CSE215
Foundations of Computer Science

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News about class format

- Following a meeting with Provost, **we need to switch back to in-person format as soon as possible.**
- This week, hybrid (so you have a week to adapt)
- Next weeks, in-person
- Absences can be excused for medical reasons with supported documents

Agenda

- Homework 06
- More on mathematical induction

Finish around 4h45

Exercise 1 (points = 10)

Prove that for all integers a, if a^5 is even, then a is even.

- This is about proving for all a, P(a) ->Q(a)
- First, think if the statement makes sense
- Proof solution 1: prove the negation is false
- Proof solution 2: Prove the contraposition
- Proof solution 3: Reduce the proof, making it shorter, before moving forward

Solution 1

Exercise 1 (points = 10)

Prove that for all integers a, if a^5 is even, then a is even.

- Negation. Suppose there's an integer a such that a^5 is even but a is odd.
- $a = 2k + 1$ (Definition of odd number)

  $\Rightarrow a^5 = (2k + 1)^5$ (^5 both sides)

  $\Rightarrow a^5 = (32k^5 + 80k^4 + 80k^3 + 40k^2 + 10k + 1)$ (Expand)

  $\Rightarrow a^5 = 2(16k^5 + 40k^4 + 40k^3 + 20k^2 + 5k) + 1$ (Taking 2 out from the terms)

  $\Rightarrow a^5 = 2m + 1$ (set $m =(16k^5 + 40k^4 + 40k^3 + 20k^2 + 5k)$)

  (m is an integer as multiplication is closed on integers)

  $\Rightarrow a^5$ = odd (Defintion of odd number)
- Contradition! Hence, the proposition is true.

Solution 2

- Proof.
  - We want to prove: for all integer a, a^5 is even -> a is even
  - We will prove by contraposition.
  - We will prove: for all integer a, a is odd -> a^5 is odd.
    - Let a be an arbitrary integer.
    - Assume a is odd. We need to prove a^5 is odd. This is true because an odd number multiplied with another odd number must be odd.
- QED

Solution 3

- Proof.
  - We want to prove: for all integer a, a^5 is even -> a is even
  - Let a be an arbitrary integer and a^5 is even. We need to prove that a is even.
    - Proof by negation / contradiction. Assume a is odd. Then a^5 must be odd because an odd number multiplied with another odd number must be odd. Thus a must be even.
- QED

Exercise 2 (points = 10)

Prove that for all integers a and b, if ab is even, then a is even or b is even.

- This is about proving for all a, P(a) ->Q(a)
- First, think if the statement makes sense
- Proof solution 1: prove the negation is false
- Proof solution 2: Prove the contraposition
- Proof solution 3: Reduce the proof, making it shorter, before moving forward

Solution 1

Exercise 2 (points = 10)

Prove that for all integers a and b, if ab is even, then a is even or b is even.

- Negation. Suppose there are odd integers a and b such that ab is even.
- $a = 2k + 1,\ b = 2n + 1$ (Definition of odd number)
  $\Rightarrow\ ab = (2k + 1)(2n + 1)$ (Multiply a and b)
  $\Rightarrow\ ab = (4kn + 2k + 2n + 1)$ (Expand)
  $\Rightarrow\ ab = 2(2kn + k + n) + 1$ (Taking 2 out from the terms)
  $\Rightarrow\ ab = 2m + 1$ (set m =$(2kn + k + n)$)
  (m is an integer as multiplication is closed on integers)
  $\Rightarrow\ ab$ = odd (Defintion of odd number)
- Contradition! Hence, the proposition is true.

A wrong solution — where is it wrong?

Exercise 2

$\forall$ a and b $\in \mathbb{Z}$, If ab is even, then a is even or b is even

$\exists\, a, b \in \mathbb{Z}$, ab is even and a, b are odd

$ab = 2K, \quad K \in \mathbb{Z}.$

$a = \frac{2K}{b} = 2\left(\frac{K}{b}\right)$ even

$b = 2\frac{K}{a} = 2\left(\frac{K}{a}\right)$ even.

$\neq$

$\therefore$ QED

Exercise 3 (points = 10)

Prove that the cube root of an irrational number is irrational.

Solution

Exercise 3 (points = 10)

Prove that the cube root of an irrational number is irrational. (The cube root of a number a is a number b such that b*b*b=a.)

- Negation. Suppose there is an irrational number a such that the cube root b is rational.
- $\Rightarrow b = \frac{m}{n}$ (m, n have no common factors, $n \neq 0$)

  $\Rightarrow a = b^3 = (\frac{m}{n})^3$ (Defintion of the cube root)

  $\Rightarrow a = b^3 = \frac{m^3}{n^3}$ (a can be expressed as a quotient of two integers with a nonzero denominator. Since m and n do not have common factor, m^3 and n^3 do not have common factor, and since $n \neq 0$, $n^3 \neq 0$)

  $\Rightarrow$ a = rational (Defintion of rational number)

Exercise 4 (points = 40)

Determine which statements are true and which are false. Prove those that are true and disprove those that are fals
disapprove means to find a counter-example).

