CSE215
Foundations of Computer Science

Instructor: Zhoulai Fu

State University of New York, Korea

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Agenda

- Homework week05 problems
- Solving exam problems on mathematical induction

Finish around 4h25

Exercise 1 (points = 40)

Determine whether the statements below are true or false.

1. 119 is a prime number.
2. 161 is a prime number.
3. $42k$ is an even number for any integer k.
4. For each integer n with $2 <= n <= 6$, $n^2 - n + 11$ is a prime number.
5. The average of any two odd integers is odd.
6. For any real number x, if $x * x >= 4$, then $x >= 2$.
7. For any real numbers x and y, $x^2 - 2xy + y^2 >= 0$.
8. There exists an integer x, such that $(2x + 1)^2$ is even.

Solution

1. 119 is a prime number.

- `false`
  - It is divisible by 7.

2. 161 is a prime number.

- `false`
  - It is also divisible by 7.

3. 42k is an even number for any integer k.

- `true`
  - `42k = 2(21k) which is even number`

4. For each integer n with 2 <= n <= 6, n^2 − n + 11 is a prime numbe

- `true`
  - 4 - 2 + 11 = 13
  - 9 - 3 + 11 = 17
  - 16 - 4 + 11 = 23
  - 25 - 5 + 11 = 31
  - 36 - 6 + 11 = 41
  - Those are all the primes

5. The average of any two odd integers is odd.

- `false`
  - (1+3)/2 = 2, it's even.

1. For any real number x, if x * x >= 4, then x >= 2.

- `false`
  - (-2) * (-2) >= 4, but -2 < 2.

7. For any real numbers x and y, x^2 - 2xy + y^2 >= 0.

- `true`
  - x^2 - 2xy + y^2 = (x-y)^2 >= 0

8. There exists an integer x, such that (2x + 1)^2 is even.

- `false`
  - (2x + 1)^2 = 4x^2 + 4x + 1 = 2(2x^2 + 2x) + 1 = 2k + 1, it's always odd.

Exercise 2 (points = 5)

Prove the following statement: There exist two integers m and n such that m > 1 and n>1 and 1/m + 1/n is an integer.

Exercise 3 (points = 5)

Prove the following statement: There is an integer n such that 2n^2 - 5n +2 is prime.

Solution

Exercise 2 (points = 5)

Prove the following statement: There exist two integers m and n such that m > 1 and n>1 and 1/m + 1/n is an integer.

Answer 2

- Suppose that m = 2 and n = 2.
- Then, 1/m + 1/n = 1/2 + 1/2 = 1.
- 1 is integer. (from the definition)
- Therefore, there exist such two integers with m = 2, n = 2.

Exercise 3 (points = 5)

Prove the following statement: There is an integer n such that 2n^2 - 5n +2 is prime.

Answer 3

- Suppose that n = 3.
- Then, 2n^2 - 5n + 2 = 2 * 9 - 15 + 2 = 5.
- 5 is prime. (from the definition)
- Therefore, there is such integer with n = 3.

Exercise 4 (points = 10)

Prove the following proposition: An even number multiplied by an integer is an even number.

Exercise 5 (points = 10)

Prove the following proposition: An odd number multiplied by an odd number is an odd number.

Solution

Exercise 4 (points = 10)

Prove the following proposition: An even number multiplied by an integer is an even number.

Answer 4

- From the definition of even number, we can rewrite all even numbers as `2k` (k ∈ Z).
- Suppose that an integer multiplier is `m` (m ∈ Z)
- Then an even number multiplied by an ineger would be `2km`. (multiplication operation on integers)
- `2km = 2p` (p ∈ Z). (k*m = p and multiplication on integers is closed)
- `2p` is even number. (because of the definition of even number)
- In other words, an even number multiplied by an integer is an even number.

Exercise 5 (points = 10)

Prove the following proposition: An odd number multiplied by an odd number is an odd number.

