

## 1 Basic Knowledge

**Useful Complex Number Properties:**  $|Re(z)|, |Im(z)| \leq |z|$   $Re(z) = \frac{z+\bar{z}}{2}, Im(z) = \frac{z-\bar{z}}{2i}, |z|^2 = z\bar{z}$

**Triangle (Reverse) Inequality:**  $|z_1 + z_2| \leq |z_1| + |z_2|$   $|z_1| - |z_2| \leq |z_1 - z_2|$   $(Re(zw) = 0 \Leftrightarrow \bar{z}\bar{w} = -zw; Im(zw) = 0 \Leftrightarrow zw = \bar{z}\bar{w})$

**Argument:**  $\arg(z) := \{\theta : z = |z|e^{i\theta}\} = \{Arg(z) + 2\pi k : k \in \mathbb{Z}\}$  **Principle Value of Argument:**  $Arg(z) \in (-\pi, \pi]$

**Operations on Argument:**  $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$   $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$   $\arg(\bar{z}) = -\arg(z)$

## 2 Holomorphic Functions

### 2.1 Open/Closed Set | Limit Point | limit of Sequence | Continuous of Function

**Open/Closed/Punctured  $\varepsilon$ -disc:**  $D_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$   $\bar{D}_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq \varepsilon\}$   $D'_\varepsilon(z_0) := \{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}$

**Open/Closed Set in  $\mathbb{C}$ :**  $U \subseteq \mathbb{C}$  is **open** if  $\forall z_0 \in U, \exists \varepsilon > 0, D_\varepsilon(z_0) \subseteq U$   $U$  is **closed** if  $\mathbb{C} \setminus U$  is open **Lemma:**  $D_\varepsilon, D'_\varepsilon$  open,  $\bar{D}_\varepsilon$  closed.

**Limit Point of  $S$ :**  $z_0 \in \mathbb{C}$  is a limit point of  $S$  if:  $\forall \varepsilon > 0, D'_\varepsilon(z_0) \cap S \neq \emptyset$  **Bounded:**  $S$  is bounded if  $\exists M > 0$  s.t.  $|z| \leq M, \forall z \in S$

**Closed of Set  $S$ :**  $\bar{S} :=$  所有  $S$  的 limit point 和  $S$  的点. **Property:** Let  $S \subseteq \mathbb{C}$ , then  $S$  is closed  $\Leftrightarrow S = \bar{S}$ .

**Limit of sequence:** Sequence  $(z_n)_{n \in \mathbb{N}}$  has limit  $z$  if  $\forall \varepsilon > 0, \exists N \in \mathbb{N}$  s.t.  $\forall n \geq N \Rightarrow |z_n - z| < \varepsilon$ . limit rules 依旧成立

1. **Lemma|Important:**  $\lim z_n = z \Leftrightarrow \lim Re(z_n) = Re(z)$  and  $\lim Im(z_n) = Im(z)$

2. **Cauchy:** Sequence  $(z_n)_{n \in \mathbb{N}}$  is cauchy if:  $\forall \varepsilon > 0, \exists N \in \mathbb{N}$  s.t.  $\forall m, n \geq N \Rightarrow |z_m - z_n| < \varepsilon$  **Lemma:** Cauchy  $\Leftrightarrow$  convergent.

3. **Lemma|Closed of Set:**  $S \subseteq \mathbb{C}, z \in \mathbb{C}. \Rightarrow [z \in \bar{S} \Leftrightarrow \exists \text{ sequence } (z_n)_{n \in \mathbb{N}} \in S \text{ s.t. } \lim z_n = z]$

4. **Bolzano-Weierstrass:** Every bounded sequence in  $\mathbb{C}$  has a convergent subsequence.

