Stochastic-alpha-beta-rho (SABR) Model
Applied Stochastic Processes (FIN 514)

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The project overview

SABR Model

- One of the most popular stochastic volatility (SV) model.
- Heavily used for pricing and risk-managing options in interest rate and FX.
- Explains volatility skew/smile with minimal and intuitive parameters.

Project Goal

- Implement option pricing with Euler/Milstein scheme
- Implement conditional MC method (and check the variance reduction)
- Compare to the approximation formula by Hagan (code provided)
- Implement a smile calibration routine

Background: volatility skew/smile

- Black Monday crash in 1987:
  DJIA -22.6% in one day!
- Overall 'short gamma' due to the *portfolio insurance* (put on equity index)
- Market values (down-side) tail event higher than before.
- Market sees volatility skew/smile

Dow Jones (19-Jul-1987 through 19-Jan-1988)

(From Wikipedia)

SMILE

SKEW

FROWN
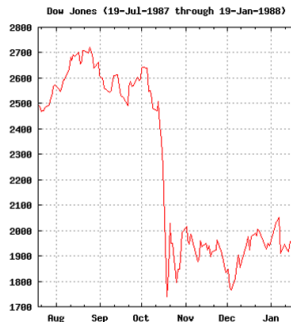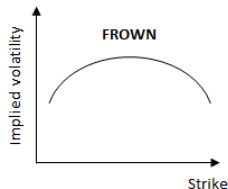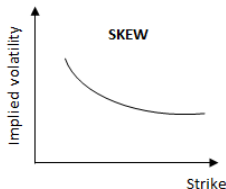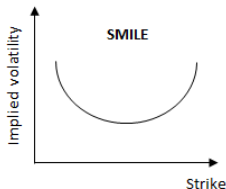
Why need model for smile? challanges in risk management

- Option trading desk (market-making/sell-side) usually accumulates option positions with different strikes.
- Under BSM model,
  - Vol σ fixed under spot change $S_0 \to S_0 + \Delta$.
  - Risk-management is easy: delta and vega clearly defined
  - One can hedge delta (with underlying stock) and vega (with ATM option)
  - However, the OTM option prices/risks are not correct!
- BSM model with different σ to each option K?
  - How do we fix the volatilities?
  - Sticky strike rule $\sigma = \sigma(K)$ vs sticky delta rule $\sigma = \sigma(S_0 - K)$.
  - Need to characterize the smile with a few minimal parameters.
- For better risk management, we need models which can capture the volatility smile.

How to model smile? Local volatility (LV)

- Volatility depending on the 'current location' of S_t:

$$\text{BSM: } \frac{dS_t}{S_t} = \sigma f(S_t) \, dW_t \qquad \text{Normal: } dS_t = \sigma_{\text{N}} f_{\text{N}}(S_t) \, dW_t$$

- BSM model: a trivial case with $f(x) = 1$. However, it is a local vol model under normal volatility ($f_{\text{N}}(x) = x$).
- Normal model: a trivial case with $f_{\text{N}}(x) = 1$. However, it is a local vol model under BSM volatility ($f(x) = 1/x$).
- What is the implied normal volatility of the Black-Scholes price on varying K? What is the relation between the implied volatility and the local vol?
- The implied volatility is the volatility average of the in-the-money paths
- **Exercise 1**: Chart the normal implied voly of the prices under BSM model for typical parameter sets. Measure the slope, $\partial\sigma(K)/\partial K$, at the money.

Case: $S_0 = 100, \sigma = 20\%(\sigma_{\mathrm{N}} = 20), r = q = 0$:

- Implied normal vol for constant BSM vol ($\sigma = 20\%$):
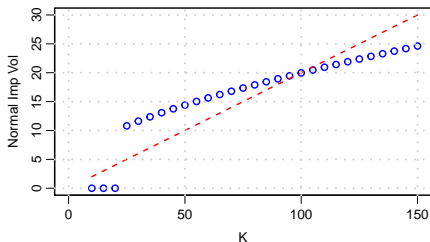
- Implied normal vol for constant normal vol ($\sigma_{\mathrm{N}} = 20$):
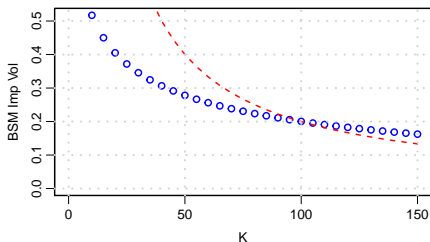
Displaced GBM (shifted BSM) model

