CSE215
Foundations of Computer Science

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May 25, 2022

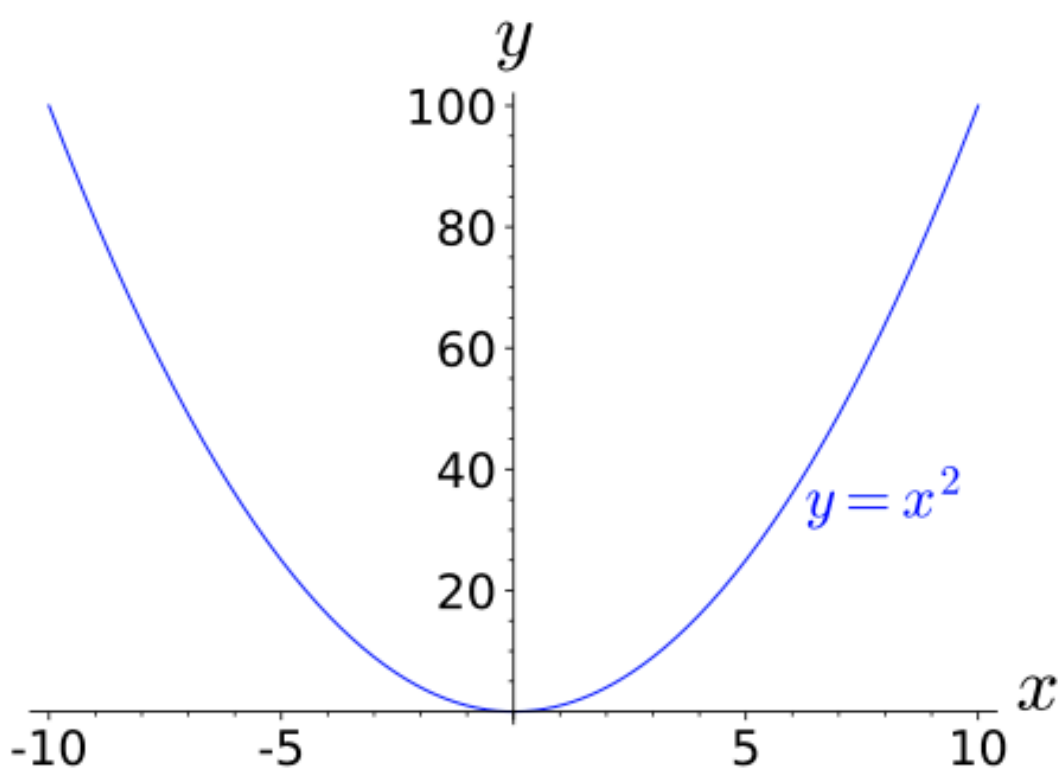
Agenda

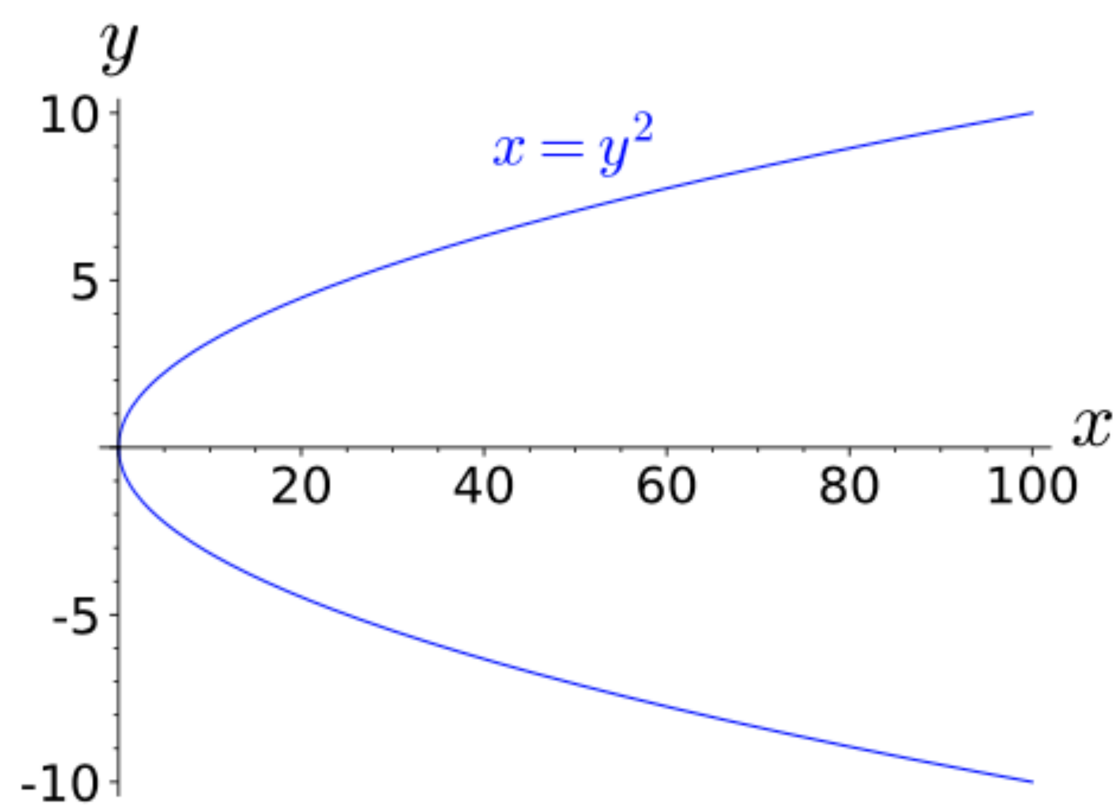
- Attendance
- Extra-credits assignment is online since last Thursday!
- Relations

To finish around 4h45

Zoom on today!

Functions vs. relations





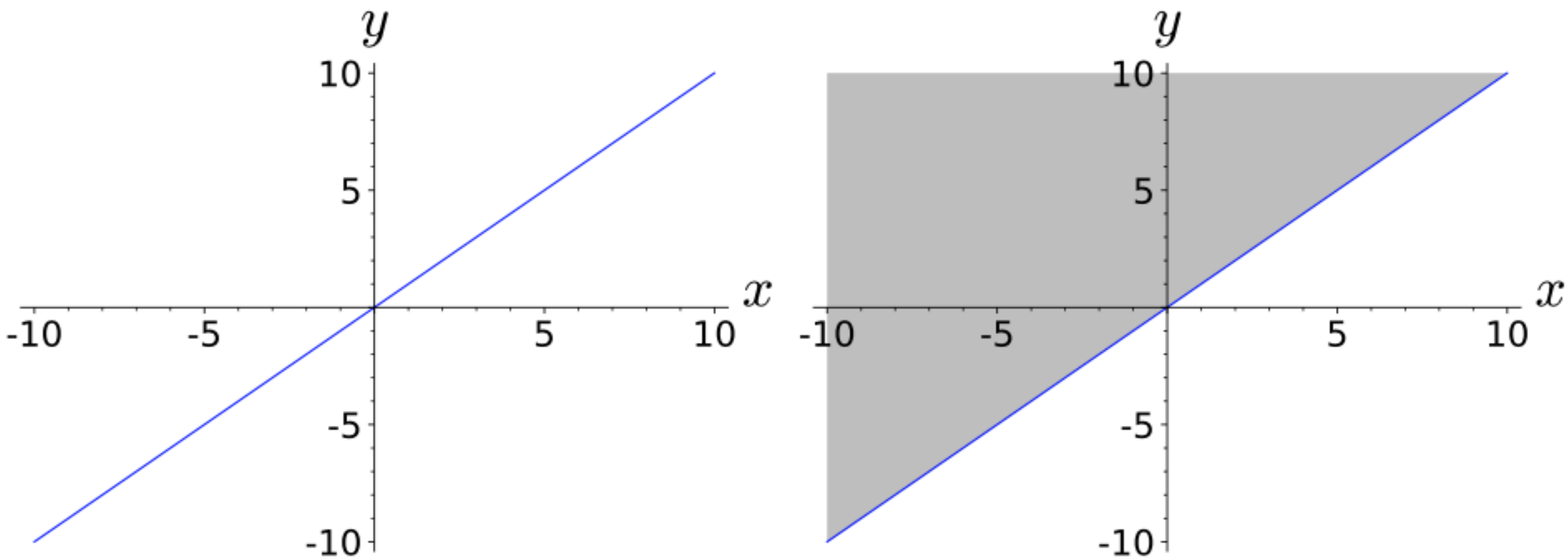
	$y = x^2$	$y = \pm\sqrt{x}$
Function?	✓	✗
Relation?	✓	✓

