HAlg Note

1 Basic Knowledge

Def of Group $(G,*)$: A set G with a operator $*$ is a group if: **Closure**: $\forall g,h \in G, g*h \in G$; **Associativity**: $\forall g,h,k \in G, (g*h)*k = g*(h*k)$; **Identity**: $\exists e \in G, \forall g \in G, e*g = g*e = g$; **Inverse**: $\forall g \in G, \exists g^{-1} \in G, g*g^{-1} = g^{-1}*g = e$. G,H groups, then $G \times H$ also.

Field (F): A set F is a field with two operators: (addition)$+ : F \times F \to F; (\lambda,\mu) \to \lambda+\mu$ (multiplication)$\cdot : F \times F \to F; (\lambda,\mu) \to \lambda\mu$ if: $(F,+)$ and $(F \setminus \{0_F\}, \cdot)$ are abelian groups with identity $0_F, 1_F$. and $\lambda(\mu+\nu) = \lambda\mu + \lambda\nu$ $e.g. Fields: \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$

F**-Vector Space (V)**: A set V over a field F is a vector space if: V is an abelian group $V = (V, \dot{+})$ and $\forall \vec{v}, \vec{w} \in V$ $\lambda,\mu \in F$ e.g. $Poly: \mathbb{R}[x]_{<n}$ $\exists$ map $F \times V \to V : (\lambda, \vec{v}) \to \lambda\vec{v}$ satisfies: **I**: $\lambda(\vec{v} \dot{+} \vec{w}) = (\lambda\vec{v}) \dot{+} (\lambda\vec{w})$ **II**: $(\lambda+\mu)\vec{v} = (\lambda\vec{v}) \dot{+} (\mu\vec{v})$ **III**: $\lambda(\mu\vec{v}) = (\lambda\mu)\vec{v}$ **IV**: $1_F\vec{v} = \vec{v}$

Vector Subspaces Criterion: $U \subseteq V$ is a subspace of V if: **I**. $\vec{0} \in U$ **II**. $\forall \vec{u}, \vec{v} \in U, \forall \lambda \in F$: $\vec{u}+\vec{v} \in U$ and $\lambda\vec{u} \in U$ (or: $\lambda\vec{u} + \mu\vec{v} \in U$)

· **property**: If U, W are subspaces of V, then $U \cap W$ and $U + W$ are also subspaces of V. ps: $U + W := \{\vec{u} + \vec{w} : \vec{u} \in U, \vec{w} \in W\}$

Complement-wise Operations: $\phi : V_1 \times V_2 \to V_1 \oplus V_2$ by **I**:$(\vec{v}_1, \vec{u}_1) + (\vec{v}_2, \vec{u}_2) := (\vec{v}_1 + \vec{v}_1, \vec{u}_1 + \vec{u}_2)$, $\lambda(\vec{v}, \vec{u}) := (\lambda\vec{v}, \lambda\vec{u})$ (ps:V_1, V_2 通过 ϕ 定义的 map 所形成的 vector space 记作 $V_1 \oplus V_2$)

Projections: $pr_i : X_1 \times \cdots \times X_n \to X_i$ by $(x_1, ..., x_n) \mapsto x_i$ **Canonical Injections**: $in_i : X_i \to X_1 \times \cdots \times X_n$ by $x \mapsto (0, ..., 0, x, 0, ..., 0)$

2 Vector Spaces/Subspaces | Generating Set | Linear Independent | Basis

Generating (subspaces) $\langle T \rangle$: $\langle T \rangle := \{\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n : \alpha_i \in F, \vec{v}_i \in T, r \in \mathbb{N}\}$ $\langle \emptyset \rangle := \{\vec{0}\}$ If T is subspace $\Rightarrow \langle T \rangle = T$.

