Stochastic-alpha-beta-rho (SABR) Model
Applied Stochastic Processes (FIN 514)

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The project overview

SABR Model

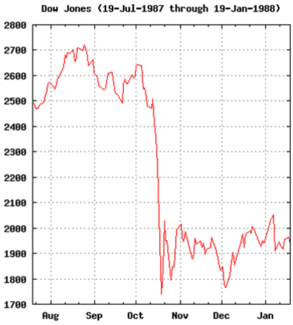
- One of the most popular stochastic volatility (SV) model.
- Heavily used for pricing and risk-managing options in interest rate and FX.
- Explains volatility skew/smile with minimal and intuitive parameters.

Project Goal

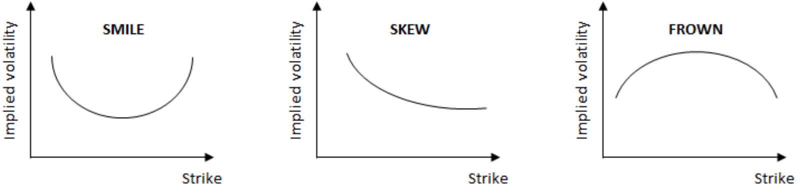
- Implement option pricing with Euler/Milstein scheme
- Implement conditional MC method (and check the variance reduction)
- Compare to the approximation formula by Hagan (code provided)
- Implement a smile calibration routine

Background: volatility skew/smile

- Black Monday crash in 1987: DJIA -22.6% in one day!
- Overall 'short gamma' due to the *portfolio insurance* (put on equity index)
- Market values (down-side) tail event higher than before.
- Market sees volatility skew/smile



(From Wikipedia)



Why need model for smile? challenges in risk management

- Option trading desk (market-making/sell-side) usually accumulates option positions with different strikes.
- Under BSM model,
  - Vol σ fixed under spot change $S_0 \to S_0 + \Delta$.
  - Risk-management is easy: delta and vega clearly defined
  - One can hedge delta (with underlying stock) and vega (with ATM option)
  - However, the OTM option prices/risks are not correct!
- BSM model with different σ to each option K?
  - How do we fix the volatilities?
  - Sticky strike rule $\sigma = \sigma(K)$ vs sticky delta rule $\sigma = \sigma(S_0 - K)$.
  - Need to characterize the smile with a few minimal parameters.
- For better risk management, we need models which can capture the volatility smile.

How to model smile? Local volatility (LV)

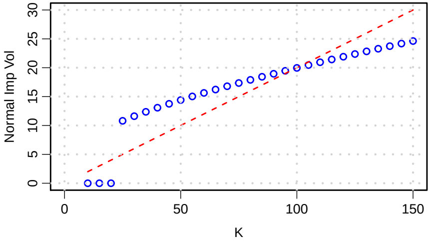
- Volatility depending on the 'current location' of S_t:

$$\text{BSM: } \frac{dS_t}{S_t} = \sigma f(S_t)\, dW_t \qquad \text{Normal: } dS_t = \sigma_\text{N} f_\text{N}(S_t)\, dW_t$$

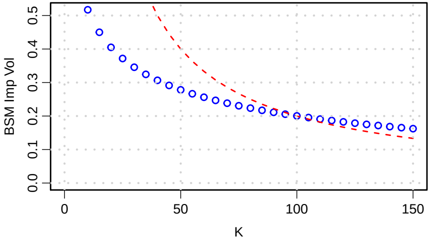
- BSM model: a trivial case with $f(x) = 1$. However, it is a local vol model under normal volatility ($f_\text{N}(x) = x$).
- Normal model: a trivial case with $f_\text{N}(x) = 1$. However, it is a local vol model under BSM volatility ($f(x) = 1/x$).
- What is the implied normal volatility of the Black-Scholes price on varying K? What is the relation between the implied volatility and the local vol?
- The implied volatility is the volatility average of the in-the-money paths
- **Exercise 1**: Chart the normal implied vol of the prices under BSM model for typical parameter sets. Measure the slope, $\partial\sigma(K)/\partial K$, at the money.

