CSE215: Lecture 06
Foundations of Computer Science

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March 7, 2022

Many slides taken from Prof. Pramod Ganapathi (SBU). Thanks!

Previous lectures

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Laws	Formula	Formula
Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Propositions:

Anything that can be assigned with “true” or “false”

Argument
Premise_1
Premise_2
$\vdots$
Premise_m
$\therefore$ Conclusion

Name	Rule	Name	Rule	
Modus Ponens	$p \to q$ p $\therefore q$	Elimination	$p \vee q$ $\sim q$ $\therefore p$	$p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	Transitivity	$p \to q$ $q \to r$ $\therefore p \to r$	

Why study propositions?

- A formal language that **express** facts (or non-facts) , and **argue** about them

- Example: Given the premises “If it rains, I will stay home” and I go outside, we are certain it does not rain

Modus Tollens	$p \to q$
	$\sim q$
	$\therefore \sim p$

- How can we express “for all integers a, b, and c, (a + b) +c = a + (b + c)?

Proposition with predicates and quantifiers

The whole is a proposition, since we can assign a truth or false value to it

$$\forall a, b, c \in I, \underbrace{(a+b)+c = a+(b+c)}_{\text{Predicate}}$$

quantifier

variables

Domain set

Today

- (~ 20 min.) Predicates, quantifiers, universal & existential propositions
    - Purpose is express, and argue about more facts
- (~30 min.) Exercises

Predicate

Definition

- A propositional function or predicate is a sentence that contains one or more variables
- A predicate is neither true nor false
- A predicate becomes a proposition when the variables are substituted with specific values
- The domain of a predicate variable is the set of all values that may be substituted for the variable

Examples

Symbol	Predicate	Domain	Propositions
$p(x)$	$x > 5$	$x \in \mathbb{R}$	$p(6), p(-3.6), p(0), \ldots$
$p(x, y)$	$x + y$ is odd	$x \in \mathbb{Z}, y \in \mathbb{Z}$	$p(4, 5), p(-4, -4), \ldots$
$p(x, y)$	$x^2 + y^2 = 4$	$x \in \mathbb{R}, y \in \mathbb{R}$	$p(-1.7, 8.9), p(-\sqrt{3}, -1), \ldots$

Universal quantifier ($\forall$)

Definition

- Let $p(x)$ be a predicate and D be the domain of x
- A universal statement is a statement of the form

$$\forall x \in D, p(x)$$

- Forms:
  - "$p(x)$ is true for all values of x"
  - "For all x, $p(x)$"
  - "For each x, $p(x)$"
  - "For every x, $p(x)$"
  - "Given any x, $p(x)$"
- It is true if $p(x)$ is true for each x in D; It is false if $p(x)$ is false for at least one x in D
- A counterexample to the universal statement is the value of x for which $p(x)$ is false

An example of an universal proposition / statement

- $\forall x \in \mathbb{R}, x^2 \geq 0$
  - Every real number has a nonnegative square
  - All real numbers have nonnegative squares
  - Any real number has a nonnegative square
  - The square of each real number is nonnegative
  - No real numbers have negative squares
  - x^2 is nonnegative for every real x
  - x^2 is not less than zero for each real number x

Existential quantifier ($\exists$)

Definition

- Let $p(x)$ be a predicate and D be the domain of x
- An existential statement is a statement of the form

$$\exists x \in D, p(x)$$

- Forms:
  - "There exists an x such that $p(x)$"
  - "For some x, $p(x)$"
  - "We can find an x, such that $p(x)$"
  - "There is some x such that $p(x)$"
  - "There is at least one x such that $p(x)$"
- It is true if $p(x)$ is true for at least one x in D; It is false if $p(x)$ is false for all x in D
- A counterproof to the existential statement is the proof to show that $p(x)$ is true is for no x

Examples of existential propositions / statements

Universal st.s	Domain
$\exists x \in D, x^2 \geq x$	$D = \{1, 2, 3\}$
$\exists x \in \mathbb{R}, x^2 \geq x$	$\mathbb{R}$
$\exists x \in \mathbb{Z}, x + 1 \leq x$	$\mathbb{Z}$

Universal conditional statement $(\forall, \rightarrow)$

Definition

- A universal conditional statement is of the form

$$\forall x, \text{ if } p(x) \text{ then } q(x)$$

Examples

- $\forall x \in \mathbb{R}$, if $x > 2$ then $x^2 > 4$
- $\forall$ real number x, if x is an integer then x is rational
  $\forall$ integer x, x is rational ▷ Logically equivalent
- $\forall x$, if x is a square then x is a rectangle
  $\forall$ square x, x is a rectangle ▷ Logically equivalent
- $\forall x \in U$, if $p(x)$ then $q(x)$
  $\forall x \in D$, $q(x)$ ▷ Logically equivalent
  (where, $D = \{x \in U \mid p(x) \text{ is true}\}$)

- Can be extended to existential conditional statement $(\exists, \rightarrow)$

Implicit quantifiers

Examples

- If a number is an integer, then it is a rational number
  Implicit meaning: $\forall$ number x, if x is an integer, x is rational
- The number 10 can be written as a sum of two prime numbers
  Implicit meaning: $\exists$ prime numbers p and q such that $10 = p+q$
- If $x > 2$, then $x^2 > 4$
  Implicit meaning: $\forall$ real x, if $x > 2$, then $x^2 > 4$

