CSE215
Foundations of Computer Science

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Agenda

- Homework08
- A bit about midterm 2

Finish around 4h45

Exercise 1 (10 points)

Prove that $n^3 + 2n$ is divisible by 3 for all integer n >= 1.

- Proof.
  - Let A(n) = n^3 + 2n
  - We will use mathematical induction to prove: for all n>=1, 3 | A(n)
  - We first prove 3 | A(1)
  - We then prove for any n>=1, 3|A(n) -> 3|A(n+1)
    - Suppose n>=1 and 3|A(n) holds, we need to show 3|A(n+1).
      - A(n+1) = (n+1)^3+2(n+1) = n^3 + 3n^2 + 5n + 3 = (n^3+2n) + 3(n^2 + n + 1).. Thus 3 |A(n+1) following the assumption
- QED.

Problematic solution

Exercise 1 (10 points)
Prove that n^3+ 2n is divisible by 3 for all integer n >= 1.

· Proof

Let $P(n)$ be the predicate $3 \mid n^3+2n$

We prove $P(1)$ holds: $P(1) = 3 \mid (1+2) = 1$ so $P(1)$ is true.

We prove for any integer $k \geq 2$, $P(k) \rightarrow P(k+1)$

Let k be an arbitrary integer and $k \geq 2$

Assume $P(k)$ holds $3 \mid k^3+2k$, namely $k^3+2k = 3m$ for some integer m

We need to prove $P(k+1)$, namely, $3 \mid (k+1)^3+2(k+1)$

$$
\begin{aligned}
(k+1)^3+2(k+1) = k^3+3k^2+3k+1+2k+2 &= k^3+3k^2+5k+3 \\
&= k^3+2k+3k^2+3k+3 \\
&= 3m+3(k^2+k+1) \\
&= 3(m+k^2+k+1)
\end{aligned}
$$

Thus, n^3+2n is divisible by 3 for all integer $n \geq 1$.

· QED.

Solution

- Proof.
  - Let $A(n) = n^3 + 2n$
  - We will use mathematical induction to prove: for all $n \geq 1$, $3 \mid A(n)$
  - We first prove $3 \mid A(1)$
  - We then prove for any $n \geq 1$, $3 \mid A(n) \rightarrow 3 \mid A(n+1)$
    - Suppose $n \geq 1$ and $3 \mid A(n)$ holds, we need to show $3 \mid A(n+1)$.
      - $A(n+1) = (n+1)^3+2(n+1) = n^3 + 3n^2 + 5n + 3 = (n^3+2n) + 3(n^2 + n + 1)$.. Thus $3 \mid A(n+1)$ following the assumption
- QED.

Exercise 2 (10 points)

Prove that if p is a real number satisfying $p > -1$, then $(1+p)^n >= 1 + np$ for any integer $n >= 1$.

Solution

Exercise 2 (10 points)

Prove that if p is a real number satisfying $p >- 1$, then $(1 + p)^n \geq 1 + np$.

Proof.

- Let $P(n)$ be the predicate $(1 + p)^n \geq 1 + np$ for any positive integer n.
- Base step: We prove $P(1)$ holds.
  - $P(1) = (1+ p)^1 \geq 1 + (1)p, True.$
- Inductive step: We prove for any integer $k \geq 1,\ P(k) \rightarrow P(k + 1)$.
  - Let k be an arbitrary integer and $k \geq 1$.
  - Assume that $P(k)$ holds, $(1 + p)^k \geq 1 + kp$.
  - We need to prove $P(k + 1)$, namely, $(1 + p)^{k+1} \geq 1 + (k + 1)p$.
    - $(1 + p)^{k+1} = (1 + p)^k(1 + p)$
    - $(1 + p)^k(1 + p) \geq (1 + kp)(1 + p)$
      - From the assumption that $P(k) = (1 + p)^k \geq 1 + kp$ holds.
    - $(1 + p)^k(1 + p) \geq (1 + kp + p + kp^2)$
    - $(1 + p)^k(1 + p) \geq 1 + (k + 1)p + kp^2$
      - Since $(1 + p)^k(1 + p) \geq 1 + (k + 1)p + kp^2$ is true and $kp^2 \geq 0$, we can safely say $(1 + p)^k(1 + p) \geq 1 + (k + 1)p$.
    - Thus, $(1 + p)^{k+1} \geq 1 + (k + 1)p$.
- QED.

