Chap 6: Manipulator Dynamics

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Acceleration of a Rigid Body -1

❑ Differentiation of a velocity vector V_Q

$$^B A_Q = \frac{d}{dt} \,^B V_Q = \lim_{\Delta t \to 0} \frac{^B V_Q(t + \Delta t) - \,^B V_Q(t)}{\Delta t}$$

Derivative of velocity vector $^B V_Q$ relative to frame $\{B\}$

$$^A(^B A_Q) = \,^A(\frac{d}{dt} \,^B V_Q)$$

Expressed in frame $\{A\}$

$$= {}_B^A R \,^B(^B A_Q) = {}_B^A R \,^B A_Q$$

When both frames are the same

$$a_C = \,^U A_{C\ ORG}$$

Acceleration of the origin of frame $\{C\}$ relative to the universe reference frame $\{U\}$
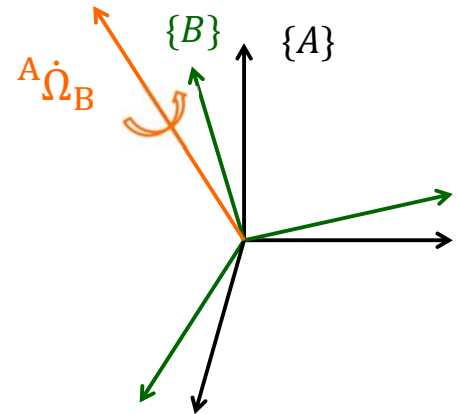
Acceleration of a Rigid Body -2

□ Angular acceleration vector $^{A}\dot{\Omega}_{B}$

$$^{A}\dot{\Omega}_{B} = \frac{d}{dt}\,^{A}\Omega_{B} = \lim_{\Delta t \to 0} \frac{^{A}\Omega_{B}(t + \Delta t) - \,^{A}\Omega_{B}(t)}{\Delta t}$$

Derivative of angular velocity of frame $\{B\}$
relative to frame $\{A\}$

$$^{C}(\,^{A}\dot{\Omega}_{B})$$

Expressed in frame $\{C\}$

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$$\dot{\omega}_{C} = \,^{U}\dot{\Omega}_{C}$$

Angular acceleration of frame $\{C\}$ relative to the universe reference frame $\{U\}$

Acceleration of a Rigid Body -3

□ Angular acceleration

$$^{A}\Omega_{C} = \,^{A}\Omega_{B} + \,^{A}_{B}R\,^{B}\Omega_{C}$$

↓ diff.

$$^{A}\dot{\Omega}_{C} = \,^{A}\dot{\Omega}_{B} + \frac{d}{dt}\,^{A}_{B}R\,^{B}\Omega_{C}$$

$$= \,^{A}\dot{\Omega}_{B} + \,^{A}_{B}R\,^{B}\dot{\Omega}_{C} + \,^{A}\Omega_{B} \times \,^{A}_{B}R\,^{B}\Omega_{C}$$
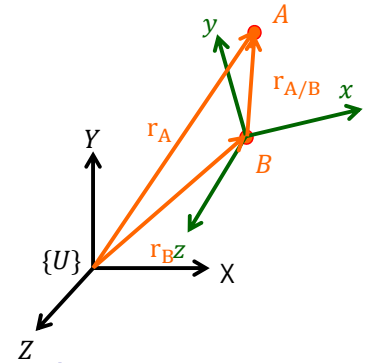
Rigid Body Motion -1

□ Freshman Dynamics

Following the materials described in Chap 5

$$\overrightarrow{v_A} = \overrightarrow{v_B} + \overrightarrow{v_{rel}} + \overrightarrow{\omega} \times \overrightarrow{r_{A/B}}$$

$$\overrightarrow{v_A} = (\dot{x}_B\hat{I} + \dot{y}_B\hat{J}) + (\dot{x}_{A/B}\hat{i} + \dot{y}_{A/B}\hat{j}) + \overrightarrow{\omega} \times (x_{A/B}\hat{I} + y_{A/B}\hat{J})$$

↓ diff.

$$\overrightarrow{a_A} = (\ddot{x}_B\hat{I} + \ddot{y}_B\hat{J})$$
$$+ (\ddot{x}_{A/B}\hat{i} + \ddot{y}_{A/B}\hat{j}) + \overrightarrow{\omega} \times (\dot{x}_{A/B}\hat{i} + \dot{y}_{A/B}\hat{j})$$
$$+ \dot{\overrightarrow{\omega}} \times (x_{A/B}\hat{I} + y_{A/B}\hat{J})$$
$$+ \overrightarrow{\omega} \times ((\dot{x}_{A/B}\hat{i} + \dot{y}_{A/B}\hat{j}) + \overrightarrow{\omega} \times (x_{A/B}\hat{I} + y_{A/B}\hat{J}))$$

Rigid Body Motion -2

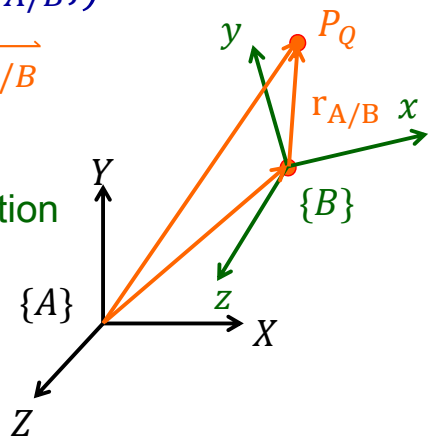$$\overrightarrow{a_A} = (\ddot{x}_B\hat{I} + \ddot{y}_B\hat{J})$$
$$+ \dot{\overrightarrow{\omega}} \times (x_{A/B}\hat{I} + y_{A/B}\hat{J}) + \overrightarrow{\omega} \times (\overrightarrow{\omega} \times (x_{A/B}\hat{I} + y_{A/B}\hat{J}))$$
$$+ 2\overrightarrow{\omega} \times (\dot{x}_{A/B}\hat{i} + \dot{y}_{A/B}\hat{j}) + (\ddot{x}_{A/B}\hat{i} + \ddot{y}_{A/B}\hat{j})$$

