Chap 5 Jacobians: Velocities and Static Forces

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Time-varying Position and Orientation -1

- Differentiation of a position vector P_Q

$$ {}^{B}V_Q = \frac{d}{dt}\,{}^{B}P_Q = \lim_{\Delta t \to 0} \frac{{}^{B}P_Q(t+\Delta t) - {}^{B}P_Q(t)}{\Delta t} $$

Derivative of position vector ${}^{B}P_Q$ relative to frame $\{B\}$

$$ {}^{A}({}^{B}V_Q) = {}^{A}(\frac{d}{dt}\,{}^{B}P_Q) $$

Expressed in frame $\{A\}$

$$ = {}^{A}_{B}R\,{}^{B}({}^{B}V_Q) = {}^{A}_{B}R\,{}^{B}V_Q $$

When both frames are the same

$$ v_C = {}^{U}V_{C\,ORG} $$

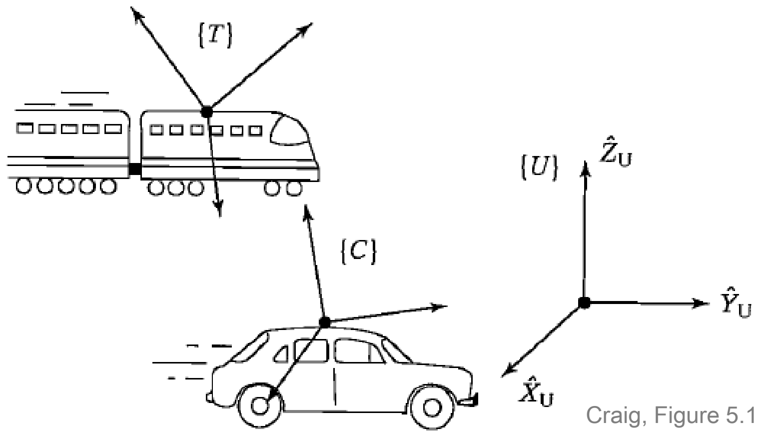
Velocity of the origin of frame $\{C\}$ relative to the universe reference frame $\{U\}$

Time-varying Position and Orientation -2

- Example

$${}^{U}V_{T} = 100\hat{\imath}$$

$${}^{U}V_{C} = 30\hat{\imath}$$



Craig, Figure 5.1

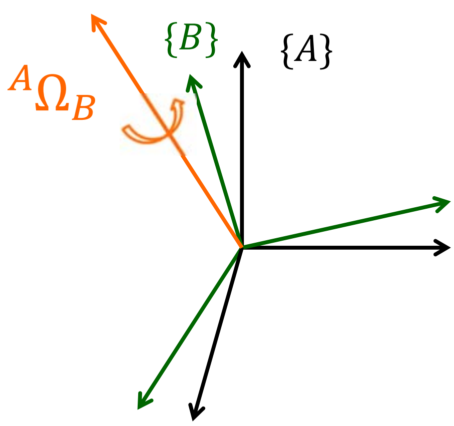
$${}^{U}\left(\frac{d}{dt}\,{}^{U}P_{C\,ORG}\right) = {}^{U}V_{C\,ORG} = v_C = 30\hat{\imath}$$

$${}^{C}({}^{U}V_{T\,ORG}) = {}^{C}v_T = {}^{C}_{U}R(v_T) = {}^{C}_{U}R(100\hat{\imath}) = {}^{U}_{C}R^{-1}100\hat{\imath}$$

$$\begin{aligned}{}^{C}({}^{T}V_{C\,ORG}) &= {}^{C}_{T}R({}^{T}({}^{T}V_{C\,ORG})) = {}^{C}_{T}R({}^{T}V_{C\,ORG}) \\ &= {}^{C}_{U}R\,{}^{U}_{T}R(-70\hat{\imath}) = -{}^{U}_{C}R^{-1}\,{}^{U}_{T}R70\hat{\imath}\end{aligned}$$

Time-varying Position and Orientation -3

- Angular velocity vector ${}^{A}\Omega_{B}$
  - The rotation of frame $\{B\}$ relative to frame $\{A\}$
  - Direction of ${}^{A}\Omega_{B}$: The instantaneous axis of rotation
  - Magnitude of ${}^{A}\Omega_{B}$: The speed of rotation



$${}^{C}({}^{A}\Omega_{B})$$

Expressed in frame $\{C\}$

$$\omega_c = {}^{U}\Omega_{C}$$

Angular velocity of frame $\{C\}$ relative to the universe reference frame $\{U\}$

