CSE215
Foundations of Computer Science

Instructor: Zhoulai Fu

State University of New York, Korea

April 4, 2022

Many slides taken from Prof. Pramod Ganapathi (SBU). Thanks!

News

FW: SUNY Korea returns to in-person classes

hamid.hefazi Mar 31

to hamid.hefazi, covid19report, Hyunsong

Dear SUNY Korea Faculty,

Remind your students of their responsibility to report infections as described below.

Report infection immediately to covid19report@sunykorea.ac.kr

When reporting please include

- Student’s name
- Student’s e-mail address
- Your class name and schedule
- Proof of positive test provided by the student

Agenda

- Quick look at exam problems on Sequences chapter
- Sequences
- Mathematical induction
- Solving some exam problems
- Recursion

Finish around 4h45

2021 Final

Problem 7. [10 points]

Use mathematical induction to prove the following identities.

(a) [5 points] For all integers $n \geq 1$,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

(b) [5 points] Consider the Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$. For all integers $n \geq 2$,

$$f_{n+2} = 1 + \sum_{i=0}^{n} f_i.$$

Problem 8. [5 points]

Write the pseudocodes of the recursive algorithms for solving the Towers of Hanoi problem and computing the greatest common divisor of two integers.

2020 Final

Problem 7. [10 points]

Use mathematical induction to prove the following identities.

(a) [5 points] For all natural numbers n,

$$1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + n^2 \times (n+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$$

(b) [5 points] For all natural numbers n,

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{n \times (n+1)} = \frac{n}{n+1}$$

Problem 8. [5 points]

Write Euclid's recursive algorithm to compute the greatest common divisor (GCD) of two whole numbers. Show the step-by-step process to compute the GCD of 46 and 14 using the algorithm.

What are sequences?

Types of sequences

- Finite sequence: $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$
  e.g.: $1^1, 2^2, 3^2, \ldots, 100^2$
- Infinite sequence: $a_m, a_{m+1}, a_{m+2}, \ldots$
  e.g.: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$

Term

- Closed-form formula: $a_k = f(k)$
  e.g.: $a_k = \frac{k}{k+1}$
- Recursive formula: $a_k = g(k, a_{k-1}, \ldots, a_{k-c})$
  e.g.: $a_k = a_{k-1} + (k-1)a_{k-2}$

Growth of sequences

- Increasing sequence

  e.g.: $2, 3, 5, 7, 11, 13, 17, \ldots$
- Decreasing sequence

  e.g.: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$

Sums and products of sequences

Sum

- Summation form:

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

where, k = index, m = lower limit, n = upper limit

e.g.: $\sum_{k=m}^{n} \frac{(-1)^k}{k+1}$

Product

- Product form:

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdot \cdots \cdot a_n$$

where, k = index, m = lower limit, n = upper limit

e.g.: $\prod_{k=m}^{n} \frac{k}{k+1}$

Properties of sums and products

- Suppose $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number

Sum

- $\sum_{k=m}^{n} a_k = \sum_{k=m}^{i} a_k + \sum_{k=i+1}^{n} a_k$ for $m \leq i < n$ where, i is between m and $n-1$ (inclusive)
- $c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} (c \cdot a_k)$
- $\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$

Product

- $(\prod_{k=m}^{n} a_k) \cdot (\prod_{k=m}^{n} b_k) = \prod_{k=m}^{n} (a_k \cdot b_k)$

Change of variable

$$\sum_{k=0}^{99} \frac{(-1)^k}{k+1} = \sum_{j=0}^{99} \frac{(-1)^j}{j+1} \qquad \text{(Set } j = k\text{)}$$

