

## 1 Basic Knowledge

**Useful Complex Number Properties:**  $|Re(z)|, |Im(z)| \leq |z|$   $Re(z) = \frac{z+\bar{z}}{2}, Im(z) = \frac{z-\bar{z}}{2i}, |z|^2 = z\bar{z}$

**Useful Tools:**  $Re(z) = 0 \Leftrightarrow z = -\bar{z}$   $Im(z) = 0 \Leftrightarrow z = \bar{z}$  In circle,  $\bar{z} = |z|^2 z^{-1}$

**Triangle (Reverse) Inequality:**  $|z_1 + z_2| \leq |z_1| + |z_2|$   $||z_1| - |z_2|| \leq |z_1 - z_2|$   $\ominus (Re(zw) = 0 \Leftrightarrow \overline{zw} = -zw; Im(zw) = 0 \Leftrightarrow zw = \overline{zw})$

**Argument:**  $arg(z) := \{\theta : z = |z|e^{i\theta}\} = \{Arg(z) + 2\pi k : k \in \mathbb{Z}\}$  **Principle Value of Argument:**  $Arg(z) \in (-\pi, \pi]$

**Operations on Argument:**  $arg(z_1 z_2) = arg(z_1) + arg(z_2)$   $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$   $arg(\bar{z}) = -arg(z)$

## 2 Holomorphic Functions

### 2.1 Open/Closed Set | Limit Point | limit of Sequence | Continuous of Function

**Open/Closed/Punctured  $\varepsilon$ -disc:**  $D_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$   $\bar{D}_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq \varepsilon\}$   $D'_\varepsilon(z_0) := \{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}$

**Open/Closed Set in  $\mathbb{C}$ :**  $U \subseteq \mathbb{C}$  is **open** if  $\forall z_0 \in U, \exists \varepsilon > 0, D_\varepsilon(z_0) \subseteq U$   $U$  is **closed** if  $\mathbb{C} \setminus U$  is open **Lemma:**  $D_\varepsilon, D'_\varepsilon$  open,  $\bar{D}_\varepsilon$  closed.

**Limit Point of  $S$ :**  $z_0 \in \mathbb{C}$  is a limit point of  $S$  if:  $\forall \varepsilon > 0, D'_\varepsilon(z_0) \cap S \neq \emptyset$  **Bounded:**  $S$  is bounded if  $\exists M > 0$  s.t.  $|z| \leq M, \forall z \in S$

**Closed of Set  $S$ :**  $\bar{S} :=$  所有  $S$  的 limit point 和  $S$  的点. **Property:** Let  $S \subseteq \mathbb{C}$ , then  $S$  is closed  $\Leftrightarrow S = \bar{S}$ .

**Limit of sequence:** Sequence  $(z_n)_{n \in \mathbb{N}}$  has limit  $z$  if  $\forall \varepsilon > 0, \exists N \in \mathbb{N}$  s.t.  $\forall n \geq N \Rightarrow |z_n - z| < \varepsilon$ . limit rules 依旧成立

1. **Lemma|Important:**  $\lim z_n = z \Leftrightarrow \lim Re(z_n) = Re(z)$  and  $\lim Im(z_n) = Im(z)$

2. **Cauchy:** Sequence  $(z_n)_{n \in \mathbb{N}}$  is cauchy if:  $\forall \varepsilon > 0, \exists N \in \mathbb{N}$  s.t.  $\forall m, n \geq N \Rightarrow |z_m - z_n| < \varepsilon$  **Lemma:** Cauchy  $\Leftrightarrow$  convergent.

3. **Lemma|Closed of Set:**  $S \subseteq \mathbb{C}, z \in \mathbb{C} \Rightarrow [z \in \bar{S} \Leftrightarrow \exists \text{ sequence } (z_n)_{n \in \mathbb{N}} \in S \text{ s.t. } \lim z_n = z]$

4. **Bolzano-Weierstrass:** Every bounded sequence in  $\mathbb{C}$  has a convergent subsequence.

**Complex Functions:**  $\forall f : \mathbb{C} \rightarrow \mathbb{C}$  we can write it as:  $f(z) = f(x + iy) = u(x, y) + iv(x, y)$  where  $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$

**Limit of Function:**  $a_0 \in \mathbb{C}$  is the limit of  $f$  at  $z_0$  if:  $\forall \varepsilon > 0, \exists \delta > 0$  s.t.  $0 < |z - z_0| < \delta \Rightarrow |f(z) - a_0| < \varepsilon$  limit rules 依旧成立

**Lemma|Important:**  $\lim_{z \rightarrow z_0} f(z) \Leftrightarrow \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = Re(a_0)$  and  $\lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = Im(a_0)$

**Useful Formula:**  $\lim_{z \rightarrow z_0} g(\bar{z}) = \lim_{z \rightarrow \bar{z}_0} g(z)$

**continuous of Function:**  $f$  is continuous at  $z_0$  if:  $\forall \varepsilon > 0, \exists \delta > 0$  s.t.  $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$  continuous rules 依旧成立

1. **Lemma|Important:**  $f$  is continuous at  $z_0 \Leftrightarrow u, v$  are continuous at  $(x_0, y_0)$

2. **'Extreme Value Theorem':**  $f$  is continuous on a closed and bounded set  $S \subseteq \mathbb{C}$ , then  $f(S)$  is closed and bounded.

3. **Lemma|continuous  $\Leftrightarrow$  open:**  $f$  is continuous  $\Leftrightarrow \forall$  open set  $U$ , preimage  $f^{-1}(U) := \{z \in \mathbb{C} | f(z) \in U\}$  is open.

### 2.2 Differentiable | Holomorphic Function | C-R Equation

**Differentiable:** Let  $z_0 \in \mathbb{C}$  and  $U \subseteq \mathbb{C}$  be neighborhood of  $z_0$ , then  $f : U \rightarrow \mathbb{C}$  is differentiable at  $z_0$  if:  $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$  exists.

**I.  $f$  is differentiable  $\Rightarrow f$  is continuous.** **II. Holomorphic  $\Leftrightarrow$  Differentiable + neighborhood** (除非是一个点时不成立,  $|z|$ ) diff rules + chain rule 成立

**Cauchy-Riemann Equations:** If  $z_0 = x_0 + iy_0, f(z) = u(x, y) + iv(x, y)$  is differentiable at  $z_0 \Rightarrow u_x = v_y, v_x = -u_y$  at  $(x_0, y_0)$ .

**If  $z_0 = x_0 + iy_0, f = u + iv$  satisfies:**  $^1 u, v$  are continuously differentiable on a neighborhood of  $(x_0, y_0)$  and:

$^2 u, v$  satisfies Cauchy-Riemann Equations at  $(x_0, y_0)$ .  $\Rightarrow f$  is differentiable at  $z_0$ .

**ps:** 常见可导复数函数:  $\exp(z), \sin z, \cos z, \log z, z^\alpha$ , polynomial,  $\sinh, \cosh, \Gamma(z), |z|^2$  (at 0), constant **ps:** 常见不可导复数函数:  $\bar{z}, |z| \cdot \bar{z}, Re(z), Im(z), Arg(z)$

**Harmonic Function:**  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  is harmonic if:  $\forall (x, y) \in \mathbb{R}^2 \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$  (Laplace Equation)

**Lemma:** If  $f = u + iv$  is holomorphic on  $\mathbb{C}$  (and  $u, v$  are twice continuously differentiable) 可以不用,  $\Rightarrow u, v$  are harmonic.  $\ominus (u, v \text{ harmonic} + CR \Leftrightarrow f \text{ holomorphic})$

**Harmonic Conjugate:** Let  $u, v : U \rightarrow \mathbb{R}, U \subseteq \mathbb{R}^2$  be harmonic functions.  $u, v$  are harmonic conjugate if:  $f = u + iv$  is holomorphic on  $U$ .

