CSE215
Foundations of Computer Science

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April 19, 2022

Agenda

- Homework 07
- A summary for proof, sequences, and mathematical induction problems

To finish around 4h20

Exercise 1 (10 points)

Suppose a_1 , a_2 , a_3 , . . . is a sequence defined as follows:

$a_1 = 1$, $a_2 = 3$, and $a_k = a_{k-2} + 2a_{k-1}$ for all integers $k \geq 3$.

Prove that a_n is odd for all integers $n \geq 1$.

Solution

- Proof.
  - Let $P(n)$ be the predicate "a_n is odd". Our goal is to prove $P(n)$ for every integer $n>=1$.
  - $P(1)$, $P(2)$ are obviously true.
  - Now we choose an arbitrary $k >= 2$ and suppose $P(1), P(2), \ldots, P(k)$ are true. We will prove $P(k+1)$, namely, a_{k+1} is odd.
    - By definition, $a_{k+1} = a_{k-1} + 2a_{k-2}$.
    - Since $P(k-1)$ is true by assumption. We know a_{k-1} is odd and a_{k+1} is the result of odd plus even, which is odd.
    - Thus $P(k+1)$ is true.
- QED

Exercise 2 (10 points)

Suppose that $g_1, g_2, g_3, \ldots$ is a sequence defined as follows:

$g_1 = 3$, $g_2 = 5$, and $g_k = 3g_{k-1} - 2g_{k-2}$ for all integers $k \geq 3$.

Prove that $g_n = 2^n + 1$ forall integers $n \geq 1$.

Solution

- Proof by Induction
- Let $P(n)$ be Predicate $g_n = 2^n + 1$ for $n \ge 1$.
- $P(1) \Rightarrow 2^1 + 1 = 3$
  $P(2) = 2^2 + 1 = 5$
- We prove for any integer $k \ge 2$, $P(1)\ P(2) \cdots P(k) \rightarrow P(k+1)$
- Let k be an arbitrary integer and ≥ 2
  and $P(1)\ P(2) \cdots P(k)$ are true
- We want to prove $P(k+1)$, namely $2^{k+1} + 1$

- From definition, $g_{k+1} = 3g_k - 2g_{k-1}$
- and $g_k = 2^k + 1$ and $g_{k-1} = 2^{k-1} + 1$ for some integer k
  from the assumption above.
- Thus $g_{k+1} = 3(2^k + 1) - 2(2^{k-1} + 1) = 2^{1+k} + 1$

QED

Exercise 3 (15 points)

The triangle inequality states that for all real numbers a and b, $|a + b| \leq |a| + |b|$. Use the the triangle inequality and mathematical induction to prove:

For any n real numbers a_1, a_2, …, and a_n,

$$|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n|$$

Solution

#3. $|a+b| \le |a| + |b|$

$\Rightarrow |a_1 + a_2 + \cdots + a_n| \le |a_1| + |a_2| + \cdots + |a_n|$

Proof

- Let $P(n)$ be predicate $|a_1 + a_2 + \cdots + a_n| \le |a_1| + |a_2| + \cdots + |a_n|$

  ↳ it can be represented as $\left|\sum_{i=1}^{n} a_i\right| \le \sum_{i=1}^{n} |a_i|$

- Basic step: $P(1)$ holds: LHS $= \left|\sum_{i=1}^{1} a_i\right| = |a_1|$

  RHS $= \sum_{i=1}^{1} |a_i| = |a_1|$

- Inductive Step: We prove for any integer $k \ge 1$, $P(k) \to P(k+1)$
- Assume $P(k)$ holds. That is $\left|\sum_{i=1}^{k} a_i\right| \le \sum_{i=1}^{k} |a_i|$ $\quad (\Rightarrow |a_1 + \cdots + a_k| \le |a_1| + \cdots + |a_k|)$
- For $P(k+1)$, $\left|\sum_{i=1}^{k+1} a_i\right| = \left|a_{k+1} + \sum_{i=1}^{k} a_i\right|$ $\quad (\Rightarrow |a_1 + \cdots + a_{k+1}| \le |a_1| + \cdots + |a_k| + |a_{k+1}|)$

  $\Rightarrow \left|a_{k+1} + \sum_{i=1}^{k} a_i\right| \le |a_{k+1}| + \sum_{i=1}^{k} |a_i|$

- Thus, $\left|\sum_{i=1}^{k+1} a_i\right| \le \sum_{i=1}^{k+1} |a_i|$

  QED

Exercise 4 (25 = 5 + 10 + 10 points)

Let f be a sequence defined recursively as follows.

$f_1 = 1$, and $f_k = f_{k-1} + 2^k$ for all integers $k \geq 2$

1. Write out f_k for $k = 1, 2, \ldots, 5$.
2. Derive an explicit form of the sequence.
3. Prove that your explcit form corresponds to the original recursive definition. [Hint: To do so, you can name your explcit form sequence as F_k, and prove: for all $k \geq 1$, $F_k = f_k$.]

