CSE215
Foundations of Computer Science

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Exercise 3

- Consider the recursively defined sequence below. Find its explicit form and prove your answer.

$$c_k = 3c_{k-1} + 1, \text{ for all integers } k \geq 2$$
$$c_1 = 1$$

Solution

- We have $c_1=1$, $c_2= 4$, $c_3=13$, b4=40… So a possible explicit form is $c_n=(3^n-1)/2$
- To prove it indeed satisfies the recursive definition, we plug it to the recursive definition:
    - The explicit form clearly satisfies $c_1=1$
    - The explicit form also satisfies_____, since LHS= ___ RHS = ___ = ____

Agenda

- Sequences part is finished for now.
- Today: Sets
- Midterm 2 will take place once we are done with sets.

Set

"a language in which we formulate and discuss the basic notions concerning collections of objects."

Roster Notation

- A = {1,2,3,4,5}
- Even = { 2, 4, 6, 8…}
- Prime = { 2, 3, 5, 7, ….}

Set-builder notation

Set builder notation describes a set that is defined by a predicate:

- {x | P(x)}
- {x : P(x)}

Where the vertical bar (or colon) is a separator that can be read as "such that", "for which", or "with the property that".

A domain E can appear on the left of the vertical bar

$$\{x \in E \mid \Phi(x)\},$$

or by adjoining it to the predicate:

$$\{x \mid x \in E \text{ and } \Phi(x)\} \quad \text{or} \quad \{x \mid x \in E \wedge \Phi(x)\}.$$

Set-builder notation and roster notation

1. $\{n : n \text{ is a prime number}\} = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$
2. $\{n \in \mathbb{N} : n \text{ is prime}\} = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$
3. $\{n^2 : n \in \mathbb{Z}\} = \{0, 1, 4, 9, 16, 25, \ldots\}$

4. $\{x \in \mathbb{R} : x^2 - 2 = 0\} = \{\sqrt{2}, -\sqrt{2}\}$
5. $\{x \in \mathbb{Z} : x^2 - 2 = 0\} = \emptyset$
6. $\{x \in \mathbb{Z} : |x| < 4\} = \{-3, -2, -1, 0, 1, 2, 3\}$

7. $\{2x : x \in \mathbb{Z}, |x| < 4\} = \{-6, -4, -2, 0, 2, 4, 6\}$
8. $\{x \in \mathbb{Z} : |2x| < 4\} = \{-1, 0, 1\}$

Subsets

Definitions

- Subset. $A \subseteq B \Leftrightarrow \forall x,$ if $x \in A$ then $x \in B$

- Not subset. $A \nsubseteq B \Leftrightarrow \exists x$ such that $x \in A$ and $x \notin B$

Exercises

Let $A = \{\{0, 10\}, \{10, 100\}, 0\}$. Determine if the following are true or false.

1. $0 \in A$
2. $\{10\} \in A$
3. $\{10\} \subseteq A$
4. $\{0, 10\} \subseteq A$

Set equality

Definition

- Given sets A and B, A equals B, written $A = B$, if, and only if, every element of A is in B and every element of B is in A.
- $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

Exercises

- $A = \{m \in \mathbb{Z} \mid m = 2a \text{ for some integer } a\}$
  $B = \{n \in \mathbb{Z} \mid n = 2b - 2 \text{ for some integer } b\}$
  Is $A = B$?

Exercises

Let Z denote the set of integers.

Let $A = \{7a+3b \mid a, b \in Z \}$

True or False: $A = Z$?

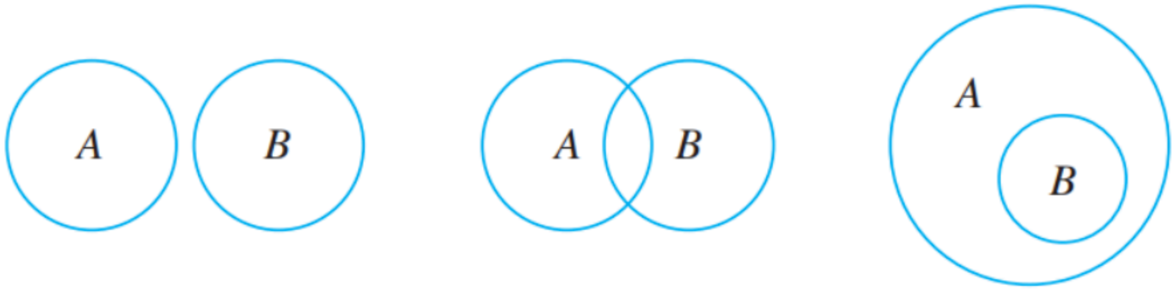
Solution: This set contains all numbers of form $7a+3b$, where a and b are integers. Each such number $7a+3b$ is an integer, so A contains only integers. But *which* integers? If n is *any* integer, then $n = 7n+3(-2n)$, so $n = 7a+3b$ where $a = n$ and $b = -2n$. Thus $n \in A$, and so $A = \mathbb{Z}$.

