機械手臂 順向運動學

Manipulator Forward Kinematics

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引言 -1

- 運動學(Kinematics)：討論運動狀態本身，未連結到產生運動的「力」
  - 位置(x)、速度(v)、加速度(a)、和時間(t)之間的關係
  - 移動/轉動
    - 位置/姿態
    - 速度/角速度
    - 加速度/角加速度

$$v = \frac{d}{dt}x \qquad a = \frac{d}{dt}v$$

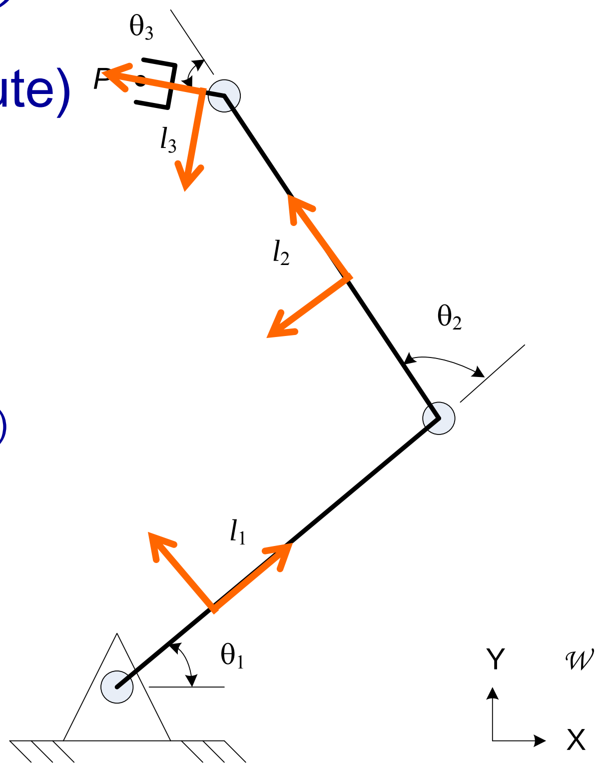
$$a = \frac{d^2}{dt^2}x \qquad vdv = ads$$

- 動力學(Dynamics)：討論力/力矩如何產生運動
  - Newton's 2nd Law $\quad \sum F = ma$
  - Work & energy $\quad T_1 + V_1 + U_{1-2}{}' = T_2 + V_2$
  - Impulse & momentum $\quad \int \sum F\,dt = G_2 - G_1$

引言 -2

- 機械手臂
  - 多個桿件(link)相串連，具有複雜的幾何外形
  - 桿件間可相對 移動(prismatic)或轉動(revolute)由致動器驅動來達成
- 對應關係
  - 需求：手臂末端點狀態（位置^{W}P、速度…）
  - 達成方式：驅動各致動器

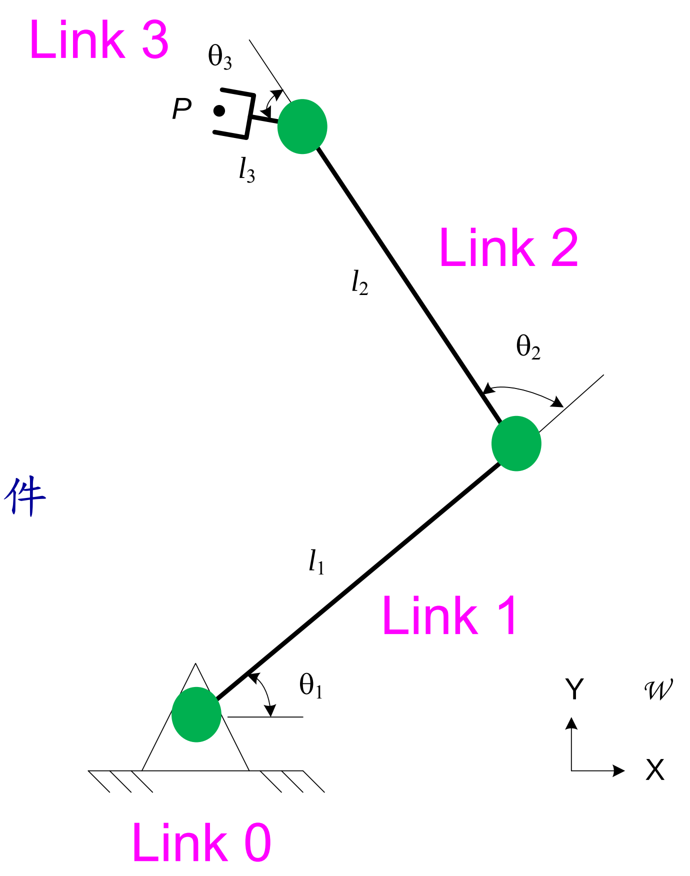
$$^{W}P = f(\theta_1, \theta_2, \ldots, \theta_n)$$



- 描述手臂狀態方法
  - 找出桿件間的相對幾何狀態
  - 在各桿件上建立frame，以frame狀態來代表桿件狀態

手臂幾何描述方式 -1

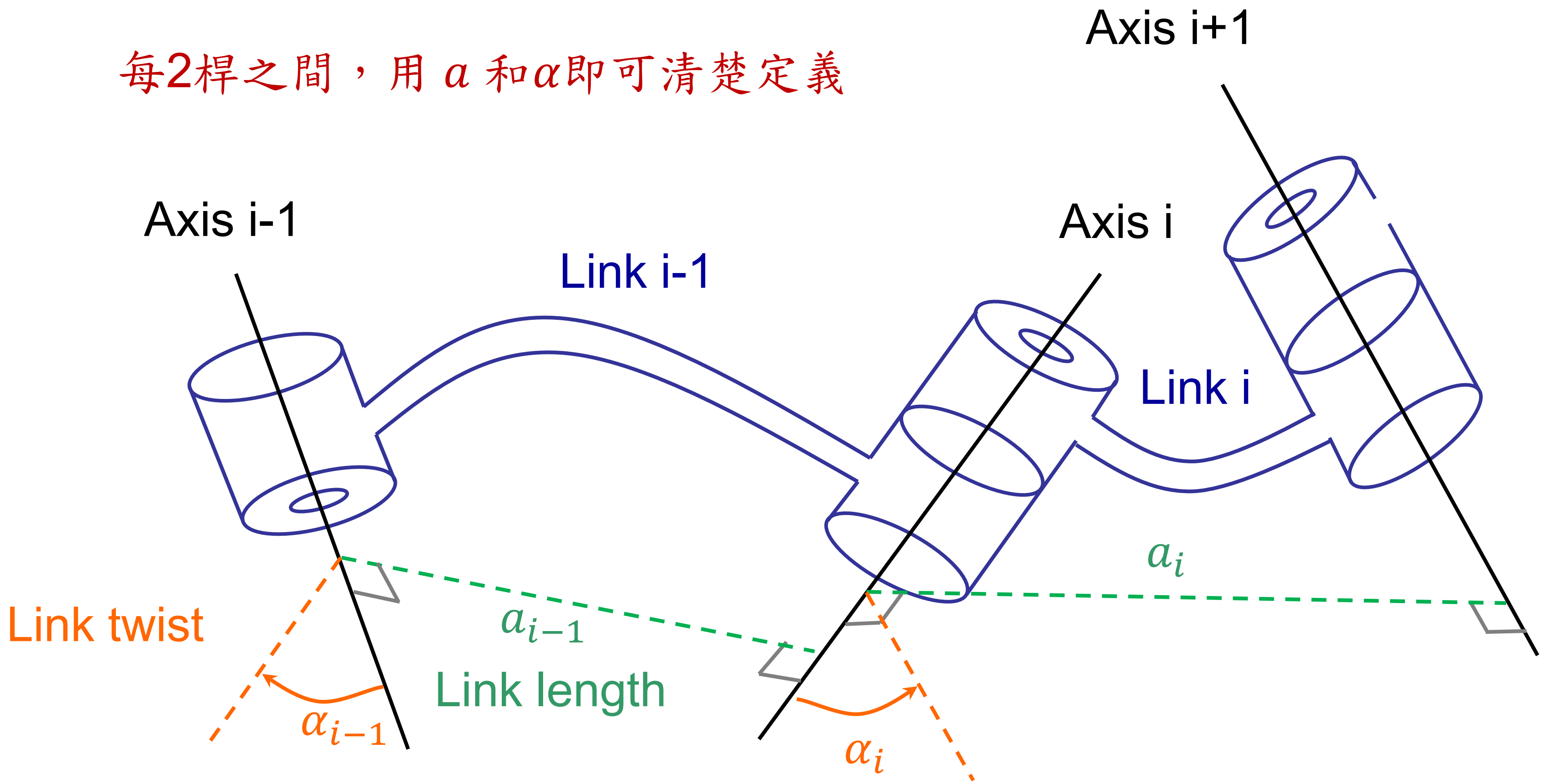
- Joint
  - 每個revolute或prismatic的joint具有 1 DOF
  - 每個joint對 某特定axis 進行rotation或translation
- Link
  - 連接joints的桿件，為剛體(rigid body)
  - 編號方式
    - Link 0: 地桿，不動的桿件
    - Link 1: 和Link 0相連，第一個可動的桿件
    - Link 2: 第二個可動的桿件
    - 依序下去…



手臂幾何描述方式 -2

- 對空間中2個任意方向的axes，兩axes之間具有一線段和此2個axes都相互垂直

每2桿之間，用 a 和α即可清楚定義



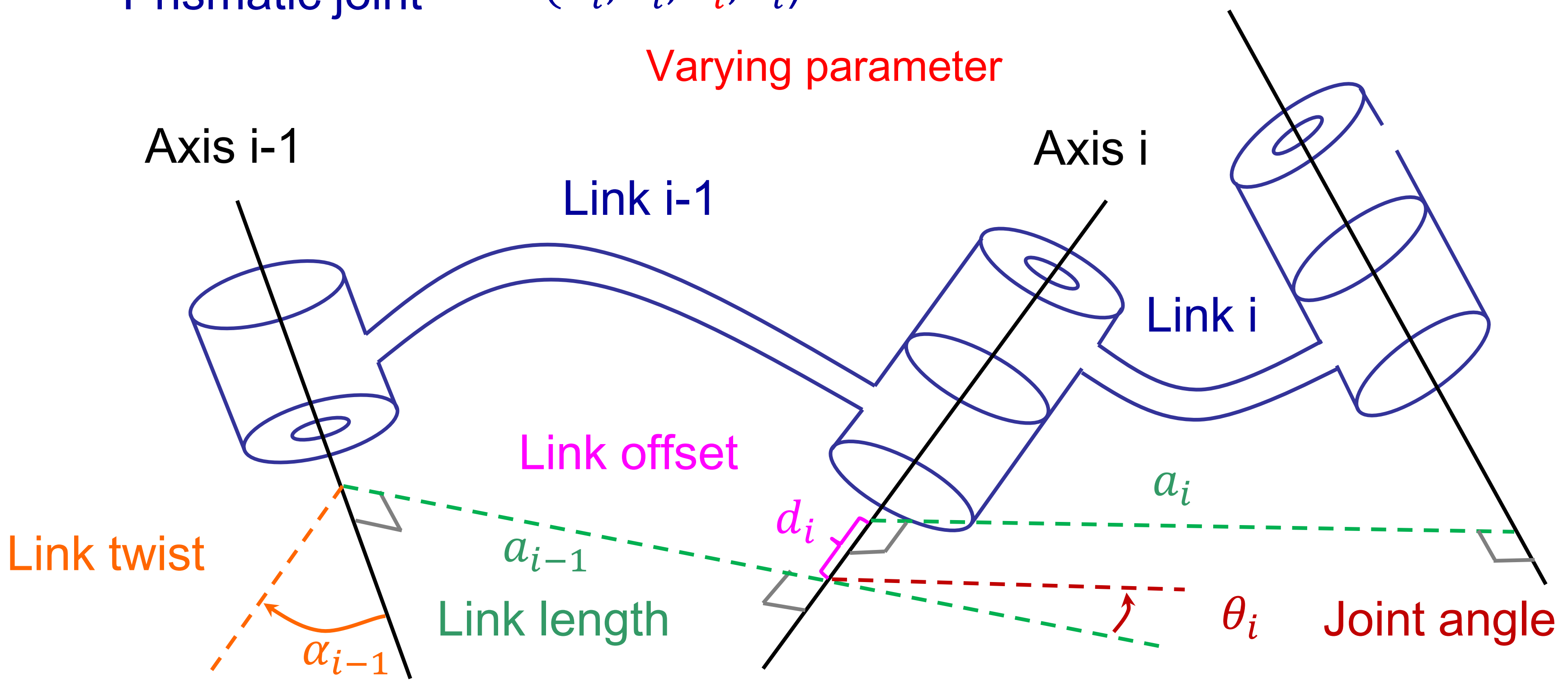
手臂幾何描述方式 -3

- 但若要多桿串連，則另需要兩個參數(d_i, θ_i)，來描述相鄰線段a_{i-1}和a_i間的相對幾何關係

Revolute joint $(\alpha_i, a_i, d_i, \theta_i)$

Prismatic joint $(\alpha_i, a_i, d_i, \theta_i)$

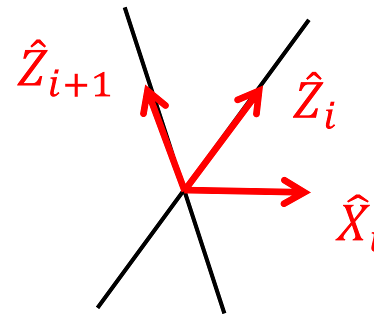
Varying parameter

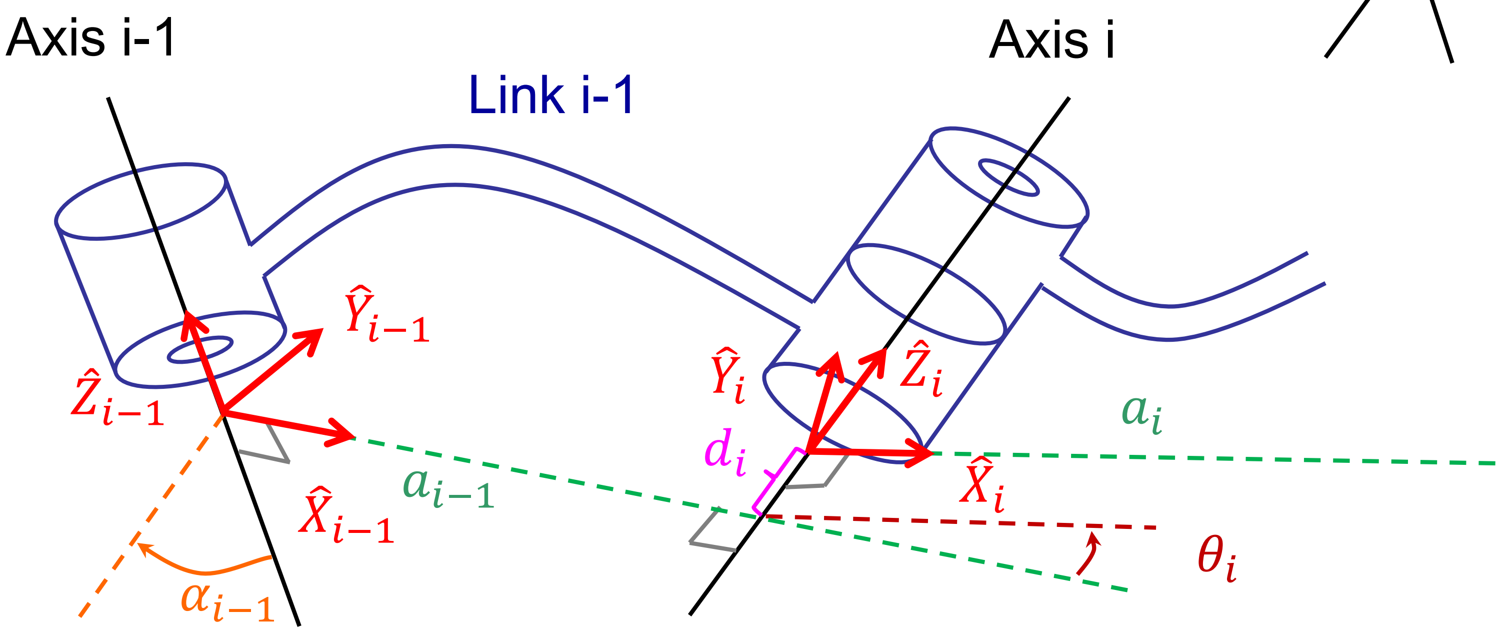


桿件上建立Frames -1

- $\hat{Z}_i$　轉動或移動axis的方向
- $\hat{X}_i$　沿著a_i方向(if $a_i \neq 0$)

