties of Integral Along Curve: Let Γ be a regular curve and $f, g : \Gamma \rightarrow \mathbb{C}$ be continuous, and $\alpha, \beta \in \mathbb{C}$

1. **M-L Lemma:** $|\int_{\Gamma} f(z)dz| \leq \max_{z \in \Gamma} |f(z)|\ell(\Gamma)$
2. **Lemma:** $\int_{\Gamma} (\alpha f + \beta g)dz = \alpha \int_{\Gamma} f(z)dz + \beta \int_{\Gamma} g(z)dz$ $\int_{-\Gamma} f(z)dz = -\int_{\Gamma} f(z)dz$ Here: $\tilde{\gamma}(t) := \gamma(b-t)$ have $\tilde{\gamma}([a, b]) = -\Gamma$
3. **Change of Variables:** If $\gamma : [a, b] \rightarrow \Gamma$, and $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \rightarrow \Gamma$ are two parametrizations of Γ ;
 $\exists \lambda : [\tilde{a}, \tilde{b}] \rightarrow [a, b]$ s.t. $\lambda'(t) > 0$ and $\tilde{\gamma}(t) = \gamma(\lambda(t))$ (防止曲线回头) $\Rightarrow \int_a^b f(\gamma(t))\gamma'(t)dt = \int_{\tilde{a}}^{\tilde{b}} f(\tilde{\gamma}(t))\tilde{\gamma}'(t)dt$.
(特别的, 如果 Γ 是 closed, f 在 Γ 上的积分与哪里选择起/终点无关)

Contour: A curve Γ is contour if it's finite union of regular curves $\Gamma_1, \Gamma_2, \dots, \Gamma_n$. Each Γ_i is regular component of Γ

Contour Integral: If $f : \Gamma \rightarrow \mathbb{C}$ is continuous and Γ is a contour. Then $\int_{\Gamma} f(z)dz := \sum_{i=1}^n \int_{\Gamma_i} f(z)dz$

4.2 Independent of Path

Domain: $D \subseteq \mathbb{C}$ is a domain if it's open and connected. (i.e. 任意两点都存在 contour(Γ) 将其连接, 并都在 D 里面)

Lemma: Let $D \subseteq \mathbb{C}$ be a domain. If $u : D \rightarrow \mathbb{C}$ is differentiable, with $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$. $\Rightarrow u$ is constant on D . \Downarrow Clearly, F is holomorphic

Antiderivative: Let D be a domain. For $f : D \rightarrow \mathbb{C}$ be continuous and $F : D \rightarrow \mathbb{C}$ s.t. $F'(z) = f(z)$ for all $z \in D$. Then F is an antiderivative of f .

Fundamental Theorem of Calculus: D domain; $f : D \rightarrow \mathbb{C}$ continuous; $F : D \rightarrow \mathbb{C}$ antiderivative of f . Contour Γ in D connecting z_0 to z_1 .

Then $\int_{\Gamma} f(z)dz = F(z_1) - F(z_0)$

1. D domain, if $f : D \rightarrow \mathbb{C}$ is holomorphic and $f'(z) = 0, \forall z \in D$. $\Rightarrow f$ is constant on D .
2. **Path-Independence Lemma:** D domain, f continuous on D . Then:
 f has antiderivative on $D \Leftrightarrow \int_{\Gamma} f(z)dz = 0 \forall$ closed contours Γ in $D \Leftrightarrow \int_{\Gamma} f(z)dz$ is path-independent.

4.3 Cauchy's Theorem

Simple: A contour Γ is simple if it doesn't intersect itself except at the endpoints. **Loop:** A contour Γ is a loop if it's simple and $\Gamma(t_0) = \Gamma(t_1)$

Jordan Curve Theorem: $\forall \Gamma$ be Loop **Interior** $Int(\Gamma)$: Γ 的内部, bounded. **Exterior** $Ext(\Gamma)$: Γ 的外部, unbounded. **Boundary** Γ 的边界, Γ itself.

And $Int(\Gamma)$ is bounded domain $Ext(\Gamma)$ is unbounded domain. **Remark:** $Int(\Gamma)$ is open and $Ext(\Gamma)$ is open also.

