HCV Note

1 Basic Knowledge

Useful Complex Number Properties: $|Re(z)|, |Im(z)| \le |z|$ $\quad Re(z) = \frac{z+\overline{z}}{2}, Im(z) = \frac{z-\overline{z}}{2i}, |z|^2 = z\overline{z}$

Useful Tools: $\mathrm{Re}(z) = 0 \Leftrightarrow z = -\overline{z}$ $\quad \mathrm{Im}(z) = 0 \Leftrightarrow z = \overline{z}$ $\quad$ In *circle*, $\overline{z} = |z|^2 z^{-1}$

Triangle (Reverse) Inequality: $|z_1 + z_2| \le |z_1| + |z_2|$ $\quad ||z_1| - |z_2|| \le |z_1 - z_2|$ $\quad ^{\ominus}(Re(zw) = 0 \Leftrightarrow \overline{zw} = -zw; Im(zw) = 0 \Leftrightarrow zw = \overline{zw})$

Argument: $\arg(z) := \{\theta : z = |z|e^{i\theta}\} = \{Arg(z) + 2\pi k : k \in \mathbb{Z}\}$ $\quad$ **Principle Value of Argument**: $Arg(z) \in (-\pi, \pi]$

· **Operations on Argument**: $arg(z_1 z_2) = arg(z_1) + arg(z_2)$ $\quad arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$ $\quad arg(\overline{z}) = -arg(z)$

2 Holomorphic Functions

2.1 Open/Closed Set | Limit Point | limit of Sequence | Continuous of Function

Open/Closed/Punctured ε-disc: $D_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ $\quad \overline{D}_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| \le \varepsilon\}$ $\quad D'_\varepsilon(z_0) := \{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}$

Open/Closed Set in $\mathbb{C}$: $U \subset \mathbb{C}$ is **open** if $\forall z_0 \in U, \exists \varepsilon > 0, D_\varepsilon(z_0) \subseteq U$ $\quad U$ is **closed** if $\mathbb{C} \setminus U$ is open $\quad$ **Lemma**: $D_\varepsilon, D'_\varepsilon$ open, $\overline{D}_\varepsilon$ closed.

Limit Point of S: $z_0 \in \mathbb{C}$ is a limit point of S if: $\forall \varepsilon > 0, D'_\varepsilon(z_0) \cap S \neq \emptyset$ 聚点 $\quad$ **Bounded**: S is bounded if $\exists M > 0$ s.t. $|z| \le M, \forall z \in S$

Closed of Set S: $\overline{S}$:= 所有 S 的 limit point 和 S 的点. $\quad$ **Property**: Let $S \subseteq \mathbb{C}$, then S is closed $\Leftrightarrow S = \overline{S}$.

Limit of sequence: Sequence $(z_n)_{n\in\mathbb{N}}$ has limit z if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \ge N \Rightarrow |z_n - z| < \varepsilon$. limit rules 依旧成立

1. **Lemma|Important**: $\lim z_n = z \Leftrightarrow \lim Re(z_n) = Re(z)$ and $\lim Im(z_n) = Im(z)$
2. **Cauchy**: Sequence $(z_n)_{n\in\mathbb{N}}$ is cauchy if: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall m, n \ge N \Rightarrow |z_m - z_n| < \varepsilon$ $\quad$ **Lemma**: Cauchy $\Leftrightarrow$ convergent.
3. **Lemma|Closed of Set**: $S \subseteq \mathbb{C}, z \in \mathbb{C}. \Rightarrow [\ z \in \overline{S} \Leftrightarrow \exists$ sequence $(z_n)_{n\in\mathbb{N}} \in S$ s.t. $\lim z_n = z\ ]$
4. **Bolzano-Weierstrass**: Every bounded sequence in $\mathbb{C}$ has a convergent subsequence.

Complex Functions: $\forall f : \mathbb{C} \to \mathbb{C}$ we can write it as: $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$

Limit of Function: $a_0 \in \mathbb{C}$ is the limit of f at z_0 if: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |z - z_0| < \delta \Rightarrow |f(z) - a_0| < \varepsilon$ $\quad$ limit rules 依旧成立

· **Lemma|Important**: $\lim_{z\to z_0} f(z) \Leftrightarrow \lim_{(x,y)\to(x_0,y_0)} u(x, y) = Re(a_0)$ and $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = Im(a_0)$

· **Useful Formula**: $\lim_{z\to z_0} g(\overline{z}) = \lim_{z\to\overline{z_0}} g(z)$

continuous of Function: f is continuous at z_0 if: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$ $\quad$ continuous rules 依旧成立

1. **Lemma|Important**: f is continuous at $z_0 \Leftrightarrow u, v$ are continuous at (x_0, y_0)
2. **'Extreme Value Theorem'**: f is continuous on a closed and bounded set $S \subseteq \mathbb{C}$, then $f(S)$ is closed and bounded.
3. **Lemma|continuous⇔open**: f is continuous $\Leftrightarrow \forall$ open set U, preimage $f^{-1}(U) := \{z \in \mathbb{C} | f(z) \in U\}$ is open.

2.2 Differentiable | Holomorphic Function | C-R Equation

Differentiable: Let $z_0 \in \mathbb{C}$ and $U \subseteq \mathbb{C}$ be neighborhood of z_0, then $f : U \to \mathbb{C}$ is differentiable at z_0 if: $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists.

· **I.** f is differentiable $\Rightarrow f$ is continuous. $\quad$ **II. Holomorphic** $\Leftrightarrow$ Differentiable + neighborhood (除非是一个点时不成立,$|z|$) $\quad$ diff rules + chain rule 成立

Cauchy-Riemann Equations: If $z_0 = x_0 + iy_0, f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 \Rightarrow u_x = v_y, v_x = -u_y$ **at** (x_0, y_0).

· If $z_0 = x_0 + iy_0, f = u + iv$ satisfies: $^1 u, v$ are *continuously differentiable* on a neighborhood of (x_0, y_0) and:
$^2 u, v$ satisfies Cauchy-Riemann Equations at (x_0, y_0). $\Rightarrow f$ is differentiable at z_0.

· ps: 常见可导复数函数: $\exp(z), \sin z, \cos z, \log z, z^\alpha$, polynomial, sinh, cosh, $\Gamma(z), |z|^2$ (at 0),constant $\quad$ ps: 常见不可导复数函数: $\overline{z}, |z|\cdot\overline{z}, Re(z), Im(z), \mathrm{Arg}(z)$

Harmonic Function: $h : \mathbb{R}^2 \to \mathbb{R}$ is harmonic if: $\forall (x, y) \in \mathbb{R}^2$ $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$ (**Laplace Equation**)

· **Lemma**: If $f = u + iv$ is holomorphic on $\mathbb{C}$ (and u, v are twice *continuously differentiable*) 可以不用, $\Rightarrow u, v$ are harmonic. $\quad ^{\ominus}(u, v$ **harmonic+CR⇔ f holomorphic**)

Harmonic Conjugate: Let $u, v : U \to \mathbb{R}, U \subseteq \mathbb{R}^2$ be harmonic functions. u, v are harmonic conjugate if: $f = u + iv$ is holomorphic on U.

