

## 剛體運動狀態描述



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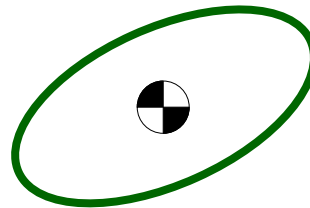
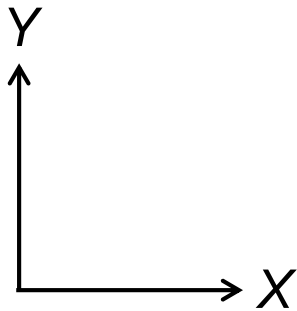
# 導讀 -1

□ 一個剛體(Rigid body)的狀態該如何描述？

◆ 平面：

移動 2 DOFs、轉動 1 DOF Degree of freedom

$\{W\}$  world frame



## 導讀 -2

□ 一個剛體(Rigid body)的狀態該如何描述？

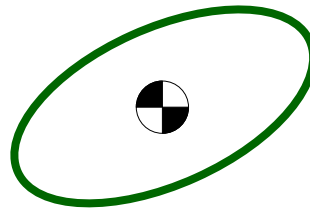
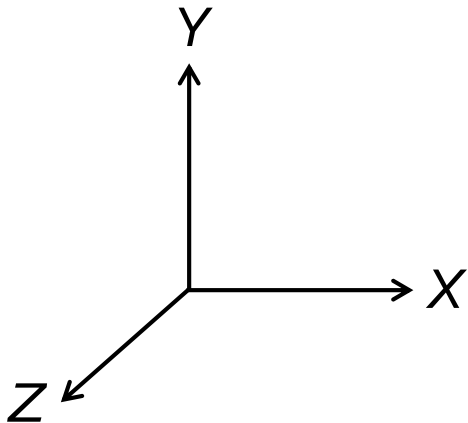
◆ 平面：