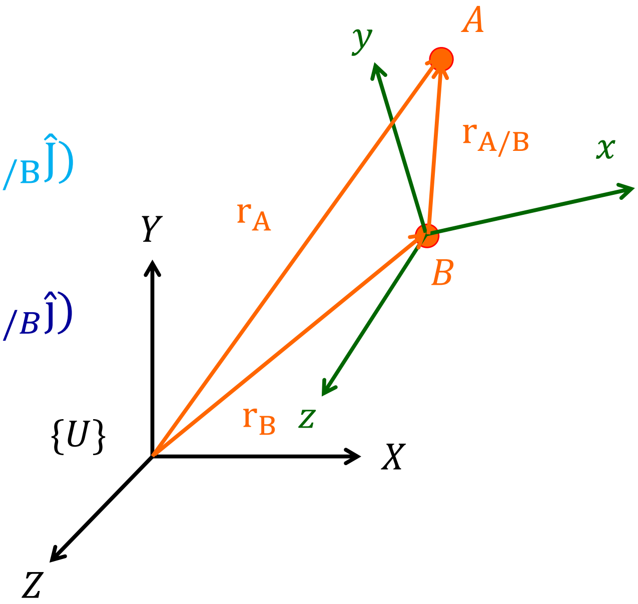
Rigid Body Motion -1

- Freshman Dynamics

$$\overrightarrow{r_A} = x_A\hat{I} + y_A\hat{J}$$
$$= \overrightarrow{r_B} + \overrightarrow{r_{A/B}}$$
$$= (x_B\hat{I} + y_B\hat{J}) + (x_{A/B}\hat{I} + y_{A/B}\hat{J})$$
$$= \overrightarrow{r_B} + \overrightarrow{r_{A/B}}$$
$$= (x_B\hat{I} + y_B\hat{J}) + (x_{A/B}\hat{\imath} + y_{A/B}\hat{\jmath})$$

$\downarrow$ diff.

$$\overrightarrow{v_A} = \dot{\overrightarrow{r_A}} = \dot{x}_A\hat{I} + \dot{y}_A\hat{J}$$
$$= \dot{\overrightarrow{r_B}} + \dot{\overrightarrow{r_{A/B}}}$$
$$= (\dot{x}_B\hat{I} + \dot{y}_B\hat{J}) + (\dot{x}_{A/B}\hat{I} + \dot{y}_{A/B}\hat{J})$$

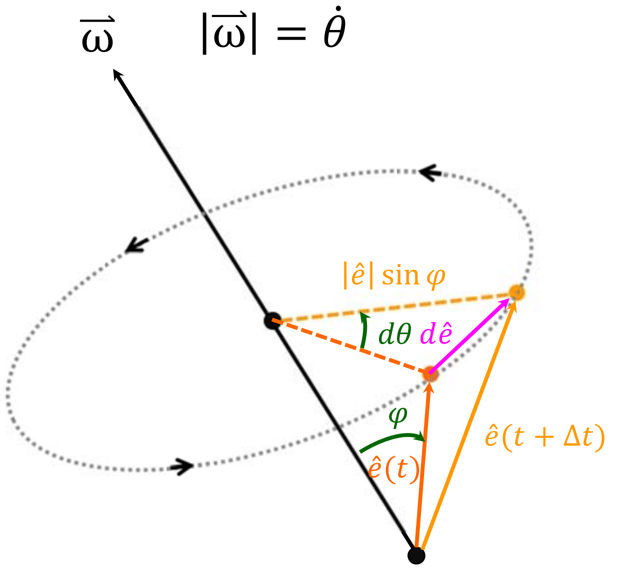


Rigid Body Motion -2

- $$\overrightarrow{v_A} = \dot{\overrightarrow{r_B}} + \dot{\overrightarrow{r_{A/B}}}$$

$$= (\dot{x}_B\hat{I} + \dot{y}_B\hat{J}) + (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) + \underline{(x_{A/B}\,\dot{\hat{\imath}} + y_{A/B}\dot{\hat{\jmath}})}$$
$$= x_{A/B}\,(\vec{\omega}\times\hat{\imath}) + y_{A/B}\,(\vec{\omega}\times\hat{\jmath})$$

$|\vec{\omega}| = \dot{\theta}$



Magnitude:

$$|d\hat{e}| = |\hat{e}|\sin\varphi\, d\theta$$
$$|\dot{\hat{e}}| = |\hat{e}|\sin\varphi\,\dot{\theta} = |\hat{e}||\vec{\omega}|\sin\varphi$$