Properties of Polynomial: The domain of rational function and polynomial are always open. $\quad$ **Lemma**: If $P(z_0) = 0$ then $P(\overline{z_0}) = 0$

First-order Operator ∂: $\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ $\quad \overline{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ $\quad || \quad f = u + iv$ satisfies C-R Equations $\Leftrightarrow \overline{\partial} f = 0$

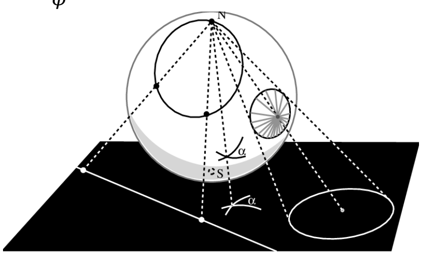
sin/cos Functions: $\sin z := \frac{e^{iz}-e^{-iz}}{2i}$ $\quad \cos z := \frac{e^{iz}+e^{-iz}}{2}$ $\quad$ **Exponential Function**: $\exp(z) = e^x(\cos(y) + i\sin(y))$

1. $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$ $\quad \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$
2. $\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$ $\quad \cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$
3. $\sin^2 z + \cos^2 z = 1$ $\quad \sin(z + \frac{\pi}{2}) = \cos(z)$ $\quad \sin(z + 2k\pi) = sin(z)$ $\quad \cos(z + 2k\pi) = \cos(z)$ $\quad \star \sin z, \cos z$ NOT bounded.

Hyperbolic Functions: $\sinh z := \frac{\exp(z)-\exp(-z)}{2}$ $\quad \cosh z := \frac{\exp(z)+\exp(-z)}{2}$ $\quad || \quad \sinh(iz) = i \sin z$ $\quad \cosh(iz) = \cos z$

Logarithm: Define *multivalued function*: $\log z := \{w \in \mathbb{C} : \exp w = z\}$ $\quad$ **Principal Branch**: $Log(z) := \ln|z| + iArg(z)$

1. **I.** $\log(z) = \ln|z| + i \arg z = \{\ln|z| + iArg(z) + i2\pi k : k \in \mathbb{Z}\}$ $\quad$ **II.** $\log(zw) = \log(z) + \log(w)$ $\quad$ **III.** $\log(1/z) = -\log(z)$
2. **Branch of Logarithm**: $Log_\phi(z) := \ln|z| + iArg_\phi(z)$ $\quad Log_\phi(z)$ is holomorphic on D_ϕ
3. If $g : U \to \mathbb{C}$, then $Log_\phi(g(z))$ is holomorphic on $g^{-1}(D_\phi) \cap U$
4. $Log(z)$ *not continuous* on $\mathbb{C}$. $\quad Log(z)$ not continuous on $Re(z) \le 0, Im(z) = 0$.
   Remark: $\log(x) + \log(x) \neq 2\log(x)$



Branch Cut|Cut Plane: *Branch Cut* $L_{z_0,\phi} := \{z \in \mathbb{C} : z = z_0 + re^{i\phi}, r \geq 0\}$

· *Cut Plane*: $D_{z_0,\phi} := \mathbb{C} \setminus L_{z_0,\phi}$ $\quad L_\phi = L_{0,\phi}; D_\phi = D_{0,\phi}$

· If $Log_\phi(z)$ is holomorphic on D_ϕ, then $Log_\phi(z-a)$ is holomorphic on $D_{a,\phi}$

Branch of Argument: $Arg_\phi(z) := z$ 的辐角, 但是角度限制在: $\phi < Arg_\phi(z) \leq \phi + 2\pi$. ps: $Arg_{-\pi}(z) = Arg(z)$

$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$
z^n	nz^{n-1}	$\exp(z)$	$\exp(z)$	$\sin(z)$	$\cos(z)$	$\cos(z)$	$-\sin(z)$	$\sinh(z)$	$\cosh(z)$	$\cosh(z)$	$\sinh(z)$	$Log_\phi z$	$\frac{1}{z}$ $_{z \in D_\phi}$

Complex Powers: $z^\alpha := \{\exp(\alpha w) : w \in \log(z)\} = \{\exp[\alpha(\ln|z| + iArg(z) + i2k\pi)] : k \in \mathbb{Z}\}$ $\quad \frac{d}{dz}z^\alpha = \alpha z^{\alpha-1}$ $_{z \in D_\phi}$

I. If $\alpha \in \mathbb{Z}$, there is one value of z^α **II.** If $\alpha = \frac{p}{q}, \gcd(p,q) = 1, p, q \in \mathbb{Z}, q \neq 0$, there are exactly q values of z^α

III. If α is *irrational* or *non-real*, there are infinitely values z^α **IV.** $1^{1/q}, q \in \mathbb{Z}, q \neq 0$ is $\{1, w, ..., w^{q-1}\}, w = \exp(i2\pi/q)$

V. We prefer use $\exp(z)$ to denote single-valued function, and e^z to denote multi-valued function.

Principal Branch: $z^\alpha := \exp(\alpha Log(z))$ **Operation**: $z^\alpha z^\beta = z^{\alpha+\beta}$ (Using Principal Branch) **NB**: $(z_1z_2)^\alpha \neq z_1^\alpha z_2^\alpha$; $(z^\alpha)^\beta \neq z^{\alpha\beta}$

3 Conformal Maps and Mobius Transformations

Conformal: Let U be open set and $f : U \to \mathbb{C}$. Then f is conformal iff: f *preserves angles*. i.e. 任意两条曲线/直线之间的角度在 f 作用下不变.

Important Theorem: If $f : U \to \mathbb{C}$ is *holomorphic*, then $\forall z_0 \in U, f'(z_0) \neq 0$, f *preserves angles*.

i.e. $\forall$ curves C_1, C_2 in U. If C_1, C_2 intersecting at a point $z_0 \in U$. C_1, C_2 在 z_0 切线的夹角与 $f(C_1), f(C_2)$ 在 $f(z_0)$ 切线的夹角一样.

Extended Complex Plane: $\widetilde{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and define that $a + \infty = \infty, b \cdot \infty = \infty, \frac{b}{0} = \infty, \frac{b}{\infty} = 0$.

Riemann Sphere: Consider $(X,Y,Z) \in \mathbb{R}^3$: ${}^1z = X + iY \in \mathbb{C}$ is the point $(X,Y,0)$ and ${}^2Z = 0$ is the complex plane.

1. Define the Riemann Sphere: $S^2 := \{(X,Y,Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1\}$ and consider the **North Pole** is point $N := (0,0,1)$
2. Define $\phi : \mathbb{C} \to S^2$ by N 点与 $z = (X,Y,0)$ 点连线与 S^2 的交点为 $\phi(z)$ $\quad$ Thus $\lim_{|z|\to\infty} \phi(z) = N$ $\quad \phi(\infty) := N$
3. Calculation shows that: $\phi(z) = \phi(x+iy) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$ $\quad \psi(X,Y,Z) = \begin{cases} \frac{X+iY}{1-Z}, (X,Y,Z) \neq N \\ \infty \quad, (X,Y,Z) = N \end{cases}$

   Remark: $\phi : \widetilde{\mathbb{C}} \to S^2$ is bijection and it's inverse $\psi : S^2 \to \widetilde{\mathbb{C}}$ is the **stereographic projection**
4. Stereographic projection $\psi(X,Y,Z)$ maps a circle to either a circle or a straight line. (见上图)

Mobius Transformation: A Mobius Transformation is a function form: $f(z) = \frac{az+b}{cz+d}$ where $a,b,c,d \in \mathbb{C}$; $ad \neq bc$

