NODEA Note

1 Basic Knowledge

Def of ODE & ODEs: (1st order) ODE: $\frac{dy}{dt} = f(t,y)$ & ODEs: $\frac{d\mathbf{y}}{dt} = \mathbf{f}(t,\mathbf{y})$, $\mathbf{y} = (y_1, ..., y_d)^T, \mathbf{f}(t,\mathbf{y}) = (f_1(t,\mathbf{y}), ..., f_d(t,\mathbf{y}))^T$

Autonomous: $\frac{d\mathbf{y}}{dt} = \mathbf{f}(\mathbf{y}) \Rightarrow$ autonomous ODE(s). || $\Downarrow$ New Autonomous ODEs: $\frac{d\mathbf{y}}{ds} = \mathbf{f}(y_{d+1}, \mathbf{y})$ and $\frac{dy_{d+1}}{ds} = 1$

· **Change to Autonomous**: For $\frac{d\mathbf{y}}{dt} = \mathbf{f}(t,\mathbf{y})$. Let $y_{d+1} = t$ and *new* independent variable s s.t. $\frac{dt}{ds} = 1 \Uparrow$

Linearity: ODE: $\frac{dy}{dt} = f(t,y)$ is linearity if $f(t,y) = a(t)y + b(t)$ || ODEs: If each ODE is linear, then the ODEs are linear.

Picard's Theorem: If $f(t,y)$ is continuous in $D := \{(t,y) : t_0 \le t \le T, |y - y_0| < K\}$ and $\exists L > 0$ (Lipschitz constant) s.t.
$\forall (t,u),(t,v) \in D \quad |f(t,u) - f(t,v)| \le L|u - v|$ (ps:Can use MVT). And Assume that $M_f(T - t_0) \le K, M_f := \max\{|f(t,u)| : (t,u) \in D\}$
$\Rightarrow$ **Then**, $\exists$ a unique continuously differentiable solution $y(t)$ to the IVP $\frac{dy}{dt} = f(t,y), y(t_0) = y_0$ on $t \in [t_0, T]$.

Existence & Uniqueness Theorem: IVP $\frac{d\mathbf{y}}{dt} = \mathbf{f}(t,\mathbf{y}), \mathbf{y}(t_0) = \mathbf{y}_0$. If $f(t,y)$ and $\frac{\partial f}{\partial y_i}$ are continuous in a neighborhood of $(t_0, \mathbf{y}_0)$.
$\Rightarrow$ **Then**, $\exists I := (t_0 - \delta, t_0 + \delta)$ s.t. $\exists$ a unique continuously differentiable solution $\mathbf{y}(t)$ to the IVP on $t \in I$.

2 Acknowledge

Notation	Meaning	Notation	Meaning
$[a,b]$	Approximate function for $t \in [a,b]$	$t_0 = a \mid t_N = b$	Assume that $t_0 = a, t_N = b$
N	number of **timesteps** (i.e. Break up interval $[a,b]$ into N equal-length sub-intervals)	h	**stepsize** $(h = \frac{b-a}{N})$
t_i	Define $N+1$ points: $t_0, t_1, ..., t_N$	t_m	$t_m = a + h \cdot m = t_0 + h \cdot m$
y_i	Approximation of y at point $t = t_i$ (Except y_0)	$y(t_i)$	Exact value of y at point $t = t_i$

3 Euler's Method and Taylor Series Method

Euler's Method Algorithm: Approximate ODE $\frac{dy}{dt} = f(t,y), y(t_0) = y_0$ with number of steps N. (Similarly for ODEs)
$\Rightarrow$ **for** $n = 0, 1, 2, ..., N-1$: $y_{n+1} = y_n + hf(t_0 + nh, y_n) = y_n + hf(t_n, y_n)$ **end** (ps: $\Downarrow$ Can get $|y''| < M$)

Boundedness Theorem: For $\frac{dy}{dt} = f(t,y), y(a) = y_0$ and suppose there exists a unique, twice differentiable, solution $y(t)$ on $[a,b]$.
Suppose: y is continuous and $|\frac{\partial f}{\partial y}| \le L$. $\Rightarrow$ the solution y_n given Euler's method satisfies: $e_n = |y_n - y(t_n)| \le Dh, D = e^{(b-a)L}\frac{M}{2L}$

· **Lemma**: If $v_{n+1} \le Av_n + B$, then $v_n \le A^n v_0 + \frac{A^n - 1}{A-1}B$ If $v_n = e_n := y_n - y(t_n)$, then $A = 1 + hL, B = h^2M/2$ (suppose $|y''| < M$)

Order Notation ($\mathcal{O}$): we write $z(h) = \mathcal{O}(h^p)$ if $\exists C, h_0 > 0$ s.t. $|z| \le Ch^p, 0 < h < h_0$

Flow Map (Φ, Ψ): $\Phi_{t_0,h}(y_0) = y(t_0 + h)$ Clearly, $\Phi(t_n + h) = y(t_n + h) = \Phi_h(y(t_n)) = y(t_{n+1})$.

