CSE215
Foundations of Computer Science

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Agenda

- Attendance
- Size of Infinite sets — review and exercises

To finish around 4h45

Zoom on today!

Main points about infinite sets (Facts part)

- From size perspective, sets can be categorized into three kinds:
  - Finite sets,
  - Countably infinite sets,
  - Uncountably infinite sets.
- Countable sets can be: Finite or countably infinite sets
- Infinite sets can be: Countably infinite, uncountably infinite
- Sets known to be countably Infinite: Integers, naturals, even, odd, rational numbers. And they are all of the same size
- Sets known to be uncountably infinite: Reals numbers, any real-number intervals (a,b) irrational numbers. Not all uncountably infinite sets are of the same size; Reals and (a, b) do have the same size

Main points about infinite sets (Proof part)

- To see if two sets are of the same size, check if it there exists a one-to-one correspondence between them
- To prove a set is countable:
  - Argue that counting all its elements is possible,
  - or argue that the set has a one-to-one correspondence with a set known to be countable
- To prove a set is uncountable:
  - argue that counting its elements is not possible (proof by contradiction)
  - or argue that the set has a one-to-one correspondence with a set known to be uncountable

Classic proof

- reals numbers on (0,1) is uncountable

In set theory, **Cantor's diagonal argument**, also called the **diagonalisation argument**, the **diagonal slash argument**, the **anti-diagonal argument**, the **diagonal method**, and **Cantor's diagonalization proof**, was published in 1891 by Georg Cantor as a mathematical proof that there are infinite sets which cannot be put into one-to-one correspondence with the infinite set of natural numbers.[1][2]:20–[3] Such sets are now known as uncountable sets, and the size of infinite sets is now treated by the theory of cardinal numbers which Cantor began.

The diagonal argument was not Cantor's first proof of the uncountability of the real numbers, which appeared in 1874.[4][5] However, it demonstrates a general technique that has since been used in a wide range of proofs,[6] including the first of Gödel's incompleteness theorems[2] and Turing's answer to the *Entscheidungsproblem*. Diagonalization arguments are often also the source of contradictions like Russell's paradox[7][8] and Richard's paradox.[2]:27

$$
\begin{array}{ll}
s_1 & = 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\ldots \\
s_2 & = 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\ldots \\
s_3 & = 0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\ldots \\
s_4 & = 1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\ldots \\
s_5 & = 1\,1\,0\,1\,0\,1\,1\,0\,1\,0\,1\ldots \\
s_6 & = 0\,0\,1\,1\,0\,1\,1\,0\,1\,1\,0\ldots \\
s_7 & = 1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\ldots \\
s_8 & = 0\,0\,1\,1\,0\,0\,1\,1\,0\,0\,1\ldots \\
s_9 & = 1\,1\,0\,0\,1\,1\,0\,0\,1\,1\,0\ldots \\
s_{10} & = 1\,1\,0\,1\,1\,1\,0\,0\,1\,0\,1\ldots \\
s_{11} & = 1\,1\,0\,1\,0\,1\,0\,0\,1\,0\,0\ldots \\
\vdots & \quad \vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\ \ddots
\end{array}
$$

Contents

Uncountable set

Example 1

Set of real numbers in $[0, 1]$ is uncountable

Problem

- Prove that the set of all real numbers between 0 and 1 is uncountable.

Solution

Proof by contradiction.

- Suppose $[0..1]$ is countable.
- We will derive a contradiction by showing that there is a number in $[0..1]$ that does not appear on this list.

$$
\begin{array}{|c|c|c|}
\boxed{\mathbb{N}} & & \boxed{[0..1]} \\
1 & \longrightarrow & 0.a_{11}a_{12}a_{13}\dots a_{1n}\dots \\
2 & \longrightarrow & 0.a_{21}a_{22}a_{23}\dots a_{2n}\dots \\
3 & \longrightarrow & 0.a_{31}a_{32}a_{33}\dots a_{3n}\dots \\
\vdots & \vdots & \vdots \\
n & \longrightarrow & 0.a_{n1}a_{n2}a_{n3}\dots a_{nn}\dots \\
\vdots & \vdots & \vdots
\end{array}
$$

