CSE215
Foundations of Computer Science

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May 30, 2022

Summary of what we have learned

- Construct truth tables for various purposes, for example, checking if an argument is valid or if two propositions are equivalent.
- Argue with rigor and prove theorems
- Understand the mathematical concepts of sequences, functions, relations.

Summary about this education

- Precision of thought and language is an critical part of doing computer science.
- It is my honor to help you develop as careful readers, clear writers, and critical thinkers.

Final Exam

- In-person
- online for legitimate reasons. Preference or convenience is not a legitimate reason.
- Open-notes, open book, open Internet
- Coverage: All lectures
- You will do well if you do homework

Agenda for the final weeks

- Final Review 1 — standard problems
- Recitation — standard problems (besides some course evaluation requirement)
- Final Review 2 next Tuesday, with Wednesday schedule — harder problems
- Final exam 06/13 (next next Monday) 3h15pm-5h45pm

Zoom on today!

Problem 1. [5 points]

Construct a truth table for the following statement form: $p \wedge (q \vee r) \leftrightarrow p \wedge (q \wedge r)$.

Definition

- Two statement forms p and q are logically equivalent, denoted by $p \equiv q$, if and only if they have the same truth values for all possible combination of truth values for the propositional variables

Problem 2. [5 points]

Construct a truth table for the following statement form: $(p \to q) \vee ((q \oplus r) \to \sim p)$.

Definition

- Conditional or implication is a compound statement of the form "if p, then q". It is denoted by $p \to q$ and read as "p implies q". It is false when p is true and q is false, and it is true, otherwise.

p ->q seen as
~p V q

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Definition

- Exclusive or of statements p and q, denoted by $p \oplus q$, is defined as p or q but not both. It is computed as $(p \vee q) \wedge \sim (p \wedge q)$

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \vee q) \wedge \sim (p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Problem 3. [5 points]

Mention whether the following statements are true or false. Reasons are not needed.

(a) [1 point] $p \vee \sim p \equiv \mathbf{c}$

(b) [1 point] $p \vee (p \wedge q) \equiv p \wedge (p \vee q)$

(c) [1 point] $\mathbf{c} \equiv p \vee \mathbf{t}$

(d) [1 point] $p \wedge p \equiv p \vee p$

(e) [1 point] $p \wedge \mathbf{c} \equiv \sim \mathbf{t}$

Laws	Formula	Formula
Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity laws	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
Negation laws	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
Double neg. law	$\sim (\sim p) \equiv p$	
Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
Uni. bound laws	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
De Morgan's laws	$\sim (p \wedge q) \equiv \sim p \vee \sim q$	$\sim (p \vee q) \equiv \sim p \wedge \sim q$
Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negations	$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

Problem 4. [5 points]

Prove that the sum of the squares of any two consecutive odd integers is even.

- Direct proof can be used to prove “for any…, if …, then …”
- Proof for “for any …” part: Take an arbitrary ….
- Proof for “if …” part: Suppose….

Problem 5. [5 points]

Suppose that x and y are real numbers. Prove that $x = y$ if and only if $xy = (x+y)^2/4$.

- To prove “A if and only if B” means to prove
  - (1) A-> B
  - (2) B-> A

Problem 7. [10 points]

Use mathematical induction to prove the following identities.

(a) [5 points] For all integers $n \geq 1$,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

(b) [5 points] Consider the Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$. For all integers $n \geq 2$,

$$f_{n+2} = 1 + \sum_{i=0}^{n} f_i.$$

- Mathematical induction
  - First identify P(n). The problem is to show P(n) for all n >= 1
  - Then show P(1)
  - Then suppose P(n) is true, show P(n+1)

Problem 9. [5 points]

Write and fill the table with ✓ or ✗. If a function is one-to-one or onto, then use ✓. On the other hand, if a function is not one-to-one or not onto, then use ✗.

