CSE215
Foundations of Computer Science

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Some slides taken from Prof. Pramod Ganapathi (Stony Brook). Thanks!

- Today: Sequences
    - Quick look at exam problems on Sequences
    - Notations on Sequences
    - Mathematical induction
    - Solving some exam problems

Sequences

"Mathematics is the art of giving the same name to different things." - Henri Poincaré

What are sequences?

Types of sequences

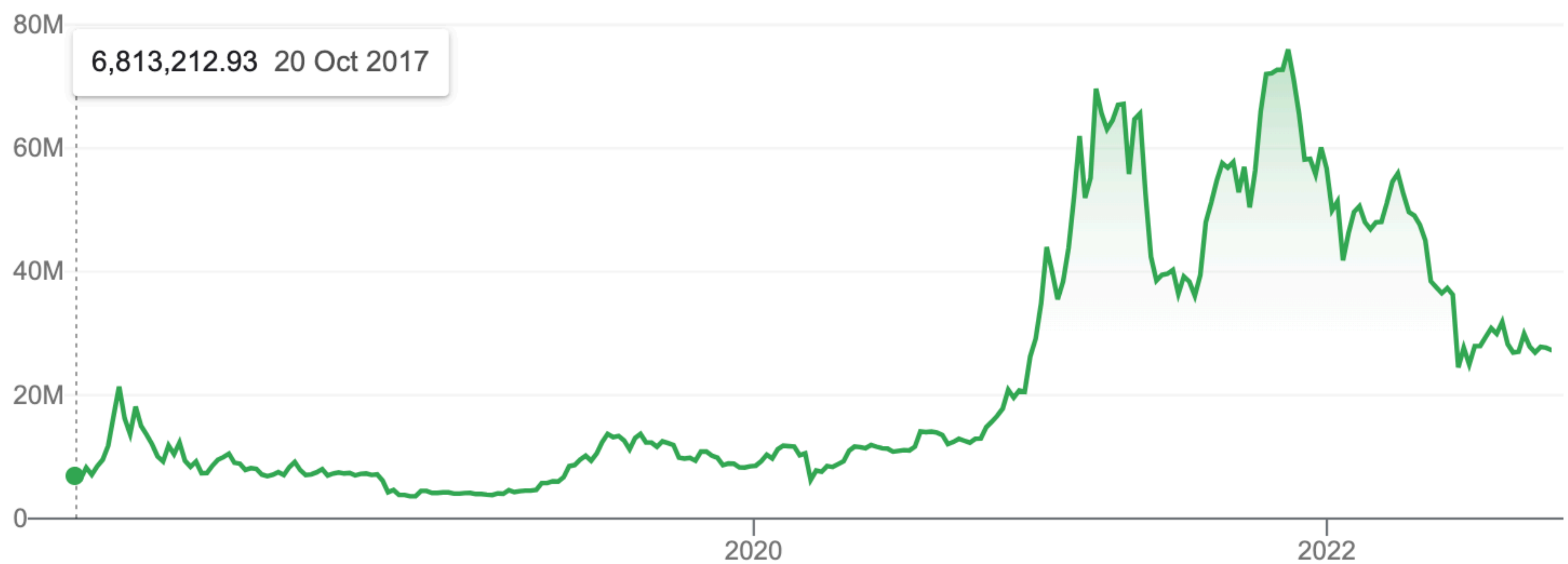
- Finite sequence: $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$
  e.g.: $1^1, 2^2, 3^2, \ldots, 100^2$
- Infinite sequence: $a_m, a_{m+1}, a_{m+2}, \ldots$
  e.g.: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$

Term

- Closed-form formula: $a_k = f(k)$
  e.g.: $a_k = \frac{k}{k+1}$
- Recursive formula: $a_k = g(k, a_{k-1}, \ldots, a_{k-c})$
  e.g.: $a_k = a_{k-1} + (k-1)a_{k-2}$

Growth of sequences

- Increasing sequence
  e.g.: $2, 3, 5, 7, 11, 13, 17, \ldots$
- Decreasing sequence
  e.g.: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$