1. **Remark**: $g(z) = \frac{f(z)}{\sqrt{ad-bc}}$ satisfies $ad - bc = 1$ | If a,b,c,d defined a mobius transformation, then $\lambda a, \lambda b, \lambda c, \lambda d$ also.
2. For Complex Matrix: $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(M) = ad - bc = 1$. We define $f_M = \frac{az+b}{cz+d}$ $\quad$ **I.**$f_{M_1M_2} = f_{M_1}f_{M_2}$ **II.**$f_{M^{-1}} = f_M^{-1}$
3. Extended $f(z)$ from $\mathbb{C}$ to $\widetilde{\mathbb{C}}$ by: $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$
4. **Translation**: $f(z) = z + b \Leftrightarrow \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ **Rotation**: $f(z) = az, a = e^{i\theta}$ $(|a| = 1) \Leftrightarrow \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & -e^{i\theta/2} \end{pmatrix}$ **Dilation**: $f(z) = rz, r > 0 \Leftrightarrow \begin{pmatrix} \sqrt{r} & 0 \\ 0 & 1/\sqrt{r} \end{pmatrix}$

   Inversion: $f(z) = 1/z \Leftrightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ $\quad$ f **fixes the point at infinity**: If $f(\infty) = \infty$ ps: 除了 inversion 其他都是 fix the point at infinity.
5. **Theorem**: $f(z) = \frac{az+b}{cz+d}$ be a Mobius Transformation. $\Rightarrow$ ^{1}If $f(\infty) = \infty$: f is a composition of finite *Translation, Rotation, Dilation* $\Rightarrow c = 0, f(z) = \frac{a}{d}z + \frac{b}{d}$

   ^{2} If $f(\infty) < \infty$: f is composition of finite *Translation, Rotation, Dilation* and only one *inversion*. $\Rightarrow f(z) = \frac{(bc-ad)/c^2}{z+d/c} + \frac{a}{c}$

Properties of Mobius Transformation: *Important*: ⋆ Möbius transformations map circlines to circlines. ⋆

1. For mobius transformation $f(z) = \frac{az+b}{cz+d}$, if: $\exists z_1, z_2, z_3 \in \mathbb{C}$ distinct points. $f(z_1) = z_1, f(z_2) = z_2, f(z_3) = z_3 \Rightarrow f$ is identity.
2. If $z_1, z_2, z_3 \in \widetilde{\mathbb{C}}$ distinct points. $\exists!$ mobius transformation $f(z)$ s.t. $f(z_1) = 1, f(z_2) = 0, f(z_3) = \infty$
3. If $(z_1, z_2, z_3), (w_1, w_2, w_3) \in \widetilde{\mathbb{C}}$ distinct points. Then $\exists!$ mobius transformation $f(z)$ s.t. $f(z_i) = w_i$, $\forall i \in \{1,2,3\}$

   ps:Method to construct 2: If $z_i < \infty, f(z) = \frac{z_1-z_3}{z_1-z_2} \cdot \frac{z-z_2}{z-z_3}$ $\quad$ If $z_i = \infty$, $f(z) = \frac{z-z_2}{z-z_3}$,$z_1 = \infty$ $f(z) = \frac{z_1-z_3}{z-z_3}$,$z_2 = \infty$; $f(z) = \frac{z-z_2}{z_1-z_2}$,$z_3 = \infty$

   ps:Method to construct 3: For 3: Let $f := h^{-1} \circ g$ where $g(z_i), h(w_i) = \{1, 0, \infty\}$ like part 2.

Geometric Meaning by using Mobius Transformation|Exponential|Complex Powers:

1. **Rotation**: $f(z) = e^{-i\theta}z$ is a rotation by θ (anticlockwise) about the origin. $\quad$ Specially, $f(z) = iz$ is a rotation by $\frac{\pi}{2}$
2. **Extend**: $f(z) = \exp(\alpha z)$ 原来的图像进行拉长, 以及旋转 (如果带 θ 带 i 时) e.g. $\{z : 0 < Im(z) < 1\}$ 可以被拉长到 $\{z : 0 < Im(z)\}$
3. **Angle Extend**: $f(z) = z^\alpha$ 原来的图像辐角范围收缩或放大
4. **Circlines**: **I.** 单位圆到实轴, $f(z) = \frac{z-i}{z+i}$ **II.** 实轴到单位圆, $f(z) = i\frac{1+z}{1-z}$ **III.** 单位圆到虚轴, $f(z) = \frac{z-1}{z+1}$ **IV.** 虚轴到单位圆, $f(z) = \frac{1+iz}{1-iz}$

Cross-Ratio: cross-ratio $[z_1, z_2, z_3, z_4] := f(z_1)$ where f is mobius transformation s.t. $f(z_2) = 1, f(z_3) = 0, f(z_4) = \infty$

1. **Formulas**: $[z_1, z_2, z_3, z_4] = \frac{z_1-z_3}{z_1-z_4}\frac{z_2-z_4}{z_2-z_3}$ $\quad [\infty, z_2, z_3, z_4] = \frac{z_2-z_4}{z_2-z_3}$ $\quad [z_1, \infty, z_3, z_4] = \frac{z_1-z_3}{z_1-z_4}$ $\quad [z_1, z_2, \infty, z_4] = \frac{z_2-z_4}{z_1-z_4}$ $\quad [z_1, z_2, z_3, \infty] = \frac{z_1-z_3}{z_2-z_3}$
2. **Theorem**: If f is a mobius transformation, $[f(z_1), f(z_2), f(z_3), f(z_4)] = [z_1, z_2, z_3, z_4]$ $\quad$ **z_i's in this "small section" are distinct.**
3. **Application**: $\exists$ mobius transformation f s.t. $f(z_1) = w_1, f(z_2) = w_2, f(z_3) = w_3, f(z_4) = w_4$ $\quad \Leftrightarrow \quad [w_1, w_2, w_3, w_4] = [z_1, z_2, z_3, z_4]$

   ps: ↑原因: $(\Rightarrow), [z_1, z_2, z_3, z_4] = [f(z_1), f(z_2), f(z_3), f(z_4)] = [w_1, w_2, w_3, w_4]$; $(\Leftarrow), [w_1, w_2, w_3, w_4] = [z_1, z_2, z_3, z_4] \Rightarrow \exists f : w_2 \to 1, w_3 \to 0, w_4 \to \infty; g : z_2 \to 1, z_3 \to 0, z_4 \to \infty; f(w_1) = g(z_1) \to g^{-1} \circ f$

4 Complex Integration

4.1 Line Integral

Integrable: $f:[a,b]\to\mathbb{C}$ as $f(t)=u(t)+iv(t)$ is integrable if: u,v are both integrable on $[a,b]$ and for $f(t)$:

1. **Def**: $\int_a^b f(t)dt := \int_a^b u(t)dt + i\int_a^b v(t)dt$
2. **Property I.** $\alpha f+\beta g$ is integrable and $\int_a^b(\alpha f+\beta g)dt=\alpha\int_a^b f(t)dt+\beta\int_a^b g(t)dt$
3. **Property II.** If f is *continuous* and $\frac{dF}{dt}=f(t)$ for $F:[a,b]\to\mathbb{C}$ is differentiable. $\Rightarrow \int_a^b f(t)dt=F(b)-F(a)$
4. **Property III.** If f is *continuous* $\Rightarrow \left|\int_a^b f(t)dt\right|\le\int_a^b|f(t)|dt$.

Parameters Curves: A parametrized curve connecting z_0 to z_1 is a *continuous* function $\gamma:[t_0,t_1]\to\mathbb{C}$ s.t. $\gamma(t_0)=z_0,\gamma(t_1)=z_1$

If $z_0=x_0+iy_0, z_1=x_1+iy_1$, then $\gamma(t)=x(t)+iy(t)$ *continuous* functions. s.t. $x(t_0)=x_0, x(t_1)=x_1, y(t_0)=y_0, y(t_1)=y_1$

Regular: γ is regular if $\gamma'(t)\neq 0$ for all $t\in[t_0,t_1]$ **Remark**: Curve $\gamma([t_0,t_1])=\Gamma$ is *closed and bdd*.