➡ $$\overrightarrow{a_A} = \overrightarrow{a_B} + \dot{\overrightarrow{\omega}} \times \overrightarrow{r_{A/B}} + \overrightarrow{\omega} \times \overrightarrow{\omega} \times \overrightarrow{r_{A/B}}$$
$$+ \underline{2\overrightarrow{\omega} \times \overrightarrow{v_{rel}}} + \underline{\overrightarrow{a_{rel}}}$$

Coriolis acceleration   "relative" acceleration

□ Thus

$$^A A_Q = {}^A A_{B\,ORG}$$
$$+ {}^A\dot{\Omega}_B \times {}^A_B R\, {}^B P_Q + {}^A\Omega_B \times \left({}^A\Omega_B \times {}^A_B R\, {}^B P_Q \right)$$
$$+ 2\, {}^A\Omega_B \times {}^A_B R\, {}^B V_Q + {}^A_B R\, {}^B A_Q$$

Mass Distribution -1

- Inertia tensor relative to frame $\{A\}$

$$^AI = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$
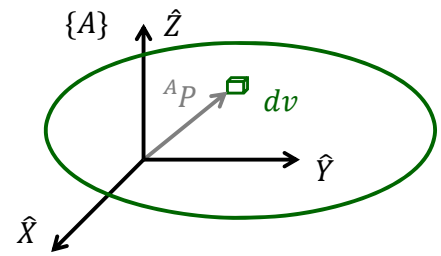
Mass moment of inertia >0

$$I_{xx} = \iiint_V (y^2 + z^2)\rho \, dv$$

$$I_{yy} = \iiint_V (x^2 + z^2)\rho \, dv$$

$$I_{zz} = \iiint_V (x^2 + y^2)\rho \, dv$$

Mass product of inertia

$$I_{xy} = \iiint_V xy\rho \, dv$$

$$I_{xz} = \iiint_V xz\rho \, dv$$

$$I_{yz} = \iiint_V yz\rho \, dv$$

Mass Distribution -2

- Inertia tensor

  - Constant real symmetric matrix (orthogonally diagonalizable)

    its eigendecomposition (i.e., $M = V\Lambda V^{-1}$)

    $$^AI = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} = R \begin{bmatrix} I_{XX} & 0 & 0 \\ 0 & I_{YY} & 0 \\ 0 & 0 & I_{ZZ} \end{bmatrix} R^T$$

    Rotation matrix,
    revealing directions of the principal axes

    principal moment of inertia

  - $I_{xx} + I_{yy} + I_{zz} = trace(^AI) = constant$

    - Trace is invariant under a similarity transformation

  - If xy-plane is plane of symmetry, then $I_{xz} = I_{yz} = 0$

Mass Distribution -3

- ❑ Parallel-axis Theorem

  - ◆ Computing how the inertia tensor changes under translations of the reference coordinate system