Case: $S_0 = 100, \sigma = 20\% (\sigma_{\mathrm{N}} = 20), r = q = 0$:

- Implied normal vol for constant BSM vol ($\sigma = 20\%$):



- Implied BSM vol for constant normal vol ($\sigma_{\mathrm{N}} = 20$):



Displaced GBM (shifted BSM) model

- A simple local vol model with analytic solution (i.e., Black-Scholes formula)
- *Displaced* (or *shifted*) asset price $S_t + L$ follows a GBM:

$$dS_t = \sigma_L(S_t + L)\, dW_t$$

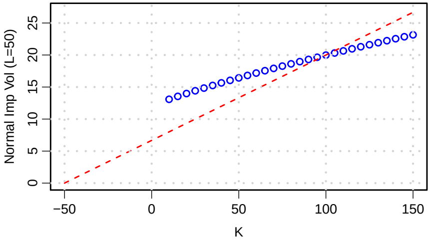
- Calibration of σ_L (ATM option price on target):

$$\sigma_\mathrm{N} \approx \sigma_L(S_0 + L) \approx \sigma\ S_0$$

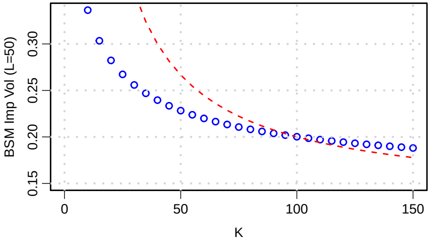
  But, needs an exact calibration of σ_L for a given σ_{BS}.
- Can reuse BS formula with $S_0 + L \rightarrow S_0$ and $K + L \rightarrow K$.
- Somewhere between normal ($L \rightarrow \infty$) and log-normal model ($L = 0$).
- **Exercise 2**: Chart the BSM implied vol of the prices under displaced GBM model. Using the implemented implied vol function, exactly calibrate σ_L to the ATM price.

Case: $S_0 = 100, L = 50, \sigma = 20\%, r = q = 0$:

- $\sigma_L = \sigma S_0/(S_0 + L) = 13.33\%$
- Implied normal vol: (red line: $\sigma_L(K + L)$)



- Implied BSM vol: (red line: $\sigma_L(K + L)/K$)



How to model smile? Stochastic volatility (SV)

- Volatility changing over time:

$$\text{BSM: } \frac{dS_t}{S_t} = \sigma_t \, dW_t \qquad \text{Normal: } dS_t = \sigma_t \, dW_t$$

- Many models proposed (mostly for BSM). For $dW_t dZ_t = \rho \, dt$,
  - Hull-White and SABR:

$$\frac{d\sigma_t}{\sigma_t} = \alpha \, dZ_t$$

  - Heston: $V_t = \sigma_t^2$ follows Cox-Ingersoll-Ross (CIR) process,

$$dV_t = \kappa(V_\infty - V_t)dt + \alpha\sqrt{V_t}dZ_t$$

- SV model correctly captures the smile, α for curvature and ρ for skewness.

SABR model: LV + SV

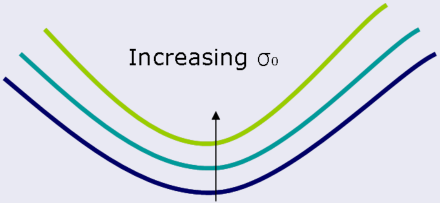
Stochastic–α, β, ρ model SDE:

$$dS_t = \sigma_t S_t^\beta dW_t$$
$$d\sigma_t = \alpha \sigma_t dZ_t$$
$$dW_t\, dZ_t = \rho\, dt$$

- Parameters: σ_0, α, β, ρ.
- σ_0: overall volatility, calibrated to ATM implied vol
- β: elasticity or 'backbone'. (Normal: $\beta = 0$, BSM: $\beta = 1$)
- α: volatility of volatility, σ following a GBM
- ρ: correlation between asset price and volatility

The impact of parameters

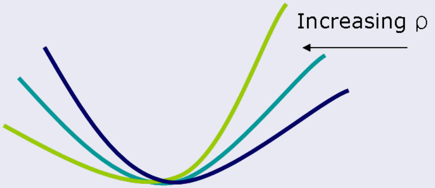
Initial vol σ_0



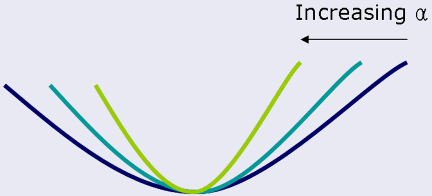
Backbone β

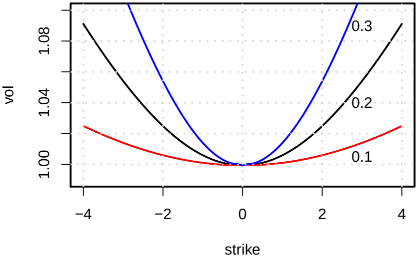
- Fixed or infrequently changed
- BSM moel: $\beta = 1$ (Equity, FX)
- Normal model: $\beta = 0$ (Interest Rate)

