

1 Basic Knowledge

Def of Group $(G, *)$: A set G with a operator $*$ is a group if: **Closure**: $\forall g, h \in G, g * h \in G$; **Associativity**: $\forall g, h, k \in G, (g * h) * k = g * (h * k)$; **Identity**: $\exists e \in G, \forall g \in G, e * g = g * e = g$; **Inverse**: $\forall g \in G, \exists g^{-1} \in G, g * g^{-1} = g^{-1} * g = e$. G, H groups, then $G \times H$ also.

Subgroup: $H \subseteq G$ is a subgroup if: $\forall h_1, h_2 \in H$ **I**: $H \neq \emptyset$; **II**: $h_1 * h_2 \in H$; **III**: $h_1^{-1} \in H$.

Field (F) : A set F is a field with two operators: (addition) $+: F \times F \rightarrow F; (\lambda, \mu) \rightarrow \lambda + \mu$ (multiplication) $\cdot: F \times F \rightarrow F; (\lambda, \mu) \rightarrow \lambda \mu$ if:
 $(F, +)$ and $(F \setminus \{0_F\}, \cdot)$ are abelian groups with identity $0_F, 1_F$. and $\lambda(\mu + \nu) = \lambda\mu + \lambda\nu$ e.g. $Fields: \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$

F-Vector Space (V) : A set V over a field F is a vector space if: V is an abelian group $V = (V, +)$ and $\forall \vec{v}, \vec{w} \in V, \lambda, \mu \in F$ e.g. $Poly: \mathbb{R}[x]_{<n}$
 $\exists \text{ map } F \times V \rightarrow V: (\lambda, \vec{v}) \rightarrow \lambda\vec{v}$ satisfies: **I**: $\lambda(\vec{v} + \vec{w}) = (\lambda\vec{v}) + (\lambda\vec{w})$ **II**: $(\lambda + \mu)\vec{v} = (\lambda\vec{v}) + (\mu\vec{v})$ **III**: $\lambda(\mu\vec{v}) = (\lambda\mu)\vec{v}$ **IV**: $1_F\vec{v} = \vec{v}$

Vector Subspaces Criterion: $U \subseteq V$ is a subspace of V if: **I**. $\vec{0} \in U$ **II**. $\forall \vec{u}, \vec{v} \in U, \forall \lambda \in F: \vec{u} + \vec{v} \in U$ and $\lambda\vec{u} \in U$ (or: $\lambda\vec{u} + \mu\vec{v} \in U$)
property: If U, W are subspaces of V , then $U \cap W$ and $U + W$ are also subspaces of V . $ps: U + W := \{\vec{u} + \vec{w} : \vec{u} \in U, \vec{w} \in W\}$

Complement-wise Operations: $\phi: V_1 \times V_2 \rightarrow V_1 \oplus V_2$ by $I: (\vec{v}_1, \vec{u}_1) + (\vec{v}_2, \vec{u}_2) := (\vec{v}_1 + \vec{v}_2, \vec{u}_1 + \vec{u}_2), \lambda(\vec{v}, \vec{u}) := (\lambda\vec{v}, \lambda\vec{u})$ (ps: V_1, V_2 通过 ϕ 定义的 map 所形成的 vector space 记作 $V_1 \oplus V_2$)

Projections: $pr_i: X_1 \times \dots \times X_n \rightarrow X_i$ by $(x_1, \dots, x_n) \mapsto x_i$ **Canonical Injections**: $in_i: X_i \rightarrow X_1 \times \dots \times X_n$ by $x \mapsto (0, \dots, 0, x, 0, \dots, 0)$

2 Vector Spaces/Subspaces | Generating Set | Linear Independent | Basis

Generating (subspaces) $\langle T \rangle$: $\langle T \rangle := \{\alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n : \alpha_i \in F, \vec{v}_i \in T, r \in \mathbb{N}\}$ $\langle \emptyset \rangle := \{\vec{0}\}$ If T is subspace $\Rightarrow \langle T \rangle = T$.

