Stability Analysis

A basic car-following model can be represented by a set of ordinary differential equations (ODEs):

$$\begin{aligned}\dot{x}_n(t) &= v_n(t)\\ \dot{v}_n(t) &= f(s_n, v_n, \Delta v_n)_t\end{aligned}$$

Δv_n is the speed difference to the leading vehicle.

In the literature, gap variation $y_n(t)$ and speed variation $u_n(t)$ with respect to the equilibrium state are frequently used.

$$\begin{aligned}y_n(t) &= s_n(t) - s_e\\ u_n(t) &= v_n(t) - v_e\end{aligned}$$

The analysis of stability is to evaluate the evolution of $y_n(t)$ or $u_n(t)$ with respect to time t(local stability) or vehicle n (string stability).

local stability

Three major methods: characteristic equation based method, Laplace transform based method, Lyapunov stability criterion.

The characteristic equation based method

Taking a multivariate first order Taylor expansion around the equilibrium point:

$$\dot{\mathrm{X}}(t) \approx \mathrm{F}\left(\mathrm{x_e}\right) + \left.\frac{\partial f}{\partial \mathrm{x}}\right|_{\mathrm{x}_e} (\mathrm{x} - \mathrm{x_e}) = \left.\frac{\partial f}{\partial \mathrm{x}}\right|_{\mathrm{x}_e} (\mathrm{x} - \mathrm{x_e})$$

The linearized CF model can be written as:

$$\dot{v}_n(t) \approx \left(f_s y_n + f_v u_n + f_{\Delta v}\left(u_{n-1} - u_n\right)\right)_t$$

The above ODE system can be rewritten with the gap variation $y_n(t)$ and the speed variation $u_n(t)$

$$\begin{aligned}\dot{y}_n(t) &= \dot{s}_n(t) = \dot{x}_{n-1}(t) - \dot{x}_n(t) = v_{n-1}(t) - v_n(t) = u_{n-1}(t) - u_n(t)\\ \dot{u}_n(t) &= \dot{v}_n(t) = (f_s y_n + (f_v - f_{\Delta v}) u_n + f_{\Delta v} u_{n-1})_t\end{aligned}$$

For the local stability, the leading vehicle could be in the equilibrium state while there is an initial perturbation of the follower,　that is $u_{n-1}(t) = 0$.

$$\begin{aligned}\dot{y}_n(t) &= -u_n(t)\\ \dot{u}_n(t) &= (f_s y_n + (f_v - f_{\Delta v}) u_n)_t\end{aligned}$$

Using the results above, the CF model is locally stable if and only if the eigenvalues of the coefficient matrix $\mathrm{J}_e = \begin{bmatrix} 0 & -1 \\ f_S & f_v - f_{\Delta v} \end{bmatrix}$ have negative real parts.

The solutions are:

$$\lambda_{\pm} = \frac{(f_v - f_{\Delta v}) \pm \sqrt{(f_v - f_{\Delta v})^2 - 4 f_s}}{2}$$

When the real parts of the two solutions are both negative, the CF model is locally stable.

Routh-Hurwitz stability criterion: the local stability criterion of CF models is

$$(f_v - f_{\Delta v}) < 0 \text{ and } f_s > 0$$

The Laplace transform based method

Take Laplace transform $U(s) = \mathscr{L}(u(t)), Y(s) = \mathscr{L}(y(t))$

$$Y(s) = \mathscr{L}\left(\int (u_{n-1} - u_n)dt\right) = \frac{1}{s}[\mathscr{L}(u_{n-1}(t)) - \mathscr{L}(u_n(t))] = \frac{1}{s}[U_{n-1}(s) - U_n(s)]$$
$$sU_n(s) = \frac{1}{s}f_s\left[U_{n-1}(s) - U_n(s)\right] + f_vU_n(s) + f_{\Delta v}\left(U_{n-1}(s) - U_n(s)\right)$$

The transfer function between the leader and the follower in the Laplace frequency domain:

$$U_n(s) = \frac{sf_{\Delta v} + f_s}{s^2 - s\left(f_v - f_{\Delta v}\right) + f_s}U_{n-1}(s)$$

The speed variation of the second vehicle in the time domain is thus obtained by taking the inverse Laplace transform:

$$u_2(t) = \mathscr{L}^{-1}\left[\frac{sf_{\Delta v} + f_s}{s^2 - s\left(f_v - f_{\Delta v}\right) + f_s}U_1(s)\right]$$

Lyapunov stability criterion

The characteristic equation based method discussed above needs to examine whether the solutions have negative real parts, which sometimes is difficult to do because of the complexity of computing the eigenvalues or applying the Routh-Hurwitz stability criterion.

If we can find a non-negative energy-like function (energy cannot be negative) $V(x)$ that always decreases along trajectories of the system, a locally stable equilibrium point can be identified simply as the minimum of the function.

It has been proved that for a two-state-variable ODEs system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

If $a \leq 0, d \leq 0, bc \leq 0$, there exists a Lyapunov function $V(x) = px_1^2 + qx_2^2$

For the local stability analysis of CF models:

$$\begin{bmatrix}\dot{y}\\ \dot{u}\end{bmatrix} = \begin{bmatrix}0 & -1\\ f_s & f_v - f_{\Delta v}\end{bmatrix}\begin{bmatrix}y\\ u\end{bmatrix}$$

When $f_s > 0$ And $f_v - f_{\Delta v} < 0$,a Lyapunov function that ensures the local stability is

$$V(x) = f_sy^2 + u^2 > 0 \text{ when } y, u \neq 0$$
$$\dot{V}(x) = 2\left(f_v - f_{\Delta v}\right)u^2 < 0$$

string stability

The definition of string stability is:

$$\|\varepsilon_1\|_\infty > \|\varepsilon_2\|_\infty > \|\varepsilon_3\|_\infty > \cdots > \|\varepsilon_n\|_\infty$$

where $\|\varepsilon_n\|_\infty = \max_t |\varepsilon_n(t)|$ is the maximum magnitude of the perturbation within infinite time.

The direct transfer function based method

We assume that the perturbation of the leading vehicle is steady oscillation $u_0(t) = e^{i\omega t}$, then $u_n(t) = G^n(s)e^{i\omega t}$. Then we have:

$$G(i\omega) = \frac{f_s + i\omega f_{\Delta v}}{-\omega^2 - i\omega\left(f_v - f_{\Delta v}\right) + f_s}$$

$$u_n(t) = |G(i\omega)|^n e^{i(\omega t + n\varphi)}$$

Only if $|G(i\omega)| < 1$, the perturbation will not amplify in a platoon. That is

$$\omega^2 f_{\Delta v}^2 + f_s^2 < \left(f_s - \omega^2\right)^2 + \omega^2\left(f_v - f_{\Delta v}\right)^2$$

The Laplace transform based method

The relationship of the perturbation between two consecutive vehicles in the frequency domain is:

$$G(s) = E_n(s)/E_{n-1}(s)$$

Similarly, the transfer function between the leader and the follower in the frequency domain is

$$G(s) = \frac{U_n(s)}{U_{n-1}(s)} = \frac{s f_{\Delta v} + f_s}{s^2 - s\left(f_v - f_{\Delta v}\right) + f_s}$$

There are also the characteristic equation based method and the root locus method that can evaluate the string stability.