CSE215
Foundations of Computer Science

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Agenda

- Attendance
- Pigeonhole principle
- Inverse functions
- Function composition

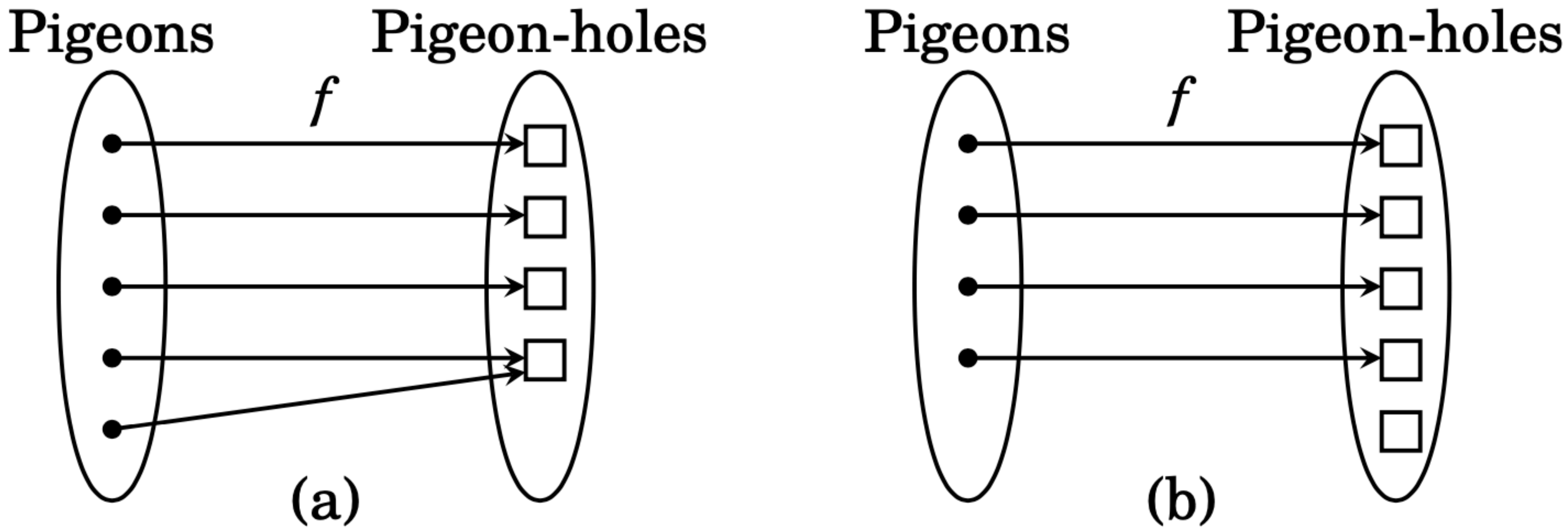
To finish around 4h45

Zoom on today!

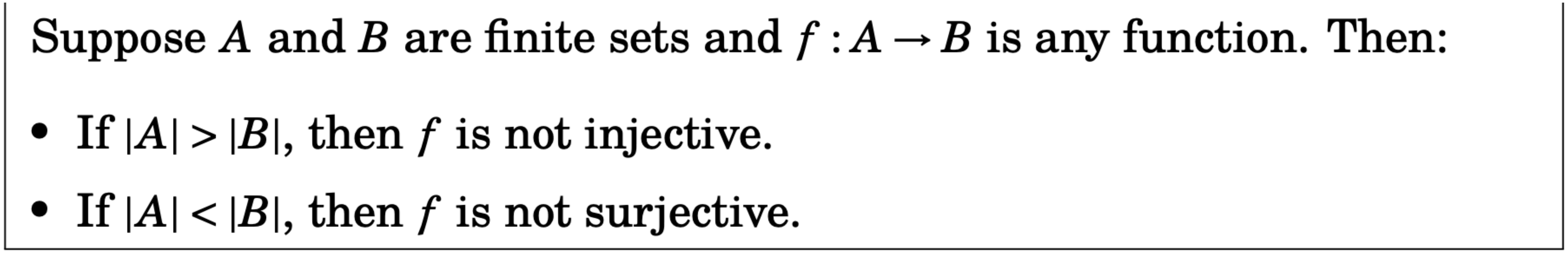
The Pigeonhole Principle

Intuition

- Imagine there is a set A of pigeons and a set B of pigeonholes, and all the pigeons fly into the pigeonholes. You can think of this as describing a function f : A → B, where pigeon X flies into pigeonhole f(X).



