

CSE215

Foundations of Computer Science

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Today

- Cardinality: Size of infinite sets
- Five classic examples

Cardinality/Size of Infinite sets

- There are as many squares as there are numbers because they are just as numerous as their roots.
 - — Galileo Galilei, 1632

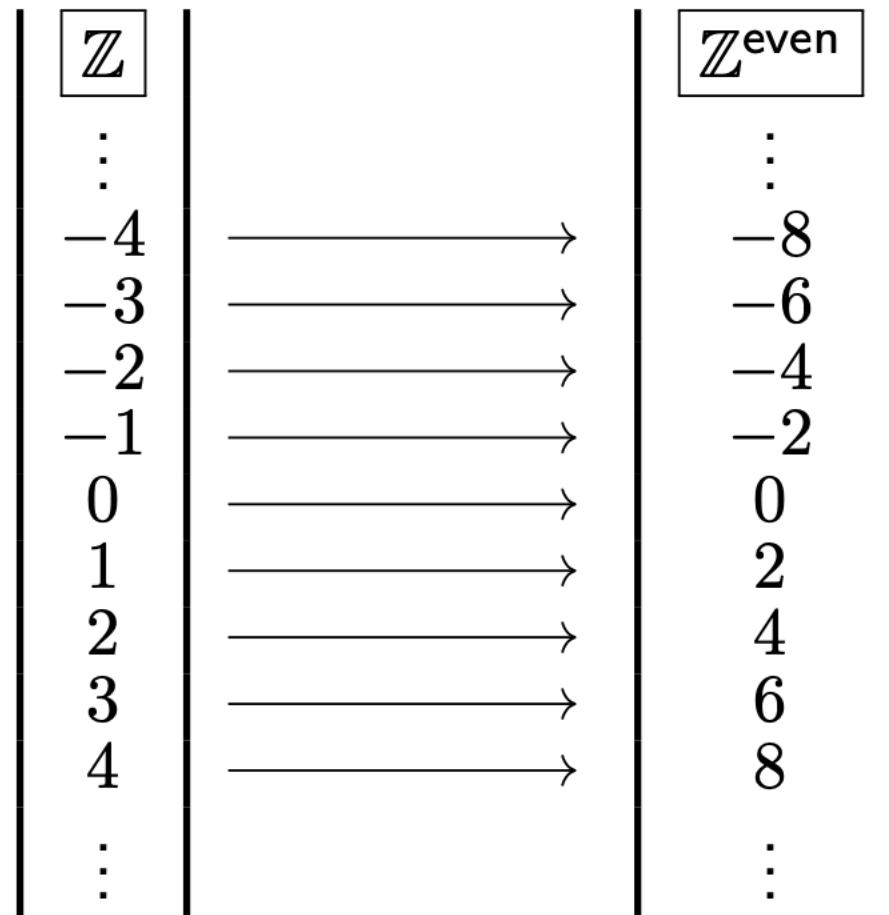
Same cardinality

Definition

- Let A and B be any sets. A has the same cardinality as B if, and only if, there is a one-to-one correspondence from A to B .
- A has the same cardinality as B if, and only if, there is a function f from A to B that is both one-to-one and onto.

Example 1

Integers and even numbers are of the same size



Proof

Problem

- Prove that the cardinality of integers and even numbers are the same.

Solution

Problem

- Prove that the cardinality of integers and even numbers are the same.

- To prove that $|\mathbb{Z}| = |\mathbb{Z}^{\text{even}}|$, we need to prove that there is a one-to-one correspondence, say f , between \mathbb{Z} and \mathbb{Z}^{even} . Suppose $f = 2n$ for all integers $n \in \mathbb{Z}$.

- **Prove that f is one-to-one.**

Suppose $f(n_1) = f(n_2)$.

$$\implies 2n_1 = 2n_2 \quad (\because \text{Defn. of } f)$$

$$\implies n_1 = n_2 \quad (\because \text{Simplify})$$

- **Prove that f is onto.**

Suppose $m \in \mathbb{Z}^{\text{even}}$.

$$\implies m \text{ is even} \quad (\because \text{Defn. of } \mathbb{Z}^{\text{even}})$$