- Suppose the list of reals starts out as follows:

$$\begin{array}{ccccccc} 0. & \boxed{9} & 0 & 1 & 4 & 8 & \ldots \\ 0. & 1 & \boxed{1} & 6 & 6 & 6 & \ldots \\ 0. & 0 & 3 & \boxed{3} & 5 & 3 & \ldots \\ 0. & 9 & 6 & 7 & \boxed{2} & 6 & \ldots \\ 0. & 0 & 0 & 0 & 3 & \boxed{1} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

- Construct a new number $d = 0.d_1 d_2 d_3 \ldots d_n \ldots$ as follows:

$$d_n = \begin{cases} 1 & a_{nn} \neq 1, \\ 2 & a_{nn} = 1. \end{cases}$$

- We have $d = 0.12112\ldots$, i.e.,

$$\begin{array}{ccccccc} 0. & \boxed{1} & \boxed{2} & \boxed{1} & \boxed{1} & \boxed{2} & \ldots \end{array}$$

Proof by making one-to-one correspondence

- Integers are countable
- Rationals are countable
- Reals are uncountable

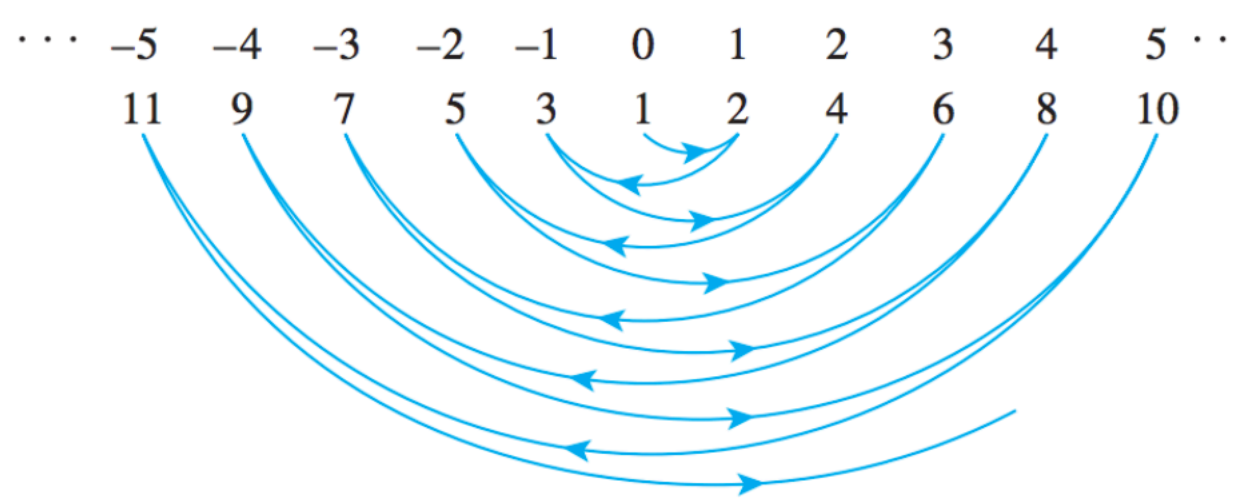
Example 2

Integers are countable

Problem

- Prove that the set of integers is countably infinite.