- Proposition**: $\langle T \rangle$ is the smallest subspace containing T . (i.e. $\langle T \rangle$ is the intersection of all subspaces containing T)
- Generating Set**: V is vector space, $T \subseteq V$. T is generating set of V if $\langle T \rangle = V$. **Finitely Generated**: \exists finite set $T, \langle T \rangle = V$
- External Direct Sum**: 一个“代数结构”, 定义为 set 是 $V_1 \oplus \dots \oplus V_n := V_1 \times \dots \times V_n$ 且有一组运算法则 component-wise operations
- Connect to Matrix**: Let $E = \{\vec{v}_1, \dots, \vec{v}_n\}$, E is GS of V . Let $A = [\vec{v}_1, \dots, \vec{v}_n] \Rightarrow \forall \vec{b} \in V, \exists \vec{x} = (x_1, \dots, x_n)^T$ s.t. $A\vec{x} = \vec{b}$ (i.e. linear map: $\phi: \vec{x} \mapsto A\vec{x}$ is surjective)

Linearly Independent: $L = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ is linearly independent if: $\forall c_1, \dots, c_r \in F, c_1 \vec{v}_1 + \dots + c_r \vec{v}_r = \vec{0} \Rightarrow c_1 = \dots = c_r = 0$.

Connect to Matrix: Let $L = \{\vec{v}_1, \dots, \vec{v}_n\}$, L is LI of V . Let $A = [\vec{v}_1, \dots, \vec{v}_n] \Rightarrow \forall \vec{x} \in F^n, A\vec{x} = 0$ (or $\vec{0}$) $\Rightarrow \vec{x} = 0$ (or $\vec{0}$) (i.e. linear map $\phi: \vec{x} \mapsto A\vec{x}$ is injective)

Basis & Dimension: If V is finitely generated. $\Rightarrow \exists$ subset $B \subseteq V$ which is both LI and GS. (B is basis) **Dim**: $\dim V := |B|$

Connect to Matrix: Let $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ is basis of V . Let $A = [\vec{v}_1, \dots, \vec{v}_n] \Rightarrow \forall \vec{x} = (x_1, \dots, x_n)^T$ s.t. $\phi: \vec{x} \mapsto A\vec{x}$ is 1-1 & onto (Bijection)

Relation|GS,LI,Basis,dim: Let V be vector space. L is linearly independent set, E is generating set, B is basis set.

- GS|LI**: $|L| \leq |E|$ (can get: dim unique) **LI \rightarrow Basis**: If V finite generate $\Rightarrow \forall L$ can extend to a basis. If $L = \emptyset$, prove $\exists B$ $kerf \cap imf = \{0\}$
- Basis|max,min**: $B \Leftrightarrow B$ is minimal GS ($E \Leftrightarrow B$ is maximal LI (L)). **Uniqueness|Basis**: 每个元素都可以由 basis 唯一表示.
- Proper Subspaces**: If $U \subset V$ is proper subspace, then $\dim U < \dim V$. \Rightarrow If $U \subseteq V$ is subspace and $\dim U = \dim V$, then $U = V$.
- Dimension Theorem**: If $U, W \subseteq V$ are subspaces of V , then $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$

Complementary: $U, W \subseteq V, U, W$ subspaces are complementary ($V = U \oplus W$) if: $\exists \phi: U \times W \rightarrow V$ by $(\vec{u}, \vec{w}) \mapsto \vec{u} + \vec{w}$
i.e. $\forall \vec{v} \in V$, we have unique $\vec{u} \in U, \vec{w} \in W$ s.t. $\vec{v} = \vec{u} + \vec{w}$. ps: It's a linear map.

3 Linear Mapping | Rank-Nullity| Matrices | Change of Basis

ps: 默认 V, W F -Vector Spaces.

Linear Mapping/Homomorphism(Hom): $f: V \rightarrow W$ is linear map if: $\forall \vec{v}_1, \vec{v}_2 \in V, \forall \lambda \in F. f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$ and $f(\lambda\vec{v}_1) = \lambda f(\vec{v}_1)$

Isomorphism: = LM & Bij. **Endomorphism(End)**: = LM & $V = W$. **Automorphism(Aut)**: = LM & $V = W$ **Monomorphism**: = LM & 1-1. **Epimorphism**: = LM & onto.

