CSE215
Foundations of Computer Science

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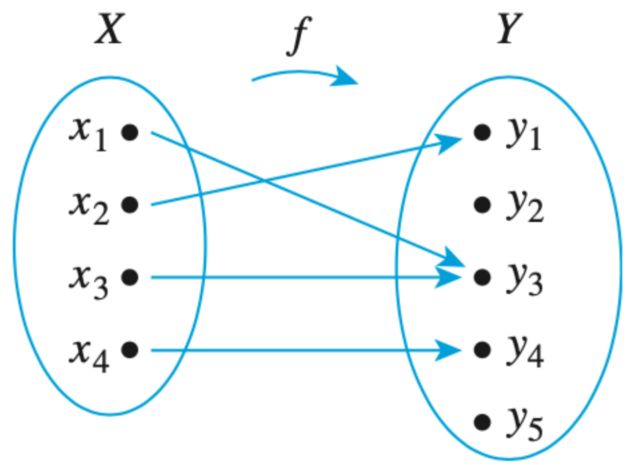
Plan

- Function concepts
- One-on-one functions
- Onto functions

Functions

Functions



• **Definition**

A **function f from a set X to a set Y**, denoted $f\colon X \to Y$, is a relation from X, the **domain**, to Y, the **co-domain**, that satisfies two properties: (1) every element in X is related to some element in Y, and (2) no element in X is related to more than one element in Y. Thus, given any element x in X, there is a unique element in Y that is related to x by f. If we call this element y, then we say that "f sends x to y" or "f maps x to y" and write $x \xrightarrow{f} y$ or $f\colon x \to y$. The unique element to which f sends x is denoted

$f(x)$ and is called **f of x**, or
the output of f for the input x, or
the value of f at x, or
the image of x under f.

Functions (cont.)

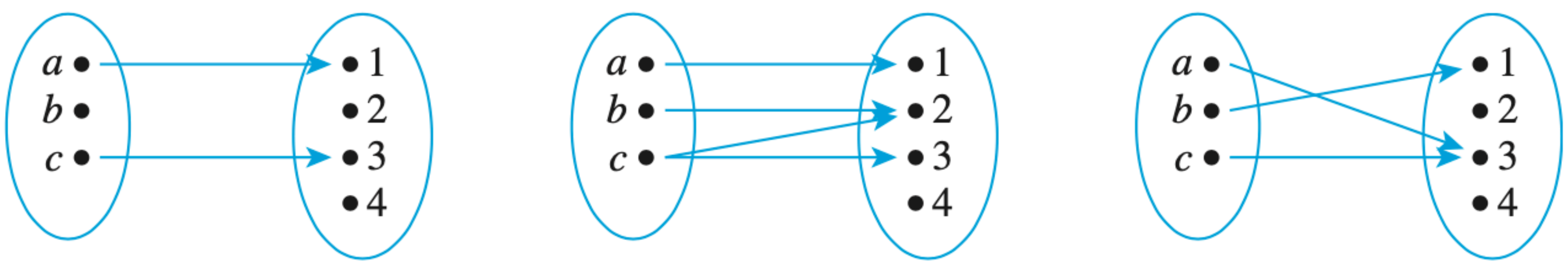
The set of all values of f taken together is called the *range of f* or the *image of X under f*. Symbolically,

$$\textbf{range of } f = \textbf{image of } X \textbf{ under } f = \{y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}.$$

Given an element y in Y, there may exist elements in X with y as their image. If $f(x) = y$, then x is called **a preimage of y** or **an inverse image of y**. The set of all inverse images of y is called *the inverse image of y*. Symbolically,

$$\textbf{the inverse image of } y = \{x \in X \mid f(x) = y\}.$$

Quiz:
Functions or non-functions



(a) (b) (c)

Quiz:
Functions or non-functions

Consider the set $f = \{(x^2, x) : x \in \mathbb{R}\}$. Is this a function from $\mathbb{R}$ to $\mathbb{R}$?

Consider the set $f = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 3x + y = 4\}$. Is this a function from $\mathbb{Z}$ to $\mathbb{Z}$?

Quiz: Other definitions

Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. Define a function f from X to Y by the arrow diagram in Figure 7.1.3.

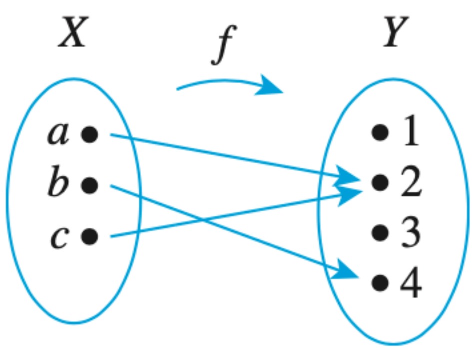


Figure 7.1.1

a. Write the domain and co-domain of f.

b. Find $f(a)$, $f(b)$, and $f(c)$.

c. What is the range of f?

d. Is c an inverse image of 2? Is b an inverse image of 3?

e. Find the inverse images of 2, 4, and 1.

f. Represent f as a set of ordered pairs.

