CSE215
Foundations of Computer Science

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Some slides taken from Prof. Pramod Ganapathi (Stony Brook). Thanks!

Plan

- Review exercises on Mathematical Induction
- Recursive sequences

Final 2020

Use mathematical induction to prove the following identities.

(a) [5 points] For all natural numbers n,

$$1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + n^2 \times (n+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$$

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Final 2020

(a) [5 points] For all natural numbers n,

$$1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + n^2 \times (n+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$$

- Proof.
  - We will prove this proposition with mathematical induction.
  - Let P(n) be the predicate 1^2 * 2 + … + n^2 * (n+1) = n (n+1) (n+2) (3n+1)/12
  - Base step: We first prove P(1) holds.
    - LHS = 1^2 * 2=2. RHS = 1*2*3*4/12 = 2.
  - Inductive step: Then, we prove P(k) -> P(k+1) for all k>=1
    - Let k be an arbitrary integer and k>=1.
    - Assume P(k) holds. That is, 1^2 * 2 + … + k^2 * (k+1) = k (k+1) (k+2) (3k+1)/12
    - We want to prove P(k+1), that is, 1^2 * 2 + … + k^2 * (k+1) + (k+1)^2 * (k+2) = (k+1) (k+2) (k+3) (3k+4)/12
      - LHS = k (k+1) (k+2) (3k+1)/12 + + (k+1)^2 * (k+2) following assumption P(k)
      - The latter can be further reduced to (k+1)(k+2)/12 * (3k^2+k + 12k+12) = RHS
- QED.

Use mathematical induction to prove the following identities.

(b) [5 points] For all natural numbers n,

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{n \times (n+1)} = \frac{n}{n+1}$$

Final 2020

Use mathematical induction to prove the following identities.

(b) [5 points] For all natural numbers n,

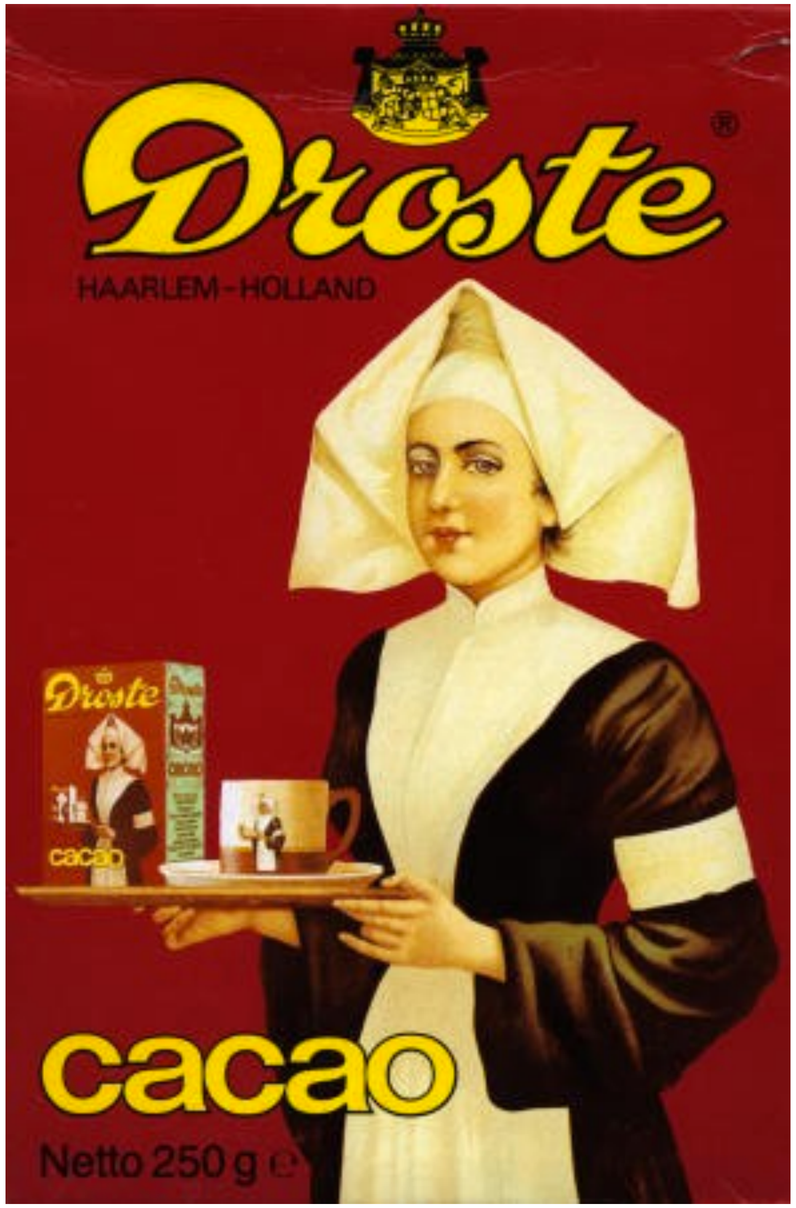
$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n\times(n+1)}=\frac{n}{n+1}$$