Direction:

$$d\hat{e} \perp \hat{e}$$
$$d\hat{e} \perp \vec{\omega}$$

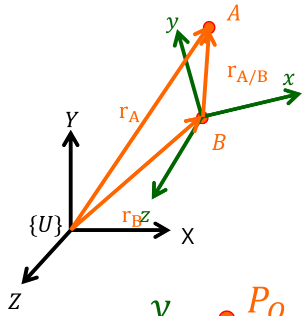
$$\Rightarrow \dot{\hat{e}} = \vec{\omega}\times\hat{e}$$

Rigid Body Motion -3

$$\overrightarrow{v_A} = (\dot{x}_B\hat{I} + \dot{y}_B\hat{J}) + (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) + \vec{\omega}\times(x_{A/B}\hat{\imath} + y_{A/B}\hat{\jmath})$$

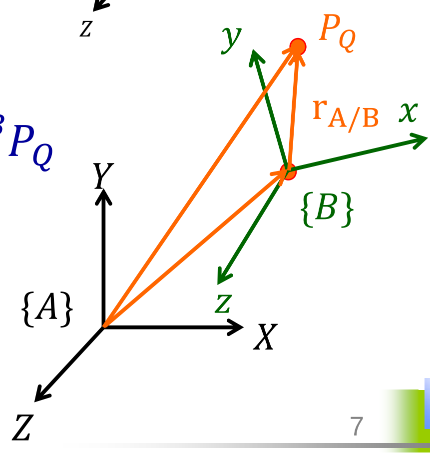
$$= (\dot{x}_B\hat{I} + \dot{y}_B\hat{J}) + (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) + \vec{\omega}\times(x_{A/B}\hat{I} + y_{A/B}\hat{J})$$

$$\Rightarrow \quad \overrightarrow{v_A} = \overrightarrow{v_B} + \underbrace{\overrightarrow{v_{rel}}}_{\text{“relative” velocity}} + \vec{\omega}\times\overrightarrow{r_{A/B}}$$



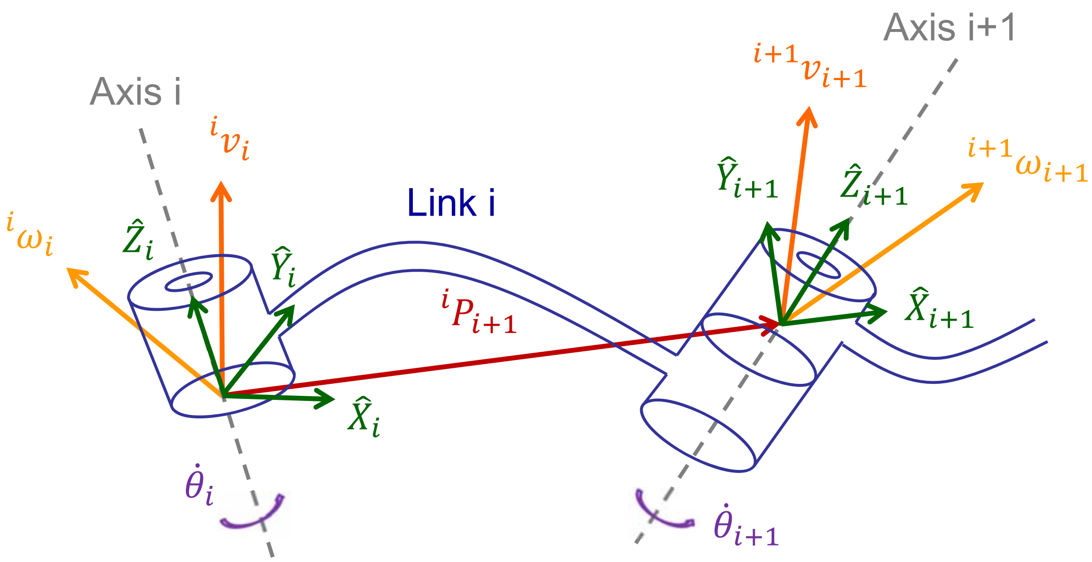
- Thus,

$$^{A}V_Q = {}^{A}V_{B\,ORG} + {}^{A}_{B}R\,\underbrace{^{B}V_Q}_{\text{“relative” velocity}} + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q$$