One-to-one functions

Definition

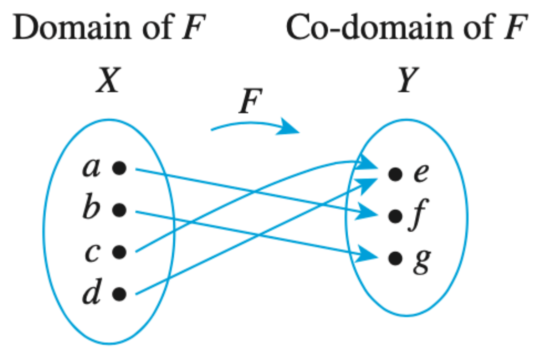
- A function $F : X \to Y$ is one-to-one (or injective) if and only if for all elements x_1 and x_2 in X,

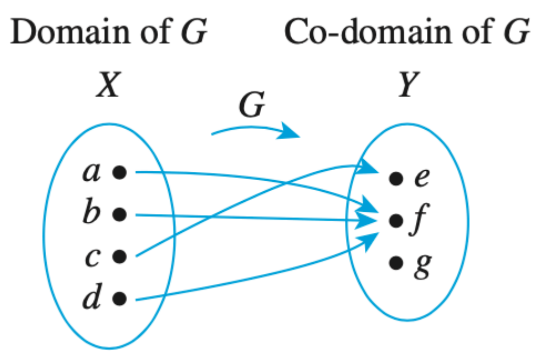
$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2, \text{ or}$$
$$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

- A function $F : X \to Y$ is one-to-one $\Leftrightarrow$
$\forall x_1, x_2 \in X$, if $F(x_1) = F(x_2)$ then $x_1 = x_2$.
A function $F : X \to Y$ is not one-to-one $\Leftrightarrow$
$\exists x_1, x_2 \in X, \quad F(x_1) = F(x_2)$ **and** $x_1 \neq x_2$.

Quiz: one-to-one functions

. Let $X = \{a, b, c, d\}$ and $Y = \{e, f, g\}$. Define functions F and G by the arrow diagrams below.





a. Is F one-to-one? Why or why not? Is it onto? Why or why not?

b. Is G one-to-one? Why or why not? Is it onto? Why or why not?

One-to-one functions: Proof technique

Problem

- Prove that a function f is one-to-one.

Proof

Direct proof.

- Suppose x_1 and x_2 are elements of X such that $f(x_1) = f(x_2)$.
- Show that $x_1 = x_2$.

Problem

- Prove that a function f is not one-to-one.

Proof

Counterexample.

- Find elements x_1 and x_2 in X so that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

One-to-one functions: Example 1

Problem

- Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$. Is f one-to-one? Prove or give a counterexample.

Proof

Direct proof.

- Suppose x_1 and x_2 are elements of X such that $f(x_1) = f(x_2)$.
  $\Longrightarrow 4x_1 - 1 = 4x_2 - 1 \qquad (\because$ Defn. of $f)$
  $\Longrightarrow 4x_1 = 4x_2 \qquad (\because$ Add 1 on both sides)
  $\Longrightarrow x_1 = x_2 \qquad (\because$ Divide by 4 on both sides)
- Hence, f is one-to-one.

One-to-one functions: Example 2

Problem

- Define $g : \mathbb{Z} \to \mathbb{Z}$ by the rule $g(n) = n^2$ for all $n \in \mathbb{Z}$. Is g one-to-one? Prove or give a counterexample.

Proof

Counterexample.

- Let $n_1 = -1$ and $n_2 = 1$.
  $\implies g(n_1) = (-1)^2 = 1$ and $g(n_2) = 1^2 = 1$
  $\implies g(n_1) = g(n_2)$ but, $n_1 \neq n_2$
- Hence, g is not one-to-one.

Onto functions

Definition

- A function $F : X \to Y$ is onto (or surjective) if and only if given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.
- A function $F : X \to Y$ is onto $\Leftrightarrow$
  $\forall y \in Y, \exists x \in X$ such that $F(x) = y$.
  A function $F : X \to Y$ is not onto $\Leftrightarrow$
  $\exists y \in Y, \forall x \in X$ such that $F(x) \neq y$.

Quiz: Onto functions

Let $X = \{1, 5, 9\}$ and $Y = \{3, 4, 7\}$.

a. Define $f\colon X \to Y$ by specifying that

$$f(1) = 4, \quad f(5) = 7, \quad f(9) = 4.$$

Is f one-to-one? Is f onto? Explain your answers.