  和$\hat{Z}_i$和$\hat{Z}_{i+1}$兩者垂直 (if $a_i = 0$)
- $\hat{Y}_i$　與$\hat{X}_i$和$\hat{Z}_i$兩者垂直，依循右手定則





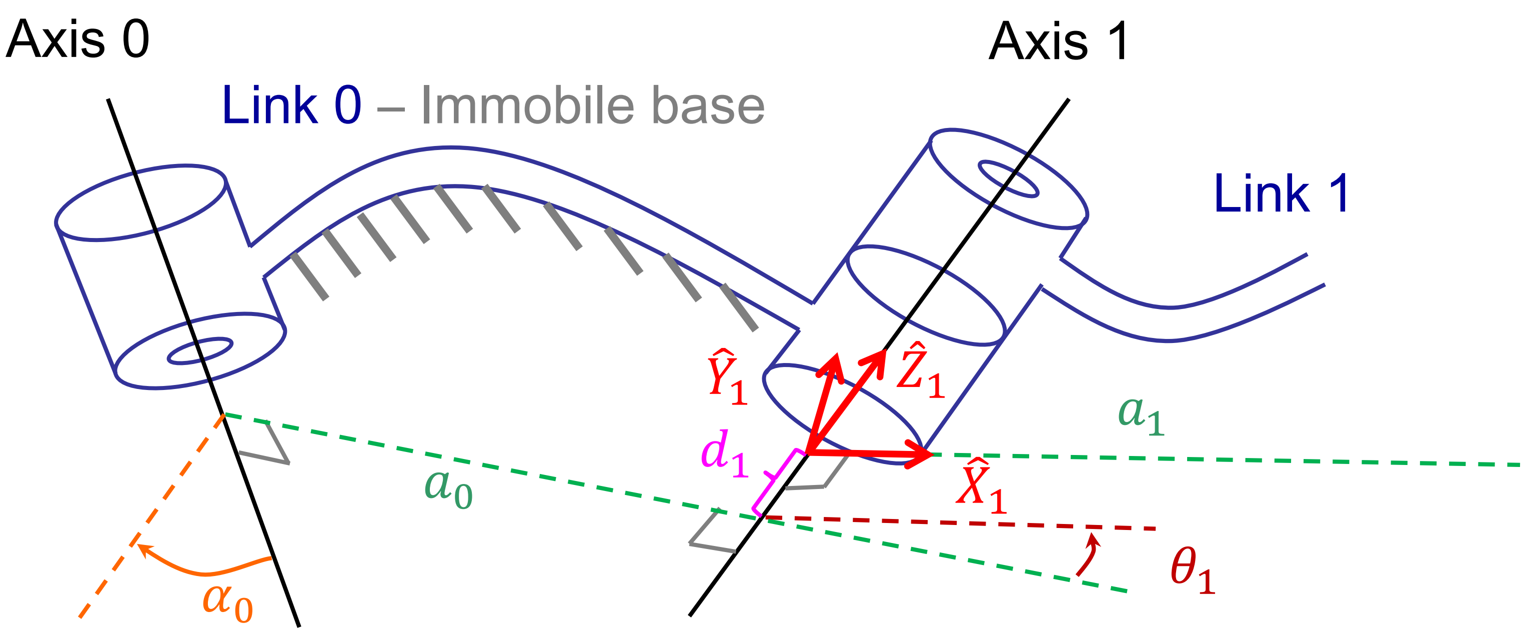
桿件上建立Frames -2

- 地桿 link (0)

Frame $\{0\}$ coincides with frame $\{1\}$ $\quad a_0 = 0 \quad \alpha_0 = 0$

Revolute joint $\quad \theta_1$ arbitrary $\quad d_1 = 0$

Prismatic joint $\quad d_1$ arbitrary $\quad \theta_1 = 0$



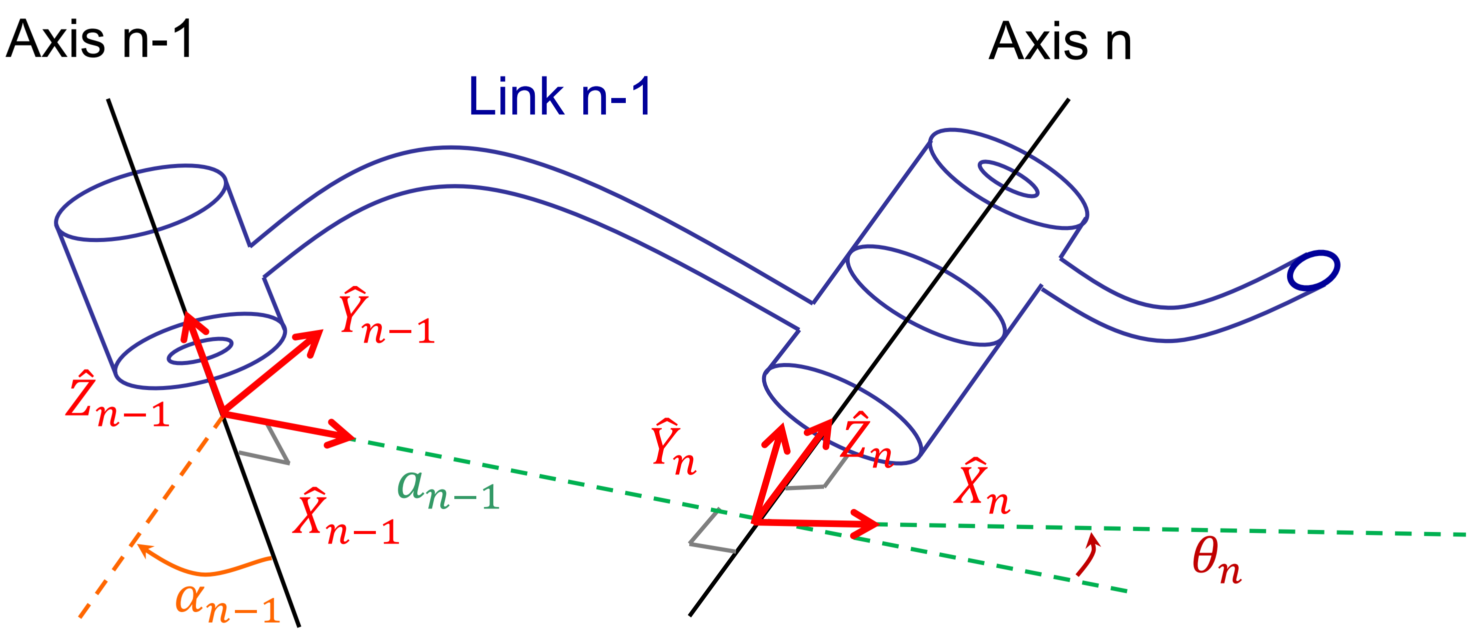
桿件上建立Frames -3

- Last link (n)

  取和 $\hat{X}_{n-1}$ 同方向　　$a_n = 0$　　$\alpha_n = 0$

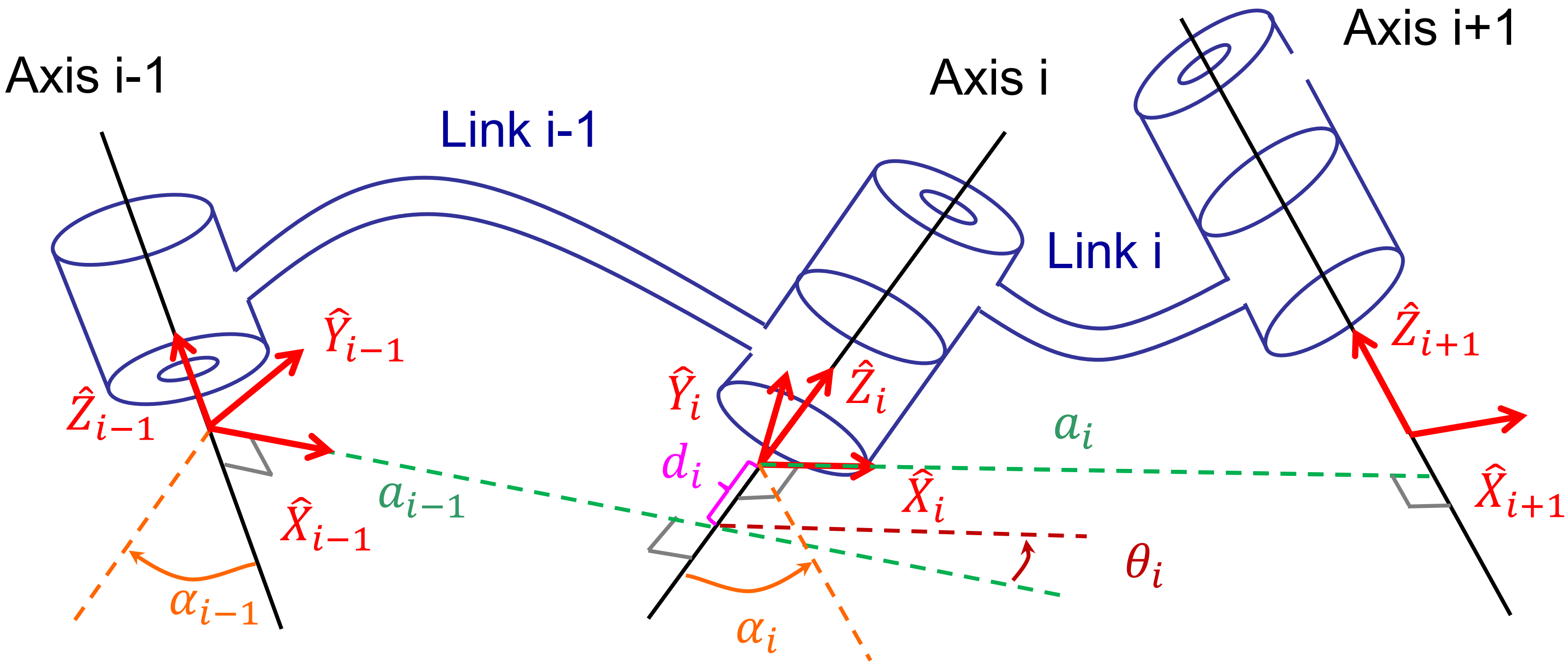
  Revolute joint　　θ_n variable　　$d_n = 0$

  Prismatic joint　　d_n variable　　$\theta_n = 0$



Denavit-Hartenberg表達法 (Craig version)

- α_{i-1}: 以$\hat{X}_{i-1}$方向看， $\hat{Z}_{i-1}$和$\hat{Z}_i$間的夾角
- a_{i-1}: 沿著$\hat{X}_{i-1}$方向，$\hat{Z}_{i-1}$和$\hat{Z}_i$間的距離 $(a_i > 0)$
- θ_i: 以$\hat{Z}_i$方向看， $\hat{X}_{i-1}$和$\hat{X}_i$間的夾角
- d_i: 沿著$\hat{Z}_i$方向，$\hat{X}_{i-1}$和$\hat{X}_i$間的距離

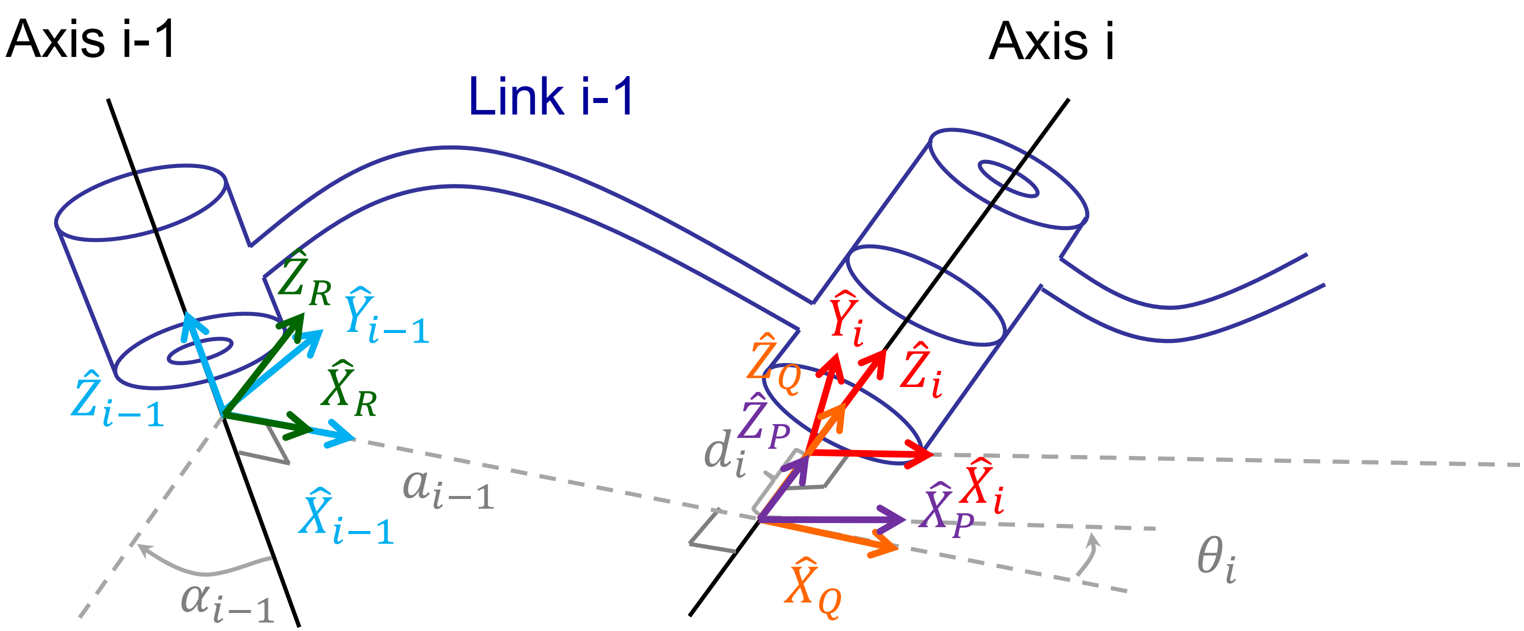


Link Transformations -1

$$^{i-1}P = {}^{i-1}_{i}T\,{}^{i}P$$

$$^{i-1}P = {}^{i-1}_{R}T\,{}^{R}_{Q}T\,{}^{Q}_{P}T\,{}^{P}_{i}T\,{}^{i}P$$

$$\begin{aligned} {}^{i-1}_{i}T &= {}^{i-1}_{R}T\,{}^{R}_{Q}T\,{}^{Q}_{P}T\,{}^{P}_{i}T \\ &= T_{\hat{X}_{i-1}}(\alpha_{i-1})\,T_{\hat{X}_R}(a_{i-1})\,T_{\hat{Z}_Q}(\theta_i)\,T_{\hat{Z}_P}(d_i) \end{aligned}$$