Answer 5

- From the definition of odd number,
  - All odd numbers are able to be rewrited as `2k+1` (k ∈ Z).
  - An integer multiplier will be rewrited as `2m+1` (m ∈ Z)
- Then an odd number multiplied by an odd ineger would be `(2k+1)(2m+1)`. (multiplication operation on integers)
- `(2k+1)(2m+1)` = `4km + 2k + 2m + 1` (expanding the binomial)
- `4km + 2k + 2m + 1` = `2(2km + k + m) + 1` (factoring 2 from first three terms)
- `2(2km + k + m) + 1` = `2j + 1` (let j = 2km + k + m)
- `j` is an integer as mult. and add. are closed on integers
- Therefore, `(2k+1)(2m+1)` is odd number.
- In other words, an odd number multiplied by an odd number is an odd number.

Exercise 6 [points = 10)

We say an integer is a perfect square if it can be expressed as a square of some integer. For example, 81 is a perfect square; 80 is not.

Prove the following statement: there is a perfect square that can be written as a sum of two other perfect squares.

Wrong proof

	Exercise 6
	Prove the following statement
	There is a perfect square that can be written as a sum of two other perfect squares
	∃ a, b, and c ∈ Z, c^2 = a^2 + b^2
	Pytagorean theorem proves that statement
	∴ True

Solution

Exercise 6 [points = 10)

We say an integer is a perfect square if it can be expressed as a square of some integer. For example, 81 is a perfect square; 80 is not.

Prove the following statement: there is a perfect square that can be written as a sum of two other perfect squares.

Answer 6

- Suppose that above values are `a`, `b`, `c`, respectively.
- prove `∃x,y,z, such that x^2 + y^2 = z^2 and x,y,z ∈ Z and x ≠ y ≠ z` (let a = x^2, b = y^2, c = z^2)
- Suppose `x=3`, `y=4`, and `z=5`
- Then, `x^2 + y^2 = z^2` = `3^2 + 4^2 = 5^2` = `9 + 16 = 25`
- 9, 16, 25 are different perfect square.
- Therefore, there is a perfect square that can be written as a sum of two other perfect squares.

Exercise 7 (points = 20)

Below are two definitions of odd numbers.

- Definition 1. An integer n is an odd number if n = 2k+1 for some integer k.
- Definition 2. An integer n is an odd number if n = 2k-1 for some integer k.

Prove the two definitions are equivalent following the two steps below.

1. First, prove any odd number n defined in the sense of Definition 1 is also an odd number defined in the sense of Definition 2.
2. Second, prove any odd number n defined in Definition 2 is also an odd number defined in the sense of Definition 1.

Wrong proof

1. First, prove any odd number n defined in the sense of Definition 1 is also an odd number defined in the sense of Definition 2.
   - An integer n is an odd number if $n = 2k + 1$ for some integer k.
   - If k is an integer, $(k - 1)$ is also an integer.
   - Substitute $(k - 1)$ to k.
   - Then, $n = 2(k - 1) + 1 = 2k - 2 + 1 = 2k - 1$.
   - Therefore, n defined in the sense of Definition 1 is also an odd number defined in the sense of Definition 2.
   - QED
2. Second, prove any odd number n defined in Definition 2 is also an odd number defined in the sense of Definition 1.
   - An integer n is an odd number if $n = 2k - 1$ for some integer k.
   - If k is an integer, $(k + 1)$ is also an integer.
   - Substitute $(k + 1)$ to k.
   - Then, $n = 2(k + 1) - 1 = 2k + 2 - 1 = 2k + 1$.

Solution

- Suppose n is an odd number in the sense of Definition 1
- Then, there exists an integer k such that $n = 2k+1$
- Thus, $n = 2(k+1)-1$
- Thus, n is an odd number in the sense of Definition 2

- Suppose n is an odd number in the sense of Definition 2
- Then, there exists an integer k such that $n = 2k-1$
- Thus $n = 2(k-1) + 1$
- Thus, n is an odd number in the sense of Definition 1

Solving exam problems on mathematical induction

Finish around 4h25

Use mathematical induction to prove the following identities.

