

1 Basic Knowledge

Def of ODE & ODEs: (1st order) ODE: $\frac{dy}{dt} = f(t, y)$ & ODEs: $\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y})$, $\mathbf{y} = (y_1, \dots, y_d)^T$, $\mathbf{f}(t, \mathbf{y}) = (f_1(t, \mathbf{y}), \dots, f_d(t, \mathbf{y}))^T$

Autonomous: $\frac{dy}{dt} = \mathbf{f}(\mathbf{y}) \Rightarrow$ autonomous ODE(s). $\parallel \Downarrow$ New Autonomous ODEs: $\frac{dy}{ds} = \mathbf{f}(y_{d+1}, \mathbf{y})$ and $\frac{dy_{d+1}}{ds} = 1$

· **Change to Autonomous:** For $\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y})$. Let $y_{d+1} = t$ and new independent variable s s.t. $\frac{dt}{ds} = 1 \uparrow$

Linearity: ODE: $\frac{dy}{dt} = f(t, y)$ is linearity if $f(t, y) = a(t)y + b(t)$ \parallel ODEs: If each ODE is linear, then the ODEs are linear.

Picard's Theorem: If $f(t, y)$ is continuous in $D := \{(t, y) : t_0 \leq t \leq T, |y - y_0| < K\}$ and $\exists L > 0$ (Lipschitz constant) s.t.

$\forall (t, u), (t, v) \in D \quad |f(t, u) - f(t, v)| \leq L|u - v|$ (ps: Can use MVT). And Assume that $M_f(T - t_0) \leq K$, $M_f := \max\{|f(t, u)| : (t, u) \in D\}$

\Rightarrow **Then**, \exists a unique continuously differentiable solution $y(t)$ to the IVP $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$ on $t \in [t_0, T]$.

Existence & Uniqueness Theorem: IVP $\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y})$, $\mathbf{y}(t_0) = \mathbf{y}_0$. If $f(t, y)$ and $\frac{\partial f}{\partial y_i}$ are continuous in a neighborhood of (t_0, \mathbf{y}_0) .

\Rightarrow **Then**, $\exists I := (t_0 - \delta, t_0 + \delta)$ s.t. \exists a unique continuously differentiable solution $\mathbf{y}(t)$ to the IVP on $t \in I$.

2 Acknowledge

Notation	Meaning	Notation	Meaning
$[a, b]$	Approximate function for $t \in [a, b]$	$t_0 = a \mid t_N = b$	Assume that $t_0 = a, t_N = b$
N	number of timesteps (i.e. Break up interval $[a, b]$ into N equal-length sub-intervals)	h	stepsize ($h = \frac{b-a}{N}$)
t_i	Define $N + 1$ points: t_0, t_1, \dots, t_N	t_m	$t_m = a + h \cdot m = t_0 + h \cdot m$
y_i	Approximation of y at point $t = t_i$ (Except y_0)	$y(t_i)$	Exact value of y at point $t = t_i$

3 Euler's Method and Taylor Series Method

Euler's Method Algorithm: Approximate ODE $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$ with number of steps N . (Similarly for ODEs)

\Rightarrow for $n = 0, 1, 2, \dots, N - 1$: $y_{n+1} = y_n + hf(t_0 + nh, y_n) = y_n + hf(t_n, y_n)$ **end** (ps: \Downarrow Can get $|y''| < M$)

Boundedness Theorem: For $\frac{dy}{dt} = f(t, y)$, $y(a) = y_0$ and suppose there exists a unique, twice differentiable, solution $y(t)$ on $[a, b]$.

Suppose: y is continuous and $|\frac{\partial f}{\partial y}| \leq L$. \Rightarrow the solution y_n given Euler's method satisfies: $e_n = |y_n - y(t_n)| \leq Dh$, $D = e^{(b-a)L} \frac{M}{2L}$

· **Lemma:** If $v_{n+1} \leq Av_n + B$, then $v_n \leq A^n v_0 + \frac{A^n - 1}{A - 1} B$ If $v_n = e_n := y_n - y(t_n)$, then $A = 1 + hL$, $B = h^2 M / 2$ (suppose $|y''| < M$)

Order Notation (\mathcal{O}): we write $z(h) = \mathcal{O}(h^p)$ if $\exists C, h_0 > 0$ s.t. $|z| \leq Ch^p$, $0 < h < h_0$

Flow Map (Φ, Ψ): $\Phi_{t_0, h}(y_0) = y(t_0 + h)$ Clearly, $\Phi(t_n + h) = y(t_n + h) = \Phi_h(y(t_n)) = y(t_{n+1})$.