Function	Domains	One-to-one function?	Onto function?
$f(x) = 3x$	$f : \mathbb{Z} \to \mathbb{Z}$		
$f(x) = 3x$	$f : \mathbb{R} \to \mathbb{R}$		
$f(x) = 3x^2$	$f : \mathbb{Z} \to \mathbb{Z}$		
$f(x) = 3x^2$	$f : \mathbb{R} \to \mathbb{R}$		

- one-to-one function (injective) if two different inputs yield different outputs

- onto function (surjective) if all co-domain is covered by possible function outputs

Problem 10. [5 points]

Consider $f : \mathbb{R}^* \to \mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R}^+ \cup \{0\}$ is the set of all nonnegative real numbers, such that f is a mapping from pounds to kilograms. A kilogram of mass in the International System of Units (SI) represents 2.2046226218 pounds. Is f a one-to-one correspondence? Prove your answer.

Problem 11. [5 points]

Let A be the set of all people. Let R be the relation defined on A as follows: For persons p and q in A, we have $p\ R\ q \Leftrightarrow p$ has the same birthday as q.
Is R an equivalence relation? Prove your answer. If R is an equivalence relation, what are the distinct equivalence classes of the relation?

In mathematics, when the elements of some set S have a notion of equivalence (formalized as an equivalence relation) defined on them, then one may naturally split the set S into **equivalence classes**. These equivalence classes are constructed so that elements a and b belong to the same **equivalence class** if, and only if, they are equivalent.

Formally, given a set S and an equivalence relation $\sim$ on S, the *equivalence class* of an element a in S, denoted by $[a]$,[1] is the set[2]

$$\{x \in S : x \sim a\}$$

of elements which are equivalent to a. It may be

Problem 1. [5 points]

Determine if the following deduction rule is valid.

$p \to (q \vee r)$

$\sim (p \to q)$

$\therefore\ r$

Problem

- Determine the validity of the argument:

  $p \to q \vee \sim r$

  $q \to p \wedge r$

  $\therefore p \to r$

p	q	r	$\sim r$	$q \vee \sim r$	$p \wedge r$	$p \to q \vee \sim r$	$q \to p \wedge r$	$p \to r$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	
T	F	T	F	F	T	F	T	
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	
F	T	F	T	T	F	T	F	
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

Definition

- A rule of inference is a valid argument form that can be used to establish logical deductions

Name	Rule		Name	Rule	
Modus Ponens	$p \to q$ p $\therefore q$		Elimination	$p \vee q$ $\sim q$ $\therefore p$	$p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$		Transitivity	$p \to q$ $q \to r$ $\therefore p \to r$	
Proof by division into cases	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$		Generalization	p $\therefore p \vee q$	q $\therefore p \vee q$
			Specialization	$p \wedge q$ $\therefore p$	$p \wedge q$ $\therefore q$
Conjunction	p q $\therefore p \wedge q$		Contradiction	$\sim p \to c$ $\therefore p$	

Problem 3. [5 points]

Verify using truth tables if the following two logical expressions are equivalent.

$(p \to q) \wedge (\sim p \to \sim q)$ and $\sim p \leftrightarrow \sim q$

- A and B are equivalent means A and B has the same value in each row of a truth table

Problem 4. [5 points]

Prove that $n^2+9n+27$ is odd for all natural numbers n. You can use any proof technique.

Problem 5. [5 points]

Prove using contradiction that the cube root of an irrational number is irrational.

Problem 6. [5 points]

Prove that if $n^2 + 8n + 20$ is odd, then n is odd for natural numbers n.

Problem 7. [10 points]

Use mathematical induction to prove the following identities.

(a) [5 points] For all natural numbers n,

$$1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + n^2 \times (n+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$$

(b) [5 points] For all natural numbers n,

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{n \times (n+1)} = \frac{n}{n+1}$$

Problem 9. [5 points]

Functions F and G are defined by formulas. Find $G \circ F$ and $F \circ G$ and determine whether $G \circ F$ equals $F \circ G$.

$F(x) = x^5$ and $G(x) = x^{1/5}$ for all real numbers x.

- F composed with G and G composed with F is generally different, unless special situations.

Problem 10. [5 points]

Prove that the following set is countable using a diagram and a formula for the one-to-one correspondence function.

$\{\pm 1^1, \pm 2^2, \pm 3^3, \pm 4^4, \pm 5^5, \ldots\}$

- To show a set is countably infinite
- First count it
- Then formulate the counting as a function
- Then prove the function is bijective

Problem 11. [5 points]

Show how to find the units digit of 1357^{7531}.