CSE215
Foundations of Computer Science

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April 6, 2022

Many slides taken from Prof. Pramod Ganapathi (SBU). Thanks!

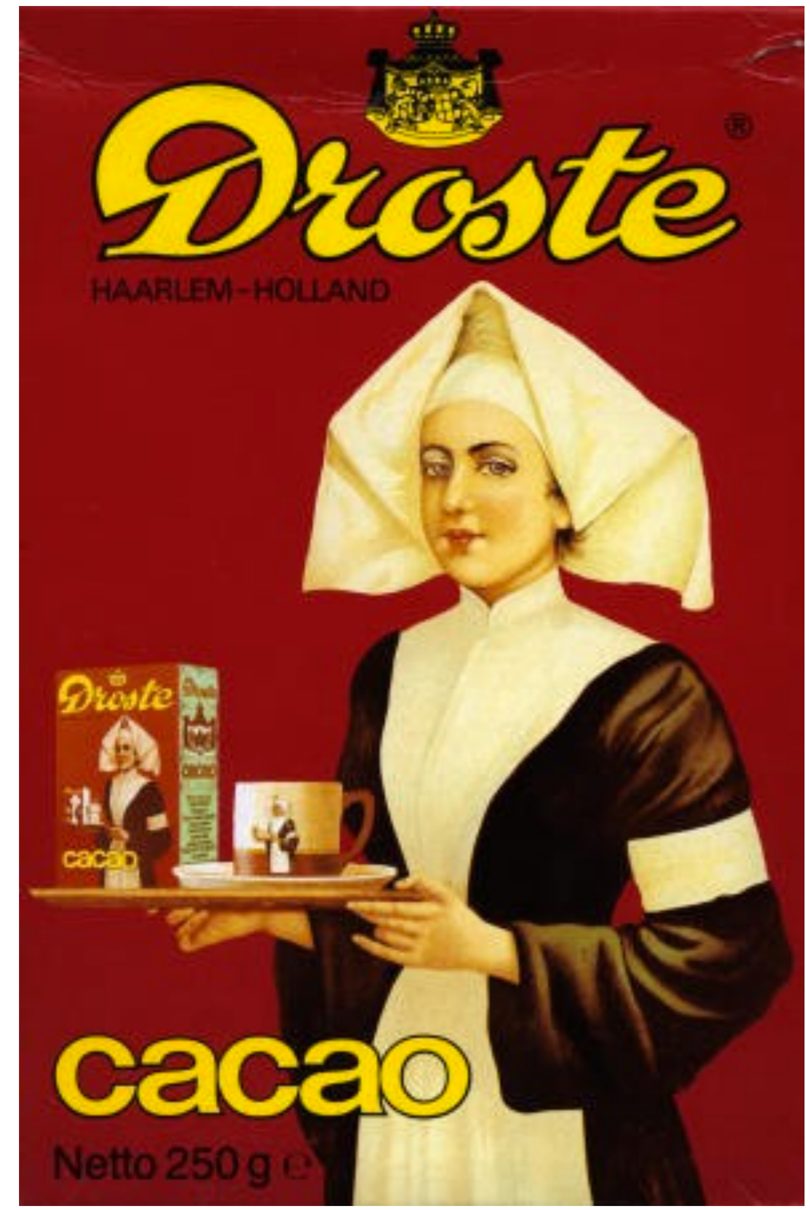
Agenda

- Recursion
- Exercises on recursion

Finish around 4h45

Recursion

- All about repeating itself
- Many forms:
  - recursive sequences
  - recursive functions
  - recursive data structures