平行軸
定理

$$^A I_{zz} = {}^C I_{zz} + m(x_c{}^2 + y_c{}^2)$$

$$^A I_{xy} = {}^C I_{xy} - m x_c y_c$$

C: at COM of the body

A: arbitrary frame

Vector-matrix form

$$^A I = {}^C I + m[P_c{}^T P_c I_3 - P_c P_c{}^T]$$

$$P_c = \begin{bmatrix} x_c & y_c & z_c \end{bmatrix}^T$$

COM relative to $\{A\}$

Newton's Equation and Euler's Equation

- ❑ Newton's equation

牛

$$F = \frac{d}{dt}(m v_c) = m \dot{v}_c$$

拉

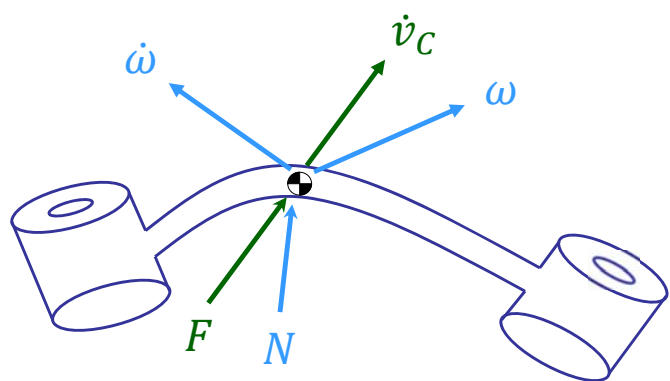- ❑ Euler's equation

方

$$N = \frac{d}{dt}(I\omega)$$

Even if using inertial frame, it can change during motion

程
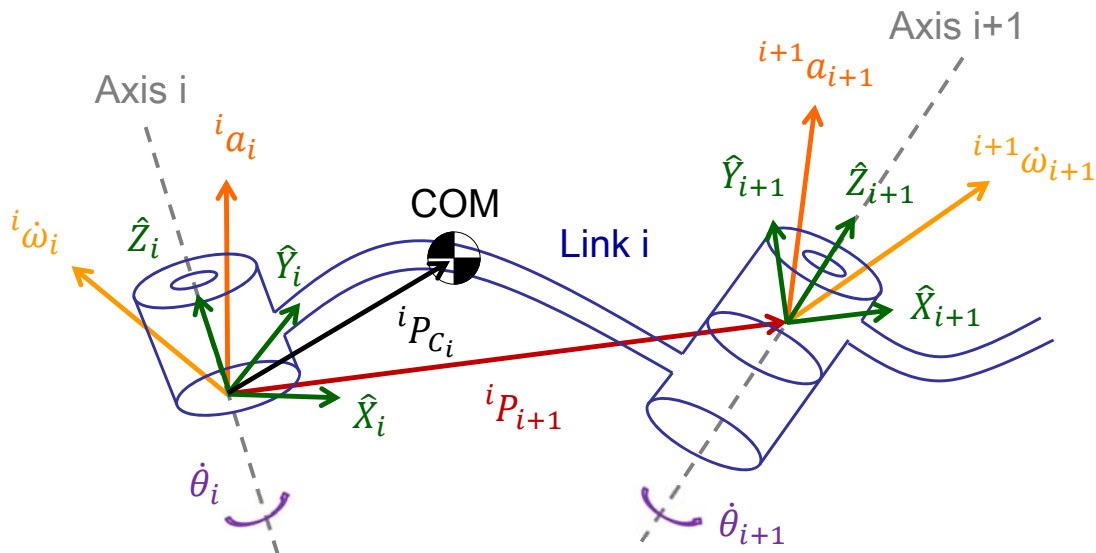
$$N = {}^C I \dot{\omega} + \omega \times {}^C I \omega$$

C: body frame, whose origin is located at COM

$^C I$: constant matrix

Acceleration "Propagation" from Link to Link -1

□ Strategy: Represent linear and angular accelerations of link i in frame $\{i\}$, and find their relationship to those of neighboring links

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Acceleration "Propagation" from Link to Link -2

□ Rotational Joint (Link i+1)

   ◆ Angular acceleration propagation

   In Chap 5

   $$^{i}\omega_{i+1} = {}^{i}\omega_i + {}_{i+1}^{i}R\dot{\theta}_{i+1}{}^{i+1}\hat{Z}_{i+1}$$

   $$\Big\downarrow \text{diff.} \qquad \dot{\theta}_{i+1}{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}$$

   $$^{i}\dot{\omega}_{i+1} = {}^{i}\dot{\omega}_i + {}^{i}\omega_i \times {}_{i+1}^{i}R\dot{\theta}_{i+1}{}^{i+1}\hat{Z}_{i+1} + {}_{i+1}^{i}R\ddot{\theta}_{i+1}{}^{i+1}\hat{Z}_{i+1}$$

   $$\Big\downarrow {}^{i+1}_{i}R$$

   $$^{i+i}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}{}^{i+1}\hat{Z}_{i+1}$$

Acceleration "Propagation" from Link to Link -3

- Linear acceleration propagation

$$^{i}a_{i+1} = {}^{i}a_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1} \right)$$

$$\downarrow {}^{i+1}_{i}R$$

$$^{i+1}a_{i+1} = {}^{i+1}_{i}R\left({}^{i}a_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1} \right)\right)$$

Acceleration "Propagation" from Link to Link -4

- Prismatic joint (Link i+1)

  - Angular acceleration propagation

  $$^{i}\dot{\omega}_{i+1} = {}^{i}\dot{\omega}_i \xrightarrow{{}^{i+1}_{i}R} {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R \; {}^{i}\dot{\omega}_i$$

  - Linear acceleration propagation

  $$^{i}a_{i+1} = {}^{i}a_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1} \right)$$
  $$+ 2\,{}^{i}\omega_i \times {}^{i}_{i+1}R\dot{d}_{i+1}{}^{i+1}\hat{Z}_{i+1} + {}^{i}_{i+1}R\ddot{d}_{i+1}{}^{i+1}\hat{Z}_{i+1}$$

  $$\ddot{d}_{i+1}{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \ddot{d}_{i+1} \end{bmatrix}$$

  $$\downarrow {}^{i+1}_{i}R$$

  $$^{i+1}a_{i+1} = {}^{i+1}_{i}R\left({}^{i}a_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1} \right)\right)$$
  $$+ 2\,{}^{i+1}\omega_{i+1} \times \dot{d}_{i+1}{}^{i+1}\hat{Z}_{i+1} + \ddot{d}_{i+1}{}^{i+1}\hat{Z}_{i+1}$$

□ COM

线加速 角加速度

$$^{i}a_{C_i} = {}^{i}a_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{C_i} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{C_i}\right)$$
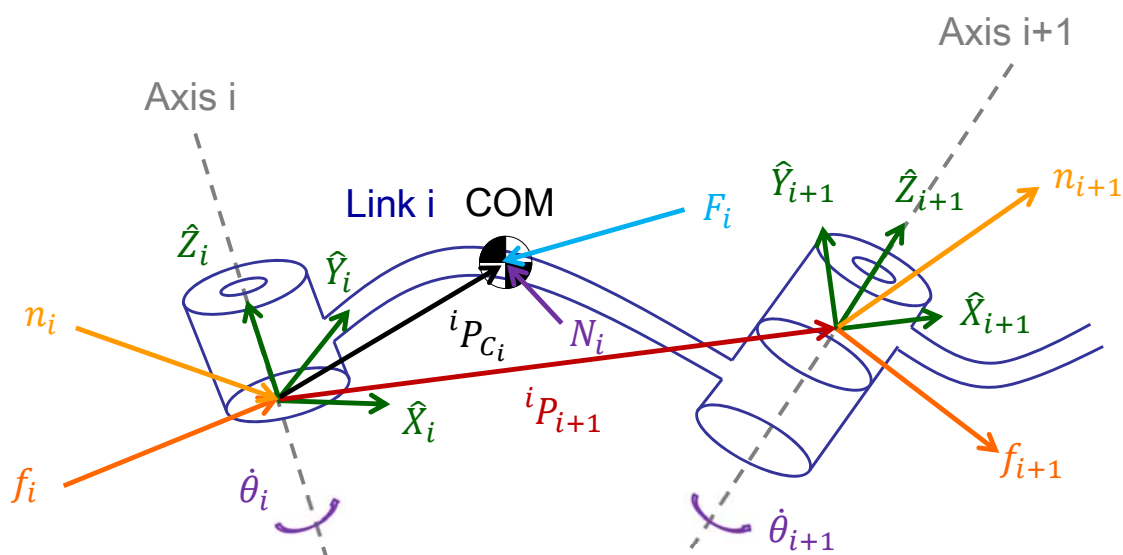
C_i: COM of the ith link