Solution



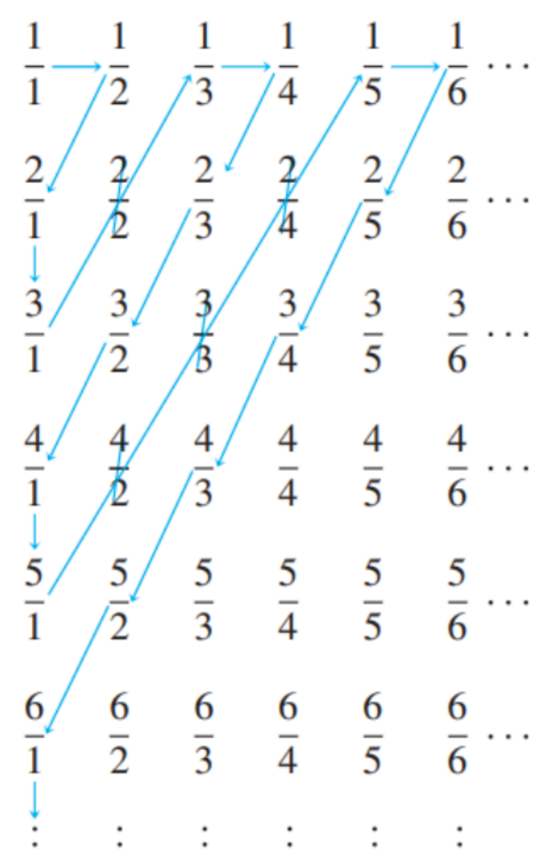
Example 3

Set of positive rationals is countable

Problem

- Prove that the set of positive rational numbers are countable.

Solution



$\boxed{\mathbb{N}}$		$\boxed{\mathbb{Q}^+}$
1	$\longrightarrow$	$1/1$
2	$\longrightarrow$	$1/2$
3	$\longrightarrow$	$2/1$
4	$\longrightarrow$	$3/1$
5	$\longrightarrow$	$1/3$
6	$\longrightarrow$	$1/4$
7	$\longrightarrow$	$2/3$
8	$\longrightarrow$	$3/2$
9	$\longrightarrow$	$4/1$
10	$\longrightarrow$	$5/1$
$\vdots$		$\vdots$

Example 4

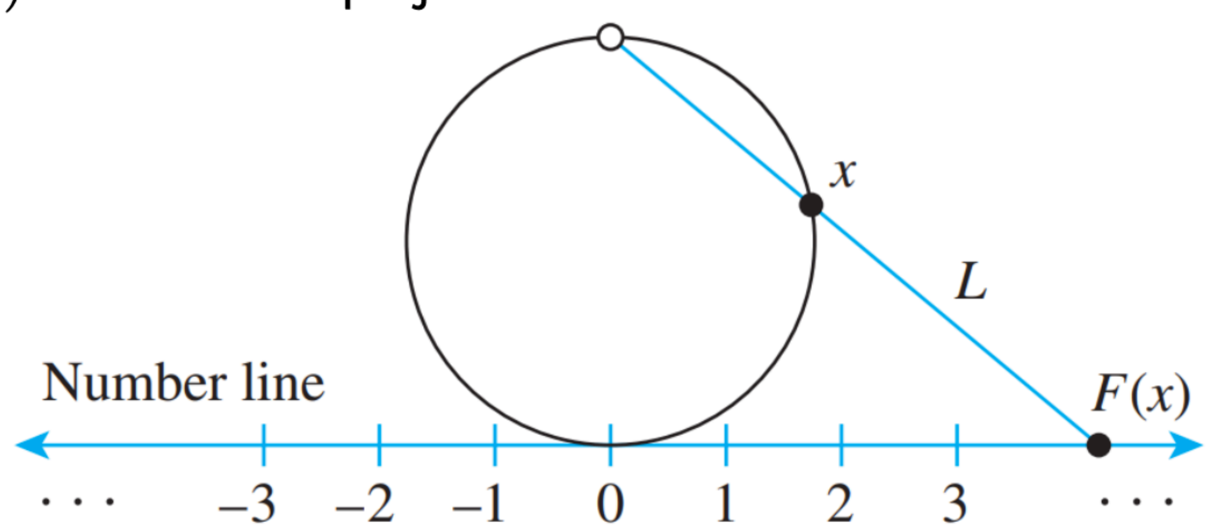
$\mathbb{R}$ and $[0, 1]$ have the same size

Problem

- Prove that the set of all real numbers has the same cardinality as the set of real numbers between 0 and 1.

Solution

- Let $S = \{x \in \mathbb{R} \mid 0 < x < 1\}$
- Bend S to create a circle as shown in the diagram.
- Define $F : S \to \mathbb{R}$ as follows.
- $F(x)$ is called the projection of x onto the number line.