- Proof.
  - We will prove this proposition with mathematical induction.
  - Let P(n) be the predicate 1/(1*2) + 1(2*3) + … + 1/(n *(n+1)) = n/(n+1)
  - Base step: We first prove P(1) holds.
    - LHS is 1/2, and RHS is 1/(1+1) = 1/2
  - Inductive step: Then, we prove P(k) -> P(k+1) for all integer k>=1
    - Let k be an arbitrary integer and k>=1.
    - Assume P(k) holds. That is 1/(1*2) + 1(2*3) + … + 1/(k*(k+1)) = k/(k+1)
    - We want to prove P(k+1), namely,1/(1*2) + 1(2*3) + … + 1/(k*(k+1)) + 1/((k+1)*(k+2))= (k+1)/(k+2)
      - LHS can be reduced to k/(k+1) + 1/((k+1)*(k+2)) following the assumption P(k)
      - The latter can be further reduced to (k^2+2k+1)/((k+2)(k+1)), which equals to RHS.
- QED.

Recursive sequences

Recursion

- All about repeating itself
- Many forms:
    - recursive sequences
    - recursive functions
    - recursive data structures



Recursive functions

Examples

- Suppose $f(n) = n!$, where $n \in \mathbb{W}$. Then,
  $$f(n) = \begin{cases} 1 & \text{if } \; n = 0, \\ n \cdot f(n-1) & \text{if } \; n \geq 1. \end{cases}$$
  Closed-form formula: $f(n) = n \cdot (n-1) \cdot \cdots \cdot 1$

Recursive functions

Examples

- Suppose $f(n) = n!$, where $n \in \mathbb{W}$. Then,
$$f(n) = \begin{cases} 1 & \text{if } n = 0, \\ n \cdot f(n-1) & \text{if } n \geq 1. \end{cases}$$
Closed-form formula: $f(n) = n \cdot (n-1) \cdot \cdots \cdot 1$
- Suppose $F(n) = n$th Fibonacci number. Then,
$$F(n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } 1, \\ F(n-1) + F(n-2) & \text{if } n \geq 2. \end{cases}$$
Closed-form formula: $F(n) = ?$

Example

Let $a_0, a_1, a_2, \ldots$ be the sequence defined recursively as follows: For all integers $k \geq 1$,

(1) $a_k = a_{k-1} + 2$ recurrence relation

(2) $a_0 = 1$ initial condition.

- Write out a_1, a_2, a_3, a_4, and a_5
- Derive an explicit formula of the sequence
- Confirm the explicit formula satisfies the recursive definition

Example: Geometric sequence (Compound interest)

Problem

- Suppose you deposit 100,000 dollars in your bank account for your newborn baby. Suppose you earn 3% interest compounded annually.
  How much will be the amount when your kid hits 21 years of age?