Relation

Definition A **relation** on a set A is a subset $R \subseteq A \times A$. We often abbreviate the statement $(x, y) \in R$ as xRy. The statement $(x, y) \notin R$ is abbreviated as $x \not R y$.

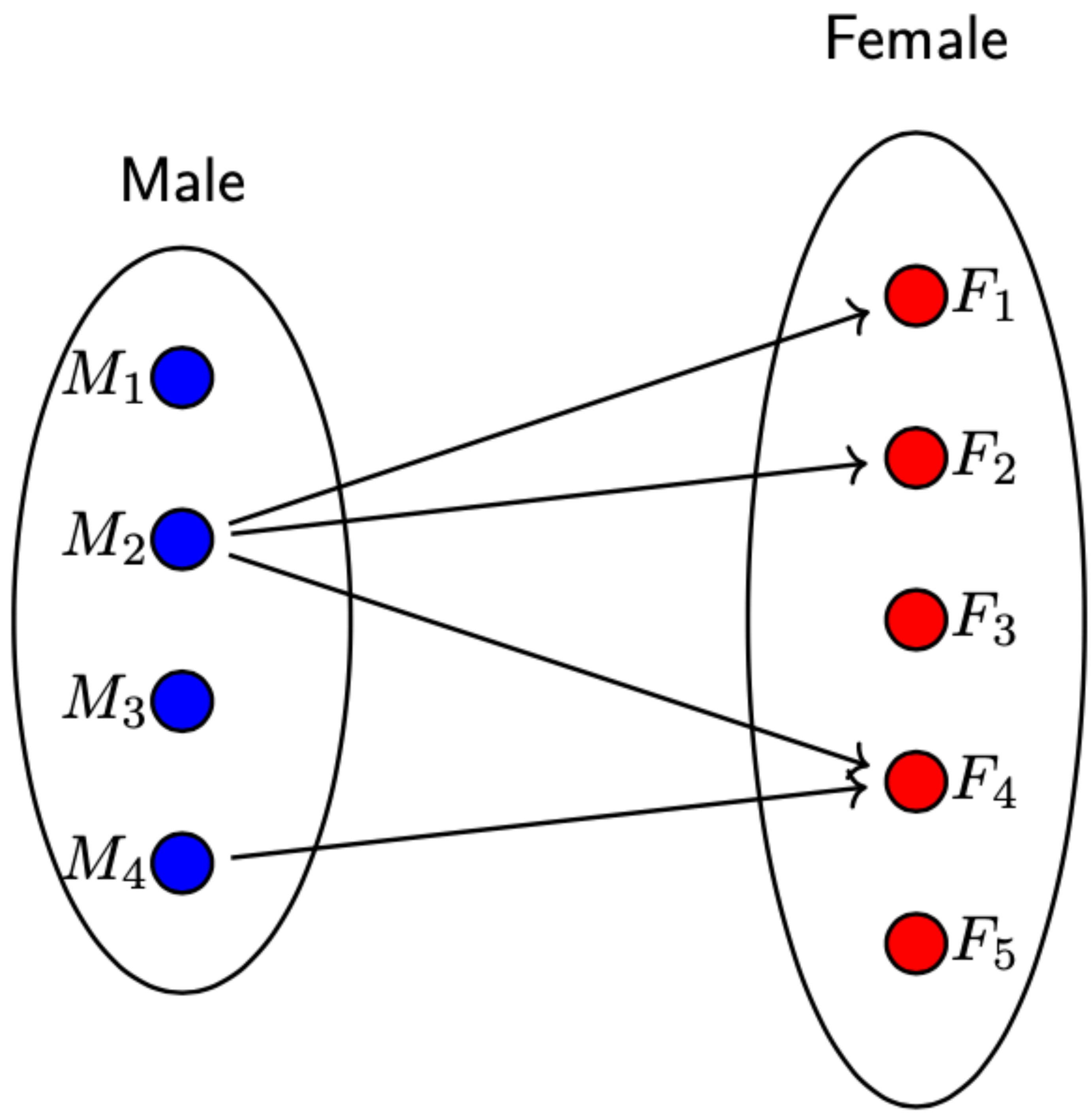
$5 < 10$	$5 \leq 5$	$6 = \frac{30}{5}$	$5 \mid 80$	$7 > 4$	$x \neq y$	$8 \nmid 3$
$a \equiv b \pmod{n}$	$6 \in \mathbb{Z}$	$X \subseteq Y$	$\pi \approx 3.14$	$0 \geq -1$	$\sqrt{2} \notin \mathbb{Z}$	$\mathbb{Z} \nsubseteq \mathbb{N}$

Functions vs. relations



	$y = x$	$y \geq x$
Function?	✓	✗
Relation?	✓	✓

Example: Marriage relation



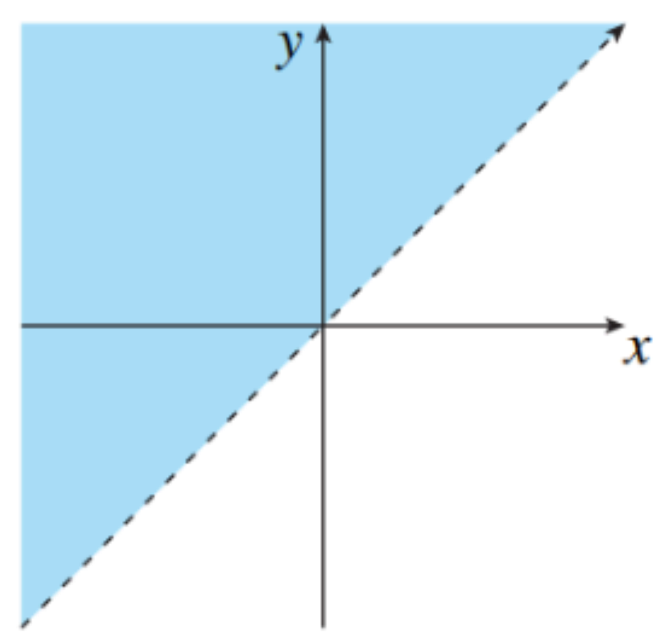
Example: Less than

Problem

- A relation $L : \mathbb{R} \to \mathbb{R}$ as follows.
  For all real numbers x and y, $(x, y) \in L \Leftrightarrow x \, L \, y \Leftrightarrow x < y$.
  Draw the graph of L as a subset of the Cartesian plane $\mathbb{R} \times \mathbb{R}$.

