CSE215
Foundations of Computer Science

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Midterm2

- We try to strategically use it for reviewing for finals
- So, results to be announced for an additional week

Plan ahead

- ~~11.14 Tu:~~ Sets - REC on Sets; no homework review in REC
- ~~11.16~~ Review Midterm 2 No homework to be announced
- ~~11.21 Tu~~ **Midterm 2** - REC on Functions
- ~~11.23~~ Function Homework to be announced
- 11.28 Tu Function **- Regular lecture in REC**
- 11.30 Function Ungraded homework to be announced
- 12.05 Tu Final Review - **Reviewing with Midterm 2**
- 12.07 Final Review No homework to be announced
- 12.10 Mon Final -

Agenda

- Review on one-to-one and onto functions
- Inverse functions

Summary of last class: one-to-one and onto functions

How to show a function $f : A \to B$ is injective:

Contrapositive approach:
Suppose $x, y \in A$ and $f(x) = f(y)$.

$\vdots$

Therefore $x = y$.

How to show a function $f : A \to B$ is surjective:

Suppose $b \in B$.
[Prove there exists $a \in A$ for which $f(a) = b$.]

Exercise 1
SBU 2021 Final

Problem 9. [5 points]

Write and fill the table with ✓ or ✗. If a function is one-to-one or onto, then use ✓. On the other hand, if a function is not one-to-one or not onto, then use ✗.

Function	Domains	One-to-one function?	Onto function?
$f(x) = 3x$	$f : \mathbb{Z} \to \mathbb{Z}$		
$f(x) = 3x$	$f : \mathbb{R} \to \mathbb{R}$		
$f(x) = 3x^2$	$f : \mathbb{Z} \to \mathbb{Z}$		
$f(x) = 3x^2$	$f : \mathbb{R} \to \mathbb{R}$		

Exercise 2 (seen last time)

A function $f : Z \times Z \rightarrow Z \times Z$ is defined as $f(m, n) = (m + n, 2m + n)$. Verify (1) if the function is one-to-one and (2) if it is onto.

Solution (skeleton)

- We verify if f is one-to-one.
  - We show, for any two pairs (m1,n1), (m2,n2) of Z x Z,   f(m1,n1) = f(m2,n2) implies (m1,n1)=(m2,n2)
  - ….
- We virify if f is onto.
  - We show, for any arbitrary element (p,q) in Z x Z, there exists m and n such that f(m,n) = (p,q)
  - Suppose an element (p,q) in Z x Z. Let (m,n) be (p,q). We have f(m,n)=(p,q)

Exercise 3

True or false?

the function $f : \mathbb{R} - \{1\} \rightarrow \mathbb{R} - \{1\}$ defined by $f(x) = \left(\dfrac{x+1}{x-1}\right)^3$ is bijective.

Solution

- We first prove f is injective

We want to prove $\forall x, y \in \mathbb{R} - \{1\}.\ f(x) = f(y) \to x = y$

Suppose $x, y \in \mathbb{R} - \{1\}$ and $f(x) = f(y)$

We have $\left(\frac{x+1}{x-1}\right)^3 = \left(\frac{y+1}{y-1}\right)^3$ Thus $\frac{x+1}{x-1} = \frac{y+1}{y-1}$

Thus $1 + \frac{2}{x-1} = 1 + \frac{2}{y-1}$. Thus $\frac{2}{x-1} = \frac{2}{y-1}$

Thus $2(y-1) = 2(x-1)$ Thus $x = y$

We have proved $\forall x, y \in \mathbb{R} - \{1\}\ f(x) = f(y) \to x = y$

- We then prove f is surjective

We want to prove $\forall y \in \mathbb{R} - \{1\}.\ \exists x \in \mathbb{R} - \{1\}.\ f(x) = y$

Suppose $y \in \mathbb{R} - \{1\}$.

We choose $x = \frac{2}{\sqrt[3]{y} - 1} + 1$

We have $f(x) = y$

We have proved $\forall y \in \mathbb{R} - \{1\}\ \exists x \in \mathbb{R} - \{1\}\ f(x) = y$

Exercise 4

- Consider the function $\theta : \{0,1\} \times \mathbb{N} \to \mathbb{Z}$ defined as $\theta(a,b) = (-1)^a b$. Is θ injective? Is it surjective? Explain.

Solution

First we show that θ is injective. Suppose $\theta(a,b)=\theta(c,d)$. Then $(-1)^a b=(-1)^c d$. As b and d are both in $\mathbb{N}$, they are both positive. Then because $(-1)^a b=(-1)^c d$, it follows that $(-1)^a$ and $(-1)^c$ have the same sign. Since each of $(-1)^a$ and $(-1)^c$ equals ± 1, we have $(-1)^a=(-1)^c$, so then $(-1)^a b=(-1)^c d$ implies $b=d$. But also $(-1)^a=(-1)^c$ means a and c have the same parity, and because $a,c\in\{0,1\}$, it follows $a=c$. Thus $(a,b)=(c,d)$, so θ is injective.
Next note that θ **is not surjective** because $\theta(a,b)=(-1)^a b$ is either positive or negative, but never zero. Therefore there exist no element $(a,b)\in\{0,1\}\times\mathbb{N}$ for which $\theta(a,b)=0\in\mathbb{Z}$.

Exercise 5
SBU 2022 Midterm

Problem 4. [5 points]

Let A and B be finite sets where $|A| = |B|$. Is it possible to define a function $f : A \to B$ that is one-to-one but not onto? Is it possible to define a function $g : A \to B$ that is onto but not one-to-one?