**Complex Functions:**  $\forall f : \mathbb{C} \rightarrow \mathbb{C}$  we can write it as:  $f(z) = f(x + iy) = u(x, y) + iv(x, y)$  where  $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$

**Limit of Function:**  $a_0 \in \mathbb{C}$  is the limit of  $f$  at  $z_0$  if:  $\forall \varepsilon > 0, \exists \delta > 0$  s.t.  $0 < |z - z_0| < \delta \Rightarrow |f(z) - a_0| < \varepsilon$  limit rules 依旧成立

**Lemma|Important:**  $\lim_{z \rightarrow z_0} f(z) \Leftrightarrow \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = Re(a_0)$  and  $\lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = Im(a_0)$

**Useful Formula:**  $\lim_{z \rightarrow z_0} g(\bar{z}) = \lim_{z \rightarrow \bar{z}_0} g(z)$

**continuous of Function:**  $f$  is continuous at  $z_0$  if:  $\forall \varepsilon > 0, \exists \delta > 0$  s.t.  $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$  continuous rules 依旧成立

1. **Lemma|Important:**  $f$  is continuous at  $z_0 \Leftrightarrow u, v$  are continuous at  $(x_0, y_0)$

2. **'Extreme Value Theorem':**  $f$  is continuous on a closed and bounded set  $S \subseteq \mathbb{C}$ , then  $f(S)$  is closed and bounded.

3. **Lemma|continuous  $\Leftrightarrow$  open:**  $f$  is continuous  $\Leftrightarrow \forall$  open set  $U$ , preimage  $f^{-1}(U) := \{z \in \mathbb{C} | f(z) \in U\}$  is open.

### 2.2 Differentiable | Holomorphic Function | C-R Equation

**Differentiable:** Let  $z_0 \in \mathbb{C}$  and  $U \subseteq \mathbb{C}$  be neighborhood of  $z_0$ , then  $f : U \rightarrow \mathbb{C}$  is differentiable at  $z_0$  if:  $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$  exists.

**I.  $f$  is differentiable  $\Rightarrow f$  is continuous.** **II. Holomorphic  $\Leftrightarrow$  Differentiable + neighborhood** (除非是一个点时不成立,  $|z|$ ) diff rules + chain rule 成立

**Cauchy-Riemann Equations:** If  $z_0 = x_0 + iy_0, f(z) = u(x, y) + iv(x, y)$  is differentiable at  $z_0 \Rightarrow u_x = v_y, v_x = -u_y$  at  $(x_0, y_0)$ .

**If  $z_0 = x_0 + iy_0, f = u + iv$  satisfies:**  $^1 u, v$  are continuously differentiable on a neighborhood of  $(x_0, y_0)$  and:

$^2 u, v$  satisfies Cauchy-Riemann Equations at  $(x_0, y_0)$ .  $\Rightarrow f$  is differentiable at  $z_0$ .

**ps:** 常见可导复数函数:  $\exp(z), \sin z, \cos z, \log z, z^\alpha$ , polynomial,  $\sinh, \cosh, \Gamma(z), |z|^2$  (at 0), constant **ps:** 常见不可导复数函数:  $\bar{z}, |z| \cdot \bar{z}, Re(z), Im(z), Arg(z)$

**Harmonic Function:**  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  is harmonic if:  $\forall (x, y) \in \mathbb{R}^2 \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$  (Laplace Equation)

**Lemma:** If  $f = u + iv$  is holomorphic on  $\mathbb{C}$  and  $u, v$  are twice continuously differentiable,  $\Rightarrow u, v$  are harmonic.

**Harmonic Conjugate:** Let  $u, v : U \rightarrow \mathbb{R}, U \subseteq \mathbb{R}^2$  be harmonic functions.  $u, v$  are harmonic conjugate if:  $f = u + iv$  is holomorphic on  $U$ .

**Properties of Polynomial:** The domain of rational function and polynomial are always open. **Lemma:** If  $P(z_0) = 0$  then  $P(\bar{z}_0) = 0$

**First-order Operator  $\partial$ :**  $\partial := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$   $\bar{\partial} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$   $\parallel f = u + iv$  satisfies C-R Equations  $\Leftrightarrow \bar{\partial} f = 0$

**sin/cos Functions:**  $\sin z := \frac{e^{iz} - e^{-iz}}{2i}$   $\cos z := \frac{e^{iz} + e^{-iz}}{2}$  **Exponential Function:**  $\exp(z) = e^x(\cos(y) + i \sin(y))$

1.  $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$   $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$

2.  $\sin(z + w) = \sin(z) \cos(w) + \cos(z) \sin(w)$   $\cos(z + w) = \cos(z) \cos(w) - \sin(z) \sin(w)$

3.  $\sin^2 z + \cos^2 z = 1$   $\sin(z + \frac{\pi}{2}) = \cos(z)$   $\sin(z + 2k\pi) = \sin(z)$   $\cos(z + 2k\pi) = \cos(z)$  \* $\sin z, \cos z$  NOT bounded.