1. **Proposition**: $\langle T \rangle$ is the smallest subspace containing T. (i.e. $\langle T \rangle$ is the intersection of all subspaces containing T)
2. **Generating Set**: V is vector space, $T \subseteq V$. T is generating set of V if $\langle T \rangle = V$. **Finitely Generated**: $\exists$ finite set T, $\langle T \rangle = V$
3. **External Direct Sum**: 一个" 代数结构", 定义为 set 是 $V_1 \oplus \cdots \oplus V_n := V_1 \times \cdots \times V_n$ 且有一组运算法则 component-wise operations
4. **Connect to Matrix**: Let $E = \{\vec{v}_1, ..., \vec{v}_n\}$, E is GS of V. Let $A = [\vec{v}_1, ..., \vec{v}_n] \Rightarrow \forall \vec{b} \in V, \exists \vec{x} = (x_1, ..., x_n)^T$ s.t. $A\vec{x} = \vec{b}$ (i.e. linear map:$\phi : \vec{x} \mapsto A\vec{x}$ is surjective)

Linearly Independent: $L = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_r\}$ is linearly independent if: $\forall c_1, ..., c_r \in F, c_1\vec{v}_1 + \cdots + c_r\vec{v}_r = \vec{0} \Rightarrow c_1 = \cdots = c_r = 0$.

· **Connect to Matrix**: Let $L = \{\vec{v}_1, ..., \vec{v}_n\}$, L is LI of V. Let $A = [\vec{v}_1, ..., \vec{v}_n] \Rightarrow \forall \vec{x} \in F^n, A\vec{x} = 0\ (or\ \vec{0}) \Rightarrow \vec{x} = 0 (or\ \vec{0})$ (i.e. linear map ϕ:$\vec{x} \mapsto A\vec{x}$ is injective)

Basis & Dimension: If V is finitely generated. $\Rightarrow \exists$ subset $B \subseteq V$ which is both LI and GS. (B is basis) **Dim**: $\dim V := |B|$

· **Connect to Matrix**: Let $B = \{\vec{v}_1, ..., \vec{v}_n\}$ is basis of V. Let $A = [\vec{v}_1, ..., \vec{v}_n] \Rightarrow \forall \vec{x} = (x_1, ..., x_n)^T$ s.t. $\phi : \vec{x} \mapsto A\vec{x}$ is 1-1 & onto (Bijection)

Relation|GS,LI,Basis,dim: Let V be vector space. L is linearly independent set, E is generating set, B is basis set.

1. **GS|LI**: $|L| \leq |E|$ (can get: dim unique) **LI→Basis**: If V finite generate $\Rightarrow \forall L$ can extend to a basis. If $L = \emptyset$, prove $\exists B$ $kerf \cap imf = \{0\}$
2. **Basis|max,min**: $B \Leftrightarrow B$ is minimal GS $(E) \Leftrightarrow B$ is maximal LI (L). **Uniqueness|Basis**: 每个元素都可以由 basis 唯一表示.
3. **Proper Subspaces**: If $U \subset V$ is proper subspace, then $\dim U < \dim V$. $\Rightarrow$ If $U \subseteq V$ is subspace and $\dim U = \dim V$, then $U = V$.
4. **Dimension Theorem**: If $U, W \subseteq V$ are subspaces of V, then $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$

Complementary: $U, W \subseteq V$,U, V subspaces are complementary $(V = U \oplus W)$ if: $\exists \phi : U \times W \to V$ by $(\vec{u}, \vec{w}) \mapsto \vec{u} + \vec{w}$ i.e. $\forall \vec{v} \in V$, we have unique $\vec{u} \in U, \vec{w} \in W$ s.t. $\vec{v} = \vec{u} + \vec{w}$. ps: It's a linear map.

3 Linear Mapping | Rank-Nullity| Matrices | Change of Basis

ps: 默认 V, W F-Vector Spaces.

Linear Mapping/Homomorphism(Hom): $f : V \to W$ is *linear map* if: $\forall \vec{v}_1, \vec{v}_2 \in V, \forall \lambda \in F$. $f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$ and $f(\lambda\vec{v}_1) = \lambda f(\vec{v}_1)$

· **Isomorphism**: = LM & Bij. **Endomorphism(End)**: = LM & $V = W$. **Automorphism(Aut)**: = LM & $V = W$ **Monomorphism**: = LM & 1-1. **Epimorphism**: = LM & onto.

· **Kernel**: $\ker f := \{\vec{v} \in V : f(\vec{v}) = \vec{0}\}$(It's subspace) **Image**: $Imf := \{f(\vec{v}) : \vec{v} \in V\}$(It's subspace) **Rank**:= $\dim(Imf)$ **Nullity**:= $\dim(\ker f)$ **Fixed Point** X^f: $X^f := \{x \in X : f(x) = x\}$

Property of Linear Map: Let $f, g \in Hom$: **a.**$f(\vec{0}) = \vec{0}$ **b.** f is 1-1 iff $\ker f = \{\vec{0}\}$ **c.** $f \circ g$ is linear map.

