

# **CSE215**

## **Foundations of Computer Science**

**Instructor: Zhoulai Fu**

**State University of New York, Korea**

# Plan ahead

- 11.14 Tu: Sets - REC on Sets; no homework review in REC
- 11.16 Review Midterm 2 No homework to be announced
- 11.21 Tu **Midterm 2** - REC on Functions
- 11.23 Function Homework to be announced
- 11.28 Tu Function -
- 11.30 Function Homework to be announced
- 12.05 Tu Final Review -
- 12.07 Final Review No homework to be announced
- 12.10 Mon Final -

- Clarification about sets 1: Membership and subsets
- Clarification about sets 2: Power sets,
- Clarification about sets 3: infinite union/intersection notation
- Clarification about sets 4: Element argument

# **Clarification on sets 1: Membership and Subsets**

# Rules for survival

- $a \in A$  if you can find the exact “a” in A’s elements
- Translate  $a \notin A$  as  $\sim (a \in A)$
- Translate  $B \subseteq A$  to for any  $b \in B$ ,  $b \in A$
- Emptset is a subset of anything

# Examples

1.  $1 \in \{1, \{1\}\}$  ..... 1 is the first element listed in  $\{1, \{1\}\}$
2.  $1 \notin \{1, \{1\}\}$  .....because 1 is not a set
3.  $\{1\} \in \{1, \{1\}\}$  .....  $\{1\}$  is the second element listed in  $\{1, \{1\}\}$

# Examples

4.  $\{1\} \subseteq \{1, \{1\}\}$  ..... make subset  $\{1\}$  by selecting 1 from  $\{1, \{1\}\}$
5.  $\{\{1\}\} \notin \{1, \{1\}\}$  ..... because  $\{1, \{1\}\}$  contains only 1 and  $\{1\}$ , and not  $\{\{1\}\}$
6.  $\{\{1\}\} \subseteq \{1, \{1\}\}$  ..... make subset  $\{\{1\}\}$  by selecting  $\{1\}$  from  $\{1, \{1\}\}$

# Examples

7.  $\mathbb{N} \notin \mathbb{N}$  .....  $\mathbb{N}$  is a set (not a number) and  $\mathbb{N}$  contains only numbers
8.  $\mathbb{N} \subseteq \mathbb{N}$  ..... because  $X \subseteq X$  for every set  $X$
9.  $\emptyset \notin \mathbb{N}$  ..... because the set  $\mathbb{N}$  contains only numbers and no sets



# Examples

- 10.  $\emptyset \subseteq \mathbb{N}$  ..... because  $\emptyset$  is a subset of every set
- 11.  $\mathbb{N} \in \{\mathbb{N}\}$  ..... because  $\{\mathbb{N}\}$  has just one element, the set  $\mathbb{N}$
- 12.  $\mathbb{N} \not\subseteq \{\mathbb{N}\}$  ..... because, for instance,  $1 \in \mathbb{N}$  but  $1 \notin \{\mathbb{N}\}$

# Examples

- 13.  $\emptyset \notin \{\mathbb{N}\}$  ..... note that the only element of  $\{\mathbb{N}\}$  is  $\mathbb{N}$ , and  $\mathbb{N} \neq \emptyset$
- 14.  $\emptyset \subseteq \{\mathbb{N}\}$  ..... because  $\emptyset$  is a subset of every set
- 15.  $\emptyset \in \{\emptyset, \mathbb{N}\}$  .....  $\emptyset$  is the first element listed in  $\{\emptyset, \mathbb{N}\}$

# Examples

- 16.  $\emptyset \subseteq \{\emptyset, \mathbb{N}\}$  ..... because  $\emptyset$  is a subset of every set
- 17.  $\{\mathbb{N}\} \subseteq \{\emptyset, \mathbb{N}\}$  ..... make subset  $\{\mathbb{N}\}$  by selecting  $\mathbb{N}$  from  $\{\emptyset, \mathbb{N}\}$
- 18.  $\{\mathbb{N}\} \not\subseteq \{\emptyset, \{\mathbb{N}\}\}$  ..... because  $\mathbb{N} \notin \{\emptyset, \{\mathbb{N}\}\}$
- 19.  $\{\mathbb{N}\} \in \{\emptyset, \{\mathbb{N}\}\}$  .....  $\{\mathbb{N}\}$  is the second element listed in  $\{\emptyset, \{\mathbb{N}\}\}$

# Quiz: True or False

$$\{(1,2), (2,2), (7,1)\} \subseteq \mathbb{N} \times \mathbb{N}$$

# Clarification on sets 2:

## Powerset

**Definition :** If  $A$  is a set, the **power set** of  $A$  is another set, denoted as  $\mathcal{P}(A)$  and defined to be the set of all subsets of  $A$ . In symbols,  $\mathcal{P}(A) = \{X : X \subseteq A\}$ .

