HAlg Note

1 Basic Knowledge

Def of Group $(G,*)$: A set G with a operator $*$ is a group if: **Closure**: $\forall g,h\in G, g*h\in G$; **Associativity**: $\forall g,h,k\in G,(g*h)*k=g*(h*k)$; **Identity**: $\exists e\in G,\forall g\in G, e*g=g*e=g$; **Inverse**: $\forall g\in G,\exists g^{-1}\in G, g*g^{-1}=g^{-1}*g=e$. G,H groups, then $G\times H$ also.

Subgroup: $H\subseteq G$ is a subgroup if: $\forall h_1,h_2\in H$ **I**: $H\neq\emptyset$; **II**: $h_1*h_2\in H$; **III**: $h_1^{-1}\in H$.

Field (F): A set F is a field with two operators: (addition)$+:F\times F\to F;(\lambda,\mu)\to\lambda+\mu$ (multiplication)$\cdot:F\times F\to F;(\lambda,\mu)\to\lambda\mu$ if: $(F,+)$ and $(F\setminus\{0_F\},\cdot)$ are abelian groups with identity $0_F,1_F$. and $\lambda(\mu+\nu)=\lambda\mu+\lambda\nu$ $e.g. Fields:\mathbb{R},\mathbb{C},\mathbb{Q},\mathbb{Z}/p\mathbb{Z}=\mathbb{F}_p$

F**-Vector Space (V)**: A set V over a field F is a vector space if: V is an abelian group $V=(V,\dotplus)$ and $\forall\vec{v},\vec{w}\in V\ \lambda,\mu\in F$ e.g. $Poly:\mathbb{R}[x]_{<n}$ $\exists$ map $F\times V\to V:(\lambda,\vec{v})\to\lambda\vec{v}$ satisfies: **I**: $\lambda(\vec{v}\dotplus\vec{w})=(\lambda\vec{v})\dotplus(\lambda\vec{w})$ **II**: $(\lambda+\mu)\vec{v}=(\lambda\vec{v})\dotplus(\mu\vec{v})$ **III**: $\lambda(\mu\vec{v})=(\lambda\mu)\vec{v}$ **IV**: $1_F\vec{v}=\vec{v}$

Vector Subspaces Criterion: $U\subseteq V$ is a subspace of V if: **I.** $\vec{0}\in U$ **II.** $\forall\vec{u},\vec{v}\in U,\forall\lambda\in F$: $\vec{u}+\vec{v}\in U$ and $\lambda\vec{u}\in U$ (or: $\lambda\vec{u}+\mu\vec{v}\in U$)

· **property**: If U,W are subspaces of V, then $U\cap W$ and $U+W$ are also subspaces of V. ps: $U+W:=\{\vec{u}+\vec{w}:\vec{u}\in U,\vec{w}\in W\}$

Complement-wise Operations: $\phi:V_1\times V_2\to V_1\oplus V_2$ by **I**:$(\vec{v_1},\vec{u_1})+(\vec{v_2},\vec{u_2}):=(\vec{v_1}+\vec{v_1},\vec{u_1}+\vec{u_2})$, $\lambda(\vec{v},\vec{u}):=(\lambda\vec{v},\lambda\vec{u})$ (ps:V_1,V_2 通过 ϕ 定义的 map 所形成的 vector space 记作 $V_1\oplus V_2$)

Projections: $pr_i:X_1\times\cdots\times X_n\to X_i$ by $(x_1,...,x_n)\mapsto x_i$ **Canonical Injections**: $in_i:X_i\to X_1\times\cdots\times X_n$ by $x\mapsto(0,...,0,x,0,...,0)$

2 Vector Spaces/Subspaces | Generating Set | Linear Independent | Basis

Generating (subspaces) $\langle T\rangle$: $\langle T\rangle:=\{\alpha_1\vec{v}_1+\cdots+\alpha_n\vec{v}_n:\alpha_i\in F,\vec{v}_i\in T,r\in\mathbb{N}\}$ $\langle\emptyset\rangle:=\{\vec{0}\}$ If T is subspace $\Rightarrow\langle T\rangle=T$.

