

# **CSE215**

# **Foundations of Computer Science**

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# Midterm2

- We will strategically use it for final review
- So, results to be announced next week

# Plan ahead

- ~~11.14 Tu~~: Sets - REC on Sets; no homework review in REC
- ~~11.16~~ Review Midterm 2 No homework to be announced
- ~~11.21 Tu~~ **Midterm 2** - REC on Functions
- ~~11.23~~ Function Homework to be announced
- 11.28 Tu Function - **Regular lecture in REC**
- 11.30 Function Ungraded homework to be announced
- 12.05 Tu Final Review - **Reviewing with Midterm 2**
- 12.07 Final Review No homework to be announced
- **12.12 Tue Final** -

# Agenda

- Review on one-to-one and onto functions
- Inverse functions

# Summary of last class: one-to-one and onto functions

**How to show a function  $f : A \rightarrow B$  is injective:**

**Contrapositive approach:**

Suppose  $x, y \in A$  and  $f(x) = f(y)$ .

$\vdots$

Therefore  $x = y$ .

**How to show a function  $f : A \rightarrow B$  is surjective:**

Suppose  $b \in B$ .

[Prove there exists  $a \in A$  for which  $f(a) = b$ .]

# Exercise 1

SBU 2021 Final

**Problem 9. [5 points]**

Write and fill the table with ✓ or ✗. If a function is one-to-one or onto, then use ✓. On the other hand, if a function is not one-to-one or not onto, then use ✗.

Function	Domains	One-to-one function?	Onto function?
$f(x) = 3x$	$f : \mathbb{Z} \rightarrow \mathbb{Z}$		
$f(x) = 3x$	$f : \mathbb{R} \rightarrow \mathbb{R}$		
$f(x) = 3x^2$	$f : \mathbb{Z} \rightarrow \mathbb{Z}$		
$f(x) = 3x^2$	$f : \mathbb{R} \rightarrow \mathbb{R}$		

# Exercise 2

True or false?

the function  $f : \mathbb{R} - \{1\} \rightarrow \mathbb{R} - \{1\}$  defined by  $f(x) = \left(\frac{x+1}{x-1}\right)^3$  is bijective.



# Solution

- We first prove  $f$  is injective.

We want to prove  $\forall x, y \in \mathbb{R} - \{1\}, f(x) = f(y) \rightarrow x = y$

Suppose  $x, y \in \mathbb{R} - \{1\}$  and  $f(x) = f(y)$

$$\text{We have } \left(\frac{x+1}{x-1}\right)^3 = \left(\frac{y+1}{y-1}\right)^3 \quad \text{Thus } \frac{x+1}{x-1} = \frac{y+1}{y-1}$$

$$\text{Thus } 1 + \frac{2}{x-1} = 1 + \frac{2}{y-1}, \quad \text{Thus } \frac{2}{x-1} = \frac{2}{y-1}$$

$$\text{Thus } 2(y-1) = 2(x-1) \quad \text{Thus } x = y$$

We have proved  $\forall x, y \in \mathbb{R} - \{1\}, f(x) = f(y) \rightarrow x = y$

- We then prove  $f$  is surjective

We want to prove  $\forall y \in \mathbb{R} - \{1\}, \exists x \in \mathbb{R} - \{1\}, f(x) = y$

Suppose  $y \in \mathbb{R} - \{1\}$ .

$$\text{We choose } x = \frac{\frac{2}{\sqrt[3]{y}-1}} + 1$$

$$\text{We have } f(x) = y$$

We have proved  $\forall y \in \mathbb{R} - \{1\}, \exists x \in \mathbb{R} - \{1\}, f(x) = y$



# Exercise 3

- Consider the function  $\theta : \{0, 1\} \times \mathbb{N} \rightarrow \mathbb{Z}$  defined as  $\theta(a, b) = (-1)^a b$ . Is  $\theta$  injective? Is it surjective? Explain.

# Solution

First we show that  $\theta$  is injective. Suppose  $\theta(a, b) = \theta(c, d)$ . Then  $(-1)^a b = (-1)^c d$ . As  $b$  and  $d$  are both in  $\mathbb{N}$ , they are both positive. Then because  $(-1)^a b = (-1)^c d$ , it follows that  $(-1)^a$  and  $(-1)^c$  have the same sign. Since each of  $(-1)^a$  and  $(-1)^c$  equals  $\pm 1$ , we have  $(-1)^a = (-1)^c$ , so then  $(-1)^a b = (-1)^c d$  implies  $b = d$ . But also  $(-1)^a = (-1)^c$  means  $a$  and  $c$  have the same parity, and because  $a, c \in \{0, 1\}$ , it follows  $a = c$ . Thus  $(a, b) = (c, d)$ , so  $\theta$  is injective.