Definition

- Let $p(x)$ and $q(x)$ be predicates and D be the common domain of x. Then implicit quant. symbols $\Rightarrow, \Leftrightarrow$ are defined as:

$$p(x) \Rightarrow q(x) \equiv \forall x, p(x) \rightarrow q(x)$$
$$p(x) \Leftrightarrow q(x) \equiv \forall x, p(x) \leftrightarrow q(x)$$

Negation of quantified statements ($\sim$)

Definition

- Formally,

$$\sim (\forall x \in D, p(x)) \equiv \exists x \in D, \sim p(x)$$
$$\sim (\exists x \in D, p(x)) \equiv \forall x \in D, \sim p(x)$$

- Negation of a universal statement ("all are") is logically equivalent to an existential statement ("there is at least one that is not")
  Negation of an existential statement ("some are") is logically equivalent to a universal statement ("all are not")

Methods

Two methods to avoid errors while finding negations:

1. Write the statements formally and then take negations
2. Ask "What exactly would it mean for the given statement to be false?"

Negation of quantified statements ($\sim$)

Examples

- All mathematicians wear glasses
  Negation (incorrect): No mathematician wears glasses
  Negation (incorrect + ambiguous): All mathematicians do not wear glasses
  Negation (correct): There is at least one mathematician who does not wear glasses
- Some snowflakes are the same
  Negation (incorrect):: Some snowflakes are different
  Negation (correct):: All snowflakes are different

Negation of quantified statements ($\sim$)

Examples

- $\forall$ primes p, p is odd
  Negation: $\exists$ primes p, p is even
- $\exists$ triangle T, sum of angles of T equals $200°$
  $\forall$ triangles T, sum of angles of T does not equal $200°$
- No politicians are honest
  Formal statement: $\forall$ politicians x, x is not honest
  Formal negation: $\exists$ politician x, x is honest
  Informal negation: Some politicians are honest
- 1357 is not divisible by any integer between 1 and 37
  Formal statement: $\forall n \in [1, 37]$, 1357 is not divisible by n
  Formal negation: $\exists n \in [1, 37]$, 1357 is divisible by n
  Informal negation: 1357 is divisible by some integer between 1 and 37

Negation of universal conditional statements

Definition

- Formally,

$$\sim (\forall x, p(x) \to q(x)) \equiv \exists x, \sim (p(x) \to q(x))$$
$$\equiv \exists x, (p(x) \wedge \sim q(x))$$

Examples

- $\forall$ real x, if $x > 10$, then $x^2 > 100$.
  Negation: $\exists$ real x such that $x > 10$ and $x^2 \leq 100$.
- If a computer program has more than 100,000 lines, then it contains a bug.
  Negation: There is at least one computer program that has more than 100,000 lines and does not contain a bug.

Relation between quantifiers $(\forall, \exists)$ and $(\wedge, \vee)$

Relation

- Universal statements are generalizations of and statements
  Existential statements are generalizations of or statements
- If $p(x)$ is a predicate and $D = \{x_1, x_2, \ldots, x_n\}$ is the domain of x, then

$$\forall x \in D, p(x) \equiv p(x_1) \wedge p(x_2) \wedge \cdots \wedge p(x_n)$$
$$\exists x \in D, p(x) \equiv p(x_1) \vee p(x_2) \vee \cdots \vee p(x_n)$$

Exercises

2021 midterm-1

Problem 3. [10 points]

Give negations of the following statements. Reasoning is not required.

(a) [1 point] $p \wedge q$

(b) [1 point] $p \vee q$

(c) [1 point] $p \oplus q$

(d) [1 point] $p \to q$

(e) [1 point] $p \leftrightarrow q$

(f) [1 point] $\forall x, \forall y$ such that $p(x, y)$

(g) [1 point] $\forall x, \exists y$ such that $p(x, y)$

(h) [1 point] $\exists x, \forall y$ such that $p(x, y)$

(i) [1 point] $\exists x, \exists y$ such that $p(x, y)$

(j) [1 point] $\exists x, \forall y, \exists z$ such that $p(x, y, z)$

Final 2021

- Students: Express the propositions we need to prove here
- Instructor: Prove

Problem 4. [5 points]

Prove that $n^2+9n+27$ is odd for all natural numbers n. You can use any proof technique.

Final 2021

- Students: Express the propositions we need to prove here
- Instructor: Prove

Problem 5. [5 points]
Prove using contradiction that the cube root of an irrational number is irrational.

Final 2021

- Students: Express the propositions we need to prove here
- Instructor: Prove

Problem 6. [5 points]

Prove that if $n^2 + 8n + 20$ is odd, then n is odd for natural numbers n.

Final 2021

- Students: Express the propositions we need to prove here
- Instructor: Prove

Problem 7. [10 points]

Use mathematical induction to prove the following identities.

(a) [5 points] For all natural numbers n,

$$1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + n^2 \times (n+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$$

Summary

- Proposition vs Predicates
- Universal and existential quantifiers
- Exercises with a bit of proof

Thank you!