· **Common Loop:** $C_r(z_0)$ is a circle of radius r centered at z_0 Corresponding $\gamma(t) = z_0 + r \exp(2\pi it) t \in [0, 1]$

· **Positive-Oriented:** If Γ is a loop, then Γ is positive-oriented if: 按方向走时, 内部在左边 (as we move along the curve in the direction of parametrization, the interior is on the left-hand side.)

Remark: Unless otherwise stated, all loops shall be positively-oriented.

Simply-Connected: A domain D is simply-connected if: \forall loop Γ in $D, Int(\Gamma) \subseteq D$ (即没有洞的 domain/open set)

Cauchy Integral Theorem: If Γ is Loop, f is holomorphic in $Int(\Gamma) \cup \Gamma$ (Inside and on Γ), then $\int_{\Gamma} f(z)dz = 0$

Corollary: If D is simply-connected domain and $f : D \rightarrow \mathbb{C}$ is holomorphic on D . Then $f(z)$ has antiderivative on D . ★

即: 在没有洞的 open set 上如果都是 holomorphic, 那么都有 antiderivative.

Remark: 如果 loop Γ 上和以內没有穿过任何非 holomorphic 点, 那么 $f(z)$ 的积分值不变.

Theorem: Let $z_0 \in \mathbb{C}, \Gamma$ be Loop. Then $\int_{\Gamma} \frac{1}{z-z_0} = \begin{cases} 2\pi i & \text{if } z_0 \in Int(\Gamma) \\ 0 & \text{otherwise} \end{cases}$

Deformation Theorem: Let Γ_1, Γ_2 be loops, and f is holomorphic on $(Int(\Gamma_1) \setminus Int(\Gamma_2)) \cup (Int(\Gamma_2) \setminus Int(\Gamma_1))$, Γ_1, Γ_2 . Then $\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$

即: 两个 loop Γ_1 和 Γ_2 及它们围成的区域中 (除公共区域) 上, 函数 $f(z)$ 全纯, 那么它们的路径积分相等 ps: 可以是内外 loop, 也可以是交叉的 loop

4.4 Cauchy's Integral Formula

Cauchy's Integral Formula: Γ Loop, $f(z)$ holomorphic inside and on Γ , $z_0 \in \text{Int}(\Gamma)$, $\Rightarrow f(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z-z_0} dz$

ps: We always use it to calculate: $\int_{\Gamma} \frac{f(z)}{z-z_0} dz$ if $f(z)$ is holomorphic on and inside Γ (loop), and $z_0 \in \text{Int}(\Gamma)$. $\Rightarrow \int_{\Gamma} \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$

Theorem: D be domain, Γ be contour in D , $g : D \rightarrow \mathbb{C}$ continuous on Γ , Then:

Function Defined as: $G : D \setminus \Gamma \rightarrow \mathbb{C}$ by $G(z) = \int_{\Gamma} \frac{g(w)}{w-z} dw$ is holomorphic on $D \setminus \Gamma$ and $G'(z) = \int_{\Gamma} \frac{g(w)}{(w-z)^2} dw$

Moreover, function $H : D \setminus \Gamma \rightarrow \mathbb{C}$ by $H(z) = \int_{\Gamma} \frac{g(w)}{(w-z)^{n+1}} dw$ is holomorphic on $D \setminus \Gamma$ and $H'(z) = n \int_{\Gamma} \frac{g(w)}{(w-z)^{n+1}} dw$

★ **Corollary:** If D is domain and f is holomorphic on D , then f is infinitely differentiable on D , and all of its derivatives are holomorphic on D .

Generalized Cauchy's Integral Formula: Γ Loop, $f(z)$ holomorphic inside and on Γ , $z \in \text{Int}(\Gamma)$, $n \in \mathbb{N}$, $\Rightarrow f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w-z)^{n+1}} dw$

ps: We always use it to calculate: $\int_{\Gamma} \frac{f(z)}{(z-z_0)^{n+1}} dz$ if $f(z)$ is holomorphic on and inside Γ (loop), and $z_0 \in \text{Int}(\Gamma)$. $\Rightarrow \int_{\Gamma} \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$

Morera Theorem: Let D is domain, if $f : D \rightarrow \mathbb{C}$ is continuous and $\int_{\Gamma} f(z) dz = 0$ for all loop Γ in D . $\Rightarrow f$ is holomorphic on D .

