CSE215
Foundations of Computer Science

Instructor: Zhoulai Fu

State University of New York, Korea

Exercise 1 (10 points)

Use mathematical induction to prove that $\Sigma_{i=1 \text{ to } n}\ i^3 = n^2(n+1)^2/4$ for every positive integer n >= 1.

Issue

Ex1. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \quad (n \in \mathbb{Z}, n \geq 1)$

Proof.

$P(n) = \frac{n^2(n+1)^2}{4}$ We want to prove that $\sum_{i=1}^{n} i^3 = P(n)$

We use mathematical induction to proceed.

Base step: $\sum_{i=1}^{1} i^3 = 1 = \frac{1 \cdot 4}{4}$

Inductive step: we need to prove $p(k) \to p(k+1)$

Suppose $p(k)$. We need to prove $p(k+1)$

$p(k) = \frac{k^2(k+1)^2}{4}$

$p(k) + (k+1)^3 = \frac{k^4+2k^3+k^2}{4} + \frac{4k^3+12k^2+12k+4}{4} = (a)$

$p(k+1) = \frac{(k+1)^2(k+2)^2}{4} = \frac{k^4+6k^3+13k^2+12k+4}{4} = (b)$

$(a) = (b)$

Thus, $p(k) \to p(k+1)$

QED.

Issue

Proof.

Let $P(n)$ denote $\sum_{i=1}^{n} i^3$. We want to prove $P(n)$ is true for $\frac{n^2(n+1)^2}{4}$. We use mathematical induction to proceed.

Base step: we want to prove $P(1)$.

LHS for $P(1)$: 1^3, RHS: $\frac{1(1+1)^2}{4} = 1$.

Inductive step: we want to prove $P(k) \to P(k+1)$.

Assume $p(k)$ is true,

$$\sum_{i=1}^{k} i^3 = 1^3 + 2^3 + \cdots + (k-1)^3 + k^3 = \frac{k^2(k+1)^2}{4}.$$

Prove $P(k+1)$., $\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2(k+2)^2}{4}$.

LHS of $P(k+1)$: $\sum_{i=1}^{k+1} i^3 = \underbrace{1^3+2^3+3^3+\cdots(k-1)^3+k^3}_{= P(k)}+(k+1)^3$

$$= P(k) + (k+1)^3.$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3.$$

$$= \frac{1}{4}(k+1)^2 \cdot (k^2 + 4(k+1))$$

$$= \frac{1}{4}(k+1)^2(k^2 + 4k + 4)$$

$$= \frac{1}{4}(k+1)^2(k+2)^2$$

$$= RHS.$$

QED.

Exercise 2 (20 points)

Use mathematical induction to prove that $\Sigma_{i=1 \text{ to } n}$ (8i-5) = $4n^2$ - n for every positive integer n >= 1.

Exercise 3 (20 points)

The triangle inequality states that for all real numbers a and b, |a + b| <= |a| + |b|. Use the triangle inequality and mathematical induction to prove:

For any n real numbers a_1, a_2, ..., and a_n,

$$|\Sigma_{i=1}^{n} a_i| \leq \Sigma_{i=1}^{n} |a_i|$$

Exercise 4 (20 points)

Let f be a sequence defined recursively as follows:

- $f_1 = 1$, and
- $f_k = f_{k-1} + 2^k$ for all integers k >= 2

1. Write out f_k for k = 1, 2,..., 5.
2. Follow the pattern in #1 to guess an explicit form of the sequence. (Remind: An explicit form is a formula that looks like a_n=n(n+1)/2, which should involve no recursions.)
3. Prove that your explicit form corresponds to the original recursive definition. (Remind: You can plug the explicit form into the recursive definition. Mathematical induction would be unnecessary for this exercise.)

Exercise 5 (20 points)

A specific computer program executes twice as many operations when it runs with an input of size k as when it runs with an input of size k – 1 (where k is an integer and k > 1). When the program runs with an input of size 1, it executes seven operations. Let a_n be the number of operations when it runs with an input of size n. Please (1) first figure out a recursive form of a_n, together with a base case. (2) Get the explicit form. (3) Confirm the explicit one is correct with regard to the recursive form, and (4) calculate a_{25}.