Solution

- $L = \{(-10.678, 30.23), (17.13, 45.98), (100/9, 200), \ldots\}$
- Graph:



Example: Congruence modulo 2

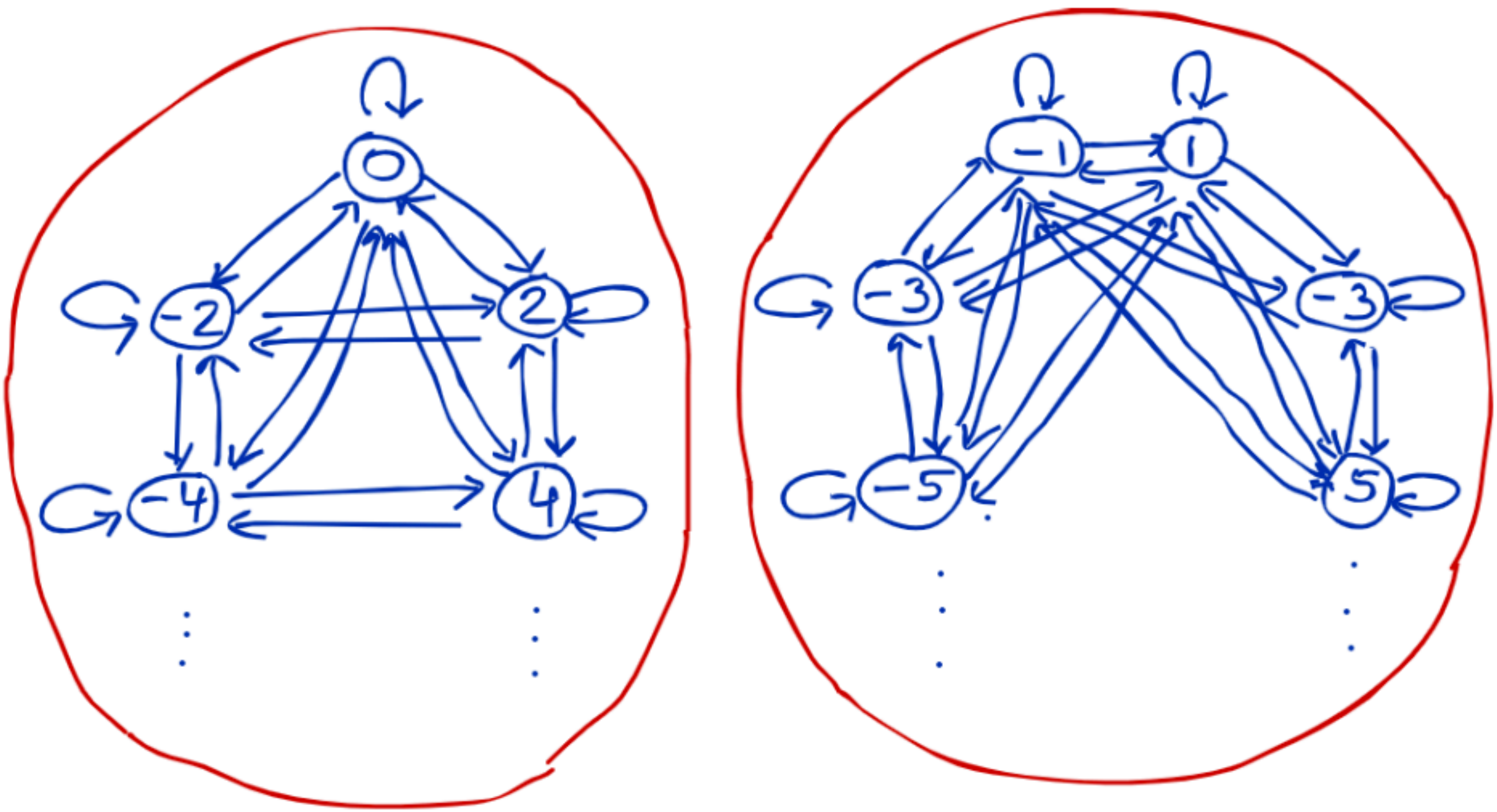
Problem

- Define a relation $C : \mathbb{Z} \to \mathbb{Z}$ as follows.
  For all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, $m \; C \; n \Leftrightarrow m - n$ is even.
- Prove that if n is any odd integer, then $n \; C \; 1$.

Solution

- $A = \{(2, 4), (56, 10), (-88, -64), \ldots\}$
  $B = \{(7, 7), (57, 11), (-87, -63), \ldots\}$
  $C = A \cup B$
- Proof. $(n, 1) \in C \Leftrightarrow n \; C \; 1 \Leftrightarrow n - 1$ is even
  Suppose n is odd i.e., $n = 2k + 1$ for some integer k.
  This implies that $n - 1 = 2k$ is even.

Example: Congruence modulo 2

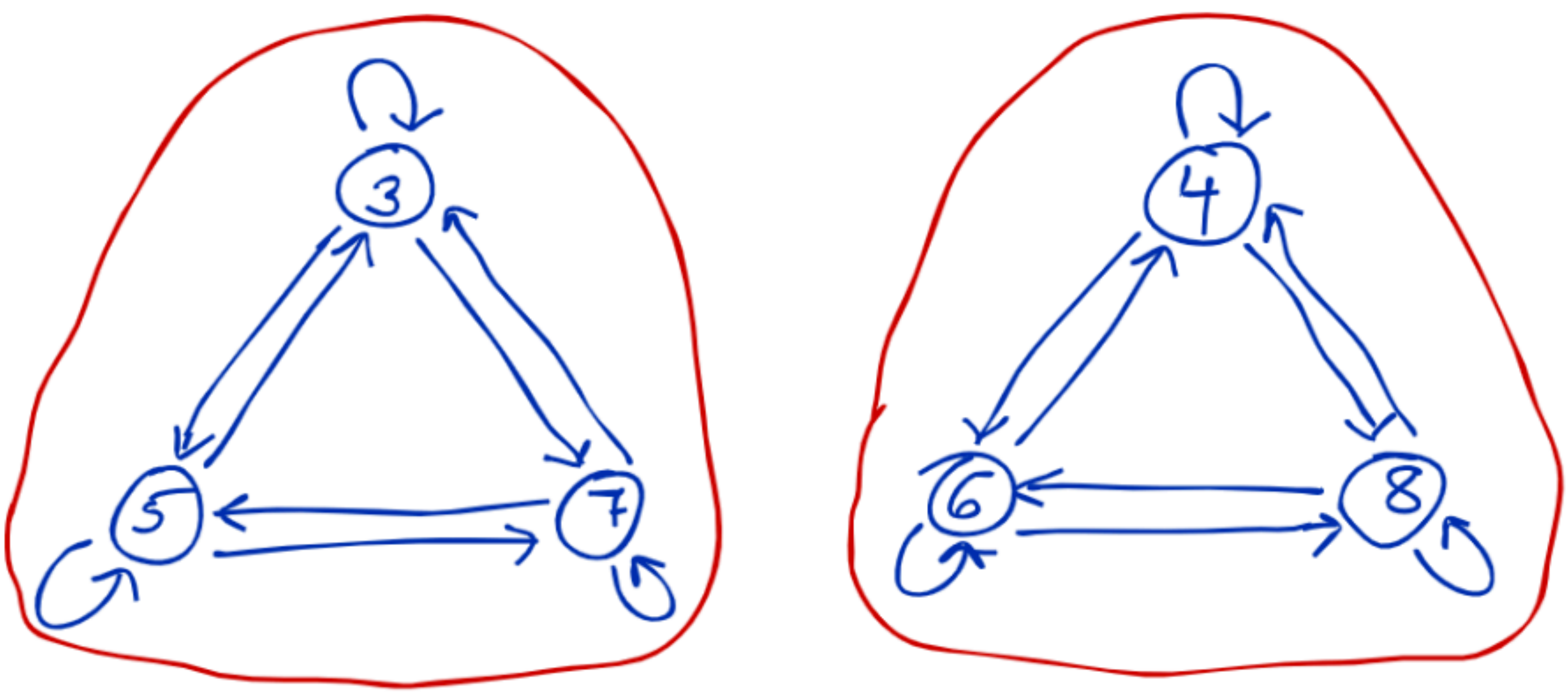


Example: Relation on a set

Problem

- Let $A = \{3, 4, 5, 6, 7, 8\}$. Define relation R on A as follows. For all $x, y \in A$, $x \; R \; y \Leftrightarrow 2|(x - y)$. Draw the graph of R.