1. **Determined**: f is determined by $f(\vec{b_i}), \vec{b_i} \in \mathcal{B}_{basis}$ ($\star$ **i.e.** $f(\sum_i \lambda_i\vec{v}_i) := \sum_i \lambda_i f(\vec{v}_i)$)
2. **Classification of Vector Spaces**: $\dim V = n \Leftrightarrow f : F^n \xrightarrow{\sim} V$ by $f(\lambda_1, ..., \lambda_n) \mapsto \sum_{i=1}^n \lambda_i\vec{v}_i$ is isomorphism.
3. **Left/Right Inverse**: f is 1-1 $\Rightarrow \exists$ left inverse g s.t. $g \circ f = id$ 考虑 direct sum f is onto $\Rightarrow \exists$ right inverse g s.t. $f \circ g = id$
4. $^{\ominus}$ **More of Left/Right Inverse**: $f \circ g = id \Rightarrow g$ is 1-1 and f is onto. 使用 kernel=0 来证明

Rank-Nullity Theorem: For linear map $f : V \to W$, $\dim V = \dim(\ker f) + \dim(Imf)$ Following are properties:

1. **Injection**: f is 1-1 $\Rightarrow \dim V \leq \dim W$ **Surjection**: f is onto $\Rightarrow \dim V \geq \dim W$ Moreover, $\dim W = \dim imf$ iff f is onto.
2. **Same Dimension**: f is isomorphism $\Rightarrow \dim V = \dim W$ **Matrix**: $\forall M$, column rank $c(M)$ = row rank $r(M)$.
3. **Relation**: If V, W finite generate, and $\dim V = \dim W$, Then: f is isomorphism $\Leftrightarrow f$ is 1-1 $\Leftrightarrow f$ is onto.

Matrix: For $A_{n\times m}, B_{m\times p}, AB_{n\times p} := (AB)_{ij} = \sum_{k=1}^m a_{ik}b_{kj}$ **Transpose**: $A_{m\times n}^T := (A^T)_{ij} = a_{ji}$

Invertible Matrices: A is invertible if $\exists B, C$ s.t. $BA = I$ and $AC = I$ || $\exists B, BA = I \Leftrightarrow \exists C, AC = I \Leftrightarrow \exists A^{-1}$ ${}_{\mathcal{B}}[f^{-1}]_{\mathcal{A}} = {}_{\mathcal{A}}[f]_{\mathcal{B}}^{-1}$

Representing matrix of linear map ${}_{\mathcal{B}}[f]_{\mathcal{A}}$: $f : V \to W$ be linear map, $\mathcal{A} = \{\vec{v}_1, ..., \vec{v}_n\}$ is basis of V, $\mathcal{B} = \{\vec{w}_1, ..., \vec{w}_m\}$ is basis of W.

1. ${}_{\mathcal{B}}[f]_{\mathcal{A}} := A$ (matrix) where $f(\vec{v}_{i\in\mathcal{A}}) = \sum_{j\in\mathcal{B}} A_{ji}\vec{w}_j$ $\exists M_{\mathcal{B}}^{\mathcal{A}} : Hom_F(V, W) \xrightarrow{\sim} Mat(n \times m; F)$
2. If $\vec{v} \in V$, then ${}_{\mathcal{A}}[\vec{v}] := \mathbf{b}$ (vector) where $\vec{v} = \sum_{i\in\mathcal{A}} \mathbf{b}_i\vec{v}_i$
3. **Theorems**: $[f \circ g] = [f] \circ [g]$ ${}_{\mathcal{C}}[f \circ g]_{\mathcal{A}} = {}_{\mathcal{C}}[f]_{\mathcal{B}} \circ {}_{\mathcal{B}}[g]_{\mathcal{A}}$ ${}_{\mathcal{B}}[f(\vec{v})] = {}_{\mathcal{B}}[f]_{\mathcal{A}} \circ {}_{\mathcal{A}}[\vec{v}]$ ${}_{\mathcal{A}}[f]_{\mathcal{A}} = I \Leftrightarrow f = id$
4. **Change of Basis**: Define *Change of Basis Matrix*$:= {}_{\mathcal{A}}[id_V]_{\mathcal{B}}$ ${}_{\mathcal{B}'}[f]_{\mathcal{A}'} = {}_{\mathcal{B}'}[id_W]_{\mathcal{B}} \circ {}_{\mathcal{B}}[f]_{\mathcal{A}} \circ {}_{\mathcal{A}}[id_V]_{\mathcal{A}'}$ ${}_{\mathcal{A}'}[f]_{\mathcal{A}'} = {}_{\mathcal{A}}[id_V]_{\mathcal{A}'}^{-1} \circ {}_{\mathcal{A}}[f]_{\mathcal{A}} \circ {}_{\mathcal{A}}[id_V]_{\mathcal{A}'}$