Venn diagrams

Definition

- Relationship between a small number of sets can be represented by pictures called Venn diagrams

(a) (b)



(a) (b) (c)

Exercises

- Draw a Venne diagram representing
  - N: the set of natural numbers
  - Z: the set of integers
  - Q: The set of rationals
  - R: the set of reals

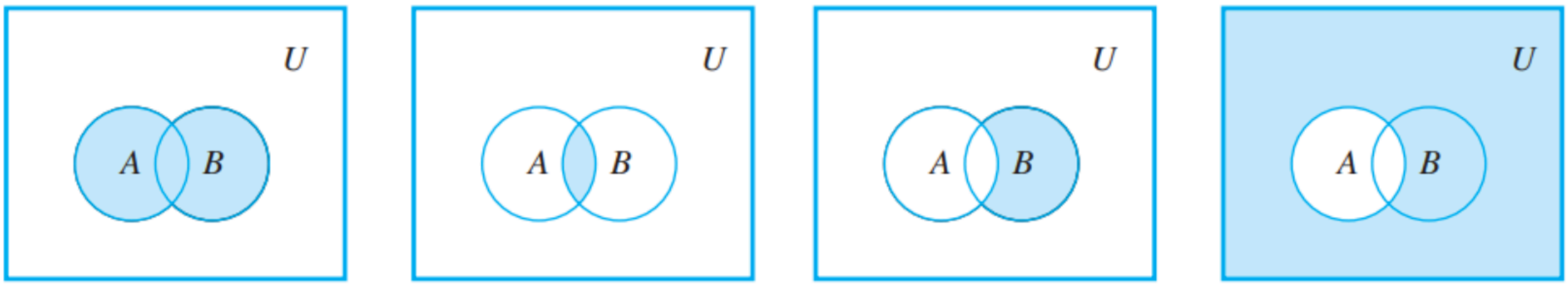
Operations on sets

Definition

Let A and B be subsets of a universal set U.

1. The union of A and B, denoted $A \cup B$, is the set of all elements that are in at least one of A or B.
   $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$
2. The intersection of A and B, denoted $A \cap B$, is the set of all elements that are common to both A and B.
   $A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$
3. The difference of B minus A (or relative complement of A in B), denoted $B - A$, is the set of all elements that are in B and not A.
   $B - A = \{x \in U \mid x \in B \text{ and } x \notin A\}$
4. The complement of A, denoted A', is the set of all elements in U that are not in A.
   $A' = \{x \in U \mid x \notin A\}$

