CSE215
Foundations of Computer Science

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Plan ahead

- 11.14 Tu: Sets   - REC on Sets; no homework review in REC

- 11.16   Review Midterm 2   No homework to be announced

- 11.21 Tu   **Midterm 2**   - REC on Functions

- 11.23   Function   Homework to be announced

- 11.28 Tu   Function   -

- 11.30   Function   Ungraded homework to be announced

- 12.05 Tu   Final Review   -

- 12.07   Final Review   No homework to be announced

- 12.10 Mon   Final   -

Today: Lec+Rec

- Clarification about sets 1: Membership and subsets

- Clarification about sets 2: Power sets

- Clarification about sets 3: infinite union/intersection notation

- Clarification about sets 4: Element argument

- Homework review postponed to midterm review time

Clarification on sets 1: Membership and Subsets

Basic Rules

- a $\in$ A if you can find the exact "a" in A

- Translate $a \notin A$ as ~ (a $\in$ A)

- Translate B $\subseteq$ A to "for any b, b$\in$ B -> b$\in$ A"

- Emptyset is a subset of any set

Examples

1. $1 \in \{1, \{1\}\}$ ....1 is the first element listed in $\{1, \{1\}\}$
2. $1 \nsubseteq \{1, \{1\}\}$ ....because 1 is not a set
3. $\{1\} \in \{1, \{1\}\}$ ....$\{1\}$ is the second element listed in $\{1, \{1\}\}$

Examples

4. $\{1\} \subseteq \{1, \{1\}\}$ . . . . make subset $\{1\}$ by selecting 1 from $\{1, \{1\}\}$

5. $\{\{1\}\} \notin \{1, \{1\}\}$ . . . . because $\{1, \{1\}\}$ contains only 1 and $\{1\}$, and not $\{\{1\}\}$

6. $\{\{1\}\} \subseteq \{1, \{1\}\}$ . . . . make subset $\{\{1\}\}$ by selecting $\{1\}$ from $\{1, \{1\}\}$

Examples

7. $\mathbb{N} \notin \mathbb{N}$ . . . . $\mathbb{N}$ is a set (not a number) and $\mathbb{N}$ contains only numbers

8. $\mathbb{N} \subseteq \mathbb{N}$ . . . . because $X \subseteq X$ for every set X

9. $\emptyset \notin \mathbb{N}$ . . . . because the set $\mathbb{N}$ contains only numbers and no sets

Examples

10. $\emptyset \subseteq \mathbb{N}$ . . . . because $\emptyset$ is a subset of every set

11. $\mathbb{N} \in \{\mathbb{N}\}$ . . . . because $\{\mathbb{N}\}$ has just one element, the set $\mathbb{N}$

12. $\mathbb{N} \nsubseteq \{\mathbb{N}\}$ . . . . because, for instance, $1 \in \mathbb{N}$ but $1 \notin \{\mathbb{N}\}$

Examples

13. $\emptyset \notin \{\mathbb{N}\}$ .... note that the only element of $\{\mathbb{N}\}$ is $\mathbb{N}$, and $\mathbb{N} \neq \emptyset$

14. $\emptyset \subseteq \{\mathbb{N}\}$ .... because $\emptyset$ is a subset of every set

15. $\emptyset \in \{\emptyset, \mathbb{N}\}$ .... $\emptyset$ is the first element listed in $\{\emptyset, \mathbb{N}\}$

Examples

16. $\emptyset \subseteq \{\emptyset, \mathbb{N}\}$ .... because $\emptyset$ is a subset of every set

17. $\{\mathbb{N}\} \subseteq \{\emptyset, \mathbb{N}\}$ .... make subset $\{\mathbb{N}\}$ by selecting $\mathbb{N}$ from $\{\emptyset, \mathbb{N}\}$

18. $\{\mathbb{N}\} \nsubseteq \{\emptyset, \{\mathbb{N}\}\}$ .... because $\mathbb{N} \notin \{\emptyset, \{\mathbb{N}\}\}$

19. $\{\mathbb{N}\} \in \{\emptyset, \{\mathbb{N}\}\}$ .... $\{\mathbb{N}\}$ is the second element listed in $\{\emptyset, \{\mathbb{N}\}\}$

Quiz: True or False

$$\{(1,2),(2,2),(7,1)\} \subseteq \mathbb{N} \times \mathbb{N}$$

Clarification on sets 2: Powerset

Definition : If A is a set, the **power set** of A is another set, denoted as $\mathscr{P}(A)$ and defined to be the set of all subsets of A. In symbols, $\mathscr{P}(A) = \{X : X \subseteq A\}$.

