CSE215
Foundations of Computer Science

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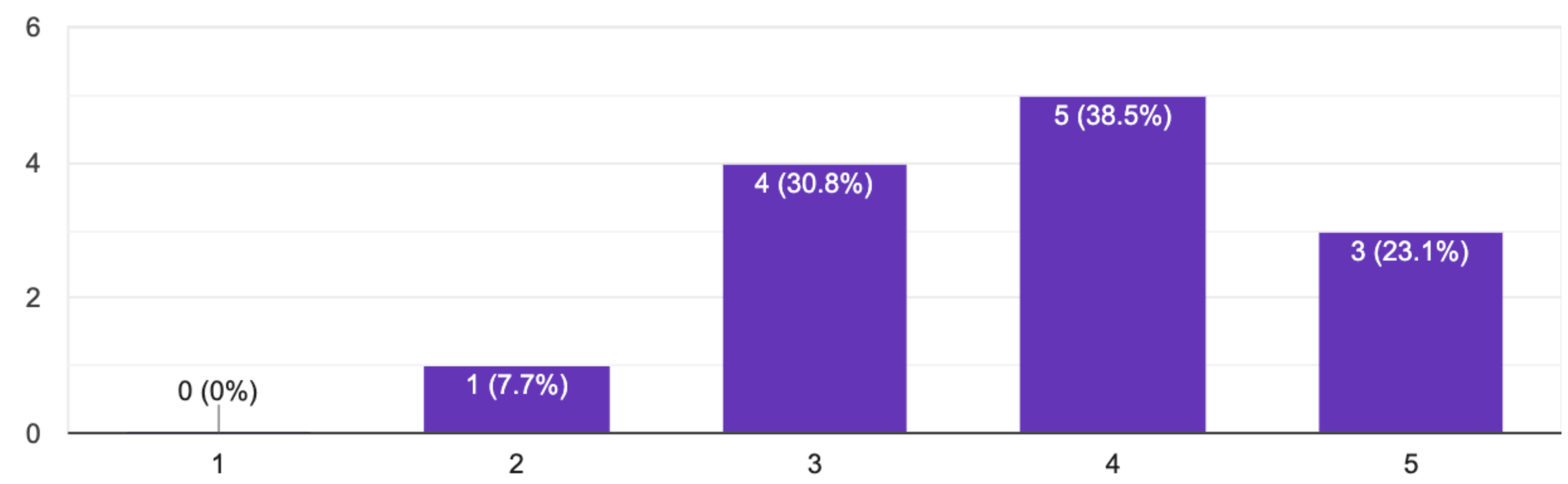
Updated!

Overall difficulty

13 responses



Needed more time

Give me some way to get extra credit please

I like the professor and TA's enthusiasm to teach and help students:)

None!

There was too much focus on sets (or too many points allocated for set-related questions in the test) when it was barely reviewed in class. I think a few more points should be given per proof question since they take a while to solve, especially with problem 9 in which the technique for proving it wasn't really covered in class that well.

nothing

I think the time was a little too short.

I think the exam questions are way more complicating than what we've covered during the class

Agenda

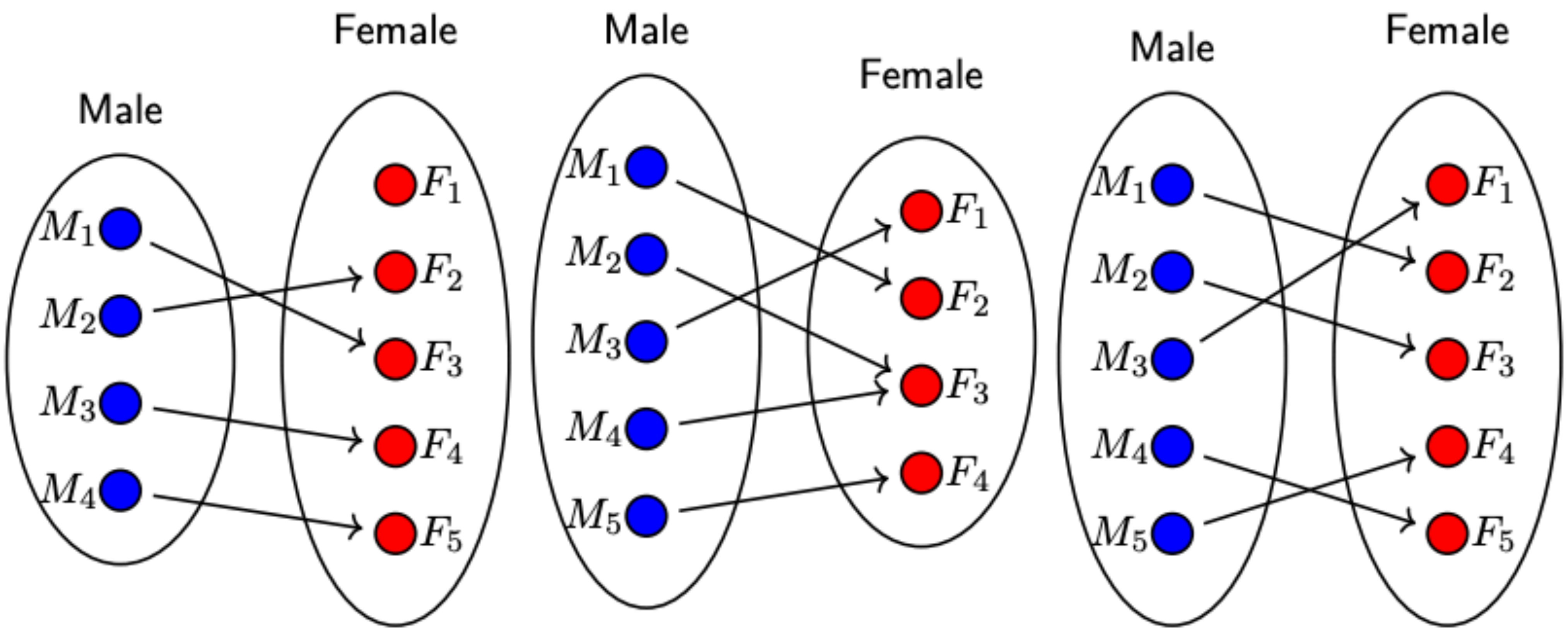
- Attendance
- One-on-one correspondence
- Exercises