(a) [5 points] For all integers $n \geq 1$,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

- Proof.
  - We will prove this proposition with mathematical induction.
  - Let P(n) be the predicate 1(1!) + 2(2!) + … + n(n!) = (n+1)! -1
  - We first prove P(1) holds.
    - LHS is 1, and RHS is 2! - 1=1
  - Then, we prove P(k) -> P(k+1) for all integer k>=1
    - Let k be an arbitrary integer and k>=1.
    - Assume P(k) holds. That is 1(1!) + 2(2!) + … + k(k!) = (k+1)! -1
    - We want to prove P(k+1), namely, 1(1!) + 2(2!) + … + k(k!) + (k+1) (k+1)! = (k+2)! -1
      - LHS can be reduced to (k+1)! -1 + (k+1) (k+1)! following assumption P(k)
      - The latter can be further reduced to (k+2)! -1, namely RHS.
- QED.

Use mathematical induction to prove the following identities.

(b) [5 points] Consider the Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$. For all integers $n \geq 2$,

$$f_{n+2} = 1 + \sum_{i=0}^{n} f_i.$$

- Proof.
  - We will prove this proposition with mathematical induction.
  - Let P(n) be the predicate f_{n+2} = 1+ f_0+f_1 +… + f_n
  - We first prove P(2) holds.
    - LHS = f_4 = 3. RHS = 1+ f_0 + f_1 + f_2 = 1+0+1+1=3
  - Then, we prove P(k) -> P(k+1) for all k>=2
    - Let k be an arbitrary integer and k>=2
    - Assume P(k) holds, namely,   f_{k+2} = 1+ f_0+f_1 +… + f_k
    - We want to prove P(k+1), that is, f_{k+3} = 1+ f_0+f_1 +… + f_k+f_{k+1}
      - LHS = f_{k+2}+f_{k+1} from Definition of Fobonacci Sequence.
      - Thus, LHS = 1+ f_0+f_1 +… + f_k + f_{k+1} following assumption P(k)
      - Thus equals RHS
- QED.=

Final 2020

Use mathematical induction to prove the following identities.

(a) [5 points] For all natural numbers n,

$$1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + n^2 \times (n+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$$

- Proof.
  - We will prove this proposition with mathematical induction.
  - Let P(n) be the predicate 1^2 * 2 + … + n^2 * (n+1) = n (n+1) (n+2) (3n+1)/12
  - We first prove P(1) holds.
    - LHS = 1^2 * 2=2. RHS = 1*2*3*4/12 = 2.
  - Then, we prove P(k) -> P(k+1) for all k>=1
    - Let k be an arbitrary integer and k>=1.
    - Assume P(k) holds. That is, 1^2 * 2 + … + k^2 * (k+1) = k (k+1) (k+2) (3k+1)/12
    - We want to prove P(k+1), that is, 1^2 * 2 + … + k^2 * (k+1) + (k+1)^2 * (k+2) = (k+1) (k+2) (k+3) (3k+4)/12
      - LHS = k (k+1) (k+2) (3k+1)/12 + + (k+1)^2 * (k+2) following assumption P(k)
      - The latter can be further reduced to (k+1)(k+2)/12 * (3k^2+k + 12k+12) = RHS
- QED.

Final 2020

Use mathematical induction to prove the following identities.

(b) [5 points] For all natural numbers n,

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{n \times (n+1)} = \frac{n}{n+1}$$

- Proof.
  - We will prove this proposition with mathematical induction.
  - Let P(n) be the predicate 1/(1*2) + 1(2*3) + … + 1/(n *(n+1)) = n/(n+1)
  - We first prove P(1) holds.
    - LHS is 1/2, and RHS is 1/(1+1) = 1/2
  - Then, we prove P(k) -> P(k+1) for all integer k>=1
    - Let k be an arbitrary integer and k>=1.
    - Assume P(k) holds. That is 1/(1*2) + 1(2*3) + … + 1/(k*(k+1)) = k/(k+1)
    - We want to prove P(k+1), namely,1/(1*2) + 1(2*3) + … + 1/(k*(k+1)) + 1/((k+1)*(k+2))= (k+1)/(k+2)
      - LHS can be reduced to k/(k+1) + 1/((k+1)*(k+2)) following the assumption P(k)
      - The latter can be further reduced to (k^2+2k+1)/((k+2)(k+1)), which equals to RHS.
- QED.

Summary question: How to prove/ disprove these kinds of propositions?

- for all x, P(x)
- for all x in Nat, P(x)
- there exists x, such that P(x)
- for all x, A(x) -> B(x)