

## 1 General LP

**General LP Problem:** Decision Variables:  $x_i$     parameters:  $a_{ij}, b_i, c_i$     Objective Function:  $f$     Constraints: subject to 后边的部分

LP problem can be written as:

$$\begin{array}{ll} \text{maximize} & f = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\ & x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \end{array}$$

**Feasible Solution:** If  $\mathbf{x}$  satisfies all constraints (i.e.  $A\mathbf{x} \leq \mathbf{b}$ ), then  $\mathbf{x}$  is a feasible solution. (可行解) **Optimal Sol:** (最优解) (可多个)

**Find Optimal Solution:** Graphical Method: 略. **Vertex Enumeration:** 所有顶点检查, 找出最优解. **Simplex Method:** 后面详述.

**Slack Variables:** For each inequality constraint, we introduce a slack variable  $x_i$  ( $i > n$ ) to convert it to an equation. (松弛变量)

LP problem can be written as: ps:  $x_i \geq 0$  ( $i > n$ ).

$$\begin{array}{ll} \text{maximize} & f = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{n+2} = b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + x_{n+m} = b_m \\ & x_1 \geq 0, x_2 \geq 0, \dots, x_{n+m} \geq 0 \end{array}$$

**Feasible Region:** It's the set  $K$  of the solutions to  $\bar{A}\mathbf{x} = \mathbf{b}$       **Convex Set:** The set  $K$  is convex if  $\forall \mathbf{x}, \mathbf{x}' \in K, \forall \theta \in [0, 1], \mathbf{x}_\theta = (1 - \theta)\mathbf{x} + \theta\mathbf{x}' \in K$ .

**Vertex on Convex Set:** A vertex of the convex set  $K$  is a point  $\mathbf{x} \in K$  which doesn't lie strictly inside any line segment connecting two points in  $K$ .

• **Theorem:** If LP has a unique optimal solution is a vertex.

**Theorem:** If LP has a non-unique solution,  $\exists$  optimal solution at vertex

We can also write the LP problem in matrix form:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{array}{ll} \text{maximize} & f = \mathbf{c}^T \mathbf{x} \\ \text{subject to} & A\mathbf{x} \leq \mathbf{b} \quad \mathbf{x} \geq 0 \end{array}$$

**Optimal Sol:** (最优解) (可多个)

We can also write the LP problem in matrix form:

$$\bar{A} = [A \ I_m], \mathbf{b} = \mathbf{b}, \bar{\mathbf{c}} = \begin{pmatrix} \mathbf{c} \\ \mathbf{0}_{m \times 1} \end{pmatrix}, \mathbf{x} = \begin{pmatrix} \mathbf{x} \\ \vdots \\ x_{n+m} \end{pmatrix}$$

$$\begin{array}{ll} \text{maximize} & f = \bar{\mathbf{c}}^T \mathbf{x} \\ \text{subject to} & \bar{\mathbf{A}}\mathbf{x} = \mathbf{b} \quad \mathbf{x} \geq 0 \end{array}$$

**Solving:**  $\mathbf{x} = \mathbf{x}_b + \mathbf{v}$  where  $A\mathbf{v} = 0$  and  $\mathbf{x}_b = [\mathbf{0} \ \mathbf{b}]^T$

## 2 Simplex Method

## 2.1 Simplex Algorithm

**Solve LP Problem:** Assume  $f = \bar{\mathbf{c}}^T \mathbf{x}$  with  $\bar{\mathbf{A}}\mathbf{x} = \mathbf{b}$  and  $\mathbf{x} \geq 0$ .

1. 修改  $x_i$ /列顺序,  $A$  中换顺序后得到可逆的  $B_{m \times m} \Rightarrow \bar{A}x = b \Leftrightarrow Bx_B + Nx_N = b \quad \bar{c} \rightarrow [c_B \ c_N]^T \quad \bar{A} \rightarrow [B \ N] \quad x \rightarrow [x_B \ x_N]^T \quad \text{Let } \hat{b} = B^{-1}b \quad \hat{N} = B^{-1}N$
2. **Solution:**  $x_B = \hat{b} - \hat{N}x_N$       If  $x_N = 0 \Rightarrow$  It's a basic solution. (But we need to check whether  $x_B \geq 0$ )      Using  $\mathcal{B}, \mathcal{N}$ : Index set of independent/else.
3. **Basic Variables:**  $x_B$       **Nonbasic Variables:**  $x_N$
4. **At Basic Solution:**  $x_B = \hat{b}, x_N = 0 \Rightarrow$  If  $\hat{b} \geq 0$ . Corresponding  $x$  is: <sup>1</sup> vertex of  $K$ ; <sup>2</sup> **Basic Feasible Solution (BFS)**
5. **Basic Costs:**  $c_B^T$       **Nonbasic Costs:**  $c_N^T$       **Reduced Costs:**  $\hat{c}_N = c_N - \hat{N}^T c_B = c_N - N^T B^{-T} c_B \quad \hat{f} = c_B^T \hat{b} \quad x_B, x_N \geq 0$
6. **Objective Value:**  $f = \bar{c}^T x = c_B^T x_B + c_N^T x_N = \hat{f} + \hat{c}_N^T x_N$       If  $\hat{c}_N \leq 0$ , then  $f \leq \hat{f} \Rightarrow$  Corresponding  $x$  is optimal.
7. If  $\hat{c}_N \leq 0$  doesn't hold, we using **Simplex Algorithm**.