Link Transformations -2

- Thus

$$ {}^{i-1}_{i}T = T_{\hat{X}_{i-1}}(\alpha_{i-1}) T_{\hat{X}_R}(a_{i-1}) T_{\hat{Z}_Q}(\theta_i) T_{\hat{Z}_P}(d_i) $$

$$ = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- 連續link transformations

$$ {}^{0}_{n}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T \ldots {}^{n-2}_{n-1}T\,{}^{n-1}_{n}T $$

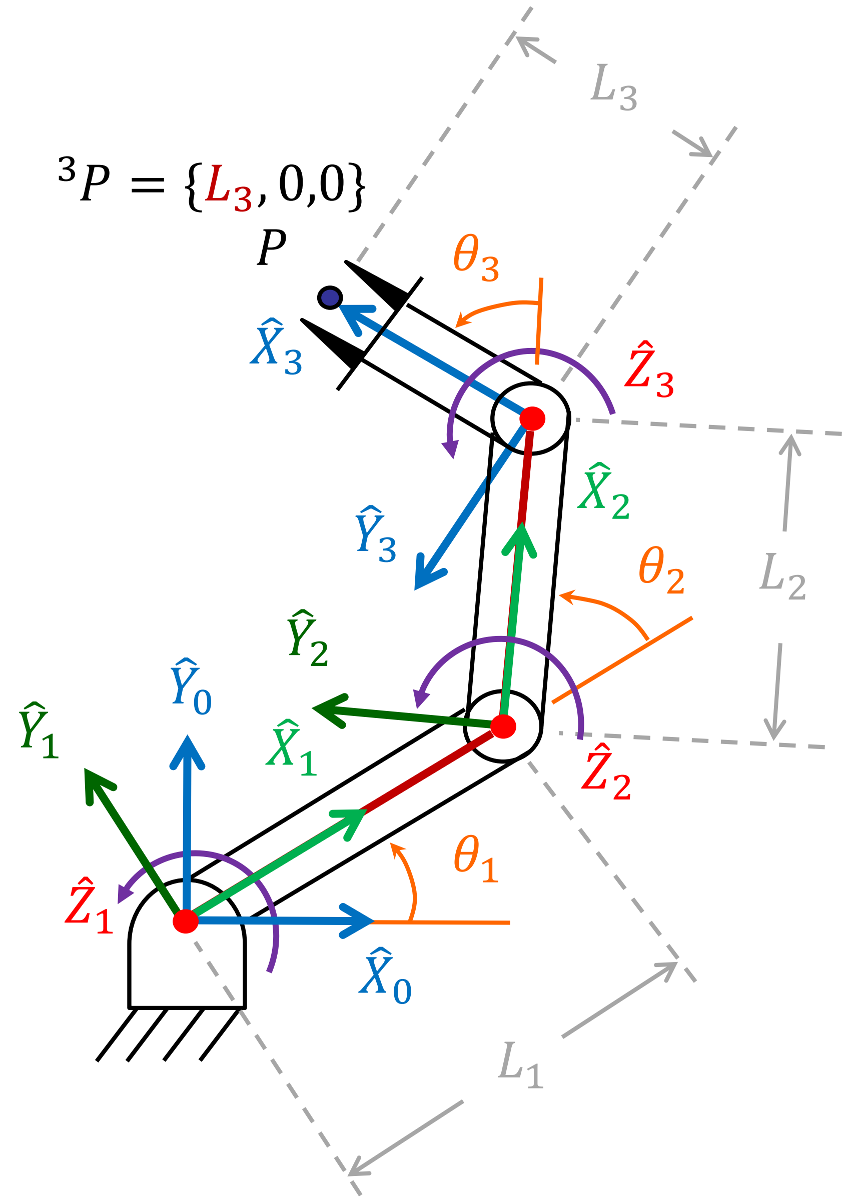
Frame {n} 相對於 Frame {0} 的空間幾何關係具清楚且量化之定義

在Frame {n} 下表達的向量可轉回 Frame {0} 下來表達

Example: A RRR Manipulator

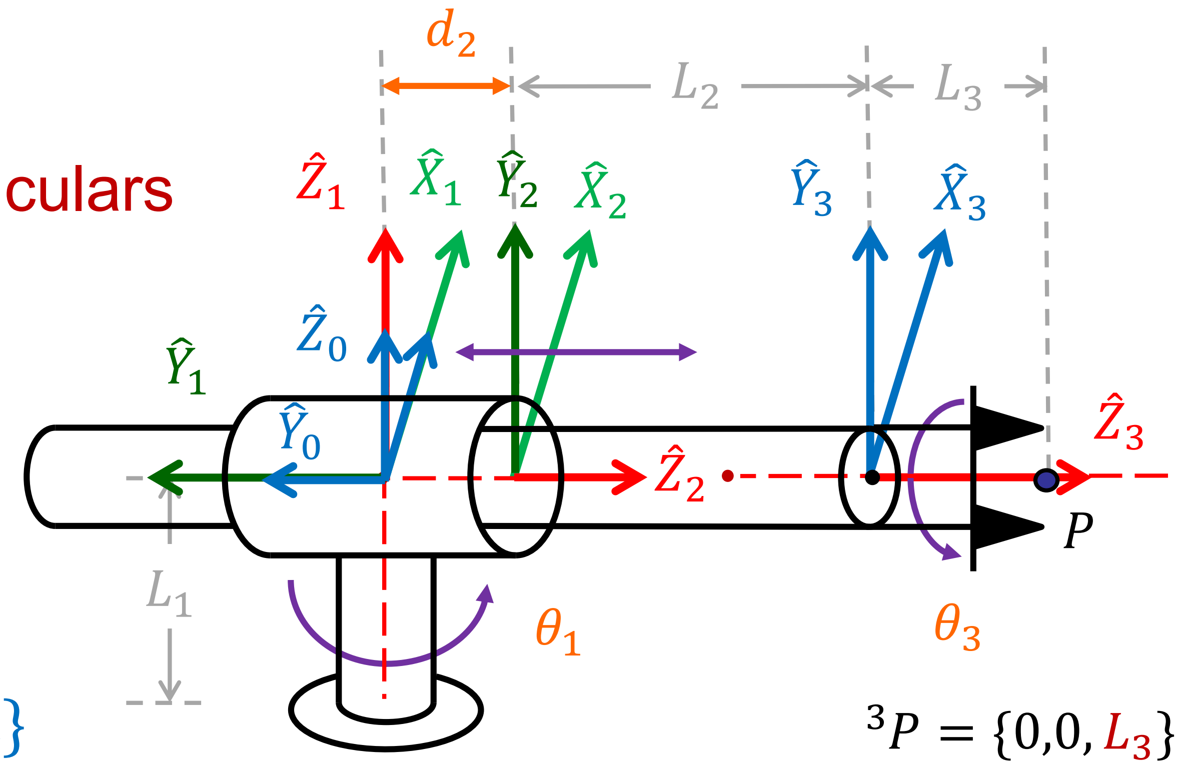
- Joint axes
- Common perpendiculars
- $\hat{Z}_i$
- $\hat{X}_i$
- $\hat{Y}_i$
- Frames $\{0\}$ and $\{n\}$

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3



Example: A RPR Manipulator -2

- Joint axes
- Common perpendiculars
- $\hat{Z}_i$
- $\hat{X}_i$
- $\hat{Y}_i$
- Frames $\{0\}$ and $\{n\}$



$^{3}P = \{0,0,L_3\}$

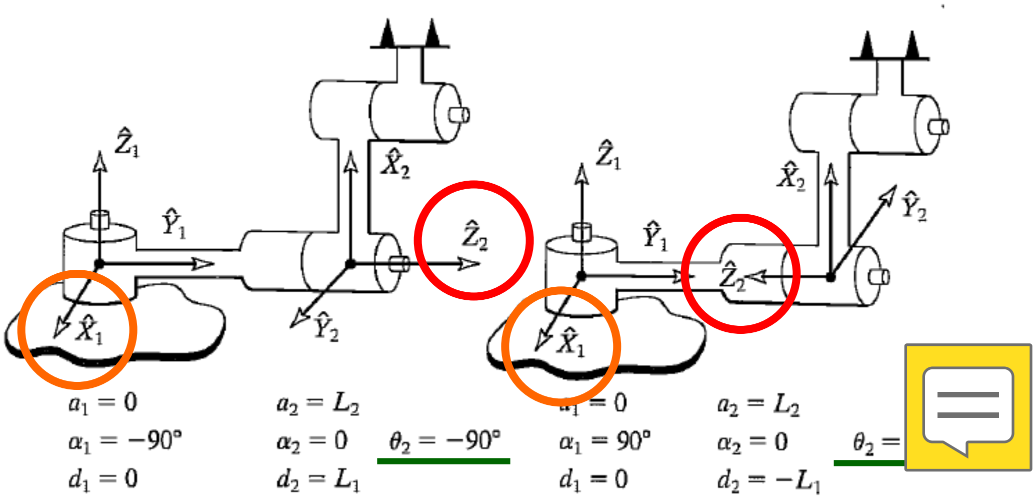
i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	90°	0	d_2	0
3	0	0	L_2	θ_3

Example: A RRR Manipulator -3

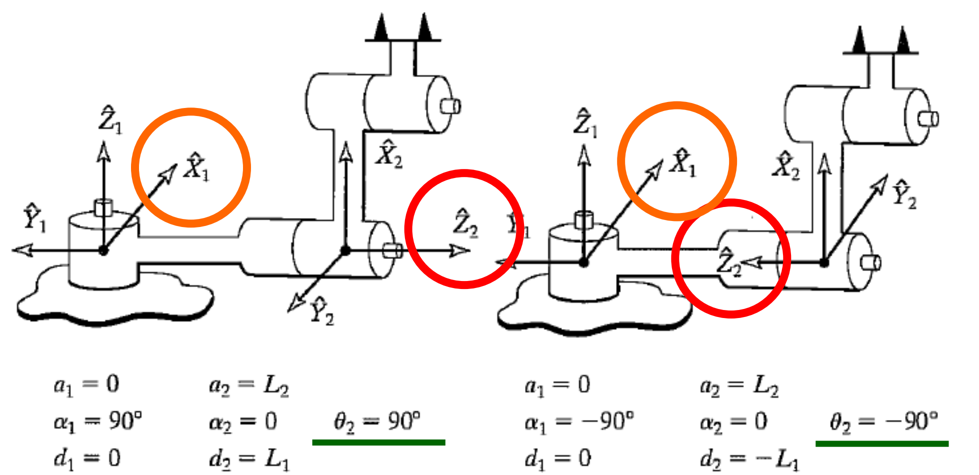
- 當$a_1 = 0$

  $\hat{Z}_1$和$\hat{Z}_2$相交

  - $\hat{Z}_2$兩個選擇



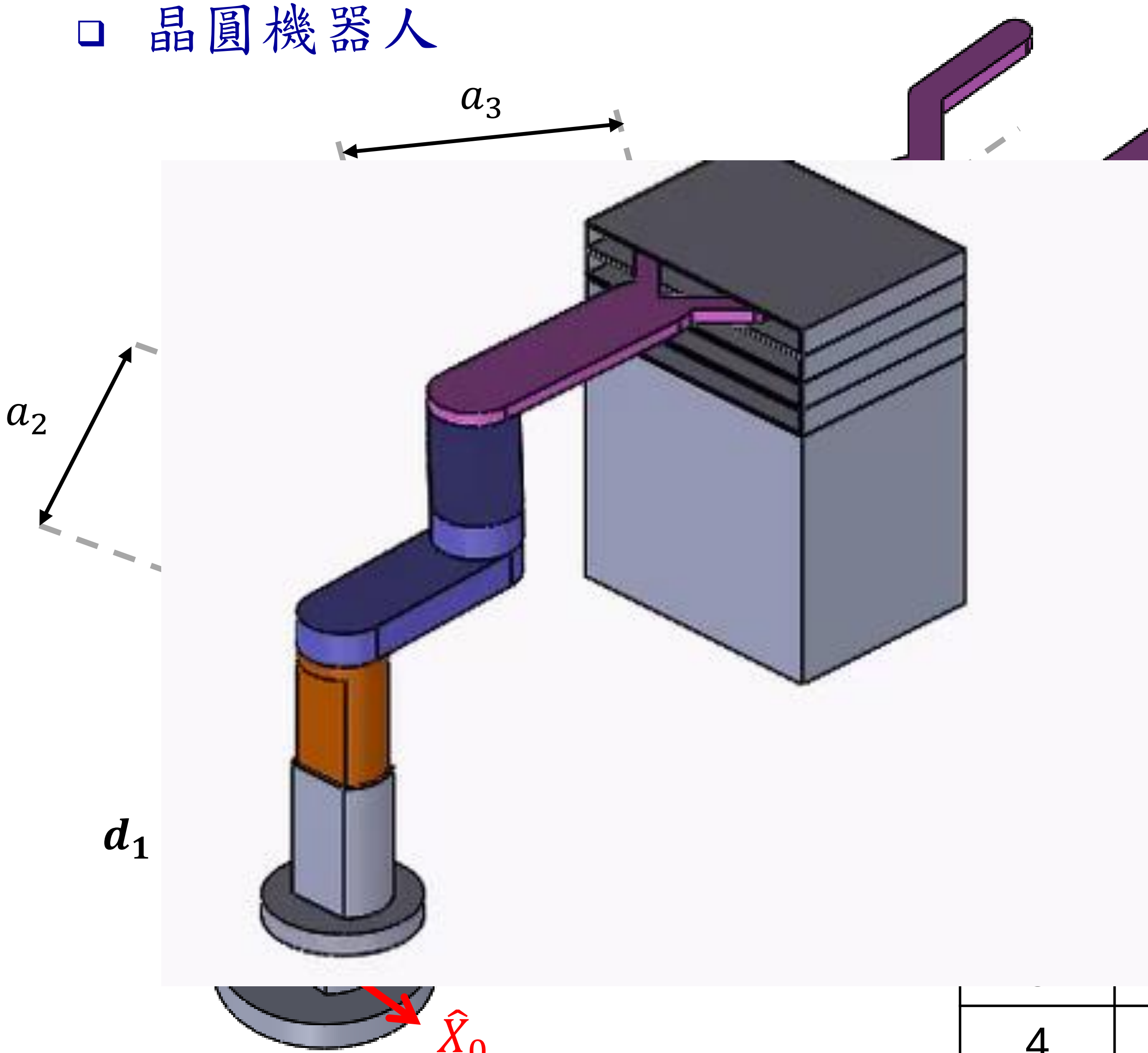
  - $\hat{X}_1$兩個選擇



John J. Craig, "Introduction to Robotics," 3rd ed., Pearson Prentice Hall, 2005, pp. 74