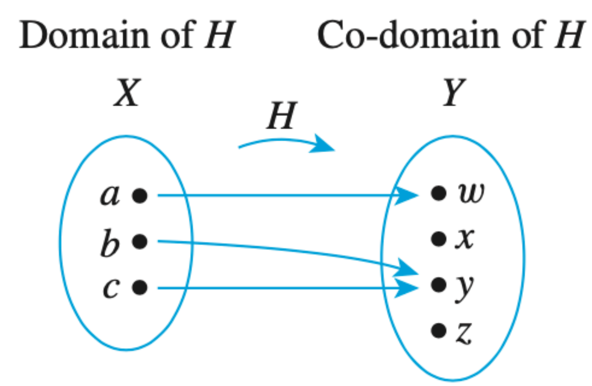
b. Define $g\colon X \to Y$ by specifying that

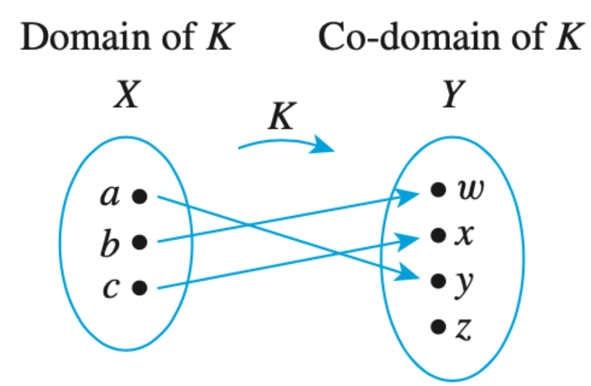
$$g(1) = 7, \quad g(5) = 3, \quad g(9) = 4.$$

Is g one-to-one? Is g onto? Explain your answers.

Quiz: Onto functions

Let $X = \{a, b, c\}$ and $Y = \{w, x, y, z\}$. Define functions H and K by the arrow diagrams below.





a. Is H one-to-one? Why or why not? Is it onto? Why or why not?

b. Is K one-to-one? Why or why not? Is it onto? Why or why not?

Onto functions: Proof technique

Problem

- Prove that a function f is onto.

Proof

Direct proof.

- Suppose that y is any element of Y
- Show that there is an element x of X with $F(x) = y$

Problem

- Prove that a function f is not onto.

Proof

Counterexample.

- Find an element y of Y such that $y \neq F(x)$ for any x in X.

Onto functions: Example 1

Problem

- Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$. Is f onto? Prove or give a counterexample.

Proof

Direct proof.

- Let $y \in \mathbb{R}$. We need to show that $\exists x$ such that $f(x) = y$.
  Let $x = \frac{y+1}{4}$. Then
  $f\left(\frac{y+1}{4}\right) = 4\left(\frac{y+1}{4}\right) - 1 \qquad (\because \text{Defn. of } f)$
  $= y \qquad (\because \text{Simplify})$
- Hence, f is onto.

Onto functions: Example 2

Problem

- Define $g : \mathbb{Z} \to \mathbb{Z}$ by the rule $g(n) = 4n - 1$ for all $n \in \mathbb{Z}$. Is g onto? Prove or give a counterexample.

Proof

Counterexample.

- We know that $0 \in \mathbb{Z}$.
- Let $g(n) = 0$ for some integer n.

  $\implies 4n - 1 = 0 \qquad (\because$ Defn. of $g)$

  $\implies n = \frac{1}{4} \qquad (\because$ Simplify$)$

  But $\frac{1}{4} \notin \mathbb{Z}$.

  So, $g(n) \neq 0$ for any integer n.
- Hence, g is not onto.

Important note

- Proof on function’s infectivity/surjectivity will not be in the exam
- But we need to be able to check if a function is injective or surjective

Exercise 1

Consider the cosine function $\cos : \mathbb{R} \to \mathbb{R}$. Decide whether this function is injective and whether it is surjective. What if it had been defined as $\cos : \mathbb{R} \to [-1, 1]$?

Exercise 2

a. Define $f: \mathbf{Z} \to \mathbf{Z}$ by the rule $f(n) = 2n$, for all integers n.

(i) Is f one-to-one? Prove or give a counterexample.

(ii) Is f onto? Prove or give a counterexample.

Exercise 3

A function $f : \mathbb{Z} \to \mathbb{Z}$ is defined as $f(n) = 2n + 1$. Verify whether this function is injective and whether it is surjective.

Exercise 4

A function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined as $f((m, n)) = 2n - 4m$. Verify whether this function is injective and whether it is surjective.

Solution

This is **not injective** because $(0,2) \neq (-1,0)$, yet $f((0,2)) = f((-1,0)) = 4$. This is **not surjective** because $f((m,n)) = 2n - 4m = 2(n - 2m)$ is always even. If $b \in \mathbb{Z}$ is odd, then $f((m,n)) \neq b$, for all $(m,n) \in \mathbb{Z} \times \mathbb{Z}$.