4.5 Liouville's Theorem, FTA and Maximum Modulus Principle

Useful Formula: If $^1 D$ domain; $^2 \exists R > 0, z_0 \in \mathbb{C}$ s.t. $\overline{D}_R(z_0) \subseteq D$; $^3 f$ is holomorphic on D

1. Then $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + R \exp(it)) dt$.

2. If $|f(z)| < M, \forall z \in D$. Then $|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}$.

3. If $\max_{z \in \overline{D}_R(z_0)} |f(z)| = |f(z_0)|$. Then f is constant on $\overline{D}_R(z_0)$.

Criteria Constant Function: If $f : \mathbb{C}$ (or D) $\rightarrow \mathbb{C}$ is holomorphic and bounded on: D domain

1. **Liouville's Theorem:** $|f(z)| < M$ bounded on $\forall z \in \mathbb{C}$, $\Rightarrow f(z)$ is constant. ps: 此定理通过 Generalised Cauchy Formula + ML Lemma 得到

2. **Maximum Modulus Principle:** $|f(z)|$ bounded on $\forall z \in D$, and $|f(z)|$ has maximum at $z_0 \in D$. $\Rightarrow f(z)$ is constant.

Remark I: 意思是对于 $f(z)$ holomorphic 且在 domain 上 bounded, 如果 $|f(z)|$ 在 domain 上有最大值 (非边界), 那么 $f(z)$ 是 constant.

Remark II: ★ If function f is holomorphic on a bounded domain D and continuous up to the boundary of D .

$\Rightarrow f$ has maximum modulus on the boundary of D .

若 f 在 D 内全纯, 且在 ∂D 上连续, 则 f 在 $D \cup \partial D$ 最大值一定在边界上. 特别地, 若 f 不是常数, 则最大值只能在边界上取到.

3. **Maximum/Minimum Principle for Harmonic Functions:** If D domain, $\phi : D \rightarrow \mathbb{R}$ is harmonic, and ϕ is bounded above/below on D by M , with $\phi(z_0) = M$ for some $z_0 \in D$. $\Rightarrow \phi$ is constant on D .

Remark: 对于调和函数 $\phi : D \rightarrow \mathbb{R}$, 如果 f 不是常数, 那么最大值/最小值, 只能在边界上取到.

Fundamental Theorem of Algebra: If $P : \mathbb{C} \rightarrow \mathbb{C}$ is a non-constant polynomial. $\Rightarrow P$ has a at least one root in \mathbb{C} .

5 infinity Series

5.1 Basic Properties, Convergence Test, Series of Functions and M-Test

Partial Sum: A Series $\sum_{n=0}^{\infty} z_n$ is convergent if partial sums $S_n = \sum_{k=0}^n z_k$ is convergent. **Remark:** $\sum z_n$ is convergent $\Rightarrow \lim z_n = 0$.

Comparison Test: If $|z_n| \leq M_n$ for all $n \in \mathbb{N}$ and $\sum M_n$ is convergent. $\Rightarrow \sum z_n$ is convergent.

Lemma|'Geometric Series': For $c \in \mathbb{C}$, $\sum_{n=0}^{\infty} c^n$ is convergent $\Leftrightarrow |c| < 1$. **Remark:** $\sum_{n=0}^{\infty} c^n = \frac{1}{1-c}$

Ratio Test: For $\sum z_n$, let $L = \lim_{n \rightarrow \infty} \left| \frac{z_{n+1}}{z_n} \right|$. If $L < 1$, then $\sum z_n$ is convergent. If $L > 1$, then $\sum z_n$ is divergent. If $L = 1$, conclude nothing.

Converge Pointwise: Seq $f_n : S \rightarrow \mathbb{C}$ pointwise convergent to $f : S \rightarrow \mathbb{C}$ if $\forall \varepsilon > 0, \forall z \in S, \exists N_{\varepsilon, z} \in \mathbb{N}$ s.t. $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N$

Uniform Convergence: Seq $f_n : S \rightarrow \mathbb{C}$ uniformly convergent to $f : S \rightarrow \mathbb{C}$ if $\forall \varepsilon > 0, \exists N_{\varepsilon} \in \mathbb{N}$ s.t. $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N$ and $\forall z \in S$

1. **Lemma|Continuous:** If $f_n : S \rightarrow \mathbb{C}$ is uniformly convergent and continuous to $f : S \rightarrow \mathbb{C}$, then f is continuous on S .