Elementary Matrix: $I + \lambda E_{ij}$ (cannot $I - E_{ii}$) 就是初等矩阵, 左乘代表 j 行乘 λ 倍加到第 i 行, 右乘代表 j 列乘 λ 倍加到第 i 列 $\Rightarrow$ Invertible!

1. 交换 i, j 列/行: $P_{ij} = diag(1, ..., 1, -1, 1, ..., 1)(I + E_{ij})(I - E_{ji})(I + E_{ij})$ where -1 in jth place.
2. **Row Echelon Form|Smith Normal Form**: $\tilde{A}$: REF 通过左乘初等矩阵可以实现 $\approx{A}$: $S(n, m, r)$ 通过 $\tilde{A}$ 右乘初等矩阵可以实现

Smith Normal Form: $\forall A, \exists$ invertible P, Q s.t. $PAQ = S(n,m,r) := n \times m$ 的矩阵, 对角线前 r 个是 1, 后面 0. **Lemma**: $r = r(A) = c(A)$

· Every linear map $f : V \to W$ can be representing by ${}_{\mathcal{B}}[f]_{\mathcal{A}} = S(n,m,r)$ for some basis $\mathcal{A}, \mathcal{B}$ of V, W.

Similar Matrices: $N = T^{-1}MT \Leftrightarrow M, N$ are similar. *Special Case*: If $N = {}_{\mathcal{B}}[f]_{\mathcal{B}}$, $M = {}_{\mathcal{A}}[f]_{\mathcal{A}}$, then $N = T^{-1}MT$. where $T = {}_{\mathcal{A}}[id_V]_{\mathcal{B}}$

1. If $A \sim B$ iff A is similar to B, then $\sim$ is an equivalence relation. ${}_{\mathcal{A}'}[f]_{\mathcal{A}'} \sim_{\mathcal{A}} [f]_{\mathcal{A}}$
2. If $\mathcal{B} = \{p(\vec{v}_1), ..., p(\vec{v}_n)\}$ and $\mathcal{A} = \{\vec{v}_1, ..., \vec{v}_n\}$ where $p : V \xrightarrow{\sim} V$. Then ${}_{\mathcal{A}}[id_V]_{\mathcal{B}} =_{\mathcal{A}} [p]_{\mathcal{A}}$
3. If V is a vector space over F, [A, B are *similar* matrices. $\Leftrightarrow$ $A =_{\mathcal{A}} [f]_{\mathcal{A}}, B =_{\mathcal{B}} [f]_{\mathcal{B}}$ for some basis $\mathcal{A}, \mathcal{B}$; $f : V \to V$]
4. Set of *Endomorphism* is in a bijection correspondence with the equivalence class of matrices under $\sim$. 一个自同态 **End** 就对应一个相似矩阵的等价类

Trace: $tr(A) := \sum_i a_{ii}$ and $tr(f) := tr({}_{\mathcal{A}}[f]_{\mathcal{A}})$ | $tr(AB) = tr(BA)$ $tr(\lambda A + \mu B) = \lambda tr(A) + \mu tr(B)$ $tr(N) = tr(M)$ if M, N similar.

4 Rings | Polynomials | Ideals | Subrings

5 Inner Product Spaces | Orthogonal Complement / Proj | Adjoints and Self-Adjoint

6 Jordan Normal Form | Spectral Theorem