· $\Psi_{t_n,h}(y_n) = y_{n+1}$:= Numerical method for ODE Clearly, $\Psi(t_n + h) = y_{n+1} = \Psi_h(y_n)$

Taylor Series Method: Approximate ODE $\frac{dy}{dt} = f(t,y), y(t_0) = y_0$ with *n-order Methods*: 用 Taylor Series 在 $t_0 + h$ 处展开保留到 n 阶

· $\Phi_{t,h}(y) = y + hf(t,y) + \frac{1}{2}h^2[f_t(t,y) + f_y(t,y)f(t,y)] + \frac{1}{6}y'''(t,y)h^3 + \cdots$ (For one variable y)

· ps: Taylor Series: $y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{h^2}{2}y''(t_0) + \cdots + \frac{h^{n-1}}{(n-1)!}y^{(n-1)}(t_0) + \frac{h^n}{n!}y^{(n)}(t^*), t^* \in [t, t+h]$ ps: $y' = f, y'' = f_t + f_y f$

4 Convergence of One-Step Methods consider for autonomous $y' = f(y)$

4.1 Convergence | Consistent | Stable

Global Error: global error after n steps: $e_n := y_n - y(t_n)$ **Local Error**: For *one-step* method is: $le(y,h) = \Psi_h(y) - \Phi_h(y)$

Consistent: If $||le(y,h)|| \le Ch^{p+1} (\le \mathcal{O}(h^{p+1}))$, $C > 0$. $\Rightarrow$ Consistent at order p. **Stable**: If $||\Psi_h(u) - \Psi_h(v)|| \le (1 + h\hat{L})||u - v||$

Convergent: A method is convergent if:$\forall T$, $\lim_{h \to 0,\, h = T/N} \max_{n = 0,1,...,N} ||e_n|| = 0$ $\Downarrow$ Then the global error satisfies: $\max_{n=0,1,...,N} ||e_n|| = \mathcal{O}(h^p)$ **p-th order**

Convergence of One-Step Method: For $y' = f(y)$, and a one-step method $\Psi_h(y)$ is [1] consistent at order p and [2] stable with $\hat{L}$ $\Uparrow$. (ps:$C = \frac{C}{\hat{L}}(e^{T\hat{L}} - 1)$)

4.2 More One-Step Methods | Runge-Kutta Methods | Collocation

Construction of More General one-step Method: For $y' = f(y), y(t_0) = y_0 \Rightarrow y(t+h) - y(t) = \int_t^{t+h} f(y(\tau))d\tau$

Trapezoidal Method: $y_{n+1} = y_n + \frac{h}{2}(f(y_n) + f(y_{n+1}))$ **Midpoint Method**: $y_{n+1} = y_n + hf(\frac{y_n + y_{n+1}}{2})$

One-Step Collocation Methods (By Lagrange Interpolating Polynomials):

1. **Lagrange Interpolating Polynomials**: $\ell_i(x) = \prod_{j=1, j\ne i}^{s} \frac{x - c_j}{c_i - c_j} \in \mathbb{P}_{s-1}$ where $c_i \in F \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$
$\Rightarrow$ **Polynomial Interpolation**: $\forall p(x) \in \mathbb{P}_s$ with $p(c_i) = g_i \in F \Rightarrow \exists!\, p(x) = \sum_{i=1}^{s} g_i \ell_i(x)$ (Can be proved by Honour Algebra)
2. **Quadrature Rule**: If $g(t) \in \mathbb{P}_{p-1} \Rightarrow$ Order p | $\int_{t_0}^{t_0+h} g(t)dt = \int_0^1 g(t_0 + hx)dx \approx h\sum_{i=1}^{s} b_i g(t_0 + hc_i), b_i := \int_0^1 \ell_i(x)dx$ ps: c_i 从 $[0,1]$ 中取不同的
3. **Collocation Methods**: For: $y(t_0) = y_0$, $y'(t_0 + c_ih) = f(y(t_0 + c_ih))$ ps: c_i 从 $[0,1]$ 中取不同的 Let: $a_{ij} := \int_0^{c_i} \ell_j(x)dx$ and $b_i := \int_0^1 \ell_i(x)dx$
$\Rightarrow F_i = f(y_n + h\sum_{j=1}^{s} a_{ij}F_j)$ and $y_{n+1} = y_n + h\sum_{i=1}^{s} b_i F_i$ where $F_i := y'(t_0 + c_ih)$

· **Remark**: For choice of c_i: The optimal choice is attained by *Gauss-Legendre collocation methods*.