POWER SETS

- If S is the set {a, b, c} then {a,c} is a subset of S. There are other subsets of S; the complete list is as follows:
    - {} (the empty set or null set)
    - {a}
    - {b}
    - {c}
    - {a, b}
    - {a, c}
    - {b, c}
    - {a, b, c} or S
- So the power set of S, written $P(S)$, is the set containing all the subsets above. Written out this would be the set:
- $P(S)$ = { {}, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c} }

Fact

If A is a finite set, then $|\mathscr{P}(A)| = 2^{|A|}$.

|A| refers to # of elements in a set A

Basic Rules

- Know how to get the powerset of N elements, for N>=0.

- Emptyset is always an element of the powerset

- The set itself is also always an element of powerset

Examples

1. $\mathscr{P}(\{0,1,3\}) = \{\ \emptyset,\ \{0\},\ \{1\},\ \{3\},\ \{0,1\},\ \{0,3\},\ \{1,3\},\ \{0,1,3\}\ \}$
2. $\mathscr{P}(\{1,2\}) = \{\ \emptyset,\ \{1\},\ \{2\},\ \{1,2\}\ \}$
3. $\mathscr{P}(\{1\}) = \{\ \emptyset,\ \{1\}\ \}$

Examples

4. $\mathscr{P}(\emptyset) = \{\,\emptyset\,\}$
5. $\mathscr{P}(\{a\}) = \{\,\emptyset,\,\{a\}\,\}$
6. $\mathscr{P}(\{\emptyset\}) = \{\,\emptyset,\,\{\emptyset\}\,\}$

Examples

7. $\mathscr{P}(\{a\}) \times \mathscr{P}(\{\emptyset\}) = \{(\emptyset,\emptyset),\ (\emptyset,\{\emptyset\}),\ (\{a\},\emptyset),\ (\{a\},\{\emptyset\})\}$

8. $\mathscr{P}(\mathscr{P}(\{\emptyset\})) = \{\emptyset,\ \{\emptyset\},\ \{\{\emptyset\}\},\ \{\emptyset,\{\emptyset\}\}\}$

9. $\mathscr{P}(\{1,\{1,2\}\}) = \{\emptyset,\ \{1\},\ \{\{1,2\}\},\ \{1,\{1,2\}\}\}$

10. $\mathscr{P}(\{\mathbb{Z},\mathbb{N}\}) = \{\emptyset,\ \{\mathbb{Z}\},\ \{\mathbb{N}\},\ \{\mathbb{Z},\mathbb{N}\}\}$

The following are wrong . Explain why

11. $\mathscr{P}(1) = \{\ \emptyset,\ \{1\}\ \}$ ....

12. $\mathscr{P}(\{1,\{1,2\}\}) = \{\emptyset,\{1\},\{1,2\},\{1,\{1,2\}\}\}$ ....

13. $\mathscr{P}(\{1,\{1,2\}\}) = \{\emptyset,\{\{1\}\},\{\{1,2\}\},\{\emptyset,\{1,2\}\}\}$

Solution

meaningless because 1 is not a set

.... wrong because $\{1,2\} \nsubseteq \{1, \{1,2\}\}$

... wrong because $\{\{1\}\} \nsubseteq \{1, \{1,2\}\}$

Clarification on sets 3:
Infinite union/intersection notation

Notations

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots = \{x : x \in A_i \text{ for at least one set } A_i \text{ with } 1 \le i\}.$$

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots = \{x : x \in A_i \text{ for every set } A_i \text{ with } 1 \le i\}.$$

Example

$$A_1 = \{-1, 0, 1\}, \quad A_2 = \{-2, 0, 2\}, \quad A_3 = \{-3, 0, 3\}, \quad \cdots, \quad A_i = \{-i, 0, i\}, \quad \cdots$$

Observe that $\displaystyle\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}$, and $\displaystyle\bigcap_{i=1}^{\infty} A_i = \{0\}$.