2. **Lemma|Integral:** If $f_n : S \rightarrow \mathbb{C}$ is uniformly convergent and continuous to $f : S \rightarrow \mathbb{C}$, then $\int_{\Gamma} f_n(z) dz$ convergent to $\int_{\Gamma} f(z) dz$.

3. **Lemma|Integral:** If $f_n : S \rightarrow \mathbb{C}$ is continuous, $\sum_{n=0}^{\infty} f_n(z)$ is uniformly convergent on S , then $\int_{\Gamma} \sum_{n=0}^{\infty} f_n(z) dz = \sum_{n=0}^{\infty} \int_{\Gamma} f_n(z) dz$.

4. **Lemma|Holomorphic:** If D is simply-connected domain, $f_n : D \rightarrow \mathbb{C}$ is holomorphic and uniformly convergent to f . $\Rightarrow f$ holomorphic on D .

Weierstrass M-Test: For $f_n : S \rightarrow \mathbb{C}$, if $\exists M_n \geq 0, n_0 \in \mathbb{N}$ s.t. $|f_n(z)| \leq M_n$ for $\forall z \in S, n \geq n_0$.

If $\sum_{n=0}^{\infty} M_n$ is convergent. $\Rightarrow \sum_{n=0}^{\infty} f_n(z)$ is uniformly convergent on S .

Power Series & Radius of Convergence: Power Series is: $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$. and there is a number $R \in [0, \infty) \cup \{\infty\}$ s.t.

1. The Series is convergent on $D_R(z_0)$.

2. The Series is divergent on $\mathbb{C} \setminus \overline{D}_R(z_0)$.

3. The Series is uniformly convergent on $\overline{D}_r(z_0)$ for all $r \in [0, R)$.

4. **Theorem|Holomorphic:** Then $f(z)$ is holomorphic on $D_R(z_0)$, where R is the radius of convergence.

Remark: By using Ratio Test, we can find $R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$. if this limit exists. 可以取 0 和 ∞

5.2 Taylor Series and Laurent Series

Taylor Series: Let $z_0 \in \mathbb{C}$ and f is holomorphic at z_0 . Then the Taylor Series of f at z_0 is: $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$

1. **Theorem|Convergence:** If f is holomorphic on $D_R(z_0)$, then 1 the Taylor Series of f at z_0 converges to $f(z)$ on $D_R(z_0)$.

2. **Theorem|Convergence:** If f is holomorphic on $D_R(z_0)$, then 2 the Taylor Series of f at z_0 converges uniformly to $f(z)$ on $\overline{D}_r(z_0)$ $r \in [0, R)$.

Analytic: Let U open, $f : U \rightarrow \mathbb{C}$ is analytic if $\forall z \in U, \exists$ some disc centered at z s.t. f can be expressed as a convergent power series centred at z .

Homo \rightarrow Analytic: If f is holomorphic on U , then f is analytic on U .

Properties of Taylor Series|Series: Let $z_0 \in \mathbb{C}, R > 0, f, g$ is holomorphic on $D_R(z_0)$, then for $\star \forall z \in D_R(z_0) \star$:

- Termwise Differentiation:** $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \quad \forall z \in D_R(z_0)$ and $f'(z) = \sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{(n-1)!} (z - z_0)^{n-1} = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(z_0)}{n!} (z - z_0)^n \quad \forall z \in D_R(z_0)$
- Lemma|Linear Combination:** $(\alpha f + \beta g)(z) = \sum_{n=0}^{\infty} \left(\frac{\alpha f^{(n)}(z_0) + \beta g^{(n)}(z_0)}{n!} \right) (z - z_0)^n \quad \forall z \in D_R(z_0)$
- Lemma|Product:** $(fg)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{f^{(k)}(z_0) g^{(n-k)}(z_0)}{k!(n-k)!} \right) (z - z_0)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^n \binom{n}{k} f^{(k)}(z_0) g^{(n-k)}(z_0) \right) (z - z_0)^n \quad \forall z \in D_R(z_0)$
- Uniqueness of Taylor series:** $f(z)$ has a power series representation at z_0 , with radius of convergence $R > 0$. \Rightarrow Then it must be the Taylor series of f at z_0 , and will equal to $f(z)$ on $D_R(z_0)$.
i.e. 假设某个函数 $f(z)$ 能够由幂级数展开, 那么这个展开是唯一的, 且在收敛区间内等于 $f(z)$.