Velocity “Propagation” from Link to Link -1

- Strategy: Represent linear and angular velocities of link i in frame $\{i\}$, and find their relationship to those of neighboring links



Velocity “Propagation” from Link to Link -2

- Rotational Joint (Link i+1)
  - Angular velocity propagation

$$ {}^{i}\omega_{i+1} = {}^{i}\omega_i + {}^{i}_{i+1}R\underline{\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}} \qquad \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$

$$ \Big\downarrow {}^{i+1}_{i}R $$

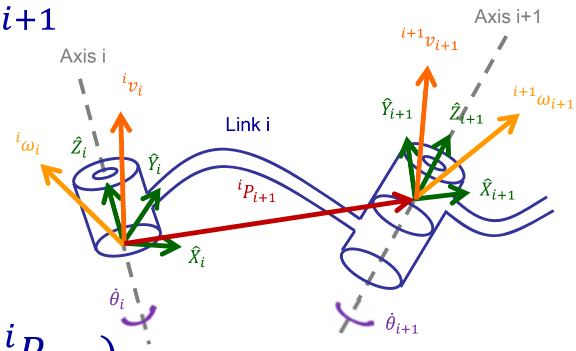
$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

  - Linear velocity propagation

$$ {}^{i}v_{i+1} = {}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1} $$

$$ \Big\downarrow {}^{i+1}_{i}R $$

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) $$



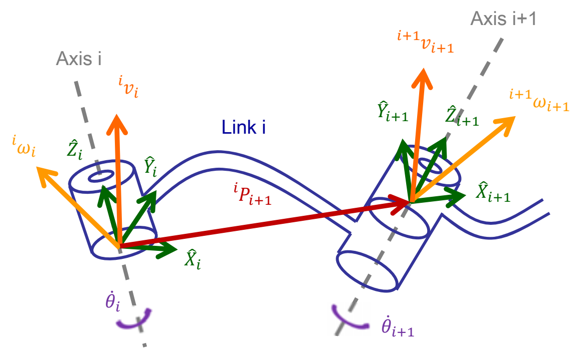
Velocity “Propagation” from Link to Link -3

- Prismatic joint (Link i+1)
  - Angular velocity propagation

$$ {}^{i}\omega_{i+1} = {}^{i}\omega_i $$

$$ \Big\downarrow {}^{i+1}_{i}R $$

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i $$



  - Linear velocity propagation

$$ {}^{i}v_{i+1} = ({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}_{i+1}R\underline{\dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}} $$

$$ \Big\downarrow {}^{i+1}_{i}R \qquad \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix} $$

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) + \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

Jacobians -1

- A multidimensional form of the derivative

$$y_1 = f_1\,(x_1, x_2, x_3, x_4, x_5, x_6)$$

$$y_2 = f_2\,(x_1, x_2, x_3, x_4, x_5, x_6)$$

$$\vdots$$

$$y_6 = f_6\,(x_1, x_2, x_3, x_4, x_5, x_6)$$

$$\Rightarrow \quad Y = F(X)$$

Jacobians -2

- Calculating the differentials of y_i as a function of differentials of x_i

$$\delta y_1 = \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_1}{\partial x_6}\delta x_6$$

$$\delta y_2 = \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_2}{\partial x_6}\delta x_6$$

$$\vdots$$

$$\delta y_6 = \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_6}{\partial x_6}\delta x_6$$

Jacobian, “linear transformation”

$$\Rightarrow \quad \delta Y = \frac{\partial F}{\partial X}\delta X = J\underline{(X)}\delta X$$

Function of X, if f_i is nonlinear

$$\Rightarrow \quad \dot{Y} = J(X)\dot{X}$$

Jacobians -3

- In robotics
  - Relating joint velocities to Cartesian velocities of the tip of the arm

$$ {}^{0}\boldsymbol{\nu} = \begin{bmatrix} {}^{0}v \\ {}^{0}\omega \end{bmatrix} = {}^{0}J(\Theta)\dot{\Theta} $$

3x1 : plane motion

6x1 : spatial motion

- Changing a Jacobian's frame of reference (spatial motion)

$$ {}^{B}\boldsymbol{\nu} = \begin{bmatrix} {}^{B}v \\ {}^{B}\omega \end{bmatrix} = {}^{B}J(\Theta)\dot{\Theta} $$

$$ {}^{A}\boldsymbol{\nu} = \begin{bmatrix} {}^{A}v \\ {}^{A}\omega \end{bmatrix} = {}^{A}J(\Theta)\dot{\Theta} = \begin{bmatrix} {}^{A}_{B}R & 0 \\ 0 & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} {}^{B}v \\ {}^{B}\omega \end{bmatrix} $$

$$ \Rightarrow \quad {}^{A}J(\Theta) = \begin{bmatrix} {}^{A}_{B}R & 0 \\ 0 & {}^{A}_{B}R \end{bmatrix} {}^{B}J(\Theta) $$

