# CSE215
# Foundations of Computer Science

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Some slides taken from Prof. Pramod Ganapathi (Stony Brook). Thanks!

- Today: Sequences
    - Notations on Sequences
    - Mathematical induction
    - Solving some exam problems

# Sequences

**"Mathematics is the art of giving the same name to different things." - Henri Poincaré**

# What are sequences?

## Types of sequences

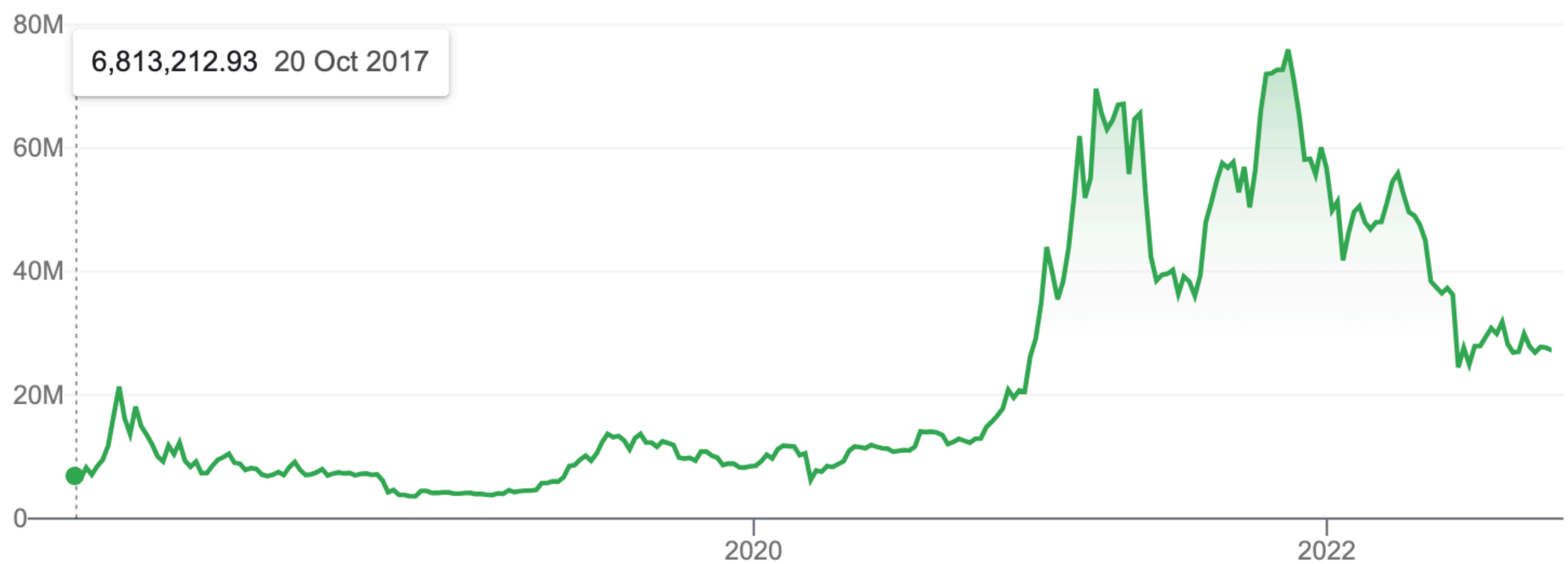
- Finite sequence: $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$
  e.g.: $1^1, 2^2, 3^2, \ldots, 100^2$
- Infinite sequence: $a_m, a_{m+1}, a_{m+2}, \ldots$
  e.g.: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$

## Term

- Closed-form formula: $a_k = f(k)$
  e.g.: $a_k = \frac{k}{k+1}$
- Recursive formula: $a_k = g(k, a_{k-1}, \ldots, a_{k-c})$
  e.g.: $a_k = a_{k-1} + (k-1)a_{k-2}$

## Growth of sequences

- Increasing sequence
  e.g.: $2, 3, 5, 7, 11, 13, 17, \ldots$
- Decreasing sequence
  e.g.: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$



# Sums and products of sequences

## Sum

- Summation form:

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

where, $k$ = index, $m$ = lower limit, $n$ = upper limit

e.g.: $\sum_{k=m}^{n} \frac{(-1)^k}{k+1}$

## Product

- Product form:

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdot \cdots \cdot a_n$$

where, $k$ = index, $m$ = lower limit, $n$ = upper limit

e.g.: $\prod_{k=m}^{n} \frac{k}{k+1}$

# Properties of sums and products

- Suppose $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and $c$ is any real number

**Sum**

- $\sum_{k=m}^{n} a_k = \sum_{k=m}^{i} a_k + \sum_{k=i+1}^{n} a_k$ for $m \leq i < n$ where, $i$ is between $m$ and $n-1$ (inclusive)
- $c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n}(c \cdot a_k)$
- $\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n}(a_k + b_k)$

**Product**

- $(\prod_{k=m}^{n} a_k) \cdot (\prod_{k=m}^{n} b_k) = \prod_{k=m}^{n}(a_k \cdot b_k)$

# Proof by Mathematical Induction

# Proof for dominos

**Core idea**

- A starting domino falls. From the starting domino, every successive domino falls. Then, every domino from the starting domino falls.