The Pigeonhole Principle (function version)

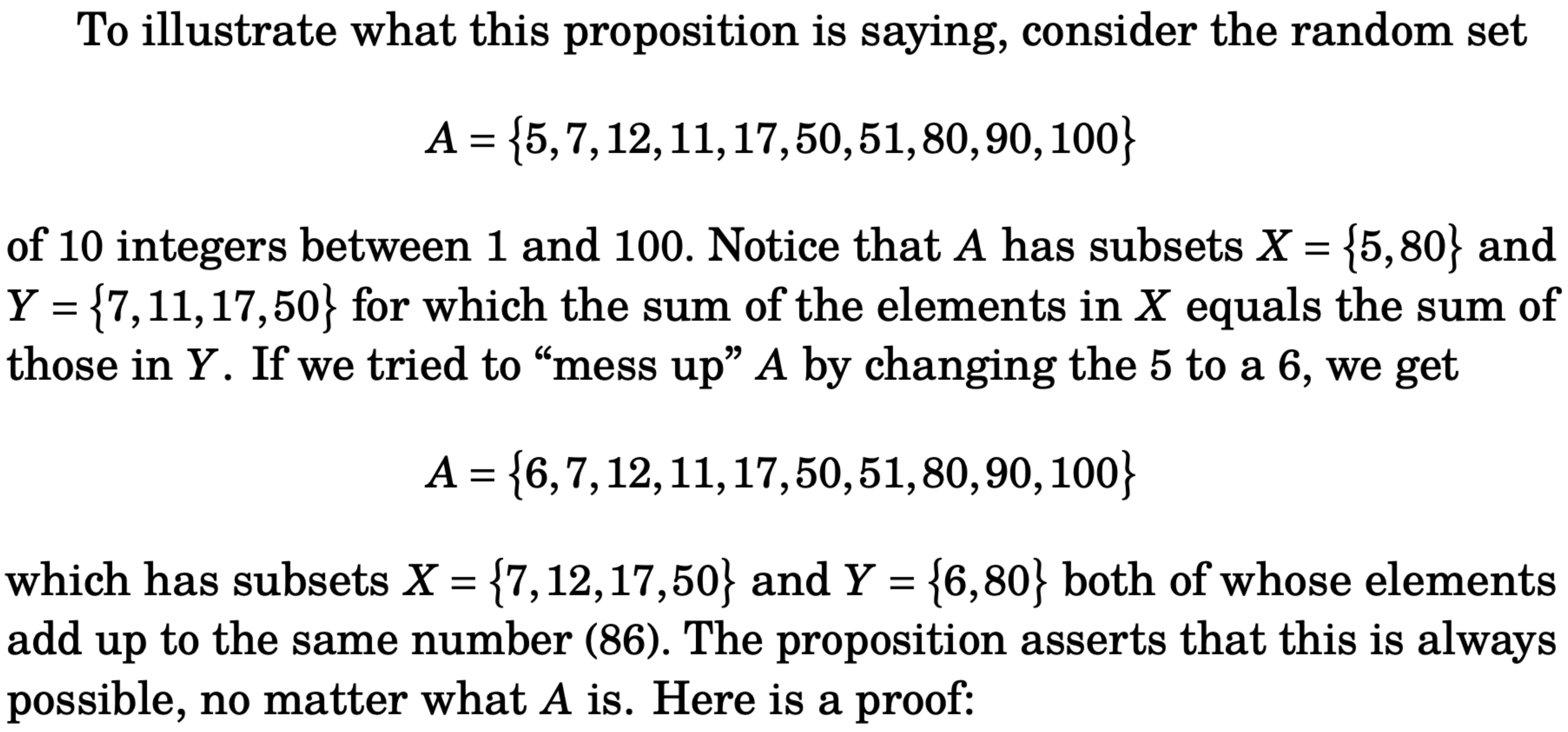


Suppose A and B are finite sets and $f : A \to B$ is any function. Then:

- If $|A| > |B|$, then f is not injective.
- If $|A| < |B|$, then f is not surjective.

Example 1

- Prove the following statement: If A is any set of 10 integers between 1 and 100, then there exist two different subsets X ⊆ A and Y ⊆ A for which the sum of elements in X equals the sum of elements in Y .