Kernel: $\ker f := \{\vec{v} \in V : f(\vec{v}) = \vec{0}\}$ (It's subspace) **Image**: $Imf := \{f(\vec{v}) : \vec{v} \in V\}$ (It's subspace) **Rank**: $\dim(Imf)$ **Nullity**: $\dim(\ker f)$ **Fixed Point** $X^f: X^f := \{x \in X : f(x) = x\}$

Property of Linear Map: Let $f, g \in Hom$: **a**. $f(\vec{0}) = \vec{0}$ **b**. f is 1-1 iff $\ker f = \{\vec{0}\}$ **c**. $f \circ g$ is linear map.

- Determined**: f is determined by $f(\vec{b}_i), \vec{b}_i \in \mathcal{B}_{basis}$ (* i.e. $f(\sum_i \lambda_i \vec{v}_i) := \sum_i \lambda_i f(\vec{v}_i)$)
- Classification of Vector Spaces**: $\dim V = n \Leftrightarrow f: F^n \xrightarrow{\sim} V$ by $f(\lambda_1, \dots, \lambda_n) \mapsto \sum_{i=1}^n \lambda_i \vec{v}_i$ is isomorphism.
- Left/Right Inverse**: f is 1-1 $\Rightarrow \exists$ left inverse g s.t. $g \circ f = id$ 考虑 direct sum f is onto $\Rightarrow \exists$ right inverse g s.t. $f \circ g = id$
- More of Left/Right Inverse**: $f \circ g = id \Rightarrow g$ is 1-1 and f is onto. 使用 kernel=0 来证明

Rank-Nullity Theorem: For linear map $f: V \rightarrow W, \dim V = \dim(\ker f) + \dim(Imf)$ Following are properties:

- Injection**: f is 1-1 $\Rightarrow \dim V \leq \dim W$ **Surjection**: f is onto $\Rightarrow \dim V \geq \dim W$ Moreover, $\dim W = \dim imf$ iff f is onto.
- Same Dimension**: f is isomorphism $\Rightarrow \dim V = \dim W$ **Matrix**: $\forall M$, column rank $c(M) = \text{row rank } r(M)$.
- Relation**: If V, W finite generate, and $\dim V = \dim W$, Then: f is isomorphism $\Leftrightarrow f$ is 1-1 $\Leftrightarrow f$ is onto.

Matrix: For $A_{n \times m}, B_{m \times p}, AB_{n \times p} := (AB)_{ij} = \sum_{k=1}^m a_{ik} b_{kj}$ **Transpose**: $A_{m \times n}^T := (A^T)_{ij} = a_{ji}$

Invertible Matrices: A is invertible if $\exists B, C$ s.t. $BA = I$ and $AC = I$ || $\exists B, BA = I \Rightarrow \exists C, AC = I \Leftrightarrow \exists A^{-1}$ $_{\mathcal{B}}[f^{-1}]_{\mathcal{A}} =_{\mathcal{A}}[f]_{\mathcal{B}}^{-1}$

Representing matrix of linear map $_{\mathcal{B}}[f]_{\mathcal{A}}: f: V \rightarrow W$ be linear map, $\mathcal{A} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is basis of $V, \mathcal{B} = \{\vec{w}_1, \dots, \vec{w}_m\}$ is basis of W .

- $_{\mathcal{B}}[f]_{\mathcal{A}} := A$ (matrix) where $f(\vec{v}_{i \in \mathcal{A}}) = \sum_{j \in \mathcal{B}} A_{ji} \vec{w}_j$ $\exists M_{\mathcal{B}}^{\mathcal{A}}: Hom_F(V, W) \xrightarrow{\sim} Mat(n \times m; F)$
- If $\vec{v} \in V$, then $_{\mathcal{A}}[\vec{v}] := \mathbf{b}$ (vector) where $\vec{v} = \sum_{i \in \mathcal{A}} \mathbf{b}_i \vec{v}_i$
- Theorems**: $[f \circ g] = [f] \circ [g]$ $_{\mathcal{C}}[f \circ g]_{\mathcal{A}} =_{\mathcal{C}}[f]_{\mathcal{B}} \circ_{\mathcal{B}}[g]_{\mathcal{A}}$ $_{\mathcal{B}}[f(\vec{v})] =_{\mathcal{B}}[f]_{\mathcal{A}} \circ_{\mathcal{A}}[\vec{v}]$ $_{\mathcal{A}}[f]_{\mathcal{A}} = I \Leftrightarrow f = id$
- Change of Basis**: Define Change of Basis Matrix: $_{\mathcal{A}}[id_V]_{\mathcal{B}} =_{\mathcal{B}'}[f]_{\mathcal{A}'} =_{\mathcal{B}'}[id_W]_{\mathcal{B}} \circ_{\mathcal{B}}[f]_{\mathcal{A}} \circ_{\mathcal{A}}[id_V]_{\mathcal{A}'}$ $_{\mathcal{A}'}[f]_{\mathcal{A}'} =_{\mathcal{A}}[id_V]_{\mathcal{A}'}^{-1} \circ_{\mathcal{A}}[f]_{\mathcal{A}} \circ_{\mathcal{A}}[id_V]_{\mathcal{A}'}$