Laurent Series: A Laurent Series centered at z_0 is the series form: $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$

Convergence: The Laurent Series is convergent if both $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $\sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$ are convergent.

Remark: If radius of convergence of $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $\sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$ are R and S . \Rightarrow Laurent Series convergent on $S^{-1} < |z - z_0| < R$.

Annulus: Open annulus: $A_{r,R}(z_0) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$ **Closed annulus:** $\bar{A}_{r,R}(z_0) = \{z \in \mathbb{C} : r \leq |z - z_0| \leq R\}$

Laurent Series|For function: Let $z_0 \in \mathbb{C}$, $0 \leq r < R \leq \infty$, f is holomorphic on $A_{r,R}(z_0)$. Then:

- f can be expressed as a Laurent Series on $A_{r,R}(z_0)$, ¹ convergent on $A_{r,R}(z_0)$. ² uniformly convergent on $\bar{A}_{r',R'}(z_0)$ for $r < r' \leq R' < R$.
- Coefficient:** $a_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz$ for any loop Γ in $A_{r,R}(z_0)$ and contain z_0 in its interior.
- Uniqueness:** If $f(z)$ has a Laurent Series on $A_{r,R}(z_0)$, then it must be the Laurent Series of f on $A_{r,R}(z_0)$, and will equal to $f(z)$ on $A_{r,R}(z_0)$.

5.3 Singularities and Identity Theorem

Singularity: A point $z_0 \in \mathbb{C}$ is a singularity of f if f is not holomorphic at z_0 . **Zero:** A point $z_0 \in \mathbb{C}$ is a zero of f if $f(z_0) = 0$.

Isolated Singularity: A singularity z_0 of f is isolated if $\exists R > 0$ s.t. f is holomorphic on $D'_R(z_0)$.

Isolated Zero: A zero z_0 is isolated if $\exists R > 0$ s.t. $f(z) \neq 0$ for all $z \in D'_R(z_0)$

Zero of finite order: If $f(z_0) = f'(z_0) = \dots = f^{(n-1)}(z_0) = 0$ and $f^{(n)}(z_0) \neq 0$, then z_0 is a zero of order n .

Simple Zero: A zero of order 1 is called a simple zero. i.e. $f(z_0) = 0$ and $f'(z_0) \neq 0$.

Properties of Zeros: Let $z_0 \in \mathbb{C}$, U be a neighborhood of z_0 , f is holomorphic on U .

- If z_0 is a zero of finite order, then z_0 is isolated (zero).
- If \exists distinct points $z_n \in U$ s.t. $z_n \rightarrow z_0$ and $f(z_n) = 0$. $\Rightarrow \exists R > 0$ s.t. $f(z) = 0$ for all $z \in D_R(z_0)$ (identically zero on some disc centred at z_0).

Remark: If \exists distinct points $z_n \in U$ s.t. $z_n \rightarrow z_0$ and z_n Zero, then z_0 cannot be a isolated zero.

Removable|Order|Essential Singularity: Let $z_0 \in \mathbb{C}$ is an isolated singularity of a function f , which is holomorphic on $D'_R(z_0)$.

- Let the Laurent Series of f at z_0 be $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ valid on $A_{0,R}(z_0)$
- removable singularity:** If $a_n = 0$ for all $n < 0$, then z_0 is a removable singularity of f . (i.e. 负的部分都是 0, 和泰勒展开很像)
- Pole of Order:** If $a_{-m} \neq 0$ and $a_{-n} = 0, \forall n > m$, then z_0 is a pole of order m of f . (i.e. 有限个负的非 0 项, 且最小的非 0 项是 $-m$)
- Essential Singularity:** If $a_n \neq 0$ for infinitely many $n < 0$. $\Rightarrow z_0$ is an essential singularity of f . (i.e. 无限多个负的非 0 项)

Remark: Poles of order 1, 2, and 3 are also known as a simple, double, and triple poles, respectively.

Properties of Singularity: Let $z_0 \in \mathbb{C}$.