Solution



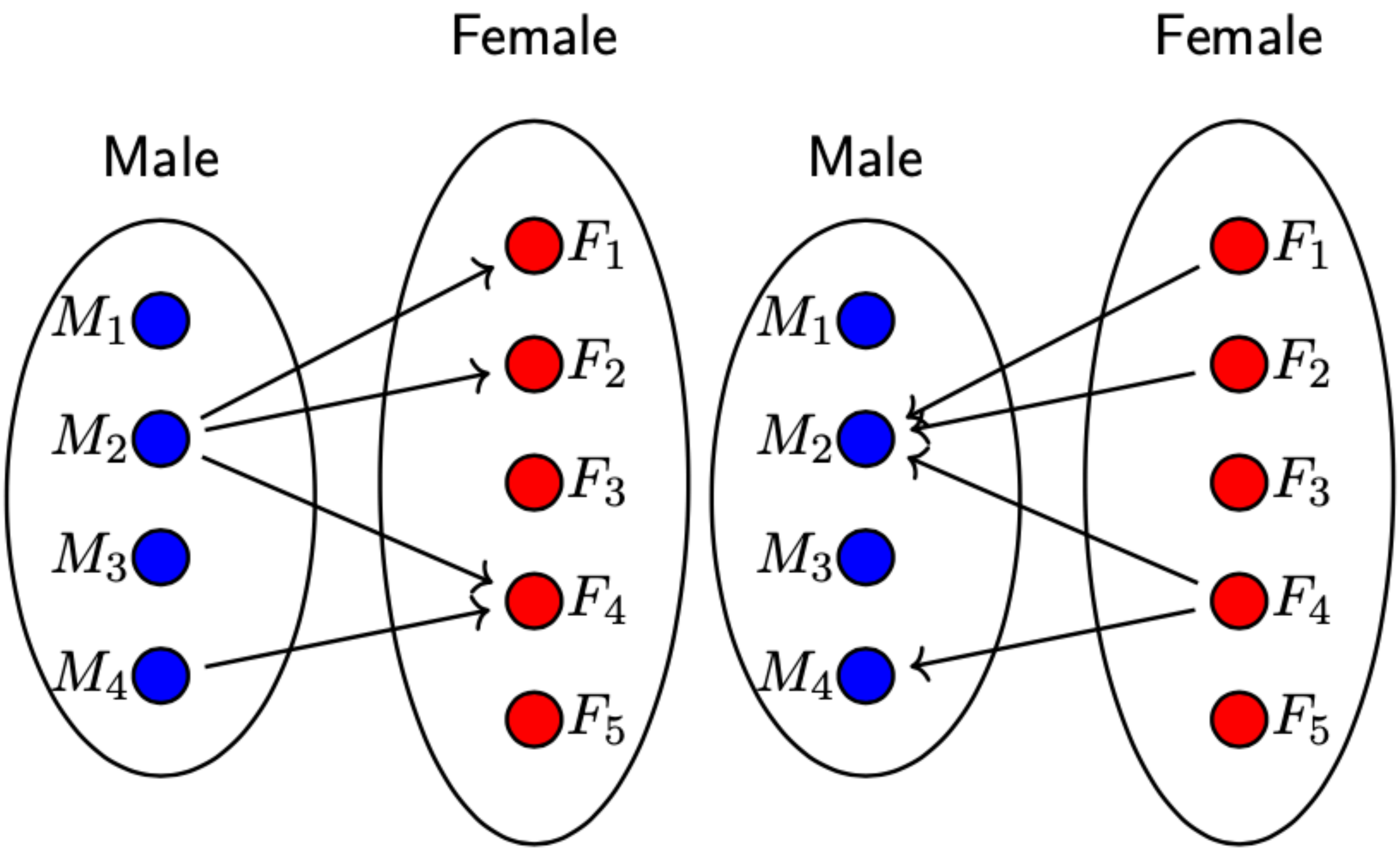
Exercise 0

The **congruence modulo 3** relation, T, is defined from $\mathbf{Z}$ to $\mathbf{Z}$ as follows: For all integers m and n,

$$m \; T \; n \quad \Leftrightarrow \quad 3 \mid (m - n).$$

a. Is $10 \; T \; 1$? Is $1 \; T \; 10$? Is $(2, 2) \in T$? Is $(8, 1) \in T$?
b. List five integers n such that $n \; T \; 0$.
c. List five integers n such that $n \; T \; 1$.
d. List five integers n such that $n \; T \; 2$.

Inverse of a relation



Inverse of a relation

Definition

- Let R be a relation from A to B.
  Then inverse relation R^{-1} from B to A is:
  $\boxed{R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}}$.
- For all $x \in A$ and $y \in B$,
  $\boxed{(x, y) \in R \Leftrightarrow (y, x) \in R^{-1}}$.