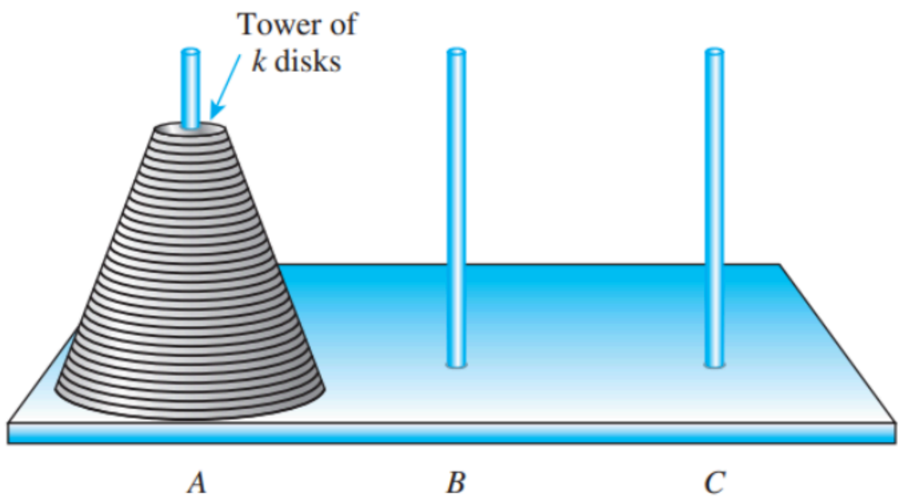
Solution

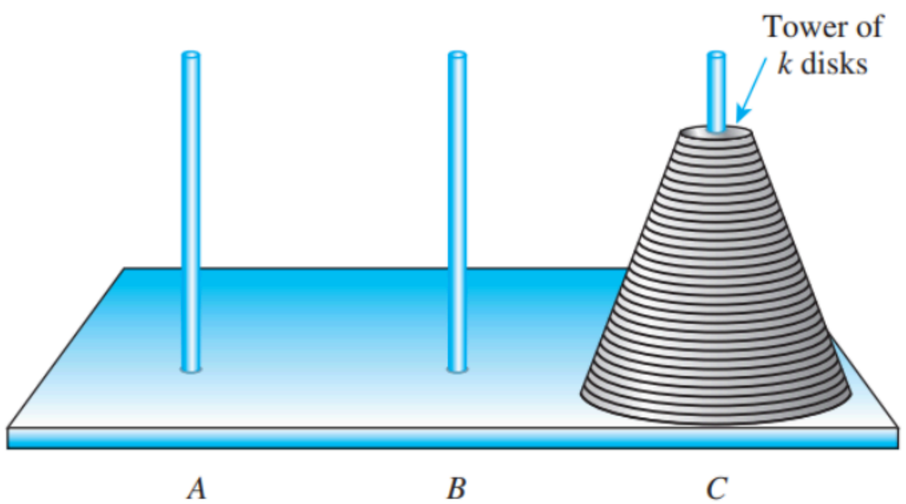
- Suppose A_k = Amount in your account after k years. Then,
  $$A_k = \begin{cases} 100,000 & \text{if } k = 0, \\ (1 + 3\%) \times A_{k-1} & \text{if } k \geq 1. \end{cases}$$
- Solving the recurrence by the method of iteration, we get
  $\boxed{A_k = ((1.03)^k \cdot 100,000) \text{ dollars}}$ ▷ How?
- Homework: Prove the formula using induction
- When your kid hits 21 years, $k = 21$, therefore
  $A_{21} = ((1.03)^{21} \cdot 100,000) \approx 186,029.46$ dollars

Example: Towers of Hanoi

Problem

- There are k disks on peg 1. Your aim is to move all k disks from peg 1 to peg 3 with the minimum number of moves. You can use peg 2 as an auxiliary peg. The constraint of the puzzle is that at any time, you cannot place a larger disk on a smaller disk.
  What is the minimum number of moves required to transfer all k disks from peg 1 to peg 3?



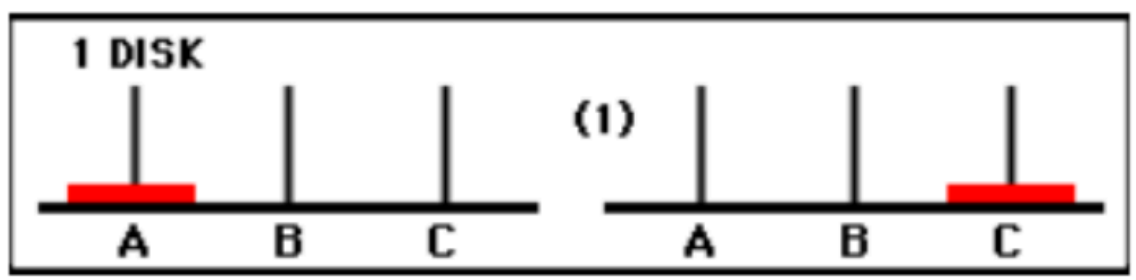


Demo: https://www.mathsisfun.com/games/towerofhanoi.html

Example: Towers of Hanoi

Solution

Suppose $k = 1$. Then, the 1-step solution is:
1. Move disk 1 from peg A to peg C.