To finish around 4h20

Zoom on today!

One-to-one correspondences

- What is the difference between the three marriage functions?



One-to-one correspondences

Definition

- A one-to-one correspondence (or bijection) from a set X to a set Y is a function $F : X \to Y$ that is both one-to-one and onto.

One-to-one correspondences: Example 1

Subset of $\{a,b,c,d\}$		4-tuple of $\{0,1\}$
$\{\}$	$\longrightarrow$	$(0,0,0,0)$
$\{a\}$	$\longrightarrow$	$(1,0,0,0)$
$\{b\}$	$\longrightarrow$	$(0,1,0,0)$
$\{c\}$	$\longrightarrow$	$(0,0,1,0)$
$\{d\}$	$\longrightarrow$	$(0,0,0,1)$
$\{a,b\}$	$\longrightarrow$	$(1,1,0,0)$
$\{a,c\}$	$\longrightarrow$	$(1,0,1,0)$
$\{a,d\}$	$\longrightarrow$	$(1,0,0,1)$
$\{b,c\}$	$\longrightarrow$	$(0,1,1,0)$
$\{b,d\}$	$\longrightarrow$	$(0,1,0,1)$
$\{c,d\}$	$\longrightarrow$	$(0,0,1,1)$
$\{a,b,c\}$	$\longrightarrow$	$(1,1,1,0)$
$\{a,b,d\}$	$\longrightarrow$	$(1,1,0,1)$
$\{a,c,d\}$	$\longrightarrow$	$(1,0,1,1)$
$\{b,c,d\}$	$\longrightarrow$	$(0,1,1,1)$
$\{a,b,c,d\}$	$\longrightarrow$	$(1,1,1,1)$

One-to-one correspondences: Example 2

Problem

- Define $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by the rule $F(x, y) = (x + y, x - y)$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}$. Is F a one-to-one correspondence? Prove or give a counterexample.

Proof

To show that F is a one-to-one correspondence, we need to show that:

1. F is one-to-one.
2. F is onto.

One-to-one correspondences: Example 2

Proof (continued)

- Proof that F is one-to-one.

  Suppose that (x_1, y_1) and (x_2, y_2) are any ordered pairs in $\mathbb{R} \times \mathbb{R}$ such that $F(x_1, y_1) = F(x_2, y_2)$.
  $\implies (x_1 + y_1, x_1 - y_1) = (x_2 + y_2, x_2 - y_2)$
  ($\because$ Defn. of F)
  $\implies x_1 + y_1 = x_2 + y_2$ and $x_1 - y_1 = x_2 - y_2$
  ($\because$ Defn. of equality of ordered pairs)
  $\implies x_1 = x_2$ and $y_1 = y_2$
  ($\because$ Solve the two simultaneous equations)
  $\implies (x_1, y_1) = (x_2, y_2)$
  ($\because$ Defn. of equality of ordered pairs)
  Hence, F is one-to-one.

One-to-one correspondences: Example 2

Proof (continued)

- Proof that F is onto.

  Suppose (u, v) is any ordered pair in the co-domain of F. We will show that there is an ordered pair in the domain of F that is sent to (u, v) by F.

  Let $r = \frac{u+v}{2}$ and $s = \frac{u-v}{2}$. The ordered pair (r, s) belongs to $\mathbb{R} \times \mathbb{R}$. Also,

  $F(r, s)$

  $= F(\frac{u+v}{2}, \frac{u-v}{2}) \qquad (\because$ Defn. of $F)$

  $= (\frac{u+v}{2} + \frac{u-v}{2}, \frac{u+v}{2} - \frac{u-v}{2}) \qquad (\because$ Substitution$)$

  $= (u, v) \qquad (\because$ Simplify$)$

  Hence, F is onto.