Recursive functions

Examples

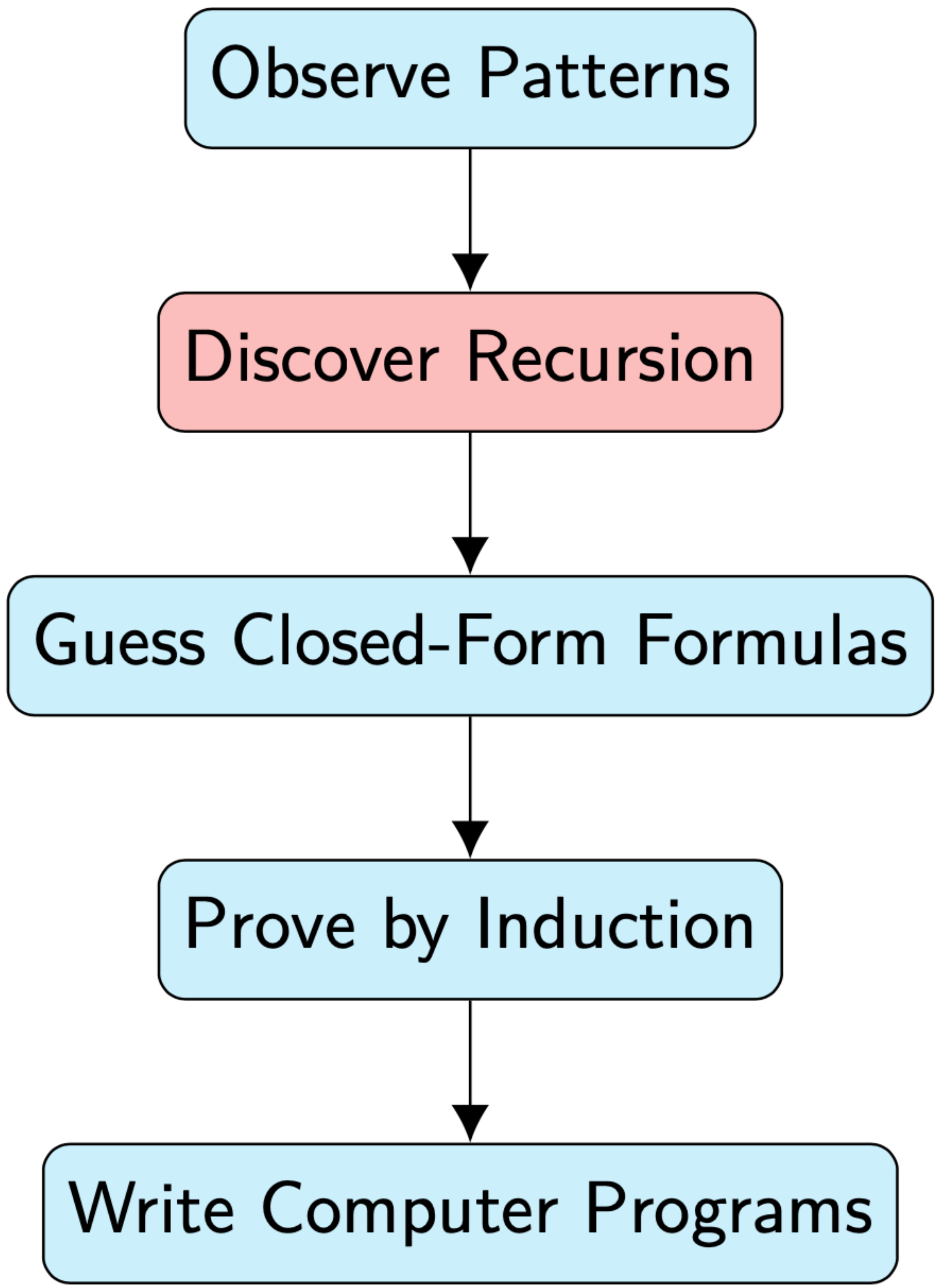
- Suppose $f(n) = n!$, where $n \in \mathbb{W}$. Then,
$$f(n) = \begin{cases} 1 & \text{if } n = 0, \\ n \cdot f(n-1) & \text{if } n \geq 1. \end{cases}$$
Closed-form formula: $f(n) = n \cdot (n-1) \cdot \cdots \cdot 1$
- Suppose $F(n) = n$th Fibonacci number. Then,
$$F(n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } 1, \\ F(n-1) + F(n-2) & \text{if } n \geq 2. \end{cases}$$
Closed-form formula: $F(n) = ?$