- Singularity of rational function|Isolated:** If z_0 is singularity of rational function f , then z_0 is isolated.
- Sequence \rightarrow Isolated:** If \exists distinct points $z_n \in U$ s.t. $z_n \rightarrow z_0$ and z_n Singularity, then z_0 cannot be a removable singularity.
- Extended:** f is holomorphic on $D'_R(z_0)$. If z_0 is a removable singularity, f can be redefined at z_0 to be holomorphic at z_0 . ($f(z_0) = a_0$)
- Functions:** If f, g holomorphic at z_0 , z_0 is a zero of g , with order m . Then:
 - If z_0 is not a zero of $f \Rightarrow \frac{f}{g}$ has a pole of order m at z_0 .
 - If z_0 is a zero of order k of f and $k < m \Rightarrow \frac{f}{g}$ has a pole of order $m - k$ at z_0 .
 - If z_0 is a zero of order k of f and $k \geq m \Rightarrow \frac{f}{g}$ has a removable singularity at z_0 .

***Singularities at Infinity:** In order to study the singularities of $f(z)$ at infinity, we can consider $f(\frac{1}{z})$ at point $z = 0$.

Analytic Continuation: Let $D \subseteq \tilde{D} \subseteq \mathbb{C}$, $f : D \rightarrow \mathbb{C}$ is holomorphic, and $F : \tilde{D} \rightarrow \mathbb{C}$ is holomorphic. $F(z) = f(z)$ for all $z \in D$.

Identity Theorem|Disk-zero: Let D domain, $z_0 \in D$, f is holomorphic on D , $f(z) = 0, \forall z \in D_R(z_0) \Rightarrow f(z) = 0, \forall z \in D$.

Corollary|Disk-func: Let D domain, f, g are holomorphic on D , $f(z) = g(z), \forall z \in D_R(z_0) \Rightarrow f(z) = g(z), \forall z \in D$.

Corollary|Sequence-zero: Let D domain, \exists distinct $z_n \in D, z_n \rightarrow z_0 \in D$ s.t. $f(z_n) = 0, \forall n \in \mathbb{N} \Rightarrow f(z) = 0, \forall z \in D$.

Corollary|Sequence-func: Let D domain, \exists distinct $z_n \in D, z_n \rightarrow z_0 \in D$ s.t. $f(z_n) = g(z_n), \forall n \in \mathbb{N} \Rightarrow f(z) = g(z), \forall z \in D$.

6 Residue Theorem

6.1 Residue and Cauchy Residue Theorem

Theorem: Let f be holomorphic on $D'_R(z_0)$. (i.e. z_0 is an isolated singularity of f). Let Γ loop in $D'_R(z_0)$, $z_0 \in \text{Int}(\Gamma)$

Then: $\int_{\Gamma} f(z) dz = 2\pi i a_{-1}$, where a_{-1} is the coefficient of $(z - z_0)^{-1}$ in the Laurent Series of f at z_0 . \leftarrow

Residue: Let f be holomorphic on $D'_R(z_0)$. (i.e. z_0 is an isolated singularity of f). Then residual: $\text{Res}(f, z_0) = a_{-1}$. where $a_{-1} \uparrow$

Properties of Residue: Let $z_0 \in \mathbb{C}$, f is holomorphic on $D'_R(z_0)$.

- If z_0 is a removable singularity, then $\text{Res}(f, z_0) = 0$.
- If z_0 is a pole of order m , then $\text{Res}(f, z_0) = a_{-1}$, where $a_{-1} = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]$.
- If f, g are holomorphic on $D_R(z_0)$, g has a simple zero at z_0 . (i.e. $g(z_0) = 0$ and $g'(z_0) \neq 0$). Then $\text{Res}\left(\frac{f}{g}, z_0\right) = \frac{f(z_0)}{g'(z_0)}$.

Cauchy Residue Theorem: Let Γ loop, f is holomorphic inside and on Γ , except for a finite isolated singularities $z_1, z_2, \dots, z_k \in \text{Int}(\Gamma)$
 Then: $\int_{\Gamma} f(z)dz = 2\pi i \sum_{j=1}^k \text{Res}(f, z_j)$ where $\text{Res}(f, z_j)$ is the residue of f at z_j .

