

1 Basic Knowledge

Useful Complex Number Properties: $|Re(z)|, |Im(z)| \leq |z|$ $Re(z) = \frac{z+\bar{z}}{2}, Im(z) = \frac{z-\bar{z}}{2i}, |z|^2 = z\bar{z}$
Triangle (Reverse) Inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$ $|z_1| - |z_2| \leq |z_1 - z_2|$ $\ominus (Re(zw) = 0 \Leftrightarrow \overline{zw} = -zw; Im(zw) = 0 \Leftrightarrow zw = \overline{zw})$
Argument: $arg(z) := \{\theta : z = |z|e^{i\theta}\} = \{Arg(z) + 2\pi k : k \in \mathbb{Z}\}$ **Principle Value of Argument:** $Arg(z) \in (-\pi, \pi]$
Operations on Argument: $arg(z_1 z_2) = arg(z_1) + arg(z_2)$ $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$ $arg(\bar{z}) = -arg(z)$

2 Holomorphic Functions

2.1 Open/Closed Set | Limit Point | limit of Sequence | Continuous of Function

Open/Closed/Punctured ε -disc: $D_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ $\bar{D}_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq \varepsilon\}$ $D'_\varepsilon(z_0) := \{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}$
Open/Closed Set in \mathbb{C} : $U \subset \mathbb{C}$ is **open** if $\forall z_0 \in U, \exists \varepsilon > 0, D_\varepsilon(z_0) \subseteq U$ U is **closed** if $\mathbb{C} \setminus U$ is open **Lemma:** $D_\varepsilon, D'_\varepsilon$ open, \bar{D}_ε closed.
Limit Point of S : $z_0 \in \mathbb{C}$ is a limit point of S if: $\forall \varepsilon > 0, D'_\varepsilon(z_0) \cap S \neq \emptyset$ **Bounded:** S is bounded if $\exists M > 0$ s.t. $|z| \leq M, \forall z \in S$
Closed of Set S : $\bar{S} :=$ 所有 S 的 limit point 和 S 的点. **Property:** Let $S \subseteq \mathbb{C}$, then S is closed $\Leftrightarrow S = \bar{S}$.
Limit of sequence: Sequence $(z_n)_{n \in \mathbb{N}}$ has limit z if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N \Rightarrow |z_n - z| < \varepsilon$. limit rules 依旧成立
1. **Lemma|Important:** $\lim z_n = z \Leftrightarrow \lim Re(z_n) = Re(z)$ and $\lim Im(z_n) = Im(z)$
2. **Cauchy:** Sequence $(z_n)_{n \in \mathbb{N}}$ is cauchy if: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N \Rightarrow |z_m - z_n| < \varepsilon$ **Lemma:** Cauchy \Leftrightarrow convergent.
3. **Lemma|Closed of Set:** $S \subseteq \mathbb{C}, z \in \mathbb{C} \Rightarrow [z \in \bar{S} \Leftrightarrow \exists \text{ sequence } (z_n)_{n \in \mathbb{N}} \in S \text{ s.t. } \lim z_n = z]$
4. **Bolzano-Weierstrass:** Every bounded sequence in \mathbb{C} has a convergent subsequence.
Complex Functions: $\forall f : \mathbb{C} \rightarrow \mathbb{C}$ we can write it as: $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ where $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$
Limit of Function: $a_0 \in \mathbb{C}$ is the limit of f at z_0 if: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |z - z_0| < \delta \Rightarrow |f(z) - a_0| < \varepsilon$ limit rules 依旧成立
Lemma|Important: $\lim_{z \rightarrow z_0} f(z) \Leftrightarrow \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = Re(a_0)$ and $\lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = Im(a_0)$
continuous of Function: f is continuous at z_0 if: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$ continuous rules 依旧成立
1. **Lemma|Important:** f is continuous at $z_0 \Leftrightarrow u, v$ are continuous at (x_0, y_0)
2. **'Extreme Value Theorem':** f is continuous on a closed and bounded set $S \subseteq \mathbb{C}$, then $f(S)$ is closed and bounded.
3. **Lemma|continuous \Leftrightarrow open:** f is continuous $\Leftrightarrow \forall$ open set U , preimage $f^{-1}(U) := \{z \in \mathbb{C} | f(z) \in U\}$ is open.