- A quick local vol model
- 'Displaced asset price' $S_t + L$ follows GBM:

$$dS_t = \sigma_L(S_t + L) \, dW_t$$

- Somewhere between normal ($L \to \infty$) and log-normal model ($L = 0$).
- Can reuse BS formula with $S_0 + L \to S_0$ and $K + L \to K$.
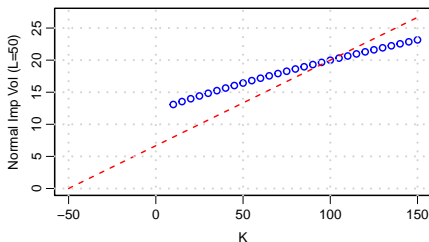- Calibration of σ_L (ATM option price on target):

$$\sigma_{\text{N}} \approx \sigma_L(S_0 + L) \approx \sigma \, S_0$$

  But, needs an exact calibration of σ_L for a given σ_{BS}.
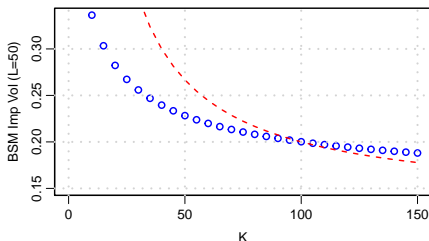
- **Exercise 2**: Chart the BSM implied vol of the prices under displaced GBM model. Using the implemented implied vol function, exactly calibrate σ_L to the ATM price.

Case: $S_0 = 100, L = 50, \sigma = 20\%, r = q = 0$:

- $\sigma_L = \sigma S_0/(S_0 + L) = 13.33\%$
- Implied normal vol: (red line: $\sigma_L(K + L)$)

- Implied BSM vol: (red line: $\sigma_L(K + L)/K$)

How to model smile? Stochastic volatility (SV)

- Volatility changing over time:

$$\text{BSM: } \frac{dS_t}{S_t} = \sigma_t \, dW_t \qquad \text{Normal: } dS_t = \sigma_t \, dW_t$$

- Many models proposed (mostly for BSM). For $dW_t dZ_t = \rho \, dt$,
  - Hull-While (SABR):
  $$\frac{d\sigma_t}{\sigma_t} = \alpha \, dZ_t$$
  - Heston: $V_t = \sigma_t^2$ follows Cox-Ingersoll-Ross (CIR) process,
  $$dV_t = \kappa(V_\infty - V_t)dt + \alpha\sqrt{V_t}dZ_t$$

- SV model correctly captures the smile, α for curvature and ρ for skewness.

Stochastic–α, β, ρ model SDE:

$$dS_t = \sigma_t S_t^{\beta} dW_t$$
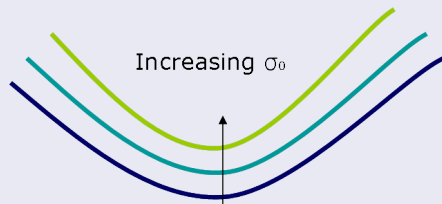$$d\sigma_t = \alpha \sigma_t dZ_t$$
$$dW_t \, dZ_t = \rho \, dt$$

