

CSE215

Foundations of Computer Science

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Agenda

- Prove set properties using
 - set identities
 - element argument

**Prove set properties
using set identities
(Algebraic proof)**

Set identities

Laws	Formula	Formula
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	$A \cup \phi = A$	$A \cap U = A$
Complement laws	$A \cup A' = U$	$A \cap A' = \phi$
Double comp. law	$(A')' = A$	
Idempotent laws	$A \cup A = A$	$A \cap A = A$
Uni. bound laws	$A \cup U = U$	$A \cap \phi = \phi$
De Morgan's laws	$(A \cup B)' = A' \cap B'$	$(A \cap B)' = A' \cup B'$
Absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
Complements	$U' = \phi$	$\phi' = U$
Set diff. laws	$A - B = A \cap B'$	

Algebraic proof: Example 1

Proposition

- Construct an algebraic proof that for all sets A , B , and C ,
 $(A \cup B) - C = (A - C) \cup (B - C)$

Proof

- $(A \cup B) - C$
 $= (A \cup B) \cap C'$ (\because set difference law)
 $= C' \cap (A \cup B)$ (\because commutative law)
 $= (C' \cap A) \cup (C' \cap B)$ (\because distributive law)
 $= (A \cap C') \cup (B \cap C')$ (\because commutative law)
 $= (A - C) \cup (B - C)$ (\because set difference law)

Algebraic proof: Example 2

Proposition

- Construct an algebraic proof that for all sets A and B , $A - (A \cap B) = A - B$

Proof

- $A - (A \cap B)$
 $= A \cap (A \cap B)'$ (\because set difference law)
 $= A \cap (A' \cup B')$ (\because De Morgan's law)
 $= (A \cap A') \cup (A \cap B')$ (\because distributive law)
 $= \phi \cup (A \cap B')$ (\because complement law)
 $= (A \cap B') \cup \phi$ (\because commutative law)
 $= A \cap B'$ (\because identity law)
 $= A - B$ (\because set difference law)

**Prove set properties using
element argument**

Reminder

$A \subseteq B$ if and only if $\forall x \in A, x \in B$

Let X and Y be subsets of a universal set U and suppose x and y are elements of U .

- $x \in X \cup Y \Leftrightarrow x \in X \text{ or } x \in Y$
- $x \in X \cap Y \Leftrightarrow x \in X \text{ and } x \in Y$
- $x \in X - Y \Leftrightarrow x \in X \text{ and } x \notin Y$
- $x \in X' \Leftrightarrow x \notin X$
- $(x, y) \in X \times Y \Leftrightarrow x \in X \text{ and } y \in Y$

Element argument – situation 1

Basic method for proving that a set is a subset of another

- Let sets X and Y be given. To prove that $X \subseteq Y$,
 1. **suppose** that x is a particular but arbitrarily chosen element of X .
 2. **show** that x is an element of Y .

Element argument – situation 2

Basic method for proving that two sets are equal

- Let sets X and Y be given. To prove that $X = Y$,
 1. Prove that $X \subseteq Y$.
 2. Prove that $Y \subseteq X$.

Element argument – situation 3

Basic method for proving a set equals the empty set

- To prove that a set X is equal to the empty set ϕ , prove that X has no elements.
- To do this, suppose X has an element and derive a contradiction.

Proof by element argument: Example 1

Proposition

- Prove that for all sets A , B , and C
 $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof

We need to prove:

1. $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
2. $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

Proof by element argument: Example 1

Proof (continued)

Proof that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Suppose $x \in A \cup (B \cap C)$.

$x \in A$ or $x \in B \cap C$ (\because defn. of union)

- **Case 1. $[x \in A.]$**

$x \in A \cup B$ (\because defn. of union)

$x \in A \cup C$ (\because defn. of union)

$x \in (A \cup B) \cap (A \cup C)$ (\because defn. of intersection)

- **Case 2. $[x \in B \cap C.]$**

$x \in B$ and $x \in C$ (\because defn. of intersection)

$x \in A \cup B$ (\because defn. of union)

$x \in A \cup C$ (\because defn. of union)

$x \in (A \cup B) \cap (A \cup C)$ (\because defn. of intersection)

Proof by element argument: Example 1

Proof (continued)

Proof that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Suppose $x \in (A \cup B) \cap (A \cup C)$.

$x \in A \cup B$ and $x \in A \cup C$ (\because defn. of intersection)

- **Case 1. $[x \in A.]$**

$x \in A \cup (B \cap C)$ (\because defn. of union)

- **Case 2. $[x \notin A.]$**

$x \in A$ or $x \in B$ (\because defn. of union)

$x \in B$ ($\because x \notin A$)

$x \in A$ or $x \in C$ (\because defn. of union)

$x \in C$ ($\because x \notin A$)

$x \in B \cap C$ (\because defn. of intersection)

$x \in A \cup (B \cap C)$ (\because defn. of union)

Proof by element argument: Example 2

Proposition

- Prove that for all sets A and B , $(A \cup B)' = A' \cap B'$.