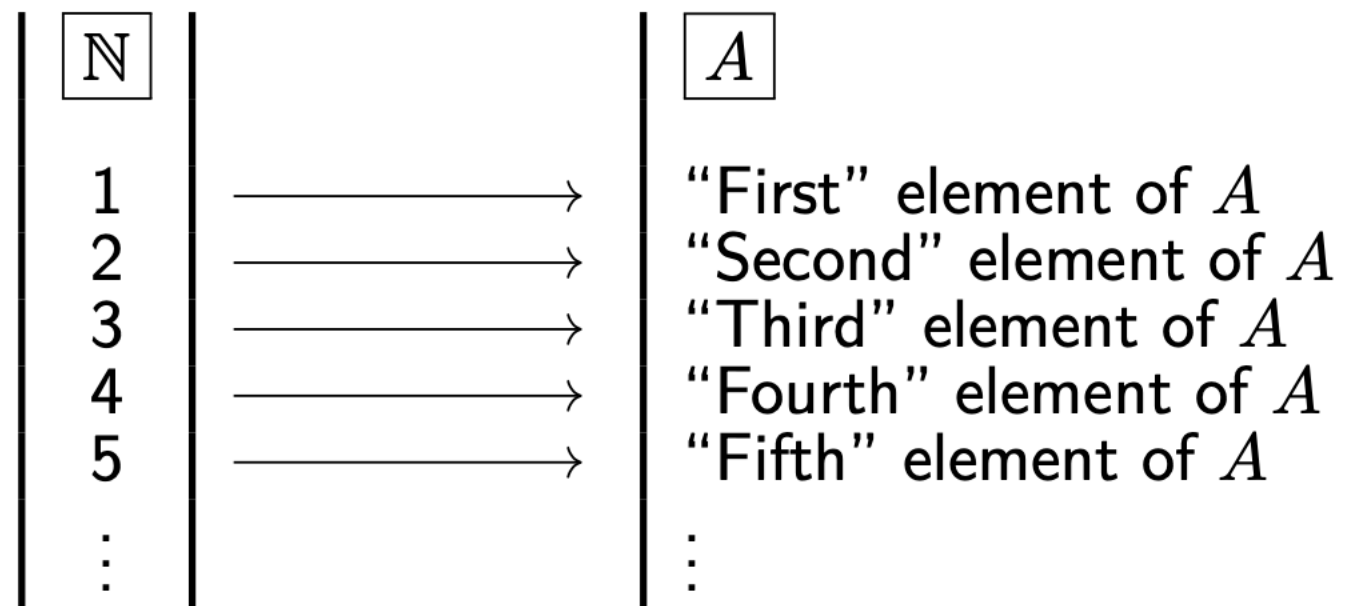
$$\implies m = 2k \text{ for } k \in \mathbb{Z} \quad (\because \text{Defn. of even})$$

$$\implies f(k) = m \quad (\because \text{Defn. of } f)$$

An infinite set and its proper subset can have the same size!



Countable sets



Definition

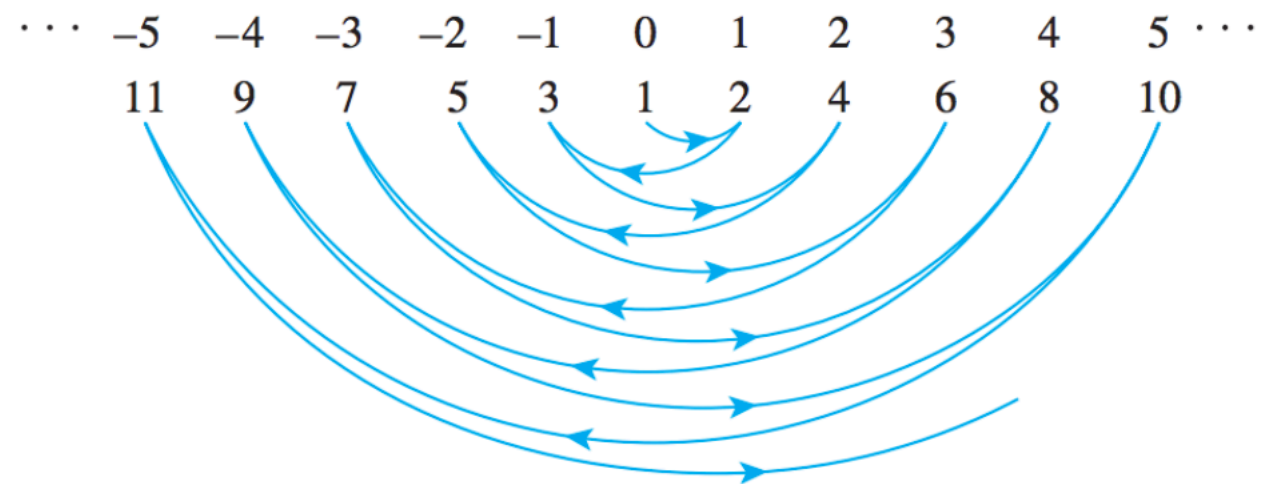
- A set is called **countably infinite** if, and only if, it has the same cardinality as the set of positive integers.
- A set is called **countable** if, and only if, it is finite or countably infinite. A set that is not countable is called **uncountable**.