To illustrate what this proposition is saying, consider the random set

$$A = \{5, 7, 12, 11, 17, 50, 51, 80, 90, 100\}$$

of 10 integers between 1 and 100. Notice that A has subsets $X = \{5, 80\}$ and $Y = \{7, 11, 17, 50\}$ for which the sum of the elements in X equals the sum of those in Y. If we tried to "mess up" A by changing the 5 to a 6, we get

$$A = \{6, 7, 12, 11, 17, 50, 51, 80, 90, 100\}$$

which has subsets $X = \{7, 12, 17, 50\}$ and $Y = \{6, 80\}$ both of whose elements add up to the same number (86). The proposition asserts that this is always possible, no matter what A is. Here is a proof:

Solution

Proof. Suppose $A \subseteq \{1,2,3,4,\ldots,99,100\}$ and $|A| = 10$, as stated. Notice that if $X \subseteq A$, then X has no more than 10 elements, each between 1 and 100, and therefore the sum of all the elements of X is less than $100 \cdot 10 = 1000$. Consider the function

$$f : \mathscr{P}(A) \to \{0,1,2,3,4,\ldots,1000\}$$

where $f(X)$ is the sum of the elements in X. (Examples: $f\big(\{3,7,50\}\big) = 60$; $f\big(\{1,70,80,95\}\big) = 246$.) As $|\mathscr{P}(A)| = 2^{10} = 1024 > 1001 = \big|\{0,1,2,3,\ldots,1000\}\big|$, it follows from the pigeonhole principle that f is not injective. Therefore there are two unequal sets $X, Y \in \mathscr{P}(A)$ for which $f(X) = f(Y)$. In other words, there are subsets $X \subseteq A$ and $Y \subseteq A$ for which the sum of elements in X equals the sum of elements in Y. ■

Example 2

- Prove the following statement: There are at least two people in Incheon with the same number of hairs on their heads.

- We accept two facts. First, the population of Incheon is around 2.93 million. Second, it is a biological fact that every human head has fewer than one million hairs.

Solution

- Let A be the set of all people of Incheon and let B = {0,1,2,3,4,…,1000000}. Let f : A → B be the function for which f (x) equals the number of hairs on the head of x. Since |A| > |B|, the pigeonhole principle asserts that f is not injective. Thus there are two people of Incheon x and y for whom f(x) = f (y), meaning that they have the same number of hairs on their heads.

Exercise 1

- Prove that if six numbers are chosen at random, then at least two of them have the same remainder when divided by 5.

Solution

- Suppose we randomly choose 6 integers.
- Let A be the set of the six integers.
- Let B the the set {0,1,2,3,4}
- Let f: A -> B be the function defined as f(a) = a mod 5
- Then f cannot be one-to-one
- Therefore there exists a1, a2 of A such that f(a1) = f(a2)

Exercise 2

- Prove that if a is a natural number, then there exist two unequal natural numbers k and l for which a^k – a^l is divisible by 10.

Solution

- Suppose we randomly choose a natural number “a”.

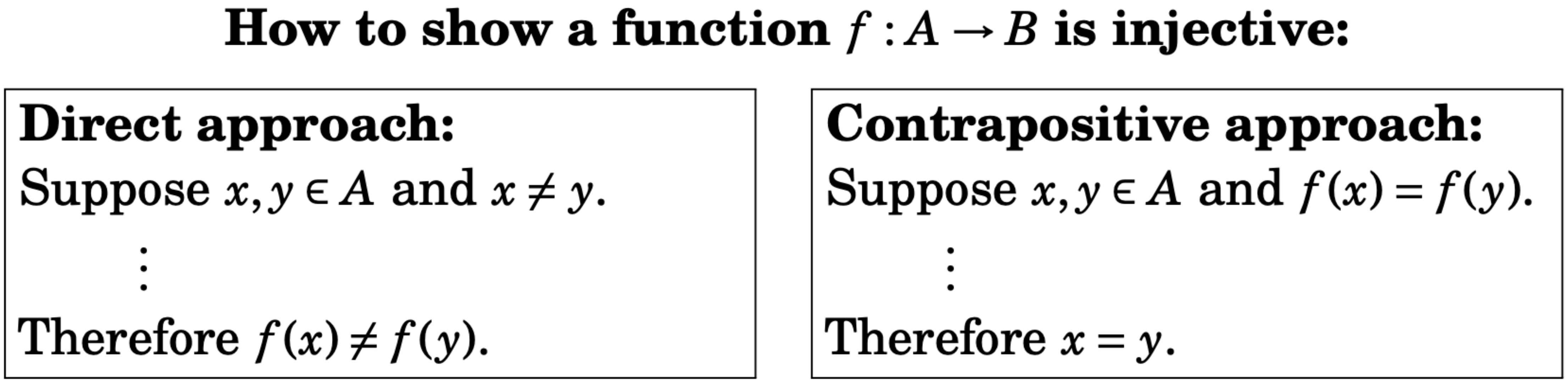
- Let f: N -> {0,1,2,…,9} be a function defined as f(k) = last digit of a^k

- Following the pigeonhole principle, f cannot be injective.