Runge-Kutta Methods: Let $y' = f(t,y)$ **Stage Values**: $Y_i = y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j)$ $i \in \{1, ..., s\}$ $F_i = f(Y_i)$

1. The RK method is the form: $y_{n+1} = y_n + h\sum_{i=1}^{s} b_i f(t_n + c_i, Y_i)$ for some values of b_i, a_{ij}, s, c_i for Autonomous: $c_i = \sum_{j=1}^{s} a_{ij}$

2. Flow-map: $\Psi_h(y) = y + h\sum_{i=1}^{s} b_i f(t_n + c_i, Y_i(y,h))$ ps: *weights*: b_i; *internal coefficients*: a_{ij}
3. We can using **Butcher Table** to represent the RK method (Appendix)   **Explicit**: $a_{ij} = 0$ for $j \ge i$ (严格下三角行)   **Implicit**: $\exists a_{ij} \ne 0$ for $j \ge i$ (Not Explicit)

4.3 Accuracy of RK Method | Order Condition

Some Notations: If $\mathbf{y} = f'(\mathbf{y})$ where $f(\mathbf{y}) : \mathbb{R}^d \to \mathbb{R}^d$.   Def $f' = (\frac{\partial f_i}{\partial y_j}), 1 \le i \le d, 1 \le j \le d$ (行向量) $f'' = (\frac{\partial^2 f_i}{\partial y_j \partial y_k})$, $1 \le i \le d, 1 \le j, k \le d$

· Def: $f''(\mathbf{a},\mathbf{b}) = \sum_{j=1}^{d}\sum_{k=1}^{d} \frac{\partial^2 f_i}{\partial y_j\,\partial y_k} a_j b_k$ $| y' = f$ $y'' = \sum_{j=1}^{d} \frac{\partial f_i}{\partial y_j} f_j = f'f$ $y''' = \sum_{j=1}^{d}\sum_{k=1}^{d} \frac{\partial^2 f_i}{\partial y_j\,\partial y_k} y_j'(t) y_k'(t) + \sum_{j=1}^{d} \frac{\partial f_i}{\partial y_j} y_j''(t) = f''(f,f) + f'f'f$

· $\Phi_h(y) = y + hf + \frac{h^2}{2} f'f + \frac{h^3}{6}[f''(f,f) + f'f'f] + \mathcal{O}(h^4)$

Order Condition: RK method: $y_{n+1} = y_n + h\sum_{i=1}^{s} b_i f(Y_i)$, Let $z(h) = \Phi_h(y)$

$\Rightarrow$ If $z'(0) = y', z''(0) = y'', ..., z^{(n)} = y^{(n)} \Rightarrow$ **Convergent at order** n

· Order 1: $\sum_{i=1}^{s} b_i = 1$ Order 2: (add) $\sum_{i=1}^{s} b_i c_i = \frac{1}{2}$ Order 3: (add) $\sum_{i=1}^{s} b_i c_i^2 = \frac{1}{3}$ and $\sum_{i=1}^{s}\sum_{j=1}^{s} b_i a_{ij} c_j = \frac{1}{6}$

5 Stability of Runge-Kutta Methods consider for autonomous $y' = f(y)$

5.1 Basic Definition for Stability

Fixed Point-Exact: For ODEs $\frac{dy}{dt} = f(y)$, point y^* is fixed point if $f(y^*) = 0 \Leftrightarrow \Phi_t(y^*) = y^*$ **Set of Fixed Points**: $\mathcal{F} = \{y^* \in \mathbb{R}^d : f(y^*) = 0\}$

Fixed Point-Numerical: *One-step* method $\Psi_h(y)$, point y^* is fixed point if $y^* = \Psi_h(y^*)$ **Set of Fixed Points**: $\mathcal{F}_h = \{y^* \in \mathbb{R}^d : y^* = \Psi_h(y^*)\}$

Theorem: For Runge-Kutta method, $\mathcal{F} \subseteq \mathcal{F}_h$ **Remark**: $\mathcal{F}_h \subseteq \mathcal{F}$ is NOT always true.