Example: A PRRR Manipulator

- 晶圓機器人



θ_3, θ_4 為驅動關節

$(X_0 // X_1)$

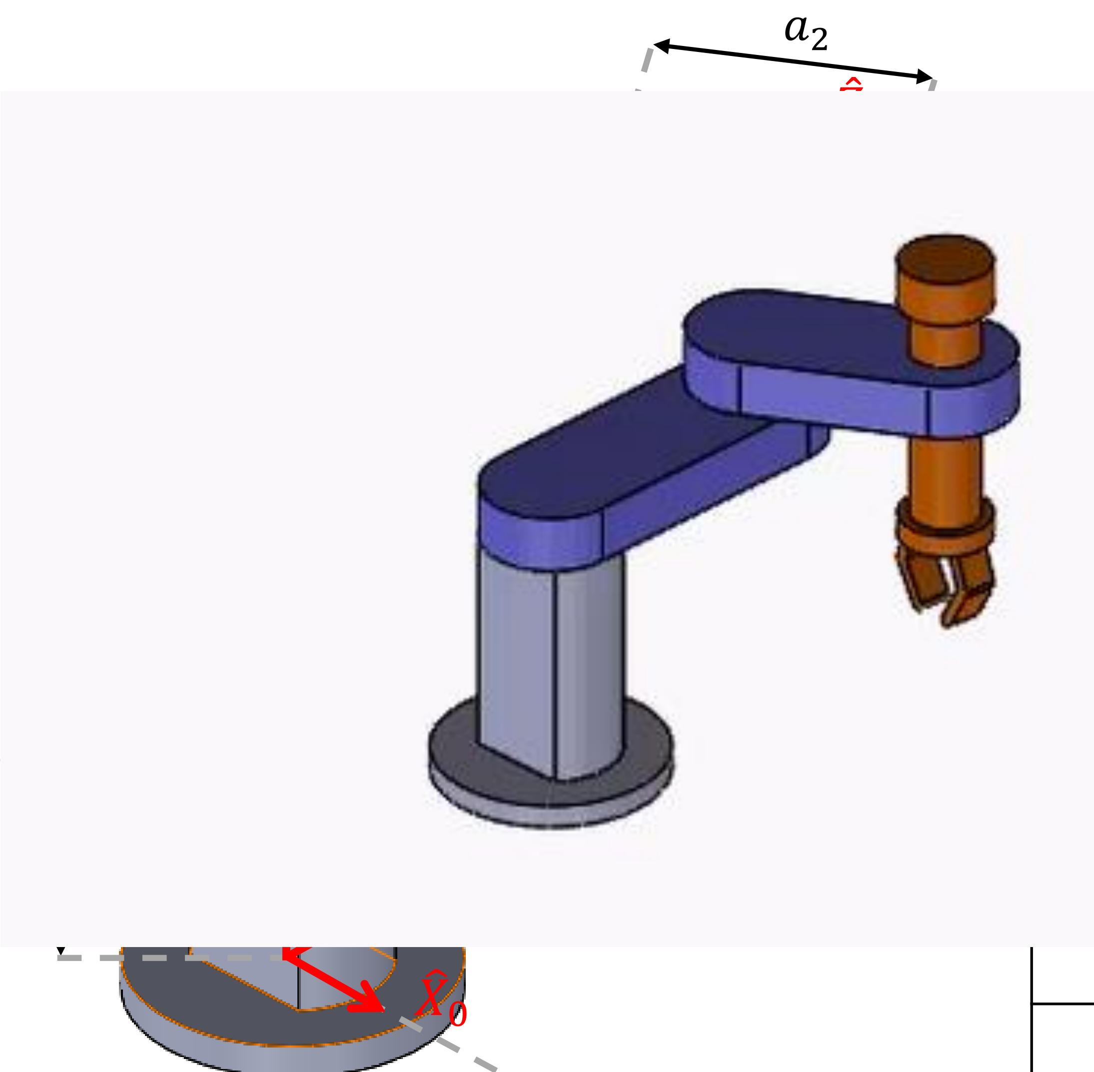
$\alpha_2, \alpha_3 = 0$（所有Z軸相互平行）

$= 0$

	α_{i-1}	a_{i-1}	d_i	θ_i
	0	0	d_1	0
	0	0	d_2	θ_2
	0	a_2	d_3	θ_3
4	0	a_3	d_4	θ_4

Example: A RRRP Manipulator

- SCARA機器人



$\theta_1, \theta_2, \theta_3, d_4$ 為驅動關節

$\theta_4 = 0$ (X_0 // X_1)

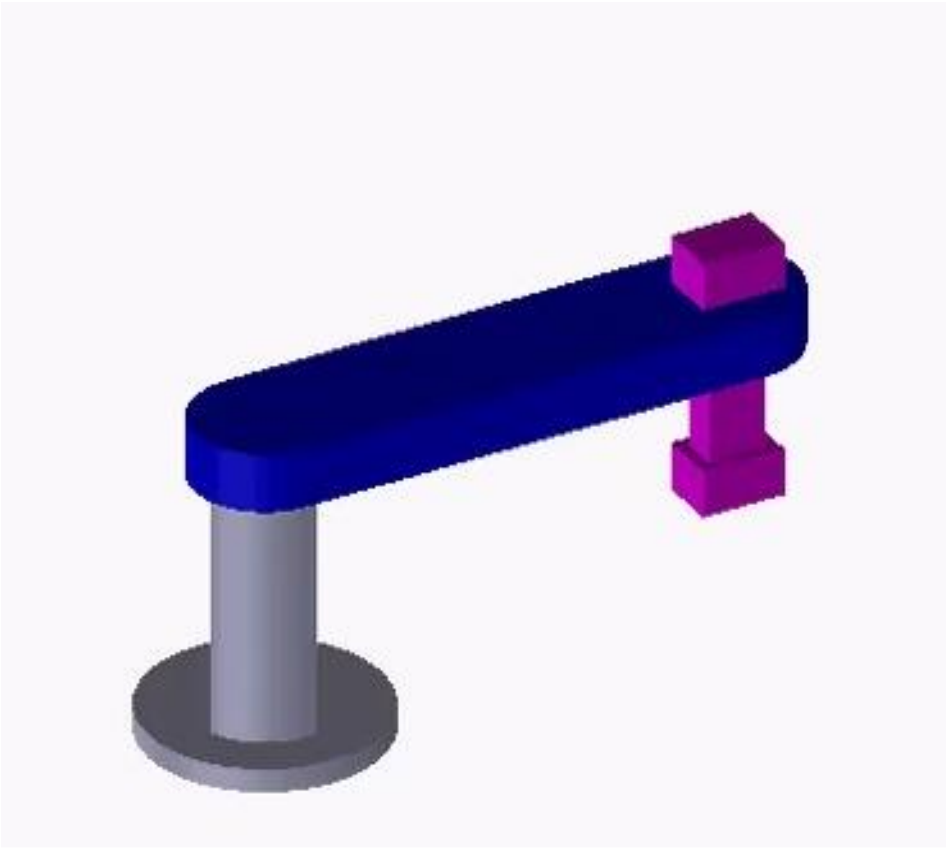
$\alpha_0, \alpha_1, \alpha_2, \alpha_3 = 0$（所有Z軸相互平行）

$a_0, a_3 = 0$

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	d_1	θ_1
2	0	a_1	d_2	θ_2
3	0	a_2	0	θ_3
4	0	0	d_4	0

Example: A RP manipulator

- In-video Quiz：下方手臂由一個revolute joint和一個prismatic joint組成，在所有的DH參數 $(a_{i-1}\ \alpha_{i-1}\ d_i\ \theta_i)$中，哪兩個參數為驅動關節？



(A) α_1, a_2

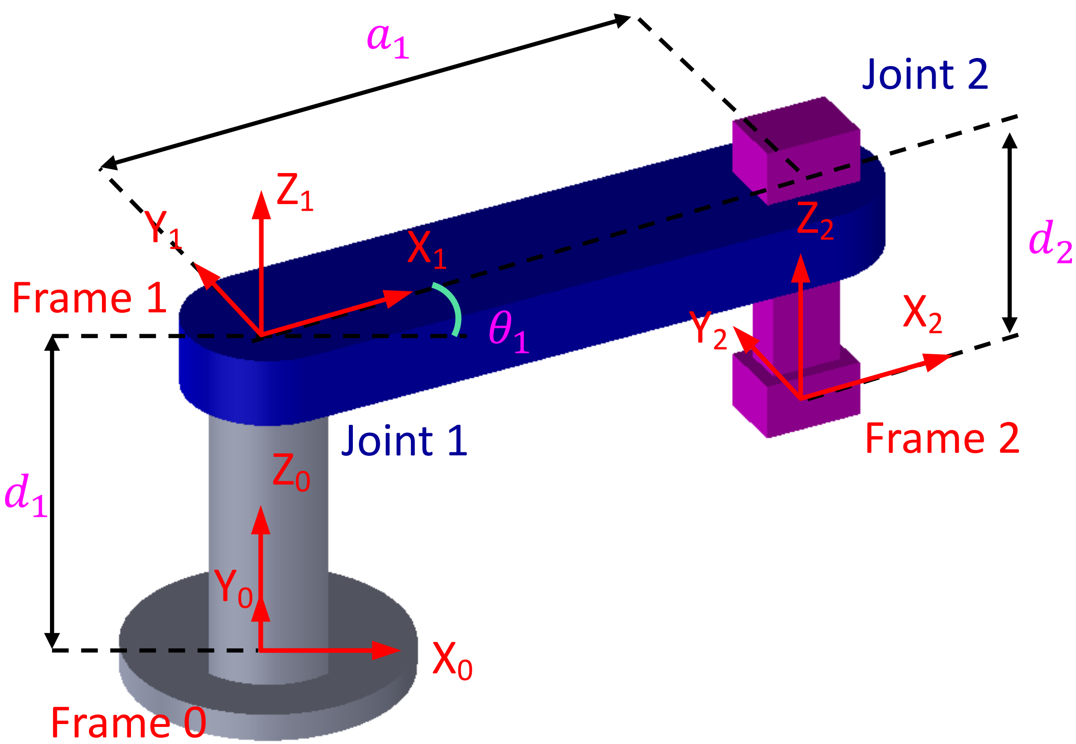
(B) α_1, d_2

(C) θ_1, a_2

(D) θ_1, d_2

Example: A RP manipulator (Solution)

- In-video Quiz：下方手臂由一個revolute joint和一個prismatic joint組成，在所有的DH參數 $(a_{i-1}\ \alpha_{i-1}\ d_i\ \theta_i)$中，哪兩個參數為驅動關節？



For Joint 1:

$a_0 = 0$ (Z_0 Z_1 are concurrent)

$\alpha_0 = 0$ (Z_0 // Z_1)

For Joint 2:

$\alpha_1 = 0$ (Z_1 // Z_2)

$\theta_2 = 0$ (X_1 // X_2)

θ_1, d_2 are varying parameters

Actuator, Joint, and Cartesian Spaces -1

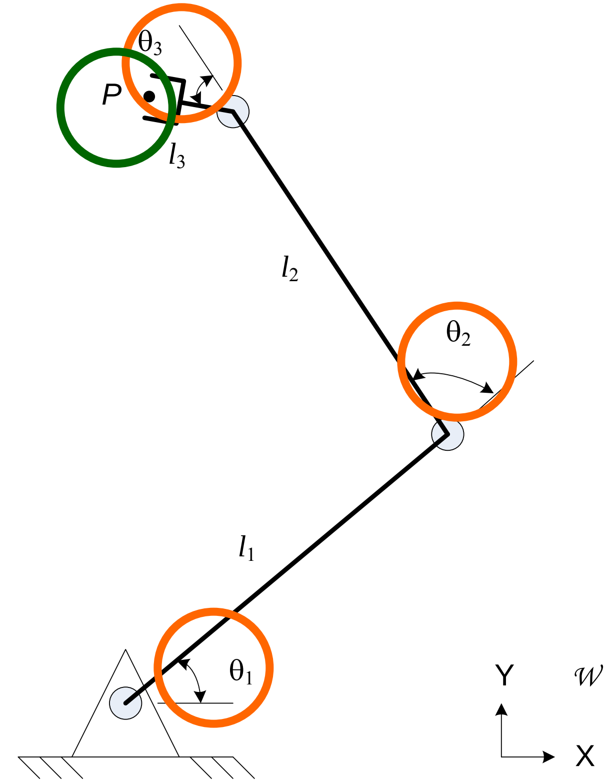
- Joint space ⬄ Cartesian space

Forward Kinematics

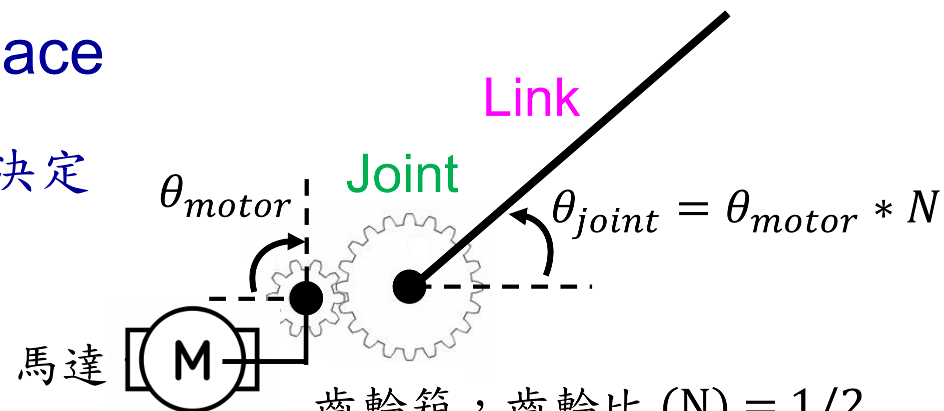
${}^{W}P$ $\theta_1, \theta_2, \ldots, \theta_n$

Cartesian space   Joint space

Inverse Kinematics



- Actuator space ⬄ joint space
  - 由連結致動器和joint的機構決定



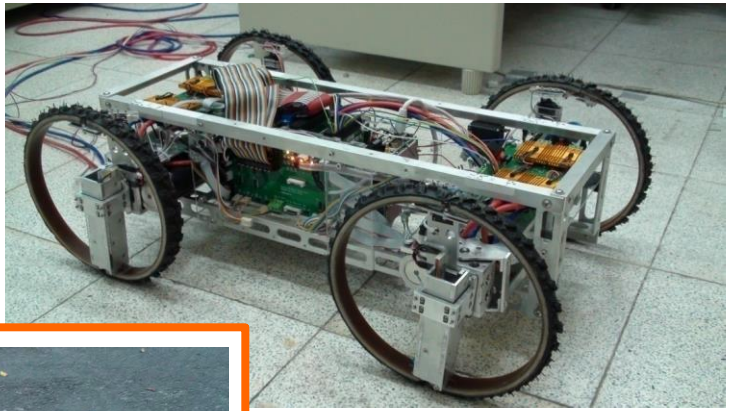
Actuator, Joint, and Cartesian Spaces -2