移動 2 DOFs、轉動 1 DOF Degree of freedom

◆ 空間：

移動 3 DOFs、轉動 3 DOFs

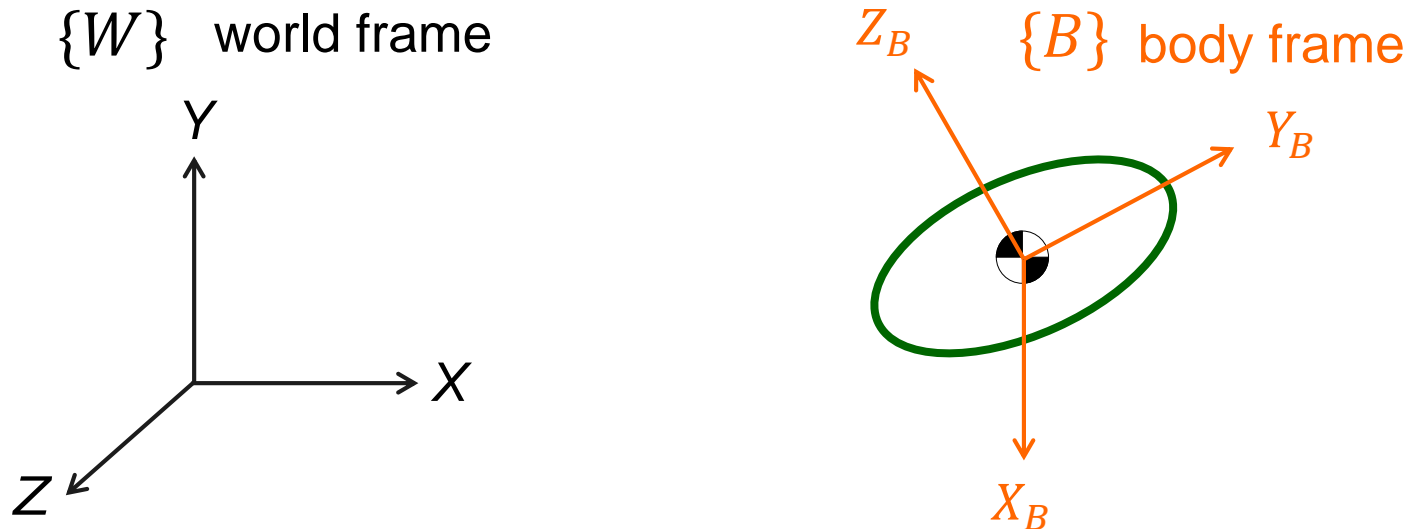
$\{W\}$  world frame



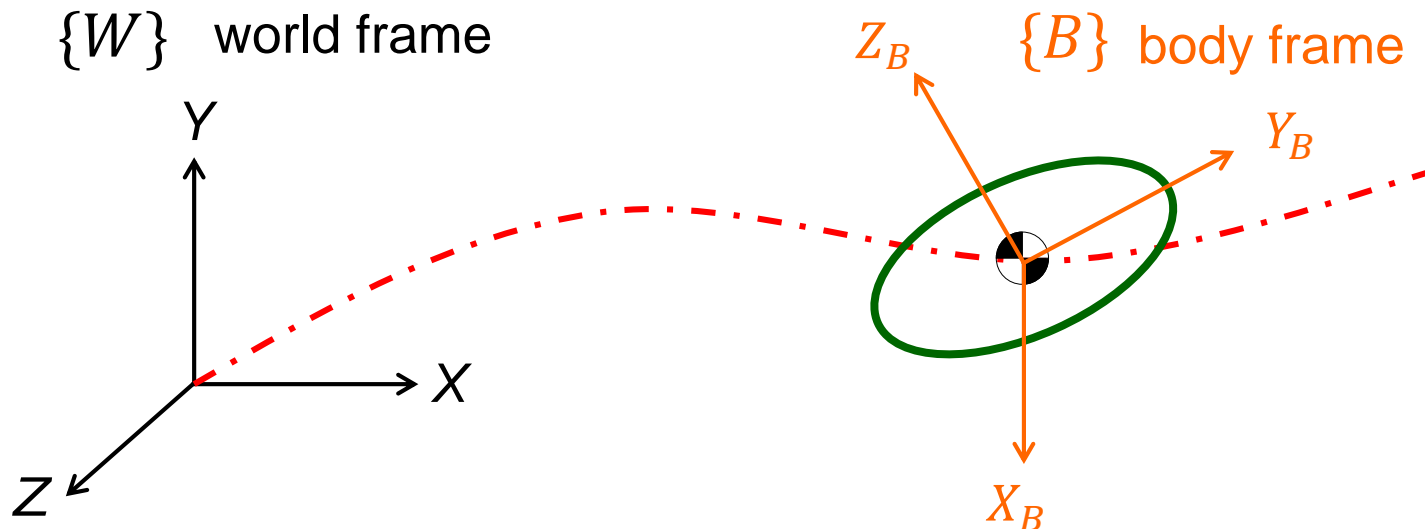
□ 該如何整合表達剛體的狀態？

⇒ 在剛體(Rigid body)上建立frame，常建立在質心上

- ◆ 移動：由body frame 的原點位置判定
- ◆ 轉動：由body frame的姿態判定



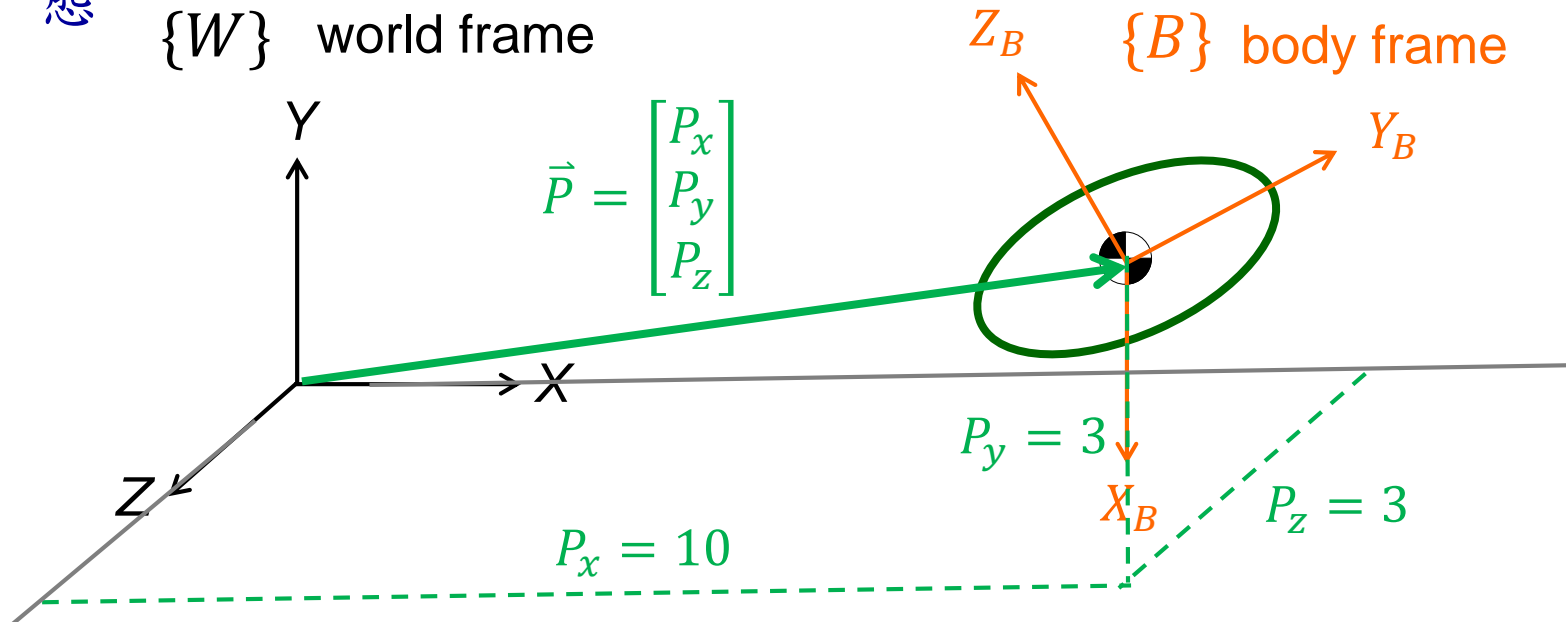
- 一個剛體(Rigid body)的「運動」狀態該如何描述？
  - ◆ 利用各個DOF的微分，將位移和姿態 (displacement / orientation) 轉換到速度 (velocity) 和加速度 (acceleration) 等運動狀態





## 移動 -1

- 移動：以向量 (vector)  $\vec{P}$  來描述{B}的原點相對於{A}的狀態



- Ex:  $\vec{P} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$

## 移動 -2

### □ 向量可表達空間關係的兩個方式

- ◆ A position in space (i.e., position vector)

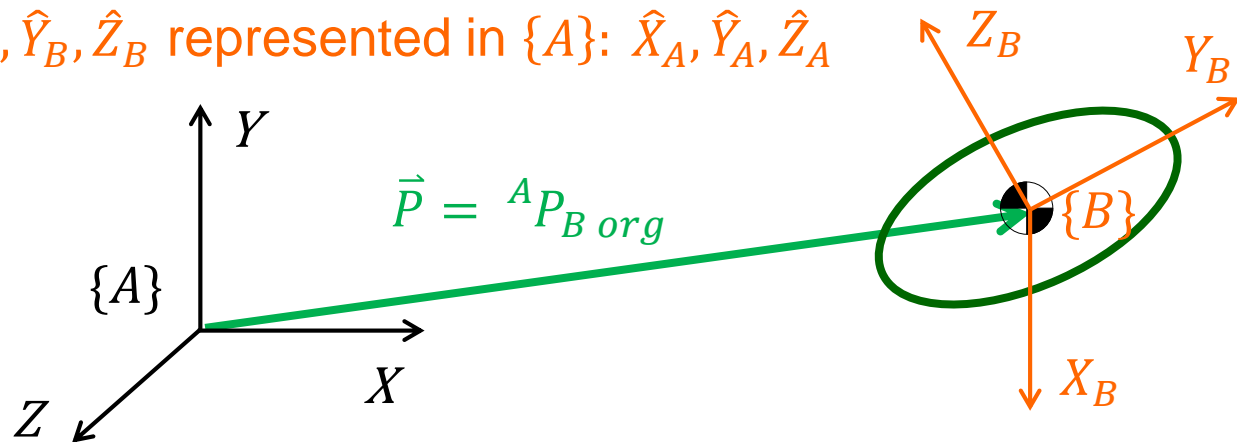
如同前一頁內容，以此方式描述body frame原點

$$\vec{P} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = {}^A P_{B \text{ org}} = \text{origin of } \{B\} \text{ represented in } \{A\}$$

- ◆ A vector (i.e., displacement, frame basis)

以此方式表達body frame上principal axes的方向

$\{B\}$ :  $\hat{X}_B, \hat{Y}_B, \hat{Z}_B$  represented in  $\{A\}$ :  $\hat{X}_A, \hat{Y}_A, \hat{Z}_A$