Recursive functions

Examples

- Suppose $M(m, n) =$ product of $m, n \in \mathbb{N}$. Then,
  $$M(m,n) = \begin{cases} m & \text{if } n = 1, \\ M(m, n-1) + m & \text{if } n \geq 2. \end{cases}$$
  Closed-form formula: $M(m, n) = m \times n$
- Suppose $E(a, n) = a^n$, where $n \in \mathbb{W}$. Then,
  $$E(a,n) = \begin{cases} 1 & \text{if } n = 0, \\ E(a, n-1) \times a & \text{if } n \geq 1. \end{cases}$$
  Closed-form formula: $E(a, n) = a^n$
- Suppose $O(n) = n$th odd number $\in \mathbb{N}$. Then,
  $$O(n) = \begin{cases} 1 & \text{if } n = 1, \\ O(n-1) + 2 & \text{if } n \geq 2. \end{cases}$$
  Closed-form formula: $O(n) = 2n - 1$

Relationship between induction and recursion



Exercise

Let $a_0, a_1, a_2, \ldots$ be the sequence defined recursively as follows: For all integers $k \geq 1$,

(1) $a_k = a_{k-1} + 2$ recurrence relation

(2) $a_0 = 1$ initial condition.

- Observe the pattern in a_1, a_2, …
- Derive an explicit formula of the sequence
- Confirm the explicit formula with mathematical induction

Exercise

- Define the sequence below in a recursive form.
    - 3, 9, 15, 21, 27, 33, ….
- Derive an explicite form
- Prove that your explicit form is right

Example: Geometric sequence (Compound interest)

Problem

- Suppose you deposit 100,000 dollars in your bank account for your newborn baby. Suppose you earn 3% interest compounded annually.
  How much will be the amount when your kid hits 21 years of age?

Solution

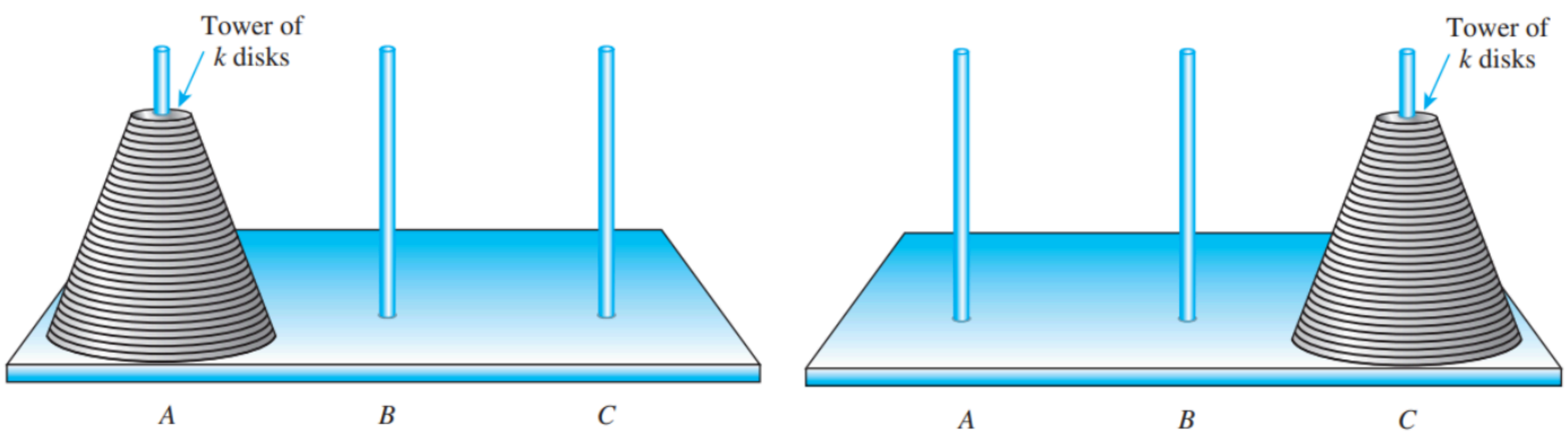
- Suppose A_k = Amount in your account after k years. Then,
  $$A_k = \begin{cases} 100,000 & \text{if } \; k = 0, \\ (1 + 3\%) \times A_{k-1} & \text{if } \; k \geq 1. \end{cases}$$
- Solving the recurrence by the method of iteration, we get
  $$\boxed{A_k = ((1.03)^k \cdot 100,000) \text{ dollars}}$$
  ▷ How?
- Homework: Prove the formula using induction
- When your kid hits 21 years, $k = 21$, therefore
  $A_{21} = ((1.03)^{21} \cdot 100,000) \approx 186,029.46$ dollars