**Hyperbolic Functions:**  $\sinh z := \frac{\exp(z) - \exp(-z)}{2}$   $\cosh z := \frac{\exp(z) + \exp(-z)}{2}$   $\parallel \sinh(iz) = i \sin z$   $\cosh(iz) = \cos z$

**Logarithm:** Define multivalued function:  $\log z := \{w \in \mathbb{C} : \exp w = z\}$  **Principal Branch:**  $Log(z) := \ln |z| + i Arg(z)$

1. **I.**  $\log(z) = \ln |z| + i \arg z = \{\ln |z| + i Arg(z) + i 2\pi k : k \in \mathbb{Z}\}$  **II.**  $\log(zw) = \log(z) + \log(w)$  **III.**  $\log(1/z) = -\log(z)$

2. **Branch of Logarithm:**  $Log_\phi(z) := \ln |z| + i Arg_\phi(z)$   $Log_\phi(z)$  is holomorphic on  $D_\phi$

3. If  $g : U \rightarrow \mathbb{C}$ , then  $Log_\phi(g(z))$  is holomorphic on  $g^{-1}(D_\phi) \cap U$

4.  $Log(z)$  not continuous on  $\mathbb{C}$ .  $Log(z)$  not continuous on  $Re(z) \leq 0, Im(z) = 0$ .

**Remark:**  $\log(x) + \log(x) \neq 2 \log(x)$

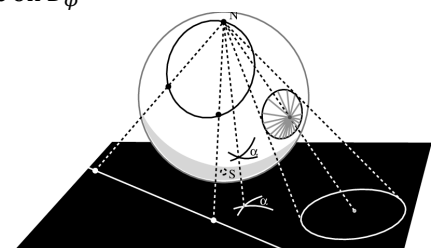
**Branch Cut|Cut Plane:** Branch Cut  $L_{z_0, \phi} := \{z \in \mathbb{C} : z = z_0 + re^{i\phi}, r \geq 0\}$

**Cut Plane:**  $D_{z_0, \phi} := \mathbb{C} \setminus L_{z_0, \phi}$   $L_\phi = L_{0, \phi}; D_\phi = D_{0, \phi}$

**If  $Log_\phi(z)$  is holomorphic on  $D_\phi$ , then  $Log_\phi(z - a)$  is holomorphic on  $D_{a, \phi}$**

**Branch of Argument:**  $Arg_\phi(z) := z$  的辐角, 但是角度限制在:  $\phi < Arg_\phi(z) \leq \phi + 2\pi$ .

**ps:**  $Arg_{-\pi}(z) = Arg(z)$



$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$
$z^n$	$nz^{n-1}$	$\exp(z)$	$\exp(z)$	$\sin(z)$	$\cos(z)$	$\cos(z)$	$-\sin(z)$	$\sinh(z)$	$\cosh(z)$	$\cosh(z)$	$\sinh(z)$	$\text{Log}_\phi z$	$\frac{1}{z} \quad z \in \mathbb{D}_\phi$

**Complex Powers:**  $z^\alpha := \{\exp(\alpha w) : w \in \log(z)\} = \{\exp[\alpha(\ln|z| + i\text{Arg}(z) + i2k\pi)] : k \in \mathbb{Z}\}$   $\frac{d}{dz} z^\alpha = \alpha z^{\alpha-1} \quad z \in \mathbb{D}_\phi$

I. If  $\alpha \in \mathbb{Z}$ , there is one value of  $z^\alpha$       II. If  $\alpha = \frac{p}{q}$ ,  $\gcd(p, q) = 1, p, q \in \mathbb{Z}, q \neq 0$ , there are exactly  $q$  values of  $z^\alpha$