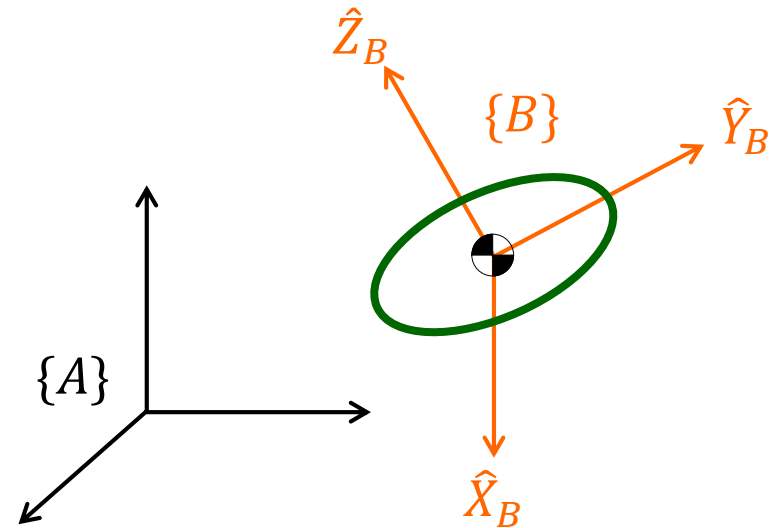


# 轉動 -1

- 轉動：描述{B}相對於{A}之姿態---Rotation Matrix

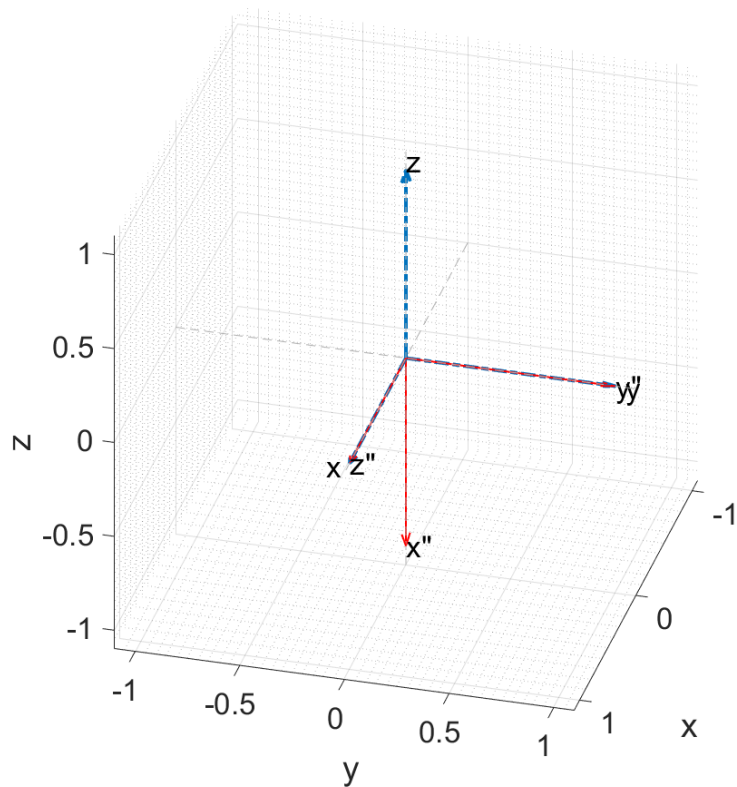
$$\begin{aligned} {}^A_B R &= \begin{bmatrix} | & | & | \\ {}^A \hat{X}_B & {}^A \hat{Y}_B & {}^A \hat{Z}_B \\ | & | & | \end{bmatrix} \\ &\quad \uparrow \text{B relative to A} \qquad \text{"column vector"} \\ &= \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} \\ &\quad \text{"direct cosines"} \end{aligned}$$

R的三個columns即為frame {B}的basis:  $\hat{X}_B, \hat{Y}_B, \hat{Z}_B$  (由{A}看)



## 轉動 -2

□ Ex:  $\{B\}$  相對於  $\{A\}$  之姿態  ${}^A_B R = ?$



藍虛線: World Frame  $\{A\}$

紅實線: Body Frame  $\{B\}$

$$\{B\} \text{ 的 } x'' \text{ 軸為 } \{A\} \text{ 的 } z \text{ 軸反向} \Rightarrow {}^A\hat{X}_B = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

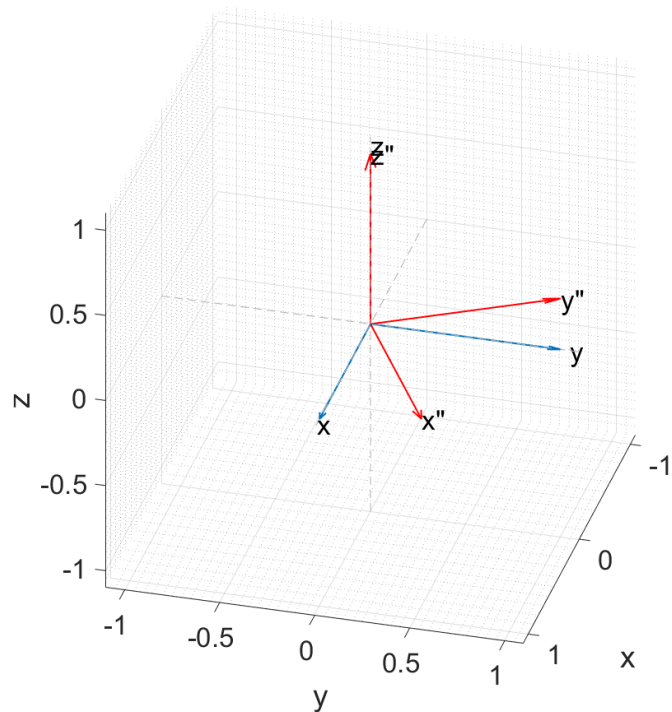
$$\{B\} \text{ 的 } y'' \text{ 軸與 } \{A\} \text{ 的 } y \text{ 軸重疊} \Rightarrow {}^A\hat{Y}_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\{B\} \text{ 的 } z'' \text{ 軸與 } \{A\} \text{ 的 } x \text{ 軸重疊} \Rightarrow {}^A\hat{Z}_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