· the point in $\mathcal{F}_h \setminus \mathcal{F}$ is called **spurious fixed point.**   As $h \to \infty$, the *spurious* fixed points will tends to infinity.

Stability of Fixed Points: Fixed point y^*, the ODEs $\frac{dy}{dt} = f(y)$ with $y(0) = y_0$.

1. **Stable in the sense of Lyapunov**: Fixed point y^* is stable if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $||y_0 - y^*|| < \delta \Rightarrow ||y(t; y_0) - y^*|| < \varepsilon\ \forall t > 0$
2. **Asymptotically Stable**: Fixed point y^* is asymptotically stable if $\exists \delta > 0$ s.t. $||y_0 - y^*|| < \delta \Rightarrow \lim_{t\to\infty} ||y(t; y_0) - y^*|| = 0$
3. **Unstable**: Fixed point y^* is unstable if it's not stable.   i.e. $\exists \epsilon > 0, \forall \delta > 0$ s.t. $||y_0 - y^*|| < \delta \Rightarrow ||y(t) - y^*|| \ge \varepsilon$ for some t.

5.2 Classification of Fixed Points

Linearization Theorem: Suppose $\frac{dy}{dt} = f(y)$, y^* is a fixed point. Let $J = f'(y^*)$ be the Jacobian matrix of f at y^*.

1. If $\forall$ eigenvalues of J in left complex half plane, then y^* is **asymptotically stable.**
2. If $\exists$ eigenvalues of J in right complex half plane, then y^* is **unstable.**

(Following is a special cases from HDE)

Classification of Critical Points at y^* (Linear): r_1, r_2 be sol of $det(J - \lambda I) = 0$. || $\mathbb{C} : r = \lambda \pm i\mu(\mu > 0)$

If J constant, write sol: $\mathbf{x} = c_1 e^{r_1 t}\xi_1 + c_2 e^{r_2 t}\xi_2$ || $GM = 1$: $\mathbf{x} = c_1 e^{rt}\xi + c_2 e^{rt}(t\xi + \eta)$ $J = \begin{pmatrix} \partial_x F(\mathbf{x}_0) & \partial_y F(\mathbf{x}_0) \\ \partial_x G(\mathbf{x}_0) & \partial_y G(\mathbf{x}_0) \end{pmatrix}$ If $f(x,y) = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}$

$\mathbb{R}/\mathbb{C}$	Condition || Stability	Type || Name	Phase Plane Description	Other	
	$r_1 < r_2 < 0$ || asy.stab	N || NSk	向原点,ξ_2 直线,ξ_1 曲线,and ξ_1 周围 $y = \pm x^3$	$c_2 \ne 0, t \to \infty$:ξ_2 主导方向; $c_2 = 0, t \to \infty$:ξ_1 主导方向	PS:
	$r_1 > r_2 > 0$ || unstable	N || NSo	原点向外,ξ_2 直线,ξ_1 曲线,and ξ_1 周围 $y = \pm x^3$	$c_1 \ne 0, t \to \infty$:ξ_1 主导方向; $c_1 = 0, t \to \infty$:ξ_2 主导方向	**N** = Node
$\mathbb{R}$	$r_1 > 0 > r_2$ || unstable	SP || SP	$t \to \infty$,ξ_1 从原点向外,ξ_2 从外向原点 and: 像 $y = \pm\frac{1}{x}$, 同进同出	$t \to \pm\infty : \lvert\mathbf{x}\rvert \to \infty$; $t \to \infty : c_1, c_2 \ne 0, \lvert\mathbf{x}\rvert \to \infty, \xi_1$ 主导; $t \to \infty : c_2 = 0, \lvert\mathbf{x}\rvert \to \infty, \xi_1$ 主导; $t \to \infty : c_1 = 0, \lvert\mathbf{x}\rvert \to 0, \xi_2$ 主导	**PN** = Proper Node
	$r_1 = r_2 < 0$, GM=2 || asy.stab	PN || PN or Stable Star	直线 向原点	直线, u_1/u_2 is t independent	**IN** = Improper
	$r_1 = r_2 > 0$, GM=2 || unstable	PN || PN or Unstable Star	直线 从原点向外	直线, u_1/u_2 is t independent	or: Degenerate Node
	$r_1 = r_2 < 0$, GM=1 || asy.stab	IN (AL:Type: SpP) || IN (Stable)	S 曲线, 向原点	$t \to \infty, \lvert\mathbf{x}\rvert \to 0, \xi$ 主导 ps: 旋转方向大体和 $\eta + c_2\xi$ 方向相同	**SP** = Saddle Point
	$r_1 = r_2 > 0$, GM=1 || unstable	IN (AL:Type: SpP) || IN (Unstable)	S 曲线, 从原点向外	$t \to \infty, \lvert\mathbf{x}\rvert \to \infty, \xi$ 主导 ps: 旋转方向大体和 $\eta + c_2\xi$ 方向相同	**SpP** = spiral point
	$\lambda \ne 0, \lambda > 0$ || unstable	SpP || Unstable Focus	向外椭圆 (elliptical) 螺旋	$t \to \infty, \lvert\mathbf{x}\rvert \to \infty$ ps: 考虑 $J = (a,b;c,d)$, 如果 bc>0, 顺时针，如果 bc<0, 逆时针	or: Focus Point
$\mathbb{C}$	$\lambda \ne 0, \lambda < 0$ || asy.stab	SpP || Stable Focus	向内椭圆 (elliptical) 螺旋	$t \to \infty, \lvert\mathbf{x}\rvert \to 0$ ps: 考虑 $J = (a,b;c,d)$, 如果 bc>0, 顺时针, 如果 bc<0, 逆时针	**C** = Center
	$\lambda = 0$ || stable (AL:Indeterminate)	C (AL:C or SpP) || C	椭圆 (elliptical) and 半长轴 ξ 实部方向	Bounded trajectory or $\exists$ Periodic Trajectories	**NSk** = Nodal Sink
					NSo = Nodal Source