Proof

We need to prove:

1. $(A \cup B)' \subseteq A' \cap B'$
2. $A' \cap B' \subseteq (A \cup B)'$

Proof by element argument: Example 2

Proof (continued)

- **Proof that $(A \cup B)' \subseteq A' \cap B'$.**

Suppose $x \in (A \cup B)'$.

$x \notin A \cup B$ (\because defn. of complement)

It is false that (x is in A or x is in B).

x is not in A and x is not in B (\because De Morgan's law of logic)

$x \notin A$ and $x \notin B$

$x \in A'$ and $x \in B'$ (\because defn. of complement)

$x \in (A' \cap B')$ (\because defn. of intersection)

Hence, $(A \cup B)' \subseteq A' \cap B'$ (\because defn. of subset)

Proof by element argument: Example 2

Proof (continued)

- **Proof that $A' \cap B' \subseteq (A \cup B)'$.**

Suppose $x \in A' \cap B'$.

$x \in A'$ and $x \in B'$ (\because defn. of intersection)

$x \notin A$ and $x \notin B$ (\because defn. of complement)

x is not in A and x is not in B

It is false that (x is in A or x is in B)

(\because De Morgan's law of logic)

$x \notin A \cup B$

$x \in (A \cup B)'$ (\because defn. of complement)

Hence, $A' \cap B' \subseteq (A \cup B)'$ (\because defn. of subset)

Proof by element argument: Example 3

Proposition

- For any sets A and B , if $A \subseteq B$, then
(a) $A \cap B = A$ and (b) $A \cup B = B$.

Proof

Part (a): We need to prove:

1. $A \cap B \subseteq A$
2. $A \subseteq A \cap B$

Part (b): We need to prove:

1. $A \cup B \subseteq B$
2. $B \subseteq A \cup B$

Proof by element argument: Example 3

Proof (continued)

Part (a).

1. **Proof that $A \cap B \subseteq A$.**

$A \cap B \subseteq A$ (\because inclusion of intersection)

2. **Proof that $A \subseteq A \cap B$.**

Suppose $x \in A$

$x \in B$ ($\because A \subseteq B$)

$x \in A$ and $x \in B$

$x \in A \cap B$ (\because defn. of intersection)

Proof by element argument: Example 3

Proof (continued)

Part (b).

1. **Proof that $A \cup B \subseteq B$.**

Suppose $x \in A \cup B$

$x \in A$ or $x \in B$ (\because defn. of union)

If $x \in A$, then $x \in B$ ($\because A \subseteq B$)

$x \in B$ (\because Modus Ponens and division into cases)

2. **Proof that $B \subseteq A \cup B$.**

$B \subseteq A \cup B$ (\because inclusion in union)

Proof by element argument: Example 4

Proposition

- If E is a set with no elements and A is any set, then $E \subseteq A$.

Proof

Proof by contradiction.

- Suppose there exists a set E with no elements and a set A such that $E \not\subseteq A$.
- $\exists x$ such that $x \in E$ and $x \notin A$ (\because defn. of a subset)
- But there can be no such element since E has no elements.
- Contradiction!
- Hence, if E is a set with no elements and A is any set, then $E \subseteq A$.

Proof by element argument: Example 5

Proposition

- There is only one set with no elements.

Proof

- Suppose E_1 and E_2 are both sets with no elements.
- $E_1 \subseteq E_2$ (\because previous proposition)
- $E_2 \subseteq E_1$ (\because previous proposition)
- Thus, $E_1 = E_2$

Proof by element argument: Example 6

Proposition

- Prove that for any set A , $A \cap \phi = \phi$

Proof

Proof by contradiction.

- Suppose there is an element x such that $x \in A \cap \phi$
- $x \in A$ and $x \in \phi$ (\because defn. of intersection)
- $x \in \phi$
- Impossible because ϕ cannot have any elements
- Hence, the supposition is incorrect.
- So, $A \cap \phi = \phi$

Proof by element argument: Example 7

Proposition

- For all sets A , B , and C , if $A \subseteq B$ and $B \subseteq C'$, then $A \cap C = \phi$.

Proof

Proof by contradiction.

- Suppose there is an element x such that $x \in A \cap C$
- $x \in A$ and $x \in C$ (\because defn. of intersection)
- $x \in A$
- $x \in B$ ($\because x \in A$ and $A \subseteq B$)
- $x \in C'$ ($\because x \in B$ and $B \subseteq C'$)
- $x \notin C$ (\because defn. of complement)
- $x \in C$ and $x \notin C$
- Contradiction!
- Hence, the supposition is incorrect.
- So, $A \cap C = \phi$