Correlation ρ



'Vol of vol' α




vol
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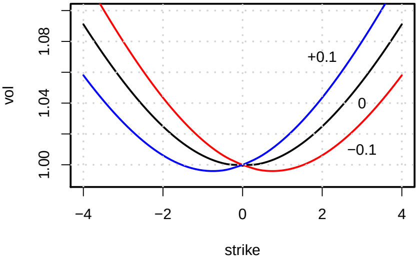
vol
1.00
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strike
+0.1
0
−0.1

Equivalent BSM-volatility formula (Hagan et al, 2002)

The first few terms of Taylor's expansion near $\alpha\sqrt{T} \approx 0$.

$$\sigma_B(K, f)$$
$$= \frac{\alpha}{(fK)^{(1-\beta)/2}\left\{1 + \frac{(1-\beta)^2}{24}\log^2 f/K + \frac{(1-\beta)^4}{1920}\log^4 f/K + \cdots\right\}} \cdot \left(\frac{z}{x(z)}\right)$$
$$\cdot \left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2\right] t_{ex} + \cdots\right. . \tag{2.17a}$$

Here

$$z = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log f/K, \tag{2.17b}$$

and $x(z)$ is defined by

$$x(z) = \log\left\{\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\right\}. \tag{2.17c}$$

Success of the SABR model

- Volatility smile information encoded into three parameters σ_0, α, ρ.
- These three parameters are parsimonious (minimal) and intuitive.
- Equivalent BSM volatility is available although not accurate for wide parameter range.
- Vega (volatility) risk managed by the three parameters rather than each individual vol.
- Three implied vols (or option prices) on the smile can calibrate the parameters. $\rightarrow$ An effective interpolation method for implied volatility (or option price)

Limitation of Hagan's formula

- Arbitrage is equivalent to some event happening with negative probability. The price of a derivative paying $\$1$ on that event is negative (should be free at most)!
- Digital call option price (probability) from call spread:

$$P(S_T > K) = D(K, \sigma(K))$$
$$= \frac{C_{\text{BS}}(K, \sigma(K)) - C_{\text{BS}}(K+\Delta K, \sigma(K+\Delta K))}{\Delta K} = -\frac{\partial C_{\text{BS}}(K, \sigma(K))}{\partial K}$$

- For positive PDF, $D(K, \sigma(K))$ should be monotonically decreasing on K. When $\alpha\sqrt{T} \gg 1$, however, Hagan's formula often implies $D(K, \sigma(K))$ increasing on K:

$$D(K, \sigma(K)) < D(K+\Delta K, \sigma(K+\Delta K)).$$

The volatility effect $\sigma(K+\Delta K)$ overcomes (should NOT!) the moneyness effect $K+\Delta K$.

Euler method (MC with time-discretization)

- Unlike normal or BSM model (as in spread/basket option project), we can not jump the simulation directly from $t=0$ to T.
- Divide the interval $[0,T]$ into N small steps, $t_k = (k/N)T$ and $\Delta t_k = T/N$ and simulate each time step with

$$S_t : \begin{cases} \beta = 0: \; S_{t_{k+1}} = S_{t_k} + \sigma_{t_k} W_1 \sqrt{\Delta t_k} \\ \beta = 1: \; \log S_{t_{k+1}} = \log S_{t_k} + \sigma_{t_k} \sqrt{\Delta t_k}\, W_1 - \frac{1}{2}\sigma_{t_k}^2 \Delta t_k, \end{cases}$$

$$\sigma_t : \sigma_{t_{k+1}} = \sigma_{t_k} \exp\left(\alpha \sqrt{\Delta t_k}\, Z_1 - \frac{1}{2}\alpha^2 \Delta t_k\right),$$

where $W_1,\ Z_1 \sim N(0,1)$ with correlation ρ.