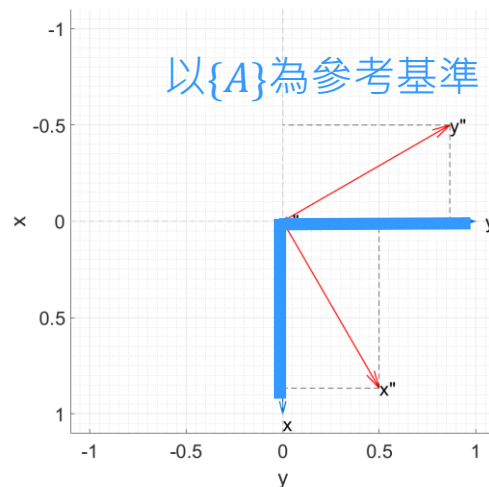
$$\text{因此, } \{B\} \text{ 相對於 } \{A\} \text{ 之姿態: } {}^A_B R = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

# 轉動 -3

□ Ex:  $\{B\}$  相對於  $\{A\}$  之姿態  ${}^A_B R = ?$



上視圖



藍虛線: World Frame  $\{A\}$

紅實線: Body Frame  $\{B\}$

$${}^A\hat{X}_B = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} 0.866 \\ 0.5 \\ 0 \end{bmatrix}$$

$${}^A\hat{Y}_B = \begin{bmatrix} \hat{Y}_B \cdot \hat{X}_A \\ \hat{Y}_B \cdot \hat{Y}_A \\ \hat{Y}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.866 \\ 0 \end{bmatrix}$$

$${}^A\hat{Z}_B = \begin{bmatrix} \hat{Z}_B \cdot \hat{X}_A \\ \hat{Z}_B \cdot \hat{Y}_A \\ \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\{B\}$  相對於  $\{A\}$  之姿態:

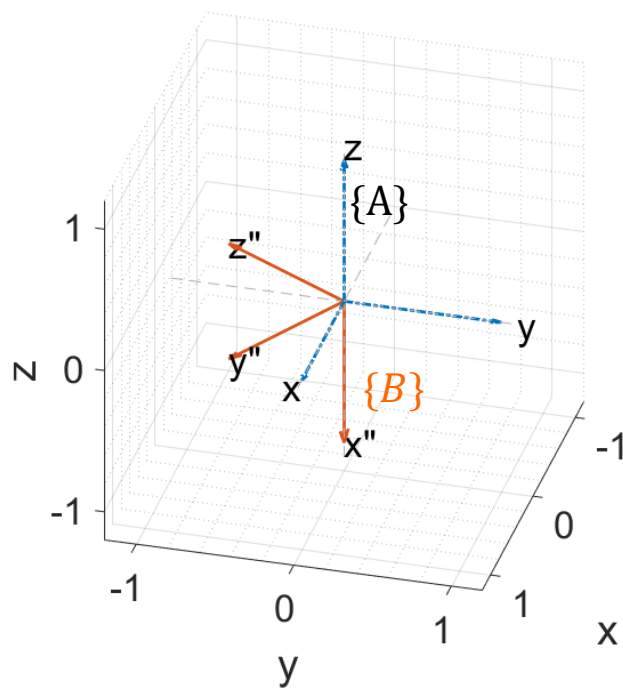
$${}^A_B R = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## 轉動 -4

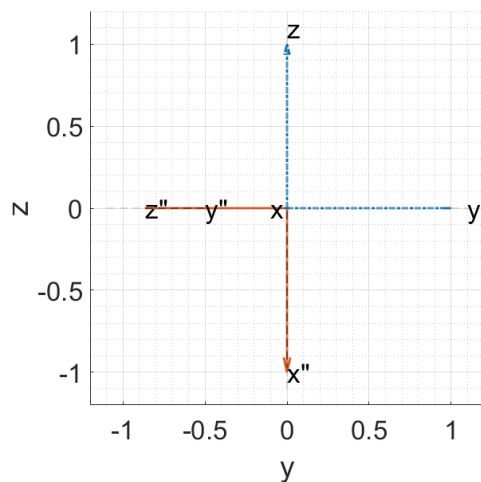
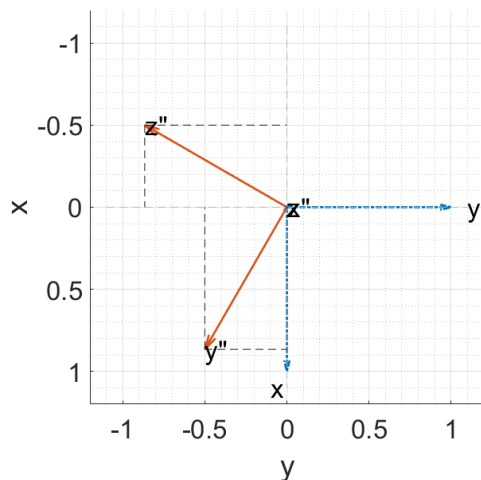
□ In-video Quiz:  $\{B\}$  相對於  $\{A\}$  之姿態  ${}^A_B R = ?$

藍虛線: World Frame  $\{A\}$

紅實線: Body Frame  $\{B\}$



投影到  
XY/YZ平面



A. 
$$\begin{bmatrix} 0 & -0.866 & 0.5 \\ 0 & 0.5 & 0.866 \\ -1 & 0 & 0 \end{bmatrix}$$

B. 
$$\begin{bmatrix} 0 & 0.866 & -0.5 \\ 0 & -0.5 & -0.866 \\ -1 & 0 & 0 \end{bmatrix}$$

C. 
$$\begin{bmatrix} 0 & 0 & -1 \\ 0.866 & -0.5 & 0 \\ -0.5 & -0.866 & 0 \end{bmatrix}$$

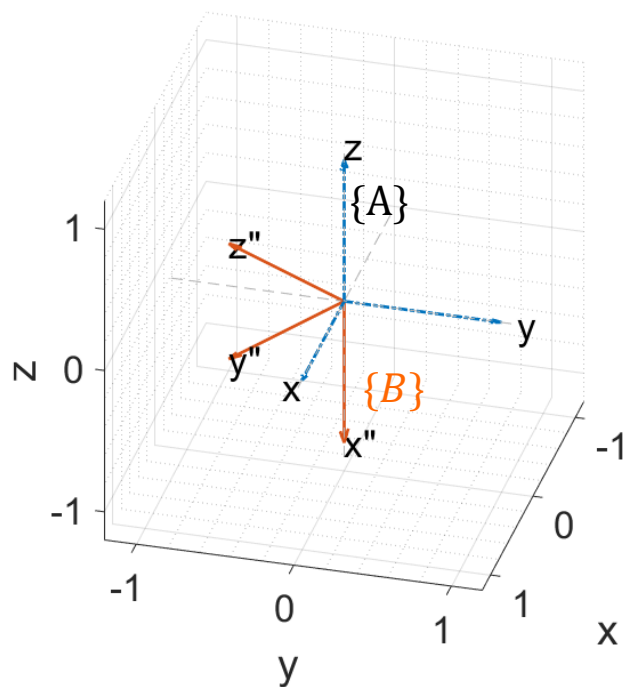
D. 
$$\begin{bmatrix} 0 & 0 & -1 \\ -0.866 & 0.5 & 0 \\ 0.5 & 0.866 & 0 \end{bmatrix}$$