Next note that  $\theta$  **is not surjective** because  $\theta(a, b) = (-1)^a b$  is either positive or negative, but never zero. Therefore there exist no element  $(a, b) \in \{0, 1\} \times \mathbb{N}$  for which  $\theta(a, b) = 0 \in \mathbb{Z}$ .

# Exercise 4

SBU 2022 Midterm

**Problem 4. [5 points]**

Let  $A$  and  $B$  be finite sets where  $|A| = |B|$ . Is it possible to define a function  $f : A \rightarrow B$  that is one-to-one but not onto? Is it possible to define a function  $g : A \rightarrow B$  that is onto but not one-to-one?

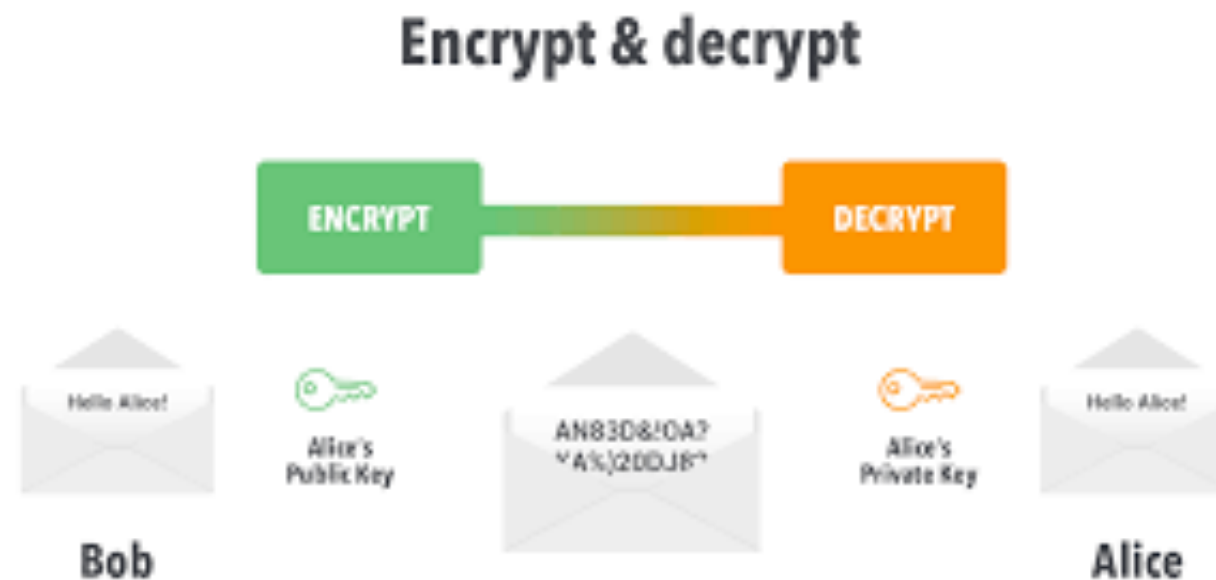
# Inverse functions

# Inverse functions

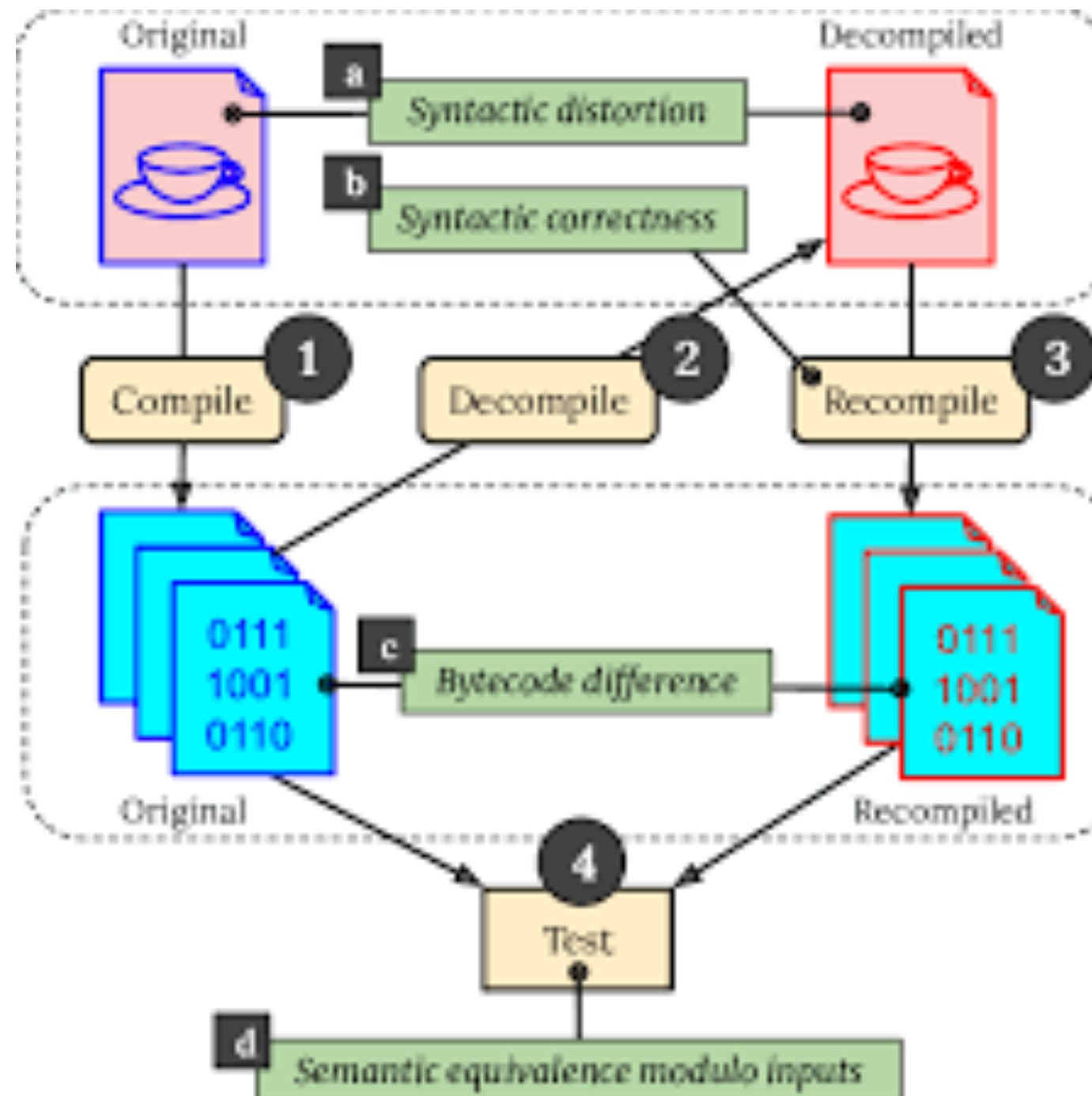
## Definition

- Suppose  $F : X \rightarrow Y$  is a one-to-one correspondence.  
Then, the **inverse function**  $F^{-1} : Y \rightarrow X$  is defined as follows:  
Given any element  $y$  in  $Y$ ,  
 $F^{-1}(y)$  = that unique element  $x$  in  $X$  such that  $F(x) = y$ .
- $F^{-1}(y) = x \Leftrightarrow y = F(x)$ .

# Does encryption have an inverse function?



# Does Java compilation have an inverse function?





# Example

## Problem

- Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by the rule  $f(x) = 4x - 1$  for all  $x \in \mathbb{R}$ . Find its inverse function.