5.3 Stability of Fixed Points of Maps (Numerical)

Definition: For flow map Ψ from $\mathbb{R}^d \to \mathbb{R}^d$. Def $y^n(y_0) :=$ the n-th iterate of y_0 under Ψ. $(y^n = y_n)$ e.g. $y^2(y_0) = \Psi(\Psi(y_0)) = y_2$

Stability of Fixed Points of Maps: Fixed point y^*, the map Ψ with $y^* = \Psi(y^*)$.

1. **Stable in the sense of Lyapunov**: y^* is stable if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $||y_0 - y^*|| < \delta \Rightarrow ||y^n(y_0) - y^*|| < \varepsilon\ \forall n \ge 0$
2. **Asymptotically Stable**: y^* is asymptotically stable if $\exists \delta > 0$ s.t. $||y_0 - y^*|| < \delta \Rightarrow \lim_{n\to\infty} ||y^n(y_0) - y^*|| = 0$
3. **Unstable**: y^* is unstable if it's not stable.   i.e. $\exists \epsilon > 0, \forall \delta > 0$ s.t. $||y_0 - y^*|| < \delta \Rightarrow ||y^n(y_0) - y^*|| \ge \varepsilon$ for some n.

6 Appendix

6.1 Useful Series | Common RK Methods

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ $\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ $(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ $\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$

$\frac{1}{1+x} = 1 - x + x^2 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$ $\ln(x) = (x-1) - \frac{(x-1)^2}{2} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$, $x > 0$

Common Runge-Kutta Methods (Butcher Table):

$$
\begin{array}{c|ccc}
c_1 & a_{11} & \cdots & a_{1s} \\
\vdots & \vdots & \ddots & \vdots \\
c_s & a_{s1} & \cdots & a_{ss} \\
\hline
 & b_1 & \cdots & b_s
\end{array}
$$

Example

$$
\begin{array}{c|c}
0 & \\
\hline
 & 1
\end{array}
$$

RK1
(Euler's Method)

$$
\begin{array}{c|cc}
0 & & \\
1 & 1 & \\
\hline
 & 1/2 & 1/2
\end{array}
$$

RK2 (Heun's Method)

$$
\begin{array}{c|ccc}
0 & & & \\
1/2 & 1/2 & & \\
1 & -1 & 2 & \\
\hline
 & 1/6 & 2/3 & 1/6
\end{array}
$$

RK3

$$
\begin{array}{c|cccc}
0 & & & & \\
1/2 & 1/2 & & & \\
1/2 & 0 & 1/2 & & \\
1 & 0 & 0 & 1 & \\
\hline
 & 1/6 & 1/3 & 1/3 & 1/6
\end{array}
$$

RK4 (Classical/Famous)