NODEA Note

1 Basic Knowledge

Def of ODE & ODEs: (1st order) ODE: $\frac{dy}{dt} = f(t, y)$ & ODEs: $\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y})$, $\mathbf{y} = (y_1, ..., y_d)^T, \mathbf{f}(t, \mathbf{y}) = (f_1(t, \mathbf{y}), ..., f_d(t, \mathbf{y}))^T$

Autonomous: $\frac{d\mathbf{y}}{dt} = \mathbf{f}(\mathbf{y}) \Rightarrow$ autonomous ODE(s). || $\Downarrow$ New Autonomous ODEs: $\frac{d\mathbf{y}}{ds} = \mathbf{f}(y_{d+1}, \mathbf{y})$ and $\frac{dy_{d+1}}{ds} = 1$

· **Change to Autonomous**: For $\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y})$. Let $y_{d+1} = t$ and *new* independent variable s s.t. $\frac{dt}{ds} = 1 \Uparrow$

Linearity: ODE: $\frac{dy}{dt} = f(t, y)$ is linearity if $f(t, y) = a(t)y + b(t)$ || ODEs: If each ODE is linear, then the ODEs are linear.

Picard's Theorem: If $f(t, y)$ is continuous in $D := \{(t, y) : t_0 \leq t \leq T, |y - y_0| < K\}$ and $\exists L > 0$ (Lipschitz constant) s.t.
$\forall (t, u), (t, v) \in D \quad |f(t, u) - f(t, v)| \leq L|u - v|$ (ps:Can use MVT). And Assume that $M_f(T - t_0) \leq K, M_f := \max\{|f(t, u)| : (t, u) \in D\}$
$\Rightarrow$ **Then**, $\exists$ a unique continuously differentiable solution $y(t)$ to the IVP $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$ on $t \in [t_0, T]$.

Existence & Uniqueness Theorem: IVP $\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y}), \mathbf{y}(t_0) = \mathbf{y}_0$. If $f(t, y)$ and $\frac{\partial f}{\partial y_i}$ are continuous in a neighborhood of $(t_0, \mathbf{y}_0)$.
$\Rightarrow$ **Then**, $\exists I := (t_0 - \delta, t_0 + \delta)$ s.t. $\exists$ a unique continuously differentiable solution $\mathbf{y}(t)$ to the IVP on $t \in I$.

2 Acknowledge

Notation	Meaning	Notation	Meaning
$[a, b]$	Approximate function for $t \in [a, b]$	$t_0 = a \mid t_N = b$	Assume that $t_0 = a, t_N = b$
N	number of **timesteps** (i.e. Break up interval $[a, b]$ into N equal-length sub-intervals)	h	**stepsize** $(h = \frac{b-a}{N})$
t_i	Define $N + 1$ points: $t_0, t_1, ..., t_N$	t_m	$t_m = a + h \cdot m = t_0 + h \cdot m$
y_i	Approximation of y at point $t = t_i$ (Except y_0)	$y(t_i)$	Exact value of y at point $t = t_i$

3 Euler's Method and Taylor Series Method

Euler's Method Algorithm: Approximate ODE $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$ with number of steps N. (Similarly for ODEs)
$\Rightarrow$ **for** $n = 0, 1, 2, ..., N - 1$: $y_{n+1} = y_n + hf(t_0 + nh, y_n) = y_n + hf(t_n, y_n)$ **end** (ps: $\Downarrow$ Can get $|y''| < M$)

Boundedness Theorem: For $\frac{dy}{dt} = f(t, y), y(a) = y_0$ and suppose there exists a unique, twice differentiable, solution $y(t)$ on $[a, b]$.
Suppose: y is continuous and $|\frac{\partial f}{\partial y}| \leq L$. $\Rightarrow$ the solution y_n given Euler's method satisfies: $e_n = |y_n - y(t_n)| \leq Dh, D = e^{(b-a)L}\frac{M}{2L}$

· **Lemma**: If $v_{n+1} \leq Av_n + B$, then $v_n \leq A^n v_0 + \frac{A^n - 1}{A - 1}B$ If $v_n = e_n := y_n - y(t_n)$, then $A = 1 + hL, B = h^2 M/2$ (suppose $|y''| < M$)

Order Notation ($\mathcal{O}$): we write $z(h) = \mathcal{O}(h^p)$ if $\exists C, h_0 > 0$ s.t. $|z| \leq Ch^p, 0 < h < h_0$

Flow Map (Φ): $\Phi_h(y)$ is a flow function if: $\Phi_{t_0,h}(y) = y(t_0 + h; t_0, y_0)$ Approx: $\Psi_h(y) := \hat{\Phi}_h(y)$ where $\Psi(y_n) = y_{n+1}$

Taylor Series Method: Approximate ODE $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$ with *n-order Methods*: 用 Taylor Series 在 $t_0 + h$ 处展开保留到 n 阶

· ps: Taylor Series: $y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{h^2}{2}y''(t_0) + \cdots + \frac{h^{n-1}}{(n-1)!}y^{(n-1)}(t_0) + \frac{h^n}{n!}y^{(n)}(t^*), t^* \in [t, t + h]$ $y' = f, y'' = f_t + f_y f$