- Typically, $\Delta t_k \approx 0.25$. For $T=30$, $N=120$, quite time-consuming.
- Any good control variate?

$$C(K) = \frac{1}{N}\sum_{i=1}^{N}(S_T^{(i)} - K)^+$$

Euler method vs Milstein method

For a stochastic process,

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,$$

the Euler scheme is given as:

$$X_{t+\Delta t} = X_t + \mu(X_t)\Delta t + \sigma(X_t)\Delta W_t$$

In Milstein scheme, an higher-order correction is added:

$$X_{t+\Delta t} = X_t + \mu(X_t)\Delta t + \sigma(X_t)\Delta W_t + \frac{\sigma(X_t)\sigma'(X_t)}{2}\left((\Delta W_t)^2 - \Delta t\right),$$

where the extra term is derived from the GBM approximation:

$$\sigma(X_t) = A\,X_t \quad \text{with} \quad A = \sigma'(X_t) \quad \text{and} \quad B = \mu(X_t)/X_t$$

$$d\log X_t = (B - \frac{1}{2}A^2)dt + A\,dW_t,$$
$$X_{t+\Delta t} = X_t \cdot \exp\left((B - \frac{1}{2}A^2)\Delta t + A\,\Delta W_t\right)$$

Stochastic integral of σ_t

From Itô's lemma,

$$\frac{d\sigma_t}{\sigma_t} = \alpha\, dZ_t \quad \Rightarrow \quad d\log\sigma_t = -\frac{1}{2}\alpha^2 dt + \alpha dZ_t$$

we can solve the volatility process:

$$\sigma_T = \sigma_0 \exp\left(-\frac{1}{2}\alpha^2 T + \alpha Z_T\right).$$

We also know

$$\alpha\int_0^T \sigma_t dZ_t = \sigma_T - \sigma_0 = \sigma_0 \exp\left(-\frac{1}{2}\alpha^2 T + \alpha Z_T\right) - \sigma_0,$$

which will be useful for the integration of S_t.

Stochastic integral of S_t (normal: $\beta = 0$)

Writing the SDE in a de-correlated form,

$$dS_t = \sigma_t \left(\rho dZ_t + \sqrt{1-\rho^2} dX_t\right) \quad \text{with} \quad dX_t\, dZ_t = 0.$$

Integrating S_t, we get so far as

$$\begin{aligned} S_T - S_0 &= \rho \int_0^T \sigma_t dZ_t + \sqrt{1-\rho^2} \int_0^T \sigma_t dX_t \\ &= \frac{\rho}{\alpha}(\sigma_T - \sigma_0) + \sqrt{1-\rho^2} \int_0^T \sigma_t dX_t \end{aligned}$$

From Itô's Isometry, the integration in blue is equivalent to

$$\int_0^T \sigma_t dX_t = X_1 \sqrt{I_T} \quad \text{where} \quad X_1 \sim N(0,1), \quad I_T := \int_0^T \sigma_t^2 dt.$$

Here, the random variable X_1 is independent from I_T and σ_T. Note that $I_T = \sigma_0^2 T$ if $\alpha = 0$ (i.e., volatility is not stochastic).

Conditional MC method (normal $\beta = 0$)

Conditional on (σ_T, I_T), S_T can be sampled from

$$S_T = S_0 + \frac{\rho}{\alpha}(\sigma_T - \sigma_0) + \sqrt{(1-\rho^2) I_T}\; X_1$$

and the option price is from the normal model:

$$C_{\mathrm{N}}\left(K, S_0 := S_0 + \frac{\rho}{\alpha}(\sigma_T - \sigma_0),\; \sigma_{\mathrm{N}} := \sqrt{(1-\rho^2) I_T / T}\right)$$

Then, the price is obtained as an expectation over (σ_T, I_T):

$$C_{\beta=0} = E\left(C_{\mathrm{N}}(\sigma_T, I_T)\right), \quad \text{where} \quad I_T = \sum_k \sigma_{t_k}^2 \Delta t_k.$$

For I_T, we can use higher-order numerical integration methods (trapezoidal rule or Simpson's rule)

$$I_T = \sum_{k=0}^{N-1} (\sigma_{t_k}^2 + \sigma_{t_{k+1}}^2) \frac{\Delta t}{2} = \left(\sigma_{t_0}^2 + 2\sigma_{t_1}^2 + \cdots + 2\sigma_{t_{N-1}}^2 + \sigma_{t_N}^2\right) \frac{\Delta t}{2}$$

Conditional MC method (BSM $\beta = 1$)