# 轉動 -4

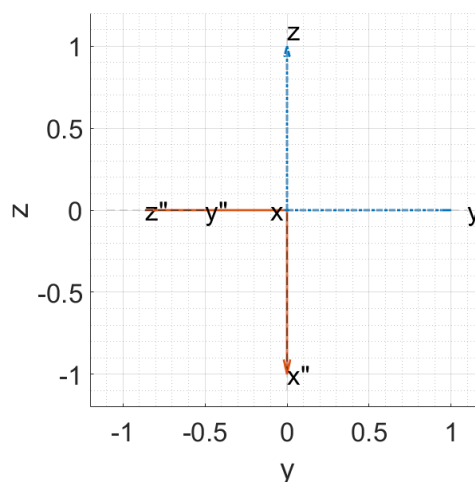
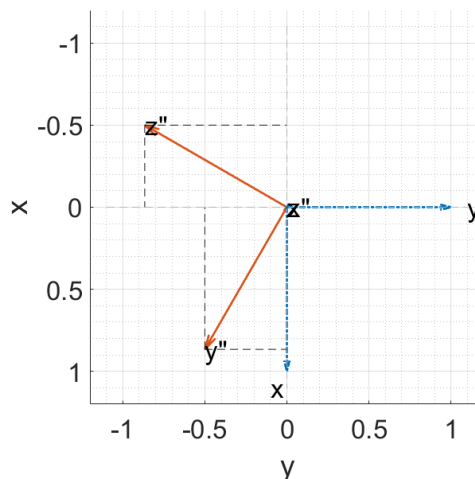
□ In-video Quiz:  $\{B\}$ 相對於 $\{A\}$ 之姿態  ${}^A_B R = ?$

藍虛線: World Frame  $\{A\}$

紅實線: Body Frame  $\{B\}$



投影到  
XY/YZ平面



A. 
$$\begin{bmatrix} 0 & -0.866 & 0.5 \\ 0 & 0.5 & 0.866 \\ -1 & 0 & 0 \end{bmatrix}$$

B. 
$$\begin{bmatrix} 0 & 0.866 & -0.5 \\ 0 & -0.5 & -0.866 \\ -1 & 0 & 0 \end{bmatrix}$$

C. 
$$\begin{bmatrix} 0 & 0 & -1 \\ 0.866 & -0.5 & 0 \\ -0.5 & -0.866 & 0 \end{bmatrix}$$

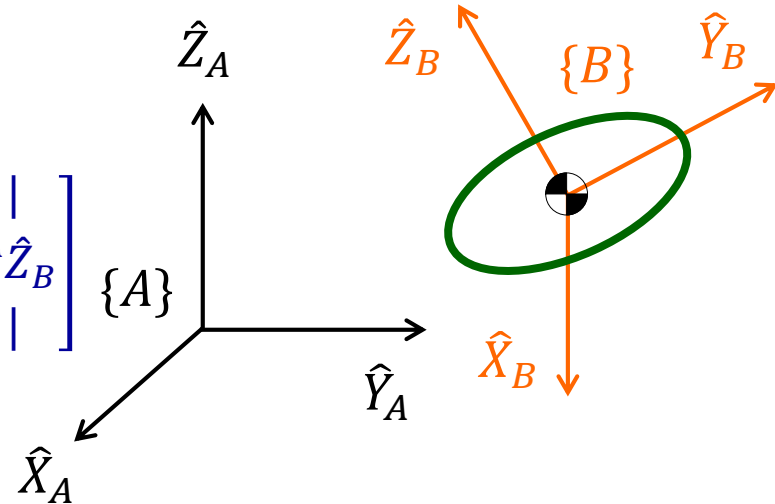
D. 
$$\begin{bmatrix} 0 & 0 & -1 \\ -0.866 & 0.5 & 0 \\ 0.5 & 0.866 & 0 \end{bmatrix}$$



# Rotation Matrix -1

## □ 特性

$${}^A_B R = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix} \{A\}$$



前後向量互換

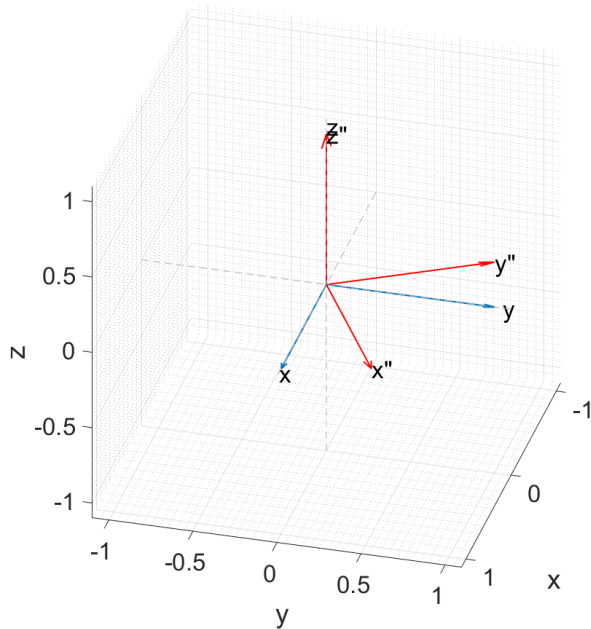
$$\begin{aligned} &= \begin{bmatrix} \hat{X}_A \cdot \hat{X}_B & \hat{X}_A \cdot \hat{Y}_B & \hat{X}_A \cdot \hat{Z}_B \\ \hat{Y}_A \cdot \hat{X}_B & \hat{Y}_A \cdot \hat{Y}_B & \hat{Y}_A \cdot \hat{Z}_B \\ \hat{Z}_A \cdot \hat{X}_B & \hat{Z}_A \cdot \hat{Y}_B & \hat{Z}_A \cdot \hat{Z}_B \end{bmatrix} = \begin{bmatrix} - & {}^B\hat{X}_A^T & - \\ - & {}^B\hat{Y}_A^T & - \\ - & {}^B\hat{Z}_A^T & - \end{bmatrix} \\ &= \begin{bmatrix} | & | & | \\ {}^B\hat{X}_A & {}^B\hat{Y}_A & {}^B\hat{Z}_A \\ | & | & | \end{bmatrix}^T = {}^B_A R^T \end{aligned}$$