□ Inertia force and torque acting at the COM

$$F_i = ma_{C_i}$$

$$N_i = {}^{C_i}I\dot{\omega}_i + \omega_i \times {}^{C_i}I\omega_i$$

Force Propagation from Link to Link -2

❑

$$^i f_i = {}^i_{i+1}R \, {}^{i+1}f_{i+1} + {}^i F_i$$

$$^i n_i = {}^i_{i+1}R \, {}^{i+1}n_{i+1} + {}^i N_i + {}^i P_{C_i} \times {}^i F_i + {}^i P_{i+1} \times {}^i_{i+1}R \, {}^{i+1}f_{i+1}$$
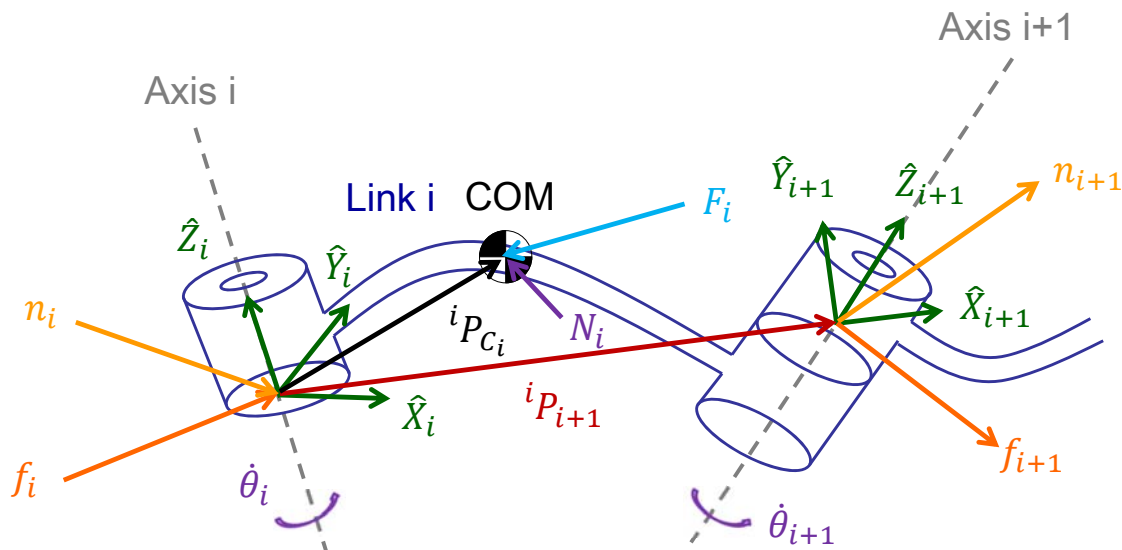
Force Propagation from Link to Link -3

❑ **Thus**

  ◆ Revolute joint

$$\tau_i = {}^i n_i^T \, {}^i\widehat{Z}_i$$

  ◆ Prismatic joint

$$\tau_i = {}^i f_i^T \, {}^i\widehat{Z}_i$$

❑ **Comments**

  ◆ Inclusion of gravity force: $^0 a_0 = g = 9.81 \, m/s$

  ◆ A manipulator moving in free space: $^{N+1}f_{N+1} = 0 \quad {}^{N+1}n_{N+1} = 0$

Iterative Newton-Euler Dynamic Formulation

- ❑ Outward iterations
  - ◆ Link 1 to link n
  - ◆ Velocities and accelerations

- ❑ Inward iterations
  - ◆ Link n to link 1
  - ◆ Forces and torques

- ❑ Revolute joint vs. prismatic joint: Choose correct equations

- ❑ General structure, can be applied to any manipulator

- ❑ Easy for numerical computation

Example: A RR Manipulator -1

- ❑ Conditions:

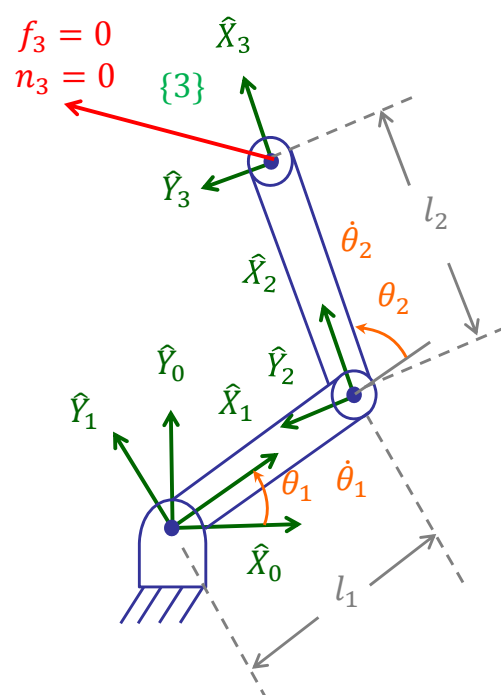$$^1P_{C_1} = l_1\hat{X}_1 \qquad ^{C_1}I_1 = 0$$

$$^2P_{C_2} = l_2\hat{X}_2 \qquad ^{C_2}I_2 = 0$$

$$m_1, m_2$$

$$\omega_0 = 0 \qquad ^0\dot{v}_0 = g\hat{Y}_0$$

$$\dot{\omega}_0 = 0$$

$$_{i+1}^{i}R = \begin{bmatrix} c_{i+1} & -s_{i+1} & 0 \\ s_{i+1} & c_{i+1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□ Velocity and acceleration propagations