Solution

#4. $\begin{bmatrix} f_1 = 1 \\ f_k = f_{k-1} + 2^k \quad \text{for } k \geq 2 \end{bmatrix}$

(1) $f_1 = 1$

$f_2 = f_{2-1} + 2^2 = 1 + 4 = 5$

$f_3 = f_{3-1} + 2^3 = 5 + 8 = 13$

$f_4 = f_{4-1} + 2^4 = 13 + 16 = 29$

$f_5 = f_{5-1} + 2^5 = 29 + 32 = 61$

(2) Explicit Form

$\therefore \underline{f_k = 2^{k+1} - 3}$ for $k \geq 1$

ex) 1) $2^2 - 3 = 1$

2) $2^3 - 3 = 8 - 3 = 5$

3) $2^4 - 3 = 16 - 3 = 13$

(3) Proof by Induction

- Let $P(k)$ be $2^{k+1} - 3$.
- $P(1) \Rightarrow 2^{1+1} - 3 = 2^2 - 3 = 1$
  $P(2) \Rightarrow 2^{2+1} - 3 = 2^3 - 3 = 5$
- $P(k) \rightarrow P(k+1)$ for all integer $k \geq 1$
  - Let k be arbitrary integer and $k \geq 1$
  - Assume $P(k)$ holds. namely $2^{k+1} - 3$
  - We want to prove $P(k+1)$, namely $\underline{2^{k+2} - 3}$

$\Rightarrow P(k) = f_{k-1} + 2^k$

$P(k+1) = f_k + 2^{k+1}$ $\updownarrow$ same

Thus, $P(k+1) = 2^{k+1} - 3 + 2^{k+1} = \underline{2^{k+2} - 3}$

QED

- Simplified proof for (3).
  - Let F_k= 2^{k+1} - 3. We will prove F_k satisfies the recursive definition of f_k.
    - f_k equals 1 if k=1 and f_{k-1} + 2^k if k >=2
    - F_k equals 1 if k=1 and equals F_{k-1} + 2^k if k >=2
    - Thus F_k is an explicit form of the recursive definition of f_k
  - QED

Exercise 5 (20 points)

A certain computer program executes twice as many operations when it runs with an input of size k as when it runs with an input of size k – 1 (where k is an integer and k > 1). When the program runs with an input of size 1, it executes seven operations. How many operations does it execute when it runs with an input of size 25?

Solution

$$
\left.\begin{array}{l}
P(1) \Rightarrow 7 \\
P(2) \Rightarrow 14 \\
P(3) \Rightarrow 28 \\
\quad \vdots \\
P(k)
\end{array}\right]
$$

$$
\begin{array}{l}
7 \times 1 = 7 \times 2^0 \\
7 \times 2 = 7 \times 2^1 \\
7 \times 4 = 7 \times 2^2 \\
\quad \vdots \qquad\quad 7 \times 2^3
\end{array}
$$

$$\underline{7 \times 2^{k-1}}$$

$$7 \times 2^{25-1} = 7 \cdot 2^{24} = \underline{117440512}$$

Exercise 6 (20 points)

Consider the two recursively defined sequences.

1. $a_1 = 0$ and $a_k = 2a_{k-1} + k - 1$ for all integers $k \geq 2$

2. $a_1 = 0$ and $a_k = (a_{k-1} + 1)^2$ for all integers $k \geq 2$

   Determine whether the two recursively defined sequences satisfies the explicit formula $a_n = (n - 1)^2$, for all integers $n \geq 1$.

Solution

#6) (1) $a_1 = 0$

$a_k = 2a_{k-1} + k - 1$ for $k \geq 2$ $\longleftrightarrow$ $a_n = (n-1)^2$ for $n \geq 1$

$a_k = 2a_{k-1} + k - 1$ $\quad$ $a_k = (k-1)^2$

$a_{k+1} = 2a_k + k + 1 - 1$ $\quad$ $a_{k+1} = \underline{k^2}$

$= 2a_k + k$

$a_{k+1} = 2(k-1)^2 + k$ $\quad$ $\neq$

$= \underline{2k^2 - 3k + 2}$

$\therefore$ recursive function doesn't satisfy explicit formula.