· $\Psi_{t_n, h}(y_n) = y_{n+1} :=$ Numerical method for ODE Clearly, $\Psi(t_n + h) = y_{n+1} = \Psi_h(y_n)$

Taylor Series Method: Approximate ODE $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$ with n -order Methods: 用 Taylor Series 在 $t_0 + h$ 处展开保留到 n 阶

· $\Phi_{t, h}(y) = y + hf(t, y) + \frac{1}{2}h^2[f_t(t, y) + f_y(t, y)f(t, y)] + \frac{1}{6}y'''(t, y)h^3 + \dots$ (For one variable y)

· ps: Taylor Series: $y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{h^2}{2}y''(t_0) + \dots + \frac{h^{n-1}}{(n-1)!}y^{(n-1)}(t_0) + \frac{h^n}{n!}y^{(n)}(t^*)$, $t^* \in [t, t + h]$ ps: $y' = f$, $y'' = f_t + f_y f$

4 Convergence of One-Step Methods consider for autonomous $y' = f(y)$

4.1 Convergence | Consistent | Stable

Global Error: global error after n steps: $e_n := y_n - y(t_n)$ **Local Error:** For one-step method is: $le(y, h) = \Psi_h(y) - \Phi_h(y)$

Consistent: If $||le(y, h)|| \leq Ch^{p+1} (\leq \mathcal{O}(h^{p+1}))$, $C > 0$. \Rightarrow Consistent at order p . **Stable:** If $||\Psi_h(u) - \Psi_h(v)|| \leq (1 + h\hat{L})||u - v||$

Convergent: A method is convergent if: $\forall T, \lim_{h \rightarrow 0} \max_{h=T/N, n=0,1,\dots,N} ||e_n|| = 0$ \Downarrow Then the global error satisfies: $\max_{n=0,1,\dots,N} ||e_n|| = \mathcal{O}(h^p)$ p -th order

Convergence of One-Step Method: For $y' = f(y)$, and a one-step method $\Psi_h(y)$ is 1 consistent at order p and 2 stable with \hat{L} . (ps: $C = \frac{C}{\hat{L}}(e^{T\hat{L}} - 1)$)

4.2 More One-Step Methods | Runge-Kutta Methods | Collocation

Construction of More General one-step Method: For $y' = f(y)$, $y(t_0) = y_0 \Rightarrow y(t + h) - y(t) = \int_t^{t+h} f(y(\tau))d\tau$

Trapezoidal Method: $y_{n+1} = y_n + \frac{h}{2}(f(y_n) + f(y_{n+1}))$ **Midpoint Method:** $y_{n+1} = y_n + hf(\frac{y_n + y_{n+1}}{2})$

One-Step Collocation Methods (By Lagrange Interpolating Polynomials):

1. **Lagrange Interpolating Polynomials:** $\ell_i(x) = \prod_{j=1, j \neq i}^s \frac{x - c_j}{c_i - c_j} \in \mathbb{P}_{s-1}$ where $c_i \in F \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$

\Rightarrow **Polynomial Interpolation:** $\forall p(x) \in \mathbb{P}_s$ with $p(c_i) = g_i \in F \Rightarrow \exists! p(x) = \sum_{i=1}^s g_i \ell_i(x)$ (Can be proved by Honour Algebra)

2. **Quadrature Rule:** If $g(t) \in \mathbb{P}_{p-1} \Rightarrow$ Order p $\mid \int_{t_0}^{t_0+h} g(t)dt = \int_0^1 g(t_0 + hx)dx \approx h \sum_{i=1}^s b_i g(t_0 + hc_i)$, $b_i := \int_0^1 \ell_i(x)dx$ ps: c_i 从 $[0, 1]$ 中取不同的

3. **Collocation Methods:** For: $y(t_0) = y_0$, $y'(t_0 + c_i h) = f(y(t_0 + c_i h))$ ps: c_i 从 $[0, 1]$ 中取不同的 Let: $a_{ij} := \int_0^{c_i} \ell_j(x)dx$ and $b_i := \int_0^1 \ell_i(x)dx$
 $\Rightarrow F_i = f(y_n + h \sum_{j=1}^s a_{ij} F_j)$ and $y_{n+1} = y_n + h \sum_{i=1}^s b_i F_i$ where $F_i := y'(t_0 + c_i h)$

· **Remark:** For choice of c_i : The optimal choice is attained by Gauss-Legendre collocation methods.

Runge-Kutta Methods: Let $y' = f(t, y)$ **Stage Values:** $Y_i = y_n + h \sum_{j=1}^s a_{ij} f(Y_j)$ $i \in \{1, \dots, s\}$ $F_i = f(Y_i)$

1. The RK method is the form: $y_{n+1} = y_n + h \sum_{i=1}^s b_i f(t_n + c_i, Y_i)$ for some values of b_i, a_{ij}, s, c_i for Autonomous: $c_i = \sum_{j=1}^s a_{ij}$

c_1	a_{11}	\cdots	a_{1s}
\vdots	\vdots	\ddots	\vdots
c_s	a_{s1}	\cdots	a_{ss}
	b_1	\cdots	b_s

Example

0	
	1

RK1
(Euler’s Method)

0		
1	1	
	1/2	1/2

RK2 (Heun’s Method)

0			
1/2	1/2		
1	-1	2	
	1/6	2/3	1/6

RK3

0				
1/2	1/2			
1/2	0	1/2		
1	0	0	1	
	1/6	1/3	1/3	1/6

RK4 (Classical/Famous)