Operations on sets

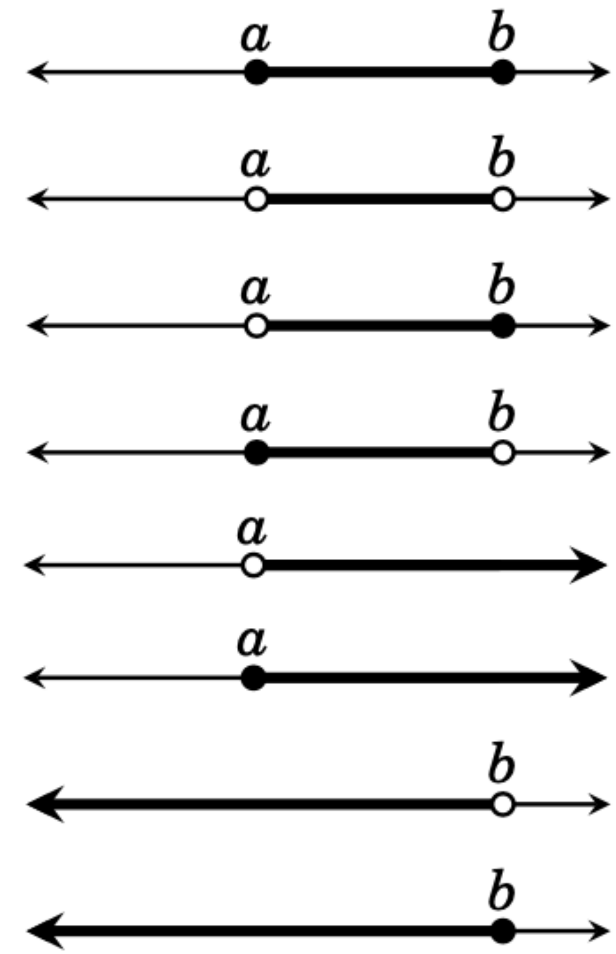


Problems

- Let the universal set $U = \{a, b, c, d, e, f, g\}$.
  Let $A = \{a, c, e, g\}$ and $B = \{d, e, f, g\}$.
  Find $A \cup B$, $A \cap B$, $B - A$, and A'.

Notations (1)

- Closed interval: $[a,b]=\{x\in\mathbb{R}:a\leq x\leq b\}$
- Open interval: $(a,b)=\{x\in\mathbb{R}:a<x<b\}$
- Half-open interval: $(a,b]=\{x\in\mathbb{R}:a<x\leq b\}$
- Half-open interval: $[a,b)=\{x\in\mathbb{R}:a\leq x<b\}$
- Infinite interval: $(a,\infty)=\{x\in\mathbb{R}:a<x\}$
- Infinite interval: $[a,\infty)=\{x\in\mathbb{R}:a\leq x\}$
- Infinite interval: $(-\infty,b)=\{x\in\mathbb{R}:x<b\}$
- Infinite interval: $(-\infty,b]=\{x\in\mathbb{R}:x\leq b\}$



Exercises

- $A = (-1, 0] = \{x \in \mathbb{R} \mid -1 < x \leq 0\}$
  $B = [0, 1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$.
  Find $A \cup B$, $A \cap B$, $B - A$, and A'.

Notations (2)

Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

- $\cup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i$ for at least one $i = 0, 1, 2, \ldots, n\}$
- $\cup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i$ for at least one whole number $i\}$
- $\cap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i$ for all $i = 0, 1, 2, \ldots, n\}$
- $\cap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i$ for all whole numbers $i\}$

Exercises

- For each positive integer i, let
  $A_i = \{x \in \mathbb{R} \mid -\frac{1}{i} < x < \frac{1}{i}\} = \left(-\frac{1}{i}, \frac{1}{i}\right)$
  Find $A_1 \cup A_2 \cup A_3$ and $A_1 \cap A_2 \cap A_3$
  Find $\cup_{i=0}^{\infty} A_i$ and $\cap_{i=0}^{\infty} A_i$

Empty set

Definition

Empty set, denoted by ϕ, is a set with no elements.

Examples

- $\{1,3\} \cap \{2,4\} = \phi$
- $\{x \in \mathbb{R} \mid x^2 = -1\} = \phi$
- $\{x \in \mathbb{R} \mid 3 < x < 2\} = \phi$

Disjoint sets

Definition

- Two sets are called disjoint if, and only if, they have no elements in common.
- A and B are disjoint $\Leftrightarrow A \cap B = \phi$

Problems

- Let $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$. Are A and B disjoint?

Mutually disjoint sets

Definition

- Sets $A_1, A_2, A_3, \ldots$ are mutually disjoint (or pairwise disjoint or nonoverlapping) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common.
- For all $i, j = 1, 2, 3, \ldots$
  $A_i \cap A_j = \phi$ whenever $i \neq j$.

Problems

Are the following sets mutually disjoint?

- $A_1 = \{3, 5\}$, $A_2 = \{1, 4, 6\}$, and $A_3 = \{2\}$.
- $B_1 = \{2, 4, 6\}$, $B_2 = \{3, 7\}$, and $B_3 = \{4, 5\}$.

Power set

Definition

- Given a set A, the power set of A, denoted $P(A)$, is the set of all subsets of A.

Problems

- Find the power set of the set $\{x, y\}$. That is, find $P(\{x, y\})$.

Cartesian product

Definition

- Given sets $A_1, A_2, \ldots, A_n$, the Cartesian product of $A_1, A_2, \ldots, A_n$ denoted $A_1 \times A_2 \times \cdots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.
- $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$.

Problems

- Let $A_1 = \{x, y\}$, $A_2 = \{1, 2, 3\}$, and $A_3 = \{a, b\}$. Find:
  (a) $A_1 \times A_2$,
  (b) $(A_1 \times A_2) \times A_3$, and