Integral Along Curve: Let $\gamma:[t_0,t_1]\to\mathbb{C}$ be a *regular* curve s.t. $\gamma([t_0,t_1])=\Gamma$ and $f:\Gamma\to\mathbb{C}$ is *continuous*.

1. ⋆ **Def**: $\int_\Gamma f(z)dz := \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)dt$ ⋆
2. **Circle at zero**: *Circle Centred at 0 with radius R*: $\gamma:[0,1]\to\mathbb{C}$ by $\gamma(t)=R\exp(2\pi it)$
3. **Constant Function**: If $f(z)=c$; $\gamma:[a,b]\to\mathbb{C}$. Then $\int_\Gamma f(z)dz=\int_b^a c\cdot\gamma'(z)dz=c\cdot(\gamma(b)-\gamma(a))$

Arclength of Curve: Let $\gamma:[t_0,t_1]\to\mathbb{C}$ be a *regular* curve. $\gamma(t)=x(t)+iy(t)$ Then arclength $\ell(\Gamma):=\int_{t_0}^{t_1}|\gamma'(t)|dt=\int_{t_0}^{t_1}\sqrt{x'(t)^2+y'(t)^2}dt$

Lemma: If Γ is an arc of a circle of radius r traced though angle θ, then $\ell(\Gamma)=r\theta$ (扇形弧长)

Properties of Integral Along Curve: Let Γ be a *regular* curve and $f,g:\Gamma\to\mathbb{C}$ be *continuous*, and $\alpha,\beta\in\mathbb{C}$

1. **M-L Lemma**: $|\int_\Gamma f(z)dz|\le\max_{z\in\Gamma}|f(z)|\ell(\Gamma)$
2. **Lemma**: $\int_\Gamma(\alpha f+\beta g)dz=\alpha\int_\Gamma f(z)dz+\beta\int_\Gamma g(z)dz$ $\int_{-\Gamma}f(z)dz=-\int_\Gamma f(z)dz$ Here: $\tilde\gamma(t):=\gamma(b-t)$ have $\tilde\gamma([a,b])=-\Gamma$
3. **Change of Variables**: If $^1\gamma:[a,b]\to\Gamma$, and $\tilde\gamma:[\tilde a,\tilde b]\to\Gamma$ are *two parametrizations* of Γ;
   $^2\ \exists\lambda:[\tilde a,\tilde b]\to[a,b]$ s.t. $\lambda'(t)>0$ and $\tilde\gamma(t)=\gamma(\lambda(t))$ (防止曲线回头) $\Rightarrow \int_a^b f(\gamma(t))\gamma'(t)dt=\int_{\tilde a}^{\tilde b}f(\tilde\gamma(t))\tilde\gamma'(t)dt$.
   (特别的, 如果 Γ 是 closed,f 在 Γ 上的积分与哪里选择起/终点无关)

Contour: A curve Γ is *contour* if it's *finite union of regular curves* $\Gamma_1,\Gamma_2,...,\Gamma_n$. Each Γ_i is **regular component** of Γ

Contour Integral: If $f:\Gamma\to\mathbb{C}$ is *continuous* and Γ is a *contour*. Then $\int_\Gamma f(z)dz:=\sum_{i=1}^n\int_{\Gamma_i}f(z)dz$

4.2 Independent of Path

Domain: $D\subseteq\mathbb{C}$ is a *domain* if it's *open* and *connected*. (i.e. 任意两点都存在 contour(Γ) 将其连接, 并都在 D 里面)

Lemma: Let $D\subseteq\mathbb{C}$ be a domain. If $u:D\to\mathbb{C}$ is *differentiable*, with $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=0$. $\Rightarrow u$ is *constant* on D. ⇓ Clearly, F is *holomorphic*

Antiderivative: Let D be a domain. For $f:D\to\mathbb{C}$ be *continuous* and $F:D\to\mathbb{C}$ s.t. $F'(z)=f(z)$ for all $z\in D$. Then F is an *antiderivative* of f.

Fundamental Theorem of Calculus: D domain; $f:D\to\mathbb{C}$ *continuous*; $F:D\to\mathbb{C}$ *antiderivative* of f. Contour Γ in D connecting z_0 to z_1.

Then $\int_\Gamma f(z)dz=F(z_1)-F(z_0)$

1. D domain, if $f:D\to\mathbb{C}$ is *holomorphic* and $f'(z)=0,\forall z\in D$. $\Rightarrow f$ is *constant* on D.
2. **Path-Independence Lemma**: D domain, f *continuous* on D. Then:
   f has *antiderivative* on D $\Leftrightarrow$ $\int_\Gamma f(z)dz=0\ \forall$ *closed contours* Γ in D $\Leftrightarrow$ $\int_\Gamma f(z)dz$ is *path-independent*.

4.3 Cauchy's Theorem

Simple: A *contour* Γ is *simple* if it doesn't intersect itself except at the endpoints. **Loop**: A *contour* Γ is a *loop* if it's *simple* and $\Gamma(t_0)=\Gamma(t_1)$

Jordan Curve Theorem: $\forall\ \Gamma$ be *Loop* **Interior** $Int(\Gamma)$: Γ 的内部,bounded. **Exterior** $Ext(\Gamma)$: Γ 的外部,unbounded. **Boundary** Γ 的边界, Γ itself.

And $Int(\Gamma)$ is bounded *domain* $Ext(\Gamma)$ is unbounded *domain*. **Remark**: $Int(\Gamma)$ is *open* and $Ext(\Gamma)$ is *open* also.

· **Common Loop**: $C_r(z_0)$ is a circle of radius r centered at z_0 Corresponding $\gamma(t)=z_0+r\exp(2\pi it)$ $t\in[0,1]$

· **Positive-Oriented**: If Γ is a *loop*, then Γ is *positive-oriented* if: 按方向走时, 内部在左边 (as we move along the curve in the direction of parametrization, the interior is on the left-hand side.)

Remark: Unless otherwise stated, all loops shall be *positively-oriented*.

Simply-Connected: A domain D is *simply-connected* if: $\forall$ *loop* Γ in D, $Int(\Gamma)\subseteq D$ (即没有洞的 domain/open set)

Cauchy Integral Theorem: If Γ is *Loop*, f is holomorphic in $Int(\Gamma)\cup\Gamma$ (Inside and on Γ), then $\int_\Gamma f(z)dz=0$

Corollary: If D is *simply-connected* domain and $f:D\to\mathbb{C}$ is *holomorphic* on D. Then $f(z)$ has *antiderivative* on D. ⋆

即: 在没有洞的 open set 上如果都是 holomorphic, 那么都有 antiderivative.

Remark: 如果 loop Γ 上和以内没有穿过任何非 holomorphic 点, 那么 $f(z)$ 的积分值不变.