- Thus there exists k and l such that f(k) = f(l)

- Thus a^k and a^l have the same last digit. Thus a^k - a^l is a multiple of 10.

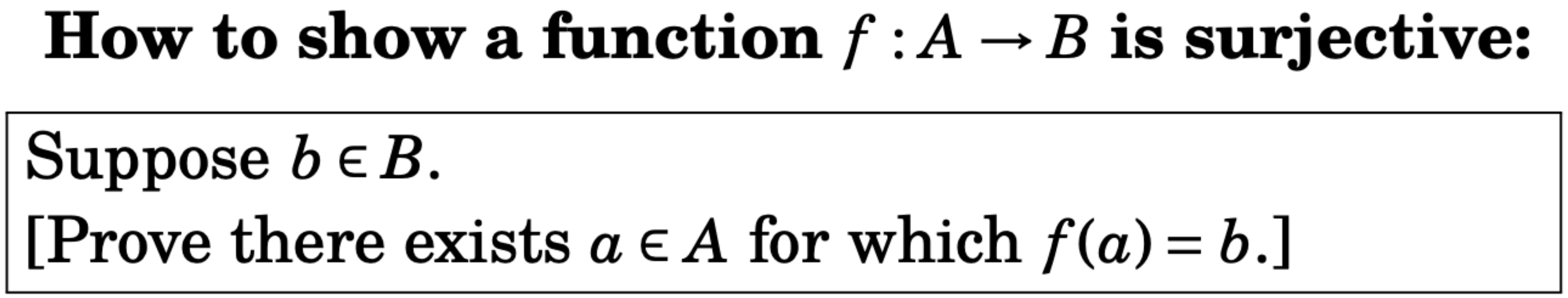
Summary: one-to-one and onto functions



How to show a function $f : A \to B$ is injective:

Direct approach:
Suppose $x, y \in A$ and $x \neq y$.
$\vdots$
Therefore $f(x) \neq f(y)$.

Contrapositive approach:
Suppose $x, y \in A$ and $f(x) = f(y)$.
$\vdots$
Therefore $x = y$.



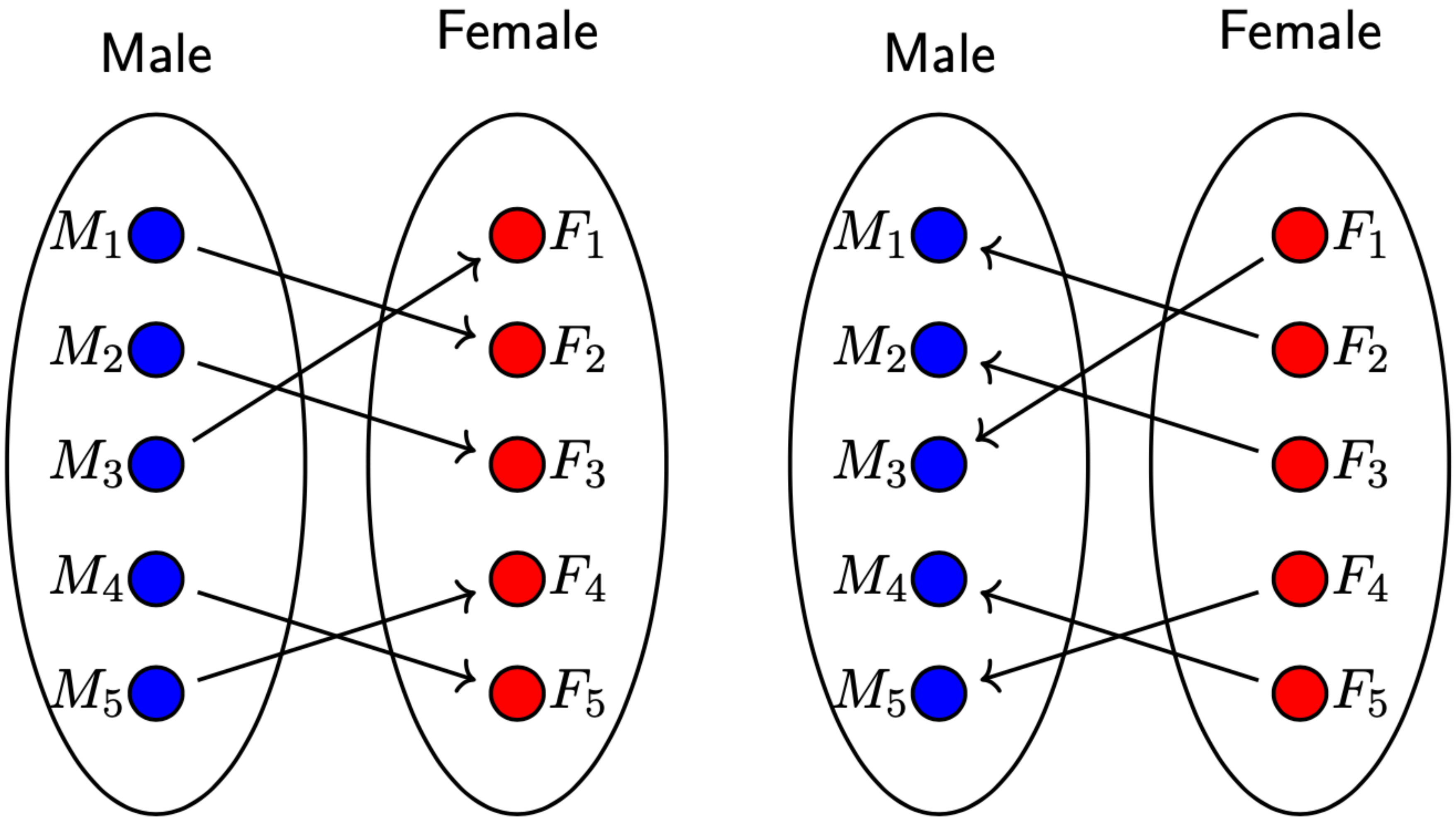
How to show a function $f : A \to B$ is surjective:

Suppose $b \in B$.
[Prove there exists $a \in A$ for which $f(a) = b$.]

Inverse functions

Inverse functions

- What is the difference between the two marriage functions?

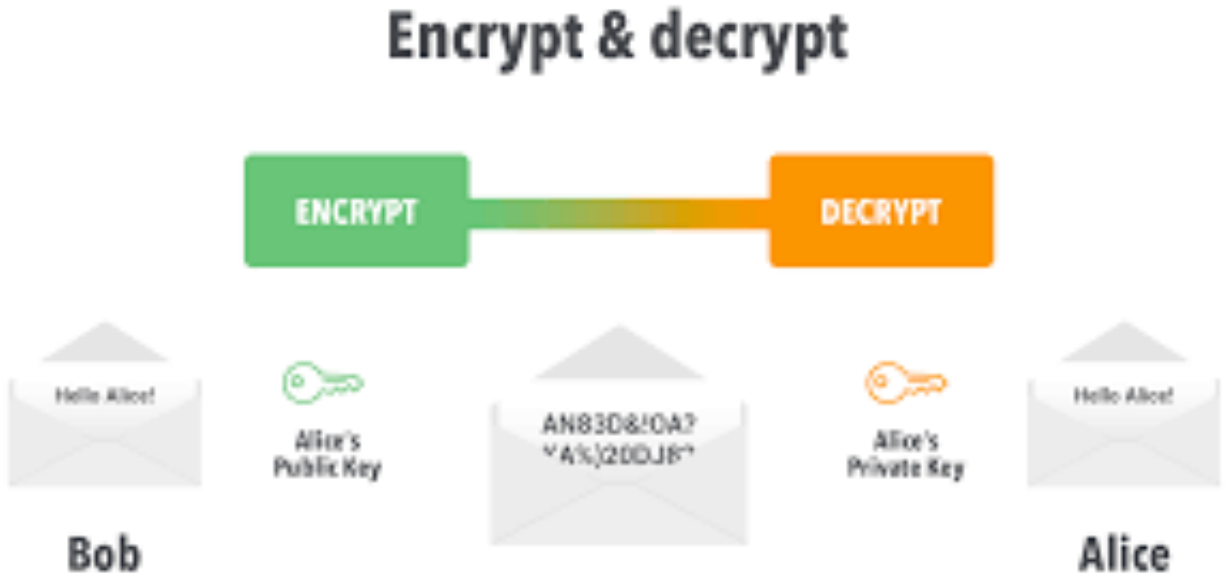


Inverse functions

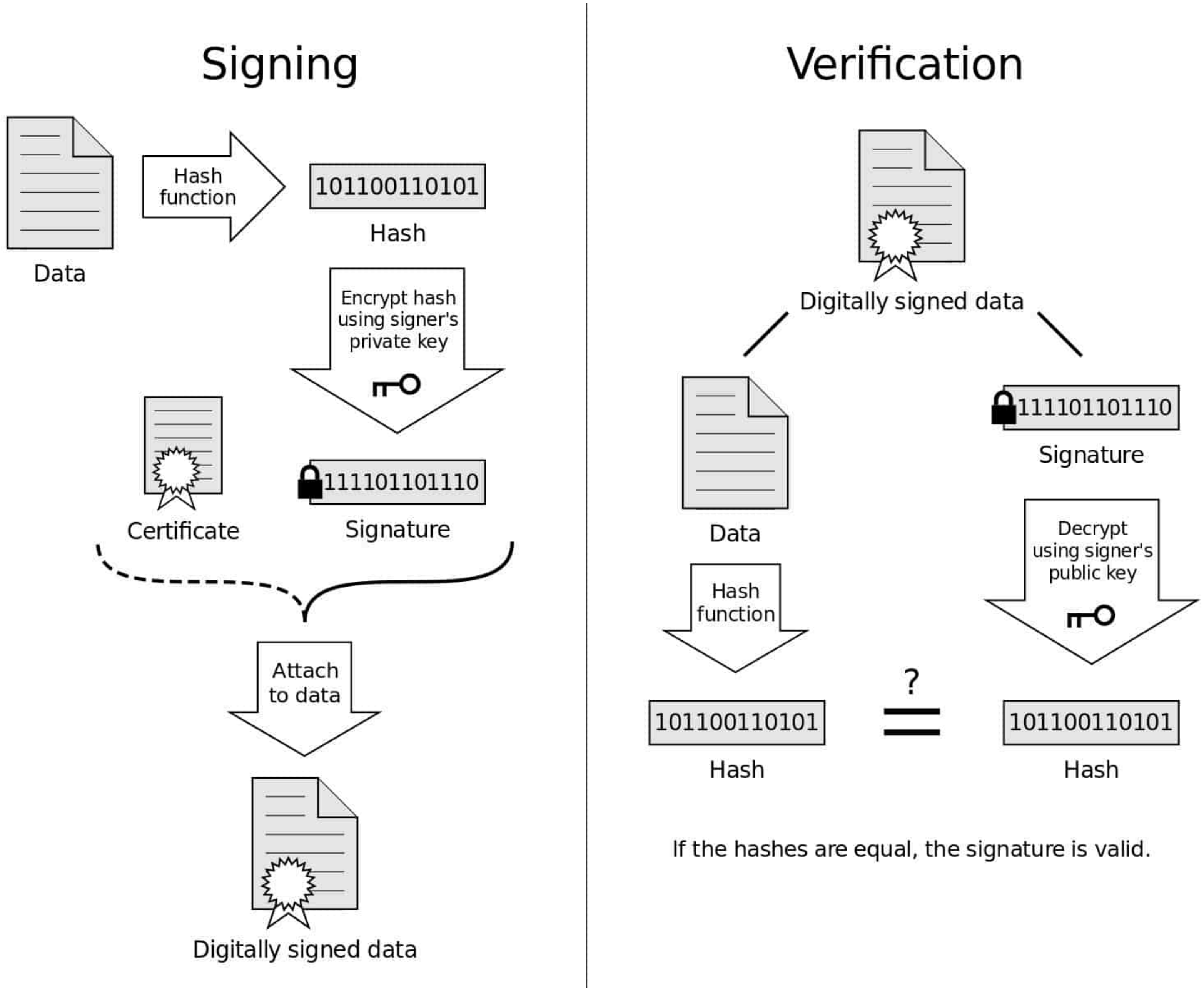
Definition

- Suppose $F : X \to Y$ is a one-to-one correspondence.
  Then, the inverse function $F^{-1} : Y \to X$ is defined as follows:
  Given any element y in Y,
  $F^{-1}(y) =$ that unique element x in X such that $F(x) = y$.
- $F^{-1}(y) = x \Leftrightarrow y = F(x)$.