Elementary Matrix: $I + \lambda E_{ij}$ (cannot $I - E_{ii}$) 就是初等矩阵, 左乘代表 j 行乘 λ 倍加到第 i 行, 右乘代表 j 列乘 λ 倍加到第 i 列 \Rightarrow Invertible!

- 交换 i, j 列/行: $P_{ij} = diag(1, \dots, 1, -1, 1, \dots, 1)(I + E_{ij})(I - E_{ji})(I + E_{ij})$ where -1 in j th place.
- Row Echelon Form|Smith Normal Form**: $\tilde{A}: REF$ 通过左乘初等矩阵可以实现 $\tilde{A}: S(n, m, r)$ 通过 \tilde{A} 右乘初等矩阵可以实现

Smith Normal Form: $\forall A, \exists$ invertible P, Q s.t. $PAQ = S(n, m, r) := n \times m$ 的矩阵, 对角线前 r 个是 1, 后面 0. **Lemma:** $r = r(A) = c(A)$

· Every linear map $f : V \rightarrow W$ can be representing by ${}_B[f]_{\mathcal{A}} = S(n, m, r)$ for some basis \mathcal{A}, \mathcal{B} of V, W .

Similar Matrices: $N = T^{-1}MT \Leftrightarrow M, N$ are similar. *Special Case:* If $N = {}_B[f]_{\mathcal{B}}, M = {}_{\mathcal{A}}[f]_{\mathcal{A}}$, then $N = T^{-1}MT$. where $T = {}_{\mathcal{A}}[id_V]_{\mathcal{B}}$

1. If $A \sim B$ iff A is similar to B , then \sim is an equivalence relation. ${}_{\mathcal{A}'}[f]_{\mathcal{A}'} \sim_{\mathcal{A}} [f]_{\mathcal{A}}$

2. If $\mathcal{B} = \{p(\vec{v}_1), \dots, p(\vec{v}_n)\}$ and $\mathcal{A} = \{\vec{v}_1, \dots, \vec{v}_n\}$ where $p : V \xrightarrow{\sim} V$. Then ${}_{\mathcal{A}}[id_V]_{\mathcal{B}} = {}_{\mathcal{A}}[p]_{\mathcal{A}}$

3. If V is a vector space over F , $[A, B]$ are similar matrices. $\Leftrightarrow A = {}_{\mathcal{A}}[f]_{\mathcal{A}}, B = {}_{\mathcal{B}}[f]_{\mathcal{B}}$ for some basis $\mathcal{A}, \mathcal{B}; f : V \rightarrow V$

4. Set of *Endomorphism* is in a bijection correspondence with the equivalence class of matrices under \sim . 一个自同态 **End** 就对应一个相似矩阵的等价类

Trace: $tr(A) := \sum_i a_{ii}$ and $tr(f) := tr({}_{\mathcal{A}}[f]_{\mathcal{A}}) \mid tr(AB) = tr(BA) \quad tr(\lambda A + \mu B) = \lambda tr(A) + \mu tr(B) \quad tr(N) = tr(M)$ if M, N similar.

4 Rings | Polynomials | Ideals | Subrings

5 Inner Product Spaces | Orthogonal Complement / Proj | Adjoints and Self-Adjoint

6 Jordan Normal Form | Spectral Theorem