Guideline

- Ungraded because we would usually review homework of week n-2 at week n.
- Important as ever homework is

Exercise 1 (20 points)

The *sigmoid* function plays a pivotal role in machine learning. Particularly, it's instrumental in classification problems where we map predicted values to probabilities. The sigmoid function can squish any real-valued number into the range between 0 and 1, making it extremely useful for converting values into probabilities.

The sigmoid function S: $\mathbb{R} \rightarrow (0, 1)$, is defined as:

$$S(x) = \frac{1}{1 + e^{-x}}$$

Your task in this exercise is to check if this sigmoid function S is bijective. In other words, determine whether it's both injective (or one-to-one) and surjective (or onto).

As a reminder:

- Injectivity: Show that if S(x) = S(y), then x = y. This means that no two different inputs will yield the same output.

- Surjectivity: Show that for any number y in the range (0, 1), there is an x in the domain of real numbers such that S(x) = y. This means that every possible output is produced by some input.

Exercise 2 (20 points)

1. We are considering the function $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by the equation $f(x, y) = (xy, x^3)$. Your task is to determine $f \circ f$, the composition of f with itself.

Hint: Function composition is essentially feeding the output of one function into the other. In this case, you're feeding the output of f back into itself.

2. Now, consider the functions $f, g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, which are defined as $f(m, n) = (mn, m^2)$ and $g(m, n) = (m + 1, m + n)$. The task here is to compute (a) $g \circ f$, and (b) $f \circ g$.

Exercise 3. (points = 10)

Let $S = \{x \in \mathbb{R} \mid 0 < x < 1\}$

1. Let $U = {x ∈ R | 0 < x <2}. Prove that S and U have the same cardinality.
2. Let V = {x ∈ R | 2 < x <5}. Prove that S and V have the same cardinality.

Exercise 4. (points = 50)

For each pair of sets given below, establish their cardinality equality by explicitly defining a bijection between them. There's no need to formally prove your function is bijective; simply provide the function's definition. For this, you can use the following template: "Let $f : A \to B$ be defined by $f(x) = \ldots$".

1. $\mathbb{R}$ and $(0, \infty)$ (Hint: Try using the exponential function in some way.)
2. $\mathbb{R}$ and $(\sqrt{2}, \infty)$
3. $\mathbb{R}$ and $(0, 1)$
4. $\mathbb{Z}$ and S where $S = \{\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \ldots\}$
5. $A = \{3k : k \in \mathbb{Z}\}$ and $B = \{7k : k \in \mathbb{Z}\}$
6. $A = \{(5n, -3n) : n \in \mathbb{Z}\}$ and $\mathbb{Z}$
7. The set of even integers, denoted by E, and the set of odd integers, denoted by O.
8. $\mathbb{Z}$ and S where $S = x \in \mathbb{R} : \sin(x) = 1$
9. $\{0, 1\} \times \mathbb{N}$ and $\mathbb{N}$
10. $\mathbb{N}$ and $\mathbb{Z}$ (Hint: create a function that interleaves positive and negative numbers as shown in our slides)