Theorem: Let $z_0\in\mathbb{C}$, Γ be *Loop*. Then $\int_\Gamma\frac{1}{z-z_0}=\begin{cases}2\pi i & if\ z_0\in\text{Int}(\Gamma)\\ 0 & \text{otherwise}\end{cases}$

Deformation Theorem: Let Γ_1,Γ_2 be *loops*, and f is *holomorphic* on $(Int(\Gamma_1)\setminus Int(\Gamma_2))\cup(Int(\Gamma_2)\setminus Int(\Gamma_1))$, Γ_1 , Γ_2. Then $\int_{\Gamma_1}f(z)dz=\int_{\Gamma_2}f(z)dz$

即: 两个 *loop* Γ_1 和 Γ_2 及它们围成的区域中 (除公共区域) 上, 函数 $f(z)$ 全纯, 那么它们的路径积分相等 ps: 可以是内外 loop, 也可以是交叉的 loop

4.4 Cauchy's Integral Formula

Cauchy's Integral Formula: Γ *Loop*, $f(z)$ *holomorphic* inside and on Γ, $z_0 \in Int(\Gamma)$, $\Rightarrow$ $f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz$

ps: We always use it to calculate: $\int_\Gamma \frac{f(z)}{z-z_0}dz$ if $f(z)$ is holomorphic on and inside Γ (*loop*), and $z_0 \in Int(\Gamma)$. $\Rightarrow$ $\int_\Gamma \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$

Theorem: D be *domain*, Γ be *contour* in D, $g : D \to \mathbb{C}$ *continuous* on Γ, Then:

Function Defined as: $G : D \setminus \Gamma \to \mathbb{C}$ by $G(z) = \int_\Gamma \frac{g(w)}{w-z}dw$ is *holomorphic* on $D \setminus \Gamma$ and $G'(z) = \int_\Gamma \frac{g(w)}{(w-z)^2}dw$

Moreover, function $H : D \setminus \Gamma \to \mathbb{C}$ by $H(z) = \int_\Gamma \frac{g(w)}{(w-z)^n}dw$ is *holomorphic* on $D \setminus \Gamma$ and $H'(z) = n\int_\Gamma \frac{g(w)}{(w-z)^{n+1}}dw$

⋆ **Corollary**: If D is *domain* and f is *holomorphic* on D, then f is *infinitely differentiable* on D, and all of its derivatives are *holomorphic* on D.

Generalized Cauchy's Integral Formula: Γ *Loop*, $f(z)$ *holomorphic* inside and on Γ, $z \in Int(\Gamma)$, $n \in \mathbb{N}$, $\Rightarrow$ $f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)^{n+1}}dw$

ps: We always use it to calculate: $\int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}dz$ if $f(z)$ is holomorphic on and inside Γ (*loop*), and $z_0 \in Int(\Gamma)$. $\Rightarrow$ $\int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(z_0)$

Morera Theorem: Let D is *domain*, if $f : D \to \mathbb{C}$ is *continuous* and $\int_\Gamma f(z)dz = 0$ for all *loop* Γ in D. $\Rightarrow$ f is *holomorphic* on D.

4.5 Liouville's Theorem, FTA and Maximum Modulus Principle

Useful Formula: If ^1D domain; $^2\exists R > 0, z_0 \in \mathbb{C}$ s.t. $\overline{D}_R(z_0) \subseteq D$; ^3f is holomorphic on D

1. Then $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + R\exp(it))dt$.
2. If $|f(z)| < M, \forall z \in D$. Then $|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}$.
3. If $\max_{z\in\overline{D}_R(z_0)} |f(z)| = |f(z_0)|$. Then f is *constant* on $\overline{D}_R(z_0)$.

Criteria Constant Function: If $f : \mathbb{C}(or\ D) \to \mathbb{C}$ is *holomorphic* and *bounded* on: D domain

1. **Liouville's Theorem**: $|f(z)| < M$ *bounded* on $\forall z \in \mathbb{C}$, $\Rightarrow$ $f(z)$ is *constant*.
2. **Maximum Modulus Principle**: $|f(z)|$ *bounded* on $\forall z \in D$, and $|f(z)|$ has *maximum* at $z_0 \in D$. $\Rightarrow$ $f(z)$ is *constant*.

   Remark I: 意思是对于 $f(z)$ holomorphic 且在 domain 上 bounded, 如果 $|f(z)|$ 在 domain 上有最大值 (非边界), 那么 $f(z)$ 是 constant.

   Remark II : ⋆ If function f is *holomorphic* on a *bounded domain* D and *continuous* up to the boundary of D.

   $\Rightarrow$ f has *maximum* modulus on the boundary of D. 若 f 在 D 内全纯, 且在 ∂D 上连续, 则 f 在 $D \cup \partial D$ 最大值一定在边界上. 特别地, 若 f 不是常数, 则最大值只能在边界上取到.
3. **Maximum/Minimum Principle for Harmonic Functions**: If D domain, $\phi : D \to \mathbb{R}$ is *harmonic*, and ϕ is *bounded above/below* on D by M, with $\phi(z_0) = M$ for some $z_0 \in D$. $\Rightarrow$ ϕ is *constant* on D.

   Remark: 对于调和函数 $\phi : D \to \mathbb{R}$, 如果 f 不是常数, 那么最大值只能在边界上取到.

Fundamental Theorem of Algebra: If $P : \mathbb{C} \to \mathbb{C}$ is a non-constant *polynomial*. $\Rightarrow$ P has a at least one *root* in $\mathbb{C}$.

5 infinity Series

5.1 Basic Properties, Convergence Test, Series of Functions and M-Test

Partial Sum: A Series $\sum_{n=0}^\infty z_n$ is *convergent* if *partial sums* $S_n = \sum_{k=0}^n z_k$ is *convergent*. **Remark**: $\sum z_n$ is *convergent* $\Rightarrow \lim z_n = 0$.

Comparison Test: If $|z_n| \leq M_n$ for all $n \in \mathbb{N}$ and $\sum M_n$ is *convergent*. $\Rightarrow$ $\sum z_n$ is *convergent*.

Lemma|'Geometric Series': For $c \in \mathbb{C}$, $\sum_{n=0}^\infty c^n$ is *convergent* $\Leftrightarrow |c| < 1$. **Remark**: $\sum_{n=0}^\infty c^n = \frac{1}{1-c}$

Ratio Test: For $\sum z_n$, let $L = \lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|$. If $L < 1$, then $\sum z_n$ is *convergent*. If $L > 1$, then $\sum z_n$ is *divergent*. If $L = 1$, conclude nothing.

Converge Pointwise: Seq $f_n : S \to \mathbb{C}$ *pointwise convergent* to $f : S \to \mathbb{C}$ if $\forall\varepsilon > 0, \forall z \in S, \exists N_{\varepsilon,z} \in \mathbb{N}$ s.t. $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N$

Uniform Convergence: Seq $f_n : S \to \mathbb{C}$ *uniformly convergent* to $f : S \to \mathbb{C}$ if $\forall\varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ s.t. $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N$ and $\forall z \in S$

1. **Lemma|Continuous**: If $f_n : S \to \mathbb{C}$ is *uniformly convergent* and *continuous* to $f : S \to \mathbb{C}$, then f is *continuous* on S.
2. **Lemma|Integral**: If $f_n : S \to \mathbb{C}$ is *uniformly convergent* and *continuous* to $f : S \to \mathbb{C}$, then $\int_\Gamma f_n(z)dz$ *convergent* to $\int_\Gamma f(z)dz$.
3. **Lemma|Integral**: If $f_n : S \to \mathbb{C}$ is *continuous*, $\sum_{n=0}^\infty f_n(z)$ is *uniformly convergent* on S, then $\int_\Gamma \sum_{n=0}^\infty f_n(z)dz = \sum_{n=0}^\infty \int_\Gamma f_n(z)dz$.
4. **Lemma|Holomorphic**: If D is *simply-connected* domain, $f_n : D \to \mathbb{C}$ is *holomorphic* and *uniformly convergent* to f. $\Rightarrow f$ *holomorphic* on D.