Example: Towers of Hanoi

Problem

- There are k disks on peg 1. Your aim is to move all k disks from peg 1 to peg 3 with the minimum number of moves. You can use peg 2 as an auxiliary peg. The constraint of the puzzle is that at any time, you cannot place a larger disk on a smaller disk.

  What is the minimum number of moves required to transfer all k disks from peg 1 to peg 3?

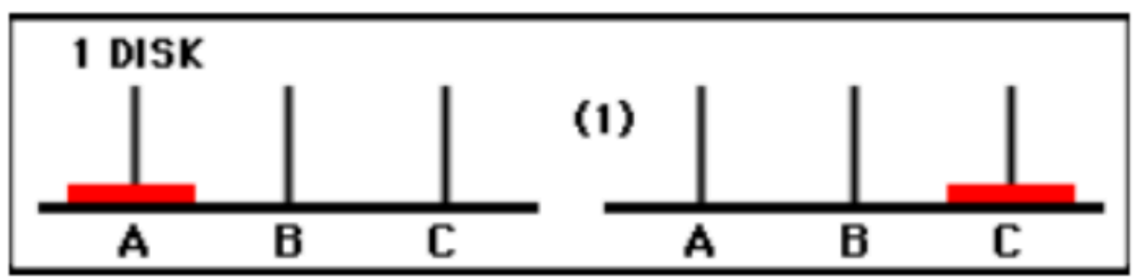


Demo: https://www.mathsisfun.com/games/towerofhanoi.html

Example: Towers of Hanoi

Solution

Suppose $k = 1$. Then, the 1-step solution is:
1. Move disk 1 from peg A to peg C.