$$^1\omega_1 = {}_0^1R\ {}^0\omega_0 + \dot\theta_1\ {}^1\hat{Z}_1 = \dot\theta_1\ {}^1\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix}$$

$$^1\dot\omega_1 = {}_0^1R\ {}^0\dot\omega_0 + {}_0^1R\ {}^0\omega_0 \times \dot\theta_1\ {}^1\hat{Z}_1 + \ddot\theta_1\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot\theta_1 \end{bmatrix}$$

$$^1a_1 = {}_0^1R(\ {}^0a_0 + \ {}^0\dot\omega_0 \times \ {}^0P_1 + \ {}^0\omega_0 \times (\ {}^0\omega_0 \times \ {}^0P_1)) = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}$$

$$^1a_{C_1} = \ {}^1a_1 + \ {}^1\dot\omega_1 \times \ {}^1P_{C_1} + \ {}^1\omega_1 \times (\ {}^1\omega_1 \times \ {}^1P_{C_1}) \qquad = \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ l_1\ddot\theta_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -l_1\dot\theta_1^2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1\dot\theta_1^2 + gs_1 \\ l_1\ddot\theta_1 + gc_1 \\ 0 \end{bmatrix}$$

$$^1F_1 = m\ {}^1a_{C_1} \begin{bmatrix} -m_1l_1\dot\theta_1^2 + m_1gs_1 \\ m_1l_1\ddot\theta_1 + m_1gc_1 \\ 0 \end{bmatrix}$$

$$^1N_1 = {}^{C_1}I\ {}^1\dot\omega_1 + \ {}^1\omega_1 \times {}^{C_1}I\ {}^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$^2\omega_2 = {}_1^2R\ {}^1\omega_1 + \dot\theta_2\ {}^2\hat{Z}_2 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 + \dot\theta_2 \end{bmatrix}$$

$$^2\dot\omega_2 = {}_1^2R\ {}^1\dot\omega_1 + {}_1^2R\ {}^1\omega_1 \times \dot\theta_2\ {}^2\hat{Z}_2 + \ddot\theta_2\ {}^2\hat{Z}_2 = \begin{bmatrix} 0 \\ 0 \\ \ddot\theta_1 + \ddot\theta_2 \end{bmatrix}$$

$$^2a_2 = {}_1^2R(\ {}^1a_1 + \ {}^1\dot\omega_1 \times \ {}^1P_2 + \ {}^1\omega_1 \times (\ {}^1\omega_1 \times \ {}^1P_2)) =$$

$$= \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -l_1\dot\theta_1^2 + gs_1 \\ l_1\ddot\theta_1 + gc_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1\ddot\theta_1 s_2 - l_1\dot\theta_1^2 c_2 + gs_{12} \\ l_1\ddot\theta_1 c_2 + l_1\dot\theta_1^2 s_2 + gc_{12} \\ 0 \end{bmatrix}$$

$${}^2 a_{C_2} = {}^2 a_2 + {}^2 \dot{\omega}_2 \times {}^2 P_{C_2} + {}^2 \omega_2 \times \left({}^2 \omega_2 \times {}^2 P_{C_2} \right)$$

$$= \begin{bmatrix} l_1 \ddot{\theta}_1 s_2 - l_1 \dot{\theta}_1{}^2 c_2 + g s_{12} \\ l_1 \ddot{\theta}_1 c_2 + l_1 \dot{\theta}_1{}^2 s_2 + g c_{12} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ l_2 (\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix} + \begin{bmatrix} -l_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ 0 \\ 0 \end{bmatrix}$$

$${}^2 F_2 = m \, {}^2 a_{C_2} = \begin{bmatrix} m_2 l_1 \ddot{\theta}_1 s_2 - m_2 l_1 \dot{\theta}_1{}^2 c_2 + m_2 g s_{12} - m_2 l_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ m_2 l_1 \ddot{\theta}_1 c_2 + m_2 l_1 \dot{\theta}_1{}^2 s_2 + m_2 g c_{12} + m_2 l_2 (\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix}$$

$${}^2 N_2 = {}^{C_2}I \, {}^2 \dot{\omega}_2 + {}^2 \omega_2 \times {}^{C_2}I \, {}^2 \omega_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

□ Force and torque propagations

$${}^2 f_2 = {}^2_3 R \, {}^3 f_3 + {}^2 F_2 = {}^2 F_2$$

$${}^2 n_2 = {}^2_3 R \, {}^3 n_3 + {}^2 N_2 + {}^2 P_{C_2} \times {}^2 F_2 + {}^2 P_3 \times {}^2_3 R \, {}^3 f_3$$

$$= \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1{}^2 + m_2 l_2 g c_{12} + m_2 l_2{}^2 (\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix}$$

$${}^1 f_1 = {}^1_2 R \, {}^2 f_2 + {}^1 F_1$$

$$= \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_2 l_1 s_2 \ddot{\theta}_1 - m_2 l_1 c_2 \dot{\theta}_1{}^2 + m_2 g s_{12} - m_2 l_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ m_2 l_1 c_2 \ddot{\theta}_1 + m_2 l_1 s_2 \dot{\theta}_1{}^2 + m_2 g c_{12} + m_2 l_2 (\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} -m_1 l_1 \dot{\theta}_1{}^2 + m_1 g s_1 \\ m_1 l_1 \ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix}$$

$$^1n_1 = \tfrac{1}{2}R\ ^2n_2 +\ ^1N_1 +\ ^1P_{C_1} \times\ ^1F_1 +\ ^1P_2 \times \tfrac{1}{2}R\ ^2f_2$$

$$= \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 c_2 \ddot\theta_1 + m_2 l_1 l_2 s_2 \dot\theta_1^{\ 2} + m_2 l_2 g c_{12} + m_2 l_2^{\ 2}(\ddot\theta_1 + \ddot\theta_2) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ m_1 l_1^{\ 2}\ddot\theta_1 + m_1 l_1 g c_1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ m_2 l_1^{\ 2}\ddot\theta_1 - m_2 l_1 l_2 s_2(\dot\theta_1 + \dot\theta_2)^2 + m_2 l_1 g s_2 s_{12} + m_2 l_1 l_2 c_2(\ddot\theta_1 + \ddot\theta_2) + m_2 l_1 g c_2 c_{12} \end{bmatrix}$$