$$\Rightarrow {}^A_B R = {}^B_A R^T$$

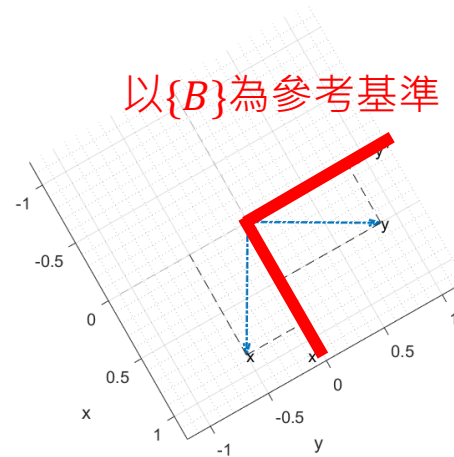


# Rotation Matrix -2

□ Ex:  $\{A\}$  相對於  $\{B\}$  之姿態  ${}^B_A R = ?$



上視圖



藍虛線: World Frame  $\{A\}$

紅實線: Body Frame  $\{B\}$

$${}^B\hat{X}_A = \begin{bmatrix} \hat{X}_A \cdot \hat{X}_B \\ \hat{Y}_A \cdot \hat{X}_B \\ \hat{Z}_A \cdot \hat{X}_B \end{bmatrix} = \begin{bmatrix} 0.866 \\ -0.5 \\ 0 \end{bmatrix}$$

$${}^B\hat{Y}_A = \begin{bmatrix} \hat{X}_A \cdot \hat{Y}_B \\ \hat{Y}_A \cdot \hat{Y}_B \\ \hat{Z}_A \cdot \hat{Y}_B \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.866 \\ 0 \end{bmatrix}$$

$${}^B\hat{Z}_A = \begin{bmatrix} \hat{X}_A \cdot \hat{Z}_B \\ \hat{Y}_A \cdot \hat{Z}_B \\ \hat{Z}_A \cdot \hat{Z}_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\{A\}$  相對於  $\{B\}$  之姿態 :

$${}^B_A R = \begin{bmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} = {}^A_B R^T$$

${}^A_B R$  , 「轉動 -3」 頁面結果

# Rotation Matrix -3

## □ 特性

$${}^A_B R^T {}^A_B R = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix}^T \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix} \{A\}$$

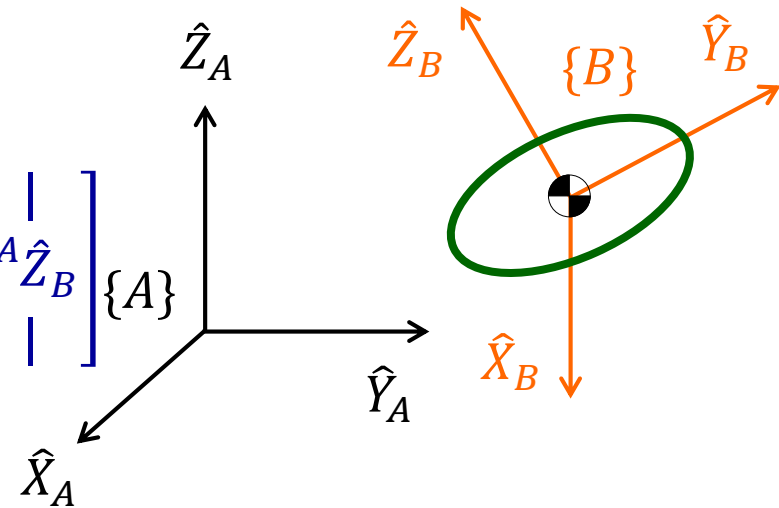
$$= \begin{bmatrix} - & {}^A\hat{X}_B^T & - \\ - & {}^A\hat{Y}_B^T & - \\ - & {}^A\hat{Z}_B^T & - \end{bmatrix} \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix}$$

$$= I_3$$

↖ 3x3 identity matrix

$$= {}^A_B R^{-1} {}^A_B R$$

$$\Rightarrow {}^A_B R^T = {}^A_B R^{-1} = {}^B_A R$$



## Rotation Matrix -4

- A 3x3 orthogonal matrix  $Q$      $QQ^T = Q^TQ = I$ 
  - ◆ Always invertible     $Q^{-1} = Q^T$
  - ◆ Columns: orthonormal basis
    - Length = 1
    - Mutually perpendicular
  - ◆ Rotation matrix (R)有9個數字，但上列兩個條件置入了6個 constraints，所以R只有3個DOFs，與空間中轉動具有3 DOFs相符
  - ◆ Determinant =1 (rotation); =-1 (reflection)

# Rotation Matrix -5

- Rotation matrix 除描述{B}相對於{A}之姿態，也可用於轉換向量之座標

original coordinate  ${}^B P = {}^B P_x \hat{X}_B + {}^B P_y \hat{Y}_B + {}^B P_z \hat{Z}_B$

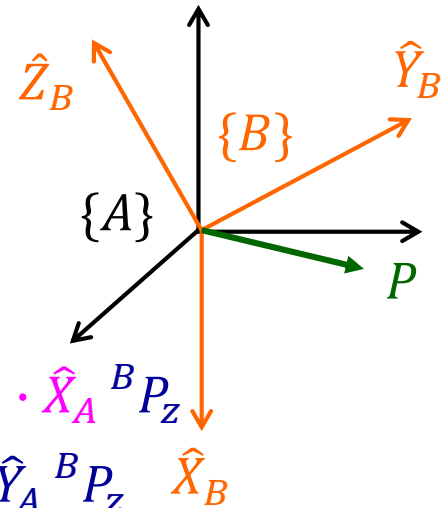
new coordinate  ${}^A P = {}^A P_x \hat{X}_A + {}^A P_y \hat{Y}_A + {}^A P_z \hat{Z}_A$

where  ${}^A P_x = {}^B P \cdot \hat{X}_A = \hat{X}_B \cdot \hat{X}_A {}^B P_x + \hat{Y}_B \cdot \hat{X}_A {}^B P_y + \hat{Z}_B \cdot \hat{X}_A {}^B P_z$

${}^A P_y = {}^B P \cdot \hat{Y}_A = \hat{X}_B \cdot \hat{Y}_A {}^B P_x + \hat{Y}_B \cdot \hat{Y}_A {}^B P_y + \hat{Z}_B \cdot \hat{Y}_A {}^B P_z$