**Properties of Polynomial:** The domain of rational function and polynomial are always open. **Lemma:** If  $P(z_0) = 0$  then  $P(\bar{z}_0) = 0$

**First-order Operator  $\partial$ :**  $\partial := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$   $\bar{\partial} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$   $|| f = u + iv$  satisfies C-R Equations  $\Leftrightarrow \bar{\partial} f = 0$

**sin/cos Functions:**  $\sin z := \frac{e^{iz} - e^{-iz}}{2i}$   $\cos z := \frac{e^{iz} + e^{-iz}}{2}$  **Exponential Function:**  $\exp(z) = e^x(\cos(y) + i \sin(y))$

1.  $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$   $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$

2.  $\sin(z + w) = \sin(z) \cos(w) + \cos(z) \sin(w)$   $\cos(z + w) = \cos(z) \cos(w) - \sin(z) \sin(w)$

3.  $\sin^2 z + \cos^2 z = 1$   $\sin(z + \frac{\pi}{2}) = \cos(z)$   $\sin(z + 2k\pi) = \sin(z)$   $\cos(z + 2k\pi) = \cos(z)$  \* $\sin z, \cos z$  NOT bounded.

**Hyperbolic Functions:**  $\sinh z := \frac{\exp(z) - \exp(-z)}{2}$   $\cosh z := \frac{\exp(z) + \exp(-z)}{2}$   $|| \sinh(iz) = i \sin z$   $\cosh(iz) = \cos z$

**Logarithm:** Define multivalued function:  $\log z := \{w \in \mathbb{C} : \exp w = z\}$  **Principal Branch:**  $Log(z) := \ln |z| + i Arg(z)$

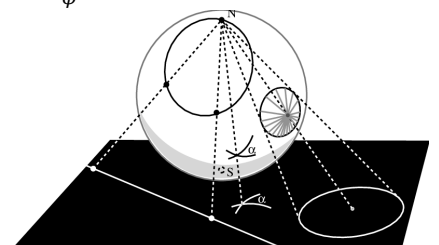
1. **I.**  $\log(z) = \ln |z| + i \arg z = \{\ln |z| + i Arg(z) + i 2\pi k : k \in \mathbb{Z}\}$  **II.**  $\log(zw) = \log(z) + \log(w)$  **III.**  $\log(1/z) = -\log(z)$

2. **Branch of Logarithm:**  $Log_\phi(z) := \ln |z| + i Arg_\phi(z)$   $Log_\phi(z)$  is holomorphic on  $D_\phi$

3. If  $g : U \rightarrow \mathbb{C}$ , then  $Log_\phi(g(z))$  is holomorphic on  $g^{-1}(D_\phi) \cap U$

4.  $Log(z)$  not continuous on  $\mathbb{C}$ .  $Log(z)$  not continuous on  $Re(z) \leq 0, Im(z) = 0$ .

**Remark:**  $\log(x) + \log(x) \neq 2 \log(x)$



**Branch Cut|Cut Plane:**  $Branch\ Cut\ L_{z_0, \phi} := \{z \in \mathbb{C} : z = z_0 + re^{i\phi}, r \geq 0\}$

· **Cut Plane:**  $D_{z_0, \phi} := \mathbb{C} \setminus L_{z_0, \phi} \quad L_{\phi} = L_{0, \phi}; D_{\phi} = D_{0, \phi}$

· If  $Log_{\phi}(z)$  is holomorphic on  $D_{\phi}$ , then  $Log_{\phi}(z - a)$  is holomorphic on  $D_{a, \phi}$

**Branch of Argument:**  $Arg_{\phi}(z) := z$  的辐角, 但是角度限制在:  $\phi < Arg_{\phi}(z) \leq \phi + 2\pi$ . ps:  $Arg_{-\pi}(z) = Arg(z)$

$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$	$f(z)$	$f'(z)$
$z^n$	$nz^{n-1}$	$\exp(z)$	$\exp(z)$	$\sin(z)$	$\cos(z)$	$\cos(z)$	$-\sin(z)$	$\sinh(z)$	$\cosh(z)$	$\cosh(z)$	$\sinh(z)$	$Log_{\phi} z$	$\frac{1}{z} \quad z \in D_{\phi}$

**Complex Powers:**  $z^{\alpha} := \{\exp(\alpha w) : w \in \log(z)\} = \{\exp[\alpha(\ln|z| + iArg(z) + i2k\pi)] : k \in \mathbb{Z}\} \quad \frac{d}{dz} z^{\alpha} = \alpha z^{\alpha-1} \quad z \in D_{\phi}$

I. If  $\alpha \in \mathbb{Z}$ , there is one value of  $z^{\alpha}$       II. If  $\alpha = \frac{p}{q}$ ,  $\gcd(p, q) = 1, p, q \in \mathbb{Z}, q \neq 0$ , there are exactly  $q$  values of  $z^{\alpha}$

III. If  $\alpha$  is irrational or non-real, there are infinitely values of  $z^{\alpha}$       IV.  $1^{1/q}, q \in \mathbb{Z}, q \neq 0$  is  $\{1, w, \dots, w^{q-1}\}, w = \exp(i2\pi/q)$

V. We prefer use  $\exp(z)$  to denote single-valued function, and  $e^z$  to denote multi-valued function.

**Principal Branch:**  $z^{\alpha} := \exp(\alpha Log(z))$       **Operation:**  $z^{\alpha} z^{\beta} = z^{\alpha+\beta}$  (Using Principal Branch)      **NB:**  $(z_1 z_2)^{\alpha} \neq z_1^{\alpha} z_2^{\alpha}; (z^{\alpha})^{\beta} \neq z^{\alpha\beta}$

### 3 Conformal Maps and Mobius Transformations

**Conformal:** Let  $U$  be open set and  $f : U \rightarrow \mathbb{C}$ . Then  $f$  is conformal iff:  $f$  preserves angles. i.e. 任意两条曲线/直线之间的角度在  $f$  作用下不变.

**Important Theorem:** If  $f : U \rightarrow \mathbb{C}$  is holomorphic, then  $\forall z_0 \in U, f'(z_0) \neq 0, f$  preserves angles.

i.e.  $\forall$  curves  $C_1, C_2$  in  $U$ . If  $C_1, C_2$  intersecting at a point  $z_0 \in U$ .  $C_1, C_2$  在  $z_0$  切线的夹角与  $f(C_1), f(C_2)$  在  $f(z_0)$  切线的夹角一样.

**Extended Complex Plane:**  $\tilde{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$  and define that  $a + \infty = \infty, b \cdot \infty = \infty, \frac{b}{\infty} = 0, \frac{\infty}{\infty} = 0$ .

**Riemann Sphere:** Consider  $(X, Y, Z) \in \mathbb{R}^3: {}^1Z = X + iY \in \mathbb{C}$  is the point  $(X, Y, 0)$  and  ${}^2Z = 0$  is the complex plane.

1. Define the Riemann Sphere:  $S^2 := \{(X, Y, Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1\}$  and consider the **North Pole** is point  $N := (0, 0, 1)$

2. Define  $\phi : \mathbb{C} \rightarrow S^2$  by  $N$  点与  $z = (X, Y, 0)$  点连线与  $S^2$  的交点为  $\phi(z)$       Thus  $\lim_{|z| \rightarrow \infty} \phi(z) = N \quad \phi(\infty) := N$

3. Calculation shows that:  $\phi(z) = \phi(x + iy) = \left( \frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right)$        $\psi(X, Y, Z) = \begin{cases} \frac{X+iY}{1-Z}, & (X, Y, Z) \neq N \\ \infty, & (X, Y, Z) = N \end{cases}$

**Remark:**  $\phi : \tilde{\mathbb{C}} \rightarrow S^2$  is bijection and it's inverse  $\psi : S^2 \rightarrow \tilde{\mathbb{C}}$  is the **stereographic projection**

4. Stereographic projection  $\psi(X, Y, Z)$  maps a circle to either a circle or a straight line. (见上图)

**Mobius Transformation:** A Mobius Transformation is a function form:  $f(z) = \frac{az+b}{cz+d}$  where  $a, b, c, d \in \mathbb{C}; ad \neq bc$

1. **Remark:**  $g(z) = \frac{f(z)}{\sqrt{ad-bc}}$  satisfies  $ad - bc = 1$  | If  $a, b, c, d$  defined a mobius transformation, then  $\lambda a, \lambda b, \lambda c, \lambda d$  also.