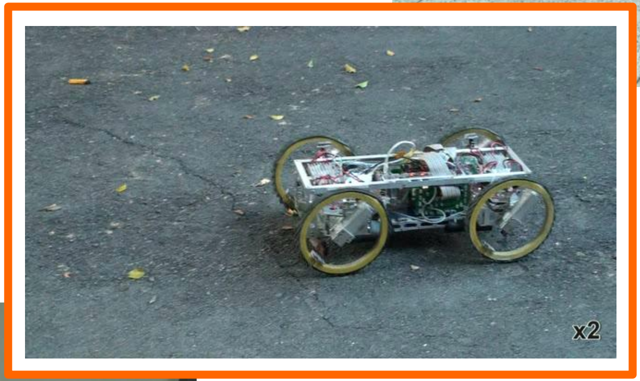
- Example: A leg-wheel transformable robot

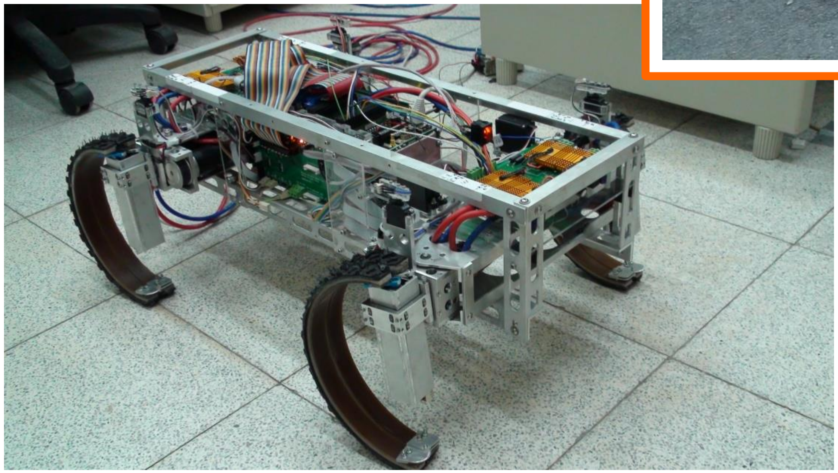
輪模式

平地上

快速、平穩、省能







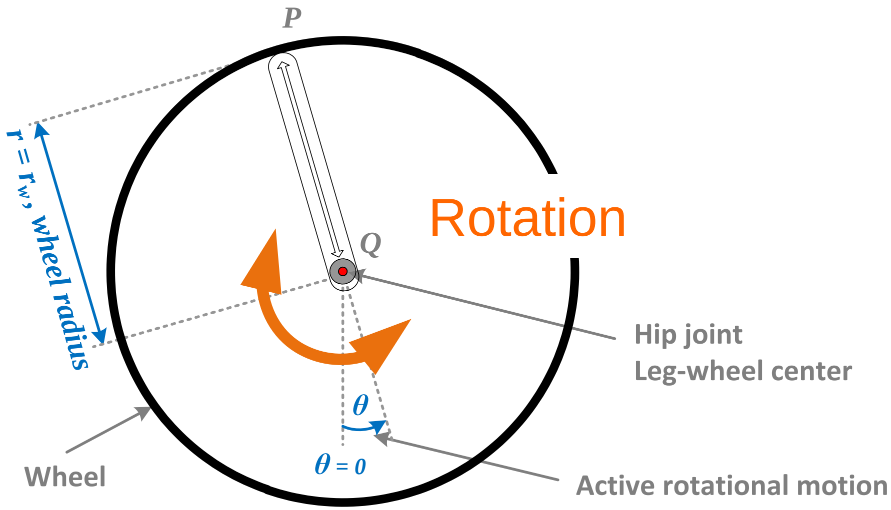
腳模式

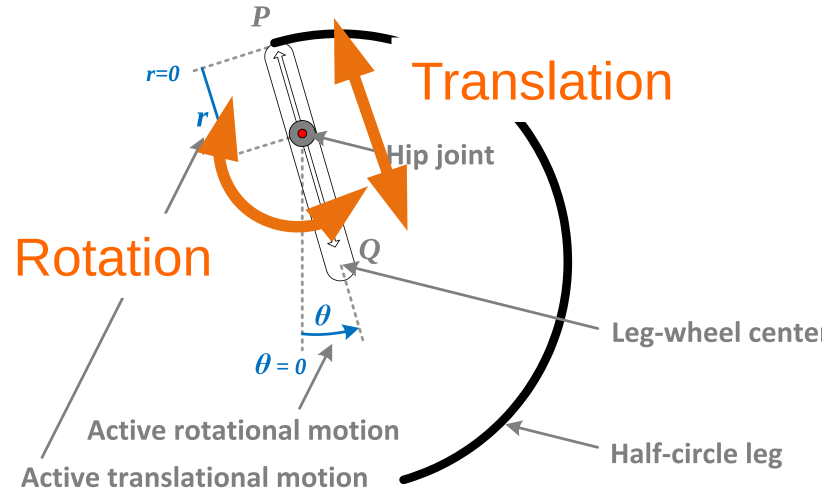
崎嶇地

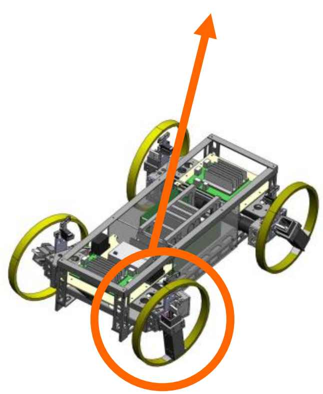
越障、動態

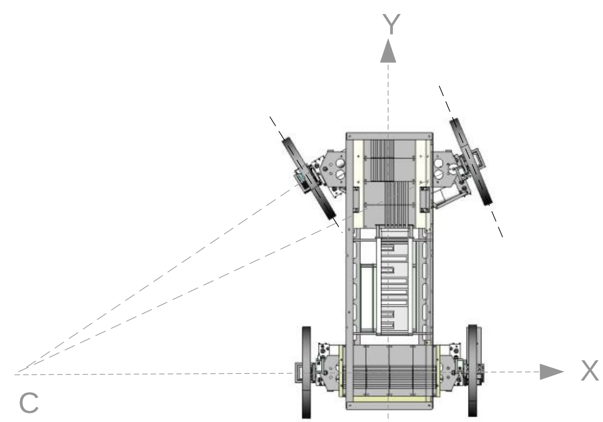
Actuator, Joint, and Cartesian Spaces -3

- 輪模式 腳模式

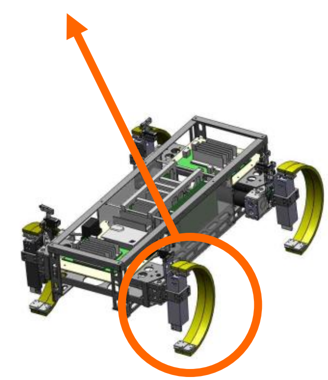






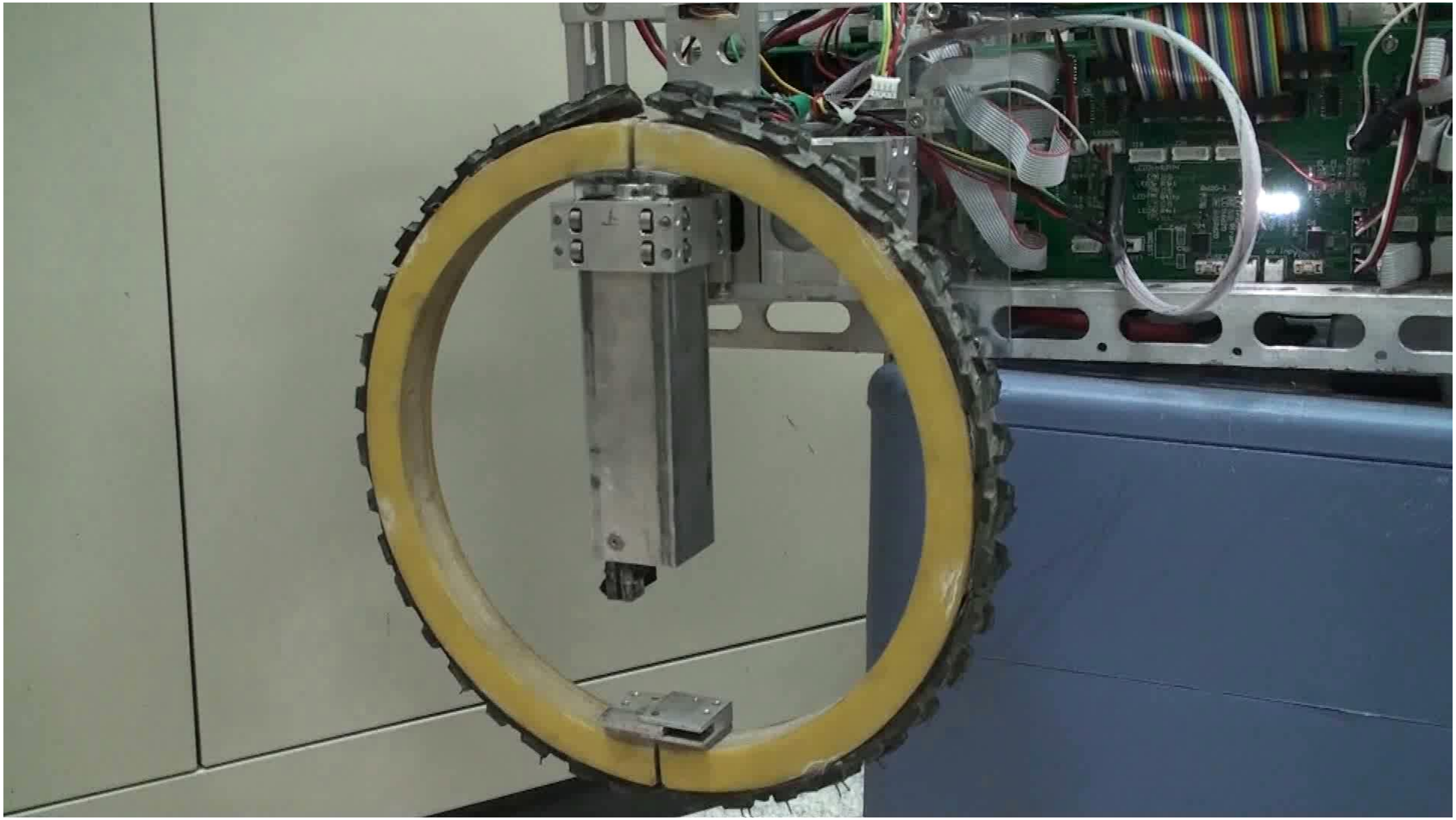


Ackermann steering



Actuator, Joint, and Cartesian Spaces -4

- Leg-wheel motion



Actuator, Joint, and Cartesian Spaces -5

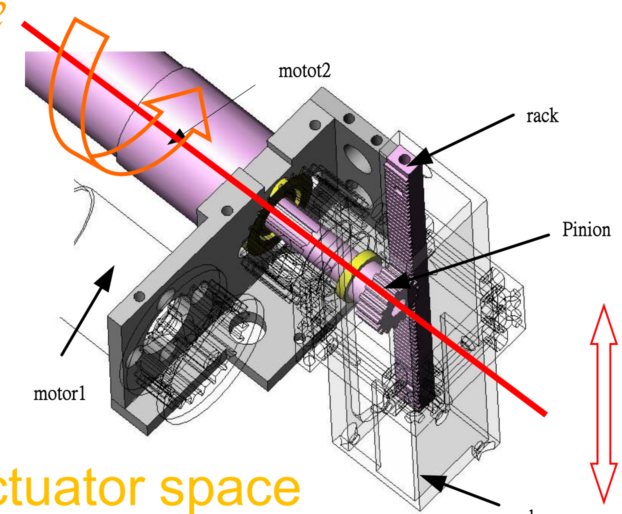
- Kinematic mapping
  - Input: Motor speeds $\dot{\phi}_1 \quad \dot{\phi}_2$
  - Output: Leg-wheel motion $\dot{\theta} \quad \dot{r}$

    in polar coordinate

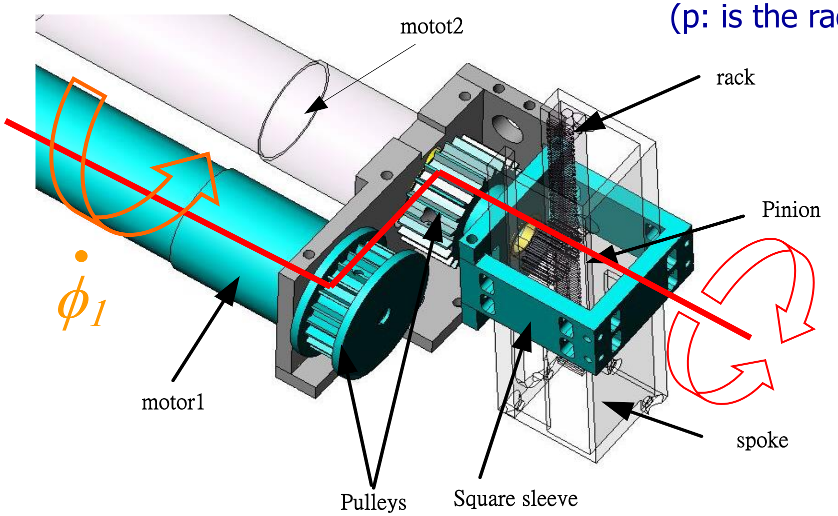
Joint space $\dot{\xi} = \begin{bmatrix} \dot{\theta} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -p & p \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$ Actuator space

(p: is the radius of pinion gear)