Example 2

Problem

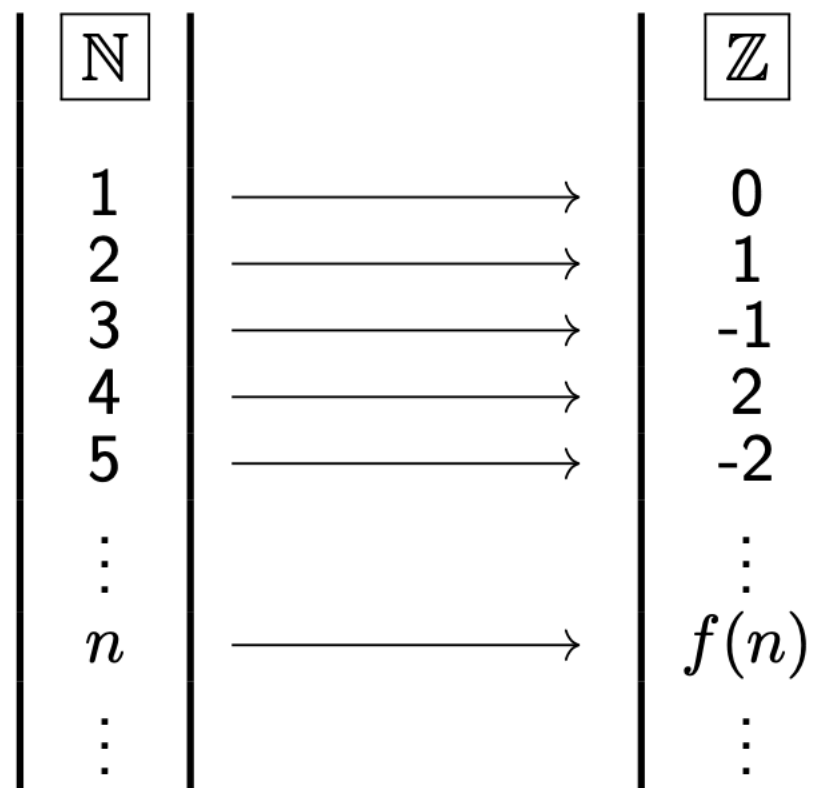
- Prove that the set of integers is countably infinite.

Intuition



Integers are countable

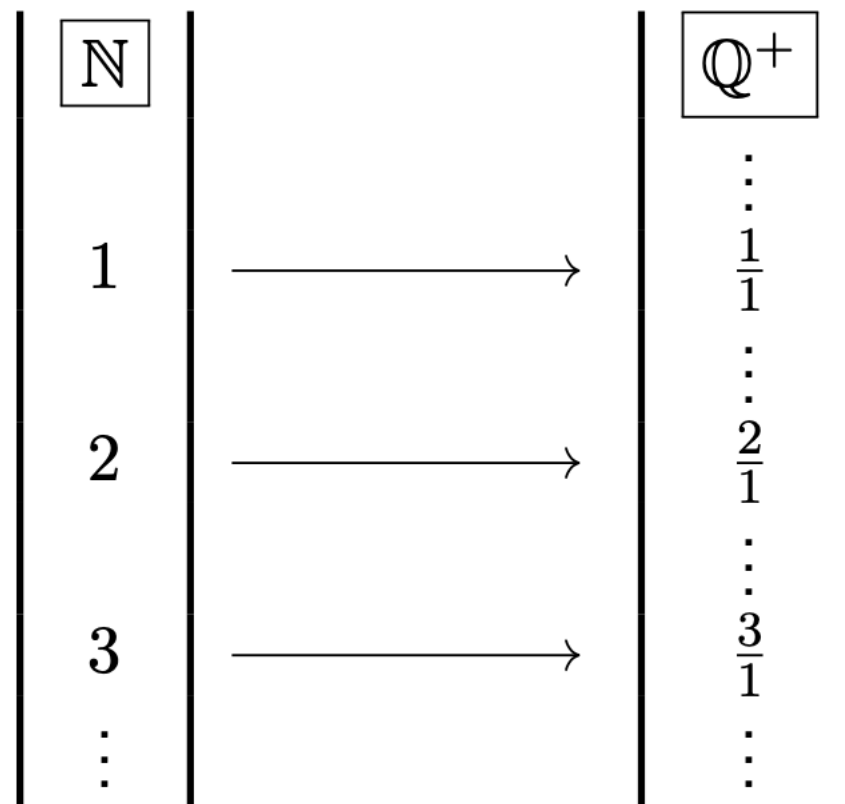
Solution (continued)

