HAlg Note

1 Basic Knowledge

Def of Matrix: A mapping from $\{1,...,n\} \times \{1,...,m\}$ to a field F is called a $n \times m$ matrix over F.

· The set of all $n \times m$ matrices over F is denoted by $Mat(n \times m; F) := Maps(\{1,...,n\} \times \{1,...,m\}, F)$. **Square Matrix**: $Mat(n; F)$

Solution Sets of Inhomogeneous Systems of Linear Equations: Solution = 特解 (Particular Solution) + 通解 (Homogeneous solution)

Def of Group $(G, *)$: A set G with a operator $*$ is a group if: **Closure**: $\forall g, h \in G, g * h \in G$; **Associativity**: $\forall g, h, k \in G, (g*h)*k = g*(h*k)$;
Identity: $\exists e \in G, \forall g \in G, e * g = g * e = g$; **Inverse**: $\forall g \in G, \exists g^{-1} \in G, g * g^{-1} = g^{-1} * g = e$.

· **Properties of Group**: If G, H are groups, then $G \times H$ also.

Field (F): A set F is a field with two operators: (addition)$+ : F \times F \to F; (\lambda, \mu) \to \lambda + \mu$ (multiplication)$\cdot : F \times F \to F; (\lambda, \mu) \to \lambda\mu$ if:
$(F, +)$ and $(F \backslash \{0_F\}, \cdot)$ are abelian groups with identity $0_F, 1_F$. and $\lambda(\mu + \nu) = \lambda\mu + \lambda\nu$ $e.g. Fields: \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}/p\mathbb{Z}$

Notation of 1-1,onto,bij: For function $f : V \to W$. **1-1**: $V \hookrightarrow W$ **onto**: $V \twoheadrightarrow W$ **bijection**: $V \xrightarrow{\sim} W$ (ps: bij 证法:1.def;2.$ff^{-1} = I, f^{-1}f = I$)

Projections (pr_i): $pr_i : X_1 \times X_2 \times \cdots \times X_n \to X_i : (x_1, ..., x_n) \mapsto x_i$ **Canonical Injections**: $in_i : X_i \to X_1 \times X_2 \times \cdots \times X_n : x \mapsto (0, ..., x, 0, ..., 0)$

2 Vector Spaces

2.1 Vector Spaces | Product of Sets | Vector Subspaces | Power, Union, Intersection of Sets

F**-Vector Space (V)**: A set V over a field F is a vector space if: V is an abelian group $V = (V, \dot{+})$ and $\forall \vec{v}, \vec{w} \in V$ $\lambda, \mu \in F$
a map $F \times V \to V : (\lambda, \vec{v}) \to \lambda\vec{v}$ satisfies: **I**: $\lambda(\vec{v} \dot{+} \vec{w}) = (\lambda\vec{v}) \dot{+} (\lambda\vec{w})$ **II**: $(\lambda + \mu)\vec{v} = (\lambda\vec{v}) \dot{+} (\mu\vec{v})$
III: $\lambda(\mu\vec{v}) = (\lambda\mu)\vec{v}$ **IV**: $1_F\vec{v} = \vec{v}$ ps:If $\lambda\vec{v} = \vec{0}$, then $\lambda = 0$ or $\vec{v} = \vec{0}$ or both. **Trivial Vector Space**: $V = \vec{0}$

· If V, W are F-vector spaces, then $V \oplus W$ is also. ps: $V \oplus W := V \times W$

Vector Subspace (U): $U \subseteq V$ is a subspace of V if: **I.** $\vec{0} \in U$ **II.** $\forall \vec{u}, \vec{v} \in U, \forall \lambda \in F$: $\vec{u} + \vec{v} \in U$ and $\lambda\vec{u} \in U$ (or: $\lambda\vec{u} + \mu\vec{v} \in U$)