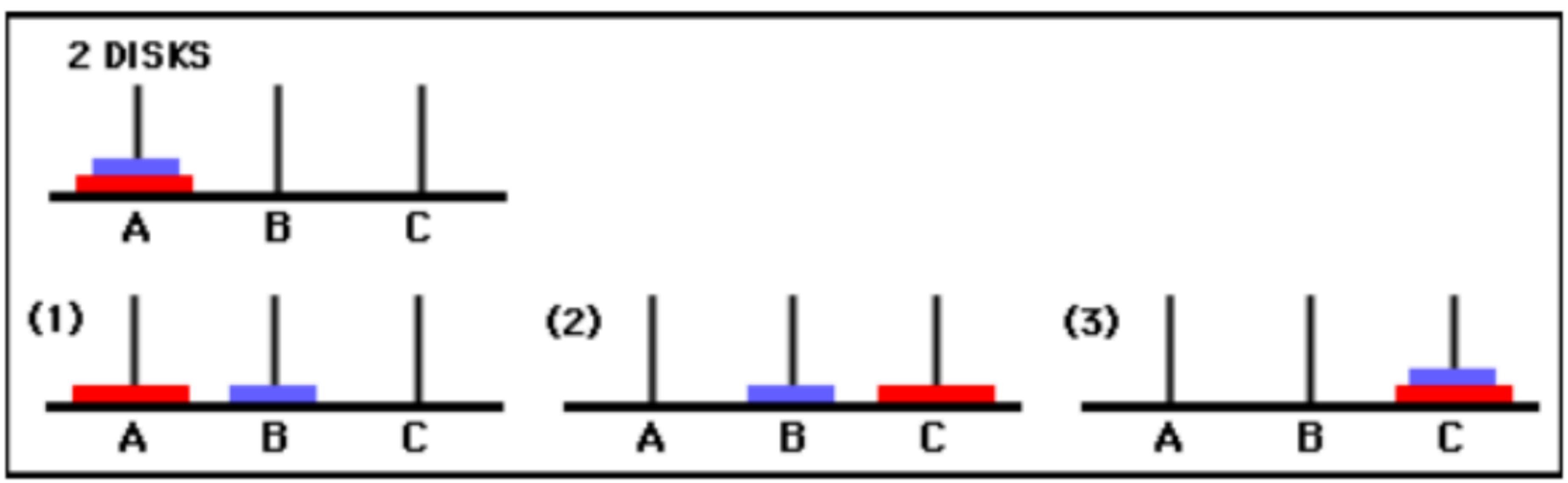


Example: Towers of Hanoi

Solution

Suppose $k = 2$. Then, the 3-step solution is:

1. Move disk 1 from peg A to peg B.
2. Move disk 2 from peg A to peg C.
3. Move disk 1 from peg B to peg C.

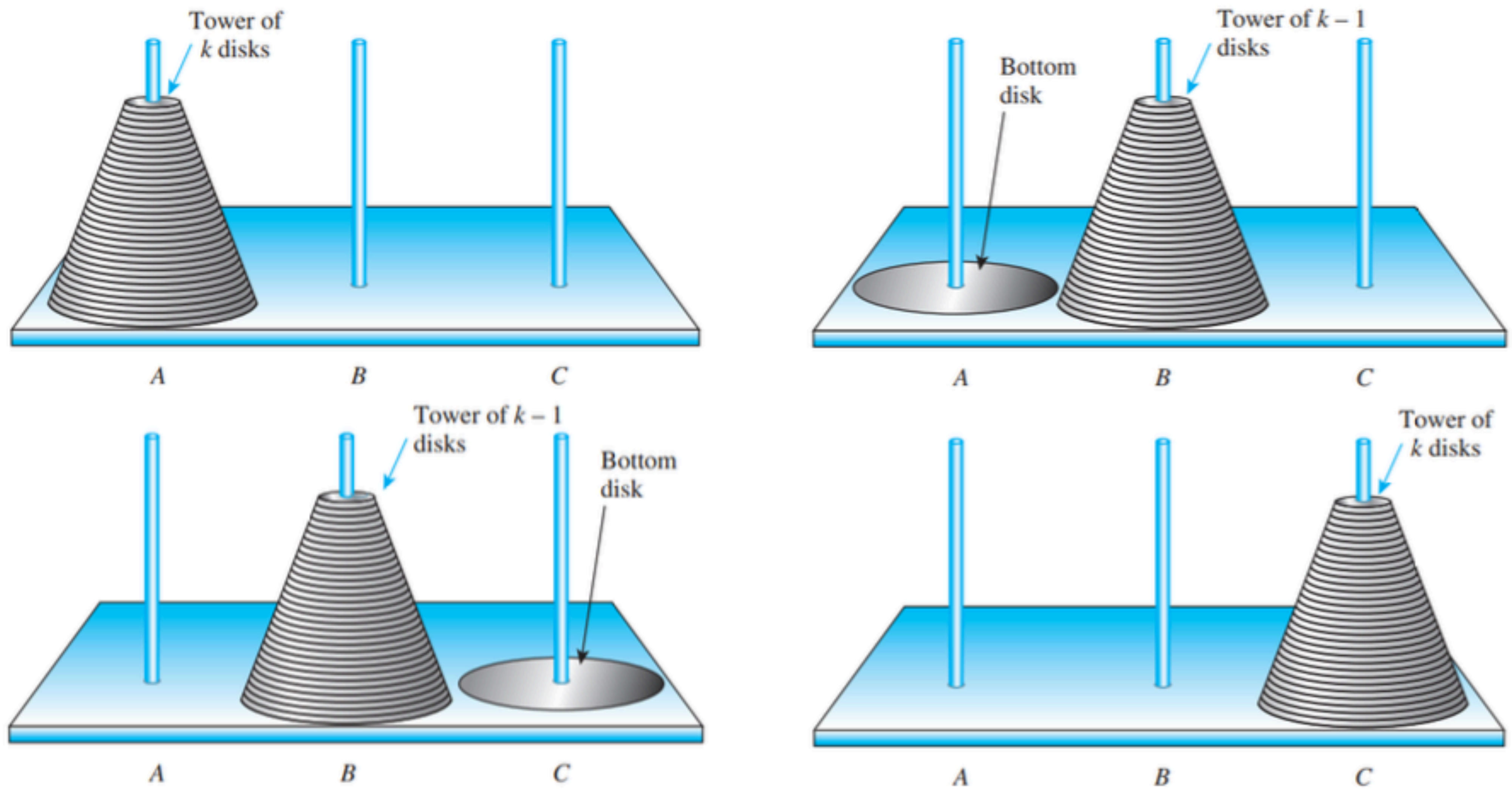


Example: Towers of Hanoi

Solution

For any $k \geq 2$, the recursive solution is:

1. Transfer the top $k-1$ disks from peg A to peg B.
2. Move the bottom disk from peg A to peg C.
3. Transfer the top $k-1$ disks from peg B to peg C.