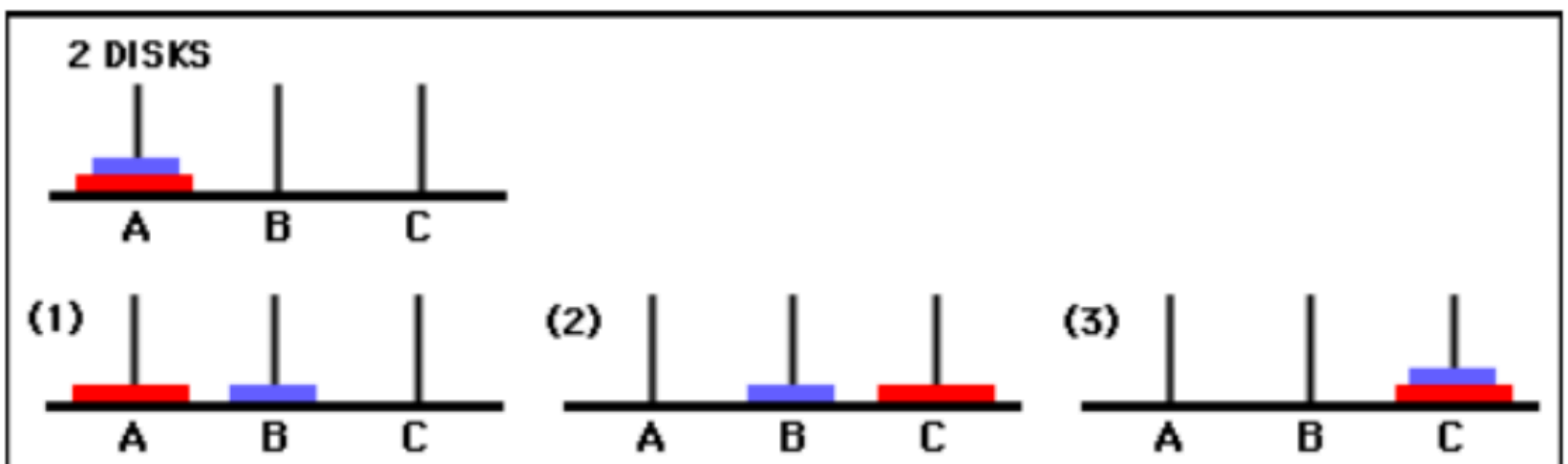


Example: Towers of Hanoi

Solution

Suppose $k = 2$. Then, the 3-step solution is:

1. Move disk 1 from peg A to peg B.
2. Move disk 2 from peg A to peg C.
3. Move disk 1 from peg B to peg C.

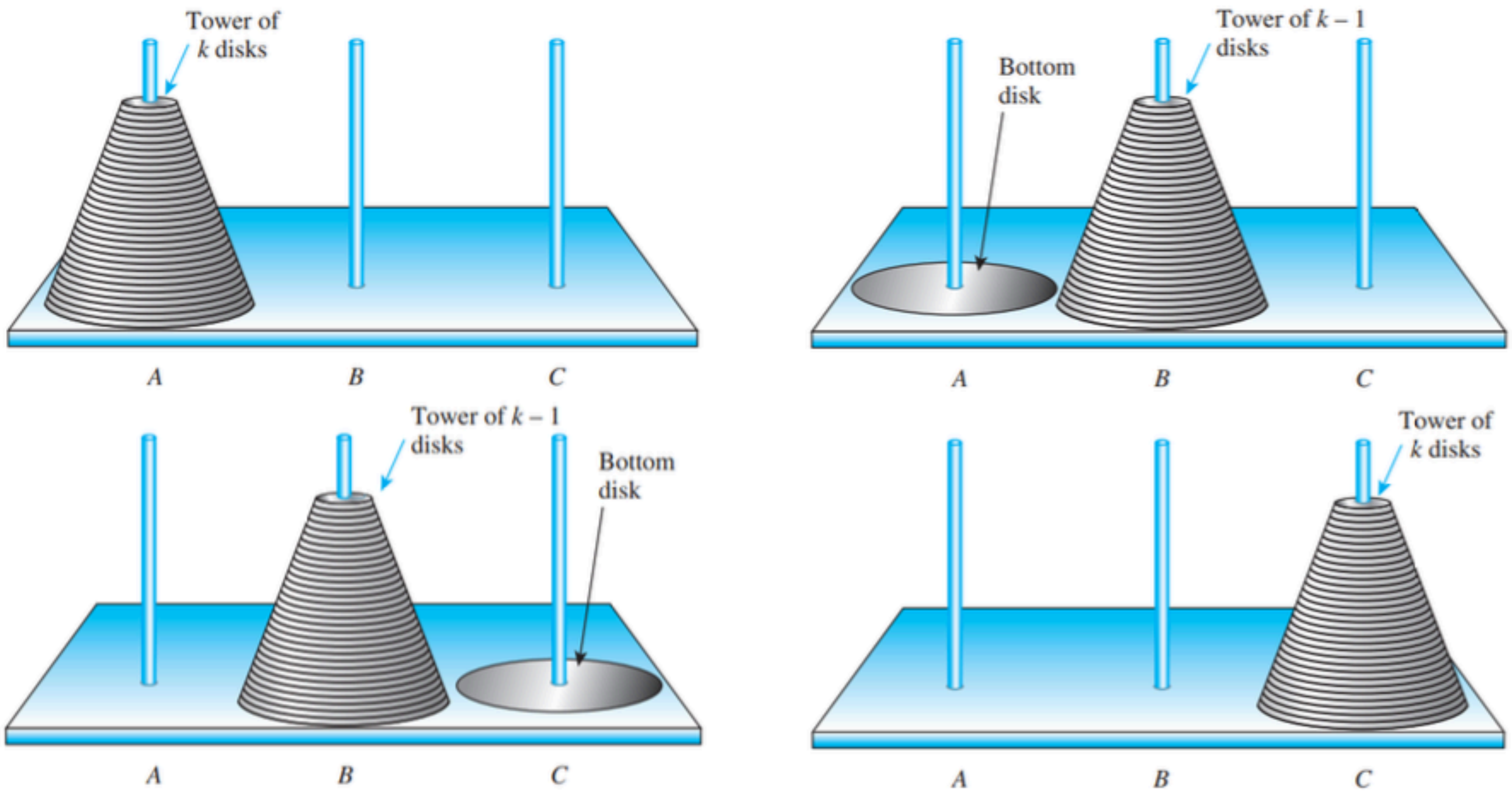


Example: Towers of Hanoi

Solution

For any $k \geq 2$, the recursive solution is:

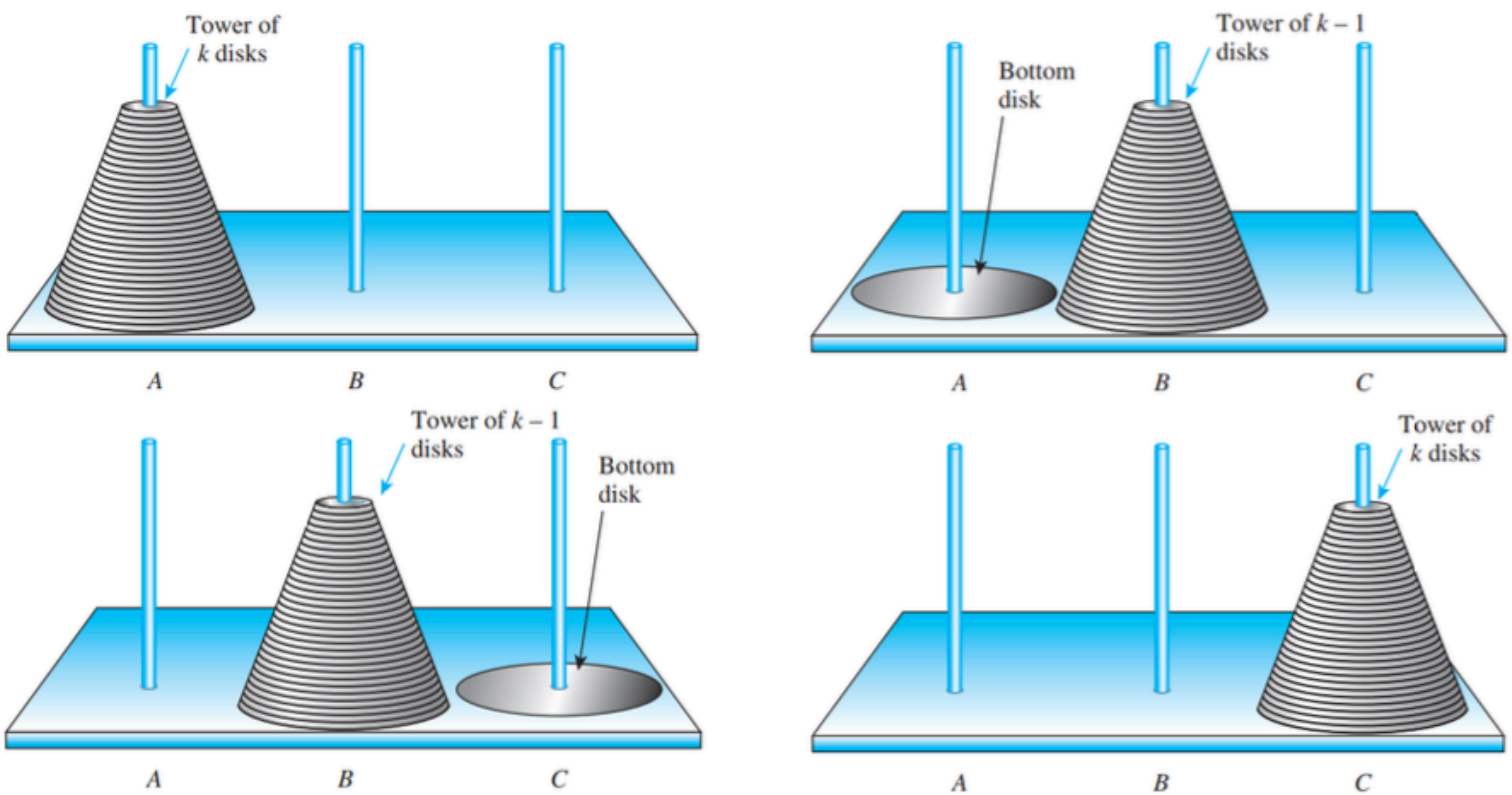
1. Transfer the top $k-1$ disks from peg A to peg B.
2. Move the bottom disk from peg A to peg C.
3. Transfer the top $k-1$ disks from peg B to peg C.