Inverse functions

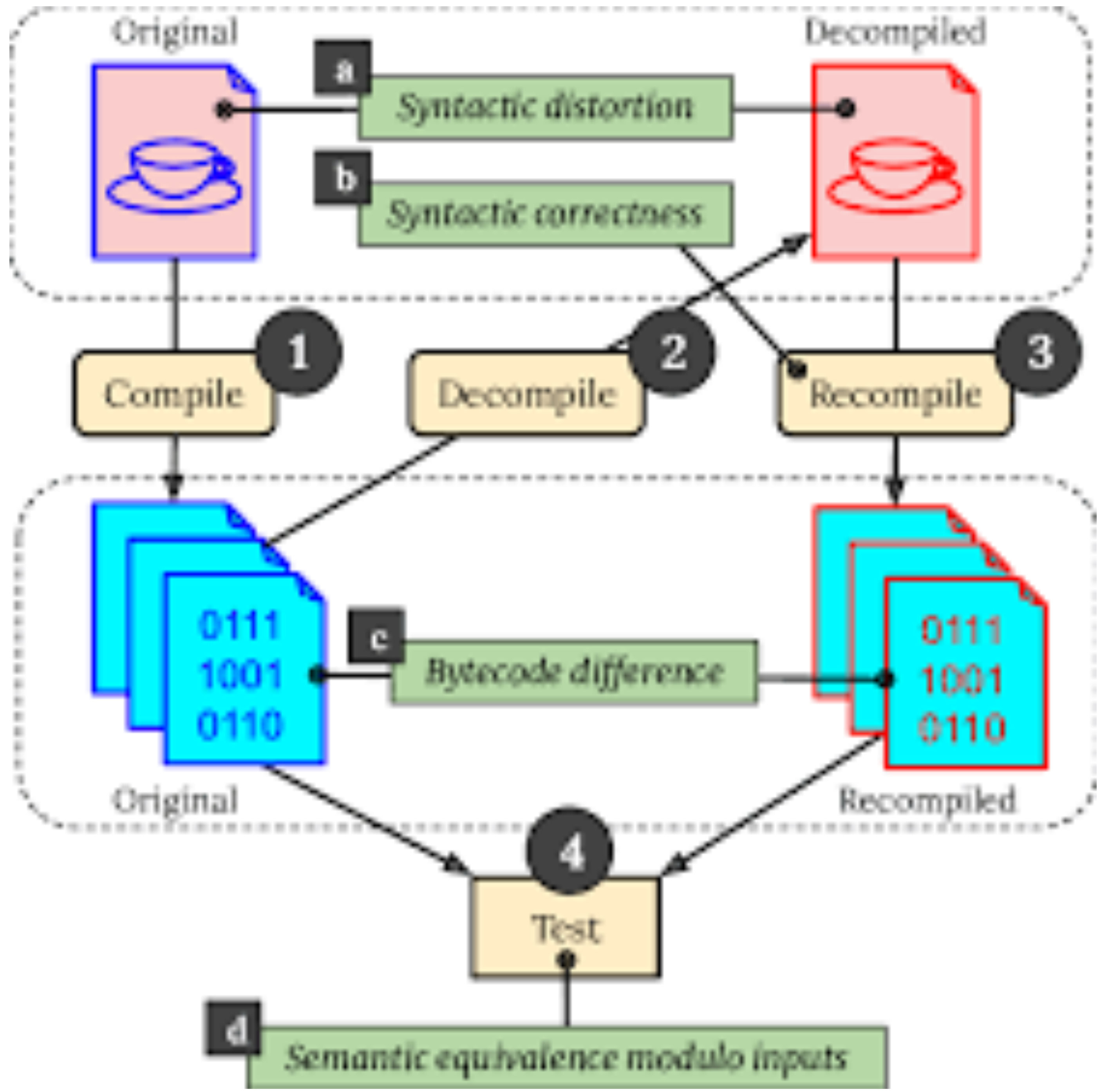
Inverse functions

Definition

- Suppose $F : X \to Y$ is a one-to-one correspondence.
  Then, the inverse function $F^{-1} : Y \to X$ is defined as follows:
  Given any element y in Y,
  $F^{-1}(y) =$ that unique element x in X such that $F(x) = y$.
- $F^{-1}(y) = x \Leftrightarrow y = F(x)$.

Does encryption have an inverse function?

Does Java compilation have an inverse function?



Example

Problem

- Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$. Find its inverse function.

Proof

For any y in R, by definition of f^{-1}

- f^{-1} = unique number x such that $f(x) = y$
  Consider $f(x) = y$
  $\implies 4x - 1 = y \qquad (\because \text{Defn. of } f)$
  $\implies x = \frac{y+1}{4} \qquad (\because \text{Simplify})$
- Hence, $f^{-1}(y) = \frac{y+1}{4}$ is the inverse function.

Exercise 0

- Check If the function $f : Z \to Z$ defined by $f(n) = 6 - n$ is one-to-one correspondence. If yes, compute its inverse.

Exercise 1

- Check if the function $f : R \rightarrow R$ defined as $f(x) = x^3 + 1$ is a one-t-one correspondence. If yes, find its inverse.

Exercise 2

- The function $f : R - \{2\} \rightarrow R - \{5\}$ defined by $f(x) = (5x+1)/(x-2)$ is bijective. Find its inverse.

Solution

- Let y be an element of $R - \{5\}$. We have $y = f(x)$ if and only if $x = 11/(y-5)+2$.

- Thus $f^{-1}(y) = 11/(y-5)+2$

Exercise 3

- The function $g: Z \times Z \rightarrow Z \times Z$ defined by the formula $g(m,n) = (m+n, m+2n)$ is a one-to-one correspondence. Find its inverse.

Solution

- Let (u,v) be an arbitrary element if Z x Z. Then f(m,n) = (u,v) if and only if m = 2u-v and n = v-u.

- Thus, f^{-1) (u,v) = (2u-v, v-u).