1. **Proposition**: $\langle T\rangle$ is the smallest subspace containing T. (i.e. $\langle T\rangle$ is the intersection of all subspaces containing T)
2. **Generating Set**: V is vector space, $T\subseteq V$. T is generating set of V if $\langle T\rangle=V$. **Finitely Generated**: $\exists$ finite set $T,\langle T\rangle=V$
3. **External Direct Sum**: 一个" 代数结构", 定义为 set 是 $V_1\oplus\cdots\oplus V_n:=V_1\times\cdots\times V_n$ 且有一组运算法则 component-wise operations
4. **Connect to Matrix**: Let $E=\{\vec{v_1},...,\vec{v_n}\}$, E is GS of V. Let $A=[\vec{v_1},...,\vec{v_n}]\Rightarrow\forall\vec{b}\in V,\exists\vec{x}=(x_1,...,x_n)^T$ s.t. $A\vec{x}=\vec{b}$ (i.e. linear map:$\phi:\vec{x}\mapsto A\vec{x}$ is surjective)

Linearly Independent: $L=\{\vec{v_1},\vec{v_2},...,\vec{v_r}\}$ is linearly independent if: $\forall c_1,...,c_r\in F, c_1\vec{v_1}+\cdots+c_r\vec{v_r}=\vec{0}\Rightarrow c_1=\cdots=c_r=0$.

· **Connect to Matrix**: Let $L=\{\vec{v_1},...,\vec{v_n}\}$, L is LI of V. Let $A=[\vec{v_1},...,\vec{v_n}]\Rightarrow\forall\vec{x}\in F^n, A\vec{x}=0\ (or\ \vec{0})\Rightarrow\vec{x}=0(or\ \vec{0})$ (i.e. linear map ϕ:$\vec{x}\mapsto A\vec{x}$ is injective)

Basis & Dimension: If V is finitely generated. $\Rightarrow\exists$ subset $B\subseteq V$ which is both LI and GS. (B is basis) **Dim**: $\dim V:=|B|$

· **Connect to Matrix**: Let $B=\{\vec{v_1},...,\vec{v_n}\}$ is basis of V. Let $A=[\vec{v_1},...,\vec{v_n}]\Rightarrow\forall\vec{x}=(x_1,...,x_n)^T$ s.t. $\phi:\vec{x}\mapsto A\vec{x}$ is 1-1 & onto (Bijection)

Relation|GS,LI,Basis,dim: Let V be vector space. L is linearly independent set, E is generating set, B is basis set.

1. **GS|LI**: $|L|\le|E|$ (can get: dim unique) **LI→Basis**: If V finite generate $\Rightarrow\forall L$ can extend to a basis. If $L=\emptyset$, prove $\exists B$ $ker f\cap im f=\{0\}$
2. **Basis|max,min**: $B\Leftrightarrow B$ is minimal GS $(E)\Leftrightarrow B$ is maximal LI (L). **Uniqueness|Basis**: 每个元素都可以由 basis 唯一表示.
3. **Proper Subspaces**: If $U\subset V$ is proper subspace, then $\dim U<\dim V$. $\Rightarrow$ If $U\subseteq V$ is subspace and $\dim U=\dim V$, then $U=V$.
4. **Dimension Theorem**: If $U,W\subseteq V$ are subspaces of V, then $\dim(U+W)=\dim U+\dim W-\dim(U\cap W)$

Complementary: $U,W\subseteq V$,U,V subspaces are complementary $(V=U\oplus W)$ if: $\exists\phi:U\times W\to V$ by $(\vec{u},\vec{w})\mapsto\vec{u}+\vec{w}$ i.e. $\forall\vec{v}\in V$, we have unique $\vec{u}\in U,\vec{w}\in W$ s.t. $\vec{v}=\vec{u}+\vec{w}$. ps: It's a linear map.