$\dot{\phi}_2$

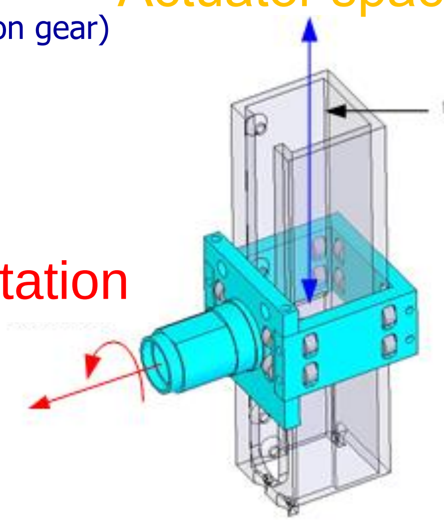


$\dot{r}$



$\dot{\phi}_1$

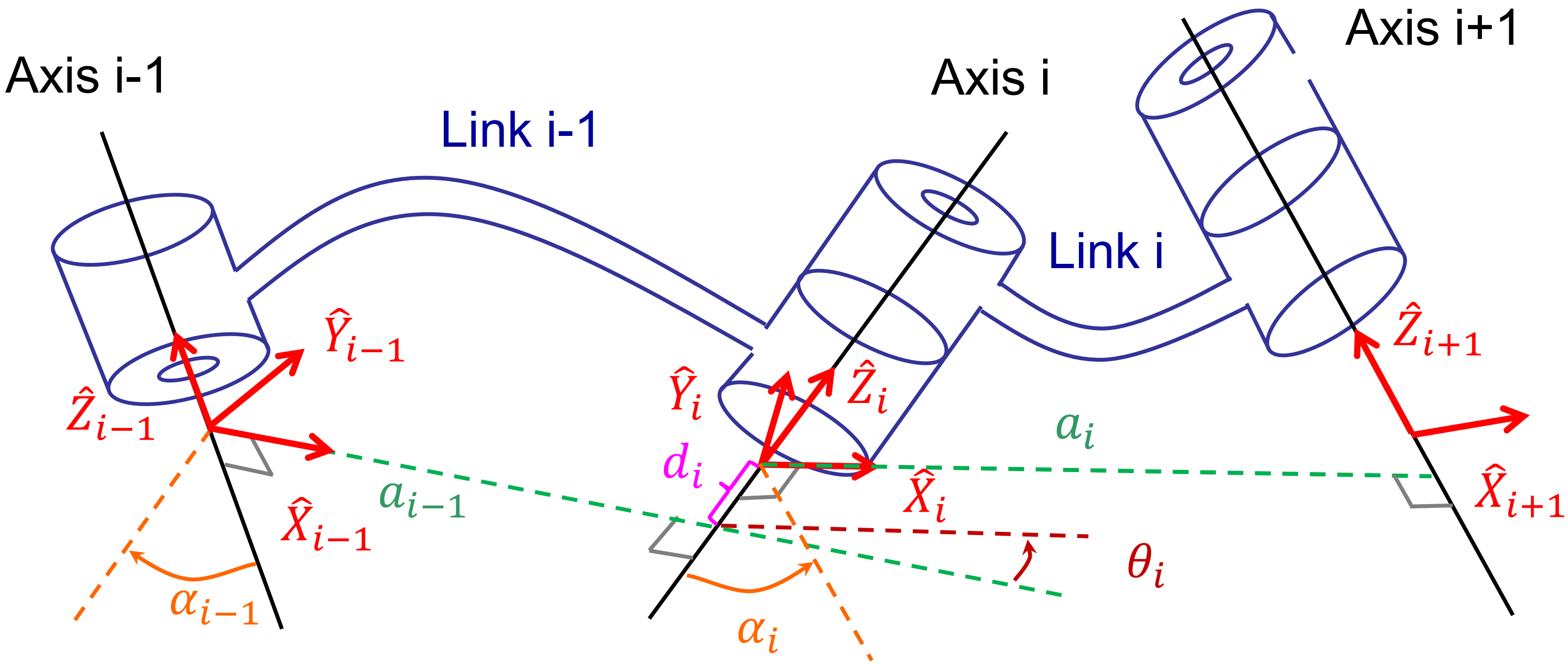
$\dot{\theta}$ Rotation



Translation $\dot{r}$

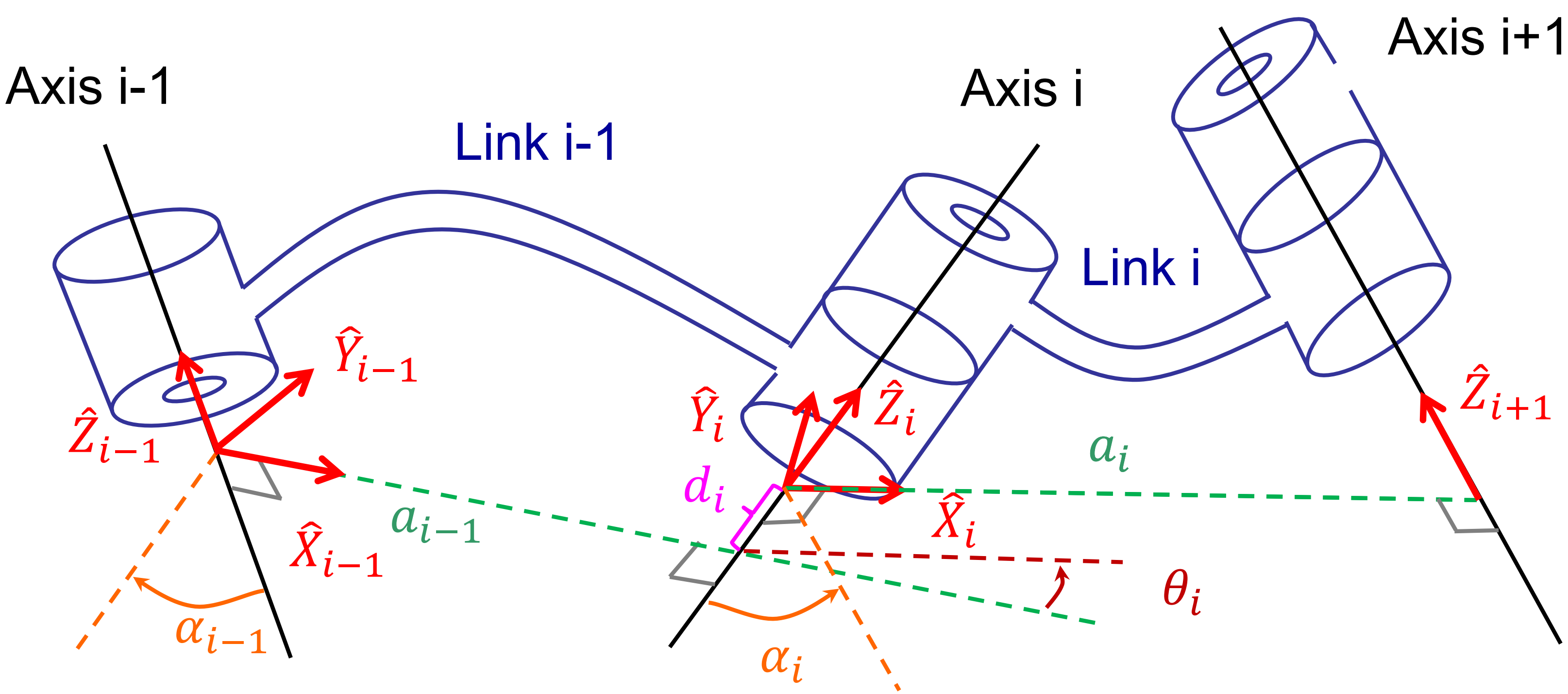
Denavit-Hartenberg表達法小結 (Craig version) -1

- α_{i-1}: 以$\hat{X}_{i-1}$方向看，$\hat{Z}_{i-1}$和$\hat{Z}_i$間的夾角
- a_{i-1}: 沿著$\hat{X}_{i-1}$方向，$\hat{Z}_{i-1}$和$\hat{Z}_i$間的距離 $(a_i > 0)$
- θ_i: 以$\hat{Z}_i$方向看，$\hat{X}_{i-1}$和$\hat{X}_i$間的夾角
- d_i: 沿著$\hat{Z}_i$方向，$\hat{X}_{i-1}$和$\hat{X}_i$間的距離



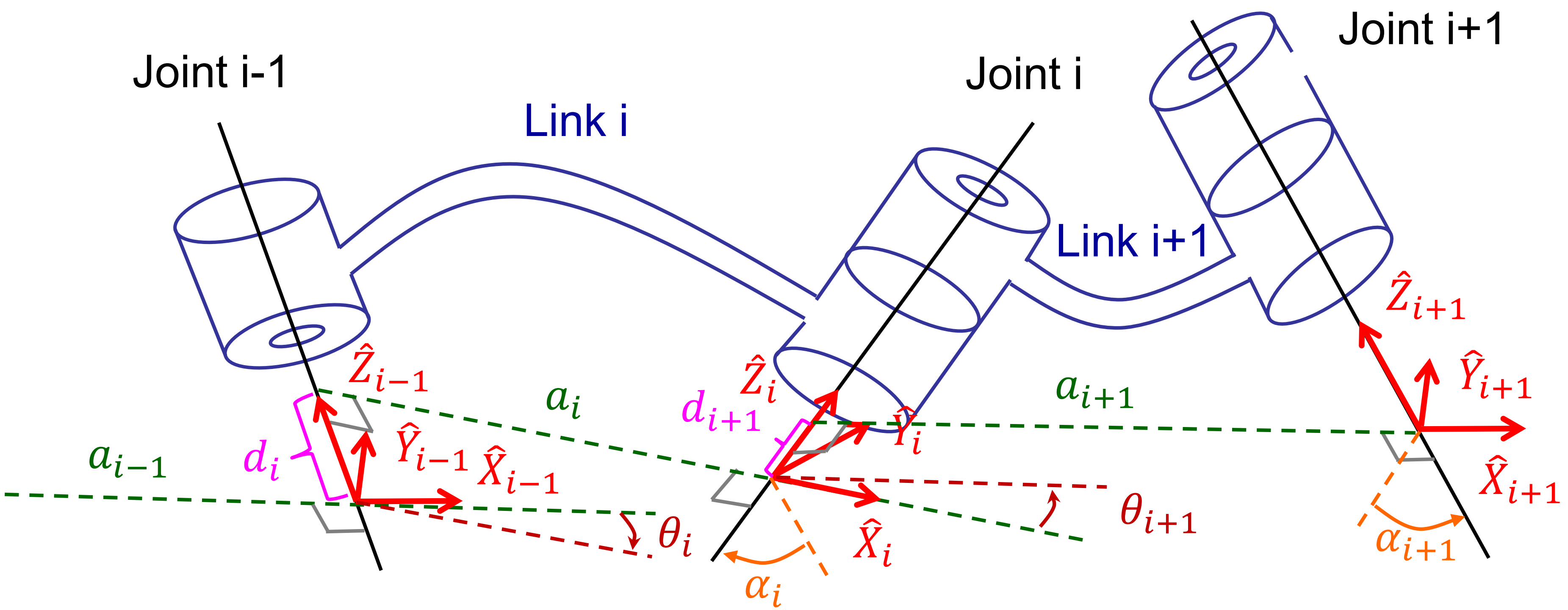
Denavit-Hartenberg表達法小結 (Craig version) -2

$$
\begin{aligned}
{}^{i-1}_{i}T &= {}^{i-1}_{R}T\,{}^{R}_{Q}T\,{}^{Q}_{P}T\,{}^{P}_{i}T \\
&= T_{\hat{X}_{i-1}}(\alpha_{i-1})T_{\hat{X}_R}(a_{i-1})T_{\hat{Z}_Q}(\theta_i)T_{\hat{Z}_P}(d_i) \\
&= \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$



Denavit-Hartenberg表達法 (Standard) -1

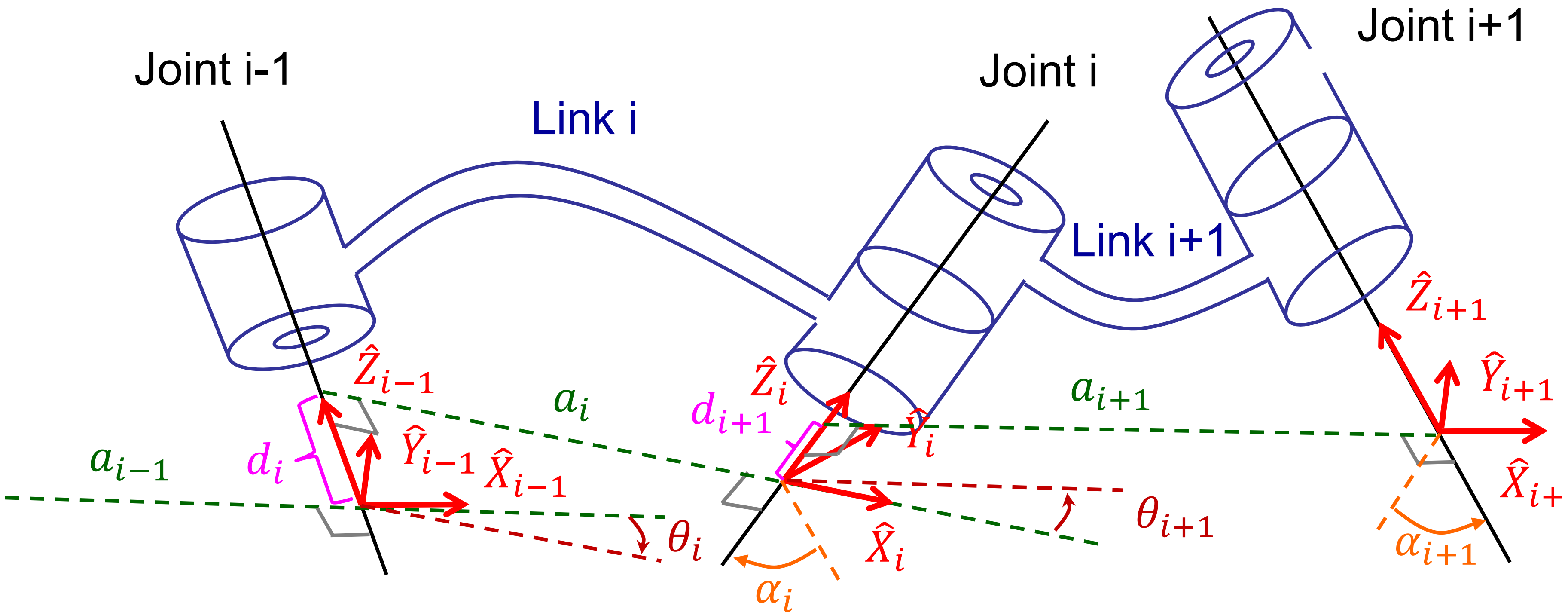
- θ_i: 以$\hat{Z}_{i-1}$方向看，$\hat{X}_{i-1}$和$\hat{X}_i$間的夾角
- d_i: 沿著$\hat{Z}_{i-1}$方向，$\hat{X}_{i-1}$和$\hat{X}_i$間的距離
- a_i: 沿著$\hat{X}_i$方向，$\hat{Z}_{i-1}$和$\hat{Z}_i$間的距離 $(a_i > 0)$
- α_i: 以$\hat{X}_i$方向看，$\hat{Z}_{i-1}$和$\hat{Z}_i$間的夾角