2. For Complex Matrix:  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with  $\det(M) = ad - bc = 1$ . We define  $f_M = \frac{az+b}{cz+d}$       I.  $f_{M_1 M_2} = f_{M_1} \circ f_{M_2}$   
II.  $f_{M^{-1}} = f_M^{-1}$

3. Extended  $f(z)$  from  $\mathbb{C}$  to  $\tilde{\mathbb{C}}$  by:  $f(-\frac{d}{c}) = \infty$  and  $f(\infty) = \frac{a}{c}$

4. **Translation:**  $f(z) = z + b \Leftrightarrow \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$       **Rotation:**  $f(z) = az, a = e^{i\theta} (|a| = 1) \Leftrightarrow \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & -e^{i\theta/2} \end{pmatrix}$       **Dilation:**  $f(z) = rz, r > 0 \Leftrightarrow \begin{pmatrix} \sqrt{r} & 0 \\ 0 & 1/\sqrt{r} \end{pmatrix}$

**Inversion:**  $f(z) = 1/z \Leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$   $f$  fixes the point at infinity: If  $f(\infty) = \infty$  ps: 除了 inversion 其他都是 fix the point at infinity.

5. **Theorem:**  $f(z) = \frac{az+b}{cz+d}$  be a Mobius Transformation.  $\Rightarrow$  1 If  $f(\infty) = \infty$ :  $f$  is a composition of finite Translation, Rotation, Dilation  $\Rightarrow c = 0, f(z) = \frac{a}{d}z + \frac{b}{d}$   
2 If  $f(\infty) < \infty$ :  $f$  is composition of finite Translation, Rotation, Dilation and only one inversion.  $\Rightarrow f(z) = \frac{(bc-ad)/c^2}{z+d/c} + \frac{a}{c}$

**Properties of Mobius Transformation: Important:** ★ Möbius transformations map circlines to circlines. ★

1. For mobius transformation  $f(z) = \frac{az+b}{cz+d}$ , if:  $\exists z_1, z_2, z_3 \in \mathbb{C}$  distinct points.  $f(z_1) = z_1, f(z_2) = z_2, f(z_3) = z_3 \Rightarrow f$  is identity.

2. If  $z_1, z_2, z_3 \in \tilde{\mathbb{C}}$  distinct points.  $\exists!$  mobius transformation  $f(z)$  s.t.  $f(z_1) = 1, f(z_2) = 0, f(z_3) = \infty$

3. If  $(z_1, z_2, z_3), (w_1, w_2, w_3) \in \tilde{\mathbb{C}}$  distinct points. Then  $\exists!$  mobius transformation  $f(z)$  s.t.  $f(z_i) = w_i, \forall i \in \{1, 2, 3\}$

**ps:Method to construct 2:** If  $z_i < \infty, f(z) = \frac{z_1-z_3}{z_1-z_2} \cdot \frac{z-z_2}{z-z_3}$       If  $z_i = \infty, f(z) = \frac{z-z_2}{z-z_3}, z_1 = \infty \Rightarrow f(z) = \frac{z_1-z_3}{z-z_3}, z_2 = \infty; f(z) = \frac{z-z_2}{z_1-z_2}, z_3 = \infty$

**ps:Method to construct 3:** For 3: Let  $f := h^{-1} \circ g$  where  $g(z_i), h(w_i) = \{1, 0, \infty\}$  like part 2.

**Geometric Meaning by using Mobius Transformation|Exponential|Complex Powers:**

1. **Rotation:**  $f(z) = e^{-i\theta}z$  is a rotation by  $\theta$  (anticlockwise) about the origin. Specially,  $f(z) = iz$  is a rotation by  $\frac{\pi}{2}$

2. **Extend:**  $f(z) = \exp(\alpha z)$  原来的图像进行拉长, 以及旋转 (如果带  $\theta$  带  $i$  时) e.g.  $\{z : 0 < Im(z) < 1\}$  可以被拉长到  $\{z : 0 < Im(z)\}$

3. **Angle Extend:**  $f(z) = z^{\alpha}$  原来的图像辐角范围收缩或放大

4. **Circlines:** I. 单位圆到实轴,  $f(z) = \frac{z-i}{z+i}$       II. 实轴到单位圆,  $f(z) = i \frac{1+z}{1-z}$       III. 单位圆到虚轴,  $f(z) = \frac{z-1}{z+1}$       IV. 虚轴到单位圆,  $f(z) = \frac{1+iz}{1-iz}$

**Cross-Ratio:** cross-ratio  $[z_1, z_2, z_3, z_4] := f(z_1)$  where  $f$  is mobius transformation s.t.  $f(z_2) = 1, f(z_3) = 0, f(z_4) = \infty$

1. **Formulas:**  $[z_1, z_2, z_3, z_4] = \frac{z_1-z_3}{z_1-z_4} \frac{z_2-z_4}{z_2-z_3}$        $[\infty, z_2, z_3, z_4] = \frac{z_2-z_4}{z_2-z_3}$        $[z_1, \infty, z_3, z_4] = \frac{z_1-z_3}{z_1-z_4}$        $[z_1, z_2, \infty, z_4] = \frac{z_2-z_4}{z_1-z_4}$        $[z_1, z_2, z_3, \infty] = \frac{z_1-z_3}{z_2-z_3}$

2. **Theorem:** If  $f$  is a mobius transformation,  $[f(z_1), f(z_2), f(z_3), f(z_4)] = [z_1, z_2, z_3, z_4]$        $z_i$ 's in this "small section" are distinct.

3. **Application:**  $\exists$  mobius transformation  $f$  s.t.  $f(z_1) = w_1, f(z_2) = w_2, f(z_3) = w_3, f(z_4) = w_4 \Leftrightarrow [w_1, w_2, w_3, w_4] = [z_1, z_2, z_3, z_4]$

**ps:** † 原因:  $(\Rightarrow), [z_1, z_2, z_3, z_4] = [f(z_1), f(z_2), f(z_3), f(z_4)] = [w_1, w_2, w_3, w_4]; (\Leftarrow), [w_1, w_2, w_3, w_4] = [z_1, z_2, z_3, z_4] \Rightarrow \exists f : w_2 \rightarrow 1, w_3 \rightarrow 0, w_4 \rightarrow \infty; g : z_2 \rightarrow 1, z_3 \rightarrow 0, z_4 \rightarrow \infty; f(w_1) = g(z_1) \Rightarrow g^{-1} \circ f$

## 4 Complex Integration

### 4.1 Line Integral

**Integrable:**  $f : [a, b] \rightarrow \mathbb{C}$  as  $f(t) = u(t) + iv(t)$  is integrable if:  $u, v$  are both integrable on  $[a, b]$  and for  $f(t)$ :

1. **Def:**  $\int_a^b f(t)dt := \int_a^b u(t)dt + i \int_a^b v(t)dt$
2. **Property I.**  $\alpha f + \beta g$  is integrable and  $\int_a^b (\alpha f + \beta g)dt = \alpha \int_a^b f(t)dt + \beta \int_a^b g(t)dt$
3. **Property II.** If  $f$  is continuous and  $\frac{dF}{dt} = f(t)$  for  $F : [a, b] \rightarrow \mathbb{C}$  is differentiable.  $\Rightarrow \int_a^b f(t)dt = F(b) - F(a)$
4. **Property III.** If  $f$  is continuous  $\Rightarrow \left| \int_a^b f(t)dt \right| \leq \int_a^b |f(t)|dt$ .

**Parameters Curves:** A parametrized curve connecting  $z_0$  to  $z_1$  is a continuous function  $\gamma : [t_0, t_1] \rightarrow \mathbb{C}$  s.t.  $\gamma(t_0) = z_0, \gamma(t_1) = z_1$

If  $z_0 = x_0 + iy_0, z_1 = x_1 + iy_1$ , then  $\gamma(t) = x(t) + iy(t)$  continuous functions. s.t.  $x(t_0) = x_0, x(t_1) = x_1, y(t_0) = y_0, y(t_1) = y_1$

**Regular:**  $\gamma$  is regular if  $\gamma'(t) \neq 0$  for all  $t \in [t_0, t_1]$  **Remark:** Curve  $\gamma([t_0, t_1]) = \Gamma$  is closed and bdd.