## Proof

For any  $y$  in  $R$ , by definition of  $f^{-1}$

- $f^{-1} =$  unique number  $x$  such that  $f(x) = y$

Consider  $f(x) = y$

$$\implies 4x - 1 = y \quad (\because \text{Defn. of } f)$$

$$\implies x = \frac{y+1}{4} \quad (\because \text{Simplify})$$

- Hence,  $f^{-1}(y) = \frac{y+1}{4}$  is the inverse function.

# Exercise 1

- Check if the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined as  $f(x) = x^3 + 1$  is a one-to-one correspondence. If yes, find its inverse.

# Exercise 2

- The function  $f : \mathbb{R} - \{2\} \rightarrow \mathbb{R} - \{5\}$  defined by  $f(x) = (5x+1)/(x-2)$  is bijective. Find its inverse.

# Solution

- Let  $y$  be an element of  $\mathbb{R} - \{5\}$ . We have  $y = f(x)$  if and only if  $x = 11/(y-5)+2$ .
- Thus  $f^{-1}(y) = 11/(y-5)+2$

# Exercise 3

- The function  $g: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$  defined by the formula  $g(m, n) = (m+n, m+2n)$  is a one-to-one correspondence. Find its inverse.

# Solution

- Let  $(u,v)$  be an arbitrary element of  $Z \times Z$ . Then  $f(m,n) = (u,v)$  if and only if  $m = 2u-v$  and  $n = v-u$ .
- Thus,  $f^{-1}(u,v) = (2u-v, v-u)$ .