Summary of DH Notation (Standard) -2

$$
\begin{aligned}
{}^{i-1}_{i}T &= {}^{i-1}_{R}T\,{}^{R}_{Q}T\,{}^{Q}_{P}T\,{}^{P}_{i}T \\
&= T_{\hat{Z}_{i-1}}(\theta_i)\,T_{\hat{Z}_R}(d_i)\,T_{\hat{X}_Q}(a_i)\,T_{\hat{X}_P}(\alpha_i)
\end{aligned}
$$

$$
= \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

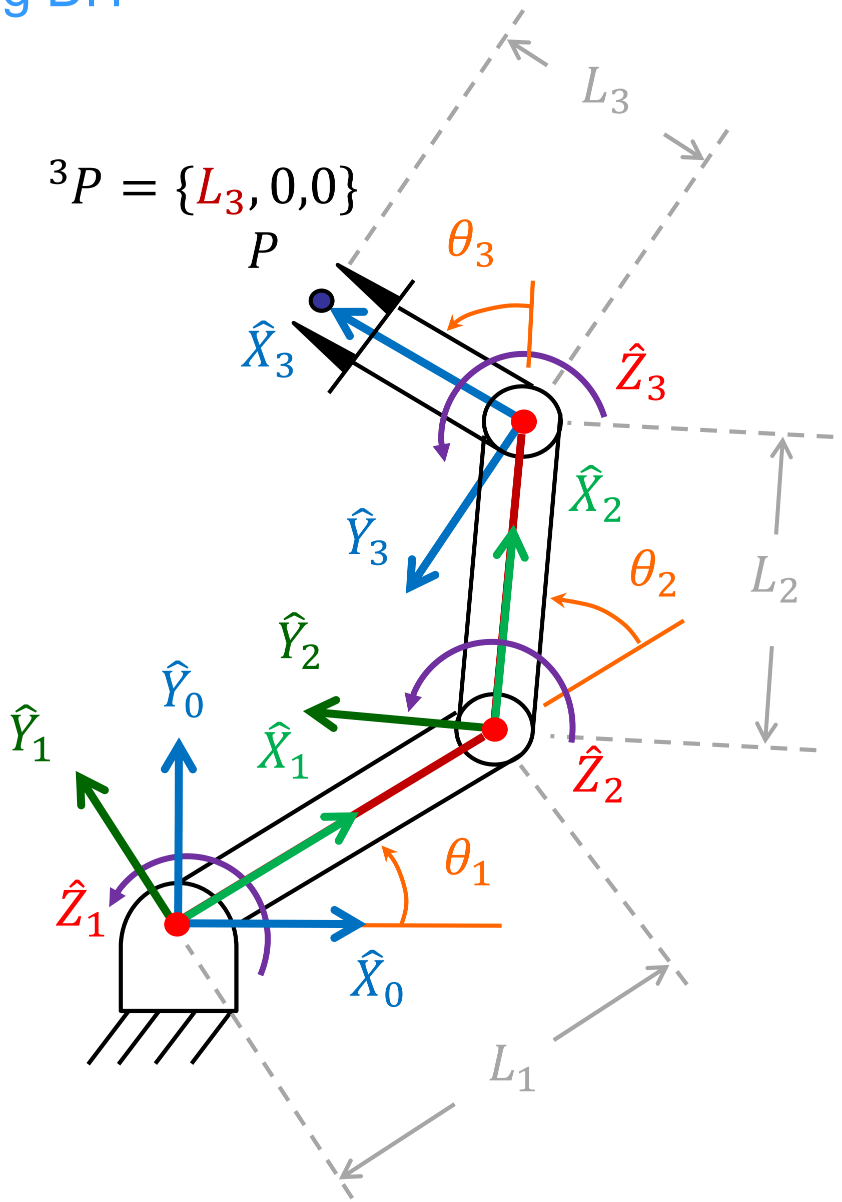


Revisit Example: A RRR Manipulator -1

Craig DH

- Joint axes
- Common perpendiculars
- $\hat{Z}_i$
- $\hat{X}_i$
- $\hat{Y}_i$
- Frames $\{0\}$ and $\{n\}$

$^3P = \{L_3, 0, 0\}$



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

Revisit Example: A RRR Manipulator -2

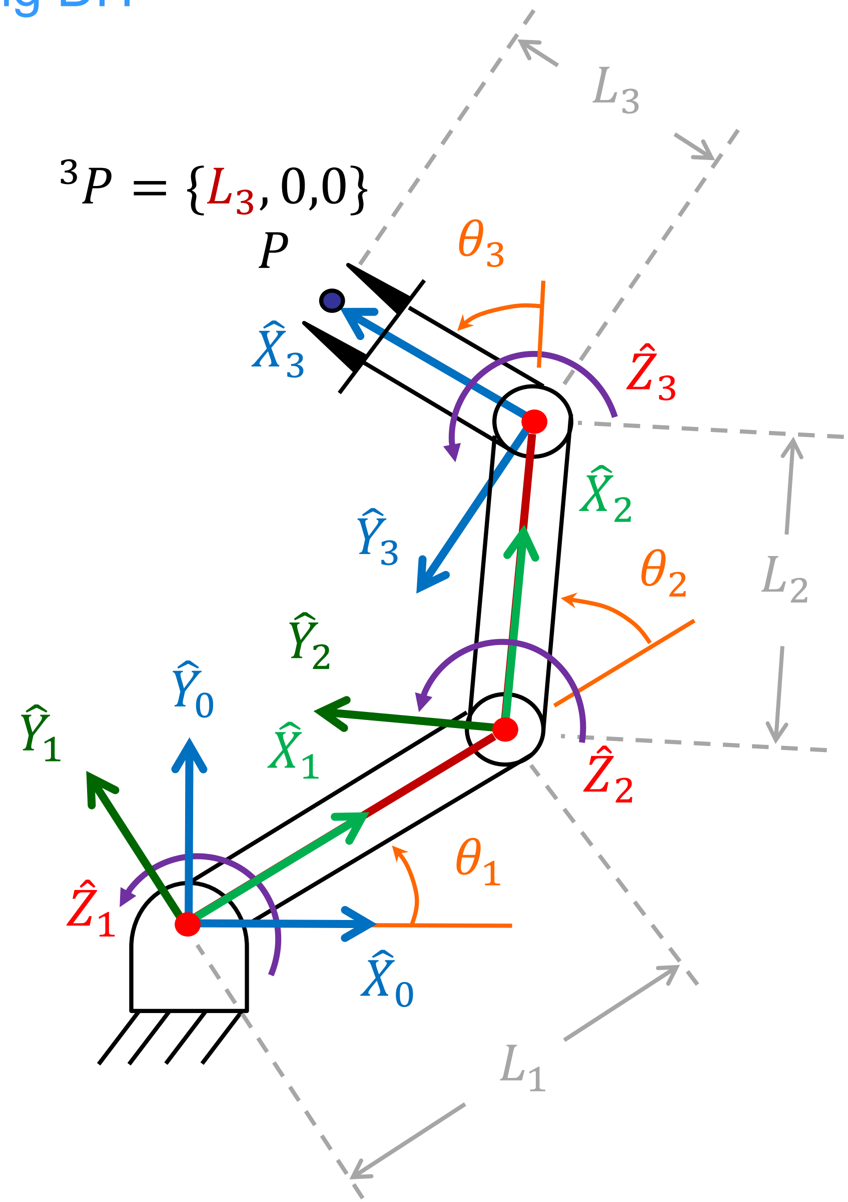
Craig DH

- Transformation matrices

$$ {}^{0}_{1}T = \begin{pmatrix} \text{Cos}[t1] & -\text{Sin}[t1] & 0 & 0 \\ \text{Sin}[t1] & \text{Cos}[t1] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

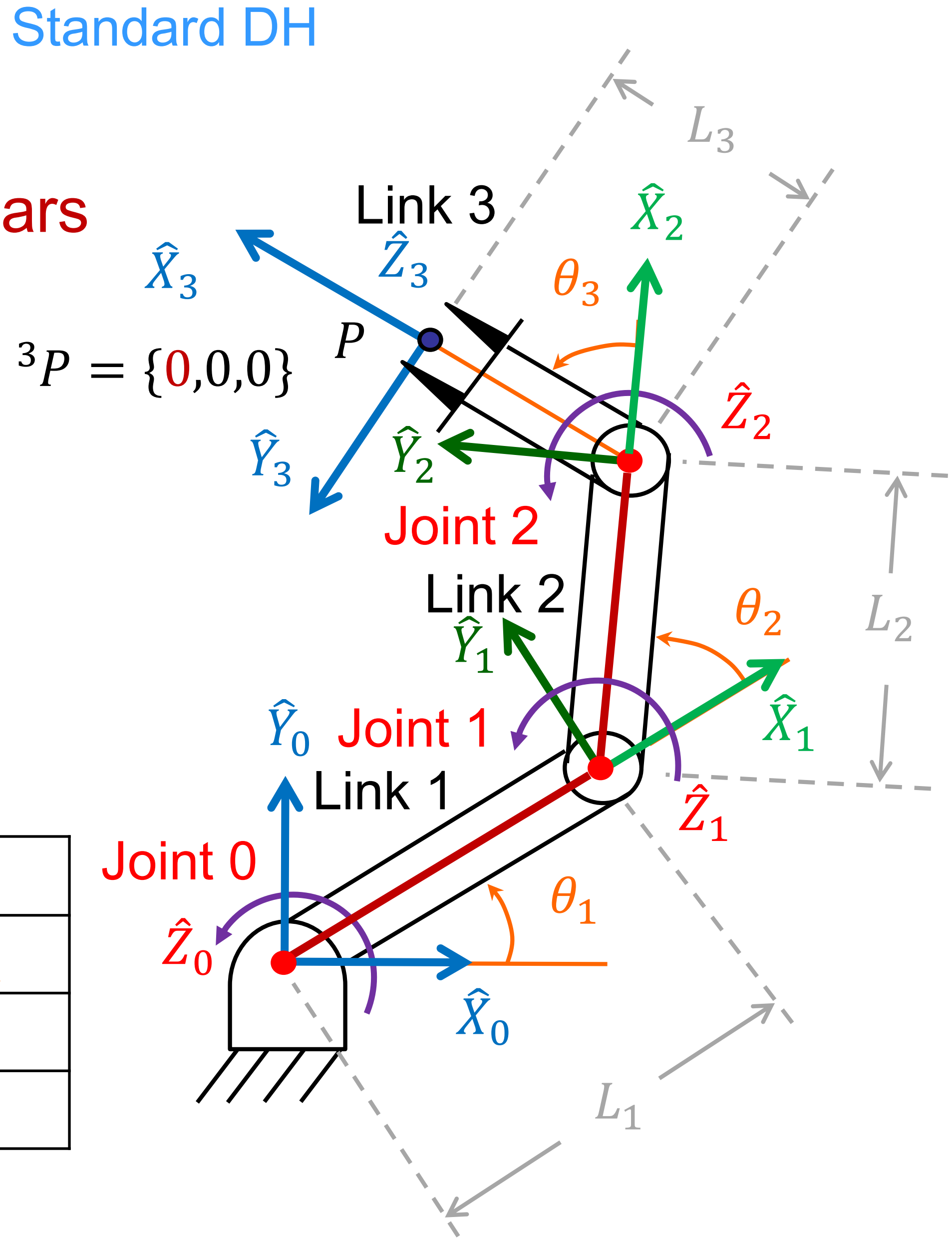
$$ {}^{1}_{2}T = \begin{pmatrix} \text{Cos}[t2] & -\text{Sin}[t2] & 0 & L1 \\ \text{Sin}[t2] & \text{Cos}[t2] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{2}_{3}T = \begin{pmatrix} \text{Cos}[t3] & -\text{Sin}[t3] & 0 & L2 \\ \text{Sin}[t3] & \text{Cos}[t3] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$



Revisit Example: A RRR Manipulator -3

- Joint axes
- Common perpendiculars
- $\hat{Z}_i$
- $\hat{X}_i$
- $\hat{Y}_i$
- Frames $\{0\}$ and $\{n\}$



i	α_i	a_i	d_i	θ_i
1	0	L_1	0	θ_1
2	0	L_2	0	θ_2
3	0	L_3	0	θ_3

Revisit Example: A RRR Manipulator -4

Standard DH

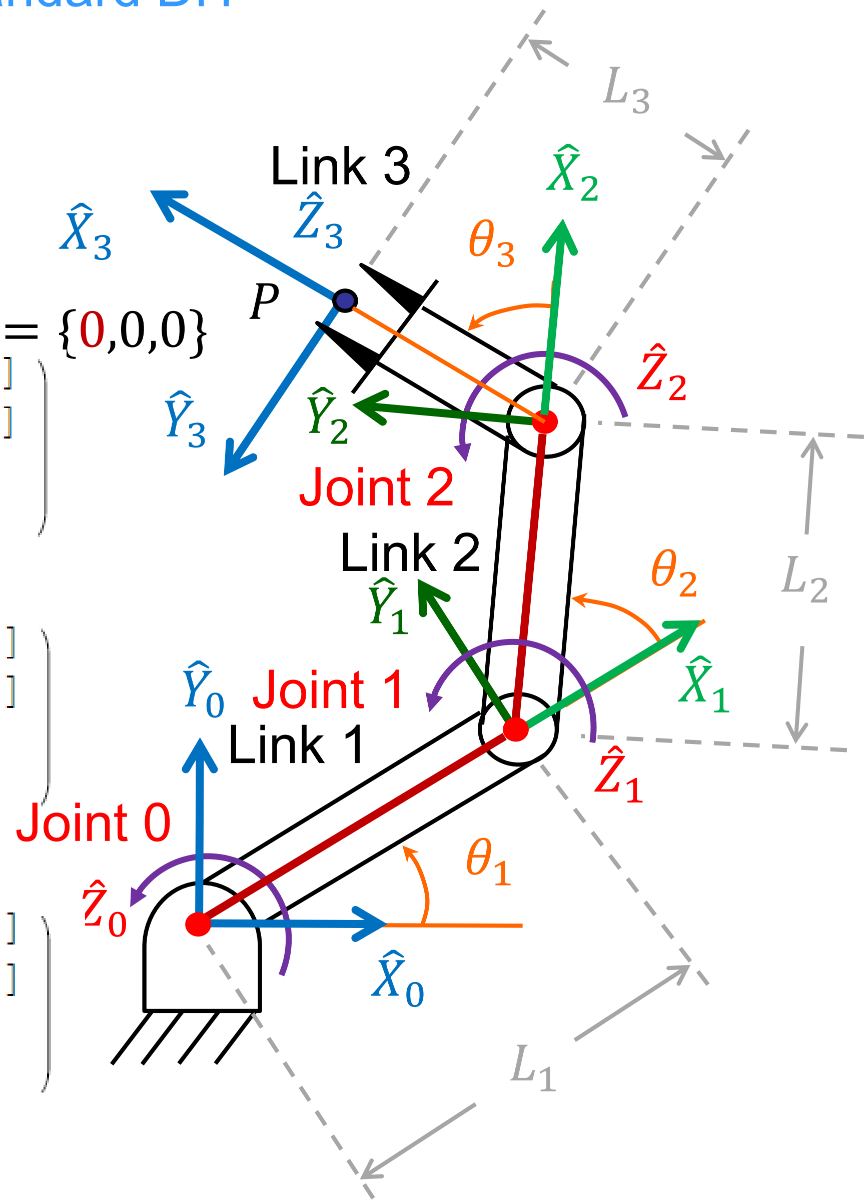
- Transformation matrices

$${}^3P = \{0,0,0\}$$

$${}^0_1T \quad \begin{pmatrix} \text{Cos[t1]} & -\text{Sin[t1]} & 0 & \text{L1 Cos[t1]} \\ \text{Sin[t1]} & \text{Cos[t1]} & 0 & \text{L1 Sin[t1]} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^1_2T \quad \begin{pmatrix} \text{Cos[t2]} & -\text{Sin[t2]} & 0 & \text{L2 Cos[t2]} \\ \text{Sin[t2]} & \text{Cos[t2]} & 0 & \text{L2 Sin[t2]} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^2_3T \quad \begin{pmatrix} \text{Cos[t3]} & -\text{Sin[t3]} & 0 & \text{L3 Cos[t3]} \\ \text{Sin[t3]} & \text{Cos[t3]} & 0 & \text{L3 Sin[t3]} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Revisit Example: A RRR Manipulator -5