**Integral Along Curve:** Let  $\gamma : [t_0, t_1] \rightarrow \mathbb{C}$  be a regular curve s.t.  $\gamma([t_0, t_1]) = \Gamma$  and  $f : \Gamma \rightarrow \mathbb{C}$  is continuous.

1. **★ Def:**  $\int_{\Gamma} f(z)dz := \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)dt$  ★
2. **Circle at zero:** Circle Centred at 0 with radius  $R$ :  $\gamma : [0, 1] \rightarrow \mathbb{C}$  by  $\gamma(t) = R \exp(2\pi it)$
3. **Constant Function:** If  $f(z) = c$ ;  $\gamma : [a, b] \rightarrow \mathbb{C}$ . Then  $\int_{\Gamma} f(z)dz = \int_a^b c \cdot \gamma'(z)dz = c \cdot (\gamma(b) - \gamma(a))$

**Arclength of Curve:** Let  $\gamma : [t_0, t_1] \rightarrow \mathbb{C}$  be a regular curve.  $\gamma(t) = x(t) + iy(t)$  Then arclength  $\ell(\Gamma) := \int_{t_0}^{t_1} |\gamma'(t)|dt = \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2}dt$

**Lemma:** If  $\Gamma$  is an arc of a circle of radius  $r$  traced through angle  $\theta$ , then  $\ell(\Gamma) = r\theta$  (扇形弧长)

**Properties of Integral Along Curve:** Let  $\Gamma$  be a regular curve and  $f, g : \Gamma \rightarrow \mathbb{C}$  be continuous, and  $\alpha, \beta \in \mathbb{C}$

1. **M-L Lemma:**  $|\int_{\Gamma} f(z)dz| \leq \max_{z \in \Gamma} |f(z)|\ell(\Gamma)$
2. **Lemma:**  $\int_{\Gamma} (\alpha f + \beta g)dz = \alpha \int_{\Gamma} f(z)dz + \beta \int_{\Gamma} g(z)dz$   $\int_{-\Gamma} f(z)dz = -\int_{\Gamma} f(z)dz$  Here:  $\tilde{\gamma}(t) := \gamma(b-t)$  have  $\tilde{\gamma}([a, b]) = -\Gamma$
3. **Change of Variables:** If  $\gamma : [a, b] \rightarrow \Gamma$ , and  $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \rightarrow \Gamma$  are two parametrizations of  $\Gamma$ ;  
 $\exists \lambda : [\tilde{a}, \tilde{b}] \rightarrow [a, b]$  s.t.  $\lambda'(t) > 0$  and  $\tilde{\gamma}(t) = \gamma(\lambda(t))$  (防止曲线回头)  $\Rightarrow \int_a^b f(\gamma(t))\gamma'(t)dt = \int_{\tilde{a}}^{\tilde{b}} f(\tilde{\gamma}(t))\tilde{\gamma}'(t)dt$ .  
(特别的, 如果  $\Gamma$  是 closed,  $f$  在  $\Gamma$  上的积分与哪里选择起/终点无关)

**Contour:** A curve  $\Gamma$  is contour if it's finite union of regular curves  $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ . Each  $\Gamma_i$  is regular component of  $\Gamma$

**Contour Integral:** If  $f : \Gamma \rightarrow \mathbb{C}$  is continuous and  $\Gamma$  is a contour. Then  $\int_{\Gamma} f(z)dz := \sum_{i=1}^n \int_{\Gamma_i} f(z)dz$

### 4.2 Independent of Path

**Domain:**  $D \subseteq \mathbb{C}$  is a domain if it's open and connected. (i.e. 任意两点都存在 contour( $\Gamma$ ) 将其连接, 并都在  $D$  里面)

**Lemma:** Let  $D \subseteq \mathbb{C}$  be a domain. If  $u : D \rightarrow \mathbb{C}$  is differentiable, with  $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$ .  $\Rightarrow u$  is constant on  $D$ . Clearly,  $F$  is holomorphic

**Antiderivative:** Let  $D$  be a domain. For  $f : D \rightarrow \mathbb{C}$  be continuous and  $F : D \rightarrow \mathbb{C}$  s.t.  $F'(z) = f(z)$  for all  $z \in D$ . Then  $F$  is an antiderivative of  $f$ .

**Fundamental Theorem of Calculus:**  $D$  domain;  $f : D \rightarrow \mathbb{C}$  continuous;  $F : D \rightarrow \mathbb{C}$  antiderivative of  $f$ . Contour  $\Gamma$  in  $D$  connecting  $z_0$  to  $z_1$ .

Then  $\int_{\Gamma} f(z)dz = F(z_1) - F(z_0)$

1.  $D$  domain, if  $f : D \rightarrow \mathbb{C}$  is holomorphic and  $f'(z) = 0, \forall z \in D$ .  $\Rightarrow f$  is constant on  $D$ .
2. **Path-Independence Lemma:**  $D$  domain,  $f$  continuous on  $D$ . Then:  
 $f$  has antiderivative on  $D \Leftrightarrow \int_{\Gamma} f(z)dz = 0 \forall$  closed contours  $\Gamma$  in  $D \Leftrightarrow \int_{\Gamma} f(z)dz$  is path-independent.

### 4.3 Cauchy's Theorem

**Simple:** A contour  $\Gamma$  is simple if it doesn't intersect itself except at the endpoints. **Loop:** A contour  $\Gamma$  is a loop if it's simple and  $\Gamma(t_0) = \Gamma(t_1)$

**Jordan Curve Theorem:**  $\forall \Gamma$  be Loop **Interior**  $Int(\Gamma)$ :  $\Gamma$  的内部, bounded. **Exterior**  $Ext(\Gamma)$ :  $\Gamma$  的外部, unbounded. **Boundary**  $\Gamma$  的边界,  $\Gamma$  itself.

And  $Int(\Gamma)$  is bounded domain  $Ext(\Gamma)$  is unbounded domain. **Remark:**  $Int(\Gamma)$  is open and  $Ext(\Gamma)$  is open also.

· **Common Loop:**  $C_r(z_0)$  is a circle of radius  $r$  centered at  $z_0$  Corresponding  $\gamma(t) = z_0 + r \exp(2\pi it) t \in [0, 1]$

· **Positive-Oriented:** If  $\Gamma$  is a loop, then  $\Gamma$  is positive-oriented if: 按方向走时, 内部在左边 (as we move along the curve in the direction of parametrization, the interior is on the left-hand side.)

**Remark:** Unless otherwise stated, all loops shall be positively-oriented.

**Simply-Connected:** A domain  $D$  is simply-connected if:  $\forall$  loop  $\Gamma$  in  $D, Int(\Gamma) \subseteq D$  (即没有洞的 domain/open set)

**Cauchy Integral Theorem:** If  $\Gamma$  is Loop,  $f$  is holomorphic in  $Int(\Gamma) \cup \Gamma$  (Inside and on  $\Gamma$ ), then  $\int_{\Gamma} f(z)dz = 0$

**Corollary:** If  $D$  is simply-connected domain and  $f : D \rightarrow \mathbb{C}$  is holomorphic on  $D$ . Then  $f(z)$  has antiderivative on  $D$ . ★

即: 在没有洞的 open set 上如果都是 holomorphic, 那么都有 antiderivative.