### Simplex Algorithm:

1. **Initial Basic Feasible Solution:** Try  $\mathcal{B} = \{n+1, \dots, n+m\}$  and  $\mathcal{N} = \{1, \dots, n\} \Rightarrow B = I, N = A, \mathbf{x}_B = \widehat{\mathbf{b}} = \mathbf{b}, \mathbf{x}_N = \mathbf{0}, \mathbf{c}_B = \mathbf{0}, \mathbf{c}_N = \mathbf{c}$
2. If  $\mathbf{b} \geq 0 \Rightarrow$  Basis is feasible + cont.      Else: Basis  $B$  is not feasible.      Go to “Cases If  $\mathbf{b} \not\geq 0$ ”
3. If  $\widehat{\mathbf{c}}_N \leq 0 \Rightarrow$  Optimal Solution      Else: cont.
4. Let  $q' \in \mathcal{N}$  对应  $\widehat{\mathbf{c}}_N$  中最大 positive 分量的 index      同理, 对应的最大正分量值为  $\widehat{c}_q$       对应的 variable 为  $x_{q'}$       对应  $N$  中的  $[q']$  列为  $\mathbf{a}_q$
5. Let  $\widehat{\mathbf{a}}_q = B^{-1}\mathbf{a}_q$ .      If  $\widehat{\mathbf{a}}_q \leq 0 \Rightarrow$  LP is unbounded.      Else: LP is bounded + cont.
6. Let  $p' \in \mathcal{B}$  be the index corresponding to  $p = \arg \min_{i=1, \dots, m; \widehat{a}_{iq} > 0} \frac{\widehat{b}_i}{\widehat{a}_{iq}}$   $p$  是对应的 index, not value      用  $\bar{\alpha} = \frac{\widehat{b}_p}{\widehat{a}_{pq}}$  代表值  
 $\widehat{b}_i, \widehat{a}_{iq}$  代表  $\widehat{\mathbf{b}}$  的第  $i$  个分量,  $\widehat{\mathbf{a}}_q$  的第  $i$  个分量  
 $p'$  代表能使  $\frac{\widehat{b}_i}{\widehat{a}_{iq}}$  的值最小的 index, 前提条件是  $\widehat{a}_{iq} > 0$       对应的 variable 为  $x_{p'}$
7. Exchange  $p'$  and  $q'$  between  $\mathcal{B}$  and  $\mathcal{N} \Rightarrow$  values of new  $\mathbf{x}_B = \widehat{\mathbf{b}} - \bar{\alpha} \widehat{\mathbf{a}}_q$       Update  $\mathcal{B}, \mathcal{N}, B, N, \widehat{\mathbf{b}}, \dots$       ps: New  $B := B + (a_q - B e_p) e_p^T$
8. Go to 3.

### Case: If $b \neq 0$ : Phase I Problem

1. Subtract **artificial variables**  $x_{n+m+1}, \dots, x_{n+m+m} \geq 0$  and change objective function  $f$  to:

$$\begin{array}{ll} \max & f = \\ \text{s. t.} & a_{11}x_1 + \dots + a_{1n}x_n + x_{n+1} - x_{n+m+1} - \dots - x_{n+2m} = b_1 \\ & \vdots \\ & a_{m1}x_1 + \dots + a_{mn}x_n + x_{n+m} - x_{n+2m} = b_m \\ & x_1 \geq 0, \dots, x_{n+2m} \geq 0 \end{array}$$