Example: Towers of Hanoi

TOWERS-OF-HANOI(k, A, C, B)

1. if $k = 1$ then
2. Move disk k from A to C.
3. elseif $k \geq 2$ then
4. TOWERS-OF-HANOI$(k-1, A, B, C)$
5. Move disk k from A to C.
6. TOWERS-OF-HANOI$(k-1, B, C, A)$



Example: Towers of Hanoi

Solution (continued)

- Let $M(k)$ denote the minimum number of moves required to move k disks from one peg to another peg. Then

$$M(k) = \begin{cases} 1 & \text{if } k = 1, \\ 2 \cdot M(k-1) + 1 & \text{if } k \geq 2. \end{cases}$$

- Solving the recurrence by the method of iteration, we get

$$\boxed{M(k) = 2^k - 1}$$

▷ How?

- Homework: Prove the formula using induction

Why minimum?

https://proofwiki.org/wiki/Tower_of_Hanoi

Exercise 1

Prove the following

If $m_1, m_2, m_3, \ldots$ is the sequence defined by

$$m_k = 2m_{k-1} + 1 \quad \text{for all integers } k \geq 2, \text{ and}$$
$$m_1 = 1,$$

then $m_n = 2^n - 1$ for all integers $n \geq 1$.

Break (if t < 4h15)

More exercises on recursion

Finish around 4h45

Exercise 2

- Consider the recursively defined sequence below. Find its explicit form and prove your answer.

$$a_k = ka_{k-1}, \text{ for all integers } k \geq 1$$
$$a_0 = 1$$

Exercise 3

- Consider the recursively defined sequence below. Find its explicit form and prove your answer.

$$b_k = \frac{b_{k-1}}{1 + b_{k-1}}, \text{ for all integers } k \geq 1$$
$$b_0 = 1$$

Exercise 4

- Consider the recursively defined sequence below. Find its explicit form and prove your answer.

$$c_k = 3c_{k-1} + 1, \text{ for all integers } k \geq 2$$
$$c_1 = 1$$