# POWER SETS

- If  $S$  is the set  $\{a, b, c\}$  then  $\{a, c\}$  is a subset of  $S$ . There are other subsets of  $S$ ; the complete list is as follows:
  - $\{\}$  (the empty set or null set)
  - $\{a\}$
  - $\{b\}$
  - $\{c\}$
  - $\{a, b\}$
  - $\{a, c\}$
  - $\{b, c\}$
  - $\{a, b, c\}$  or  $S$
- So the power set of  $S$ , written  $P(S)$ , is the set containing all the subsets above. Written out this would be the set:
- $P(S) = \{ \{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \}$

# Fact

If  $A$  is a finite set, then  $|\mathcal{P}(A)| = 2^{|A|}$ .

$|A|$  refers to # of elements in a set  $A$



# Rules for survival

- Know how to get the powerset of  $N$  elements, for  $N \geq 0$ .
- `\emptyset` is always an element of the powerset
- The set itself is also always an element of powerset

# Examples

1.  $\mathcal{P}(\{0,1,3\}) = \{ \emptyset, \{0\}, \{1\}, \{3\}, \{0,1\}, \{0,3\}, \{1,3\}, \{0,1,3\} \}$
2.  $\mathcal{P}(\{1,2\}) = \{ \emptyset, \{1\}, \{2\}, \{1,2\} \}$
3.  $\mathcal{P}(\{1\}) = \{ \emptyset, \{1\} \}$

# Examples

4.  $\mathcal{P}(\emptyset) = \{ \emptyset \}$

5.  $\mathcal{P}(\{a\}) = \{ \emptyset, \{a\} \}$

6.  $\mathcal{P}(\{\emptyset\}) = \{ \emptyset, \{\emptyset\} \}$

# Examples

7.  $\mathcal{P}(\{a\}) \times \mathcal{P}(\{\emptyset\}) = \{(\emptyset, \emptyset), (\emptyset, \{\emptyset\}), (\{a\}, \emptyset), (\{a\}, \{\emptyset\})\}$
8.  $\mathcal{P}(\mathcal{P}(\{\emptyset\})) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
9.  $\mathcal{P}(\{1, \{1, 2\}\}) = \{\emptyset, \{1\}, \{\{1, 2\}\}, \{1, \{1, 2\}\}\}$
10.  $\mathcal{P}(\{\mathbb{Z}, \mathbb{N}\}) = \{\emptyset, \{\mathbb{Z}\}, \{\mathbb{N}\}, \{\mathbb{Z}, \mathbb{N}\}\}$

# The following are wrong .

## Explain why

11.  $\mathcal{P}(1) = \{ \emptyset, \{1\} \}$  .....
12.  $\mathcal{P}(\{1, \{1, 2\}\}) = \{ \emptyset, \{1\}, \{1, 2\}, \{1, \{1, 2\}\} \}$  ....
13.  $\mathcal{P}(\{1, \{1, 2\}\}) = \{ \emptyset, \{\{1\}\}, \{\{1, 2\}\}, \{ \emptyset, \{1, 2\} \} \}$

# Solution

meaningless because 1 is not a set

....wrong because  $\{1,2\} \not\subseteq \{1,\{1,2\}\}$

... wrong because  $\{\{1\}\} \not\subseteq \{1,\{1,2\}\}$

# **Clarification on sets 3: Infinite union/intersection notation**

# Notations

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots = \{x : x \in A_i \text{ for at least one set } A_i \text{ with } 1 \leq i\}.$$

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots = \{x : x \in A_i \text{ for every set } A_i \text{ with } 1 \leq i\}.$$



# Example

$$A_1 = \{-1, 0, 1\}, \quad A_2 = \{-2, 0, 2\}, \quad A_3 = \{-3, 0, 3\}, \quad \dots, \quad A_i = \{-i, 0, i\}, \quad \dots$$

Observe that  $\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}$ , and  $\bigcap_{i=1}^{\infty} A_i = \{0\}$ .