Exercise 3 (10 points)

Prove that $9^n - 2^n$ is divisible by 7 for any integer n >= 1.

Solution

Exercise 3 (10 points)

Prove that $9^n - 2^n$ is divisible by 7 for any integer $n \geq 1$.

Proof.

- Let $P(n)$ be the predicate $7|9^n - 2^n$.
- Base step: We prove $P(1)$ holds.
  - $P(1) = (9)^1 - (2)^1 = 7,\ 7|7 = True.$
- Inductive step: We prove for any integer $k \geq 1,\ P(k) \rightarrow P(k+1)$.
  - Let k be an arbitrary integer and $k \geq 1$.
  - Assume that $P(k)$ holds, $7|9^k - 2^k$, namely $9^k - 2^k = 7m$ for some integer m.
  - We need to prove $P(k+1)$, namely, $7|9^{(k+1)} - 2^{(k+1)}$
    - $= 9 \times 9^k - 2 \times 2^k = (7+2)9^k - (2)2^k = (7)9^k + (2)9^k - (2)2^k$
    - $= (7)9^k + (2)(9^k - 2^k)$
    - Following assumption $P(k),\ 7|(9^k - 2^k)$ and $7|(7)9^k$
    - Thus, $7|(7)9^k + (2)(9^k - 2^k)$ and $7|9^{(k+1)} - 2^{(k+1)}$
- QED.

Exercise 4 (30 points)

Prove:

(1) For all integer n >= 3, n^2 >= 2n + 1.

(2) For all integer n >= 4, 2^n >= n^2. [Hint: you may use the result established in #1]

Solution

Exercise 4 (30 points)

Prove:

(1) For all integer $n \geq 3$, $n^2 \geq 2n + 1$.

Proof.

- Let $P(n)$ be the predicate $n^2 - 2n - 1 \geq 0$.
- Base step: We prove $P(3)$ holds.
  - $P(3) = 3^2 - 2(3) - 1 \geq, 2 \geq 0, True.$
- Inductive step: We prove for any integer $k \geq 3$, $P(k) \rightarrow P(k + 1)$.
  - Let k be an arbitrary integer and $k \geq 3$.
  - Assume that $P(k)$ holds, $k^2 - 2k - 1 \geq 0$, for some integer k.
  - We need to prove $P(k + 1)$, namely, $(k + 1)^2 - 2(k + 1) - 1 \geq 0$.
    - $k^2 + 2k + 1 - 2k - 3 \geq 0$
    - $k^2 - 2 \geq 0$
    - $k^2 \geq 2$
      - Since $k \geq 3$, k^2 will always be bigger than 2.
    - Thus, $(k + 1)^2 - 2(k + 1) - 1 \geq 0$ and the proposition is true.
- QED.

(2) For all integer $n \geq 4$, $2^n \geq n^2$. [Hint: you may use the result established in #1]

Proof.

- Let $P(n)$ be the predicate $2^n \geq n^2$.
- Base step: We prove $P(4)$ holds.
  - $P(4) = 2^4 \geq 4^2$, $16 \geq 16$, $True$.
- Inductive step: We prove for any integer $k \geq 4$, $P(k) \rightarrow P(k + 1)$.
  - Let k be an arbitrary integer and $k \geq 4$.
  - Assume that $P(k)$ holds, $2^k \geq k^2$, for some integer k.
  - We need to prove $P(k + 1)$, namely, $2^{k+1} \geq (k + 1)^2$.
    - $2^k \times 2 \geq 2k^2$
    - $2k^2 \geq 2(2k + 1)$
      - From #1: $k^2 \geq 2k + 1$.
    - $k^2 - 2k - 1 \geq 0$
    - $(k^2 - 2k + 1) - 1 - 1 \geq 0$
    - $(k - 1)^2 - 2 \geq 0$
    - $(k - 1)^2 \geq 2$
      - Since $k \geq 4$, $(k - 1)^2$ will always be bigger than 2.
    - Thus, $2^{k+1} \geq (k + 1)^2$ and the proposition is true.
- QED.