↳ex) $a_1 = 0$ $\quad$ $a_1 = 0$

$a_2 = 2 \cdot 0 + 2 - 1 = 1$ $\quad$ $a_2 = (2-1)^2 = 1$

$a_3 = 2 \cdot 1 + 3 - 1 = 4$ $\quad$ $a_3 = (3-1)^2 = 4$

$a_4 = 2 \cdot 4 + 4 - 1 = 11$ $\longleftrightarrow$ $a_4 = (4-1)^2 = 9$

(2) $a_1 = 0$

$a_k = (a_{k-1} + 1)^2$ for $k \geq 2$ $\longleftrightarrow$ $a_n = (n-1)^2$ for $n \geq 1$

$a_1 = 0$ $\quad$ $a_1 = (1-1)^2 = 0^2 = 0$

$a_2 = (0+1)^2 = 1$ $\quad$ $a_2 = (2-1)^2 = 1$

$a_3 = (1+1)^2 = 4$ $\quad$ $a_3 = (3-1)^2 = 4$

$a_4 = (4+1)^2 = 5^2 = 25$ $\longleftrightarrow$ $a_4 = (4-1)^2 = 9$

$\therefore$ recursive function doesn't satisfy explicit formula

$a_k = (a_{k-1} + 1)^2$ $\quad$ $\underline{a_k = (k-1)^2}$

$a_{k+1} = (\underline{a_k} + 1)^2$ $\quad$ $a_{k+1} = k^2$

↳ $a_{k+1} = ((k-1)^2 + 1)^2$ $\quad$ $\neq$

$= (k^2 - 2k + 2)^2$

A summary for proof, sequences, and mathematical induction problems

To finish around 4h20

Major Species

- Show a sequence satisfies a certain property
- Show a sequence does not satisfy a certain property
- Show a for-all statement is true
- Show a for-all statement is false
- Show a there-exists statement is true
- Show a there-exists statement is false

Show a sequence satisfies a certain property

Use mathematical induction to prove the following identities.

(a) [5 points] For all integers $n \geq 1$,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

(b) [5 points] Consider the Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$. For all integers $n \geq 2$,

$$f_{n+2} = 1 + \sum_{i=0}^{n} f_i.$$

Show a sequence does not satisfy a certain property

Exercise 6 (20 points)

Consider the two recursively defined sequences.

1. $a_1 = 0$ and $a_k = 2a_{k-1} + k - 1$ for all integers $k \geq 2$
2. $a_1 = 0$ and $a_k = (a_{k-1} + 1)^2$ for all integers $k \geq 2$

Determine whether the two recursively defined sequences satisfies the explicit formula $a_n = (n - 1)^2$, for all integers $n \geq 1$.

Show a for-all statement is true
(= show a there-exists statement is false)

Exercise 5 (points = 10)

Prove the following proposition: An odd number multiplied by an odd number is an odd number.

Exercise 5 (points = 15)

Prove that there are no integers x and y such that x^3 = 4y + 6.

Exercise 6 (points = 15)

Prove that the product of any four consecutive integers is a multiple of 8.
(Note this statement is stronger than the one we discussed in the class.)

Show a there-exists statement is true (= show a for-all statement is false)

Exercise 2 (points = 5)

Prove the following statement: There exist two integers m and n such that m > 1 and n>1 and 1/m + 1/n is an integer.

Exercise 3 (points = 5)

Prove the following statement: There is an integer n such that 2n^2 - 5n +2 is prime.

Exercise 6 [points = 10)

We say an integer is a perfect square if it can be expressed as a square of some integer. For example, 81 is a perfect square; 80 is not.

Prove the following statement: there is a perfect square that can be written as a sum of two other perfect squares.

Variations from the major species

- Check if two sequences are the same
- Check if a for-all statement is true or false
- Check if there exists statement is true

Check if two sequences are the same

Exercise 6 (20 points)

Consider the two recursively defined sequences.

1. $a_1 = 0$ and $a_k = 2a_{k-1} + k - 1$ for all integers $k \geq 2$

2. $a_1 = 0$ and $a_k = (a_{k-1} + 1)^2$ for all integers $k \geq 2$

Determine whether the two recursively defined sequences satisfies the explicit formula $a_n = (n - 1)^2$, for all integers $n \geq 1$.

Check if a for-all statement is true/false

Exercise 4 (points = 40)

Determine which statements are true and which are false. Prove those that are true and disprove those that are false. (points = To disapprove means to find a counter-example).

1. rational/irrational is irrational.
2. Irrational*irrational is irrational.
3. The sum of any two positive irrational numbers is irrational.
4. The square root of any rational number is irrational.

Check if an there-exists statement is true/false

Exercise 1 (points = 40)

Determine whether the statements below are true or false.

1. 119 is a prime number.
2. 161 is a prime number.
3. 42k is an even number for any integer k.
4. For each integer n with 2 <= n <= 6, n^2 – n + 11 is a prime number.
5. The average of any two odd integers is odd.
6. For any real number x, if x * x >= 4, then x >= 2.
7. For any real numbers x and y, x^2 - 2xy + y^2 >= 0.
8. There exists an integer x, such that (2x + 1)^2 is even.