# Extended Notation

Here is a useful twist on our new notation. We can write

$$\bigcup_{i=1}^3 A_i = \bigcup_{i \in \{1,2,3\}} A_i,$$

as this takes the union of the sets  $A_i$  for  $i = 1, 2, 3$ . Likewise:

$$\bigcap_{i=1}^3 A_i = \bigcap_{i \in \{1,2,3\}} A_i$$

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i \in \mathbb{N}} A_i$$

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i \in \mathbb{N}} A_i$$

# Extended Notations

Here we are taking the union or intersection of a collection of sets  $A_i$  where  $i$  is an element of some set, be it  $\{1,2,3\}$  or  $\mathbb{N}$ . In general, the way this works is that we will have a collection of sets  $A_i$  for  $i \in I$ , where  $I$  is the set of possible subscripts. The set  $I$  is called an **index set**.

If  $A_\alpha$  is a set for every  $\alpha$  in some index set  $I$ , then

$$\bigcup_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for at least one set } A_\alpha \text{ with } \alpha \in I\}$$

$$\bigcap_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for every set } A_\alpha \text{ with } \alpha \in I\}.$$

# Exercise 1

-----

**(a)**  $\bigcup_{i \in \mathbb{N}} [0, i + 1] =$

-----

**(b)**  $\bigcap_{i \in \mathbb{N}} [0, i + 1] =$

# Solution

- $[0, \text{infinity})$
- $[0, 2]$

# Exercise 2

(a)  $\bigcup_{i \in \mathbb{N}} \mathbb{R} \times [i, i+1] =$

(b)  $\bigcap_{i \in \mathbb{N}} \mathbb{R} \times [i, i+1] =$

# Solution

- $\mathbb{R}^* [1, \text{infinity})$
- `\emptyset`

# Exercise 3

**(a)**  $\bigcup_{\alpha \in \mathbb{R}} \{\alpha\} \times [0, 1] =$

**(b)**  $\bigcap_{\alpha \in \mathbb{R}} \{\alpha\} \times [0, 1] =$



# Solution

- $\mathbb{R}^* [0,1]$
- `\emptyset`

**Writing element arguments more  
efficiently for proving some proof  
properties**

# Example 1

- Prove De Morgan law: Complement of A intersection B = Complement of A union Complement of B

$$\begin{aligned}(A \cap B)^c &= \{x | x \notin A \cap B\} && \text{(Def. set complement)} \\ &= \{x | \sim (x \in A \cap B)\} && \text{(Def. of } \notin \text{)} \\ &= \{x | \sim (x \in A \wedge x \in B)\} && \text{(Def. } \cap \text{)} \\ &= \{x | x \notin A \vee x \notin B\} && \text{(De Morgan in logic)} \\ &= \{x | x \in A^c \vee x \in B^c\} && \text{(Def. set complement)} \\ &= A^c \cup B^c && \text{(Def. } \cup \text{)}\end{aligned}$$

# Example 2

Given sets  $A$ ,  $B$ , and  $C$ , prove  $A \times (B \cap C) = (A \times B) \cap (A \times C)$ .

- Proof

$$\begin{aligned} A \times (B \cap C) &= \{(x, y) : (x \in A) \wedge (y \in B \cap C)\} && \text{(def. of } \times) \\ &= \{(x, y) : (x \in A) \wedge (y \in B) \wedge (y \in C)\} && \text{(def. of } \cap) \\ &= \{(x, y) : (x \in A) \wedge (x \in A) \wedge (y \in B) \wedge (y \in C)\} && (P = P \wedge P) \\ &= \{(x, y) : ((x \in A) \wedge (y \in B)) \wedge ((x \in A) \wedge (y \in C))\} && \text{(rearrange)} \\ &= \{(x, y) : (x \in A) \wedge (y \in B)\} \cap \{(x, y) : (x \in A) \wedge (y \in C)\} && \text{(def. of } \cap) \\ &= (A \times B) \cap (A \times C) && \text{(def. of } \times) \end{aligned}$$

- QED.

# Exercise

- Prove De Morgan law: Complement of A union B = Complement of A intersection Complement of B

# Example (This one has to be proved by the old-style element argument)

- Let  $Z$  be the set of integers
- Let  $A$  be the set of  $\{7a + 8b \mid a \in Z \text{ and } b \in Z\}$
- Prove  $Z = A$

# Solution

- Proof.
  - A is clearly a subset of Z
  - So we only need to prove Z is a subset of A
  - Suppose n is arbitrary element of Z, n can be written as  $7(-n) + 8(n)$ , Therefore Z is a subset of A
- QED.

# Exercise (This one has to be proved by the old-style element argument)

- Let  $Z$  be the set of integers
- Let  $A$  be the set of  $\{7a + 3b \mid a \in Z \text{ and } b \in Z\}$
- Prove  $Z = A$