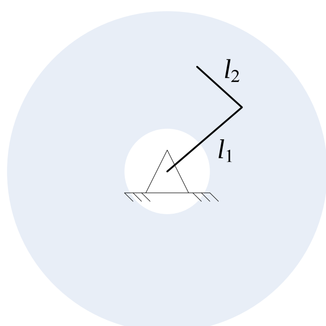
Jacobians -4

- Invertibility

$$ \dot{\Theta} = J^{-1}(\Theta)\boldsymbol{\nu} $$

  - Singular: When the Jacobian J is NOT invertible
    - Workspace-boundary singularities

      Ex: When the manipulator is fully stretch out or folded back on itself
    - Workspace-interior singularities
  - When a manipulator is in a singular configuration
    - Lost one or more DOF



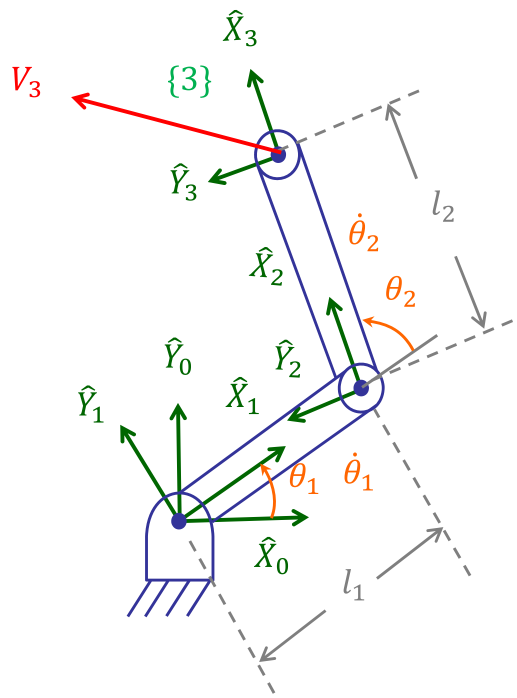
Example: A RR Manipulator -1

- Method 1: Velocity “propagation” from link to link

$$
{}^{0}_{1}T=\begin{bmatrix} c_1 & -s_1 & 0 & 0\\ s_1 & c_1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

$$
{}^{1}_{2}T=\begin{bmatrix} c_2 & -s_2 & 0 & l_1\\ s_2 & c_2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

$$
{}^{2}_{3}T=\begin{bmatrix} 1 & 0 & 0 & l_2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}
$$



Example: A RR Manipulator -2

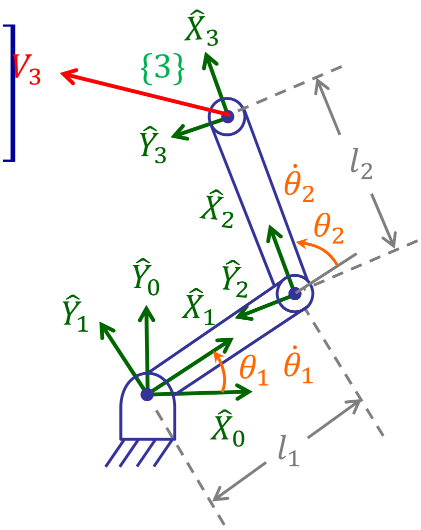
- Link “propagation”

$$
{}^{1}\omega_1={}^{1}_{0}R\ {}^{0}\cancel{\omega_0}+\dot{\theta}_1\ {}^{1}\hat{Z}_1=\dot{\theta}_1\ {}^{1}\hat{Z}_1=\begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}
$$

$$
{}^{1}v_1={}^{1}_{0}R(\ {}^{0}\cancel{v_0}+\ {}^{0}\cancel{\omega_0}\times\ {}^{0}\cancel{P_1})=\begin{bmatrix}0\\0\\0\end{bmatrix}
$$

$$
{}^{2}\omega_2={}^{2}_{1}R\ {}^{1}\omega_1+\dot{\theta}_2\ {}^{2}\hat{Z}_2=\begin{bmatrix}0\\0\\\dot{\theta}_1+\dot{\theta}_2\end{bmatrix}
$$

$$
{}^{2}v_2={}^{2}_{1}R(\ {}^{1}\cancel{v_1}+\ {}^{1}\omega_1\times\ {}^{1}P_2)=\begin{bmatrix} c_2 & s_2 & 0\\ -s_2 & c_2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\ l_1\dot{\theta}_1\\ 0\end{bmatrix}=\begin{bmatrix} l_1 s_2\dot{\theta}_1\\ l_1 c_2\dot{\theta}_1\\ 0\end{bmatrix}
$$