III. If  $\alpha$  is *irrational* or *non-real*, there are infinitely values of  $z^\alpha$       IV.  $1^{1/q}, q \in \mathbb{Z}, q \neq 0$  is  $\{1, w, \dots, w^{q-1}\}, w = \exp(i2\pi/q)$

V. We prefer use  $\exp(z)$  to denote single-valued function, and  $e^z$  to denote multi-valued function.

**Principal Branch:**  $z^\alpha := \exp(\alpha \text{Log}(z))$       **Operation:**  $z^\alpha z^\beta = z^{\alpha+\beta}$  (Using Principal Branch)      **NB:**  $(z_1 z_2)^\alpha \neq z_1^\alpha z_2^\alpha; (z^\alpha)^\beta \neq z^{\alpha\beta}$

## 3 Conformal Maps and Mobius Transformations

### 3.1 Conformal Maps & Definition of Mobius Transformations

**Conformal:** Let  $U$  be open set and  $f : U \rightarrow \mathbb{C}$ . Then  $f$  is conformal iff:  $f$  preserves angles. i.e. 任意两条曲线/直线之间的角度在  $f$  作用下不变.

**Important Theorem:** If  $f : U \rightarrow \mathbb{C}$  is holomorphic, then  $\forall z_0 \in U, f'(z_0) \neq 0, f$  preserves angles.

i.e.  $\forall$  curves  $C_1, C_2$  in  $U$ . If  $C_1, C_2$  intersecting at a point  $z_0 \in U$ .  $C_1, C_2$  在  $z_0$  切线的夹角与  $f(C_1), f(C_2)$  在  $f(z_0)$  切线的夹角一样.

**Extended Complex Plane:**  $\tilde{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$  and define that  $a + \infty = \infty, b \cdot \infty = \infty, \frac{b}{0} = \infty, \frac{0}{\infty} = 0$ .

**Riemann Sphere:** Consider  $(X, Y, Z) \in \mathbb{R}^3$ :  ${}^1z = X + iY \in \mathbb{C}$  is the point  $(X, Y, 0)$  and  ${}^2Z = 0$  is the complex plane.

1. Define the Riemann Sphere:  $S^2 := \{(X, Y, Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1\}$  and consider the **North Pole** is point  $N := (0, 0, 1)$
2. Define  $\phi : \mathbb{C} \rightarrow S^2$  by  $N$  点与  $z = (X, Y, 0)$  点连线与  $S^2$  的交点为  $\phi(z)$       Thus  $\lim_{|z| \rightarrow \infty} \phi(z) = N$        $\phi(\infty) := N$

$$3. \text{ Calculation shows that: } \phi(z) = \phi(x + iy) = \left( \frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right) \quad \psi(X, Y, Z) = \begin{cases} \frac{X+iY}{1-Z}, & (X, Y, Z) \neq N \\ \infty, & (X, Y, Z) = N \end{cases}$$

**Remark:**  $\phi : \tilde{\mathbb{C}} \rightarrow S^2$  is bijection and it's inverse  $\psi : S^2 \rightarrow \tilde{\mathbb{C}}$  is the **stereographic projection**

4. Stereographic projection  $\psi(X, Y, Z)$  maps a circle to either a circle or a straight line. (见上图)

**Mobius Transformation:** A Mobius Transformation is a function form:  $f(z) = \frac{az+b}{cz+d}$  where  $a, b, c, d \in \mathbb{C}; ad \neq bc$

1. **Remark:**  $g(z) = \frac{f(z)}{\sqrt{ad-bc}}$  satisfies  $ad - bc = 1$       If  $a, b, c, d$  defined a mobius transformation, then  $\lambda a, \lambda b, \lambda c, \lambda d$  also.