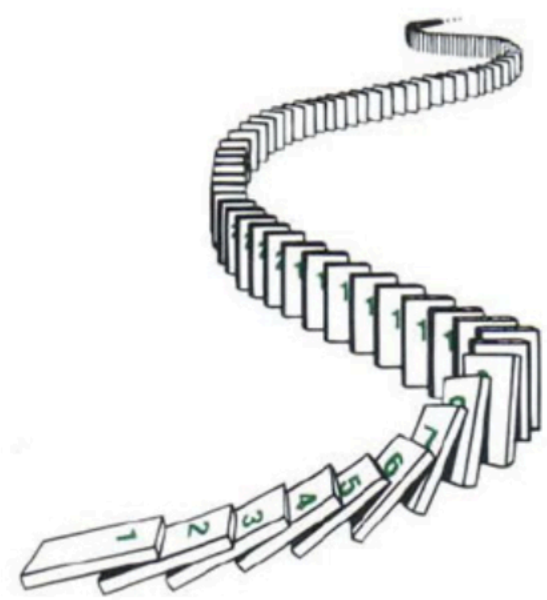
$$= \sum_{i=1}^{100} \frac{(-1)^{i-1}}{i} \qquad \text{(Set } i = j+1\text{)}$$

Proof by Mathematical Induction

Proof for dominos

Core idea

- A starting domino falls. From the starting domino, every successive domino falls. Then, every domino from the starting domino falls.



- Prove P(n) for all integers n>=0
- First prove P(0)
- Then, prove P(k) => P(k+1) for any K>=0

Proof by mathematical induction: Example 0

Proposition

- $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all integers $n \geq 1$.

Proof

Let $P(n)$ denote $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

- Basis step. $P(1)$ is true because $1 = 1(1+1)/2$.
- Induction step. Suppose that $P(k)$ is true for some $k \geq 1$.
  Now, we want to show that $P(k+1)$ is true. That is,
  Assume $P(k)$: $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ for some $k \geq 1$
  Prove $P(k+1)$: $1 + 2 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}$
  LHS of $P(k+1)$
  $= (1 + 2 + \cdots + k) + (k+1)$
  $= \frac{k(k+1)}{2} + (k+1)$ $\quad$ ($\because$ $P(k)$ is true)
  $= \frac{(k+1)(k+2)}{2}$ $\quad$ ($\because$ distributive law)
  $=$ RHS of $P(k+1)$

Proof by mathematical induction: Example 1

Proposition

- $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$ for all integers $n \geq 2$.

Proof

Let $P(n)$ denote $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$.

- Basis step. $P(2)$ is true. ▷ How?
- Induction step.

  Assume $P(k)$: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)=\frac{1}{k}$ for some $k \geq 2$.

  Prove $P(k+1)$: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k+1}\right)=\frac{1}{k+1}$

  LHS of $P(k+1)$

  $=\left[\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)\right]\left(1-\frac{1}{k+1}\right)$

  $=\frac{1}{k}\left(1-\frac{1}{k+1}\right)$ ($\because P(k)$ is true)

  $=\frac{1}{k}\cdot\frac{k}{k+1}$ ($\because$ common denominator)

  $=\frac{1}{k+1}$ ($\because$ remove common factor)

  $=$ RHS of $P(k+1)$

Proof by mathematical induction: Example 2

Proposition

- $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$ for all integers $n \geq 1$.

Proof

Let $P(n)$ denote $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$.

- Basis step. $P(1)$ is true. $\triangleright$ How?
- Induction step.

  Assume $P(k)$: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k\cdot(k+1)} = \frac{k}{k+1}$ for some $k \geq 1$

  Prove $P(k+1)$: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(k+1)\cdot(k+2)} = \frac{k+1}{k+2}$

  LHS of $P(k+1)$

  $= \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k\cdot(k+1)}\right) + \frac{1}{(k+1)\cdot(k+2)}$

  $= \frac{k}{k+1} + \frac{1}{(k+1)\cdot(k+2)}$ $(\because P(k)$ is true$)$

  $= \frac{k^2+2k+1}{(k+1)\cdot(k+2)}$ $(\because$ common denominator$)$

  $= \frac{(k+1)^2}{(k+1)\cdot(k+2)}$ $(\because$ simplify$)$

  $= \frac{k+1}{k+2}$ $(\because$ remove common factor$)$

  $=$ RHS of $P(k+1)$

Proof by mathematical induction: Example 3

Proposition

- Fibonacci sequence is: $F(0) = 1$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Prove that: $F(0)^2 + F(1)^2 + \cdots + F(n)^2 = F(n)F(n+1)$ for all $n \geq 0$.

