CSE215
Foundations of Computer Science

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Agenda

- Final exercises on mathematical induction

Finish around 4h45

2020 Midterm-2

Problem 1. [20 points]

Use mathematical induction to prove the following identities.

(a) [5 points] For integers $n \geq 1$,

$$\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$$

(b) [5 points] For integers $n \geq 1$,

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}\frac{n(n+1)}{2}$$

(c) [5 points] $9^n + 3$ is divisible by 4 for integers $n \geq 1$.

(d) [5 points] Suppose that $g_1, g_2, g_3, \ldots$ is a sequence defined as follows:

$$g_1 = 3, g_2 = 5$$
$$g_k = 3g_{k-1} - 2g_{k-2} \qquad \text{for all integers } k \geq 3.$$

Prove that $g_n = 2^n + 1$ for all integers $n \geq 1$.

(a) [5 points] For integers $n \geq 1$,

$$\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$$

- Proof.
    - Let P(n) be the predicate 1/(1*3) + 1/(3*5) +…1/((2n-1)(2n+1)) = n/(2n+1)
    - Base step: We prove P(1).
    - Inductive step: We prove for any integer k>=1, P(k) -> P(k+1)
        - Let k be an arbitrary integer and k>=1.
        - Assume P(k) holds. That is 1/(1*3) + 1/(3*5) +…1/((2k-1)(2k+1)) = k/(2k+1)
        - We need to prove P(k+1), namely, 1/(1*3) + 1/(3*5) +…1/((2k-1)(2k+1)) + 1/((2k+1)(2k+3))= (k+2)/(2k+3)
            - Following assumption P(k), LHS = k/(2k+1) + 1/((2k+1)(2k+3))
            - k/(2k+1) + 1/((2k+1)(2k+3)) = … = (k+2)/(2k+3)
- QED.

(b) [5 points] For integers $n \geq 1$,

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}\frac{n(n+1)}{2}$$

- Proof.
  - Let P(n) be the predicate 1+(-1)^2*2^2+…+(-1)^n*n^2=(-1)^{n-1}* n(n+1)/2
  - Base step: We prove P(1).
  - Inductive step: We prove for any integer k>=1, P(k) -> P(k+1)
    - Let k be an arbitrary integer and k>=1.
    - Assume P(k) holds. That is 1+…+(-1)^k*k^2=(-1)^{k-1}* k(k+1)/2
    - We need to prove P(k+1), namely, 1+…+(-1)^k*k^2+(-1)^{k+1} *(k+1)^2=(-1)^k* (k+1)(k+2)/2
      - Following assumption P(k), LHS=(-1)^{k-1}* k(k+1)/2+(-1)^{k+1}*(k+1)^2 = … = RHS
- QED.

(c) [5 points] $9^n + 3$ **is divisible by 4 for integers** $n \geq 1$.

- Proof.
  - Let P(n) be the predicate 4|9^n+3
  - Base step: We prove P(1) holds.
  - Inductive step: We prove for any integer k>=1, P(k) -> P(k+1)
    - Let k be an arbitrary integer and k>=1.
    - Assume P(k) holds 4|9^k+3, namely 9^k+3 = 4m for some integer m.
    - We need to prove P(k+1), namely, 4|9^{k+1}+3
      - 9^{k+1}+3 = 9 * 9^k+3 = 9 *(9^k+3) - 24
      - Following assumption P(k), 9 *(9^k+3) - 24 = 36m - 24 = 4 * (9m - 6)
- QED.

2021 Midterm-2

Problem 1. [20 points]

Use mathematical induction to prove the following identities.

(a) [5 points] For all integers $n \geq 1$,

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

(b) [5 points] For all integers $n \geq 1$, $n(n^2+5)$ is a multiple of 6.

(c) [5 points] For all integers $n \geq 0$,

$$1 + \frac{2}{3} + \frac{4}{9} + \cdots + \left(\frac{2}{3}\right)^n = 3\left[1 - \left(\frac{2}{3}\right)^{n+1}\right]$$

(d) [5 points] Suppose that $c_1, c_2, c_3, \ldots$ is a sequence defined as follows:

$$c_1 = 3, c_2 = -9$$
$$c_k = 7c_{k-1} - 10c_{k-2} \qquad \text{for all integers } k \geq 3$$

Prove that $c_n = 4 \cdot 2^n - 5^n$ for all integers $n \geq 1$.