- Joint torques

$$\tau_1 =\ ^1n_1^T\ ^1\widehat{Z_1}$$
$$= m_2 l_2^{\ 2}(\ddot\theta_1 + \ddot\theta_2) + m_2 l_1 l_2 c_2(2\ddot\theta_1 + \ddot\theta_2) + (m_1 + m_2)l_1^{\ 2}\ddot\theta_1$$
$$\qquad - m_2 l_1 l_2 s_2 \dot\theta_2^{\ 2} - 2m_2 l_1 l_2 s_2 \dot\theta_1 \dot\theta_2 + m_2 l_2 g c_{12} + (m_1 + m_2)l_1 g c_1$$

$$\tau_2 =\ ^2n_2^T\ ^2\widehat{Z_2}$$
$$= m_2 l_1 l_2 c_2 \ddot\theta_1 + m_2 l_1 l_2 s_2 \dot\theta_1^{\ 2} + m_2 l_2 g c_{12} + m_2 l_2^{\ 2}(\ddot\theta_1 + \ddot\theta_2)$$

The Structure of Dynamic Equations -1

□ The state-space equation

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta)$$

$n \times 1$ $n \times n$ $n \times 1$ $n \times 1$ $n \times 1$

       Mass   Centrifugal   gravity

       matrix   Coriolis

□ Revisit the RR manipulator

$$M(\Theta) = \begin{bmatrix} l_2{}^2 m_2 + 2l_1 l_2 m_2 c_2 + l_1{}^2(m_1 + m_2) & l_2{}^2 m_2 + l_1 l_2 m_2 c_2 \\ l_2{}^2 m_2 + l_1 l_2 m_2 c_2 & l_2{}^2 m_2 \end{bmatrix}$$

$$V(\Theta,\dot{\Theta}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2{}^2 - 2m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1{}^2 \end{bmatrix}$$

$$G(\Theta) = \begin{bmatrix} m_2 l_2 g c_{12} + (m_1 + m_2)l_1 g c_1 \\ m_2 l_2 g c_{12} \end{bmatrix}$$

The Structure of Dynamic Equations -2

□ The configuration-space equation

$$\tau = M(\Theta)\ddot{\Theta} + B(\Theta)[\dot{\Theta}\dot{\Theta}] + C(\Theta)[\dot{\Theta}^2] + G(\Theta)$$

$n \times 1$ $n \times n$ $n \times \dfrac{n(n-1)}{2}$ $n \times n$ $n \times 1$

       Mass   Coriolis   Centrifugal   gravity

       matrix

          $n \times 1$

$$[\dot{\Theta}\dot{\Theta}] = [\dot{\theta}_1 \dot{\theta}_2 \quad \dot{\theta}_1 \dot{\theta}_3 \quad ... \quad \dot{\theta}_{n-1} \dot{\theta}_n]^T \qquad [\dot{\Theta}^2] = [\dot{\theta}_1{}^2 \quad \dot{\theta}_2{}^2 \quad ... \quad \dot{\theta}_n{}^2]^T$$

$\dfrac{n(n-1)}{2} \times 1$ $n \times 1$

The Structure of Dynamic Equations -3

❑ Revisit the RR manipulator

$$V(\Theta, \dot{\Theta}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2{}^2 - 2m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1{}^2 \end{bmatrix} = B(\Theta)[\dot{\Theta}\dot{\Theta}] + C(\Theta)[\dot{\Theta}^2]$$

$$B(\Theta) = \begin{bmatrix} -2m_2 l_1 l_2 s_2 \\ 0 \end{bmatrix} \qquad [\dot{\Theta}\dot{\Theta}] = [\dot{\theta}_1 \dot{\theta}_2]$$

$$C(\Theta) = \begin{bmatrix} 0 & -m_2 l_1 l_2 s_2 \\ m_2 l_1 l_2 s_2 & 0 \end{bmatrix} \qquad [\dot{\Theta}^2] = \begin{bmatrix} \dot{\theta}_1{}^2 \\ \dot{\theta}_2{}^2 \end{bmatrix}$$

Lagrangian Formulation of Manipulator Dynamics -1

❑ Newton-Euler: Force-moment-based analysis

  Lagrange: Energy-based analysis

❑ Of course, for a system, both methods should yield the same equations of motion

Lagrangian Formulation of Manipulator Dynamics -2

- Kinetic energy

$$k_i = \frac{1}{2} m_i v_{c_i}^T v_{c_i} + \frac{1}{2}\ {}^i\omega_i^{T}\ {}^{C_i}I_i\ {}^i\omega_i$$

$$k = \sum_{i=1}^{n} k_i \qquad k = k\left(\Theta, \dot{\Theta}\right) = \frac{1}{2}\dot{\Theta}^{T}\mathrm{M}(\Theta)\dot{\Theta}$$