${}^A P_z = {}^B P \cdot \hat{Z}_A = \hat{X}_B \cdot \hat{Z}_A {}^B P_x + \hat{Y}_B \cdot \hat{Z}_A {}^B P_y + \hat{Z}_B \cdot \hat{Z}_A {}^B P_z$

$$\Rightarrow {}^A P = \begin{bmatrix} {}^A P_x \\ {}^A P_y \\ {}^A P_z \end{bmatrix} = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} {}^B \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = {}^A_B R {}^B P$$

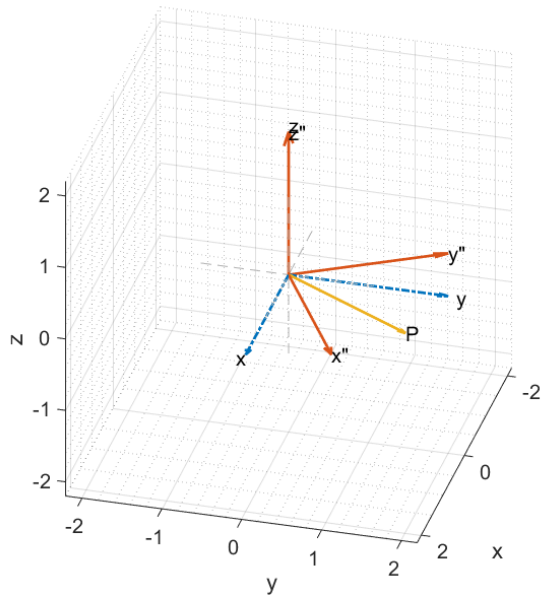


和「轉動-1」頁matrix相同，為rotation matrix

# Rotation Matrix -6

□ Ex: 若{B}和{A}的相對狀態同「轉動 -3」頁面所示，假設

$${}^B P = [1.732 \quad 1 \quad 0]^T, \quad {}^A P = ?$$



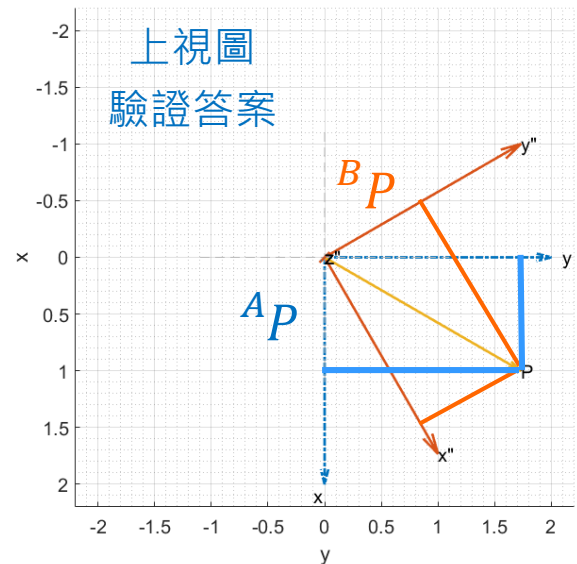
$${}^A P = {}^A_B R {}^B P$$

$${}^A P = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.732 \\ 1 \\ 0 \end{bmatrix}$$

$${}^A P = \begin{bmatrix} 1 \\ 1.732 \\ 0 \end{bmatrix}$$

藍虛線: World Frame {A}

紅實線: Body Frame {B}

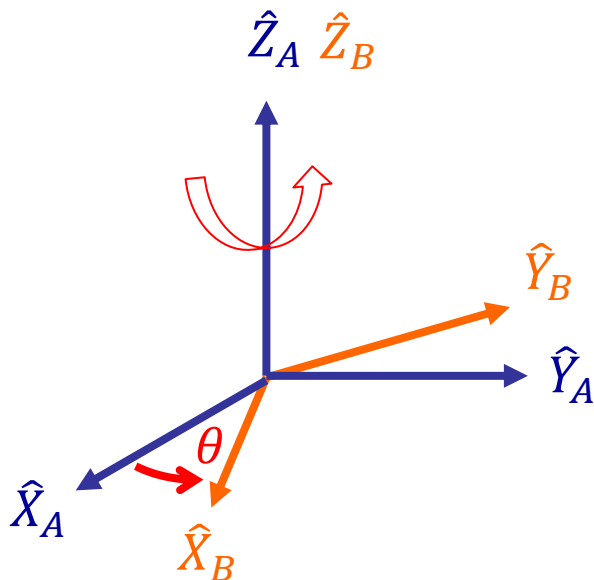


# Rotation Matrix -7

□ Rotation matrix的第三個功能，可進一步來描述物體「轉動」的狀態

□ 以對三個principal axes旋轉的matrix為基礎

□ About  $\hat{Z}_A$  with  $\theta$



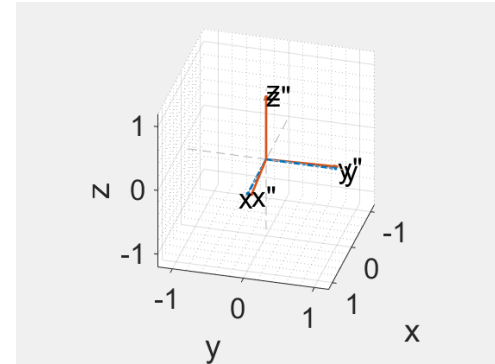
旋轉角度

$$R_{\hat{Z}_A}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

旋轉軸

$$= \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note:  ${}^A_B R = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix}$