Conditional on (σ_T, I_T), S_T can be sampled from

$$\log\left(\frac{S_T}{S_0}\right) = \frac{\rho}{\alpha}(\sigma_T - \sigma_0) - \frac{1}{2}I_T + \sqrt{1-\rho^2}\, X_1\sqrt{I_T}$$

and the option price is from the BSM formula:

$$C_{\text{BS}}\left(K, S_0 e^{\frac{\rho}{\alpha}\left(\sigma_T - \sigma_0\right) - \frac{\rho^2}{2}I_T}, \sqrt{(1-\rho^2)I_T/T}\right)$$

Then, the price is obtained as an expectation over (σ_T, I_T):

$$C_{\beta=1} = E\left(C_{\text{BS}}(\sigma_T, I_T)\right), \quad \text{where} \quad I_T = \sum_k \sigma_{t_k}^2 \Delta t_k$$

For I_T, we can use higher-order numerical integration methods (trapezoidal rule or Simpson's rule)

$$I_T = \left(\sigma_{t_0}^2 + 4\sigma_{t_1}^2 + 2\sigma_{t_2}^2 + \cdots + 4\sigma_{t_{N-2}}^2 + 2\sigma_{t_{N-1}}^2 + \sigma_{t_N}^2\right)\frac{\Delta t}{3} \quad \text{for even } N$$

Advantages of conditional MC method

- No need to simulate S_t: less computation, less memory use.
- Given (σ_T, I_T), the option price is exact. Therefore, MC variance is much smaller than that of the MC simulating both σ_t and S_t.
- Can obtain correct option value for extreme strike values: If we have so simulate S_T, no simulation path arrives at $S_T > K$ for very big or small K, option value from MC is zero. The conditional MC method result in very small (correct) option value because the price comes from (analytic) BSM formula.

Conditional distribution for the Heston model

Conditional MC method can apply to other SV models. Key step is to express the final price S_T conditional on (σ_T, I_T).
For example, Heston model is given by

$$\frac{dS_t}{S_t} = \sigma_t(\rho dZ_t + \rho_* dX_t), \quad \text{for} \quad \rho_* = \sqrt{1-\rho^2}$$
$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}dZ_t \quad (v_t = \sigma_t^2).$$

Integrating v_t,

$$v_T - v_0 = \kappa(\theta T - I_T) + \xi \int_0^T \sqrt{v_t}dZ_t$$
$$\int_0^T \sigma_t\, dZ_t = \frac{1}{\xi}\Big(v_T - v_0 - \kappa(T\theta - I_T)\Big)$$

Finally we obtain

$$\log\left(\frac{S_T}{S_0}\right) \;=\; \frac{\rho}{\xi}\big(v_T - v_0 - \kappa(\theta - I_T)\big) - \frac{1}{2}I_T + \rho_*\sqrt{I_T}\,X_1.$$

Conditional distribution for the Heston model

It is known that v_T is distributed as a noncentral chi-square distribution, $\textsc{ncx2}(\delta, \lambda)$:

$$v_T \;=\; \frac{\xi^2(1-e^{-\kappa T})}{4\kappa}\textsc{ncx2}(\delta,\lambda) = \frac{e^{-\kappa T/2}}{2\phi(\kappa)}\textsc{ncx2}(\delta,\lambda),$$

where the degrees of freedom δ and the noncentrality λ are

$$\delta = \frac{4\kappa\theta}{\xi^2}, \quad \lambda = \frac{4v_0\kappa\, e^{-\kappa T}}{\xi^2(1-e^{-\kappa T})} = 2v_0 e^{-\kappa T/2}\phi(\kappa) \quad \text{for} \quad \phi(\kappa) = \frac{2\kappa/\xi^2}{\sinh(\kappa T/2)}.$$

Therefore, the procedure for conditional MC is similar:

1. sample v_T from $\textsc{ncx2}(\delta,\lambda)$ (library available)
2. simulate paths of v_t to compute I_T
3. use BS formula for given (v_T, I_T).

We can apply this scheme to many well-known SV models: SABR, Heston, 3/2, etc.