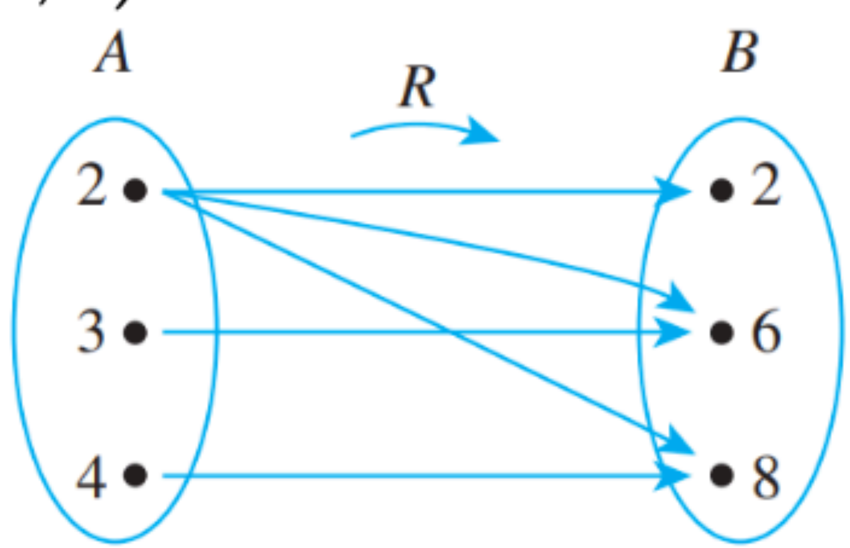
Example: Inverse of a finite relation

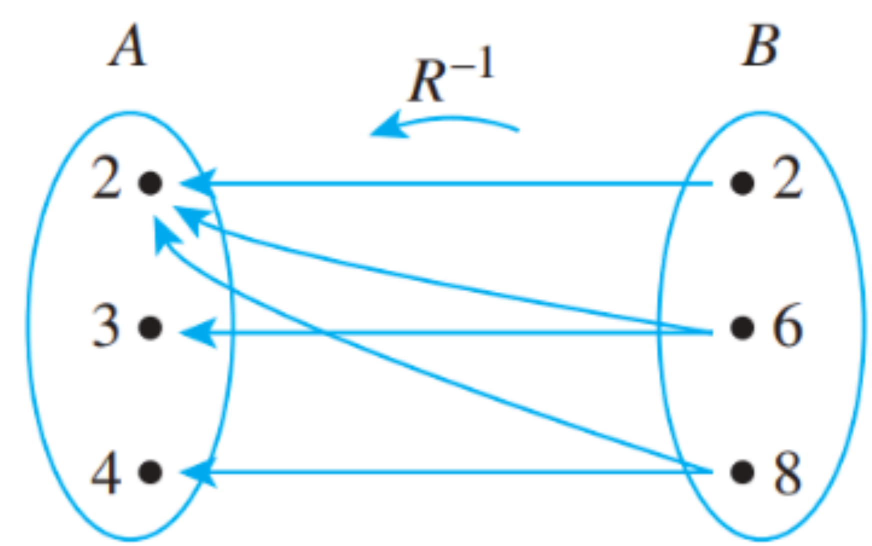
Problem

- Let $A = \{2, 3, 4\}$ and $B = \{2, 6, 8\}$.
  Let $R : A$ to B. For all $(a, b) \in A \times B$, $a\,R\,b \Leftrightarrow a \mid b$
- Determine R and R^{-1}. Draw arrow diagrams for both.
  Describe R^{-1} in words.

Solution

- $R = \{(2,2), (2,6), (2,8), (3,6), (4,8)\}$
  $R^{-1} = \{(2,2), (6,2), (8,2), (6,3), (8,4)\}$
- For all $(b, a) \in B \times A$,
  $(b, a) \in R^{-1} \Leftrightarrow b$ is a multiple of a





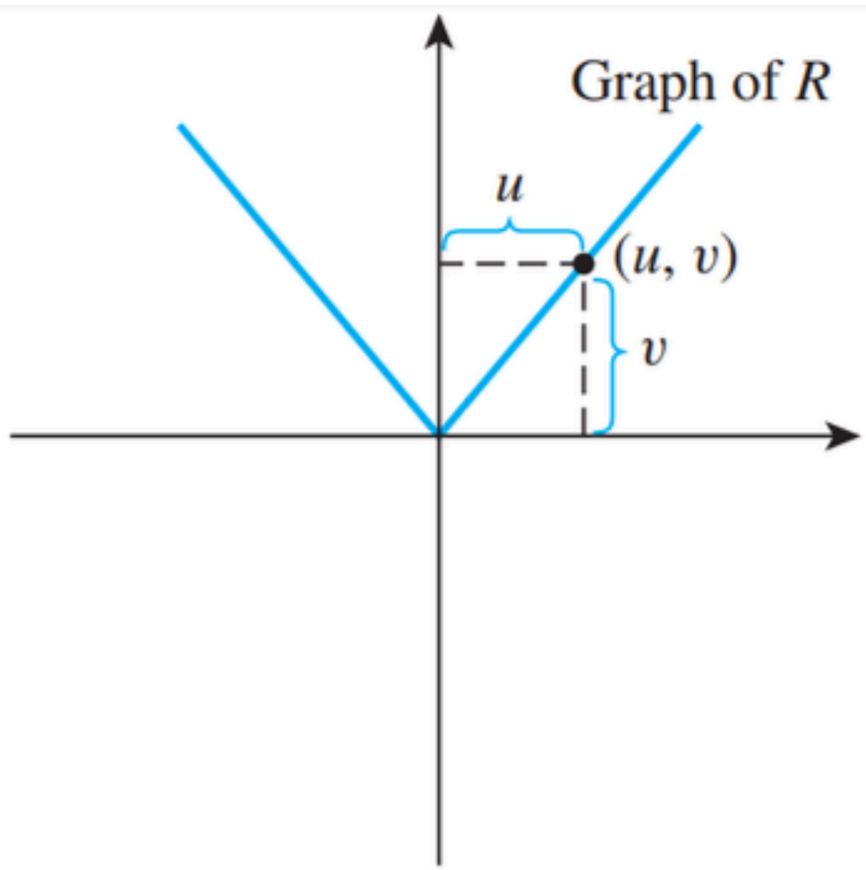
Example: Inverse of an infinite relation

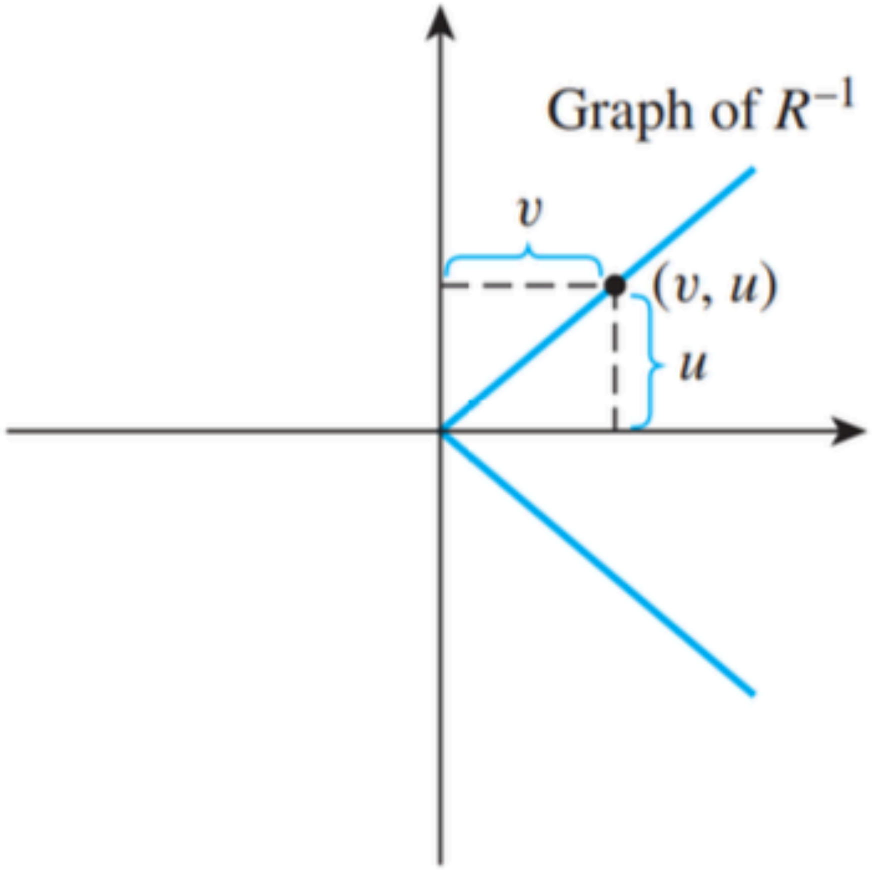
Problem

- Define a relation R from $\mathbb{R}$ to $\mathbb{R}$ as follows:
  For all $(u, v) \in \mathbb{R} \times \mathbb{R}$, $u \, R \, v \Leftrightarrow v = 2|u|$.
- Draw the graphs of R and R^{-1} in the Cartesian plane.
  Is R^{-1} a function?

Solution

- R^{-1} is not a function. Why?