Proof

Let $P(n)$ denote $F(0)^2 + F(1)^2 + \cdots + F(n)^2 = F(n)F(n+1)$.

- Basis step. $P(0)$ is true. ▷ How?
- Induction step. Suppose that $P(k)$ is true for some $k \geq 0$.
  Now, we want to show that $P(k+1)$ is true.
  LHS of $P(k+1)$
  $= (F(0)^2 + F(1)^2 + \cdots + F(k)^2) + F(k+1)^2$
  $= F(k)F(k+1) + F(k+1)^2$ ($\because$ $P(k)$ is true)
  $= F(k+1)(F(k) + F(k+1))$ ($\because$ distributive law)
  $= F(k+1)F(k+2)$ ($\because$ recursive definition)
  $=$ RHS of $P(k+1)$

Proof by strong mathematical induction

Proposition

- For all integers $n \geq a$, a property $P(n)$ is true.

Proof

- Basis step.
  Show that $P(a), P(a+1), \ldots, P(b)$ are true.
- Induction step.
  Assume $\{P(a), P(a+1), \ldots, P(k)\}$ are true for some $k \geq b$.
  (This supposition is called the inductive hypothesis.)
  Now, prove that $P(k+1)$ is true.

Proposition

- Any integer greater than 1 is divisible by a prime number.

Proof

Let $P(n)$ denote "n is divisible by a prime number."

- Basis step. $P(2)$ is true. ▷ How?
- Induction step. Suppose that $P(i)$ is true for some $k \geq 2$ and any $i \in [2, k]$. We want to show that $P(k+1)$ is true.

  Two cases:

  Case 1: [$k+1$ is prime.] $P(k+1)$ is true. ▷ How?

  Case 2: [$k+1$ is not prime.] We can write $k+1 = ab$ such that both $a, b \in [2, k]$ using the definition of a composite. This means, $k+1$ is divisible by a. We see that a is divisible by a prime due to the inductive hypothesis. As $k+1$ is divisible by a and a is divisible by a prime, $k+1$ is divisible by a prime, due to the transitivity of divisibility. Hence, $P(k+1)$ is true.

If (t < 4h) break

Solving some exam problems

Finish around 4h45

Use mathematical induction to prove the following identities.

(a) [5 points] For all integers $n \geq 1$,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

- Proof.
  - We will prove this proposition with mathematical induction.
  - Let P(n) be the predicate _______
  - We first prove P(1) holds.
    - _______
  - Then, we prove P(n) -> P(n+1) for all n>=1
    - Assume P(n) holds. We want to prove P(n+1), that is, ______
    - Argue why, under the assumption, P(n+1) holds.
      - _________
- QED.

Use mathematical induction to prove the following identities.

(b) [5 points] Consider the Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$. For all integers $n \geq 2$,

$$f_{n+2} = 1 + \sum_{i=0}^{n} f_i.$$

- Proof.
  - We will prove this proposition with mathematical induction.
  - Let P(n) be the predicate _______
  - We first prove P(2) holds.
    - _______
  - Then, we prove P(n) -> P(n+1) for all n>=2
    - Assume P(n) holds. We want to prove P(n+1), that is, ______
    - Argue why, under the assumption, P(n+1) holds.
      - _________
- QED.

Final 2020

Use mathematical induction to prove the following identities.

(a) [5 points] For all natural numbers n,

$$1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + n^2 \times (n+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$$

- Proof.
  - We will prove this proposition with mathematical induction.
  - Let P(n) be the predicate _______
  - We first prove P(1) holds.
    - _______
  - Then, we prove P(n) -> P(n+1) for all n>=1s
    - Assume P(n) holds. We want to prove P(n+1), that is, ______
    - Argue why, under the assumption, P(n+1) holds.
      - _________
- QED.

Use mathematical induction to prove the following identities.

(b) [5 points] For all natural numbers n,

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n\times(n+1)}=\frac{n}{n+1}$$

- Proof.
  - ____________
- QED.