Exact simulation of the SV models (for project)

We can even avoid the time-discretized simulation of σ_t (v_t): jumping from $t = 0$ to T is possible.
The Laplace transform of I_T conditional on σ_T is available.

$$E(e^{-sI_T}|v_t) = \int_{I_T=0}^{\infty} e^{-sI_T} dI_T = f(s, v_T) = \cdots$$

The CDF of I_T is the inverse-Laplace transform which can be computed numerically. Then, I_T (conditional on v_T) can be sample from the CDF. [Broadie & Kaya, 2006. *Exact Simulation of Stochastic Volatility and Other Affine Jump Diffusion Processes.* Operations Research 54, 217–231].
A similar approach is possible for other processes:

- SABR: Cai, Song, Chen, 2017. Exact Simulation of the SABR Model. Operations Research 65, 931–951.
- 3/2 model: Baldeaux, 2012. Exact simulation of the 3/2 model. Int. J. Theor. Appl. Finan. 15, 1250032.

Smile Calibration

- When β is given (0 or 1), three parameters, σ_0, ρ and α, can be calibrated to three option prices (or implied volatilities), typically at $K = S_0$ (ATM), $S_0 - \Delta$ and $S_0 + \Delta$.

$$\text{SABR}(\sigma_0, \rho, \alpha) \rightarrow \sigma(S_0), \sigma(S_0 - \Delta), \sigma(S_0 + \Delta)$$

- Write a calibration routine in R to solve σ_0, ρ and α in homework.

The conditional distribution of I_T on σ_T (Optional)

Kennedy et al method: The conditional mean of I_T on σ_T is known as

$$E(I_T|\sigma_T) = \frac{\sigma_0^2\sqrt{T}}{2\alpha}\frac{N(d_\alpha+\alpha\sqrt{T}) - N(d_\alpha-\alpha\sqrt{T})}{n(d_\alpha+\alpha\sqrt{T})}$$

$$\text{for}\quad d_\alpha = \log(\sigma_T/\sigma_0)/(\alpha\sqrt{T}).$$

The distribution of S_T is approximated

$$S_T = S_0 + \frac{\rho}{\alpha}\big(\sigma_T - \sigma_0\big) + \sqrt{1-\rho^2}\,\eta(\sigma_T) W\sqrt{T}$$

for $\eta(\sigma_T) = E(I_T|\sigma_T)/\sqrt{T}$. For a given σ_T, S_T follows a normal distribution, so we now the option

$$C_\mathrm{N}(\sigma_T) = C_\mathrm{N}\left(S_0 := S_0 + \frac{\rho}{\alpha}(\sigma_T - \sigma_0),\ \sigma_\mathrm{N} := \sqrt{1-\rho^2}\,\eta(\sigma_T)\right)$$

Option price as an integration (Optional)

$$C_{\beta=0} = E\Big((S_T - K)^+\Big) = E\Big((S_T - K)^+ | \sigma_T\Big) = E\Big(C_{\text{N}}(\sigma_T)\Big)$$

$$= \int_{-\infty}^{\infty} C_{\text{N}}\left(S_0 + \frac{\rho}{\alpha}(\sigma_T(z) - \sigma_0), \sqrt{1-\rho^2}\,\eta(\sigma_T(z))\right) n(z)\, dz$$

$$\text{where} \quad \sigma_T(z) = \sigma_0 \exp\left(-\frac{1}{2}\alpha^2 T + \alpha\sqrt{T}\, z\right)$$

Using Gauss-Hermite quadrature (GHQ), [Py Demo]

$$C_{\beta=0} = \sum_m C_{\text{N}}\left(S_0 + \frac{\rho}{\alpha}(\sigma_T(z_m) - \sigma_0), \sqrt{1-\rho^2}\,\eta(z_m)\right) w_m$$

for some points $\{z_m\}$ and weights $\{w_m\}$, and $\eta(z_m) := \eta(\sigma_T(z_m))$.

BSM: $\beta = 1$

The results are similar:

$$\log(S_T/S_0) = \frac{\rho}{\alpha}(\sigma_T - \sigma_0) - \frac{1}{2}I_T + \sqrt{1-\rho^2}\, W\sqrt{I_T}$$

$$C_{\beta=1} = \sum_m C_{BS}\left(S_0 e^{\frac{\rho}{\alpha}\left(\sigma_T(z_m)-\sigma_0\right)-\frac{\rho^2}{2}\eta(z_m)}, \sqrt{1-\rho^2}\,\eta(z_m)\right) w_m$$

for some points $\{z_m\}$ and weights $\{w_m\}$.

- Implement the method of Kennedy et al and compare it against the Monte Carlo result for both normal ($\beta = 0$) and BSM backbone ($\beta = 1$).