Exercises

To finish around 4h20

Exercise 0
SBU 2022 Midterm

Problem 3. [5 points]

Let $X = \{1, 2, 3\}$, $Y = \{1, 2, 3, 4\}$ and $Z = \{1, 2\}$. Use arrow diagrams to define functions.

1. Define a function $f : X \to Y$ that is one-to-one but not onto.
2. Define a function $g : X \to Z$ that is onto but not one-to-one.
3. Define a function $h : X \to X$ that is neither one-to-one nor onto.
4. Define a function $k : X \to X$ that is one-to-one and onto but is not the identity function on X.

Exercise 1

A function $f : Z \times Z \to Z \times Z$ is defined as $f(m, n) = (m + n, 2m + n)$. Verify (1) if the function is one-to-one and (2) if it is onto.

Solution (partial)

- We prove f is one-to-one. Let (m1,n1), (m2,n2) are two elements of Z x Z, we show f(m1,n1) = f(m2,n2) implies (m1,n1)=(m2,n2) ….

- We show f is onto. Let (p,q) be an arbitrary element in Z x Z. We show there exists m and n such that f(m,n)= (p,q)

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Exercise 2

Prove that the function $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{5\}$ defined by $f(x) = \dfrac{5x+1}{x-2}$ is bijective.

Solution

Proof. First, let's check that f is injective. Suppose $f(x) = f(y)$. Then

$$\begin{aligned}
\frac{5x+1}{x-2} &= \frac{5y+1}{y-2} \\
(5x+1)(y-2) &= (5y+1)(x-2) \\
5xy - 10x + y - 2 &= 5yx - 10y + x - 2 \\
-10x + y &= -10y + x \\
11y &= 11x \\
y &= x.
\end{aligned}$$

Since $f(x) = f(y)$ implies $x = y$, it follows that f is injective.
Next, let's check that f is surjective. For this, take an arbitrary element $b \in \mathbb{R} - \{5\}$. We want to see if there is an $x \in \mathbb{R} - \{2\}$ for which $f(x) = b$, or $\frac{5x+1}{x-2} = b$. Solving this for x, we get:

$$\begin{aligned}
5x + 1 &= b(x-2) \\
5x + 1 &= bx - 2b \\
5x - xb &= -2b - 1 \\
x(5-b) &= -2b - 1.
\end{aligned}$$

Since we have assumed $b \in \mathbb{R} - \{5\}$, the term $(5-b)$ is not zero, and we can divide with impunity to get $x = \dfrac{-2b-1}{5-b}$. This is an x for which $f(x) = b$, so f is surjective.
Since f is both injective and surjective, it is bijective. ■

Exercise 3

Prove the function $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ defined by $f(x) = \left(\dfrac{x+1}{x-1}\right)^3$ is bijective.

Exercise 4

- Consider the function $\theta : \{0,1\} \times \mathbb{N} \to \mathbb{Z}$ defined as $\theta(a,b) = (-1)^a b$. Is θ injective? Is it surjective? Explain.

Solution

First we show that θ is injective. Suppose $\theta(a,b)=\theta(c,d)$. Then $(-1)^a b=(-1)^c d$. As b and d are both in $\mathbb{N}$, they are both positive. Then because $(-1)^a b=(-1)^c d$, it follows that $(-1)^a$ and $(-1)^c$ have the same sign. Since each of $(-1)^a$ and $(-1)^c$ equals ± 1, we have $(-1)^a=(-1)^c$, so then $(-1)^a b=(-1)^c d$ implies $b=d$. But also $(-1)^a=(-1)^c$ means a and c have the same parity, and because $a,c\in\{0,1\}$, it follows $a=c$. Thus $(a,b)=(c,d)$, so θ is injective.
Next note that θ **is not surjective** because $\theta(a,b)=(-1)^a b$ is either positive or negative, but never zero. Therefore there exist no element $(a,b)\in\{0,1\}\times\mathbb{N}$ for which $\theta(a,b)=0\in\mathbb{Z}$.

Exercise 5
SBU 2022 Midterm

Problem 4. [5 points]

Let A and B be finite sets where $|A| = |B|$. Is it possible to define a function $f : A \to B$ that is one-to-one but not onto? Is it possible to define a function $g : A \to B$ that is onto but not one-to-one?

Exercise 6
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Problem 5. [5 points]

We know that $\mathcal{P}(X)$ is a power set of set X. Let $A = \{1, 2, 3, \ldots, 10\}$. Consider the function $f : \mathcal{P}(A) \to \mathbb{W}$ given by $f(B) = |B|$, where $\mathbb{W}$ is a set of whole numbers (which includes 0). That is, f takes a subset of A as an input and it outputs the cardinality of that set.

Is f one-to-one? Prove your answer.
Is f onto? Prove your answer.