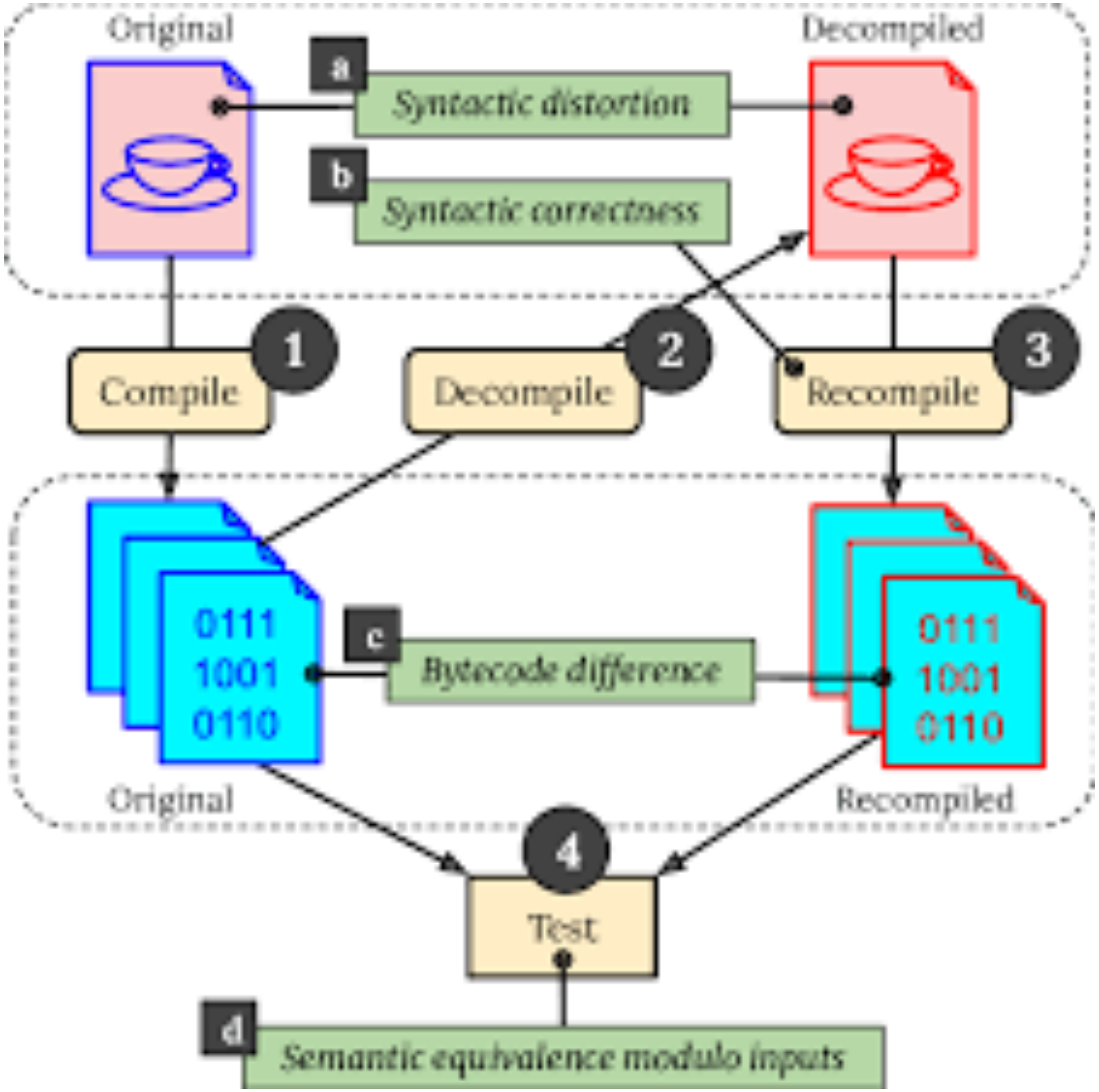
Does encryption have an inverse function?



Does digital signing have an inverse function?



Does Java compilation have an inverse function?



Inverse functions: Example 1

Subset of $\{a, b, c, d\}$		4-tuple of $\{0, 1\}$
$\{\}$	$\longleftarrow$	$(0, 0, 0, 0)$
$\{a\}$	$\longleftarrow$	$(1, 0, 0, 0)$
$\{b\}$	$\longleftarrow$	$(0, 1, 0, 0)$
$\{c\}$	$\longleftarrow$	$(0, 0, 1, 0)$
$\{d\}$	$\longleftarrow$	$(0, 0, 0, 1)$
$\{a, b\}$	$\longleftarrow$	$(1, 1, 0, 0)$
$\{a, c\}$	$\longleftarrow$	$(1, 0, 1, 0)$
$\{a, d\}$	$\longleftarrow$	$(1, 0, 0, 1)$
$\{b, c\}$	$\longleftarrow$	$(0, 1, 1, 0)$
$\{b, d\}$	$\longleftarrow$	$(0, 1, 0, 1)$
$\{c, d\}$	$\longleftarrow$	$(0, 0, 1, 1)$
$\{a, b, c\}$	$\longleftarrow$	$(1, 1, 1, 0)$
$\{a, b, d\}$	$\longleftarrow$	$(1, 1, 0, 1)$
$\{a, c, d\}$	$\longleftarrow$	$(1, 0, 1, 1)$
$\{b, c, d\}$	$\longleftarrow$	$(0, 1, 1, 1)$
$\{a, b, c, d\}$	$\longleftarrow$	$(1, 1, 1, 1)$

Inverse functions: Example 2

Problem

- Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$. Find its inverse function.