1. If U_1, U_2 are subspaces of V. Then $U_1 \cap U_2$ and $U_1 + U_2$ are also. ps: $U_1 + U_2 := \langle U_1 \cup U_2 \rangle$
2. **Vector Subspace Generated by T** $(\langle T \rangle)$: If T is a subset of a F-vector space V. $\Rightarrow$ $\langle T \rangle$ is the smallest subspace of V containing T.
   Also, we can get: $\langle T \rangle = span(T) := \{\sum_i c_i\vec{v}_i : \vec{v}_i \in T, c_i \in F\}$ $\forall \vec{v} \in \langle T \rangle, \langle T \cup \{\vec{v}\} \rangle = \langle T \rangle$
3. **Generating/Spanning Set**: If $\langle T \rangle = V$. $\Rightarrow$ T is a generating set of V. **Finitely Generated**: $\exists$ T finite set, s.t. $V = \langle T \rangle$

Free Vector Space on the Set X: Set X, 将 X 中每一个元素都视为基, then $\{\sum_{x \in X} a_x x : a_x \in F, F\ is\ field\}$ is FVS on X.

Functional Vector Space: If X be a set and F be field. Then $Maps(X, F)$ is a F-Vector Space. ps: 'almost all': all but finitely many (全部, 但可以有有限个除外)

· $F\langle X \rangle := \{f : X \to F \mid f(x) = 0\ for\ almost\ all\ x \in X\}$ ps: $F\langle X \rangle$ is a subspace of $Maps(X, F)$ ？没写完！

Power of Set $\mathcal{P}(X)$: If X is a set, then $\mathcal{P}(X) := \{U : U \subseteq X\}$ (set of all subsets) ps: $\mathcal{U} \subseteq \mathcal{P}(X) \Rightarrow U$ is called a **system of subsets of** X.

1. **Empty System of subsets of X**: Empty System of subsets of X $:= \emptyset \in \mathcal{P}(X)$ (NOT $\{\emptyset\}$) $\star \bigcap \emptyset = X$ and $\bigcup \emptyset = \emptyset$ $\star$
2. **Union**: For $\mathcal{U} \subseteq \mathcal{P}(X)$, $\bigcup_{U \in \mathcal{U}} U := \{x \in X : \exists U \in \mathcal{U}\ s.t.\ x \in U\}$ **Intersection**: For $\mathcal{U} \subseteq \mathcal{P}(X)$, $\bigcap_{U \in \mathcal{U}} U := \{x \in X : \forall U \in \mathcal{U}, x \in U\}$

2.2 Linear Independence | Basis | Dimension

Linearly Independent: $L = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_r\}$ is linearly independent if: $\forall c_1, ..., c_r \in F, c_1\vec{v}_1 + ... + c_r\vec{v}_r = \vec{0} \Rightarrow c_1 = ... = c_r = 0$.

Linearly Dependent: L is linearly dependent if: $\exists \alpha_1, ..., \alpha_r$ not all zero s.t. $\alpha_1\vec{v}_1 + ... + \alpha_r\vec{v}_r = \vec{0}$

Basis: A basis of a vector space V is a linearly *independent generating set* of V. (Finitely generated $\Leftrightarrow \exists$ finite basis.)

1. subset E is a basis $\Leftrightarrow$ E is minimal generating sets $\Leftrightarrow$ E is maximal linearly independent sets.
2. **Fundamental Estimate of Linear Algebra**: Linearly independent sets $\subseteq$ basis $\subseteq$ generating sets.
3. $(\vec{v}_i)_{i \in I}$ is a basis of V $\Leftrightarrow$ $\forall \vec{v} \in V, \exists!\ c_i \in F$ (almost all of c_i are zero) s.t. $\vec{v} = \sum_{i \in I} c_i\vec{v}_i$

Family of Elements of A **Indexed by** I: $(a_i)_{i \in I} := func\ f : I \to A$ with $i \mapsto a_i$. e.g. $f(0) = 1, f(1) = 2, f(2) = 3$ 可以用 $(a_i)_{i \in \{0,1,2\}}, a_0 = 1, a_1 = 2, a_2 = 3$ 代替