Weierstrass M-Test: For $f_n : S \to \mathbb{C}$, if $\exists M_n \geq 0, n_0 \in \mathbb{N}$ s.t. $|f_n(z)| \leq M_n$ for $\forall z \in S, n \geq n_0$.

If $\sum_{n=0}^\infty M_n$ is *convergent*. $\Rightarrow$ $\sum_{n=0}^\infty f_n(z)$ is *uniformly convergent* on S.

Power Series & Radius of Convergence: Power Series is: $f(z) = \sum_{n=0}^\infty a_n(z-z_0)^n$. and there is a number $R \in [0,\infty) \cup \{\infty\}$ s.t.

1. The Series is *convergent* on $D_R(z_0)$.
2. The Series is *divergent* on $\mathbb{C} \setminus \overline{D}_R(z_0)$.
3. The Series is *uniformly convergent* on $\overline{D}_r(z_0)$ for all $r \in [0, R)$.
4. **Theorem|Holomorphic**: Then $f(z)$ is *holomorphic* on $D_R(z_0)$, where R is the *radius of convergence*.

   Remark: By using Ratio Test, we can find $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$. if this limit exists. 可以取 0 和 ∞

5.2 Taylor Series and Laurent Series

Taylor Series: Let $z_0 \in \mathbb{C}$ and f is holomorphic at z_0. Then the *Taylor Series* of f at z_0 is: $\sum_{n=0}^\infty \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$

1. **Theorem|Convergence**: If f is holomorphic on $D_R(z_0)$, then ^{1} the *Taylor Series* of f at z_0 *converges* to $f(z)$ on $D_R(z_0)$.
2. **Theorem|Convergence**: If f is holomorphic on $D_R(z_0)$, then ^{2} the *Taylor Series* of f at z_0 *converges uniformly* to $f(z)$ on $\overline{D}_r(z_0)$ $r \in [0, R)$.

Analytic: Let U open, $f : U \to \mathbb{C}$ is *analytic* if $\forall z \in U$, $\exists$ some *disc centered at* z s.t. f can be expressed as a *convergent power series centred at* z.

Homo→Analytic: If f is *holomorphic* on U, then f is *analytic* on U.

Properties of Taylor Series|Series: Let $z_0 \in \mathbb{C}$, $R > 0$, f, g is holomorphic on $D_R(z_0)$, then for ⋆ $\forall z \in D_R(z_0)$ ⋆:

1. **Termwise Differentiation**: $f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n\ {\scriptstyle\forall z\in D_R(z_0)}$ and $f'(z)=\sum_{n=1}^{\infty}\frac{f^{(n)}(z_0)}{(n-1)!}(z-z_0)^{n-1}=\sum_{n=0}^{\infty}\frac{f^{(n+1)}(z_0)}{n!}(z-z_0)^n\ {\scriptstyle\forall z\in D_R(z_0)}$
2. **Lemma|Linear Combination**: $(\alpha f+\beta g)(z)=\sum_{n=0}^{\infty}\left(\frac{\alpha f^{(n)}(z_0)+\beta g^{(n)}(z_0)}{n!}\right)(z-z_0)^n\ {\scriptstyle\forall z\in D_R(z_0)}$
3. **Lemma|Product**: $(fg)(z)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{f^{(k)}(z_0)g^{(n-k)}(z_0)}{k!(n-k)!}\right)(z-z_0)^n=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{k=0}^{n}\binom{n}{k}f^{(k)}(z_0)g^{(n-k)}(z_0)\right)(z-z_0)^n\ {\scriptstyle\forall z\in D_R(z_0)}$
4. **Uniqueness of Taylor series**: $f(z)$ has a *power series representation* at z_0, with radius of convergence $R>0$. $\Rightarrow$ Then it must be the *Taylor series* of f at z_0, and will equal to $f(z)$ on $D_R(z_0)$. i.e. 假设某个函数 $f(z)$ 能够由幂级数展开，那么这个展开是唯一的, 且在收敛区间内等于 $f(z)$.

Laurent Series: A *Laurent Series* centered at z_0 is the series form: $\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}a_{-n}(z-z_0)^{-n}$

Convergence: The *Laurent Series* is *convergent* if both $\sum_{n=0}^{\infty}a_n(z-z_0)^n$ and $\sum_{n=1}^{\infty}a_{-n}(z-z_0)^{-n}$ are *convergent*.

Remark: If *radius of convergence* of $\sum_{n=0}^{\infty}a_n(z-z_0)^n$ and $\sum_{n=1}^{\infty}a_{-n}(z-z_0)^{-n}$ are R and S. $\Rightarrow$ *Laurent Series convergent* on $S^{-1}<|z-z_0|<R$.

Annulus: **Open annulus**: $A_{r,R}(z_0)=\{z\in\mathbb{C}: r<|z-z_0|<R\}$ **Closed annulus**: $\overline{A}_{r,R}(z_0)=\{z\in\mathbb{C}: r\le|z-z_0|\le R\}$

Laurent Series|For function: Let $z_0\in\mathbb{C}$, $0\le r<R\le\infty$, f is holomorphic on $A_{r,R}(z_0)$. Then:
1. f can be expressed as a *Laurent Series* on $A_{r,R}(z_0)$, ^{1} *convergent* on $A_{r,R}(z_0)$. ^{2} *uniformly convergent* on $\overline{A}_{r',R'}(z_0)$ for $r<r'\le R'<R$.
2. **Coefficient**: $a_n=\frac{1}{2\pi i}\int_\Gamma\frac{f(z)}{(z-z_0)^{n+1}}dz$ for any *loop* Γ in $A_{r,R}(z_0)$ and contain z_0 in its interior.
3. **Uniqueness**: If $f(z)$ has a *Laurent Series* on $A_{r,R}(z_0)$, then it must be the *Laurent Series* of f on $A_{r,R}(z_0)$, and will equal to $f(z)$ on $A_{r,R}(z_0)$.

5.3 Singularities and Identity Theorem

Singularity: A point $z_0\in\mathbb{C}$ is a *singularity* of f if f is *not holomorphic* at z_0. **Zero**: A point $z_0\in\mathbb{C}$ is a *zero* of f if $f(z_0)=0$.

Isolated Singularity: A *singularity* z_0 of f is *isolated* if $\exists R>0$ s.t. f is *holomorphic* on $D_R'(z_0)$.

Isolated Zero: A zero z_0 is *isolated* if $\exists R>0$ s.t. $f(z)\ne0$ for all $z\in D_R'(z_0)$

Zero of finite order: If $f(z_0)=f'(z_0)=\cdots=f^{(n-1)}(z_0)=0$ and $f^{(n)}(z_0)\ne0$, then z_0 is a *zero of order* n.

Simple Zero: A zero of order 1 is called a *simple zero*. i.e. $f(z_0)=0$ and $f'(z_0)\ne0$.

Properties of Zeros: Let $z_0\in\mathbb{C}$, U be a *neighborhood* of z_0, f is holomorphic on U.
1. If z_0 is a *zero of finite order*, then z_0 is *isolated* (zero).
2. If $\exists$ *distinct* points $z_n\in U$ s.t. $z_n\to z_0$ and $f(z_n)=0$. $\Rightarrow$ $\exists R>0$ s.t. $f(z)=0$ for all $z\in D_R(z_0)$ (identically zero on some disc centred at z_0).

   Remark: If $\exists$ *distinct* points $z_n\in U$ s.t. $z_n\to z_0$ and z_n *Zero*, then z_0 cannot be a *isolated zero*.