1. rational/irrational is irrational.
2. Irrational*irrational is irrational.
3. The sum of any two positive irrational numbers is irrational.
4. The square root of any rational number is irrational.

- To disprove a for-all statement = To prove a there-exists statement —> find an example

Solution

1. rational/irrational is irrational.
    - False.
    - Counter-example: zero is a rational number. Dividing zero by an irrational number equals zero. There are rational a and irrational number b such that a/b is rational. Therefore, the proposition is not true.

2. Irrational*irrational is irrational.
    - False.
    - Counter-example: $\sqrt{2}$ is an irrational number. $\sqrt{2} \times \sqrt{2} = 2$ and 2 is a rational number. there is an irrational number n such that n multiplied by an irrational number is rational. Therefore, the proposition is not true.

3. The sum of any two positive irrational numbers is irrational.
    - False.
    - Counter-example: π is a positive irrational number and $8 - \pi$ is also a positive irrational number. The sum of the two irrational numbers, $\pi + (8 - \pi) = 8$, and 8 is a rational number. There are positive irrational numbers that the sum of them is irrational. Therefore, the proposition is not true.

4. The square root of any rational number is irrational.
    - False
    - Counter-example: $\sqrt{\frac{1}{4}} = \frac{1}{2}$. A sqaure root of a rational number $\frac{1}{4}$ is a rational number $\frac{1}{2}$. There is a rational number such that the square root of it is rational. Therefore, the proposition is not true.

Exercise 5 (points = 15)

Prove that there are no integers x and y such that x^3 = 4y + 6.

- The problem is to prove “There is no x,y, such that P(x,y)”
- Its negation is there exists x,y, such that P(x,y)
- We start from the negation, trying to prove the negation is false

Solution

Exercise 5
Prove that there are no integers x and y such that $x^3 = 4y + 6$.

Negation of the given statement: Suppose that there are integers x and y such that $x^3 = 4y + 6$.

We can know that x^3 is even because $4y + 6$ is even. Thus, x is also even.

Let $x = 2k$ (since x is an even number),
$x^3 = (2k)^3 = 8k^3 = 2(2y + 3) \Rightarrow 4k^3 = 2y + 3$

$2y + 3$ is odd, but $4k^3$ is even, which is a contradiction.

By the proof by contradiction, then the given statement is true.

Exercise 6 (points = 15)

Prove that the product of any four consecutive integers is a multiple of 8.

- Intuition?

Solution

Exercise 6

Proof

- We want to prove: $\forall n \in \mathbb{Z}$, $n(n+1)(n+2)(n+3)$ is a multiple of 8
- proof by division into cases
  - If n is even, $n=2k$ for some k,

    $n(n+1)(n+2)(n+3) = 2k(2k+1)(2k+2)(2k+3) = 4k(2k+1)(k+1)(2k+3)$
    - If k is even, $k=2\ell$ for some ℓ, $8\ell(4\ell+1)(2\ell+1)(4\ell+3)$
    - If k is odd, $k=2\ell+1$ for some ℓ, $(8\ell+4)(4\ell+3)(2\ell+2)(4\ell+5)$

      $= 8(2\ell-1)(4\ell+3)(\ell+1)(4\ell+5)$
  - If n is odd, $n=2k+1$ for some k,

    $n(n+1)(n+2)(n+3) = (2k+1)(2k+2)(2k+3)(2k+4) = 4(2k+1)(k+1)(2k+3)(k+2)$
    - If k is even, $k=2\ell$ for some ℓ, $4(4\ell+1)(2\ell+1)(4\ell+3)(2\ell+2)$

      $= 8(4\ell+1)(2\ell+1)(4\ell+3)(\ell+1)$
    - If k is odd, $k=2\ell+1$ for some ℓ, $4(4\ell+3)(2\ell+2)(4\ell+5)(2\ell+3)$

      $= 8(4\ell+3)(\ell+1)(4\ell+5)(2\ell+3)$

QED

Solution 2

We have four consecutive integers n, n+1, n+2, n+3.
We prove $n(n+1)(n+2)(n+3)$ is a multiple of 8 by division into cases.
By the quotient-remainder theorem, n can be written as 4k, (4k + 1), (4k + 2), or (4k + 3) for some integer k.

