

# 1 General Linear Programming Problem

**General LP Problem:** Decision Variables:  $x_i$  parameters:  $a_{ij}, b_i, c_i$  Objective Function:  $f$  Constraints: subject to 后边的部分

LP problem can be written as:

$$\begin{aligned} \text{maximize} \quad & f = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\ & x_1 \geq 0, x_2 \geq 0, \cdots, x_n \geq 0 \end{aligned}$$

We can also write the LP problem in matrix form:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{aligned} \text{maximize} \quad & f = \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \quad \mathbf{x} \geq 0 \end{aligned}$$

**Feasible Solution:** If  $\mathbf{x}$  satisfies all constraints (i.e.  $A\mathbf{x} \leq \mathbf{b}$ ), then  $\mathbf{x}$  is a feasible solution. (可行解) **Optimal Sol:** (最优解) (可多个)

**Find Optimal Solution: Graphical Method:** 略. **Vertex Enumeration:** 所有顶点检查, 找出最优解. **Simplex Method:** 后面详述.

**Slack Variables:** For each inequality constraint, we introduce a slack variable  $x_i$  ( $i > n$ ) to convert it to an equation. (松弛变量)

LP problem can be written as: ps:  $x_i \geq 0$  ( $i > n$ ).

$$\begin{aligned} \text{maximize} \quad & f = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{n+2} = b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + x_{n+m} = b_m \\ & x_1 \geq 0, x_2 \geq 0, \cdots, x_{n+m} \geq 0 \end{aligned}$$

We can also write the LP problem in matrix form:

$$\bar{A} = [A \ I_m], \mathbf{b} = \mathbf{b}, \bar{\mathbf{c}} = \begin{pmatrix} \mathbf{c} \\ \mathbf{0}_{m \times 1} \end{pmatrix}, \mathbf{x} = \begin{pmatrix} \mathbf{x} \\ x_{n+m} \end{pmatrix}$$

$$\begin{aligned} \text{maximize} \quad & f = \bar{\mathbf{c}}^T \mathbf{x} \\ \text{subject to} \quad & \bar{A}\mathbf{x} = \mathbf{b} \quad \mathbf{x} \geq 0 \end{aligned}$$

**Solving:**  $\mathbf{x} = \mathbf{x}_b + \mathbf{v}$  where  $\bar{A}\mathbf{v} = 0$  and  $\mathbf{x}_b = [\mathbf{0} \ \mathbf{b}]^T$

**Feasible Region:** It's the set  $K$  of the solutions to  $\bar{A}\mathbf{x} = \mathbf{b}$  **Convex Set:** The set  $K$  is convex if  $\forall \mathbf{x}, \mathbf{x}' \in K, \forall \theta \in [0, 1], \mathbf{x}_\theta = (1-\theta)\mathbf{x} + \theta\mathbf{x}' \in K$ .

**Vertex on Convex Set:** A vertex of the convex set  $K$  is a point  $\mathbf{x} \in K$  which doesn't lie strictly inside any line segment connecting two points in  $K$ .

**Theorem:** If LP has a unique optimal solution is a vertex. **Theorem:** If LP has a non-unique solution,  $\exists$  optimal solution at vertex

**Solve LP Problem:** Assume  $f = \bar{\mathbf{c}}^T \mathbf{x}$  with  $\bar{A}\mathbf{x} = \mathbf{b}$  and  $\mathbf{x} \geq 0$ .

1. 修改  $x_i$ /列顺序,  $A$  中换顺序后得到可逆的  $B_{m \times m} \Rightarrow \bar{A}\mathbf{x} = \mathbf{b} \Leftrightarrow B\mathbf{x}_B + N\mathbf{x}_N = \mathbf{b} \quad \mathbf{c} \rightarrow [\mathbf{c}_B \ \mathbf{c}_N]^T \quad \bar{A} \rightarrow [B \ N] \quad \mathbf{x} \rightarrow [\mathbf{x}_B \ \mathbf{x}_N]^T$  Let  $\hat{\mathbf{b}} = B^{-1}\mathbf{b} \quad \hat{N} = B^{-1}N$
2. **Solution:**  $\mathbf{x}_B = \hat{\mathbf{b}} - \hat{N}\mathbf{x}_N$  If  $\mathbf{x}_N = 0 \Rightarrow$  It's a basic solution. (But we need to check whether  $\mathbf{x}_B \geq 0$ ) Using  $B, N$ : Index set of independent/else.
3. **At Basic Solution:**  $\mathbf{x}_B = \hat{\mathbf{b}}, \mathbf{x}_N = \mathbf{0} \Rightarrow$  If  $\hat{\mathbf{b}} \geq 0$ . Corresponding  $\mathbf{x}$  is: <sup>1</sup> vertex of  $K$ ; <sup>2</sup> **Basic Feasible Solution (BFS)**
4. **Basic Costs:**  $\mathbf{c}_B^T$  **Nonbasic Costs:**  $\mathbf{c}_N^T$  **Reduced Costs:**  $\hat{\mathbf{c}}_N = \mathbf{c}_N - \hat{N}^T \mathbf{c}_B = \mathbf{c}_N - N^T B^{-T} \mathbf{c}_B \quad \hat{f} = \mathbf{c}_B^T \hat{\mathbf{b}} \quad \mathbf{x}_B, \mathbf{x}_N \geq 0$
5. **Objective Value:**  $f = \bar{\mathbf{c}}^T \mathbf{x} = \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N = \hat{f} + \hat{\mathbf{c}}_N^T \mathbf{x}_N$  If  $\hat{\mathbf{c}}_N \leq 0$ , then  $f \leq \hat{f} \Rightarrow$  Corresponding  $\mathbf{x}$  is optimal.
6. If  $\hat{\mathbf{c}}_N \leq 0$  doesn't hold, we using **Simplex Algorithm**.