(a) [5 points] For all integers $n \geq 1$,

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

- Proof.
  - Let P(n) be the predicate 1*2*3+…+n(n+1)(n+2)=n(n+1)(n+2)(n+3)/4
  - Base step: We prove P(1) holds: LHS=6, RHS=6.
  - Inductive step: We prove for any integer k>=1, P(k) -> P(k+1)
    - Let k be an arbitrary integer and k>=1.
    - Assume P(k) holds. That is 1*2*3+…+k(k+1)(k+2)=k(k+1)(k+2)(k+3)/4
    - We need to prove P(k+1), namely, 1*2*3+…+k(k+1)(k+2) + (k+1)(k+2)(k+3)=(k+1)(k+2)(k+3)(k+4)/4
      - LHS=k(k+1)(k+2)(k+3)/4 + (k+1)(k+2)(k+3)=RHS
- QED.

(b) [5 points] For all integers $n \geq 1$, $n(n^2+5)$ is a multiple of 6.

- Proof.
  - Let P(n) be the predicate 6|n(n^2+5)
  - Base step: We prove P(1) holds: 6|6 is true.
  - Inductive step: We prove for any integer k>=1, P(k) -> P(k+1)
    - Let k be an arbitrary integer and k>=1.
    - Assume P(k) holds. That is 6|k(k^2+5)
    - We need to prove P(k+1), namely, 6|(k+1)(k^2+2k+6)
      - (k+1)(k^2+2k+6)= k^3+3k^2+8k+6= (k^3+5k)+3(k^2+k+2)
      - Above is a multiple of 6 because 6 | (k^3+5k) and 2 | (k^2+k+2)
- QED.

(c) [5 points] For all integers $n \geq 0$,

$$1 + \frac{2}{3} + \frac{4}{9} + \cdots + \left(\frac{2}{3}\right)^n = 3\left[1 - \left(\frac{2}{3}\right)^{n+1}\right]$$

- Proof.
  - Let P(n) be the predicate 1+ (2/3)^1+ … + (2/3)^n= 3[1-(2/3)^{n+1}]
  - Base step: We prove P(0) holds: LHS=1, RHS=1
  - Inductive step: We prove for any integer k>=1, P(k) -> P(k+1)
    - Let k be an arbitrary integer and k>=1.
    - Assume P(k) holds. That is 1+ (2/3)^1+ … + (2/3)^k= 3[1-(2/3)^{k+1}]
    - We need to prove P(k+1), namely, 1+ (2/3)^1+ … + (2/3)^k + (2/3)^{k+1} = 3[1-(2/3)^{k+2}]
      - LHS= 3[1-(2/3)^{k+1}] + (2/3)^{k+1}=RHS
- QED.

(d) **[5 points] Suppose that $c_1, c_2, c_3, \ldots$ is a sequence defined as follows:**

$$
\begin{aligned}
&c_1 = 3, c_2 = -9 \\
&c_k = 7c_{k-1} - 10c_{k-2} \qquad \textbf{for all integers } k \geq 3
\end{aligned}
$$

Prove that $c_n = 4 \cdot 2^n - 5^n$ for all integers $n \geq 1$.

- Proof.
  - Let P(n) be the predicate $c_n=4*2^n-5^n$
  - Base step: We prove P(1) and P(2) hold.
  - Inductive step: We prove for any integer k>=2, P(1),…, P(k) -> P(k+1)
    - Let k be an arbitrary integer and k>=2.
    - Assume P(1), P(2),…, P(k) hold
    - We need to prove P(k+1), namely, $c_{k+1}=4*2^{k+1} - 5^{k+1}$
      - By definition. LHS=$7c_k - 10c_{k-1}=7(4*2^k - 5^k) -10 (4*2^{k-1} - 5^{k-1})$ = …. RHS
- QED.