- Craig

$$
{}^0_3T \quad \begin{pmatrix} \text{Cos}[t1+t2+t3] & -\text{Sin}[t1+t2+t3] & 0 & \text{L1 Cos}[t1]+\text{L2 Cos}[t1+t2] \\ \text{Sin}[t1+t2+t3] & \text{Cos}[t1+t2+t3] & 0 & \text{L1 Sin}[t1]+\text{L2 Sin}[t1+t2] \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^0_3T . T_{\hat{X}_3}([L_3,0,0])
$$

$$
\begin{pmatrix} \text{Cos}[t1+t2+t3] & -\text{Sin}[t1+t2+t3] & 0 & \text{L1 Cos}[t1]+\text{L2 Cos}[t1+t2]+\text{L3 Cos}[t1+t2+t3] \\ \text{Sin}[t1+t2+t3] & \text{Cos}[t1+t2+t3] & 0 & \text{L1 Sin}[t1]+\text{L2 Sin}[t1+t2]+\text{L3 Sin}[t1+t2+t3] \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

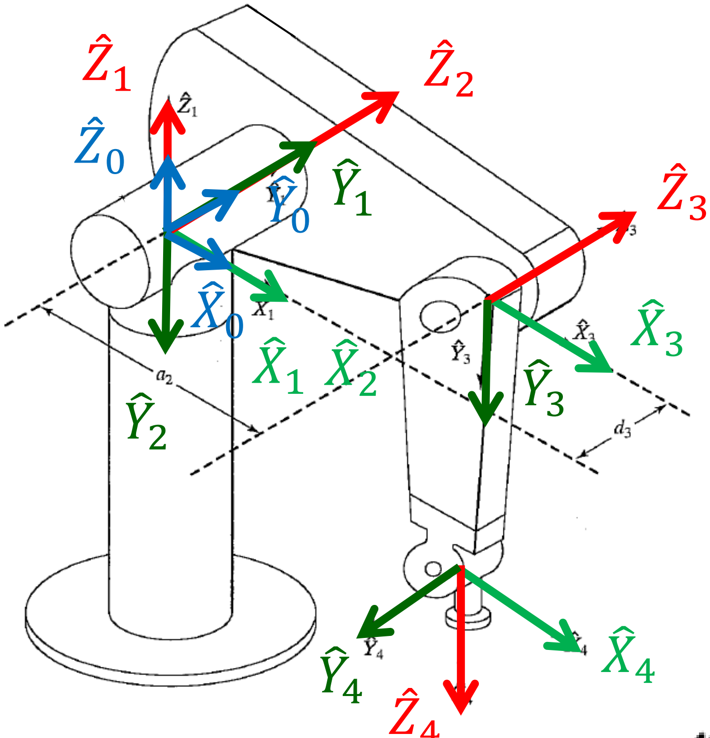
- Standard

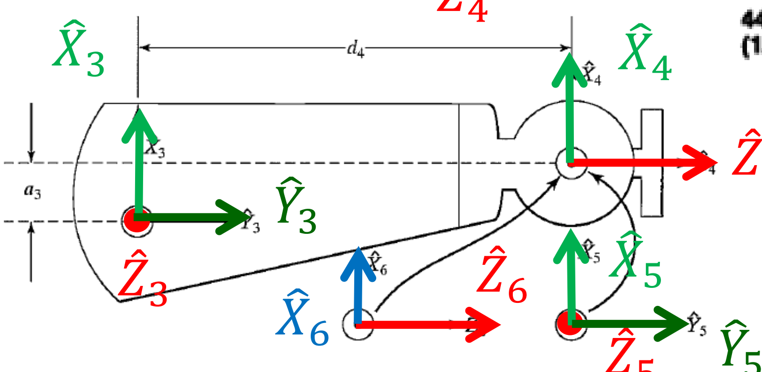
$$
{}^0_3T
$$

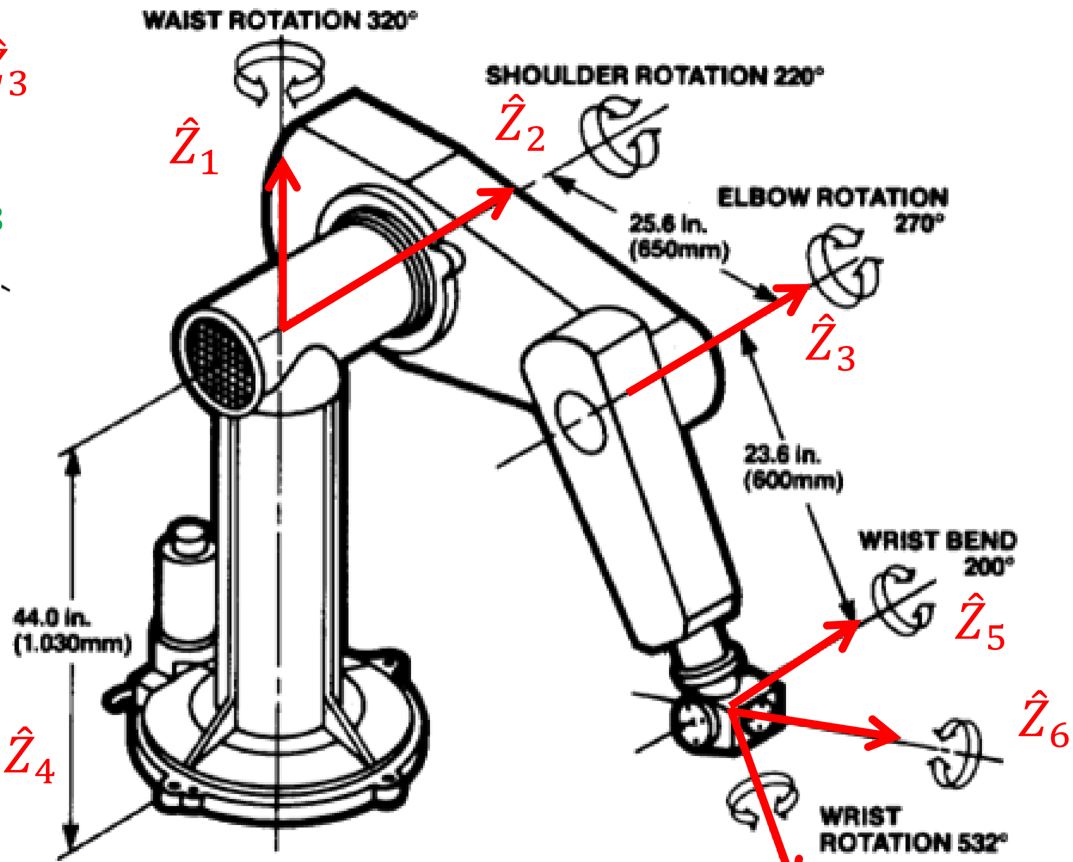
$$
\begin{pmatrix} \text{Cos}[t1+t2+t3] & -\text{Sin}[t1+t2+t3] & 0 & \text{L1 Cos}[t1]+\text{L2 Cos}[t1+t2]+\text{L3 Cos}[t1+t2+t3] \\ \text{Sin}[t1+t2+t3] & \text{Cos}[t1+t2+t3] & 0 & \text{L1 Sin}[t1]+\text{L2 Sin}[t1+t2]+\text{L3 Sin}[t1+t2+t3] \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Example: PUMA 560 -1

- Frames (Craig)





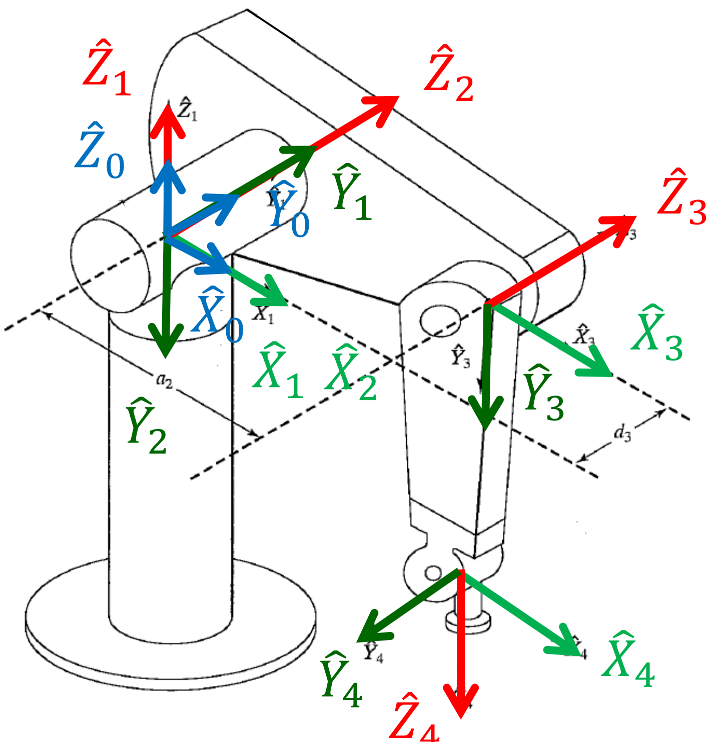


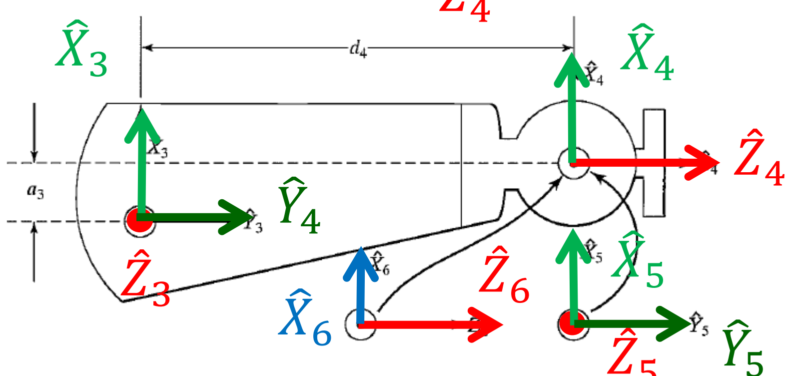
From www.iisartonline.org

John J. Craig, "Introduction to Robotics," 3rd ed., Pearson Prentice Hall, 2005, pp. 79

Example: PUMA 560 -2

- DH parameters (Craig)





i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0°	0	0	θ_1
2	-90°	0	0	θ_2
3	0°	a_2	d_3	θ_3
4	-90°	a_3	d_4	θ_4
5	90°	0	0	θ_5
6	-90°	0	0	θ_6

John J. Craig, "Introduction to Robotics," 3rd ed., Pearson Prentice Hall, 2005, pp. 79

Example: PUMA 560 -3

- Transformation matrices

$$ {}^0_1T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^3_4T = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_2 & -c\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^4_5T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^2_3T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^5_6T = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_6 & -c\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Example: PUMA 560 -4

- Combining transformation matrices -1

$$ {}^4_6T = {}^4_5T\,{}^5_6T = \begin{bmatrix} c_5c_6 & -c_5s_6 & -s_5 & 0 \\ s_6 & c_6 & 0 & 0 \\ s_5c_6 & -s_5s_6 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^3_6T = {}^3_4T\,{}^4_6T = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & -c_4s_5 & a_3 \\ s_5c_6 & -s_5s_6 & c_5 & d_4 \\ -s_4c_5c_6 - c_4s_6 & s_4c_5s_6 - c_4c_6 & s_4s_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_3T = {}^1_2T\,{}^2_3T = \begin{bmatrix} c_{23} & -s_{23} & 0 & a_2c_2 \\ 0 & 0 & 1 & d_3 \\ -s_{23} & -c_{23} & 0 & -a_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Example: PUMA 560 -5

- Combining transformation matrices -2

$$ {}^{1}_{6}T = {}^{1}_{3}T\,{}^{3}_{6}T = \begin{bmatrix} {}^{1}r_{11} & {}^{1}r_{12} & {}^{1}r_{13} & {}^{1}p_x \\ {}^{1}r_{21} & {}^{1}r_{22} & {}^{1}r_{23} & {}^{1}p_y \\ {}^{1}r_{31} & {}^{1}r_{32} & {}^{1}r_{33} & {}^{1}p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}r_{11} = c_{23}[c_4c_5c_6 - s_4s_6] - s_{23}s_5s_6 $$
$$ {}^{1}r_{21} = -s_4c_5c_6 - c_4s_6 $$
$$ {}^{1}r_{31} = -s_{23}[c_4c_5c_6 - s_4s_6] - c_{23}s_5c_6 $$
$$ {}^{1}r_{12} = -c_{23}[c_4c_5s_6 + s_4c_6] + s_{23}s_5s_6 $$
$$ {}^{1}r_{22} = s_4c_5s_6 - c_4c_6 $$
$$ {}^{1}r_{32} = s_{23}[c_4c_5s_6 + s_4c_6] + c_{23}s_5s_6 $$
$$ {}^{1}r_{13} = -c_{23}c_4s_5 - s_{23}c_5 $$
$$ {}^{1}r_{23} = s_4s_5 $$
$$ {}^{1}r_{33} = s_{23}c_4s_5 - c_{23}c_5 $$
$$ {}^{1}p_x = a_2c_2 + a_3c_{23} - d_4s_{23} $$
$$ {}^{1}p_y = d_3 $$
$$ {}^{1}p_z = -a_3s_{23} - a_2s_2 - d_4c_{23} $$

Example: PUMA 560 -6

- Combining transformation matrices -3

$$ {}^{0}_{6}T = {}^{0}_{1}T\,{}^{1}_{6}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ r_{11} = c_1[c_{23}(c_4c_5c_6 - s_4s_5) - s_{23}s_5c_5] + s_1(s_4c_5c_6 + c_4s_6) $$

$$ r_{21} = s_1[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6] - c_1(s_4c_5c_6 + c_4s_6) $$

$$ r_{31} = -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_5c_6 $$

$$ r_{12} = c_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] + s_1(c_4c_6 - s_4c_5s_6) $$

$$ r_{22} = s_1[s_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] - c_1(c_4c_6 - s_4c_5s_6) $$

$$ r_{32} = -s_{23}(-c_4c_5s_6 - s_4c_6) + c_{23}s_5s_6 $$

$$ r_{13} = -c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5 $$

$$ r_{23} = -s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5 $$

$$ r_{33} = s_{23}c_4s_5 - c_{23}c_5 $$

$$ p_x = c_1[a_2c_2 + a_3c_{23} - d_4s_{23}] - d_3s_1 $$

$$ p_y = s_1[a_2c_2 + a_3c_{23} - d_4s_{23}] + d_3c_1 $$

$$ p_z = -a_3s_{23} - a_2s_2 - d_4c_{23} $$