**Remark:** 如果 loop  $\Gamma$  上和以内部没有穿过任何非 holomorphic 点, 那么  $f(z)$  的积分值不变.

**Theorem:** Let  $z_0 \in \mathbb{C}, \Gamma$  be Loop. Then  $\int_{\Gamma} \frac{1}{z-z_0} = \begin{cases} 2\pi i & \text{if } z_0 \in Int(\Gamma) \\ 0 & \text{otherwise} \end{cases}$

**Deformation Theorem:** Let  $\Gamma_1, \Gamma_2$  be loops, and  $f$  is holomorphic on  $(Int(\Gamma_1) \setminus Int(\Gamma_2)) \cup (Int(\Gamma_2) \setminus Int(\Gamma_1))$ ,  $\Gamma_1, \Gamma_2$ . Then  $\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$

即: 两个 loop  $\Gamma_1$  和  $\Gamma_2$  及它们围成的区域中 (除公共区域) 上, 函数  $f(z)$  全纯, 那么它们的路径积分相等 ps: 可以是内外 loop, 也可以是交叉的 loop

## 4.4 Cauchy's Integral Formula

**Cauchy's Integral Formula:**  $\Gamma$  Loop,  $f(z)$  holomorphic inside and on  $\Gamma$ ,  $z_0 \in \text{Int}(\Gamma)$ ,  $\Rightarrow f(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z-z_0} dz$

ps: We always use it to calculate:  $\int_{\Gamma} \frac{f(z)}{z-z_0} dz$  if  $f(z)$  is holomorphic on and inside  $\Gamma$  (loop), and  $z_0 \in \text{Int}(\Gamma)$ .  $\Rightarrow \int_{\Gamma} \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$

**Theorem:**  $D$  be domain,  $\Gamma$  be contour in  $D$ ,  $g: D \rightarrow \mathbb{C}$  continuous on  $\Gamma$ , Then:

Function Defined as:  $G: D \setminus \Gamma \rightarrow \mathbb{C}$  by  $G(z) = \int_{\Gamma} \frac{g(w)}{w-z} dw$  is holomorphic on  $D \setminus \Gamma$  and  $G'(z) = \int_{\Gamma} \frac{g(w)}{(w-z)^2} dw$

Moreover, function  $H: D \setminus \Gamma \rightarrow \mathbb{C}$  by  $H(z) = \int_{\Gamma} \frac{g(w)}{(w-z)^{n+1}} dw$  is holomorphic on  $D \setminus \Gamma$  and  $H'(z) = n \int_{\Gamma} \frac{g(w)}{(w-z)^{n+1}} dw$

★ **Corollary:** If  $D$  is domain and  $f$  is holomorphic on  $D$ , then  $f$  is infinitely differentiable on  $D$ , and all of its derivatives are holomorphic on  $D$ .

**Generalized Cauchy's Integral Formula:**  $\Gamma$  Loop,  $f(z)$  holomorphic inside and on  $\Gamma$ ,  $z \in \text{Int}(\Gamma)$ ,  $n \in \mathbb{N}$ ,  $\Rightarrow f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w-z)^{n+1}} dw$

ps: We always use it to calculate:  $\int_{\Gamma} \frac{f(z)}{(z-z_0)^{n+1}} dz$  if  $f(z)$  is holomorphic on and inside  $\Gamma$  (loop), and  $z_0 \in \text{Int}(\Gamma)$ .  $\Rightarrow \int_{\Gamma} \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$

**Morera Theorem:** Let  $D$  is domain, if  $f: D \rightarrow \mathbb{C}$  is continuous and  $\int_{\Gamma} f(z) dz = 0$  for all loop  $\Gamma$  in  $D$ .  $\Rightarrow f$  is holomorphic on  $D$ .

## 4.5 Liouville's Theorem, FTA and Maximum Modulus Principle

**Useful Formula:** If  $^1 D$  domain;  $^2 \exists R > 0, z_0 \in \mathbb{C}$  s.t.  $\overline{D}_R(z_0) \subseteq D$ ;  $^3 f$  is holomorphic on  $D$

1. Then  $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + R \exp(it)) dt$ .

2. If  $|f(z)| < M, \forall z \in D$ . Then  $|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}$ .

3. If  $\max_{z \in \overline{D}_R(z_0)} |f(z)| = |f(z_0)|$ . Then  $f$  is constant on  $\overline{D}_R(z_0)$ .

**Criteria Constant Function:** If  $f: \mathbb{C}(\text{or } D) \rightarrow \mathbb{C}$  is holomorphic and bounded on:  $D$  domain

1. **Liouville's Theorem:**  $|f(z)| < M$  bounded on  $\forall z \in \mathbb{C}$ ,  $\Rightarrow f(z)$  is constant.

2. **Maximum Modulus Principle:**  $|f(z)|$  bounded on  $\forall z \in D$ , and  $|f(z)|$  has maximum at  $z_0 \in D$ .  $\Rightarrow f(z)$  is constant.

**Remark I:** 意思是对于  $f(z)$  holomorphic 且在 domain 上 bounded, 如果  $|f(z)|$  在 domain 上有最大值 (非边界), 那么  $f(z)$  是 constant.

**Remark II:** ★ If function  $f$  is holomorphic on a bounded domain  $D$  and continuous up to the boundary of  $D$ .

$\Rightarrow f$  has maximum modulus on the boundary of  $D$ .

若  $f$  在  $D$  内全纯, 且在  $\partial D$  上连续, 则  $f$  在  $D \cup \partial D$  最大值一定在边界上. 特别地, 若  $f$  不是常数, 则最大值只能在边界上取到.

3. **Maximum/Minimum Principle for Harmonic Functions:** If  $D$  domain,  $\phi: D \rightarrow \mathbb{R}$  is harmonic, and  $\phi$  is bounded above/below on  $D$  by  $M$ , with  $\phi(z_0) = M$  for some  $z_0 \in D$ .  $\Rightarrow \phi$  is constant on  $D$ .

**Remark:** 对于调和函数  $\phi: D \rightarrow \mathbb{R}$ , 如果  $f$  不是常数, 那么最大值只能在边界上取到.

**Fundamental Theorem of Algebra:** If  $P: \mathbb{C} \rightarrow \mathbb{C}$  is a non-constant polynomial.  $\Rightarrow P$  has a at least one root in  $\mathbb{C}$ .

## 5 infinity Series

### 5.1 Basic Properties, Convergence Test, Series of Functions and M-Test

**Partial Sum:** A Series  $\sum_{n=0}^{\infty} z_n$  is convergent if partial sums  $S_n = \sum_{k=0}^n z_k$  is convergent. **Remark:**  $\sum z_n$  is convergent  $\Rightarrow \lim z_n = 0$ .

**Comparison Test:** If  $|z_n| \leq M_n$  for all  $n \in \mathbb{N}$  and  $\sum M_n$  is convergent.  $\Rightarrow \sum z_n$  is convergent.

**Lemma|'Geometric Series':** For  $c \in \mathbb{C}$ ,  $\sum_{n=0}^{\infty} c^n$  is convergent  $\Leftrightarrow |c| < 1$ . **Remark:**  $\sum_{n=0}^{\infty} c^n = \frac{1}{1-c}$

**Ratio Test:** For  $\sum z_n$ , let  $L = \lim_{n \rightarrow \infty} \left| \frac{z_{n+1}}{z_n} \right|$ . If  $L < 1$ , then  $\sum z_n$  is convergent. If  $L > 1$ , then  $\sum z_n$  is divergent. If  $L = 1$ , conclude nothing.

**Converge Pointwise:** Seq  $f_n: S \rightarrow \mathbb{C}$  pointwise convergent to  $f: S \rightarrow \mathbb{C}$  if  $\forall \varepsilon > 0, \forall z \in S, \exists N_{\varepsilon, z} \in \mathbb{N}$  s.t.  $|f_n(z) - f(z)| < \varepsilon$  for all  $n \geq N$

**Uniform Convergence:** Seq  $f_n: S \rightarrow \mathbb{C}$  uniformly convergent to  $f: S \rightarrow \mathbb{C}$  if  $\forall \varepsilon > 0, \exists N_{\varepsilon} \in \mathbb{N}$  s.t.  $|f_n(z) - f(z)| < \varepsilon$  for all  $n \geq N$  and  $\forall z \in S$

1. **Lemma|Continuous:** If  $f_n: S \rightarrow \mathbb{C}$  is uniformly convergent and continuous to  $f: S \rightarrow \mathbb{C}$ , then  $f$  is continuous on  $S$ .

2. **Lemma|Integral:** If  $f_n: S \rightarrow \mathbb{C}$  is uniformly convergent and continuous to  $f: S \rightarrow \mathbb{C}$ , then  $\int_{\Gamma} f_n(z) dz$  convergent to  $\int_{\Gamma} f(z) dz$ .

3. **Lemma|Integral:** If  $f_n: S \rightarrow \mathbb{C}$  is continuous,  $\sum_{n=0}^{\infty} f_n(z)$  is uniformly convergent on  $S$ , then  $\int_{\Gamma} \sum_{n=0}^{\infty} f_n(z) dz = \sum_{n=0}^{\infty} \int_{\Gamma} f_n(z) dz$ .