Sums and products of sequences

Sum

- Summation form:

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

where, $k =$ index, $m =$ lower limit, $n =$ upper limit

e.g.: $\sum_{k=m}^{n} \frac{(-1)^k}{k+1}$

Product

- Product form:

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdot \cdots \cdot a_n$$

where, $k =$ index, $m =$ lower limit, $n =$ upper limit

e.g.: $\prod_{k=m}^{n} \frac{k}{k+1}$

Properties of sums and products

- Suppose $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number

Sum

- $\sum_{k=m}^{n} a_k = \sum_{k=m}^{i} a_k + \sum_{k=i+1}^{n} a_k$ for $m \leq i < n$ where, i is between m and $n-1$ (inclusive)
- $c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} (c \cdot a_k)$
- $\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$

Product

- $(\prod_{k=m}^{n} a_k) \cdot (\prod_{k=m}^{n} b_k) = \prod_{k=m}^{n} (a_k \cdot b_k)$

Proof by Mathematical Induction

Proof for dominos

Core idea

- A starting domino falls. From the starting domino, every successive domino falls. Then, every domino from the starting domino falls.

Mathematical Induction

- Let P(n) denote a predicate, and our objective is to prove P(n) for all integers n>=0. Mathematical induction works in two steps.
- Base step: prove P(0)
- Inductive step: prove P(k) -> P(k+1) for any k>=0

Proof by mathematical induction: Example 1

Proposition

- $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$ for all integers $n \geq 2$.

Proof by mathematical induction: Example 1

Proposition

- $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$ for all integers $n \geq 2$.

Proof

Let $P(n)$ denote $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$.

- Basis step. $P(2)$ is true. ▷ How?
- Induction step.

  Assume $P(k)$: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)=\frac{1}{k}$ for some $k \geq 2$.

  Prove $P(k+1)$: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k+1}\right)=\frac{1}{k+1}$

  LHS of $P(k+1)$

  $=\left[\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)\right]\left(1-\frac{1}{k+1}\right)$

  $=\frac{1}{k}\left(1-\frac{1}{k+1}\right)$ $\quad$ ($\because P(k)$ is true)

  $=\frac{1}{k}\cdot\frac{k}{k+1}$ $\quad$ ($\because$ common denominator)

  $=\frac{1}{k+1}$ $\quad$ ($\because$ remove common factor)

  $=$ RHS of $P(k+1)$

Proof by mathematical induction: Example 2

Proposition

- Fibonacci sequence is: $F(0) = 1$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Prove that: $F(0)^2 + F(1)^2 + \cdots + F(n)^2 = F(n)F(n+1)$ for all $n \geq 0$.

Proof by Mathematical Induction — Guideline

- The problem usually looks like this:
  - Prove: For any integer n >=1, P(n) holds
- Check P(1), P(2) … before the proof
- Try to solve P(k) -> P(k+1)
- Write a proof that has:
  - A notation for predicate P(n)
  - A base step
  - An inductive step

Use mathematical induction to prove the following identities.

(a) [5 points] For all integers $n \geq 1$,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

Use mathematical induction to prove the following identities.

(a) [5 points] For all integers $n \geq 1$,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

- **Proof**.
  - We will prove this proposition with mathematical induction.
  - **Let P(n) denote the predicate** 1(1!) + 2(2!) + … + n(n!) = (n+1)! -1
  - **Base step**: **We prove** P(1).
    - LHS is 1, and RHS is 2! - 1=1
  - **Inductive step: We prove** P(k) -> P(k+1) for all integer k>=1
    - Let k be an arbitrary integer and k>=1.
    - Assume P(k) holds. That is 1(1!) + 2(2!) + … + k(k!) = (k+1)! -1
    - We want to prove P(k+1), namely, 1(1!) + 2(2!) + … + k(k!) + (k+1) (k+1)! = (k+2)! -1
      - LHS can be reduced to (k+1)! -1 + (k+1) (k+1)! following assumption P(k)
      - The latter can be further reduced to (k+2)! -1, namely RHS.
- **QED**.