4 Convergence of One-Step Methods **consider for autonomous** $y' = f(y)$

4.1 Convergence | Consistent | Stable

Global Error: global error after n steps: $e_n := y_n - y(t_n)$ **Local Error**: For *one-step* method is: $le(y, h) = \Psi_h(y) - \Phi_h(y)$

Consistent: If $||le(y, h)|| < Ch^{p+1} (< \mathcal{O}(h^{p+1}))$, $C > 0$. $\Rightarrow$ Consistent at order p. **Stable**: If $||\Psi_h(u) - \Psi_h(v)|| \leq (1 + h\hat{L})||u - v||$

Convergent: A method is convergent if: $\forall T$, $\lim_{h \to 0,\, h = T/N} \max_{n=0,1,...,N} ||e_n|| = 0$ $\Downarrow$ Then the global error satisfies: $\max_{n=0,1,...,N} ||e_n|| = \mathcal{O}(h^p)$ **p-th order**

Convergence of One-Step Method: For $y' = f(y)$, and a one-step method $\Psi_h(y)$ is [1] consistent at order p and [2] stable with $\hat{L}$ $\Uparrow$. (ps: $C = \frac{C}{\hat{L}}(e^{T\hat{L}} - 1)$)

4.2 More One-Step Methods | Runge-Kutta Methods | Collocation

Construction of More General one-step Method: For $y' = f(y), y(t_0) = y_0 \Rightarrow y(t + h) - y(t) = \int_t^{t+h} f(y(\tau))d\tau$

Trapezoidal Method: $y_{n+1} = y_n + \frac{h}{2}(f(y_n) + f(y_{n+1}))$ **Midpoint Method**: $y_{n+1} = y_n + hf(\frac{y_n + y_{n+1}}{2})$

One-Step Collocation Methods (By Lagrange Interpolating Polynomials):

1. **Lagrange Interpolating Polynomials**: $\ell_i(x) = \prod_{j=1, j \neq i}^{s} \frac{x - c_j}{c_i - c_j} \in \mathbb{P}_{s-1}$ where $c_i \in F \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$
$\Rightarrow$ **Polynomial Interpolation**: $\forall p(x) \in \mathbb{P}_s$ with $p(c_i) = g_i \in F \Rightarrow \exists!\, p(x) = \sum_{i=1}^{s} g_i \ell_i(x)$ (Can be proved by Honour Algebra)
2. **Quadrature Rule**: If $g(t) \in \mathbb{P}_{p-1} \Rightarrow$ Order p | $\int_{t_0}^{t_0+h} g(t)dt = \int_0^1 g(t_0 + hx)dx \approx h\sum_{i=1}^{s} b_i g(t_0 + hc_i), b_i := \int_0^1 \ell_i(x)dx$ ps: c_i 从 $[0,1]$ 中取不同的
3. **Collocation Methods**: For: $y(t_0) = y_0$, $y'(t_0 + c_i h) = f(y(t_0 + c_i h))$ ps: c_i 从 $[0,1]$ 中取不同的 Let: $a_{ij} := \int_0^{c_i} \ell_j(x)dx$ and $b_i := \int_0^1 \ell_i(x)dx$
$\Rightarrow F_i = f(y_n + h\sum_{j=1}^{s} a_{ij}F_j)$ and $y_{n+1} = y_n + h\sum_{i=1}^{s} b_i F_i$ where $F_i := y'(t_0 + c_i h)$

· **Remark**: For choice of c_i: The optimal choice is attained by *Gauss-Legendre collocation methods*.

Runge-Kutta Methods: Let $y' = f(y)$ **Stage Values**: $Y_i = y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j)$ $i \in \{1, ..., s\}$ $F_i = f(Y_i)$ ps:*weights*: b_i; *internal coefficients*: a_{ij}

1. The RK method is the form: $y_{n+1} = y_n + h\sum_{i=1}^{s} b_i f(Y_i)$ for some values of b_i, a_{ij}, s
2. RK Method can be viewed as: $y_{n+1} = y_n + h\sum_{i=1}^{s} b_i f(Y_i(y_n, h)) \Rightarrow$ Flow-map: $\Psi_h(y) = y + h\sum_{i=1}^{s} b_i f(Y_i(y, h))$
3. We can using **Butcher Table** to represent the RK method (Appendix) **Explicit**: $a_{ij} = 0$ for $j \geq i$ (严格下三角行) **Implicit**: $\exists a_{ij} \neq 0$ for $j \geq i$ (Not Explicit)

4.3 Accuracy of RK Method | Order Condition

5 Appendix

5.1 Useful Series | Common RK Methods

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ $\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ $(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ $\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$

$\frac{1}{1+x} = 1 - x + x^2 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$ $\ln(x) = (x-1) - \frac{(x-1)^2}{2} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, x > 0$

Common Runge-Kutta Methods (Butcher Table):

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}$$

Example

$$\begin{array}{c|c} 0 & \\ \hline & 1 \end{array}$$

RK1
(Euler's Method)

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array}$$

RK2 (Heun's Method)

$$\begin{array}{c|ccc} 0 & & & \\ 1/2 & 1/2 & & \\ 1 & -1 & 2 & \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

RK3

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

RK4 (Classical/Famous)