# Mathematical Induction

- Let P(n) denote a predicate, and our objective is to prove P(n) for all integers n>=0. Mathematical induction works in two steps.
- Base step: prove P(0)
- Inductive step: prove P(k) -> P(k+1) for any k>=0

# Proof by mathematical induction: Example 1

**Proposition**

- $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$ for all integers $n \geq 2$.

# Proof by mathematical induction: Example 1

**Proposition**

- $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$ for all integers $n \geq 2$.

**Proof**

Let $P(n)$ denote $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$.

- Basis step. $P(2)$ is true. ▷ How?
- Induction step.

  Assume $P(k)$: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)=\frac{1}{k}$ for some $k \geq 2$.

  Prove $P(k+1)$: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k+1}\right)=\frac{1}{k+1}$

  LHS of $P(k+1)$

  $=\left[\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)\right]\left(1-\frac{1}{k+1}\right)$

  $=\frac{1}{k}\left(1-\frac{1}{k+1}\right)$ $\quad(\because P(k)$ is true$)$

  $=\frac{1}{k}\cdot\frac{k}{k+1}$ $\quad(\because$ common denominator$)$

  $=\frac{1}{k+1}$ $\quad(\because$ remove common factor$)$

  $=$ RHS of $P(k+1)$

# Proof by mathematical induction: Example 2

**Proposition**

- Fibonacci sequence is: $F(0) = 1$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Prove that: $F(0)^2 + F(1)^2 + \cdots + F(n)^2 = F(n)F(n+1)$ for all $n \geq 0$.

# Proof by Mathematical Induction — Guideline

- The problem usually looks like this:
  - Prove: For any integer n >=1, P(n) holds
- Check P(1), P(2) … before the proof
- Try to solve P(k) -> P(k+1)
- Write a proof that has:
  - A notation for predicate P(n)
  - A base step
  - An inductive step

**Use mathematical induction to prove the following identities.**

**(a) [5 points] For all integers $n \geq 1$,**

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

**Use mathematical induction to prove the following identities.**

**(a) [5 points] For all integers $n \geq 1$,**

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

- **Proof**.
  - We will prove this proposition with mathematical induction.
  - **Let P(n) denote the predicate** 1(1!) + 2(2!) + … + n(n!) = (n+1)! -1
  - **Base step**: **We prove** P(1).
    - LHS is 1, and RHS is 2! - 1=1
  - **Inductive step: We prove** P(k) -> P(k+1) for all integer k>=1
    - Let k be an arbitrary integer and k>=1.
    - Assume P(k) holds. That is 1(1!) + 2(2!) + … + k(k!) = (k+1)! -1
    - We want to prove P(k+1), namely, 1(1!) + 2(2!) + … + k(k!) + (k+1) (k+1)! = (k+2)! -1
      - LHS can be reduced to (k+1)! -1 + (k+1) (k+1)! following assumption P(k)
      - The latter can be further reduced to (k+2)! -1, namely RHS.
- **QED**.

**Use mathematical induction to prove the following identities.**

(b) **[5 points] Consider the Fibonacci sequence:** $f_0 = 0$, $f_1 = 1$, **and** $f_n = f_{n-1} + f_{n-2}$, **for** $n \geq 2$. **For all integers** $n \geq 2$,

$$f_{n+2} = 1 + \sum_{i=0}^{n} f_i.$$

**Final 2021**

# Use mathematical induction to prove the following identities.

(b) [5 points] Consider the Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$. For all integers $n \geq 2$,

$$f_{n+2} = 1 + \sum_{i=0}^{n} f_i.$$

- Proof.
    - We will prove this proposition with mathematical induction.
    - Let P(n) be the predicate f_{n+2} = 1+ f_0+f_1 +… + f_n
    - Base step: We first prove P(2) holds.
        - LHS = f_4 = 3. RHS = 1+ f_0 + f_1 + f_2 = 1+0+1+1=3
    - Inductive step:: Then we prove P(k) -> P(k+1) for all k>=2
        - Let k be an arbitrary integer and k>=2
        - Assume P(k) holds, namely,   f_{k+2} = 1+ f_0+f_1 +… + f_k
        - We want to prove P(k+1), that is,  f_{k+3} = 1+ f_0+f_1 +… + f_k+f_{k+1}
            - LHS = f_{k+2}+f_{k+1} from Definition of Fobonacci Sequence.
            - Thus, LHS = 1+ f_0+f_1 +… + f_k + f_{k+1} following assumption P(k)
            - Thus equals RHS
- QED.=