6.2 Argument Principle and Rouché's Theorem

Meromorphic: Let D domain, $f: D \rightarrow \mathbb{C}$ is meromorphic if $\forall z \in D$ f is holomorphic or has a pole of some finite order.

Lemma|Finite Zero & Pole: Let D domain, Γ loop in D , f is meromorphic on D , $f \neq 0$. $\Rightarrow f$ has finite number of zeroes and poles $\in \text{Int}(\Gamma)$.

Definition of N_0 and N_{∞} : Let Γ loop, f be meromorphic on $\text{Int}(\Gamma)$, with zeros: z_1, \dots, z_n and poles: p_1, \dots, p_m .

Then: $N_0(f) := \sum_j \text{order of } z_j$ and $N_{\infty}(f) := \sum_j \text{order of } p_j$.

Argument Principle: Let Γ loop, f is meromorphic in $\text{Int}(\Gamma)$, f is holomorphic and non-zero on Γ . Then: $\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} dz = N_0(f) - N_{\infty}(f)$

Corollary: Let Γ loop, if f is holomorphic inside and on Γ , and non-zero on Γ . 可能在里面为 0 Then: $\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} dz = N_0(f)$

Rouché's Theorem: Let Γ loop, f, g are holomorphic inside and on Γ . If $|f(z) - g(z)| < |f(z)|$ for all $z \in \Gamma$. $\Rightarrow N_0(f) = N_0(g)$.

Open Mapping Theorem: D domain, f non-constant and holomorphic on $D \Rightarrow f(D)$ is open in \mathbb{C} .

Corollary: Let D domain, f holomorphic on D . If $\text{Re}(f)$ or $\text{Im}(f)$ or $|f|$ or $\text{Arg}(f)$ is constant on D . $\Rightarrow f$ is constant on D .

• **Trigonometric Integrals:** If R is rational function. Then $\int_0^{2\pi} R(\cos \theta, \sin \theta) d\theta = \int_{C_1(0)} f(z) dz$ where $f(z) = \frac{1}{iz} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)$ LHS is real int.

6.3 Improper Integral

Def of Improper Integral: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be continuous, then:

$$\int_{-\infty}^0 f(x) dx = \lim_{R \rightarrow -\infty} \int_R^0 f(x) dx \quad \int_0^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_0^R f(x) dx \quad \int_{-\infty}^{\infty} f(x) dx = \lim_{R_1 \rightarrow -\infty} \lim_{R_2 \rightarrow \infty} \int_{R_1}^{R_2} f(x) dx \quad R_1, R_2 \text{ 不一定一样}$$

Remark: If $\int_{-\infty}^{\infty} f(x) dx$ is convergent, then $\int_{-\infty}^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$.

Cauchy Principal Value of the integral: p.v. $\int_{-\infty}^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$

Jordan Lemma: Let P, Q be polynomial, $\deg(Q) > \deg(P)$. $a \neq 0$ Then: $\lim_{R \rightarrow \infty} \int_{C_R^{\pm}} \frac{P(z)}{Q(z)} \exp(iaz) dz = 0$ for $+$: $a > 0$; $-$: $a < 0$.

Useful Method in Improper Integral: For $\cos(nx)$, consider $\text{Re}(\exp(inx))$; for $\sin(nx)$, consider $\text{Im}(\exp(inx))$. (i.e. $e^{inx} = \cos(nx) + i \sin(nx)$)

Useful Method in Improper Integral: If denominator has $e^x \Rightarrow$ Consider choose $\Gamma: -R \rightarrow R \rightarrow R + 2\pi i \rightarrow -R + 2\pi i \rightarrow -R$. 相当于通过纵向限制 pole 的数量

Def of Improper Integral: Let $c \in (a, b) \subseteq \mathbb{R}$, f is continuous on $[a, b] \setminus \{c\}$, then:

$$\int_a^c f(x) dx = \lim_{r \downarrow 0} \int_a^{c-r} f(x) dx \quad \int_c^b f(x) dx = \lim_{s \downarrow 0} \int_{c+s}^b f(x) dx \quad \int_a^b f(x) dx = \lim_{r \downarrow 0} \int_a^{c-r} f(x) dx + \lim_{s \downarrow 0} \int_{c+s}^b f(x) dx$$

Remark: If $\int_a^b f(x) dx$ is convergent, then $\int_a^b f(x) dx = \lim_{r \downarrow 0} \left(\int_a^{c-r} f(x) dx + \int_{c+r}^b f(x) dx \right)$

Cauchy Principal Value of the integral: p.v. $\int_a^b f(x) dx = \lim_{r \downarrow 0} \left(\int_a^{c-r} f(x) dx + \int_{c+r}^b f(x) dx \right)$

Lemma: Let D domain, f is meromorphic on D , with simple pole at c .