Use mathematical induction to prove the following identities.

(b) [5 points] Consider the Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$. For all integers $n \geq 2$,

$$f_{n+2} = 1 + \sum_{i=0}^{n} f_i.$$

Use mathematical induction to prove the following identities.

(b) [5 points] Consider the Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$. For all integers $n \geq 2$,

$$f_{n+2} = 1 + \sum_{i=0}^{n} f_i.$$

- Proof.
  - We will prove this proposition with mathematical induction.
  - Let P(n) be the predicate f_{n+2} = 1+ f_0+f_1 +… + f_n
  - Base step: We first prove P(2) holds.
    - LHS = f_4 = 3. RHS = 1+ f_0 + f_1 + f_2 = 1+0+1+1=3
  - Inductive step:: Then we prove P(k) -> P(k+1) for all k>=2
    - Let k be an arbitrary integer and k>=2
    - Assume P(k) holds, namely, f_{k+2} = 1+ f_0+f_1 +… + f_k
    - We want to prove P(k+1), that is, f_{k+3} = 1+ f_0+f_1 +… + f_k+f_{k+1}
      - LHS = f_{k+2}+f_{k+1} from Definition of Fobonacci Sequence.
      - Thus, LHS = 1+ f_0+f_1 +… + f_k + f_{k+1} following assumption P(k)
      - Thus equals RHS
- QED.=

Use mathematical induction to prove the following identities.

(a) [5 points] For all natural numbers n,

$$1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + n^2 \times (n+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$$

Use mathematical induction to prove the following identities.

(a) [5 points] For all natural numbers n,

$$1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + n^2 \times (n+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$$

- Proof.
  - We will prove this proposition with mathematical induction.
  - Let P(n) be the predicate 1^2 * 2 + … + n^2 * (n+1) = n (n+1) (n+2) (3n+1)/12
  - Base step: We first prove P(1) holds.
    - LHS = 1^2 * 2=2. RHS = 1*2*3*4/12 = 2.
  - Inductive step: Then, we prove P(k) -> P(k+1) for all k>=1
    - Let k be an arbitrary integer and k>=1.
    - Assume P(k) holds. That is, 1^2 * 2 + … + k^2 * (k+1) = k (k+1) (k+2) (3k+1)/12
    - We want to prove P(k+1), that is, 1^2 * 2 + … + k^2 * (k+1) + (k+1)^2 * (k+2) = (k+1) (k+2) (k+3) (3k+4)/12
      - LHS = k (k+1) (k+2) (3k+1)/12 + + (k+1)^2 * (k+2) following assumption P(k)
      - The latter can be further reduced to (k+1)(k+2)/12 * (3k^2+k + 12k+12) = RHS
- QED.

Use mathematical induction to prove the following identities.

(b) [5 points] For all natural numbers n,

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{n \times (n+1)} = \frac{n}{n+1}$$

Use mathematical induction to prove the following identities.

(b) [5 points] For all natural numbers n,

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{n \times (n+1)} = \frac{n}{n+1}$$

- Proof.
  - We will prove this proposition with mathematical induction.
  - Let P(n) be the predicate 1/(1*2) + 1(2*3) + … + 1/(n *(n+1)) = n/(n+1)
  - Base step: We first prove P(1) holds.
    - LHS is 1/2, and RHS is 1/(1+1) = 1/2
  - Inductive step: Then, we prove P(k) -> P(k+1) for all integer k>=1
    - Let k be an arbitrary integer and k>=1.
    - Assume P(k) holds. That is 1/(1*2) + 1(2*3) + … + 1/(k*(k+1)) = k/(k+1)
    - We want to prove P(k+1), namely,1/(1*2) + 1(2*3) + … + 1/(k*(k+1)) + 1/((k+1)*(k+2))= (k+1)/(k+2)
      - LHS can be reduced to k/(k+1) + 1/((k+1)*(k+2)) following the assumption P(k)
      - The latter can be further reduced to (k^2+2k+1)/((k+2)(k+1)), which equals to RHS.
- QED.