4. **Lemma|Holomorphic:** If  $D$  is simply-connected domain,  $f_n: D \rightarrow \mathbb{C}$  is holomorphic and uniformly convergent to  $f$ .  $\Rightarrow f$  holomorphic on  $D$ .

**Weierstrass M-Test:** For  $f_n: S \rightarrow \mathbb{C}$ , if  $\exists M_n \geq 0, n_0 \in \mathbb{N}$  s.t.  $|f_n(z)| \leq M_n$  for  $\forall z \in S, n \geq n_0$ .

If  $\sum_{n=0}^{\infty} M_n$  is convergent.  $\Rightarrow \sum_{n=0}^{\infty} f_n(z)$  is uniformly convergent on  $S$ .

**Power Series & Radius of Convergence:** Power Series is:  $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ . and there is a number  $R \in [0, \infty) \cup \{\infty\}$  s.t.

1. The Series is convergent on  $D_R(z_0)$ .

2. The Series is divergent on  $\mathbb{C} \setminus \overline{D}_R(z_0)$ .

3. The Series is uniformly convergent on  $\overline{D}_r(z_0)$  for all  $r \in [0, R)$ .

4. **Theorem|Holomorphic:** Then  $f(z)$  is holomorphic on  $D_R(z_0)$ , where  $R$  is the radius of convergence.

**Remark:** By using Ratio Test, we can find  $R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$ . if this limit exists. 可以取 0 和  $\infty$

### 5.2 Taylor Series and Laurent Series

**Taylor Series:** Let  $z_0 \in \mathbb{C}$  and  $f$  is holomorphic at  $z_0$ . Then the Taylor Series of  $f$  at  $z_0$  is:  $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$

1. **Theorem|Convergence:** If  $f$  is holomorphic on  $D_R(z_0)$ , then  $^1$  the Taylor Series of  $f$  at  $z_0$  converges to  $f(z)$  on  $D_R(z_0)$ .

2. **Theorem|Convergence:** If  $f$  is holomorphic on  $D_R(z_0)$ , then  $^2$  the Taylor Series of  $f$  at  $z_0$  converges uniformly to  $f(z)$  on  $\overline{D}_r(z_0)$   $r \in [0, R)$ .

**Analytic:** Let  $U$  open,  $f: U \rightarrow \mathbb{C}$  is analytic if  $\forall z \in U, \exists$  some disc centered at  $z$  s.t.  $f$  can be expressed as a convergent power series centred at  $z$ .

**Homo  $\rightarrow$  Analytic:** If  $f$  is holomorphic on  $U$ , then  $f$  is analytic on  $U$ .

**Properties of Taylor Series|Series:** Let  $z_0 \in \mathbb{C}, R > 0, f, g$  is holomorphic on  $D_R(z_0)$ , then for  $\star \forall z \in D_R(z_0) \star$ :

- Termwise Differentiation:**  $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \quad \forall z \in D_R(z_0)$  and  $f'(z) = \sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{(n-1)!} (z - z_0)^{n-1} = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(z_0)}{n!} (z - z_0)^n \quad \forall z \in D_R(z_0)$
- Lemma|Linear Combination:**  $(\alpha f + \beta g)(z) = \sum_{n=0}^{\infty} \left( \frac{\alpha f^{(n)}(z_0) + \beta g^{(n)}(z_0)}{n!} \right) (z - z_0)^n \quad \forall z \in D_R(z_0)$
- Lemma|Product:**  $(fg)(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \frac{f^{(k)}(z_0) g^{(n-k)}(z_0)}{k!(n-k)!} \right) (z - z_0)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{k=0}^n \binom{n}{k} f^{(k)}(z_0) g^{(n-k)}(z_0) \right) (z - z_0)^n \quad \forall z \in D_R(z_0)$
- Uniqueness of Taylor series:**  $f(z)$  has a *power series representation* at  $z_0$ , with radius of convergence  $R > 0$ .  $\Rightarrow$  Then it must be the *Taylor series* of  $f$  at  $z_0$ , and will equal to  $f(z)$  on  $D_R(z_0)$ .  
i.e. 假设某个函数  $f(z)$  能够由幂级数展开, 那么这个展开是唯一的, 且在收敛区间内等于  $f(z)$ .

**Laurent Series:** A *Laurent Series* centered at  $z_0$  is the series form:  $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$

**Convergence:** The *Laurent Series* is *convergent* if both  $\sum_{n=0}^{\infty} a_n (z - z_0)^n$  and  $\sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$  are *convergent*.

**Remark:** If *radius of convergence* of  $\sum_{n=0}^{\infty} a_n (z - z_0)^n$  and  $\sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$  are  $R$  and  $S$ .  $\Rightarrow$  *Laurent Series convergent* on  $S^{-1} < |z - z_0| < R$ .

**Annulus: Open annulus:**  $A_{r,R}(z_0) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$  **Closed annulus:**  $\bar{A}_{r,R}(z_0) = \{z \in \mathbb{C} : r \leq |z - z_0| \leq R\}$

**Laurent Series|For function:** Let  $z_0 \in \mathbb{C}$ ,  $0 \leq r < R \leq \infty$ ,  $f$  is holomorphic on  $A_{r,R}(z_0)$ . Then:

- $f$  can be expressed as a *Laurent Series* on  $A_{r,R}(z_0)$ , <sup>1</sup> *convergent* on  $A_{r,R}(z_0)$ . <sup>2</sup> *uniformly convergent* on  $\bar{A}_{r',R'}(z_0)$  for  $r < r' \leq R' < R$ .
- Coefficient:**  $a_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz$  for any *loop*  $\Gamma$  in  $A_{r,R}(z_0)$  and contain  $z_0$  in its interior.
- Uniqueness:** If  $f(z)$  has a *Laurent Series* on  $A_{r,R}(z_0)$ , then it must be the *Laurent Series* of  $f$  on  $A_{r,R}(z_0)$ , and will equal to  $f(z)$  on  $A_{r,R}(z_0)$ .

## 5.3 Singularities and Identity Theorem

**Singularity:** A point  $z_0 \in \mathbb{C}$  is a *singularity* of  $f$  if  $f$  is not holomorphic at  $z_0$ . **Zero:** A point  $z_0 \in \mathbb{C}$  is a *zero* of  $f$  if  $f(z_0) = 0$ .

**Isolated Singularity:** A *singularity*  $z_0$  of  $f$  is *isolated* if  $\exists R > 0$  s.t.  $f$  is holomorphic on  $D_R'(z_0)$ .

**Isolated Zero:** A zero  $z_0$  is *isolated* if  $\exists R > 0$  s.t.  $f(z) \neq 0$  for all  $z \in D_R'(z_0)$

**Zero of finite order:** If  $f(z_0) = f'(z_0) = \dots = f^{(n-1)}(z_0) = 0$  and  $f^{(n)}(z_0) \neq 0$ , then  $z_0$  is a *zero of order*  $n$ .

**Simple Zero:** A zero of order 1 is called a *simple zero*. i.e.  $f(z_0) = 0$  and  $f'(z_0) \neq 0$ .

**Properties of Zeros:** Let  $z_0 \in \mathbb{C}$ ,  $U$  be a *neighborhood* of  $z_0$ ,  $f$  is holomorphic on  $U$ .

- If  $z_0$  is a *zero of finite order*, then  $z_0$  is *isolated* (zero).
- If  $\exists$  *distinct* points  $z_n \in U$  s.t.  $z_n \rightarrow z_0$  and  $f(z_n) = 0$ .  $\Rightarrow \exists R > 0$  s.t.  $f(z) = 0$  for all  $z \in D_R(z_0)$  (identically zero on some disc centred at  $z_0$ ).

**Remark:** If  $\exists$  *distinct* points  $z_n \in U$  s.t.  $z_n \rightarrow z_0$  and  $z_n$  *Zero*, then  $z_0$  cannot be a *isolated zero*.

**Removable|Order|Essential Singularity:** Let  $z_0 \in \mathbb{C}$  is an isolated singularity of a function  $f$ , which is holomorphic on  $D_R'(z_0)$ .