- Potential energy

$$u_i = -m_i\ {}^0g^T\ {}^0P_{C_i} + \underline{u_{ref_i}}$$

Shift the zero reference height

$$u = \sum_{i=1}^{n} u_i \qquad u = u(\Theta)$$

Lagrangian Formulation of Manipulator Dynamics -3

- Lagrangian

$$\mathcal{L}\left(\Theta, \dot{\Theta}\right) = k\left(\Theta, \dot{\Theta}\right) - u(\Theta)$$
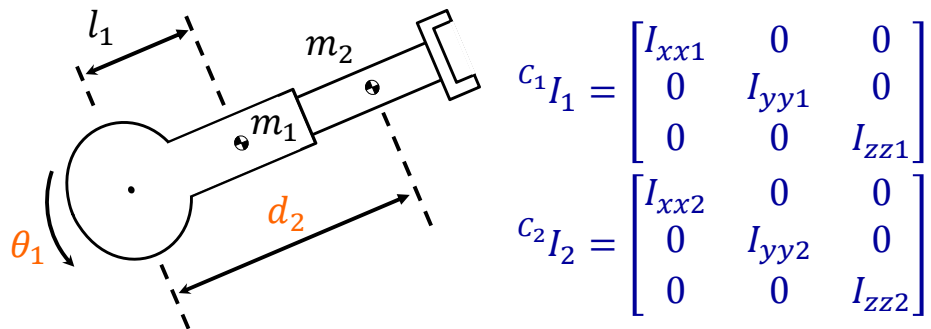
- Equation of motion for the manipulator

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Theta}} - \frac{\partial \mathcal{L}}{\partial \Theta} = \tau$$

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{\Theta}} - \frac{\partial k}{\partial \Theta} + \frac{\partial u}{\partial \Theta} = \tau$$

Example: An RP Manipulator -1

$$^{C_1}I_1 = \begin{bmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix}$$

$$^{C_2}I_2 = \begin{bmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{bmatrix}$$

- Kinetic energy

$$k_1 = \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}I_{zz1}\dot{\theta}_1^2$$

$$k_2 = \frac{1}{2}m_2(d_2^2\dot{\theta}_1^2 + \dot{d}_2^2) + \frac{1}{2}I_{zz2}\dot{\theta}_1^2$$

$$k(\Theta, \dot{\Theta}) = \frac{1}{2}(m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2)\dot{\theta}_1^2 + \frac{1}{2}m_2\dot{d}_2^2$$

Example: An RP Manipulator -2

- Potential energy

$$u_1 = m_1 g l_1 sin\theta_1 + m_1 g l_1$$

$$u_2 = m_2 g d_2 sin\theta_1 + m_2 g d_{2max}$$

$$u(\Theta) = (m_1 l_1 + m_2 d_2)g sin\theta_1 + \underline{m_1 g l_1 + m_2 g d_{2max}}$$

Shift the zero reference height

- Lagrangian

$$\frac{\partial k}{\partial \dot{\Theta}} = \begin{bmatrix} (m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2)\dot{\theta}_1 \\ m_2 \dot{d}_2 \end{bmatrix}$$

$$\frac{\partial k}{\partial \Theta} = \begin{bmatrix} 0 \\ m_2 d_2\,\dot{\theta}_1^2 \end{bmatrix}$$

$$\frac{\partial u}{\partial \Theta} = \begin{bmatrix} (m_1 l_1 + m_2 d_2)g cos\theta_1 \\ m_2 g sin\,\theta_1 \end{bmatrix}$$

Example: An RP Manipulator -3

- Equations of motion

$$\tau_1 = (m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2)\ddot{\theta}_1 + 2m_2 d_2 \dot{\theta}_1 \dot{d}_2$$
$$+ (m_1 l_1 + m_2 d_2)gcos\theta_1$$

$$\tau_2 = m_2 \ddot{d}_2 - m_2 d_2 \dot{\theta}_1^2 + m_2 gsin\theta_1$$

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state-space representation

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

$$M(\Theta) = \begin{bmatrix} m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$V(\Theta, \dot{\Theta}) = \begin{bmatrix} 2m_2 d_2 \dot{\theta}_1 \dot{d}_2 \\ -m_2 d_2 \dot{\theta}_1^2 \end{bmatrix}$$

$$G(\Theta) = \begin{bmatrix} (m_1 l_1 + m_2 d_2)gcos\theta_1 \\ m_2 gsin\theta_1 \end{bmatrix}$$

Manipulator Dynamics in Cartesian Space -1

- Dynamic equations

  - In joint space

    $$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

  - In Cartesian space

    $$F = M_x(\Theta)\ddot{X} + V_x(\Theta, \dot{\Theta}) + G_x(\Theta)$$

- Formulation

$$\tau = J^T(\Theta)F$$

$$F = J^{-T}\tau = J^{-T}M(\Theta)\ddot{\Theta} + J^{-T}V(\Theta, \dot{\Theta}) + J^{-T}G(\Theta)$$

$$\dot{X} = J\dot{\Theta} \qquad \ddot{X} = \dot{J}\dot{\Theta} + J\ddot{\Theta} \qquad \ddot{\Theta} = J^{-1}\ddot{X} - J^{-1}\dot{J}\dot{\Theta}$$

Manipulator Dynamics in Cartesian Space -2

$$F = J^{-T}M(\Theta)J^{-1}\ddot{X} - J^{-T}M(\Theta)J^{-1}\dot{J}\dot{\Theta} + J^{-T}V(\Theta,\dot{\Theta}) + J^{-T}G(\Theta)$$
$$= M_x(\Theta)\ddot{X} + V_x(\Theta,\dot{\Theta}) + G_x(\Theta)$$

$$M_x(\Theta) = J^{-T}(\Theta)M(\Theta)J^{-1}(\Theta)$$

$$V_x(\Theta,\dot{\Theta}) = J^{-T}(\Theta)\left(V(\Theta,\dot{\Theta}) - M(\Theta)J^{-1}(\Theta)\dot{J}(\Theta)\dot{\Theta}\right)$$