2. Let Basic Variables:  $\mathbf{x}_B =$  第  $i$  个元素是:  $x_{n+i} = b_i$  (if  $b_i \geq 0$ ) 是:  $x_{n+m+i} = -b_i$  (if  $b_i < 0$ ) \*
- Let Nonbasic Variables:  $\mathbf{x}_N =$  第  $i$  个元素是:  $x_{n+m+i} = 0$  (if  $b_i \geq 0$ ) 是:  $x_{n+i} = b_i$  (if  $b_i < 0$ )
- Let Basic Matrix  $B =$  第  $i$  列是:  $\mathbf{e}_i$  (if  $b_i \geq 0$ ) 否则:  $-\mathbf{e}_i$  其他列是  $N$  的对应列      Let  $\hat{\mathbf{b}} = B^{-1}\mathbf{b} = |\mathbf{b}| \geq 0$
- Let  $\mathbf{c}_B = \mathbf{x}_B$  对应的  $f$  中的系数       $\mathbf{c}_N =$  同理       $\hat{\mathbf{c}}_N = \mathbf{c}_N - N^T B^{-T} \mathbf{c}_B$
3. Size of Infeasibility: is  $\sum_{i=1}^m x_{n+m+i} \quad f = -\sum_{i=1}^m x_{n+m+i} \leq 0$
4. If  $f = 0$ ,  $\mathbf{x}$  is a BFS.  $\Rightarrow$  Go to **Phase II Problem** (恢复到之前的  $f$  并删除 artificial variables)
- Else: Go to **Simplex Algorithm** 得到辅助问题 Phase I 的最优解.    If  $f < 0$ : LP is infeasible.    If  $f = 0$ : LP is feasible.  $\Rightarrow$  **Phase II**
- Case: If  $\mathbf{b} \not\geq 0$ : Phase II Problem**

## 2.2 More thing about Simplex Algorithm

**Degeneracy:** If  $\hat{\mathbf{b}}$  has any zero component, then  $\mathbf{x}$  is a **degenerate vertex**.      如果  $\hat{\mathbf{b}}$  的第  $p$  个分量  $\hat{b}_p = 0$ , 那么单纯形法可能会陷入循环.

**Theorem:** If no degenerate situation, then Simplex Algorithm will terminate in a finite number of steps.

**Examples:** 1. The worst case of Simplex Algorithm.    2. Cyclic case of Simplex Algorithm.

- Klee-Minty Problem:**  $\max f = \sum_{j=1}^n 10^{j-1} x_j$  ; s.t.  $x_i + 2 \sum_{j=i+1}^n 10^{j-i} x_j \leq 100^{n-i}$  for  $i = 1, \dots, n$  ;  $x_i \geq 0$   
has  $2^n$  vertices.    **Worst Case:**  $2^n - 1$  iterations.
- Hall-McKinnon Problem:**  $\max f = x_1 - 5.5x_2 + 0.75x_3 - 5.75x_4$  ; s.t.  $2.5x_1 - 19.5x_2 - 3.5x_3 + 19.5x_4 + x_5 = 0$ ;  $0.5x_1 - 3.5x_2 - 0.5x_3 + 3.5x_4 + x_6 = 0$ ;  $x_i \geq 0$

## 2.3 Sparsity LP Problem

**Implementation:** 计算/编程中的计算化简      Consider:  $\hat{\mathbf{b}} = B^{-1}\mathbf{b}$     $\hat{\mathbf{c}}_N = \mathbf{c}_N - N^T B^{-T} \mathbf{c}_B$     $\hat{\mathbf{a}}_q = B^{-1} \mathbf{a}_q$

- Solve:  $B\hat{\mathbf{b}} = \mathbf{b}$  to get:  $\hat{\mathbf{b}} = B^{-1}\mathbf{b}$
- Solve:  $B^T \pi = \mathbf{c}_B$  to get:  $\pi = B^{-T} \mathbf{c}_B$
- Solve:  $\hat{\mathbf{c}}_N = \mathbf{c}_N - N^T \pi$  to get:  $\hat{\mathbf{c}}_N = \mathbf{c}_N - N^T B^{-T} \mathbf{c}_B$
- Solve:  $B^T \hat{\mathbf{a}}_q = \mathbf{a}_q$  to get:  $\hat{\mathbf{a}}_q = B^{-T} \mathbf{a}_q$
- Matrix  $B$  is **sparse** and **invertible**. And in Simplex Algorithm, the next matrix  $B$  obtained by  $B := B + (\mathbf{a}_q - B\mathbf{e}_p)\mathbf{e}_p^T$

**Sparse LP Problem:** For LP problem:  $\max f = \mathbf{c}^T \mathbf{x}$  s.t.  $A\mathbf{x} = \mathbf{b}$     $\mathbf{x} \geq 0$

It is **sparse** if the matrix  $A$  is sparse. (i.e. Most of the elements of  $A$  are zero)

**Find Inverse of  $B$ :** Use **Gaussian Elimination** to decomposition  $B$  into  $LU$  where  $L$  is lower triangular and  $U$  is upper triangular.

Then we can solve  $B\mathbf{x} = \mathbf{b}$  by solving  $^1 L\mathbf{y} = \mathbf{b}$  and  $^2 U\mathbf{x} = \mathbf{y}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} & l_{21}u_{14} + u_{24} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} & l_{31}u_{14} + l_{32}u_{24} + u_{34} \\ l_{41}u_{11} & l_{41}u_{12} + l_{42}u_{22} & l_{41}u_{13} + l_{42}u_{23} + l_{43}u_{33} & l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + u_{44} \end{bmatrix}$$

By Solving ■  $\rightarrow$  ■  $\rightarrow$  ■  $\rightarrow$  ■  $\rightarrow$  ■  $\rightarrow$  ■