Removable|Order|Essential Singularity: Let $z_0\in\mathbb{C}$ is an isolated singularity of a function f, which is holomorphic on $D_R'(z_0)$.
1. Let the *Laurent Series* of f at z_0 be $\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$ valid on $A_{0,R}(z_0)$
2. **removable singularity**: If $a_n=0$ for all $n<0$, then z_0 is a *removable singularity* of f. (i.e. 负的部分都是 0, 和泰勒展开很像)
3. **Pole of Order**: If $a_{-m}\ne0$ and $a_{-n}=0,\forall n>m$, then z_0 is a *pole of order* m of f. (i.e. 有限个负的非 0 项, 且最小的非 0 项是 $-m$)
4. **Essential Singularity**: If $a_n\ne0$ for infinitely many $n<0$. $\Rightarrow$ z_0 is an *essential singularity* of f. (i.e. 无限多个负的非 0 项)

   Remark: Poles of order 1, 2, and 3 are also known as a simple, double, and triple poles, respectively.

Properties of Singularity: Let $z_0\in\mathbb{C}$.
1. **Singularity of rational function|Isolated**: If z_0 is *singularity* of *rational function* f, then z_0 is *isolated*.
2. $^\ominus$**Sequence→Isolated**: If $\exists$ *distinct* points $z_n\in U$ s.t. $z_n\to z_0$ and z_n *Singularity*, then z_0 cannot be a *removable singularity*.
3. **Extended**: f is holomorphic on $D_R'(z_0)$. If z_0 is a *removable singularity*, f can be *redefined* at z_0 to be *holomorphic* at z_0. $(f(z_0)=a_0)$
4. **Functions**: If f,g holomorphic at z_0, z_0 is a zero of g, with order m. Then:
   (a) If z_0 is not a zero of f $\Rightarrow$ $\frac{f}{g}$ has a *pole of order* m at z_0.
   (b) If z_0 is a zero of order k of f and $k<m$ $\Rightarrow$ $\frac{f}{g}$ has a *pole of order* $m-k$ at z_0.
   (c) If z_0 is a zero of order k of f and $k\ge m$ $\Rightarrow$ $\frac{f}{g}$ has a *removable singularity* at z_0.

Analytic Continuation: Let $D\subseteq\widetilde{D}\subseteq\mathbb{C}$, $f:D\to\mathbb{C}$ is *holomorphic*, and $F:\widetilde{D}\to\mathbb{C}$ is *holomorphic*. $F(z)=f(z)$ for all $z\in D$.

Identity Theorem|Disk-zero: Let D *domain*, $z_0\in D$, f is *holomorphic* on D, $f(z)=0,\forall z\in D_R(z_0)$ $\Rightarrow$ $f(z)=0,\forall z\in D$.

Corollary|Disk-func: Let D *domain*, f,g are *holomorphic* on D, $f(z)=g(z),\forall z\in D_R(z_0)$ $\Rightarrow$ $f(z)=g(z),\forall z\in D$.

Corollary|Sequence-zero: Let D *domain*, $\exists$ distinct $z_n\in D$, $z_n\to z_0\in D$ s.t. $f(z_n)=0,\forall n\in\mathbb{N}$ $\Rightarrow$ $f(z)=0,\forall z\in D$.

Corollary|Sequence-func: Let D *domain*, $\exists$ distinct $z_n\in D$, $z_n\to z_0\in D$ s.t. $f(z_n)=g(z_n),\forall n\in\mathbb{N}$ $\Rightarrow$ $f(z)=g(z),\forall z\in D$.

6 Residue Theorem

6.1 Residue and Cauchy Residue Theorem

Theorem: Let f be holomorphic on $D_R'(z_0)$. (i.e. z_0 is an isolated singularity of f). Let Γ *loop* in $D_R'(z_0)$, $z_0\in\operatorname{Int}(\Gamma)$

Then: $\int_\Gamma f(z)dz=2\pi i a_{-1}$, where a_{-1} is the coefficient of $(z-z_0)^{-1}$ in the *Laurent Series* of f at z_0. ←

Residue: Let f be holomorphic on $D_R'(z_0)$. (i.e. z_0 is an isolated singularity of f). Then *residual*: $\operatorname{Res}(f,z_0)=a_{-1}$. where a_{-1} ↑

Properties of Residue: Let $z_0\in\mathbb{C}$, f is *holomorphic* on $D_R'(z_0)$.
1. If z_0 is a *removable singularity*, then $\operatorname{Res}(f,z_0)=0$.
2. If z_0 is a *pole of order* m, then $\operatorname{Res}(f,z_0)=a_{-1}$, where $a_{-1}=\frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]$.
3. If f,g are *holomorphic* on $D_R(z_0)$, g has a *simple zero* at z_0. (i.e. $g(z_0)=0$ and $g'(z_0)\ne0$). Then $\operatorname{Res}\left(\frac{f}{g},z_0\right)=\frac{f(z_0)}{g'(z_0)}$.

Cauchy Residue Theorem: Let Γ *loop*, f is *holomorphic inside and on* Γ, expect for a *finite isolated singularities* $z_1, z_2, \cdots, z_k \in \text{Int}(\Gamma)$

Then: $\int_\Gamma f(z)dz = 2\pi i \sum_{j=1}^{k} \text{Res}(f, z_j)$ where $\text{Res}(f, z_j)$ is the *residue* of f at z_j.

6.2 Argument Principle and Rouché's Theorem

Meromorphic: Let D *domain*, $f: D \to \mathbb{C}$ is *meromorphic* if $\forall z \in D$ f is *holomorphic* or has a *pole* of some *finite order*.

Lemma|Finite Zero & Pole: Let D *domain*, Γ *loop* in D, f is *meromorphic* on D, $f \neq 0$. $\Rightarrow$ f has finite number of *zeroes* and *poles* $\in \text{Int}(\Gamma)$.

Definition of N_0 and N_∞: Let Γ *loop*, f be *meromorphic* on $\text{Int}(\Gamma)$, with *zeros*: $z_1, ..., z_n$ and *poles*: $p_1, ..., p_m$.

Then: $N_0(f) := \sum_j$ order of z_j and $N_\infty(f) := \sum_j$ order of p_j.

Argument Principle: Let Γ *loop*, f is *meromorphic* in $\text{Int}(\Gamma)$, f is *holomorphic and non-zero* on Γ. Then: $\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}dz = N_0(f) - N_\infty(f)$

Corollary: Let Γ *loop*, if f is holomorphic inside and on Γ, and *non-zero* on Γ. 可能在里面为 0 *Then*: $\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}dz = N_0(f)$

Rouché's Theorem: Let Γ *loop*, f, g are *holomorphic* inside and on Γ. If $|f(z) - g(z)| < |f(z)|$ for all $z \in \Gamma$. $\Rightarrow$ $N_0(f) = N_0(g)$.

Open Mapping Theorem: D *domain*, f *non-constant* and *holomorphic* on $D \Rightarrow f(D)$ is *open* in $\mathbb{C}$.

Corollary: Let D *domain*, f *holomorphic* on D. If $\text{Re}(f)$ or $\text{Im}(f)$ or $|f|$ or $\text{Arg}(f)$ is *constant* on D. $\Rightarrow$ f is *constant* on D.

- **Trigonometric Integrals**: If R is *rational* function. Then $\int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta = \int_{C_1(0)} f(z)dz$ where $f(z) = \frac{1}{iz}R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)$ LHS is real int.

