

## 1 Basic Knowledge

**Def of ODE & ODEs:** (1st order) ODE:  $\frac{dy}{dt} = f(t, y)$  & ODEs:  $\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y}), \mathbf{y} = (y_1, \dots, y_d)^T, \mathbf{f}(t, \mathbf{y}) = (f_1(t, \mathbf{y}), \dots, f_d(t, \mathbf{y}))^T$

**Autonomous:**  $\frac{dy}{dt} = \mathbf{f}(\mathbf{y}) \Rightarrow$  autonomous ODE(s).  $\parallel \Downarrow$  New Autonomous ODEs:  $\frac{dy}{ds} = \mathbf{f}(y_{d+1}, \mathbf{y})$  and  $\frac{dy_{d+1}}{ds} = 1$

· **Change to Autonomous:** For  $\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y})$ . Let  $y_{d+1} = t$  and new independent variable  $s$  s.t.  $\frac{dt}{ds} = 1 \uparrow$

**Linearity:** ODE:  $\frac{dy}{dt} = f(t, y)$  is linearity if  $f(t, y) = a(t)y + b(t)$   $\parallel$  ODEs: If each ODE is linear, then the ODEs are linear.

**Picard's Theorem:** If  $f(t, y)$  is continuous in  $D := \{(t, y) : t_0 \leq t \leq T, |y - y_0| < K\}$  and  $\exists L > 0$  (Lipschitz constant) s.t.

$\forall (t, u), (t, v) \in D \quad |f(t, u) - f(t, v)| \leq L|u - v|$  (ps: Can use MVT). And Assume that  $M_f(T - t_0) \leq K, M_f := \max\{|f(t, u)| : (t, u) \in D\}$

$\Rightarrow$  **Then,**  $\exists$  a unique continuously differentiable solution  $y(t)$  to the IVP  $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$  on  $t \in [t_0, T]$ .

**Existence & Uniqueness Theorem:** IVP  $\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y}), \mathbf{y}(t_0) = \mathbf{y}_0$ . If  $f(t, y)$  and  $\frac{\partial f}{\partial y_i}$  are continuous in a neighborhood of  $(t_0, \mathbf{y}_0)$ .

$\Rightarrow$  **Then,**  $\exists I := (t_0 - \delta, t_0 + \delta)$  s.t.  $\exists$  a unique continuously differentiable solution  $\mathbf{y}(t)$  to the IVP on  $t \in I$ .

## 2 Acknowledge

Notation	Meaning	Notation	Meaning
$[a, b]$	Approximate function for $t \in [a, b]$	$t_0 = a \mid t_N = b$	Assume that $t_0 = a, t_N = b$
$N$	number of <b>timesteps</b> (i.e. Break up interval $[a, b]$ into $N$ equal-length sub-intervals)	$h$	<b>stepsize</b> ( $h = \frac{b-a}{N}$ )
$t_i$	Define $N + 1$ points: $t_0, t_1, \dots, t_N$	$t_m$	$t_m = a + h \cdot m = t_0 + h \cdot m$
$y_i$	Approximation of $y$ at point $t = t_i$ (Except $y_0$ )	$y(t_i)$	Exact value of $y$ at point $t = t_i$

## 3 Euler's Method and Taylor Series Method

**Euler's Method Algorithm:** Approximate ODE  $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$  with number of steps  $N$ . (Similarly for ODEs)

$\Rightarrow$  for  $n = 0, 1, 2, \dots, N - 1$ :  $y_{n+1} = y_n + hf(t_n + nh, y_n) = y_n + hf(t_n, y_n)$  **end** (ps:  $\Downarrow$  Can get  $|y''| < M$ )

**Boundedness Theorem:** For  $\frac{dy}{dt} = f(t, y), y(a) = y_0$  and suppose there exists a unique, twice differentiable, solution  $y(t)$  on  $[a, b]$ .