· If $\{\vec{v}_i : i \in I\}$ is generating set of V, then $(\vec{v}_i)_{i \in I}$ is called a generating set. (同理对: $(\vec{v}_i)_{i \in I}$ is **basis indexed by** $i \in I$)

Linear Combinations of Basis: Let F be a field, family $(\vec{v}_i)_{1 \le i \le r}$, V is vector space. $\Phi : F^r \to V$ with $(c_1, ..., c_r) \mapsto c_1\vec{v}_1 + ... + c_r\vec{v}_r$:

1. **I.**$(\vec{v}_i)_{1 \le i \le r}$ is generating set $\Leftrightarrow \Phi$ is onto. $(F^r \twoheadrightarrow V)$ **II.**$(\vec{v}_i)_{1 \le i \le r}$ is linearly independent $\Leftrightarrow \Phi$ is 1-1. $(F^r \hookrightarrow V)$
2. $(\vec{v}_i)_{1 \le i \le r}$ is basis $\Leftrightarrow \Phi$ is bijection. $(F^r \xrightarrow{\sim} V)$

Steinitz Exchange Theorem: Let V be vector space. L is linearly independent set, E is generating set. $\Rightarrow \exists$ 1-1 $\phi : L \hookrightarrow E$ s.t.
$(E \setminus \phi(L)) \cup L$ is generating set. i.e. E 中的一部分元素可以完全由 L 中的元素线性表示出来.(即 L 中的元素可以替换 E 中的部分元素)

Dimension: Dimension of F-vector space is $\dim_F V := \#$ basis (i.e. cardinality of basis). e.g. $\dim_F F^n = n$

· Let V: Vector Space. L LI set, E generating set. **I.** $\dim L \le \dim V \le \dim E$ **II.** If $|L| = \dim V$ ($|E| = \dim V$), then L (E) is basis.

· **Dimension Theorem**: Let V: Vector Space. U, W: Subspaces. **I.** $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$ **II.** $\dim U \le \dim V$

2.3 Linear Maps | Rank-Nullity Theorem

Linear Maps: $f : V \to W$ (V,W vector spaces) is F**-linear** or **homomorphism** if: $f(\vec{v} + \vec{w}) = f(\vec{v}) + f(\vec{w})$ $f(\lambda\vec{v}) = \lambda f(\vec{v})$ $\forall \vec{v}, \vec{w} \in V; \forall \lambda \in F$

Isomorphism: Linear map $f : V \to W$ is bij. **Endomorphism (End)**: Linear map $f : V \to V$. **Automorphism (Aut)**: Isomorphism. $f : V \to V$.

General Linear Group / Automorphism Group: $GL(V) = Aut(V) := \{f : V \to V \mid f\ is\ isomorphism\}$ (subspace)

Fixed Point: For $f : X \to X$, if $f(x) = x$, then x is fixed point of f. **Set of Fixed Points**: $X^f := \{x \in X : f(x) = x\}$

Notation $\oplus$: Let U, W be subspaces of V. $V_1, ..., V_n$ be subspaces of V. $V_1 + \cdots + V_n := \langle V_1 \cup \cdots \cup V_n \rangle$:

1. **Basic Case**: If $f : U \times W \xrightarrow{\sim} V$ by $(\vec{u}, \vec{v}) \mapsto \vec{u} + \vec{v}$ then we write: $V = U \oplus W$ and called U, W **Complementary**.
2. **Special Case**: If $f : V_1 + \cdots + V_n \to V$ by $x \mapsto x$？不对把？P18 没写完

Classification of Vector Spaces: For Vector Space V, $\dim V = \dim F^n = n \Leftrightarrow \exists \phi : F^n \xrightarrow{\sim} V$ is isomorphism.