Break

Exercises

To finish around 4h45

Exercise 1

- Prove this: “There are as many squares as there are numbers”. (Galileo Galilei, 1632)

Exercise 2

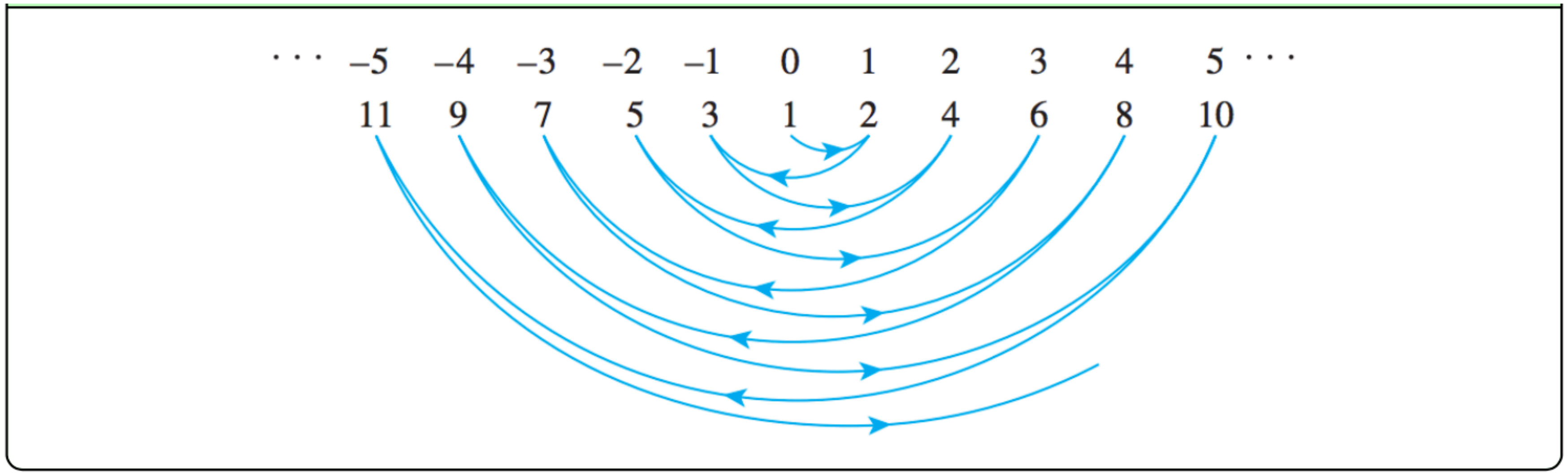
- Prove the set of real numbers (a, b) is uncountable

Exercise 3

Problem 6. [5 points]

Prove that $\mathbb{Z}$ is countable. Come up with a function $f : \mathbb{N} \to \mathbb{Z}$ that can map a unique number of $\mathbb{N}$ to a unique number $f(n)$ of $\mathbb{Z}$.

- Step 1: Count
- Step 2: Formulate
- Step 3: Prove



Solution

- One-to-one correspondance: $f(n) = (n+1)/2$ if n is odd, or $-n/2$ if n is even

Exercise 4

Problem 10. [5 points]

Prove that the following set is countable using a diagram and a formula for the one-to-one correspondence function.

$\{\pm 1^1, \pm 2^2, \pm 3^3, \pm 4^4, \pm 5^5, \ldots\}$

Solution

- $f(n) = -(n/2)^2$ if n is even, or $((n+1)/2)^2$ if n is odd

Exercise 5

True or False

(j) [1 point] The size of the set of real numbers in the range $[1, 2]$ is the same or larger than the size of the set of real numbers in the range $[1, 4]$.

Exercise 6

- Prove that positive real numbers are uncountable.

Exercise 7

- A binary string is a sequence of 0 and 1. A binary string can be of arbitrary length including infinite.
- Determine if the following sets are finite, countably infinite, or uncountable
    - Set of binary strings of length <=100000
    - Set of binary strings of finite length
    - Set of binary string of any length including infinite