

CSE215

Foundations of Computer Science

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Today

- Revision on proof of dividing into cases
- Disproof

Prove the following statement

- Suppose $x, y \in \mathbb{R}$. If $x^2 + 5y = y^2 + 5x$, then $x = y$ or $x + y = 5$.

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Proof. Suppose $x^2 + 5y = y^2 + 5x$.

Then $x^2 - y^2 = 5x - 5y$, and factoring gives $(x - y)(x + y) = 5(x - y)$.

Now consider two cases.

Case 1. If $x - y \neq 0$ we can divide both sides of $(x - y)(x + y) = 5(x - y)$ by the non-zero quantity $x - y$ to get $x + y = 5$.

Case 2. If $x - y = 0$, then $x = y$. (By adding y to both sides.)

Thus $x = y$ or $x + y = 5$. ■

Prove the following statement

If $n \in \mathbb{Z}$, then $n^2 + 3n + 4$ is even.

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Proof. Suppose $n \in \mathbb{Z}$. We consider two cases.

Case 1. Suppose n is even. Then $n = 2a$ for some $a \in \mathbb{Z}$.

Therefore $n^2 + 3n + 4 = (2a)^2 + 3(2a) + 4 = 4a^2 + 6a + 4 = 2(2a^2 + 3a + 2)$.

So $n^2 + 3n + 4 = 2b$ where $b = 2a^2 + 3a + 2 \in \mathbb{Z}$, so $n^2 + 3n + 4$ is even.

Case 2. Suppose n is odd. Then $n = 2a + 1$ for some $a \in \mathbb{Z}$.

Therefore $n^2 + 3n + 4 = (2a + 1)^2 + 3(2a + 1) + 4 = 4a^2 + 4a + 1 + 6a + 3 + 4 = 4a^2 + 10a + 8 = 2(2a^2 + 5a + 4)$. So $n^2 + 3n + 4 = 2b$ where $b = 2a^2 + 5a + 4 \in \mathbb{Z}$, so $n^2 + 3n + 4$ is even.

In either case $n^2 + 3n + 4$ is even. ■

Problem 4. [5 points]

Prove that $n^2 + 9n + 27$ is odd for all natural numbers n . You can use any proof technique.

Solution

- Proof.
 - We want to prove
 - for any natural number n , $n^2+9n+27$ is odd
 - $n^2+9n+27 = n(n+9) + 27$
 - We consider two cases
 - If n is even, $n(n+9) + 27$ is even+odd, thus odd
 - if n is odd, $n(n+9) + 27$ is also even+odd, thus odd
 - Thus $n^2+9n+27$ is odd for whatever n .
- QED.

**Disproof = Proof for
something being false**

Principle of disproof

- Suppose you want to disprove a statement P . In other words you want to prove that P is false.
- The way to do this is to prove that $\sim P$ is true, for if $\sim P$ is true, it follows immediately that P has to be false.

How to disprove P : Prove $\sim P$.

Disproving Universal Statements: Counterexamples

- How to disprove a universally quantified statement such as
 - $\forall x \in S, P(x)$?
- To disprove this statement, we must prove its negation. Its negation is
 - $\sim(\forall x \in S, P(x)) = \exists x \in S, \sim P(x)$.

How to disprove $\forall x \in S, P(x)$.

Produce an example of an $x \in S$
that makes $P(x)$ false.

Disproving Conditional Statements

- How to disprove a conditional statement $P(x) \Rightarrow Q(x)$?
- This statement asserts that for every x that makes $P(x)$ true, $Q(x)$ will also be true.
- The statement can only be false if there is an x that makes $P(x)$ true and $Q(x)$ false.
- Formally, to disprove this statement, we must prove its negation. Its negation is
 - $\sim(\forall x \in S, P(x) \Rightarrow Q(x)) = \exists x \in S, P(x) \wedge \sim Q(x)$.

How to disprove $P(x) \Rightarrow Q(x)$.

Produce an example of an x that makes $P(x)$ true and $Q(x)$ false.

**There is a special name for an example that disproves a statement:
It is called a counterexample.**

Example: Prove or disprove the following conjecture

Conjecture: For every $n \in \mathbb{Z}$, the integer $f(n) = n^2 - n + 11$ is prime.

- In resolving the truth or falsity of a conjecture, it's a good idea to gather as much information about the conjecture as possible. In this case let's start by making a table that tallies the values of $f(n)$ for some integers n .

n	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
$f(n)$	23	17	13	11	11	13	17	23	31	41	53	67	83	101

- $f(11) = 11^2 - 11 + 11 = 121$ is not prime. The conjecture is false because $n = 11$ is a counterexample. We summarize our disproof as follows:

Disproof. The statement “For every $n \in \mathbb{Z}$, the integer $f(n) = n^2 - n + 11$ is prime,” is **false**. For a counterexample, note that for $n = 11$, the integer $f(11) = 121 = 11 \cdot 11$ is not prime. ■

Disproving by contradiction

- suppose we wish to disprove a statement P .
- We know that to disprove P , we must prove $\sim P$.
- To prove $\sim P$ with contradiction, we assume $\sim\sim P$ is true and deduce a contradiction

How to disprove P with contradiction:

Assume P is true, and deduce a contradiction.

Disproving existential Statements

- How to disprove a existential statement $\exists x \in S, P(x)$?
- To disprove it, we have to prove its negation $\sim (\exists x \in S, P(x)) = \forall x \in S, \sim P(x)$.
- Note: this negation is universally quantified. Proving it involves showing that $\sim P(x)$ is true for all $x \in S$, and for this an example does not suffice.

Example: Prove or disprove the following conjecture

Conjecture: There is a real number x for which $x^4 < x < x^2$.

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Conjecture: There is a real number x for which $x^4 < x < x^2$.

Disproof. Suppose for the sake of contradiction that this conjecture is true. Let x be a real number for which $x^4 < x < x^2$. Then x is positive, since it is greater than the non-negative number x^4 . Dividing all parts of $x^4 < x < x^2$ by the positive number x produces $x^3 < 1 < x$. Now subtract 1 from all parts of $x^3 < 1 < x$ to obtain $x^3 - 1 < 0 < x - 1$ and reason as follows:

$$\begin{aligned}x^3 - 1 &< 0 < x - 1 \\(x - 1)(x^2 + x + 1) &< 0 < (x - 1) \\x^2 + x + 1 &< 0 < 1\end{aligned}$$

Now we have $x^2 + x + 1 < 0$, which is a contradiction because x is positive. Thus the conjecture must be false. ■

Prove or disprove

- If $x, y \in \mathbb{R}$, then $|x+y| = |x| + |y|$.
- For every natural number n , the integer $2n^2 - 4n + 31$ is prime.
- If $a, b \in \mathbb{N}$, then $a + b < ab$
- Every odd integer is the sum of three odd integers.
- Rational + Irrational = Irrational
- Rational * Irrational = Irrational

Solution

- If $x, y \in \mathbb{R}$, then $|x+y| = |x| + |y|$.
 - False. Counterexample: $x = 2, y = -2$
- For every natural number n , the integer $2n^2 - 4n + 31$ is prime.
 - False. Counterexample: $n = 31$
- If $a, b \in \mathbb{N}$, then $a + b < ab$
 - False counterexample: $a=1, b=1$.
- Every odd integer is the sum of three odd integers.
 - True. Any odd number n can be written as $(n-2)=+1+1$
- Rational + Irrational = Irrational
 - True Proof by contradiction. ...
- Rational * Irrational = Irrational
 - False. Counterexample: $0 * \sqrt{2} = 0$