Let S_r circular arc parametrized by $\gamma(t) = c + r \exp(it)$, $t \in [\theta_0, \theta_1]$, $0 \leq \theta_0 < \theta_1 \leq 2\pi$. Then: $\lim_{r \downarrow 0} \int_{S_r} f(z) dz = i(\theta_1 - \theta_0) \text{Res}(f, c)$

Infinite Series: (这里提供一个例子, 但一般的大体思路都差不多: 1. 构建一个矩形 $\Gamma_N(N+1/2)(\pm 1 \pm i)$, 2. 通过两种方式: LM 计算 $\int_{\Gamma} f(z) dz$, 3. 通过 Residue 定理计算 $\int_{\Gamma} f(z) dz$, 4. 结合两者的结果, 得到级数的值)

e.g. Consider $f(z) = \frac{\cot(\pi z)}{z^2}$ to calculate $\sum_{n=1}^{\infty} \frac{1}{n^2}$: Calculate $\text{Res}(f, n) = \dots = \frac{1}{n^2}$, $n \neq 0$ Calculate $\text{Res}(f, 0)$ = 通过直接计算它的 Laurent 展开式得到.

Important: By using Geometric Expansion: $\frac{1}{1-(\square+\dots)} = 1 + \square + \square^2 + \dots$. By L-M calculate $|\int_{\Gamma} f(z) dz| \leq \dots \rightarrow 0$ **Important:** 寻找 bounded: e.g. $\cot(\pi z)$ is bounded at Γ .

ps: 如果使用 $\Gamma_N: (N+1/2)(\pm 1 \pm i)$, $\ell(\Gamma_N) = 4(2N+1)$ and $|z| \geq N + \frac{1}{2}$

Theorem: Let Γ be loop with $0 \in \text{Int}(\Gamma)$. Then $\binom{n}{k} = \frac{1}{2\pi i} \int_{\Gamma} \frac{(1+z)^n}{z^{k+1}} dz$

7 Appendix

7.1 Series

Technical to write Taylor Series: Since $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$. for any $\frac{1}{b-cx} \Rightarrow \frac{a}{b} \cdot \frac{1}{1-\frac{c}{b}x} = \frac{a}{b} \sum_{n=0}^{\infty} \left(\frac{c}{b}x\right)^n$ 2 for $\frac{1}{(1-z)^2} \Rightarrow = \frac{d}{dz} \left(\frac{1}{1-z}\right)$

Moreover, need try to construct $\frac{1}{1-(\frac{z-z_0}{R})}$ if it's holomorphic on $D_R(z_0)$. or: $\frac{1}{1-\frac{z-z_0}{R}}$ if it's holomorphic on $A_{1,\infty}(z_0)$.

Taylor Series for Familiar functions: $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$ $\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$ all of them have infinite radius of convergence.

Check sin/cos answer: If $a^2 > b^2 + c^2$ $\int_0^{2\pi} \frac{1}{a+b \sin(\theta)+c \cos(\theta)} d\theta = \frac{2\pi}{\sqrt{a^2-b^2-c^2}}$ If $a > |b|$, $\int_0^{\pi} \frac{1}{(a+b \cos(\theta))^2} d\theta = \frac{\pi a}{(a^2-b^2)^{\frac{3}{2}}}$

7.2 Other

If $|f(z) - g(z)| < |f(z)|$ for all $z \in S$. $\Rightarrow f, g(z) \neq 0$ for all $z \in S$. (f, g non-zero on S)

Trigonometric: $\sin(x) = \text{Im}(\exp(ix))$ $\cos(x) = \text{Re}(\exp(ix))$. $\cos(n\pi) = (-1)^n$ $\sin(n\pi) = 0$ $\cos((n+1)\pi) = (-1)^{n+1}$