Suppose:  $y$  is continuous and  $|\frac{\partial f}{\partial y}| \leq L$ .  $\Rightarrow$  the solution  $y_n$  given Euler's method satisfies:  $e_n = |y_n - y(t_n)| \leq Dh, D = e^{(b-a)L} \frac{M}{2L}$

· **Lemma:** If  $v_{n+1} \leq Av_n + B$ , then  $v_n \leq A^n v_0 + \frac{A^n - 1}{A - 1} B$  If  $v_n = e_n := y_n - y(t_n)$ , then  $A = 1 + hL, B = h^2 M / 2$  (suppose  $|y''| < M$ )

**Order Notation ( $\mathcal{O}$ ):** we write  $z(h) = \mathcal{O}(h^p)$  if  $\exists C, h_0 > 0$  s.t.  $|z| \leq Ch^p, 0 < h < h_0$

**Flow Map ( $\Phi, \Psi$ ):**  $\Phi_{t_0, h}(y_0) = y(t_0 + h)$  Clearly,  $\Phi(t_n + h) = y(t_n + h) = \Phi_h(y(t_n)) = y(t_{n+1})$ .

·  $\Psi_{t_n, h}(y_n) = y_{n+1} :=$  Numerical method for ODE Clearly,  $\Psi(t_n + h) = y_{n+1} = \Psi_h(y_n)$

**Taylor Series Method:** Approximate ODE  $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$  with  $n$ -order Methods: 用 Taylor Series 在  $t_0 + h$  处展开保留到  $n$  阶

·  $\Phi_{t, h}(y) = y + hf(t, y) + \frac{1}{2}h^2 f[f_t(t, y) + f_y(t, y)f(t, y)] + \frac{1}{6}y'''(t, y)h^3 + \dots$

· ps: Taylor Series:  $y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{h^2}{2}y''(t_0) + \dots + \frac{h^{n-1}}{(n-1)!}y^{(n-1)}(t_0) + \frac{h^n}{n!}y^{(n)}(t^*), t^* \in [t, t + h]$  ps:  $y' = f, y'' = f_t + f_y f$

## 4 Convergence of One-Step Methods consider for autonomous $y' = f(y)$

### 4.1 Convergence | Consistent | Stable

**Global Error:** global error after  $n$  steps:  $e_n := y_n - y(t_n)$  **Local Error:** For one-step method is:  $le(y, h) = \Psi_h(y) - \Phi_h(y)$

**Consistent:** If  $||le(y, h)|| < Ch^{p+1} (< \mathcal{O}(h^{p+1}))$ ,  $C > 0 \Rightarrow$  Consistent at order  $p$ . **Stable:** If  $||\Psi_h(u) - \Psi_h(v)|| \leq (1 + h\hat{L})||u - v||$

**Convergent:** A method is convergent if:  $\forall T, \lim_{h \rightarrow 0, h=T/N} \max_{n=0,1,\dots,N} ||e_n|| = 0$   $\Downarrow$  Then the global error satisfies:  $\max_{n=0,1,\dots,N} ||e_n|| = \mathcal{O}(h^p)$   $p$ -th order

**Convergence of One-Step Method:** For  $y' = f(y)$ , and a one-step method  $\Psi_h(y)$  is <sup>1</sup> consistent at order  $p$  and <sup>2</sup> stable with  $\hat{L}$ . (ps:  $C = \frac{C}{L}(e^{T\hat{L}} - 1)$ )

### 4.2 More One-Step Methods | Runge-Kutta Methods | Collocation

**Construction of More General one-step Method:** For  $y' = f(y), y(t_0) = y_0 \Rightarrow y(t + h) - y(t) = \int_t^{t+h} f(y(\tau))d\tau$

**Trapezoidal Method:**  $y_{n+1} = y_n + \frac{h}{2}(f(y_n) + f(y_{n+1}))$  **Midpoint Method:**  $y_{n+1} = y_n + hf(\frac{y_n + y_{n+1}}{2})$