3 Linear Mapping | Rank-Nullity| Matrices | Change of Basis

ps: 默认 V,W F**-Vector Spaces.**

3.1 Linear Mapping | Rank-Nullity

Linear Mapping/Homomorphism(Hom): $f:V\to W$ is *linear map* if: $\forall\vec{v}_1,\vec{v}_2\in V,\forall\lambda\in F$. $f(\vec{v}_1+\vec{v}_2)=f(\vec{v}_1)+f(\vec{v}_2)$ and $f(\lambda\vec{v}_1)=\lambda f(\vec{v}_1)$

· **Isomorphism**: = LM & Bij. **Endomorphism(End)**: = LM & $V=W$. **Automorphism(Aut)**: = LM & $V=W$ **Monomorphism**: = LM & 1-1. **Epimorphism**: = LM & onto.

· **Kernel**: $\ker f:=\{\vec{v}\in V:f(\vec{v})=\vec{0}\}$(It's subspace) **Image**: $Imf:=\{f(\vec{v}):\vec{v}\in V\}$(It's subspace) **Rank**:$=\dim(Imf)$ **Nullity**:$=\dim(\ker f)$ **Fixed Point** X^f: $X^f:=\{x\in X:f(x)=x\}$

Property of Linear Map: Let $f,g\in Hom$: **a.**$f(\vec{0})=\vec{0}$ **b.** f is 1-1 iff $\ker f=\{\vec{0}\}$ **c.** $f\circ g$ is linear map.

1. **Determined**: f is determined by $f(\vec{b_i}),\vec{b_i}\in\mathcal{B}_{basis}$ ($\star$ **i.e.** $f(\sum_i\lambda_i\vec{v_i}):=\sum_i\lambda_i f(\vec{v_i})$)
2. **Classification of Vector Spaces**: $\dim V=n\Leftrightarrow f:F^n\xrightarrow{\sim}V$ by $f(\lambda_1,...,\lambda_n)\mapsto\sum_{i=1}^n\lambda_i\vec{v_i}$ is isomorphism.
3. **Left/Right Inverse**: f is 1-1 $\Rightarrow\exists$ left inverse g s.t. $g\circ f=id$ 考虑 direct sum f is onto $\Rightarrow\exists$ right inverse g s.t. $f\circ g=id$
4. $^\ominus$**More of Left/Right Inverse**: $f\circ g=id\Rightarrow g$ is 1-1 and f is onto. 使用 kernel=0 来证明

Rank-Nullity Theorem: For linear map $f:V\to W$, $\dim V=\dim(\ker f)+\dim(Imf)$ Following are properties:

1. **Injection**: f is 1-1 $\Rightarrow\dim V\le\dim W$ **Surjection**: f is onto $\Rightarrow\dim V\ge\dim W$ Moreover, $\dim W=\dim imf$ iff f is onto.
2. **Same Dimension**: f is isomorphism $\Rightarrow\dim V=\dim W$ **Matrix**: $\forall M$, column rank $c(M)$ = row rank $r(M)$.
3. **Relation**: If V,W finite generate, and $\dim V=\dim W$, Then: f is isomorphism $\Leftrightarrow f$ is 1-1 $\Leftrightarrow f$ is onto.

3.2 Matrices | Change of Basis | Similar Matrices | Trace

Matrix: For $A_{n\times m},B_{m\times p},AB_{n\times p}:=(AB)_{ij}=\sum_{k=1}^m a_{ik}b_{kj}$ **Transpose**: $A_{m\times n}^T:=(A^T)_{ij}=a_{ji}$

Invertible Matrices: A is invertible if $\exists B,C$ s.t. $BA=I$ and $AC=I$ || $\exists B,\ BA=I\Leftrightarrow\exists C,AC=I\Leftrightarrow\exists A^{-1}$ ${}_{\mathcal{B}}[f^{-1}]_{\mathcal{A}}={}_{\mathcal{A}}[f]_{\mathcal{B}}^{-1}$

Representing matrix of linear map ${}_{\mathcal{B}}[f]_{\mathcal{A}}$: $f:V\to W$ be linear map, $\mathcal{A}=\{\vec{v}_1,...,\vec{v}_n\}$ is basis of V, $\mathcal{B}=\{\vec{w}_1,...,\vec{w}_m\}$ is basis of W.