Extended Notation

Here is a useful twist on our new notation. We can write

$$\bigcup_{i=1}^{3} A_i \quad = \quad \bigcup_{i \in \{1,2,3\}} A_i,$$

as this takes the union of the sets A_i for $i = 1, 2, 3$. Likewise:

$$\bigcap_{i=1}^{3} A_i \quad = \quad \bigcap_{i \in \{1,2,3\}} A_i$$

$$\bigcup_{i=1}^{\infty} A_i \quad = \quad \bigcup_{i \in \mathbb{N}} A_i$$

$$\bigcap_{i=1}^{\infty} A_i \quad = \quad \bigcap_{i \in \mathbb{N}} A_i$$

Extended Notations

Here we are taking the union or intersection of a collection of sets A_i where i is an element of some set, be it $\{1,2,3\}$ or $\mathbb{N}$. In general, the way this works is that we will have a collection of sets A_i for $i \in I$, where I is the set of possible subscripts. The set I is called an **index set**.

If A_α is a set for every α in some index set I, then

$$\bigcup_{\alpha \in I} A_\alpha \;=\; \{x : x \in A_\alpha \text{ for at least one set } A_\alpha \text{ with } \alpha \in I\}$$

$$\bigcap_{\alpha \in I} A_\alpha \;=\; \{x : x \in A_\alpha \text{ for every set } A_\alpha \text{ with } \alpha \in I\}.$$

Exercise 1

(a) $\displaystyle\bigcup_{i\in\mathbb{N}}[0, i+1] =$

(b) $\displaystyle\bigcap_{i\in\mathbb{N}}[0, i+1] =$

Solution

- [0, infinity)

- [0, 2]

Exercise 2

(a) $\displaystyle\bigcup_{i \in \mathbb{N}} \mathbb{R} \times [i, i+1] =$

(b) $\displaystyle\bigcap_{i \in \mathbb{N}} \mathbb{R} \times [i, i+1] =$

Solution

- R * [1, infinity)

- \emptyset

Exercise 3

(a) $\bigcup_{\alpha \in \mathbb{R}} \{\alpha\} \times [0,1] =$

(b) $\bigcap_{\alpha \in \mathbb{R}} \{\alpha\} \times [0,1] =$

Solution

- R * [0,1]

- \emptyset

Writing element arguments more efficiently for proving some proof properties

Example 1

- Prove De morgen law: Complement of A intersection B = Complement of A union Complement of B

$$(A \cap B)^c = \{x | x \notin A \cap B\} \qquad \text{(Def. set complement)}$$
$$= \{x | \sim (x \in A \cap B)\} \qquad \text{(Def. of } \notin)$$
$$= \{x | \sim (x \in A \wedge x \in B)\} \qquad \text{(Def. } \cap)$$
$$= \{x | x \notin A \vee x \notin B)\} \qquad \text{(De Morgen in logic)}$$
$$= \{x | x \in A^c \vee x \in B^c)\} \qquad \text{(Def. set complement)}$$
$$= A^c \cup B^c \qquad \text{(Def. } \cup)$$

Example 2

Given sets A, B, and C, prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

- Proof

$$
\begin{aligned}
A \times (B \cap C) \quad &= \{(x,y) : (x \in A) \wedge (y \in B \cap C)\} & \text{(def. of } \times) \\
&= \{(x,y) : (x \in A) \wedge (y \in B) \wedge (y \in C)\} & \text{(def. of } \cap) \\
&= \{(x,y) : (x \in A) \wedge (x \in A) \wedge (y \in B) \wedge (y \in C)\} & (P = P \wedge P) \\
&= \{(x,y) : ((x \in A) \wedge (y \in B)) \wedge ((x \in A) \wedge (y \in C))\} & \text{(rearrange)} \\
&= \{(x,y) : (x \in A) \wedge (y \in B)\} \cap \{(x,y) : (x \in A) \wedge (y \in C)\} & \text{(def. of } \cap) \\
&= (A \times B) \cap (A \times C) & \text{(def. of } \times)
\end{aligned}
$$

- QED.

Exercise

- Prove De morgen law: Complement of A union B = Complement of A intersection Complement of B

Example (This one has to be proved by the old-style element argument)

- Let Z be the set of integers

- Let A be the set of {7a + 8b | a \in Z and b \in Z}

- Prove Z = A

Solution

- Proof.

  - A is clearly a subset of Z

  - So we only need to prove Z is a subset of A

  - Suppose n is arbitrary element of Z, n can be written as 7 (-n) + 8 (n), Therefore Z is a subset of A

- QED.

Exercise (This one has to be proved by the old-style element argument)

- Let Z be the set of integers

- Let A be the set of {7a + 3b | a \in Z and b \in Z}

- Prove Z = A