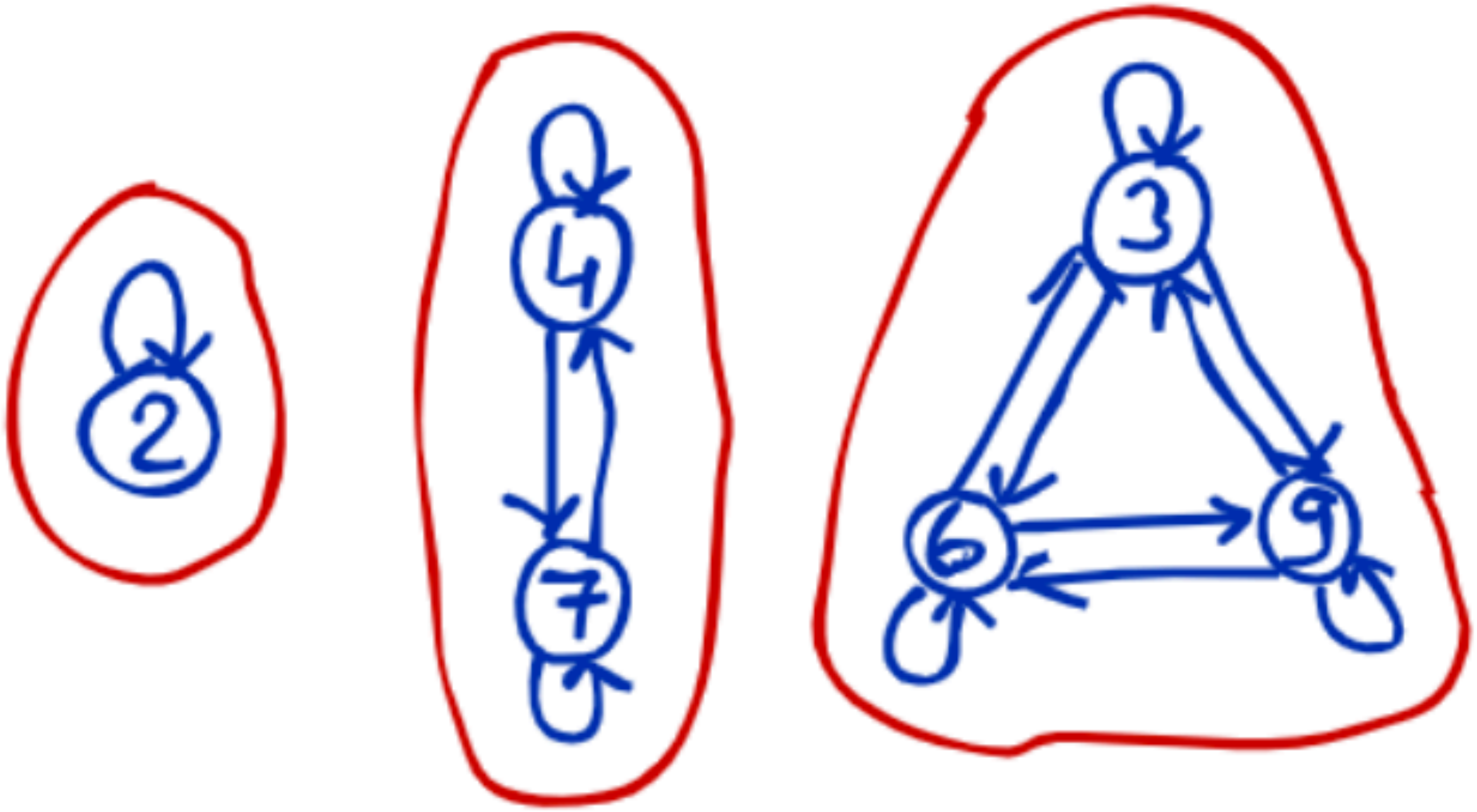




Reflexivity, symmetry, and transitivity

Properties

- Set $A = \{2, 3, 4, 6, 7, 9\}$
  Relation R on set A is: $\forall x, y \in A,\ x \mathrel{R} y \Leftrightarrow 3 \mid (x - y)$



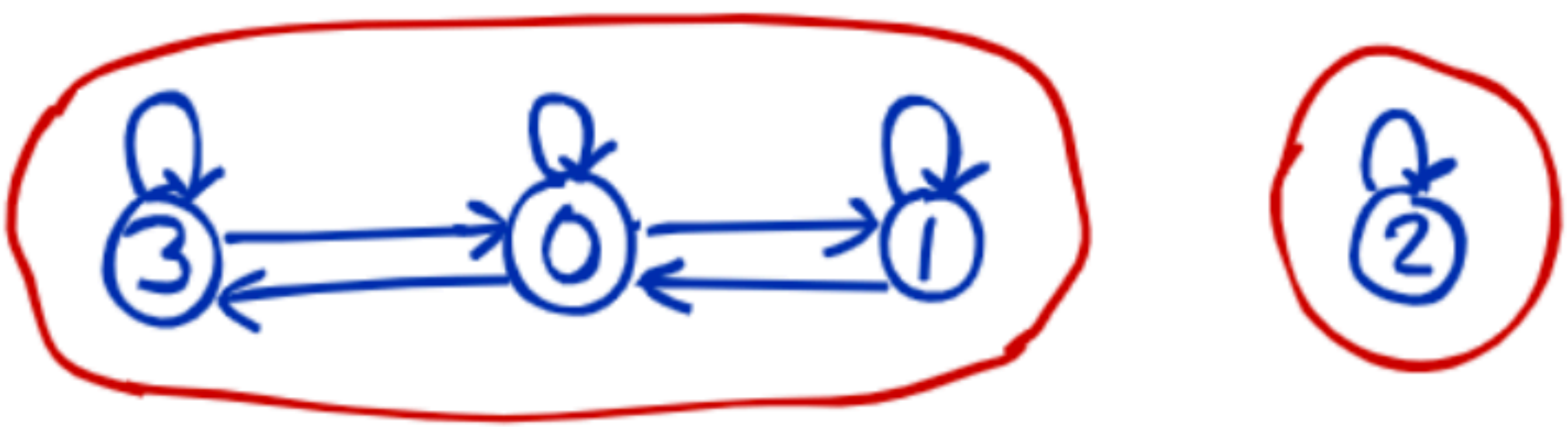
- Reflexivity. $\forall x \in A, (x, x) \in R$.
- Symmetry. $\forall x, y \in A$, if $(x, y) \in R$, then $(y, x) \in R$.
- Transitivity.
  $\forall x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Example

Problem

- $A = \{0, 1, 2, 3\}$.
  $R = \{(0,0), (0,1), (0,3), (1,0), (1,1), (2,2), (3,0), (3,3)\}$.
  Is R reflexive, symmetric, and transitive?