Proof

For any y in R, by definition of f^{-1}

- $f^{-1} =$ unique number x such that $f(x) = y$

  Consider $f(x) = y$

  $\implies 4x - 1 = y \qquad (\because$ Defn. of $f)$

  $\implies x = \frac{y+1}{4} \qquad (\because$ Simplify$)$

- Hence, $f^{-1}(y) = \frac{y+1}{4}$ is the inverse function.

Exercise 0

- Check that the function f : Z → Z defined by f (n) = 6−n is one-to-one correspondence. Then compute its inverse.

Exercise 1

- The function $f : R \to R$ defined as $f (x) = x^3 + 1$ is a one-t-one correspondence. Find its inverse.

Exercise 2

- Earlier, you proved that $f : R–\{2\} \rightarrow R–\{5\}$ defined by $f(x) = (5x+1)/(x-2)$ is bijective. Now find its inverse.

Solution

- Earlier, you proved that f : R–{2} → R–{5} defined by f(x) = (5x+1)/(x-2) is bijective. Now find its inverse.

- Let y be an element of R - {5}. We have y = f(x) if and only if x = 11/(y-5)+2.

- Thus f^{-1} (x) = 11/(y-5)+2

Exercise 3

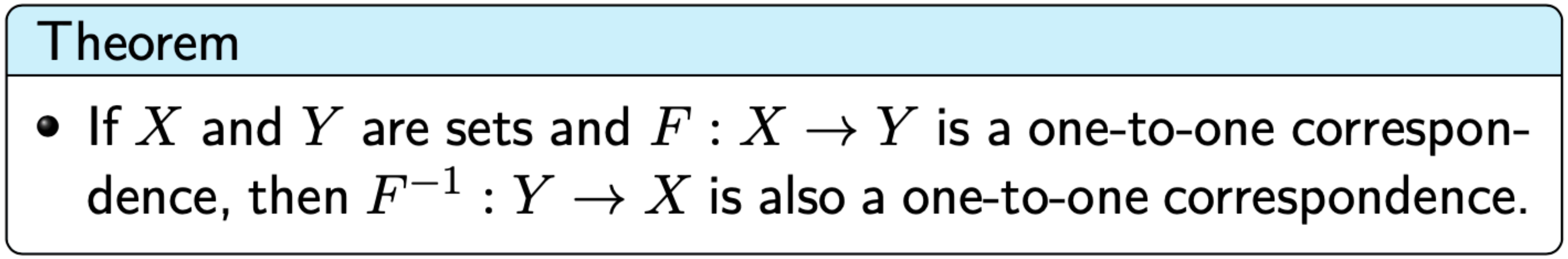
- The function $g: Z \times Z \rightarrow Z \times Z$ defined by the formula $g(m,n) = (m+n, m+2n)$ is a one-to-one correspondence. Find its inverse.

Solution

- Let (u,v) be an arbitrary element if Z x Z. Then f(m,n) = (u,v) if and only if m = 2u-v and n = v-u.

- Thus, f^{-1) (u,v) = (2u-v, v-u).

Exercise 4
Prove the following theorem



Theorem

- If X and Y are sets and $F : X \to Y$ is a one-to-one correspondence, then $F^{-1} : Y \to X$ is also a one-to-one correspondence.

Inverse functions

Theorem

- If X and Y are sets and $F : X \to Y$ is a one-to-one correspondence, then $F^{-1} : Y \to X$ is also a one-to-one correspondence.

Proof

- F^{-1} is one-to-one.
  Suppose $F^{-1}(y_1) = F^{-1}(y_2)$ for some $y_1, y_2 \in Y$.
  We must show that $y_1 = y_2$.
  Let $F^{-1}(y_1) = F^{-1}(y_2) = x \in X$. Then
  $y_1 = F(x)$ since $F^{-1}(y_1) = x$ and
  $y_2 = F(x)$ since $F^{-1}(y_2) = x$.
  So, $y_1 = y_2$.
- F^{-1} is onto.
  We must show that for any $x \in X$, there exists an element y in Y such that $F^{-1}(y) = x$.
  For any $x \in X$, we consider $y = F(x)$.
  We see that $y \in Y$ and $F^{-1}(y) = x$.