**One-Step Collocation Methods (By Lagrange Interpolating Polynomials):**

1. **Lagrange Interpolating Polynomials:**  $\ell_i(x) = \prod_{j=1, j \neq i}^s \frac{x - c_j}{c_i - c_j} \in \mathbb{P}_{s-1}$  where  $c_i \in F \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$

$\Rightarrow$  **Polynomial Interpolation:**  $\forall p(x) \in \mathbb{P}_s$  with  $p(c_i) = g_i \in F \Rightarrow \exists! p(x) = \sum_{i=1}^s g_i \ell_i(x)$  (Can be proved by Honour Algebra)

2. **Quadrature Rule:** If  $g(t) \in \mathbb{P}_{p-1} \Rightarrow$  Order  $p$   $\mid \int_{t_0}^{t_0+h} g(t)dt = \int_0^1 g(t_0 + hx)dx \approx h \sum_{i=1}^s b_i g(t_0 + hc_i), b_i := \int_0^1 \ell_i(x)dx$  ps:  $c_i$  从  $[0, 1]$  中取不同的

3. **Collocation Methods:** For:  $y(t_0) = y_0, y'(t_0 + c_i h) = f(y(t_0 + c_i h))$  ps:  $c_i$  从  $[0, 1]$  中取不同的 Let:  $a_{ij} := \int_0^{c_i} \ell_j(x)dx$  and  $b_i := \int_0^1 \ell_i(x)dx$

$\Rightarrow F_i = f(y_n + h \sum_{j=1}^s a_{ij} F_j)$  and  $y_{n+1} = y_n + h \sum_{i=1}^s b_i F_i$  where  $F_i := y'(t_0 + c_i h)$

· **Remark:** For choice of  $c_i$ : The optimal choice is attained by Gauss-Legendre collocation methods.

**Runge-Kutta Methods:** Let  $y' = f(t, y)$  **Stage Values:**  $Y_i = y_n + h \sum_{j=1}^s a_{ij} f(Y_j) \quad i \in \{1, \dots, s\} \quad F_i = f(Y_i)$

1. The RK method is the form:  $y_{n+1} = y_n + h \sum_{i=1}^s b_i f(t_n + c_i, Y_i)$  for some values of  $b_i, a_{ij}, s, c_i$  for Autonomous:  $c_i = \sum_{j=1}^s a_{ij}$

2. Flow-map:  $\Psi_h(y) = y + h \sum_{i=1}^s b_i f(t_n + c_i, Y_i(y, h))$  ps: weights:  $b_i$ ; internal coefficients:  $a_{ij}$

3. We can using **Butcher Table** to represent the RK method (Appendix) **Explicit:**  $a_{ij} = 0$  for  $j \geq i$  (严格下三角行) **Implicit:**  $\exists a_{ij} \neq 0$  for  $j \geq i$  (Not Explicit)

4.3 Accuracy of RK Method | Order Condition

5 Appendix

5.1 Useful Series | Common RK Methods

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad \arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$
$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad (1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \dots = \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad \frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n$$
$$\frac{1}{1+x} = 1 - x + x^2 + \dots = \sum_{n=0}^{\infty} (-1)^n x^n \quad \ln(x) = (x-1) - \frac{(x-1)^2}{2} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad x > 0$$

Common Runge-Kutta Methods (Butcher Table):

$c_1$	$a_{11}$	$\cdots$	$a_{1s}$	$0$	$0$	$0$	$0$	$0$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$1$	$1/2$	$1/2$	$1/2$	$1/2$
$c_s$	$a_{s1}$	$\cdots$	$a_{ss}$	$1$	$1$	$-1$	$2$	$1$
	$b_1$	$\cdots$	$b_s$		$1/2$	$2/3$	$1/6$	
Example				RK1 (Euler's Method)	RK2 (Heun's Method)		RK3	
							RK4 (Classical/Famous)	
							$1/6$	$1/3$
							$1/3$	$1/6$