Solution for HW10

1

Proof

- Let $P(n)$ denote $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$. We need to prove $P(n)$ for every positive integer $n \geq 1$
- We use mathematical induction
- Base step: Prove $P(1)$
  - LHS $= \sum_{i=1}^{1} i^3 = 1^3 = 1$
  - RHS $= \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$
  - Therefore, $P(1)$ is true
- Inductive step: Prove $P(k) \to P(k+1)$
  - Suppose $P(k)$
  - We need to prove $P(k+1)$
  - LHS of $P(k+1) = \sum_{i=1}^{k} i^3 + (k+1)^3$

  - We have $\sum_{i=1}^{k} i^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$ $= \frac{k^2(k+1)^2 + 4(k+1)(k+1)^2}{4}$ by algebraic identities.
  - We have $\frac{(k+1)^2(k^2+4(k+1))}{4}$ by distribution.
  - We have $\frac{(k+1)^2(k^2+4(k+1))}{4} = \frac{(k+1)^2(k^2+4k+4)}{4} = \frac{(k+1)^2(k+2)^2}{4}$ by algebraic identities and factoring.
  - Therefore, $P(k) \to P(k+1)$.
- Therefore, $P(n)$ for every positive integer $n \geq 1$

QED

Exercise 2

Proof

- Let $P(n)$ denote $\sum_{i=1}^{n}(8i-5)=4n^2-n$. We need to prove $P(n)$ for every positive integer $n \geq 1$.
- We use mathematical induction.
- Base step: prove $P(1)$
  - LHS = $\sum_{i=1}^{1}(8i-5)=(8(1)-5)=3$
  - RHS = $4(1)^2-1=3$
  - Therefore, $P(1)$ is true
- Inductive step: Prove $P(k) \to P(k+1)$
  - Suppose $P(k)$
  - We need to prove: $P(k+1)$
  - LHS of $P(k+1) = \sum_{i=1}^{k}(8i-5)+(8(k+1)-5)$

  - We have $\sum_{i=1}^{k}(8i-5)+(8(k+1)-5)=(4k^2-k)+(8(k+1)-5)$ $=(4k^2-k)+(8k+3)=4k^2+7k+3$ by algebraic identities
  - RHS of $P(k+1)=4(k+1)^2-(k+1)$
  - We have $4(k+1)^2-(k+1)=4(k^2+2k+1)-(k+1)=$ $4k^2+8k+4-(k+1)=4k^2+7k+3$ by algebraic identities
  - $4k^2+7k+3=4k^2+7k+3$, therefore LHS = RHS
  - Therefore, $P(k) \to P(k+1)$
- Therefore, $P(n)$ is true for every $n \geq 1$

QED

3.

proof.

Let $P(n)$ denote the statement : For any n real numbers, $a_1, a_2, \dots, a_n$, $\left|\sum_{i=1}^{n} a_i\right| \le \sum_{i=1}^{n} |a_i|$

We want to prove that $P(n)$ is true for all positive integers n.

We use mathematical induction to proceed.

Base step : We want to prove $P(1)$, and that LHS is $|a_1|$, and RHS is also $|a_1|$, which means $P(1)$ is true.

Inductive step: We want to prove that for any positive integer k, if $P(k)$ is true, then $P(k+1)$ is also true.

Assume $P(k)$ is the statement for any k real numbers $a_1, a_2, \dots, a_k$, $\left|\sum_{i=1}^{k} a_i\right| \le \sum_{i=1}^{k} |a_i|$

We want to prove $P(k+1)$ which is for any $k+1$ real numbers $a_1, a_2, \dots, a_k, a_{k+1}$, $\left|\sum_{i=1}^{k+1} a_i\right| \le \sum_{i=1}^{k+1} |a_i|$

LHS: $\left|\sum_{i=1}^{k+1} a_i\right| = \left|\sum_{i=1}^{k} a_i + a_{k+1}\right|$

(By triangle inequality) $\left|\sum_{i=1}^{k} a_i + a_{k+1}\right| \le \left|\sum_{i=1}^{k} a_i\right| + |a_{k+1}|$.

$\left|\sum_{i=1}^{k} a_i\right| + |a_{k+1}| \le \sum_{i=1}^{k} |a_i| + |a_{k+1}|$

Now we shown that $\left|\sum_{i=1}^{k} a_i\right| + |a_{k+1}| \le \sum_{i=1}^{k+1} |a_i|$ (which is RHS)

QED.