2.2 Differentiable | Holomorphic Function

Differentiable: Let $z_0 \in \mathbb{C}$ and $U \subseteq \mathbb{C}$ be neighborhood of z_0 , then $f : U \rightarrow \mathbb{C}$ is differentiable at z_0 if: $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.
I. f is differentiable $\Rightarrow f$ is continuous. **II. Holomorphic \Leftrightarrow Differentiable + neighborhood** (除非是一个点时不成立, $|z|$) diff rules + chain rule 成立
Cauchy-Riemann Equations: If $z_0 = x_0 + iy_0, f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 \Rightarrow u_x = v_y, v_x = -u_y$ at (x_0, y_0) .
If $z_0 = x_0 + iy_0, f = u + iv$ satisfies: $^1 u, v$ are continuously differentiable on a neighborhood of (x_0, y_0) and:
 $^2 u, v$ satisfies Cauchy-Riemann Equations at (x_0, y_0) . $\Rightarrow f$ is differentiable at z_0 .
ps: 常见可导复数函数: $e^z, \sin z, \cos z, \log z, z^\alpha$, polynomial, sinh, cosh, $\Gamma(z), |z|^2$ (at 0), constant **ps:** 常见不可导复数函数: $\bar{z}, |z| \cdot \bar{z}, Re(z), Im(z), Arg(z)$
Harmonic Function: $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ is harmonic if: $\forall (x, y) \in \mathbb{R}^2 \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$ (Laplace Equation)
Lemma: If $f = u + iv$ is holomorphic on \mathbb{C} and u, v are twice continuously differentiable, $\Rightarrow u, v$ are harmonic.
Harmonic Conjugate: Let $u, v : U \rightarrow \mathbb{R}, U \subseteq \mathbb{R}^2$ be harmonic functions. u, v are harmonic conjugate if: $f = u + iv$ is holomorphic on U .
Properties of Polynomial: The domain of rational function and polynomial are always open. **Lemma:** If $P(z_0) = 0$ then $P(\bar{z}_0) = 0$
First-order Operator ∂ : $\partial := \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$ $\bar{\partial} := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$ || $f = u + iv$ satisfies C-R Equations $\Leftrightarrow \bar{\partial} f = 0$
sin/cos Functions: $\sin z := \frac{e^{iz} - e^{-iz}}{2i}$ $\cos z := \frac{e^{iz} + e^{-iz}}{2}$ **Exponential Function:** $e^z = e^{x+iy} = e^x(\cos(y) + i \sin(y))$
1. $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$ $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$
2. $\sin(z + w) = \sin(z) \cos(w) + \cos(z) \sin(w)$ $\cos(z + w) = \cos(z) \cos(w) - \sin(z) \sin(w)$
3. $\sin^2 z + \cos^2 z = 1$ $\sin(z + \frac{\pi}{2}) = \cos(z)$ $\sin(z + 2k\pi) = \sin(z)$ $\cos(z + 2k\pi) = \cos(z)$ * $\sin z, \cos z$ NOT bounded.
Hyperbolic Functions: $\sinh z := \frac{e^z - e^{-z}}{2}$ $\cosh z := \frac{e^z + e^{-z}}{2}$ || $\sinh(iz) = i \sin z$ $\cosh(iz) = \cos z$

$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$
z^n	nz^{n-1}	e^z	e^z	$\sin(z)$	$\cos(z)$	$\cos(z)$	$-\sin(z)$	$\sinh(z)$	$\cosh(z)$	$\cosh(z)$	$\sinh(z)$	$\ominus \log z$	$\ominus 1/z$