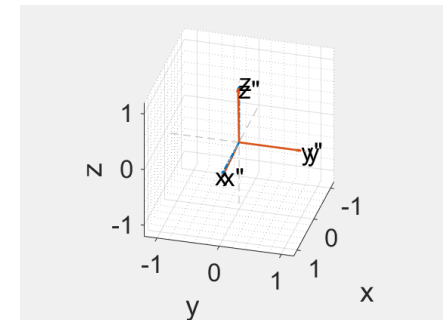
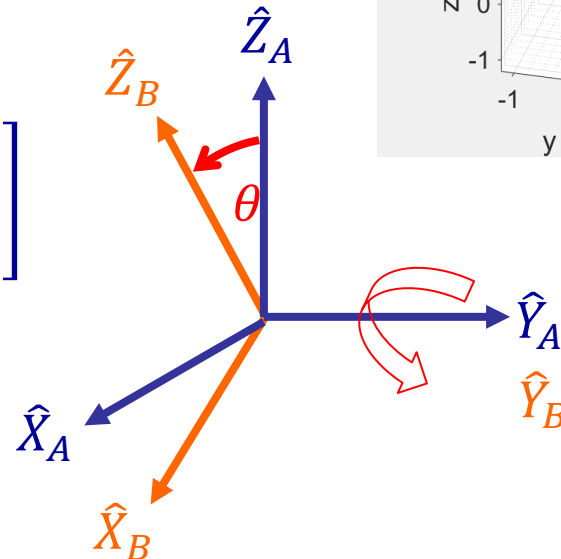
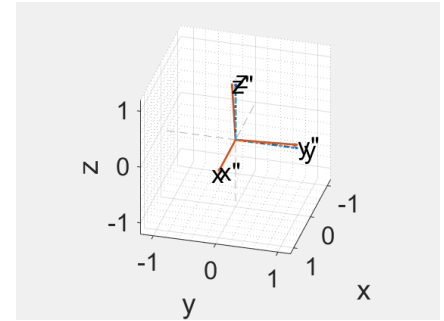
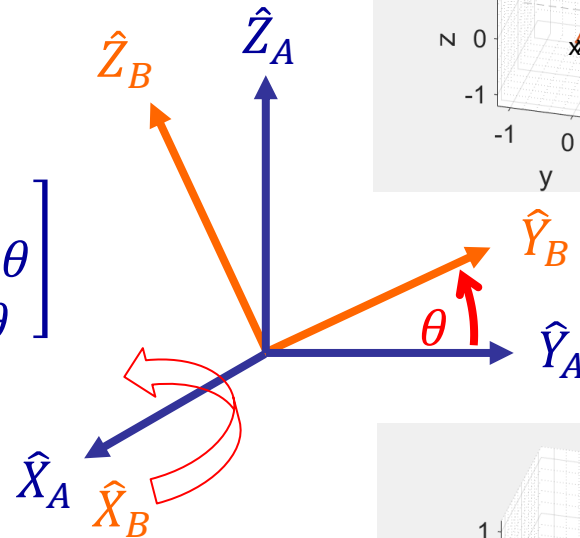
# Rotation Matrix -8

- About  $\hat{X}_A$  with  $\theta$

$$R_{\hat{X}_A}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$$

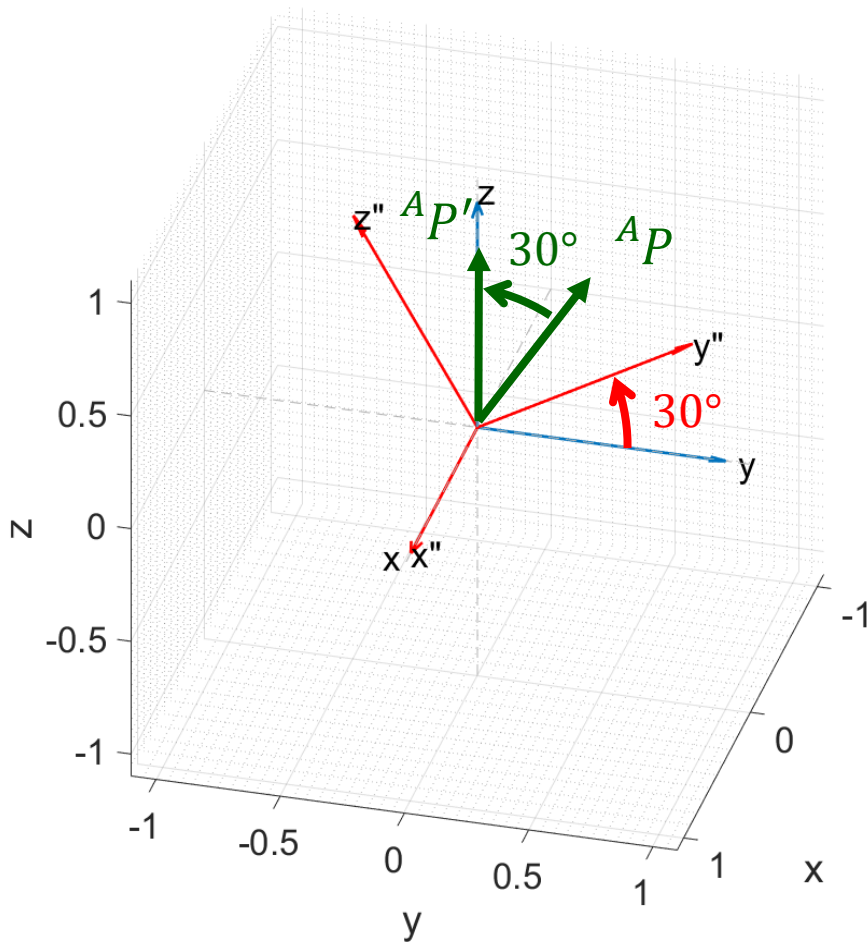
- About  $\hat{Y}_A$  with  $\theta$

$$R_{\hat{Y}_A}(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$



# Rotation Matrix -9

□ Ex:  ${}^A P = [0 \quad 1 \quad 1.732]^T$  對  $\hat{X}_A$  軸旋轉  $30^\circ$  ,  ${}^A P' = ?$



$$R_{\hat{X}_A}(\theta)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30^\circ & -\sin 30^\circ \\ 0 & \sin 30^\circ & \cos 30^\circ \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.866 & -0.5 \\ 0 & 0.5 & 0.866 \end{bmatrix}$$

$${}^A P' = R_{\hat{X}_A}(\theta) {}^A P$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.866 & -0.5 \\ 0 & 0.5 & 0.866 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1.732 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

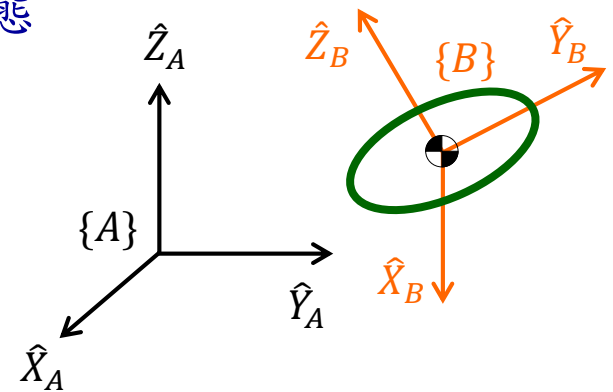


# Rotation Matrix -10

## □ Rotation matrix 的三種用法

- ◆ 描述一個frame(相對於另一個frame)的姿態

$${}^A_B R = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix}$$



- ◆ 將point由某一個frame的表達換到另一個和此frame僅有相對轉動的frame來表達

$${}^A P = {}^A_B R {}^B P$$

- ◆ 將point(vector)在同一個frame中進行轉動

$${}^A P' = R(\theta) {}^A P$$