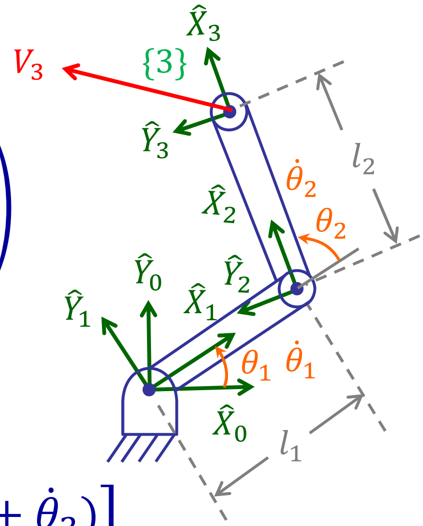
Example: A RR Manipulator -3

$$^{3}\omega_3 = {}^{2}\omega_2$$

$$^{3}v_3 = {}^{3}_{2}R\left({}^{2}v_2 + {}^{2}\omega_2 \times {}^{2}P_3\right)$$

$$= I\left(\begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$



$$^{0}v_3 = \underline{{}^{0}_{3}R}\;{}^{3}v_3 = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$

$$= {}^{0}_{1}R\,{}^{1}_{2}R\,{}^{2}_{3}R$$

$$= \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: A RR Manipulator -4

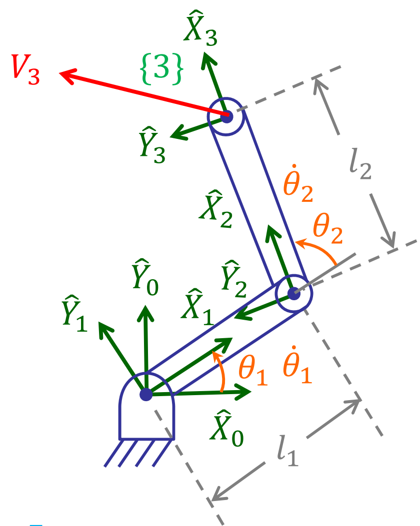
- Therefore

$$^{3}\boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$= {}^{3}J(\Theta)\dot{\Theta}$$

$$det \begin{vmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{vmatrix} = l_1 l_2 s_2 = 0$$

$\Rightarrow$ $\theta_2 = 0\ or\ 180$

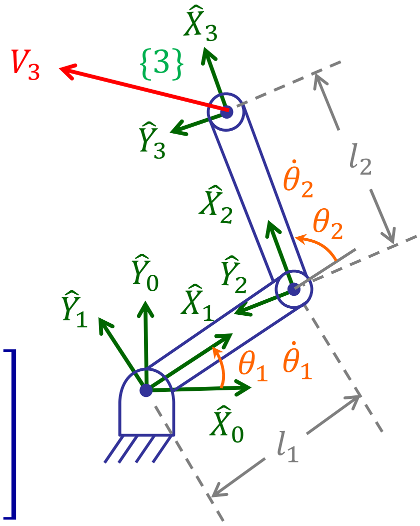


$$^{0}\boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$= {}^{0}J(\Theta)\dot{\Theta}$$

Example: A RR Manipulator -5

- Method 2: Direct differentiation



$$^{0}\begin{bmatrix} p_x \\ p_y \\ \theta \end{bmatrix} = {}^{0}\begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ \theta_1 + \theta_2 \end{bmatrix}$$

$$\Big\downarrow \text{diff.}$$

$$^{0}\begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} = {}^{0}\begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

$$= {}^{0}\begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\dot{X} = {}^{0}J(\Theta)\dot{\Theta}$$