- Let the *Laurent Series* of  $f$  at  $z_0$  be  $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$  valid on  $A_{0,R}(z_0)$
- removable singularity:** If  $a_n = 0$  for all  $n < 0$ , then  $z_0$  is a *removable singularity* of  $f$ . (i.e. 负的部分都是 0, 和泰勒展开很像)
- Pole of Order:** If  $a_{-m} \neq 0$  and  $a_{-n} = 0, \forall n > m$ , then  $z_0$  is a *pole of order*  $m$  of  $f$ . (i.e. 有限个负的非 0 项, 且最小的非 0 项是  $-m$ )
- Essential Singularity:** If  $a_n \neq 0$  for infinitely many  $n < 0$ .  $\Rightarrow z_0$  is an *essential singularity* of  $f$ . (i.e. 无限多个负的非 0 项)

**Remark:** Poles of order 1, 2, and 3 are also known as a simple, double, and triple poles, respectively.

**Properties of Singularity:** Let  $z_0 \in \mathbb{C}$ .

- Singularity of rational function|Isolated:** If  $z_0$  is *singularity* of *rational function*  $f$ , then  $z_0$  is *isolated*.
- Sequence $\rightarrow$ Isolated:** If  $\exists$  *distinct* points  $z_n \in U$  s.t.  $z_n \rightarrow z_0$  and  $z_n$  *Singularity*, then  $z_0$  cannot be a *removable singularity*.
- Extended:**  $f$  is holomorphic on  $D_R'(z_0)$ . If  $z_0$  is a *removable singularity*,  $f$  can be *redefined* at  $z_0$  to be *holomorphic* at  $z_0$ . ( $f(z_0) = a_0$ )
- Functions:** If  $f, g$  holomorphic at  $z_0$ ,  $z_0$  is a zero of  $g$ , with order  $m$ . Then:

- If  $z_0$  is not a zero of  $f \Rightarrow \frac{f}{g}$  has a *pole of order*  $m$  at  $z_0$ .
- If  $z_0$  is a zero of order  $k$  of  $f$  and  $k < m \Rightarrow \frac{f}{g}$  has a *pole of order*  $m - k$  at  $z_0$ .
- If  $z_0$  is a zero of order  $k$  of  $f$  and  $k \geq m \Rightarrow \frac{f}{g}$  has a *removable singularity* at  $z_0$ .

**Analytic Continuation:** Let  $D \subseteq \tilde{D} \subseteq \mathbb{C}$ ,  $f : D \rightarrow \mathbb{C}$  is *holomorphic*, and  $F : \tilde{D} \rightarrow \mathbb{C}$  is *holomorphic*.  $F(z) = f(z)$  for all  $z \in D$ .

**Identity Theorem|Disk-zero:** Let  $D$  domain,  $z_0 \in D$ ,  $f$  is *holomorphic* on  $D$ ,  $f(z) = 0, \forall z \in D_R(z_0) \Rightarrow f(z) = 0, \forall z \in D$ .

**Corollary|Disk-func:** Let  $D$  domain,  $f, g$  are *holomorphic* on  $D$ ,  $f(z) = g(z), \forall z \in D_R(z_0) \Rightarrow f(z) = g(z), \forall z \in D$ .

**Corollary|Sequence-zero:** Let  $D$  domain,  $\exists$  *distinct*  $z_n \in D$ ,  $z_n \rightarrow z_0 \in D$  s.t.  $f(z_n) = 0, \forall n \in \mathbb{N} \Rightarrow f(z) = 0, \forall z \in D$ .

**Corollary|Sequence-func:** Let  $D$  domain,  $\exists$  *distinct*  $z_n \in D$ ,  $z_n \rightarrow z_0 \in D$  s.t.  $f(z_n) = g(z_n), \forall n \in \mathbb{N} \Rightarrow f(z) = g(z), \forall z \in D$ .

## 6 Residue Theorem

### 6.1 Residue and Cauchy Residue Theorem

**Theorem:** Let  $f$  be holomorphic on  $D_R'(z_0)$ . (i.e.  $z_0$  is an isolated singularity of  $f$ ). Let  $\Gamma$  loop in  $D_R'(z_0)$ ,  $z_0 \in \text{Int}(\Gamma)$

Then:  $\int_{\Gamma} f(z) dz = 2\pi i a_{-1}$ , where  $a_{-1}$  is the coefficient of  $(z - z_0)^{-1}$  in the *Laurent Series* of  $f$  at  $z_0$ .  $\leftarrow$

**Residue:** Let  $f$  be holomorphic on  $D_R'(z_0)$ . (i.e.  $z_0$  is an isolated singularity of  $f$ ). Then *residual*:  $\text{Res}(f, z_0) = a_{-1}$ . where  $a_{-1} \uparrow$

**Properties of Residue:** Let  $z_0 \in \mathbb{C}$ ,  $f$  is *holomorphic* on  $D_R'(z_0)$ .

- If  $z_0$  is a *removable singularity*, then  $\text{Res}(f, z_0) = 0$ .
- If  $z_0$  is a *pole of order*  $m$ , then  $\text{Res}(f, z_0) = a_{-1}$ , where  $a_{-1} = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]$ .
- If  $f, g$  are *holomorphic* on  $D_R(z_0)$ ,  $g$  has a *simple zero* at  $z_0$ . (i.e.  $g(z_0) = 0$  and  $g'(z_0) \neq 0$ ). Then  $\text{Res}\left(\frac{f}{g}, z_0\right) = \frac{f(z_0)}{g'(z_0)}$ .

**Cauchy Residue Theorem:** Let  $\Gamma$  loop,  $f$  is holomorphic inside and on  $\Gamma$ , except for a finite isolated singularities  $z_1, z_2, \dots, z_k \in \text{Int}(\Gamma)$   
 Then:  $\int_{\Gamma} f(z)dz = 2\pi i \sum_{j=1}^k \text{Res}(f, z_j)$  where  $\text{Res}(f, z_j)$  is the residue of  $f$  at  $z_j$ .

## 6.2 Argument Principle and Rouché's Theorem

**Meromorphic:** Let  $D$  domain,  $f: D \rightarrow \mathbb{C}$  is meromorphic if  $\forall z \in D$   $f$  is holomorphic or has a pole of some finite order.

**Lemma|Finite Zero & Pole:** Let  $D$  domain,  $\Gamma$  loop in  $D$ ,  $f$  is meromorphic on  $D$ ,  $f \neq 0$ .  $\Rightarrow f$  has finite number of zeroes and poles  $\in \text{Int}(\Gamma)$ .

**Definition of  $N_0$  and  $N_{\infty}$ :** Let  $\Gamma$  loop,  $f$  be meromorphic on  $\text{Int}(\Gamma)$ , with zeros:  $z_1, \dots, z_n$  and poles:  $p_1, \dots, p_m$ .

Then:  $N_0(f) := \sum_j \text{order of } z_j$  and  $N_{\infty}(f) := \sum_j \text{order of } p_j$ .

**Argument Principle:** Let  $\Gamma$  loop,  $f$  is meromorphic in  $\text{Int}(\Gamma)$ ,  $f$  is holomorphic and non-zero on  $\Gamma$ . Then:  $\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} dz = N_0(f) - N_{\infty}(f)$

**Corollary:** Let  $\Gamma$  loop, if  $f$  is holomorphic inside and on  $\Gamma$ , and non-zero on  $\Gamma$ . 可能在里面为0 Then:  $\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} dz = N_0(f)$

**Rouché's Theorem:** Let  $\Gamma$  loop,  $f, g$  are holomorphic inside and on  $\Gamma$ . If  $|f(z) - g(z)| < |f(z)|$  for all  $z \in \Gamma$ .  $\Rightarrow N_0(f) = N_0(g)$ .

**Open Mapping Theorem:**  $D$  domain,  $f$  non-constant and holomorphic on  $D \Rightarrow f(D)$  is open in  $\mathbb{C}$ .

**Corollary:** Let  $D$  domain,  $f$  holomorphic on  $D$ . If  $\text{Re}(f)$  or  $\text{Im}(f)$  or  $|f|$  or  $\text{Arg}(f)$  is constant on  $D$ .  $\Rightarrow f$  is constant on  $D$ .

