CSE215
Foundations of Computer Science

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Zoom is needed for today

This week

- Some other proof tech (cont.)
- Organizing thoughts into clear writings
- SBU exam exercises
- Mock exam will be announced as ungraded homework.
- Use it for review; midterm 1 can be different.

Some other proof tech. (cont)

Non constructive proof

Proposition There exist irrational numbers x, y for which x^y is rational.

Solution

Proposition There exist irrational numbers x, y for which x^y is rational.

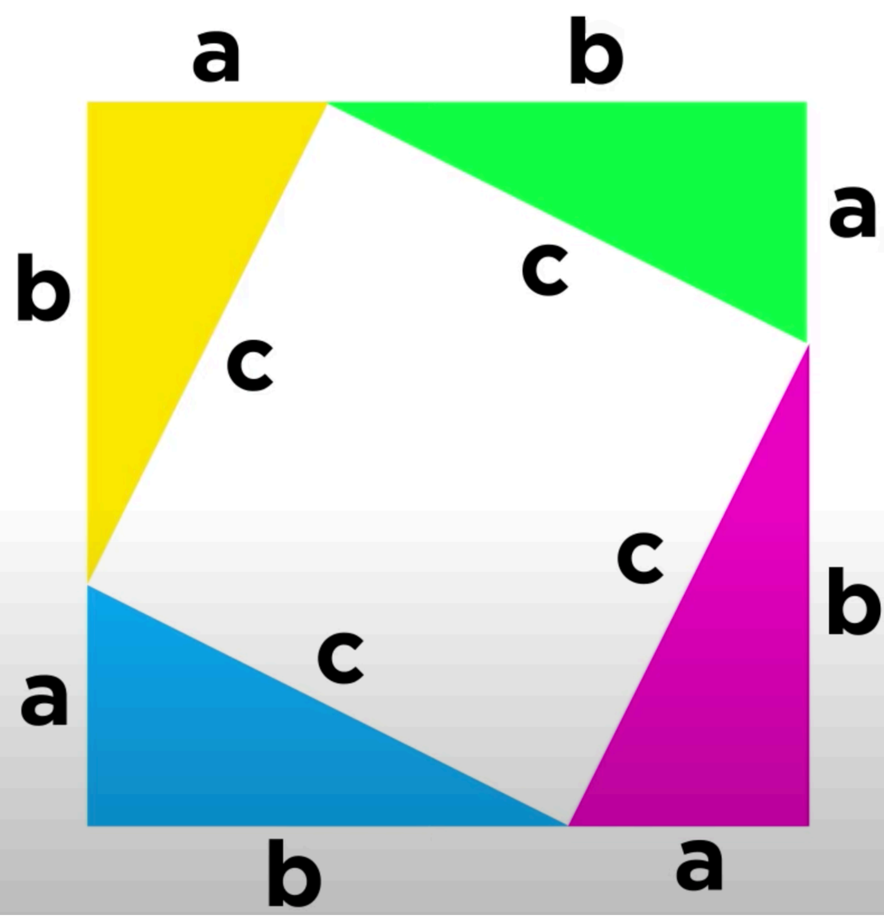
Proof. Let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$. We know y is irrational, but it is not clear whether x is rational or irrational. On one hand, if x is irrational, then we have an irrational number to an irrational power that is rational:

$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2.$$

On the other hand, if x is rational, then $y^y = \sqrt{2}^{\sqrt{2}} = x$ is rational. Either way, we have a irrational number to an irrational power that is rational. ■

Non-analytical, geometry-based proof

Proof of the Pythagorean Theorem



area of one triangle $= \frac{1}{2}ab$

area of large square $= (a + b)^2$

area of small square $= c^2$

large square = triangles + small square

$$(a + b)^2 = 4\left(\frac{1}{2}ab\right) + c^2$$

Organizing thoughts into clear writings

SBU 2020 Midterm

Problem 8. [5 points]

Prove that for all integers a, if a^3 is even, then a is even.