- Parameters: σ_0, α, β, ρ.
- σ_0: ovarall volatility, calibrated to ATM implied vol
- β: elasticity or 'backbone'. (Normal: $\beta = 0$, BSM: $\beta = 1$)
- α: volatility of volatility, σ following a GBM
- ρ: correlation between asset price and volatility
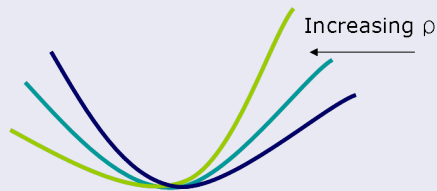
The impact of parameters
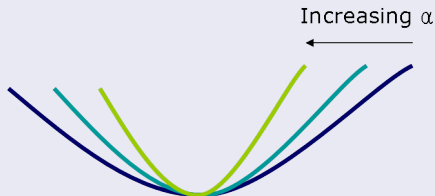
Starting vol σ_0

Increasing σ_0

Backbone β

- Fixed or infrequently changed
- BSM moel: $\beta = 1$ (Equity, FX)
- Normal model: $\beta = 0$ (Interest Rate)

Correlation ρ

Increasing ρ

'Vol of vol' α

Increasing α

Implied vol formula (Hagan et al, 2002)

The first few terms of the 'Taylor expansion' near $\alpha\sqrt{T} \approx 0$.

$$\sigma_B(K, f)$$

$$= \frac{\alpha}{(fK)^{(1-\beta)/2} \left\{ 1 + \frac{(1-\beta)^2}{24} \log^2 f/K + \frac{(1-\beta)^4}{1920} \log^4 f/K + \cdots \right\}} \cdot \left(\frac{z}{x(z)} \right)$$

$$\cdot \left\{ 1 + \left[\frac{(1-\beta)^2}{24} \frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4} \frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24} \nu^2 \right] t_{ex} + \cdots \right. .$$

$$(2.17a)$$

Here

$$z = \frac{\nu}{\alpha} (fK)^{(1-\beta)/2} \log f/K, \qquad (2.17b)$$

and $x(z)$ is defined by

$$x(z) = \log \left\{ \frac{\sqrt{1 - 2\rho z + z^2} + z - \rho}{1 - \rho} \right\}. \qquad (2.17c)$$

Success of SABR model

- Volatility smile information encoded into three parameters σ_0, α, ρ !!
- Vega (volatility) risk managed by the three parameters rather than each individual vol.
- Three implied vols (or option prices) on the smile can calibrate the parameters. $\rightarrow$ An effective interpolation method for implied volatility (or option price)

Limitation of Hagan's formula

- Arbitrage is equivalent to some event happening with negative probability. The price of a derivative paying \$1 on that event is negative (should be free at most)!

- Digital (Call/Put) option from call spread

$$\text{Prob}(S_T > K) = D(K) = \frac{C_{\text{BS}}(K) - C_{\text{BS}}(K + \Delta K)}{\Delta K} = -\frac{\partial C_{\text{BS}}(K)}{\partial K}$$

- When $\alpha\sqrt{T} \gg 1$, Hagan's formula sometimes implies $P(S_T > K) < 0$ from

$$C_{\text{BS}}(K, \sigma(K)) < C_{\text{BS}}(K + \Delta K, \sigma(K + \Delta K)).$$

The volatility effect $\sigma(K + \Delta K)$ overcomes (should NOT!) the moneyness effect $K + \Delta K$.

MC Simulation (Euler method)

- Unlike normal or BSM model (as in spread/basket option project), we can not jump directly from $t = 0$ to T.

- Divide the interval $[0, T]$ into N small steps, $t_k = (k/N)T$ and $\Delta t_k = T/N$ and simulate each time step,

$$S_t : \begin{cases} \beta = 0 : \ S_{t_{k+1}} = S_{t_k} + \sigma_{t_k} Z_1 \sqrt{\Delta t_k} \\ \beta = 1 : \ \log S_{t_{k+1}} = \log S_{t_k} + \sigma_{t_k} \sqrt{\Delta t_k} \, Z_1 - \frac{1}{2}\sigma_{t_k}^2 \Delta t_k, \end{cases}$$

$$\sigma_t : \sigma_{t_{k+1}} = \sigma_{t_k} \exp\left(\alpha \sqrt{\Delta t_k} \, Z_2 - \frac{1}{2}\alpha^2 \Delta t_k \right),$$

  where $Z_1, \ Z_2 \sim N(0, 1)$ with correlation ρ.