Exercise 4

1. $f_1 = 1$

$f_2 = 1 + 2^2 = 5$

$f_3 = 5 + 2^3 = 13$

$f_4 = 13 + 2^4 = 29$

$f_5 = 29 + 2^5 = 61$

2. $f_n = 2^{n+1} - 3$

question 3 on next page

3.

Proof

- We need to prove $f_n = 2^{n+1} - 3$ for every integer $n \geq 1$
- We use direct proof
- Suppose $n = 1$, then:
  - LHS $= f_1 = 1$
  - RHS $= 2^{1+1} - 3 = 1$
  - Therefore, $f_1 = 2^{1+1} - 3$
- Suppose $n > 1$, then:
  - LHS $= f_n = (2^n - 3) + 2^n$
  - We have $(2^n - 3) + 2^n = 2^n + 2^n - 3 = 2^{n+1} - 3$
  - Therefore, $f_n = 2^{n+1} - 3$ for every integer $n > 1$
- Therefore, $f_n = 2^{n+1} - 3$ for every integer $n \geq 1$

QED

Exercise 5

1. Recursive form a_n

$a_1 = 7$

$a_n = 2(a_{n-1})$ for every integer $n > 1$

2. Explicit form

$a_n = 7(2^{n-1})$

3.

Proof

- We need to prove $a_n = 7(2^{n-1})$ for every integer $n \geq 1$
- We use direct proof
- Suppose $n = 1$, then:
  - LHS $= a_1 = 7$
  - RHS $= 7(2^{1-1}) = 7$
  - Therefore, $a_1 = 7(2^{1-1})$
- Suppose $n > 1$, then:
  - LHS $= a_n = 2(7(2^{n-2}))$
  - We have $2(7(2^{n-2})) = 7(2^{n-2} \cdot 2) = 7(2^{n-1})$
  - Therefore, $a_n = 7(2^{n-1})$ for all integers $n > 1$
- Therefore, $a_n = 7(2^{n-1})$ for every integer $n \geq 1$

QED

4. $a_{25} = 7(2^{24}) = 7(16,777,216)$

$= 117,440,512$

Mock Midterm 2

- Mock midterms2 cover proof, sequences and sets
- Midterm2 will cover the same
- But questions will be different

Problem 1 (points = 12)

Determine if the following statements are true or false. You do not need to explain the reasons.

1. Let $A = \{1, 2, 3, 4, 5\}$ and $\mathbf{Z}$ be the set of integers. Then A is a proper subset of $\mathbf{Z}$.
2. $\{x \in \mathbf{Z} \mid x^3 - 1 = 0\} = \{x \in \mathbf{Z} \mid x^2 - x = 0\}$.
3. Suppose x is a real number. If $x^{100} + 5x^9 < 0$ then $x < 0$.
4. Suppose a and b are two real numbers. If $a - b$ is irrational and a is rational, then b is irrational.

Problem 2 (points = 21)

Let $A = \{0, 2, 4\}$, $B = \{1, 2, 3, 4\}$ and the universal set $U = \{0, 1, 2, 3, 4\}$. Find:

1. A'
2. B'
3. $A \cap A'$
4. $A \cup A'$
5. $A - B$
6. $B - A$
7. $(A - B) \times (B - A)$

Problem 3 (points = 15)

Let A be the set $\{\{0, 100\}, \{10, 100\}, 10\}$. First, determine if the following (1-6) are true or false. You do not need to explain the reasons.

1. $100 \in A$
2. $\{10\} \in A$
3. $\{0, 10\} \in A$
4. $\{0, 100\} \subseteq A$
5. $\{10, 100\} \subseteq A$

Problem 4 (points = 14)

1. Prove that for any sets A and B, $A \cup (A' \cap B) = A \cup B$. [Hint: Use set identities]
2. Prove that for any sets A and B, $A - (A - B) = A \cap B$. [Hint: Use set identities]

Problem 5 (points = 14)

1. Prove that there are no integers a and b such that $10a + 1 = 6b$.
2. Prove that there exist integers a and b such that $10a + 2 = 6b$.

Problem 6 (points = 24)

1. Prove that $\frac{1}{1*2} + \frac{1}{2*3} + \ldots + \frac{1}{n*(n+1)} = 1 - \frac{1}{n+1}$ for every integer $n \geq 1$.
2. Prove that that $2^n + 1 \leq 3^n$ for every positive integer $n \geq 1$.
3. Prove that $9|(4^{3n}+8)$ for every integer $n \geq 0$.