Note: NO 3x1 orientation vector whose derivative is ω

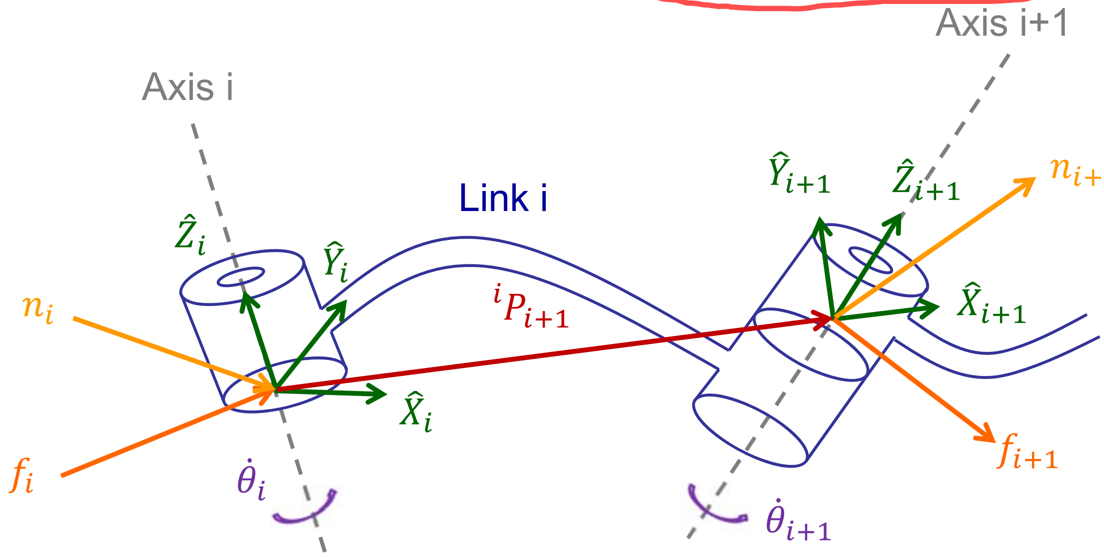
Static Forces in Manipulators -1

- When considering static forces
  - Lock all the joints
  - Write force-moment relationship
  - Compute static torque (ignore gravity)

Static Forces in Manipulators -2

- $^{i}f_i = {}^{i}f_{i+1}$ $\quad\Big\downarrow {}^{i}_{i+1}R$ $\quad {}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}$ 力

$^{i}n_i = {}^{i}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i+1}$ $\quad\Big\downarrow {}^{i}_{i+1}R$ $\quad {}^{i}n_i = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i$ 力矩



Static Forces in Manipulators -3

- The joint toque required to maintain the static equilibrium

驱动力

  - Revolute joint

$$\tau_i = {}^{i}n_i^T\,{}^{i}\hat{Z}_i$$

  - Prismatic joint

$$\tau_i = {}^{i}f_i^T\,{}^{i}\hat{Z}_i$$

Example: A RR Manipulator -1

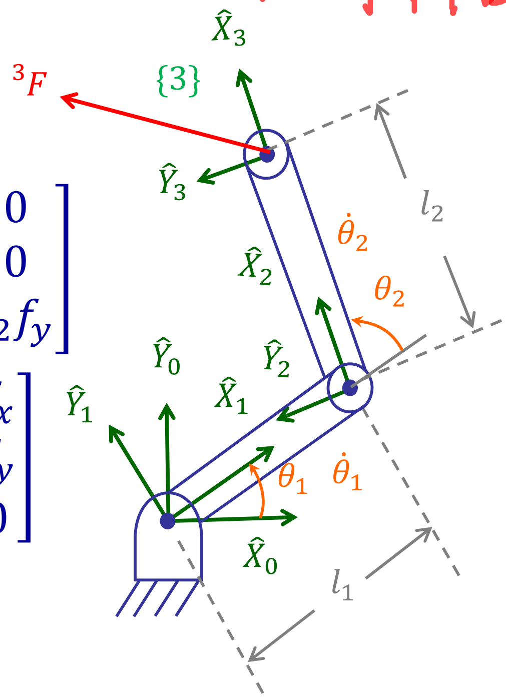
计算力从高杆件到低杆

- Force "propagation" from link to link

$$ {}^2f_2 = {}^2_3R\ {}^3f_3 = I\ {}^3F = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} $$

$$ {}^2n_2 = {}^2_3R\ {}^3n_3 + {}^2P_3 \times {}^2f_2 = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix} $$

$$ {}^1f_1 = {}^1_2R\ {}^2f_2 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} c_2 f_x - s_2 f_y \\ s_2 f_x + c_2 f_y \\ 0 \end{bmatrix} $$