• **Trigonometric Integrals:** If  $R$  is rational function. Then  $\int_0^{2\pi} R(\cos \theta, \sin \theta) d\theta = \int_{C_1(0)} f(z) dz$  where  $f(z) = \frac{1}{iz} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)$  LHS is real int.

## 6.3 Improper Integral

**Def of Improper Integral:** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be continuous, then:

$$\int_{-\infty}^0 f(x) dx = \lim_{R \rightarrow -\infty} \int_R^0 f(x) dx \quad \int_0^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_0^R f(x) dx \quad \int_{-\infty}^{\infty} f(x) dx = \lim_{R_1 \rightarrow -\infty} \lim_{R_2 \rightarrow \infty} \int_{R_1}^{R_2} f(x) dx \quad R_1, R_2 \text{ 不一定一样}$$

**Remark:** If  $\int_{-\infty}^{\infty} f(x) dx$  is convergent, then  $\int_{-\infty}^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$ .

**Cauchy Principal Value of the integral:** p.v.  $\int_{-\infty}^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$

**Jordan Lemma:** Let  $P, Q$  be polynomial,  $\deg(Q) > \deg(P)$ .  $a \neq 0$  Then:  $\lim_{R \rightarrow \infty} \int_{C_R^{\pm}} \frac{P(z)}{Q(z)} \exp(iaz) dz = 0$  for  $+$ :  $a > 0$ ;  $-$ :  $a < 0$ .

**Useful Method in Improper Integral:** For  $\cos(nx)$ , consider  $\text{Re}(\exp(inx))$ ; for  $\sin(nx)$ , consider  $\text{Im}(\exp(inx))$ . (i.e.  $e^{inx} = \cos(nx) + i \sin(nx)$ )

**Useful Method in Improper Integral:** If denominator has  $e^x \Rightarrow$  Consider choose  $\Gamma: -R \rightarrow R \rightarrow R + 2\pi i \rightarrow -R + 2\pi i \rightarrow -R$ . 相当于通过纵向限制 pole 的数量

**Def of Improper Integral:** Let  $c \in (a, b) \subseteq \mathbb{R}$ ,  $f$  is continuous on  $[a, b] \setminus \{c\}$ , then:

$$\int_a^c f(x) dx = \lim_{r \downarrow 0} \int_a^{c-r} f(x) dx \quad \int_c^b f(x) dx = \lim_{s \downarrow 0} \int_{c+s}^b f(x) dx \quad \int_a^b f(x) dx = \lim_{r \downarrow 0} \int_a^{c-r} f(x) dx + \lim_{s \downarrow 0} \int_{c+s}^b f(x) dx$$

**Remark:** If  $\int_a^b f(x) dx$  is convergent, then  $\int_a^b f(x) dx = \lim_{r \downarrow 0} \left( \int_a^{c-r} f(x) dx + \int_{c+r}^b f(x) dx \right)$

**Cauchy Principal Value of the integral:** p.v.  $\int_a^b f(x) dx = \lim_{r \downarrow 0} \left( \int_a^{c-r} f(x) dx + \int_{c+r}^b f(x) dx \right)$

**Lemma:** Let  $D$  domain,  $f$  is meromorphic on  $D$ , with simple pole at  $c$ .

Let  $S_r$  circular arc parametrized by  $\gamma(t) = c + r \exp(it)$ ,  $t \in [\theta_0, \theta_1]$ ,  $0 \leq \theta_0 < \theta_1 \leq 2\pi$ . Then:  $\lim_{r \downarrow 0} \int_{S_r} f(z) dz = i(\theta_1 - \theta_0) \text{Res}(f, c)$

**Infinite Series:** (这里提供一个例子, 但一般的大体思路都差不多: 1. 构建一个矩形  $\Gamma_N(N+1/2)(\pm 1 \pm i)$ , 2. 通过两种方式: LM 计算  $\int_{\Gamma} f(z) dz$ , 3. 通过 Residue 定理计算  $\int_{\Gamma} f(z) dz$ , 4. 结合两者的结果, 得到级数的值)

e.g. Consider  $f(z) = \frac{\cot(\pi z)}{z^2}$  to calculate  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ : Calculate  $\text{Res}(f, n) = \dots = \frac{1}{n^2}$ ,  $n \neq 0$  Calculate  $\text{Res}(f, 0)$  = 通过直接计算它的 Laurent 展开式得到.

**Important:** By using Geometric Expansion:  $\frac{1}{1-(\square+\dots)} = 1 + \square + \square^2 + \dots$ . By L-M calculate  $|\int_{\Gamma} f(z) dz| \leq \dots \rightarrow 0$  **Important:** 寻找 bounded: e.g.  $\cot(\pi z)$  is bounded at  $\Gamma$ .

ps: 如果使用  $\Gamma_N: (N+1/2)(\pm 1 \pm i)$ ,  $\ell(\Gamma_N) = 4(2N+1)$  and  $|z| \geq N + \frac{1}{2}$

**Theorem:** Let  $\Gamma$  be loop with  $0 \in \text{Int}(\Gamma)$ . Then  $\binom{n}{k} = \frac{1}{2\pi i} \int_{\Gamma} \frac{(1+z)^n}{z^{k+1}} dz$

## 7 Appendix

### 7.1 Convergence Test for Real Series

**Divergence Test:** If  $\lim a_n \neq 0 \Rightarrow \sum a_n$  diverges. (If  $\sum a_n$  convergent  $\Rightarrow \lim a_n = 0$ .) **p-Test:**  $\sum \frac{1}{n^p}$  convergent iff  $p > 1$

**Comparison Test:** If  $0 < a_n < b_n$ ,  $\sum b_n$  convergent  $\Rightarrow \sum a_n$  also;  $\sum a_n$  divergent  $\Rightarrow \sum b_n$  also.

**Integral Test:** Let  $f: [1, \infty) \rightarrow \mathbb{R}$  is 非负递减,  $a_n = f(n)$ . Then  $\sum a_n$  converges iff  $\int_1^{\infty} f(x) dx < \infty$ .

**Absolutely Convergence:**  $\sum a_n$  convergent absolutely iff  $\sum |a_n|$  convergent. **If convergent abs  $\Rightarrow$  convergent.**

**Alternating Series Test:** If  $a_n$  decreasing,  $a_n \geq 0$ ,  $\lim a_n = 0$ . Then  $\sum (-1)^{n-1} a_n$  convergent.

**Cauchy's Condensation Test:** If  $a_n \geq 0$ ,  $a_n$  decreasing,  $\Rightarrow [\sum a_n \text{ convergent} \Leftrightarrow \sum 2^n a_{2^n} \text{ also}]$

### 7.2 Series

**Technical to write Taylor Series:** Since  $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$  for  $|z| < 1$ . for any  $\frac{1}{b-cx} \Rightarrow \frac{a}{b} \cdot \frac{1}{1-\frac{c}{b}x} = \frac{a}{b} \sum_{n=0}^{\infty} \left(\frac{c}{b}\right)^n x^n$  2 for  $\frac{1}{(1-z)^2} \Rightarrow \frac{d}{dz} \left(\frac{1}{1-z}\right)$

Moreover, need try to construct  $\frac{1}{1-(\frac{z-z_0}{R})}$  if it's holomorphic on  $D_R(z_0)$ . or:  $\frac{1}{1-\frac{z-z_0}{R}}$  if it's holomorphic on  $A_{1,\infty}(z_0)$ .

**Taylor Series for Familiar functions:**  $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$   $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$   $\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$  all of them have infinite radius of convergence.

### 7.3 Other

If  $|f(z) - g(z)| < |f(z)|$  for all  $z \in S$ .  $\Rightarrow f, g(z) \neq 0$  for all  $z \in S$ . ( $f, g$  non-zero on  $S$ )

**Trigonometric:**  $\sin(x) = \text{Im}(\exp(ix))$   $\cos(x) = \text{Re}(\exp(ix))$ .  $\cos(n\pi) = (-1)^n$   $\sin(n\pi) = 0$   $\cos((n+1)\pi) = (-1)^{n+1}$