1. ${}_{\mathcal{B}}[f]_{\mathcal{A}}:=A$ (matrix) where $f(\vec{v}_{i\in\mathcal{A}})=\sum_{j\in\mathcal{B}}A_{ji}\vec{w}_j$ $\exists M_{\mathcal{B}}^{\mathcal{A}}:Hom_F(V,W)\xrightarrow{\sim}Mat(n\times m;F)$
2. If $\vec{v}\in V$, then ${}_{\mathcal{A}}[\vec{v}]:=\mathbf{b}$ (vector) where $\vec{v}=\sum_{i\in\mathcal{A}}\mathbf{b}_i\vec{v}_i$
3. **Theorems**: $[f\circ g]=[f]\circ[g]$ ${}_{\mathcal{C}}[f\circ g]_{\mathcal{A}}={}_{\mathcal{C}}[f]_{\mathcal{B}}\circ{}_{\mathcal{B}}[g]_{\mathcal{A}}$ ${}_{\mathcal{B}}[f(\vec{v})]={}_{\mathcal{B}}[f]_{\mathcal{A}}\circ{}_{\mathcal{A}}[\vec{v}]$ ${}_{\mathcal{A}}[f]_{\mathcal{A}}=I\Leftrightarrow f=id$
4. **Change of Basis**: Define *Change of Basis Matrix*:$={}_{\mathcal{A}}[id_V]_{\mathcal{B}}$ ${}_{\mathcal{B}'}[f]_{\mathcal{A}'}={}_{\mathcal{B}'}[id_W]_{\mathcal{B}}\circ{}_{\mathcal{B}}[f]_{\mathcal{A}}\circ{}_{\mathcal{A}}[id_V]_{\mathcal{A}'}$ ${}_{\mathcal{A}'}[f]_{\mathcal{A}'}={}_{\mathcal{A}}[id_V]_{\mathcal{A}'}^{-1}\circ{}_{\mathcal{A}}[f]_{\mathcal{A}}\circ{}_{\mathcal{A}}[id_V]_{\mathcal{A}'}$

Elementary Matrix: $I + \lambda E_{ij}$ (cannot $I - E_{ii}$) 就是初等矩阵, 左乘代表 j 行乘 λ 倍加到第 i 行, 右乘代表 j 列乘 λ 倍加到第 i 列 $\Rightarrow$ Invertible!

1. 交换 i, j 列/行: $P_{ij} = diag(1, ..., 1, -1, 1, ..., 1)(I + E_{ij})(I - E_{ji})(I + E_{ij})$ where -1 in jth place.
2. **Row Echelon Form|Smith Normal Form**: $\tilde{A}$: REF 通过左乘初等矩阵可以实现 $\tilde{\tilde{A}}$: $S(n, m, r)$ 通过 $\tilde{A}$ 右乘初等矩阵可以实现

Smith Normal Form: $\forall A, \exists$ invertible P, Q s.t. $PAQ = S(n, m, r) := n \times m$ 的矩阵, 对角线前 r 个是 1, 后面 0. **Lemma**: $r = r(A) = c(A)$

· Every linear map $f : V \to W$ can be representing by ${}_{\mathcal{B}}[f]_{\mathcal{A}} = S(n, m, r)$ for some basis $\mathcal{A}, \mathcal{B}$ of V, W.