- Typically, $\Delta t_k \approx 0.25$. For $T = 30$, $N = 120$, quite time-consuming.

- Any good control variate?

$$C(K) = \frac{1}{N} \sum_{i=1}^{N} (S_T^{(i)} - K)^+$$

Stochastic integral of σ_t

From Itô's lemma,

$$d\sigma_t = \alpha\sigma_t \, dZ_t \quad \rightarrow \quad d\log\sigma_t = -\frac{1}{2}\alpha^2 dt + \alpha dZ_t$$

and we know the final distribution,

$$\sigma_T = \sigma_0 \exp\left(-\frac{1}{2}\alpha^2 T + \alpha Z_T\right).$$

We also know

$$\alpha\int_0^T \sigma_t dZ_t = \sigma_T - \sigma_0 = \sigma_0 \exp\left(-\frac{1}{2}\alpha^2 T + \alpha Z_T\right) - \sigma_0,$$

which will be useful for the integration of S_t.

Stochastic integral of S_t (normal: $\beta = 0$)

Writing the SDE in a de-correlated form,

$$dS_t = \sigma_t \left(\rho dZ_t + \sqrt{1-\rho^2} dW_t \right) \quad \text{for} \quad dW_t \, dZ_t = 0.$$

Integrating S_t, we get so far as

$$S_T - S_0 = \rho \int_0^T \sigma_t dZ_t + \sqrt{1-\rho^2} \int_0^T \sigma_t dW_t$$

$$= \frac{\rho}{\alpha} (\sigma_T - \sigma_0) + \sqrt{1-\rho^2} \int_0^T \sigma_t dW_t$$

From Itô's Isometry, the integrated variance is $I_T := \int_0^T \sigma_t^2 dt$. With some more work, the box can be expressed as

$$\int_0^T \sigma_t dW_t = W \sqrt{I_T}, \text{ where } W \sim N(0,1) \text{ intependent from } I_T \text{ and } \sigma_T$$

Conditional MC method (normal $\beta = 0$)

Given (σ_T, I_T), the distribution of S_T is

$$S_T = S_0 + \frac{\rho}{\alpha}(\sigma_T - \sigma_0) + \sqrt{(1-\rho^2)I_T} \, W$$

and the option price is from the normal model:

$$C_N \left(K, S_0 := S_0 + \frac{\rho}{\alpha}(\sigma_T - \sigma_0), \, \sigma_N := \sqrt{(1-\rho^2)I_T/T} \, \right)$$

Then, the price is obtained as an expectation over MC simulation of I_T:

$$C_{\beta=0} = E\left(C_N(\sigma_T, I_T) \right), \quad \text{where} \quad I_T = \sum_k \sigma_{t_k}^2 \Delta t_k$$

In this way, no need for simulating S_t. Given (σ_T, I_T) the price is exact, therefore MC variance is much smaller than that of brute-force MC.

Conditional MC method (BSM $\beta = 1$)

Given (σ_T, I_T), the distribution of S_T is

$$\log(S_T/S_0) = \frac{\rho}{\alpha}(\sigma_T - \sigma_0) - \frac{1}{2}I_T + \sqrt{1 - \rho^2}\, W\sqrt{I_T}$$

and the option price is from the BSM model:

$$C_{\text{BS}}\left(K, S_0 e^{\frac{\rho}{\alpha}(\sigma_T - \sigma_0) - \frac{\rho^2}{2}I_T}, \sqrt{(1 - \rho^2)I_T/T}\right)$$

Then, the price is obtained as an expectation over MC simulation of I_T:

$$C_{\beta=1} = E\left(C_{\text{BS}}(\sigma_T, I_T)\right), \quad \text{where} \quad I_T = \sum_k \sigma_{t_k}^2 \Delta t_k$$

In this way, no need for simulating S_t. Given (σ_T, I_T) the price is exact, therefore MC variance is much smaller than that of brute-force MC.