Exercise 5 (20 points)

Each of n guests (where n >= 2) who meets at a party wants to shake hands with all the others. How many handshakes will there be? Explain your results.

(Hint) Imagine n people in the room and one more arrives and shakes hands with all the others.

Solution

Exercise 5 (20 points)

Each of n guests (where $n \geq 2$) who meets at a party wants to shake hands with all the others. How many handshakes will there be? Explain your results.
(Hint) Imagine n people in the room and one more arrives and shakes hands with all the others.

- 2 guests: a-b = 1 handshake
- 3 guests: a-b, a-c, b-c = 3 handsakes
- 4 guests: a-b, a-c, a-d, b-c, b-d, c-d = 6 handsakes
- 5 guests: a-b, a-c, a-d, a-e, b-c, b-d, b-e, c-d, c-e, d-e = 10 handshakes
- It has a pattern: a shakes hand for (n-1) times, b does for (n-2), until it gets to the second to the last guest, who shakes hand for one time.
- (n-1)+(n-2)+...+1 = $\frac{n(n-1)}{2}$ handshakes for n guests.

Exercise 6 (20 points)

Prove that for any integer n >= 1, $3^{2n+2} - 8n - 9$ can be divisible by 64.

Solution

Exercise 6 (20 points)

Prove that for any integer $n \geq 1,\ 3^{2n+2} - 8n - 9$ can be divisible by 64.

Proof.

- Let $P(n)$ be the predicate $64 | 3^{2n+2} - 8n - 9$.
- Base step: We prove $P(1)$ holds.
  - $P(1) = 3^{2(1)+2} - 8(1) - 9 = 64,\ 64|64 = True.$
- Inductive step: We prove for any integer $k \geq 1,\ P(k) \rightarrow P(k+1)$.

  - Let k be an arbitrary integer and $k \geq 1$.
  - Assume that $P(k)$ holds, $64|3^{2k+2} - 8k - 9$, namely $3^{2k+2} - 8k - 9 = 64m$, for some integer m.
  - We need to prove $P(k+1)$, namely, $64|3^{2(k+1)+2} - 8(k+1) - 9$
    - $= 3^{2k+4} - 8k - 8 - 9 = 3^{2k} \times 3^{4} - 8k - 17$
    - $= 3^{2k} \times 3^{4} - 8k - 17$
    - Following assumption $P(k)$,
    - Thus, $7|(7)9^{k} + (2)(9^{k} - 2^{k})$ and $7|9^{(k+1)} - 2^{(k+1)}$
- QED.

Break;
Midterm exam 2 info

- Format

- Content: Proof techniques, sequences, sets

- Difficulty ~ Homework

Summary on proof techniques

- Direct proof
- Proof by contradiction
- Proof by cases

- problems on for-all
- problems on there-exists
- problems on checking if a statement is true or false

Proof techniques 2022 SBU

Problem 6. [5 points]

Let $a_1, a_2, \ldots, a_n$ be real numbers for $n \geq 1$. Prove that at least one of these numbers is greater or equal to the average of the numbers.

Problem 7. [5 points]

Prove or disprove the following statement. If x and y are rational, then x^y is rational.

Problem 8. [5 points]

Prove that for any two integers a and b, if ab is odd, then a and b are both odd.

Proof techniques 2021 SBU

Problem 6. [5 points]

Prove or disprove the following statement. For all integers, if a is odd, then a^4 is odd.