Similar Matrices: $N = T^{-1}MT \Leftrightarrow M, N$ are similar. *Special Case*: If $N = {}_{\mathcal{B}}[f]_{\mathcal{B}}, M = {}_{\mathcal{A}}[f]_{\mathcal{A}}$, then $N = T^{-1}MT$. where $T = {}_{\mathcal{A}}[id_V]_{\mathcal{B}}$

1. If $A \sim B$ iff A is similar to B, then $\sim$ is an equivalence relation. ${}_{\mathcal{A}'}[f]_{\mathcal{A}'} \sim_{\mathcal{A}} [f]_{\mathcal{A}}$
2. If $\mathcal{B} = \{p(\vec{v}_1), ..., p(\vec{v}_n)\}$ and $\mathcal{A} = \{\vec{v}_1, ..., \vec{v}_n\}$ where $p : V \xrightarrow{\sim} V$. Then ${}_{\mathcal{A}}[id_V]_{\mathcal{B}} = {}_{\mathcal{A}}[p]_{\mathcal{A}}$
3. If V is a vector space over F, $[A, B$ are *similar* matrices. $\Leftrightarrow$ $A = {}_{\mathcal{A}}[f]_{\mathcal{A}}, B = {}_{\mathcal{B}}[f]_{\mathcal{B}}$ for some basis $\mathcal{A}, \mathcal{B}; f : V \to V]$
4. Set of *Endomorphism* is in a bijection correspondence with the equivalence class of matrices under $\sim$. 一个自同态 **End** 就对应一个相似矩阵的等价类

Trace: $tr(A) := \sum_i a_{ii}$ and $tr(f) := tr({}_{\mathcal{A}}[f]_{\mathcal{A}})$ | $tr(AB) = tr(BA)$ $tr(\lambda A + \mu B) = \lambda tr(A) + \mu tr(B)$ $tr(N) = tr(M)$ if M, N similar.

4 Rings | Polynomials | Ideals | Subrings

4.1 Rings | Polynomial Rings

Ring $(R, +, \cdot)$: A set R with two operators $+, \cdot$ is a ring if: e.g. $Mat(n, F)$; $R[X]$; $\mathbb{Z}/m\mathbb{Z}$; $\mathbb{Z}$

1. $(R, +)$ is an *abelian group* with identity 0_R. **Commutative Ring**: add: $\forall a, b \in R, ab = ba$.
2. $(R, \cdot)$ is a **monoid** with identity 1_R. i.e. **Associativity**: $\forall a, b, c \in R, (a \cdot b) \cdot c = a \cdot (b \cdot c)$. **Identity**: $\forall a \in R, 1_R \cdot a = a \cdot 1_R = a$.
3. **Distributive**: $\forall a, b, c \in R$: $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$. ps: 默认 monoid 是 closure 的

· If R is ring $\Rightarrow [1_R = 0_R \Leftrightarrow R = \{0\}]$ i.e. For any non-zero ring, $1_R \neq 0_R$ **Field**: Commutative ring + multiplicative inverse = Field.

Properties of Ring: $\forall a, b \in R$. **I.** $0 \cdot a = a \cdot 0 = 0$ **II.** $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$ **III.** $(-a) \cdot (-b) = a \cdot b$

Unit: $a \in R$ is unit if it's *Invertible*. i.e. $\exists a^{-1} \in R$ s.t. $aa^{-1} = a^{-1}a = 1_R$ **Group of Unit** $(R^\times, \cdot) := \{a \in R : a \text{ is unit}\}$

Zero-divisors: $a \in R$ is *zero-divisor* if $\exists b \in R, b \neq 0$ s.t. $ab = 0$ or $ba = 0$ *Field has no zero-divisors.* · e.g. $\mathbb{Z}^\times = \{-1, 1\}$; 1_R is a unit.

Integral Domain: A *commutative* ring R is an integral domain if it has no zero-divisors. e.g. $\mathbb{Z}/p\mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}, ...$

Properties of Integral Domain: $\forall a, b \in R$. **I.** $ab = 0 \Rightarrow a = 0$ or $b = 0$ **II.** $a, b \neq 0 \Rightarrow ab \neq 0$ **III.** $ac = bc, a \neq 0 \Rightarrow b = c$

· *Field is Integral Domain* **Every finite integral domain is a field** $\mathbb{Z}/p\mathbb{Z}$ is field iff p is prime. e.g.(integral domain) $\mathbb{Z}$; $\mathbb{Z}/p\mathbb{Z}$