2. For Complex Matrix:  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with  $\det(M) = ad - bc = 1$ . We define  $f_M = \frac{az+b}{cz+d}$       I.  $f_{M_1 M_2} = f_{M_1} \circ f_{M_2}$   
II.  $f_{M^{-1}} = f_M^{-1}$

3. Extended  $f(z)$  from  $\mathbb{C}$  to  $\tilde{\mathbb{C}}$  by:  $f(-\frac{d}{c}) = \infty$  and  $f(\infty) = \frac{a}{c}$

4. **Translation:**  $f(z) = z + b \Leftrightarrow \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$       **Rotation:**  $f(z) = az, a = e^{i\theta} (|a| = 1) \Leftrightarrow \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & -e^{i\theta/2} \end{pmatrix}$       **Dilation:**  $f(z) = rz, r > 0 \Leftrightarrow \begin{pmatrix} \sqrt{r} & 0 \\ 0 & 1/\sqrt{r} \end{pmatrix}$

**Inversion:**  $f(z) = 1/z \Leftrightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$        $f$  fixes the point at infinity: If  $f(\infty) = \infty$  ps: 除了 inversion 其他都是 fix the point at infinity.

5. **Theorem:**  $f(z) = \frac{az+b}{cz+d}$  be a Mobius Transformation.  $\Rightarrow$  <sup>1</sup> If  $f(\infty) = \infty$ :  $f$  is a composition of finite Translation, Rotation, Dilation  $\Rightarrow c = 0, f(z) = \frac{a}{d}z + \frac{b}{d}$

<sup>2</sup> If  $f(\infty) < \infty$ :  $f$  is composition of finite Translation, Rotation, Dilation and only one inversion.  $\Rightarrow f(z) = \frac{(bc-ad)/c^2}{z+d/c} + \frac{a}{c}$

**Properties of Mobius Transformation: Important:** ★ Möbius transformations map circlines to circlines. ★

1. For mobius transformation  $f(z) = \frac{az+b}{cz+d}$ , if:  $\exists z_1, z_2, z_3 \in \mathbb{C}$  distinct points.  $f(z_1) = z_1, f(z_2) = z_2, f(z_3) = z_3 \Rightarrow f$  is identity.
2. If  $z_1, z_2, z_3 \in \tilde{\mathbb{C}}$  distinct points.  $\exists!$  mobius transformation  $f(z)$  s.t.  $f(z_1) = 1, f(z_2) = 0, f(z_3) = \infty$
3. If  $(z_1, z_2, z_3), (w_1, w_2, w_3) \in \tilde{\mathbb{C}}$  distinct points. Then  $\exists!$  mobius transformation  $f(z)$  s.t.  $f(z_i) = w_i, \forall i \in \{1, 2, 3\}$

**ps:Method to construct 2:** If  $z_i < \infty, f(z) = \frac{z_1 - z_3}{z_1 - z_2} \cdot \frac{z - z_2}{z - z_3}$       If  $z_i = \infty, f(z) = \frac{z - z_2}{z - z_3}, z_1 = \infty \Rightarrow f(z) = \frac{z_1 - z_3}{z_1 - z_2}, z_2 = \infty; f(z) = \frac{z - z_2}{z_1 - z_2}, z_3 = \infty$

**ps:Method to construct 3:** For 3: Let  $f := h^{-1} \circ g$  where  $g(z_i), h(w_i) = \{1, 0, \infty\}$  like part 2.

**Cross-Ratio:** cross-ratio  $[z_1, z_2, z_3, z_4] := f(z_1)$  where  $f$  is mobius transformation s.t.  $f(z_2) = 1, f(z_3) = 0, f(z_4) = \infty$

1. **Formulas:**  $[z_1, z_2, z_3, z_4] = \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3}$        $[\infty, z_2, z_3, z_4] = \frac{z_2 - z_4}{z_2 - z_3}$        $[z_1, \infty, z_3, z_4] = \frac{z_1 - z_3}{z_1 - z_4}$        $[z_1, z_2, \infty, z_4] = \frac{z_2 - z_4}{z_1 - z_4}$        $[z_1, z_2, z_3, \infty] = \frac{z_1 - z_3}{z_2 - z_3}$

2. **Theorem:** If  $f$  is a mobius transformation,  $[f(z_1), f(z_2), f(z_3), f(z_4)] = [z_1, z_2, z_3, z_4]$        $z_i$ 's in this "small section" are distinct.