Example: A RR Manipulator -2

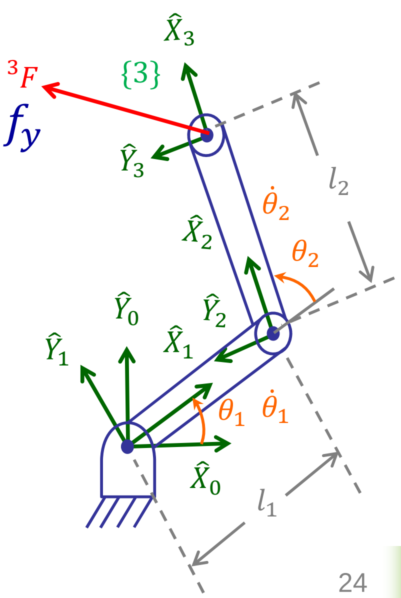
$$ {}^1n_1 = {}^1_2R\ {}^2n_2 + {}^1P_2 \times {}^1f_1 = \begin{bmatrix} 0 \\ 0 \\ l_1 s_2 f_x + l_1 c_2 f_y + l_2 f_y \end{bmatrix} $$

- Therefore,

$$ \tau_1 = {}^1n_1^T\ {}^1\widehat{Z_1} = l_1 s_2 f_x + (l_1 c_2 + l_2) f_y $$

$$ \tau_2 = {}^2n_2^T\ {}^2\widehat{Z_2} = l_2 f_y $$

$$ \Rightarrow \tau = \begin{bmatrix} l_1 s_2 & l_1 c_2 + l_2 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix} $$



Jacobian in the Force Domain

- The principal of virtual work

$$F \cdot \delta \mathcal{X} = \Gamma \cdot \delta \Theta$$

功

忽略重力

$$F^T \delta \mathcal{X} = F^T J \delta \Theta = \Gamma^T \delta \Theta$$

$$\Gamma = J^T F$$

Respect to frame $\{0\}$

$$\Rightarrow \quad \Gamma = {}^0J^T \, {}^0F$$

"inverse" Cartesian torque to joint torque without using IK technique

Cartesian Transformation -1

- General velocity and force representations

$$\boldsymbol{\nu} = \begin{bmatrix} v \\ \omega \end{bmatrix} \qquad \boldsymbol{\mathcal{F}} = \begin{bmatrix} F \\ N \end{bmatrix}$$

- Frame transformation

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R \; {}^{i}\omega_i + \dot{\theta}_{i+1} \, {}^{i+1}\hat{Z}_{i+1}$$

$${}^{i+1}v_{i+1} = {}^{i+1}_{i}R(\; {}^{i}v_i + \; {}^{i}\omega_i \times \; {}^{i}P_{i+1})$$

$$\Big\downarrow \; i = A, i+1 = B, \dot{\theta} = 0$$

$$\begin{bmatrix} {}^{A}v_A \\ {}^{A}\omega_A \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{B\,ORG} \times {}^{A}_{B}R \\ 0 & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} {}^{B}v_B \\ {}^{B}\omega_B \end{bmatrix}$$

$${}^{A}\boldsymbol{\nu}_A = {}^{A}_{B}T_v \, {}^{B}\boldsymbol{\nu}_B \qquad P \times = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}$$

Cartesian Transformation -2

$$\downarrow \text{“}inverse\text{"}$$

$$\begin{bmatrix} {}^{B}v_B \\ {}^{B}\omega_B \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R & -{}^{B}_{A}R\,{}^{A}P_{BORG}\times \\ 0 & {}^{B}_{A}R \end{bmatrix} \begin{bmatrix} {}^{A}v_A \\ {}^{A}\omega_A \end{bmatrix}$$

$$ {}^{B}\boldsymbol{v}_B = {}^{B}_{A}T_v\,{}^{A}\boldsymbol{v}_A $$

- Similarly,

$$\begin{bmatrix} {}^{A}F_A \\ {}^{A}N_A \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & 0 \\ {}^{A}P_{BORG}\times{}^{A}_{B}R & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} {}^{B}F_B \\ {}^{B}N_B \end{bmatrix}$$

$$ {}^{A}\boldsymbol{\mathcal{F}}_A = {}^{A}_{B}T_f\,{}^{B}\boldsymbol{\mathcal{F}}_B $$

$$\Rightarrow\quad {}^{A}_{B}T_f = {}^{A}_{B}T_v{}^{T}$$

The End

- Questions?