Solution



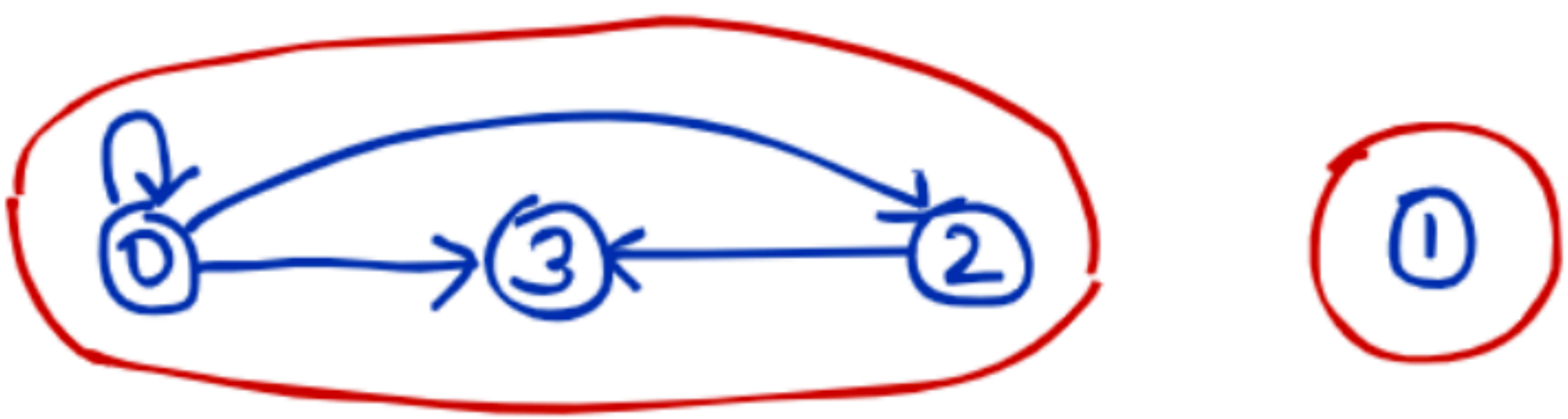
- Reflexive. $\forall x \in A, (x, x) \in R$.
- Symmetric. $\forall x, y \in A$, if $(x, y) \in R$, then $(y, x) \in R$.
- Not transitive. e.g.: $(1, 0), (0, 3) \in R$ but $(1, 3) \notin R$.
  $\exists x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \notin R$.

Exercise 1

Problem

- $A = \{0,1,2,3\}$. $R = \{(0,0),(0,2),(0,3),(2,3)\}$.
  Is R reflexive, symmetric, and transitive?

Solution



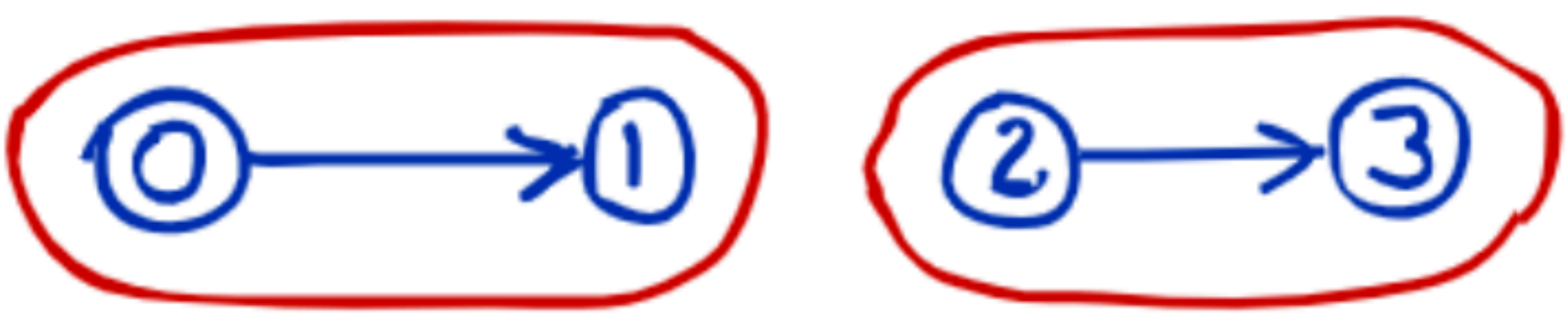
- Not reflexive. e.g.: $(1,1) \notin R$. $\exists x \in A, (x,x) \notin R$.
- Not symmetric. e.g.: $(0,3) \in R$ but $(3,0) \notin R$.
  $\exists x, y \in A$, if $(x,y) \in R$, then $(y,x) \notin R$.
- Transitive.
  $\forall x, y, z \in A$, if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$.

Exercise 2

Problem

- $A = \{0, 1, 2, 3\}$. $R = \{(0, 1), (2, 3)\}$.
  Is R reflexive, symmetric, and transitive?

Solution



- Not reflexive. e.g.: $(0, 0) \notin R$. $\exists x \in A, (x, x) \notin R$.
- Not symmetric. e.g.: $(0, 1) \in R$ but $(1, 0) \notin R$.
  $\exists x, y \in A$, if $(x, y) \in R$, then $(y, x) \notin R$.
- Transitive. Why?
  $\forall x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Equivalence relation and equivalence class

Definition

- Relation R on set A is an equivalence relation iff
  $\boxed{R \text{ is reflexive, symmetric, and transitive.}}$
- Equivalence class of element a, denoted by $[a]$, for an equivalence relation is defined as:
  $\boxed{[a] = \{x \in A \mid (x, a) \in R\}.}$

Example: Less than

Problem

- Suppose R is a relation on $\mathbb{R}$ such that $x \; R \; y \Leftrightarrow x < y$.
  Is R an equivalence relation?

Solution

- Not reflexive. e.g.: $0 \not< 0$. $\exists x \in \mathbb{R}, x \not< x$.
- Not symmetric. e.g.: $0 < 1$ but $1 \not< 0$.
  $\exists x, y \in \mathbb{R}$, if $x < y$, then $y \not< x$.
- Transitive. $\forall x, y, z \in \mathbb{R}$, if $x < y$ and $y < z$, then $x < z$.

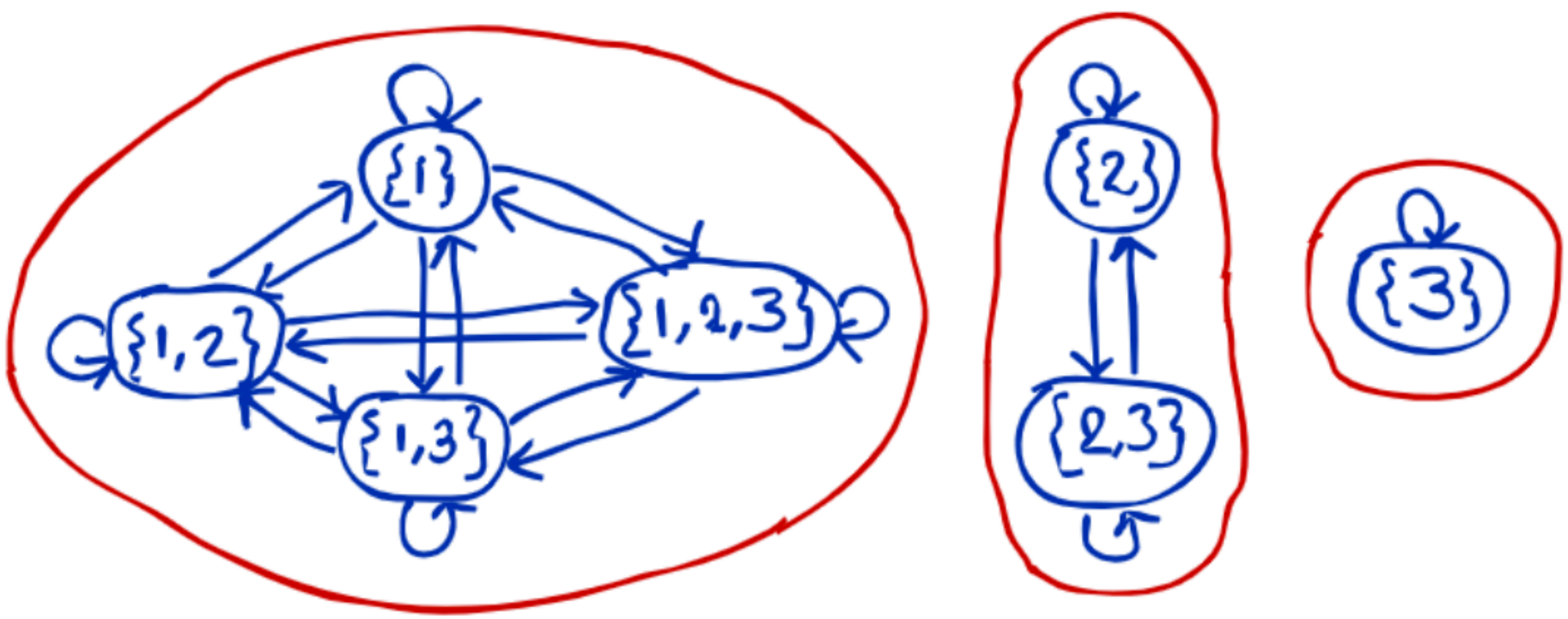
So, R is not an equivalence relation.