6.3 Improper Integral

Def of Improper Integral: Let $f: \mathbb{R} \to \mathbb{R}$ be *continuous*, then:

$$\int_\infty^0 f(x)dx = \lim_{R\to-\infty}\int_R^0 f(x)dx \quad \int_0^\infty f(x)dx = \lim_{R\to\infty}\int_0^R f(x)dx \quad \int_\infty^{-\infty} f(x)dx = \lim_{\substack{R_1\to\infty \\ R_2\to-\infty}}\int_{R_2}^{R_1} f(x)dx$$

R_1, R_2 不一定一样

Remark: If $\int_{-\infty}^\infty f(x)dx$ is *convergent*, then $\int_{-\infty}^\infty f(x) = \lim_{R\to\infty}\int_{-R}^R f(x)dx$.

Cauchy Principal Value of the integral: p.v. $\int_{-\infty}^\infty f(x)dx = \lim_{R\to\infty}\int_{-R}^R f(x)dx$

Jordan Lemma: Let P, Q be *polynomial*, $\deg(Q) > \deg(P)$. $a \neq 0$ Then: $\lim_{R\to\infty}\int_{C_R^\pm}\frac{P(z)}{Q(z)}\exp(iaz)dz = 0$ for $+: a > 0; -: a < 0$.

Useful Method in Improper Integral: For $cos(nx)$, consider $\text{Re}(\exp(inx))$; for $sin(nx)$, consider $\text{Im}(\exp(inx))$. (i.e. $e^{inx} = \cos(nx) + i\sin(nx)$)

Useful Method in Improper Integral: If denominator has $e^x \Rightarrow$ Consider choose $\Gamma: -R \to R \to R + 2\pi i \to -R + 2\pi i \to -R$. 相当于通过纵向限制 pole 的数量

Def of Improper Integral: Let $c \in (a, b) \subseteq \mathbb{R}$, f is *continuous* on $[a, b] \setminus \{c\}$, then:

$$\int_a^c f(x)dx = \lim_{r\downarrow0}\int_a^{c-r} f(x)dx \quad \int_c^b f(x)dx = \lim_{s\downarrow0}\int_{c+s}^b f(x)dx \quad \int_a^b f(x)dx = \lim_{r\downarrow0}\int_a^{c-r} f(x)dx + \lim_{s\downarrow0}\int_{c+s}^b f(x)dx$$

Remark: If $\int_a^b f(x)dx$ is *convergent*, then $\int_a^b f(x)dx = \lim_{r\downarrow0}\left(\int_a^{c-r} f(x)dx + \int_{c+r}^b f(x)dx\right)$

Cauchy Principal Value of the integral: p.v. $\int_a^b f(x)dx = \lim_{r\downarrow0}\left(\int_a^{c-r} f(x)dx + \int_{c+r}^b f(x)dx\right)$

Lemma: Let D domain, f is *meromorphic* on D, with *simple pole* at c.

Let S_r circular arc parametrized by $\gamma(t) = c + r\exp(it)$, $t \in [\theta_0, \theta_1]$, $0 \leq \theta_0 < \theta_1 \leq 2\pi$. Then: $\lim_{r\downarrow0}\int_{S_r} f(z)dz = i(\theta_1 - \theta_0)\text{Res}(f, c)$

Infinite Series: (这里提供一个例子, 但一般的大体思路都差不多: 1. 构建一个矩形 Γ:$(N + 1/2)(\pm1 \pm i)$, 2. 通过两种方式: LM 计算 $\int_\Gamma f(z)dz$, 3. 通过 Residue 定理计算 $\int_\Gamma f(z)dz$, 4. 结合两者的结果, 得到级数的值)

e.g. Consider $f(z) = \frac{cot(\pi z)}{z^2}$ to calculate $\sum_{n=1}^\infty \frac{1}{n^2}$: Calculate $\text{Res}(f, n) = \cdots = \frac{1}{n^2}, n \neq 0$ Calculate $\text{Res}(f, 0) =$ 通过直接计算它的 Laurent 展开式得到.

Important: By using Geometric Expansion: $\frac{1}{1-(\square+\cdots)} = 1 + \square + \square^2 + \cdots$. By L-M calculate $|\int_\Gamma f(z)dz| \leq \cdots \to 0$ **Important**: 寻找 bounded: e.g. $cot(\pi z)$ is bounded at Γ.

ps: 如果使用 Γ_N:$(N + 1/2)(\pm1 \pm i)$, $\ell(\Gamma_N) = 4(2N + 1)$ and $|z| \geq N + \frac{1}{2}$

Theorem: Let Γ be *loop* with $0 \in \text{Int}(\Gamma)$. Then $\binom{n}{k} = \frac{1}{2\pi i}\int_\Gamma \frac{(1+z)^n}{z^{k+1}}dz$

7 Appendix

7.1 Convergence Test for Real Series

Divergence Test: If $\lim a_n \neq 0 \Rightarrow \sum a_n$ diverges. (If $\sum a_n$ convergent $\Rightarrow \lim a_n = 0$.) **p-Test**: $\sum \frac{1}{n^p}$ convergent iff $p > 1$

Comparison Test: If $0 < a_n < b_n$, $\sum b_n$ convergent $\Rightarrow \sum a_n$ also ; $\sum a_n$ divergent $\Rightarrow \sum b_n$ also.

Integral Test: Let $f: [1, \infty) \to \mathbb{R}$ is 非负递减, $a_n = f(n)$. Then $\sum a_n$ converges iff $\int_1^\infty f(x)dx < \infty$.

Absolutely Convergence: $\sum a_n$ convergent absolutely iff $\sum |a_n|$ convergent. **If convergent abs $\Rightarrow$ convergent.**

Alternating Series Test: If a_n decreasing, $a_n \geq 0$, $\lim a_n = 0$. Then $\sum(-1)^{n-1}a_n$ convergent.

Cauchy's Condensation Test: If $a_n \geq 0$, a_n decreasing, $\Rightarrow$ $[\sum a_n convergent \Leftrightarrow \sum 2^n a_{2^n}\ also]$

7.2 Series

Technical to write Taylor Series: Since $\frac{1}{1-z} = \sum_{n=0}^\infty z^n$ for $|z| < 1$. for any [1] $\frac{a}{b-cx}$, $\Rightarrow \frac{a}{b}\cdot\frac{1}{1-\frac{c}{b}x} = \frac{a}{b}\sum_{n=0}^\infty\left(\frac{c}{b}x\right)^n$ [2] for $\frac{1}{(1-z)^2}$ $\Rightarrow = \frac{d}{dz}\left(\frac{1}{1-z}\right)$

Moreover, need try to construct $\frac{1}{1-(\frac{z-z_0}{R})}$ if it's holomorphic on $D_R(z_0)$. or: $\frac{1}{1-\frac{1}{z-z_0}}$ if it's holomorphic on $A_{1,\infty}(z_0)$.

Taylor Series for Familiar functions: $\exp(z) = \sum_{n=0}^\infty \frac{z^n}{n!}$ $\sin(z) = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}z^{2n+1}$ $\cos(z) = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!}z^{2n}$ all of them have *infinite radius of convergence*.

7.3 Other

If $|f(z) - g(z)| < |f(z)|$ for all $z \in S$. $\Rightarrow$ $f, g(z) \neq 0$ for all $z \in S$. (f, g non-zero on S)

Trigonometric: $sin(x) = \text{Im}(\exp(ix))$ $cos(x) = \text{Re}(\exp(ix))$. $cos(n\pi) = (-1)^n$ $sin(n\pi) = 0$ $cos((n+1)\pi) = (-1)^{n+1}$