Polynomial Ring $R[X]$: $R[X] := \{a_nX^n + \cdots + a_1X + a_0 : a_i \in R, n \in \mathbb{N}\}$ where X is **indeterminate** $\Leftarrow X \notin R$ and $\forall x \in R, Xa = aX$

1. **Degree**: $\deg(P) := \max\{n \in \mathbb{N} : a_n \neq 0\}$ **Leading Coefficient**: a_n **Monic**: $a_n = 1$ ps: Polynomial NOT a function
2. **Lemma**: ^{1} *R integral domain|no zero-divisors* $\Rightarrow R[X]$ also. ^{2} *R integral domain or no zero-divisor* $\Rightarrow \deg(PQ) = \deg(P) + \deg(P)$
3. **Division and Remainder**: If R is *integral domain* and $P, Q \in R[X]$, Q **monic** $\exists! A, B \in R[X]$ s.t. $P = AQ + B$ and $\deg(B) < \deg(Q)$
4. **Function | Factorize**: If R is *commutative ring* $\Rightarrow$ ^{1} $R[X] \to Maps(R, R)$ (可以视作函数) ^{2} $\lambda \in R$ is root of $P \Leftrightarrow (X - \lambda) \mid P(X)$
5. **Roots**: If R is *Integral domain*: P has at most $\deg(P)$ roots.

Algebraically Closed: $R = F$ field is *algebraically closed* if every non-constant polynomial has a root in F. e.g. $\mathbb{C}$

· **Decomposes**: If F field is *algebraically closed* $\Rightarrow P$ decomposes into: $P(X) = a(X - \lambda_1) \cdots (X - \lambda_n), a \in F^\times$ i.e. $a \neq 0$

4.2 Homomorphism | Ideals | Subrings

Ideal ($I \trianglelefteq R$): A subset $I \subseteq R$ (ring) is an ideal if: **I.** $I \neq \emptyset$ **II.** $\forall a, b \in I, a - b \in I$ **III.** $\forall i \in I, \forall r \in R, ri, ir \in I$

1. **Generated Ideal** ${}_R\langle T\rangle$: $T \subseteq R$ (ring), where R is *commutative ring*. We define ${}_R\langle T\rangle := \{\sum_{i=1}^n r_it_i : n \in \mathbb{N}, r_i \in R, t_i \in T\}$
   · **Lemma**: ${}_R\langle T\rangle$ is the smallest ideal containing T. **Principal Ideal**: ${}_R\langle a\rangle$ **Proper Ideal**: $I \neq R$ ps: R 一定是 commutative ring
2. If I, J are *ideals* of R. Then $I + J$; $I \cap J$ are also ideals.

Subring Test: $R' \subseteq R$ (ring) is a subring if: **I.** $1_R \in R'$ **II.** $\forall a, b \in R', a - b \in R'$ **III.** $\forall a, b \in R', ab \in R'$

· If $f : R \to S$ is *ring homomorphism*, and R' is subring of R. $\Rightarrow$ $f(R')$ is subring of S.

Ring Homomorphism: R, S are rings, $f : R \to S$ is *ring homomorphism* if: **I.** $f(a + b) = f(a) + f(b)$ **II.** $f(ab) = f(a)f(b)$ $f(1_R) = 1_S$ is NOT need

1. **Second Def**: f is ring homomorphism if: $f : (R, +) \to (S, +)$ is group homomorphism and $f(xy) = f(x)f(y)$.
2. **I.** $f(0_R) = 0_S$ **II.** $f(-a) = -f(a)$ **III.** $f(a^n) = (f(a))^n$ **IV.** $f(x - y) = f(x) - f(y)$ **V.** $f(mx) = mf(x)$
3. **Kernel**: $\ker f := \{a \in R : f(a) = 0_S\}$ is an *ideal* **Image**: $Imf := \{f(a) : a \in R\}$ is a *subring*. **1-1**: f is 1-1 $\Leftrightarrow \ker f = \{0_R\}$

4.3 Equivalence Relation

5 Inner Product Spaces | Orthogonal Complement / Proj | Adjoints and Self-Adjoint

6 Jordan Normal Form | Spectral Theorem