Problem 7. [5 points]

Prove or disprove the following statement. The difference of two perfect squares is not a prime number. Here is the reasoning for the claim: $a^2 - b^2 = (a+b)(a-b)$, which is a composite number.

Problem 8. [5 points]

Prove by contradiction that there are no integers x and y such that $x^2 = 4y + 2$. (Hint: For this problem, you can assume without giving proof that if x^2 is even, then x is even.)

Proof techniques: 2020 SBU

Problem 6. [5 points]

Prove that the product of any four consecutive integers is a multiple of 4.

Problem 7. [5 points]

Determine which statements are true and which are false. Prove those that are true and disprove those that are false.

(a) [2 points] If r is any rational number and s is any irrational number, then r/s is irrational.

(b) [2 points] The sum of any two positive irrational numbers is irrational.

(c) [1 point] The square root of any rational number is irrational.

Problem 8. [5 points]

Prove that for all integers a, if a^3 is even, then a is even.

Summary on Sequences

- Mathematical Induction
- Why MI works
- Problems on guessing and proving an explicit form of sequences
- Problems on checking if two sequences are the same

Sequence 2021 SBU

Problem 1. [20 points]

Use mathematical induction to prove the following identities.

(a) [5 points] For all integers $n \geq 1$,

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

(b) [5 points] For all integers $n \geq 1$, $n(n^2+5)$ is a multiple of 6.

(c) [5 points] For all integers $n \geq 0$,

$$1 + \frac{2}{3} + \frac{4}{9} + \cdots + \left(\frac{2}{3}\right)^n = 3\left[1 - \left(\frac{2}{3}\right)^{n+1}\right]$$

(d) [5 points] Suppose that $c_1, c_2, c_3, \ldots$ is a sequence defined as follows:

$$c_1 = 3, c_2 = -9$$
$$c_k = 7c_{k-1} - 10c_{k-2} \qquad \text{for all integers } k \geq 3$$

Prove that $c_n = 4 \cdot 2^n - 5^n$ for all integers $n \geq 1$.

Sequences 2020 SBU

Problem 1. [20 points]

Use mathematical induction to prove the following identities.

(a) [5 points] For integers $n \geq 1$,

$$\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$$

(b) [5 points] For integers $n \geq 1$,

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2}$$

(c) [5 points] $9^n + 3$ is divisible by 4 for integers $n \geq 1$.

(d) [5 points] Suppose that $g_1, g_2, g_3, \ldots$ is a sequence defined as follows:

$$\begin{aligned} &g_1 = 3, g_2 = 5 \\ &g_k = 3g_{k-1} - 2g_{k-2} \qquad \text{for all integers } k \geq 3. \end{aligned}$$

Prove that $g_n = 2^n + 1$ for all integers $n \geq 1$.

Summary on Sets

- Concepts about subsets, set equality, empty set
- Prove set properties with set identities
- Prove set properties with element argument

- problems on guessing and proving an explicit form of sequences
- problems on checking if two sequences are the same

Sets SBU 2021

Problem 2. [5 points]

Mention whether the following statements are true or false without giving any reasons. Assume all sets are subsets of a universal set U.

(a) [1 point] $(A \cap B) \cap (A \cap C) = A \cap (B \cup C)$

(b) [1 point] $A = A \cup (A \cap B)$

(c) [1 point] $A \subseteq A \cup B$

(d) [1 point] $A \cap (A \cup B) = A \cap B$

(e) [1 point] $A \subseteq B$ if and only if $A \cup B = B$

Sets SBU 2020

Problem 2. [5 points]

Find if the statement is true or false. If the statement is true, prove it. If the statement is false, find a counterexample. Assume all sets are subsets of a universal set U. Let set A' be the complement of set A.

For all sets A and B, if $A' \subseteq B$, then $A \cup B = U$.

Summary of summary

- Proof on integer properties — even, odd; prime, composite, multiples
- proof on sequence properties — sum, explicit form,
- proof on set properties — subsets, equality
- Concepts: Truthfulness of for-all, there-exists statement; why mathematical induction works, element argument