The conditional distribution of I_T on σ_T (Kennedy et al)

The conditional mean of I_T on σ_T is known as

$$E(I_T|\sigma_T) = \frac{\sigma_0^2 \sqrt{T}}{2\alpha} \frac{N(d_\alpha + \alpha\sqrt{T}) - N(d_\alpha - \alpha\sqrt{T})}{n(d_\alpha + \alpha\sqrt{T})}$$

$$\text{for} \quad d_\alpha = \log(\sigma_T/\sigma_0)/(\alpha\sqrt{T}).$$

The distribution of S_T is approximated

$$S_T = S_0 + \frac{\rho}{\alpha}\big(\sigma_T - \sigma_0\big) + \sqrt{1-\rho^2}\,\eta(\sigma_T)W\sqrt{T}$$

for $\eta(\sigma_T) = E(I_T|\sigma_T)/\sqrt{T}$. For a given σ_T, S_T follows a normal distribution, so we now the option

$$C_{\mathrm{N}}(\sigma_T) = C_{\mathrm{N}}\left(S_0 := S_0 + \frac{\rho}{\alpha}\big(\sigma_T - \sigma_0\big),\ \sigma_{\mathrm{N}} := \sqrt{1-\rho^2}\,\eta(\sigma_T)\right)$$

Option price as an integration (Kennedy et al)

$$C_{\beta=0} = E\Big((S_T - K)^+\Big) = E\Big((S_T - K)^+|\sigma_T\Big) = E\Big(C_{\mathrm{N}}(\sigma_T)\Big)$$

$$= \int_{-\infty}^{\infty} C_{\mathrm{N}}\left(S_0 + \frac{\rho}{\alpha}\big(\sigma_T(z) - \sigma_0\big), \sqrt{1-\rho^2}\,\eta(\sigma_T(z))\right) n(z)\,dz$$

$$\text{where} \quad \sigma_T(z) = \sigma_0 \exp\left(-\frac{1}{2}\alpha^2 T + \alpha\sqrt{T}\,z\right)$$

Using Gauss-Hermite quadrature (GHQ), [Py Demo]

$$C_{\beta=0} = \sum_m C_{\mathrm{N}}\left(S_0 + \frac{\rho}{\alpha}\big(\sigma_T(z_m) - \sigma_0\big), \sqrt{1-\rho^2}\,\eta(z_m)\right) w_m$$

for some points $\{z_m\}$ and weights $\{w_m\}$, and $\eta(z_m) := \eta(\sigma_T(z_m))$.

The results are similar:

$$\log(S_T/S_0) = \frac{\rho}{\alpha}(\sigma_T - \sigma_0) - \frac{1}{2}I_T + \sqrt{1-\rho^2}\,W\sqrt{I_T}$$

$$C_{\beta=1} = \sum_m C_{BS}\left(S_0 e^{\frac{\rho}{\alpha}\left(\sigma_T(z_m)-\sigma_0\right)-\frac{\rho^2}{2}\eta(z_m)}, \sqrt{1-\rho^2}\,\eta(z_m)\right) w_m$$

for some points $\{z_m\}$ and weights $\{w_m\}$.

- Implement the method of Kennedy et al and compare it against the Monte Carlo result for both normal ($\beta = 0$) and BSM backbone ($\beta = 1$).

Smile Calibration

- When β is given (0 or 1), three parameters, σ_0, ρ and α, can be calibrated to three option prices (or implied volatilities), typically at $K = S_0$ (ATM), $S_0 - \Delta$ and $S_0 + \Delta$.

$$\text{SABR}(\sigma_0, \rho, \alpha) \to \sigma(S_0), \sigma(S_0 - \Delta), \sigma(S_0 + \Delta)$$

- Write a calibration routine in R to solve σ_0, ρ and α.

Projects

Validation

- $C_{\beta=0}$ should converge to the normal model price when α is very small.
- Test against the result from Korn & Tang (Wilmott)

Homework

- First focus on the normal backbone $\beta = 0$.
- Make sure to use antithetic method (create Z, then add $-Z$).
- Short (1 page) write-up briefly explaining the code.
- Reproduce the graphs in 'The impact of parameters'. Fix your normal implied vol at ATM.
- Your script should be self-complete and should run without error.