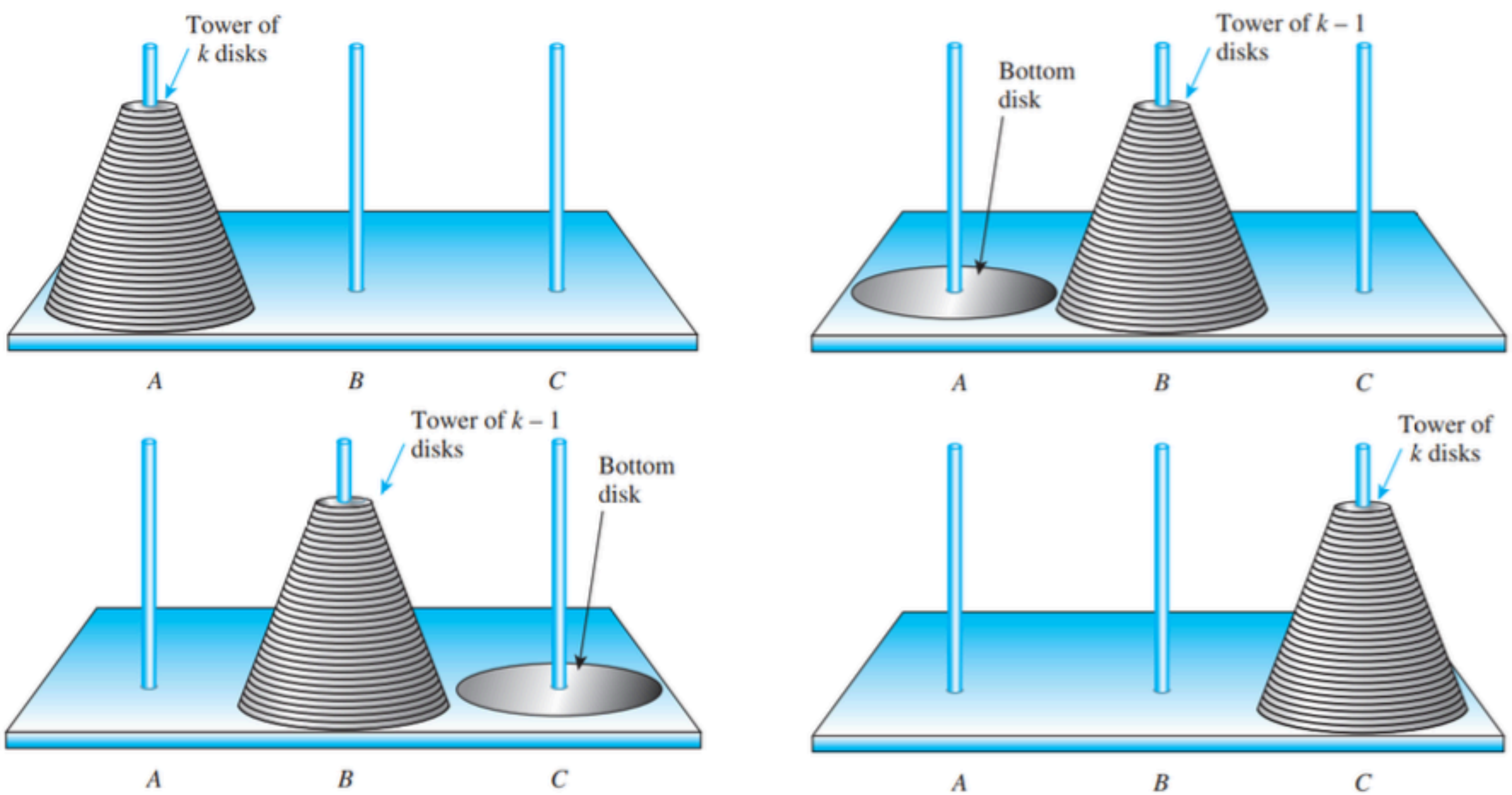


Example: Towers of Hanoi

Code Demo

TOWERS-OF-HANOI(k, A, C, B)

if $k = 1$ then
  Move disk k from A to C.
elseif $k \geq 2$ then
  TOWERS-OF-HANOI$(k-1, A, B, C)$
  Move disk k from A to C.
  TOWERS-OF-HANOI$(k-1, B, C, A)$



Example: Towers of Hanoi

Solution (continued)

- Let $M(k)$ denote the minimum number of moves required to move k disks from one peg to another peg. Then
$$M(k) = \begin{cases} 1 & \text{if } k = 1, \\ 2 \cdot M(k-1) + 1 & \text{if } k \geq 2. \end{cases}$$
- Solving the recurrence by the method of iteration, we get
$$\boxed{M(k) = 2^k - 1}$$
▷ How?

Why minimum?
https://proofwiki.org/wiki/Tower_of_Hanoi

Exercise 1

- Consider the recursively defined sequence below. Find its explicit form and prove your answer.

$$a_k = ka_{k-1}, \text{ for all integers } k \geq 1$$
$$a_0 = 1$$

Solution

We have: $a_k = k \cdot a_{k-1} = k \cdot (k-1) \cdot a_{k-2} = k(k-1)(k-2) \cdots 1 \cdot a_0 = k!$

- To prove it indeed satisfies the recursive definition, we plug it to the recursive definition:
    - The explicit form clearly satisfies a_0=1
    - The explicit form also satisfies a_k = k a_{k-1}, since the left is k!, the right is k * (k-1)! which is also k!

Exercise 2

- Consider the recursively defined sequence below. Find its explicit form and prove your answer.

$$b_k = \frac{b_{k-1}}{1 + b_{k-1}}, \text{ for all integers } k \geq 1$$
$$b_0 = 1$$

Solution

- We have $b_0=1$, $b_1= 1/2$, $b_2=1/3$, $b3=1/4$… So a possible explicit form is $b_n=1/(n+1)$
- To prove it indeed satisfies the recursive definition, we plug it to the recursive definition:
  - The explicit form clearly satisfies $b_0=1$
  - The explicit form also satisfies $b_k = b_{k-1} / (1+b_{k-1})$, since LHS=$1/(k+1)$, RHS = $1/(k+1) / (1 + 1/(k+1)) = 1/(k+1)$

Exercise 3

- Consider the recursively defined sequence below. Find its explicit form and prove your answer.

$$c_k = 3c_{k-1} + 1, \text{ for all integers } k \geq 2$$
$$c_1 = 1$$

Solution

- We have c_1=1, c_2= 4, c_3=13, b4=40… So a possible explicit form is c_n=(3^n-1)/2
- To prove it indeed satisfies the recursive definition, we plug it to the recursive definition:
    - The explicit form clearly satisfies c_1=1
    - The explicit form also satisfies_____, since LHS= ___ RHS = ___ = ____