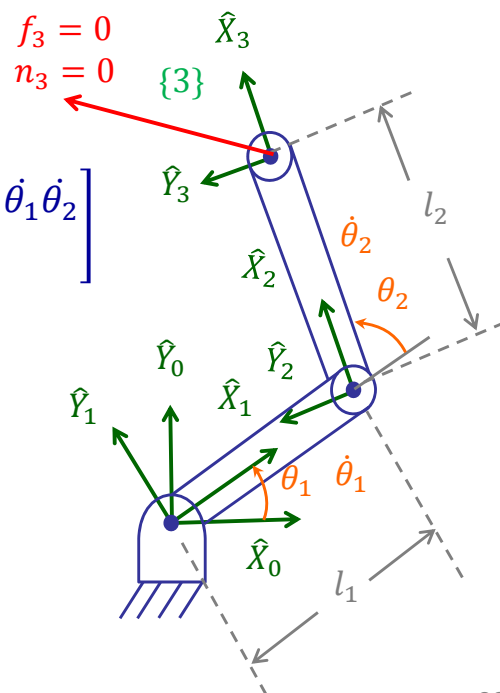$$G_x(\Theta) = J^{-T}(\Theta)G(\Theta)$$

Revisit Example: A RR Manipulator -1

◻ In joint space

$$M(\Theta) = \begin{bmatrix} l_2{}^2 m_2 + 2l_1 l_2 m_2 c_2 + l_1{}^2(m_1 + m_2) & l_2{}^2 m_2 + l_1 l_2 m_2 c_2 \\ l_2{}^2 m_2 + l_1 l_2 m_2 c_2 & l_2{}^2 m_2 \end{bmatrix}$$

$$V(\Theta,\dot{\Theta}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2{}^2 - 2m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1{}^2 \end{bmatrix}$$

$$G(\Theta) = \begin{bmatrix} m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{bmatrix}$$

❑ Jacobian

$$J(\Theta) = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix} \quad J^{-1}(\Theta) = \frac{1}{l_1 l_2 s_2} \begin{bmatrix} l_2 & 0 \\ -l_1 c_2 - l_2 & l_1 s_2 \end{bmatrix}$$

$$\dot{J}(\Theta) = \begin{bmatrix} l_1 c_2 \dot{\theta}_2 & 0 \\ -l_1 s_2 \dot{\theta}_2 & 0 \end{bmatrix}$$

❑ In Cartesian space

$$M_x(\Theta) = J^{-T}(\Theta)M(\Theta)J^{-1}(\Theta) = \begin{bmatrix} m_2 + \frac{m_1}{s_2{}^2} & 0 \\ 0 & m_2 \end{bmatrix}$$

$$V_x(\Theta, \dot{\Theta}) = J^{-T}(\Theta)\left(V(\Theta, \dot{\Theta}) - M(\Theta)J^{-1}(\Theta)\dot{J}(\Theta)\dot{\Theta}\right)$$

$$= \begin{bmatrix} -(m_2 l_1 c_2 + m_2 l_2)\dot{\theta}_1{}^2 - m_2 l_2 \dot{\theta}_2{}^2 - (2m_2 l_2 + m_2 l_1 c_2 + m_1 l_1 \frac{c_2}{s_2{}^2})\dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 s_2 \dot{\theta}_1{}^2 + m_2 l_1 s_2 \dot{\theta}_1 \dot{\theta}_2 \end{bmatrix}$$

$$G_x(\Theta) = J^{-T}(\Theta)G(\Theta) = \begin{bmatrix} m_1 g \frac{c_1}{s_2} + m_2 g s_{12} \\ m_2 g c_{12} \end{bmatrix}$$

❑ In Cartesian space

$$\tau = J^T(\Theta)F = J^T(\Theta)\left(M_x(\Theta)\ddot{X} + V_x(\Theta, \dot{\Theta}) + G_x(\Theta)\right)$$

$$\tau = J^T(\Theta)M_x(\Theta)\ddot{X} + B_x(\Theta)[\dot{\Theta}\dot{\Theta}] + C_x(\Theta)[\dot{\theta}^2] + G(\Theta)$$

拆解

$$\Rightarrow \quad J^T(\Theta)V_x(\Theta, \dot{\Theta}) = B_x(\Theta)[\dot{\Theta}\dot{\Theta}] + C_x(\Theta)[\dot{\theta}^2]$$

Revisit Example: A RR Manipulator

$$J^T(\Theta)V_x(\Theta,\dot{\Theta}) = B_x(\Theta)[\dot{\Theta}\dot{\Theta}] + C_x(\Theta)[\dot{\theta}^2]$$

$$= \begin{bmatrix} l_1 s_2 & l_1 c_2 + l_2 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} -(m_2 l_1 c_2 + m_2 l_2)\dot{\theta}_1^2 - m_2 l_2 \dot{\theta}_2^2 - (2m_2 l_2 + m_2 l_1 c_2 + m_1 l_1 \frac{c_2}{s_2^2})\dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 s_2 \dot{\theta}_1^2 + l_1 m_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \end{bmatrix}$$

$$B_x(\Theta) = \begin{bmatrix} -m_1 l_1^2 \dfrac{c_2}{s_2} - m_2 l_1 l_2 s_2 \\ m_2 l_1 l_2 s_2 \end{bmatrix}$$

$$C_x(\Theta) = \begin{bmatrix} 0 & -m_2 l_1 l_2 s_2 \\ m_2 l_1 l_2 s_2 & 0 \end{bmatrix}$$

Friction

❑ Viscous friction

$$\tau_{friction} = c\dot{\theta}$$

❑ Coulomb friction

$$\tau_{friction} = c\,sgn\dot{\theta}$$

$$\dot{\theta} = 0,\ c = \text{"static coefficient"}$$

$$\dot{\theta} \neq 0,\ c = \text{"dynamic coefficient"}$$

The End

- Questions?