Case 1: $n = 4k$,
$n(n+1)(n+2)(n+3) = 4k(4k+1)(4k+2)(4k+3) = 8k(4k+1)(2k+1)(4k+3) = 8m$ (let $m = k(4k+1)(2k+1)(4k+3)$).
We have $n(n+1)(n+2)(n+3) = 8m$ where m is an integer. Hence, $n(n+1)(n+2)(n+3)$ is a multiple of 8.

Case 2: $n = 4k + 1$,
$n(n+1)(n+2)(n+3) = (4k+1)(4k+2)(4k+3)(4k+4) = 8(4k+1)(2k+1)(4k+3)(k+1) = 8m$ (let $m = (4k+1)(2k+1)(4k+3)(k+1)$).
We have $n(n+1)(n+2)(n+3) = 8m$ where m is an integer. Hence, $n(n+1)(n+2)(n+3)$ is a multiple of 8.

Case 3: $n = 4k + 2$,
$n(n+1)(n+2)(n+3) = (4k+2)(4k+3)(4k+4)(4k+5) = 8(2k+1)(4k+3)(k+1)(4k+5) = 8m$ (let $m = (2k+1)(4k+3)(k+1)(4k+5)$).
We have $n(n+1)(n+2)(n+3) = 8m$ where m is an integer. Hence, $n(n+1)(n+2)(n+3)$ is a multiple of 8.

Case 4: $n = 4k + 3$,
$n(n+1)(n+2)(n+3) = (4k+3)(4k+4)(4k+5)(4k+6) = 8(4k+3)(k+1)(4k+5)(2k+3) = 8m$ (let $m = (4k+3)(k+1)(4k+5)(2k+3)$).
We have $n(n+1)(n+2)(n+3) = 8m$ where m is an integer. Hence, $n(n+1)(n+2)(n+3)$ is a multiple of 8.

Thus, we prove the product of any consecutive four integers is a multiple of 8 by division into cases.
QED.

A wrong solution to Ex6 — where was it wrong?

Ex6,

Date No

$\forall n \in \mathbb{Z}$, $n(n+1)(n+2)(n+3) = 8N$ $(\exists N \in \mathbb{Z})$

if $n=1$,

$1 \times 2 \times 3 \times 4 = 24 = 8(3)$ => True

if $n=k$,

$k(k+1)(k+2)(k+3) = 8p$ $(\exists p \in \mathbb{Z})$
(supposely true)

if $n=k+1$

$(k+1)(k+2)(k+3)(k+4) = 8q$ $(\exists q \in \mathbb{Z})$

↳ $\frac{8p}{k}$

$= \frac{8p}{k}(k+4) = 8q$ $\qquad \frac{8p(k+4)}{k} = 8q$

LHS has a multiple of 8 and so does RHS

∴ QED.

Agenda

- Homework 06
- **More on mathematical induction**

Finish around 4h45

Proof by strong mathematical induction

- Proposition to prove: for all n >=1, P(n)
- Base step: P(1), P(2), … P(b)
- Inductive step: for any k>=b, P(1), P(2) …, P(k) -> P(k+1)

Suppose $b_1, b_2, b_3, \ldots$ is a sequence defined as follows:

$$b_1 = 4, \; b_2 = 12$$
$$b_k = b_{k-2} + b_{k-1} \quad \text{for all integers } k \geq 3.$$

Prove that b_n is divisible by 4 for all integers $n \geq 1$.

- The proof is to show 4|b1, 4|b2, 4|b3, 4|b4, 4|b5, ….
- b1 = 4, b2= 12, b3 = 16, b4 = 28, b5 = 44
- So apparently, the statement makes sense
- How to prove 4|b_k for all k?
- Choose an arbitrary k. Since b_k = b_{k-2}+b_{k-1}, if we can prove 4|b_{k-2} and 4|b_{k-1} then we will prove 4|b_k

Suppose $b_1, b_2, b_3, \ldots$ is a sequence defined as follows:

$$b_1 = 4, \; b_2 = 12$$

$$b_k = b_{k-2} + b_{k-1} \quad \text{for all integers } k \geq 3.$$

Prove that b_n is divisible by 4 for all integers $n \geq 1$.

- Proof
  - We prove this statement with mathematical induction
  - Let P(n) be the 4|b_n
  - We first show P(1) and P(2) are true
  - Then, we show that for any k>=2, P(1), P(2) …. P(k) are true implies P(k+1) is true
  - Let k be an arbitrary integer, k>=2, and P(1),P(2) …. P(k) are true
  - We want to prove P(k+1), namely, 4|b_{k+1}
    - From definition, b_{k+1} = b_k + b_{k-1}.
    - And b_k=4s for some integer s and b{k-1}=4s’ for some integer s’ from the assumption above
    - Thus, we have b_{k+1} = 4(s+s’).
    - Thus 4|b_{k+1}.
- QED