Example: Least element

Problem

- Let X denote the power set of $\{1, 2, 3\}$.
  Suppose R is a relation on X such that $\forall A, B \in X$
  $A \; R \; B \Leftrightarrow$ Least element of A is same as that of B.
  Is R an equivalence relation?

Solution



- R is reflexive, symmetric, and transitive.
- So, R is an equivalence relation.
- Equivalence classes: $[\{1\}]$, $[\{2\}]$, and $[\{3\}]$.

Example: Congruence modulo 3

Problem

- Suppose R is a relation on $\mathbb{Z}$ such that $m \, R \, n \Leftrightarrow 3 \mid (m - n)$. Is R an equivalence relation?

Solution

- Reflexive. $\forall m \in A, 3 \mid (m - m)$.
- Symmetric. $\forall m, n \in A$, if $3 \mid (m - n)$, then $3 \mid (n - m)$.
- Transitive.
  $\forall m, n, p \in A$, if $3 \mid (m - n)$ and $3 \mid (n - p)$, then $3 \mid (m - p)$.

So, R is an equivalence relation.

Example: Congruence modulo 3

Solution

- Equivalence classes.
  Three distinct equivalence classes are $[0]$, $[1]$, and $[2]$.
  $[0] = \{a \in \mathbb{Z} \mid a \equiv 0 \pmod 3\} = \{0, \pm 3, \pm 6, \pm 9, \ldots\}$
  $[1] = \{a \in \mathbb{Z} \mid a \equiv 1 \pmod 3\} = \{1, 1 \pm 3, 1 \pm 6, 1 \pm 9, \ldots\}$
  $[2] = \{a \in \mathbb{Z} \mid a \equiv 2 \pmod 3\} = \{2, 2 \pm 3, 2 \pm 6, 2 \pm 9, \ldots\}$

  Intuition.
  $[0] =$ Set of integers when divided by 3 leave a remainder of 0.
  $[1] =$ Set of integers when divided by 3 leave a remainder of 1.
  $[2] =$ Set of integers when divided by 3 leave a remainder of 2.

Congruence modulo n

Definition

Let a and b be integers and n be a positive integer.
The following statements are equivalent:

- a and b leave the same remainder when divided by n.
  $\boxed{a \bmod n = b \bmod n.}$
- $\boxed{n \mid (a-b).}$
- a is congruent to b modulo n.
  $\boxed{a \equiv b \pmod{n}}$
- $\boxed{a = b + kn}$ for some integer k.

Examples

- $12 \equiv 7 \pmod{5}$
- $6 \equiv -6 \pmod{4}$
- $3 \equiv 3 \pmod{7}$

Exercise 3

Problem

- Suppose R is a relation on $\mathbb{Z}$ such that
  $a \; R \; b \Leftrightarrow a \equiv b \pmod{n}$.
  Is R an equivalence relation?

Solution

- Reflexive. $\forall a \in \mathbb{Z}, a \equiv a \pmod{n}$.
- Symmetric.
  $\forall a, b \in \mathbb{Z}$, if $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$.
- Transitive.
  $\forall a, b, c \in \mathbb{Z}$, if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.

So, R is an equivalence relation.

Equivalence classes: $[0], [1], \ldots, [n-1]$.

Exercise 3 (cont.)

Solution

- R is Reflexive. Show that $\forall a \in \mathbb{Z}, n \mid (a-a)$. We know that $a - a = 0$ and $n \mid 0$. Hence, $n \mid (a-a)$.
- R is Symmetric. Show that $\forall a, b \in \mathbb{Z}$, if $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$. We see that $a \equiv b \pmod{n}$ means $n \mid (a-b)$.
  Let $(a-b) = nk$, for some integer k.
  $\implies -(a-b) = -nk$ (multiply both sides by -1)
  $\implies (b-a) = n(-k)$ (simplify)
  $\implies n \mid (b-a)$ ($-k$ is an integer; use defn. of divisibility)
  In other words, $b \equiv a \pmod{n}$.

Exercise 3 (cont.)

Solution

- R is transitive. Show that $\forall a, b, c \in \mathbb{Z}$, if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.
  We see that $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ imply that $n \mid (a-b)$ and $n \mid (b-c)$, respectively.
  Let $(a-b) = nk$ and $(b-c) = n\ell$, for some integers k and ℓ.
  Adding the two equations, we get
  $(a-c) = (k+\ell)n$, where $k+\ell$ is an integer because addition is closed on integers.
  By definition of divisibility, $n \mid (a-c)$ or $a \equiv c \pmod{n}$.

More Exercises

Exercise 4

- Let $A = \{3, 4, 5\}$ and $B = \{4, 5, 6\}$ and let R be the "less than" relation. That is, for all $(x, y) \in A \times B$,

$$x \ R \ y \quad \Leftrightarrow \quad x < y.$$

State explicitly which ordered pairs are in R and R^{-1}.

Exercise 5

Define relations R and S on $\mathbf{R}$ as follows:

$$R = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid x < y\} \quad \text{and}$$
$$S = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid x = y\}.$$

That is, R is the "less than" relation and S is the "equals" relation on $\mathbf{R}$. Graph R, S, $R \cup S$, and $R \cap S$ in the Cartesian plane.

Exercise 5

- Determine if the following relations is reflective, symmetric, and transitive

9. R is the "greater than or equal to" relation on the set of real numbers: For all $x, y \in \mathbf{R}$, $x \; R \; y \Leftrightarrow x \geq y$.

10. C is the circle relation on the set of real numbers: For all $x, y \in \mathbf{R}$, $x \; C \; y \Leftrightarrow x^2 + y^2 = 1$.

11. D is the relation defined on $\mathbf{R}$ as follows: For all $x, y \in \mathbf{R}$, $x \; D \; y \Leftrightarrow xy \geq 0$.

12. E is the congruence modulo 2 relation on $\mathbf{Z}$: For all $m, n \in \mathbf{Z}$, $m \; E \; n \Leftrightarrow 2 \mid (m - n)$.

13. F is the congruence modulo 5 relation on $\mathbf{Z}$: For all $m, n \in \mathbf{Z}$, $m \; F \; n \Leftrightarrow 5 \mid (m - n)$.

14. O is the relation defined on $\mathbf{Z}$ as follows: For all $m, n \in \mathbf{Z}$, $m \; O \; n \Leftrightarrow m - n$ is odd.

15. D is the "divides" relation on $\mathbf{Z}^+$: For all positive integers m and n, $m \; D \; n \Leftrightarrow m \mid n$.