- Proof.
    - We want to prove ________________
    - We use **proof by contraposition** to prove the statement above.
    - That is, we want to prove __________
    - Assume ______
    - _______
    - Therefore _______
- QED.

SBU 2022 Midterm

Problem 8. [5 points]

Prove that for any two integers a and b, if ab is odd, then a and b are both odd.

- Proof.
    - We want to prove ________________
    - We use **proof by contradiction** to prove the statement above.
    - That is, we assume __________ (A)
    - From this assumption, ______
    - So, we get a contradiction with (A)
- QED.

SBU 2020 Midterm

Problem 6. [5 points]

Prove that the product of any four consecutive integers is a multiple of 4.

- Proof.
  - Suppose ____. We want to prove ________________ (Q)
  - We prove the statement by **division into cases.**
  - Case 1: _____. In this case, we have ____.
  - Case 2: _____. In this case, we have ____.
  - Thus, (Q) holds in either case.
- QED.

SBU 2021 Midterm

Problem 8. [5 points]

Prove by contradiction that there are no integers x and y such that $x^2 = 4y + 2$.

- Proof.
  - (formal statement of proof objectives)
  - (our proof strategy, derived proof objectives, assumptions)
  - (core proof)
- QED.

Solution

- Proof.
  - **(formal statement of proof objectives)** We want to prove: ~(\exists x,y\in Z, such that x^2 = 4 y + 2)
  - **(proof strategy, derived proof objectives, assumptions)** We use proof by contradiction. Assume
    - (A) \exists x, y\in Z, such that x^2 = 4 y + 2
  - **(core proof)**
    - x^2 must be even since x^2 = 4 y + 2. Thus x must be even. Let x = 2k for some integer k.
    - Then 4 k^2 = 4 y + 2. Thus, 2 * k^2 = 2y + 1.
    - This is a contradiction, since 2 * k^2 is even and 2y + 1 is odd. Therefore (A) must be false.
- QED.

Solution

- The statement is false.
- To disprove it, we choose x=2, y=1/2. Then both x and y are rational, but x^y is irrational.

SBU exam problems

- Prove: Given an integer a, then $a^3 + a^2 + a$ is even if and only if a is even.

Solution

- Suppose a is an integer.
- We first prove $a^3 + a^2 + a$ is even -> a is even.
  - We only need to show a is odd -> $a^3 + a^2 + a$ is odd
  - Suppose a is odd ….
- We then prove a is even -> $a^3 + a^2 + a$ is even
  - Suppose a is even ….

Problem 6. [5 points]

Prove that if $n^2 + 8n + 20$ is odd, then n is odd for natural numbers n.

Solution

- Proof.
  - We want to prove: for all natural numbers n, n^2+8n+20 is odd -> n is odd.
  - We use proof by contradiction to prove the statement above.
  - That is, we assume there exists a natural number n such that n^2+8n+20 is odd, and n is even.
  - From “n is even”, we know n^2 must be even, and 8n must be even
  - Therefore n^2+8n+20 must be even, which contradicts with the assumption above.
- QED.

SBU 2022 Midterm

Problem 6. [5 points]

Let $a_1, a_2, \ldots, a_n$ be real numbers for $n \geq 1$. Prove that at least one of these numbers is greater or equal to the average of the numbers.

Solution

- Proof.
  - We want to prove: for any real numbers a1,…a_n, there exists an a_i where 1<=i<=n, such that a_i>= (a1+a2+…a_n)/n.
  - We use proof by contradiction to prove the statement above.
  - That is, we assume: there exists some real numbers a1,…a_n, such that for all a_i, 1<=i<=n, a_i<(a1+a2+…a_n)/n.
  - From this assumption, we know (a1+a2+…a_n) < n * (a1+a2+…a_n)/n, which is a contradiction.
- QED.

SBU 2022 Midterm

Problem 7. [5 points]

Prove or disprove the following statement. If x and y are rational, then x^y is rational.