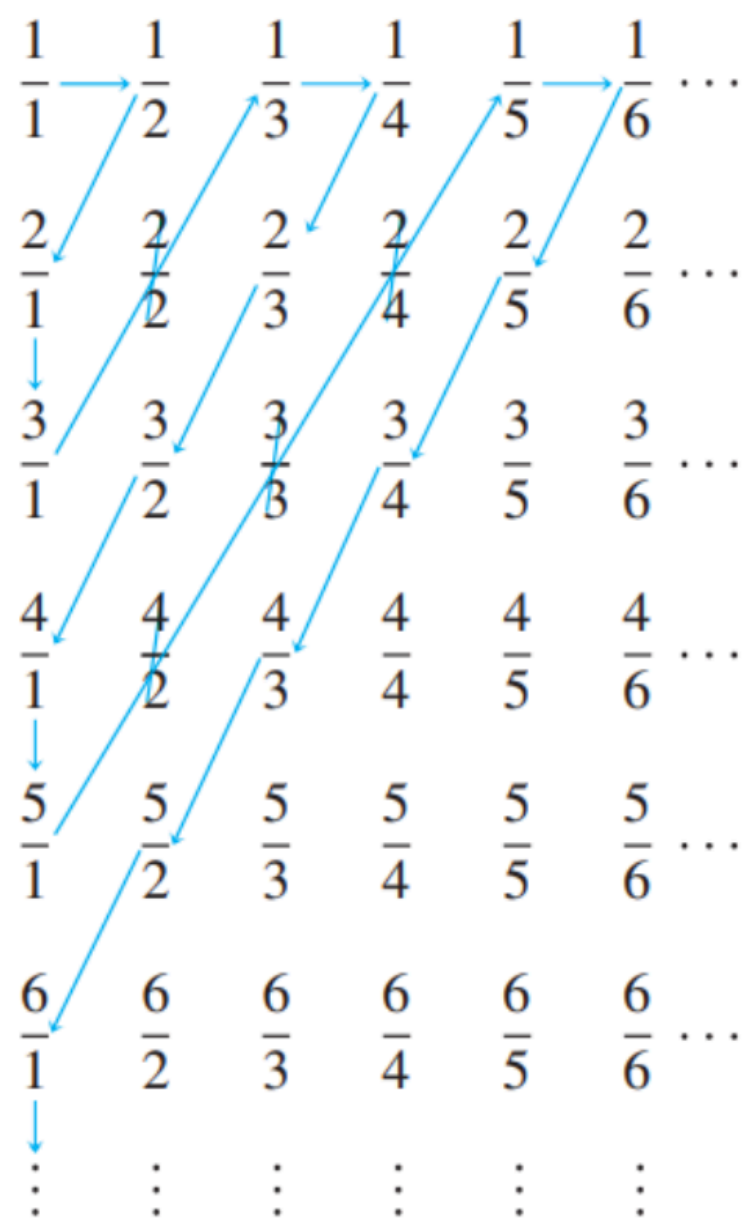


- Define a function $f(n) : \mathbb{N} \rightarrow \mathbb{Z}$ such that
$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is an even natural number,} \\ -\left(\frac{n-1}{2}\right) & \text{if } n \text{ is an odd natural number.} \end{cases}$$
- As f is a one-to-one correspondence between \mathbb{N} and \mathbb{Z} , the set of integers is countably infinite.

Example 3: True or False?

Set of positive rationals is uncountable





Set of positive rationals is countable

Problem

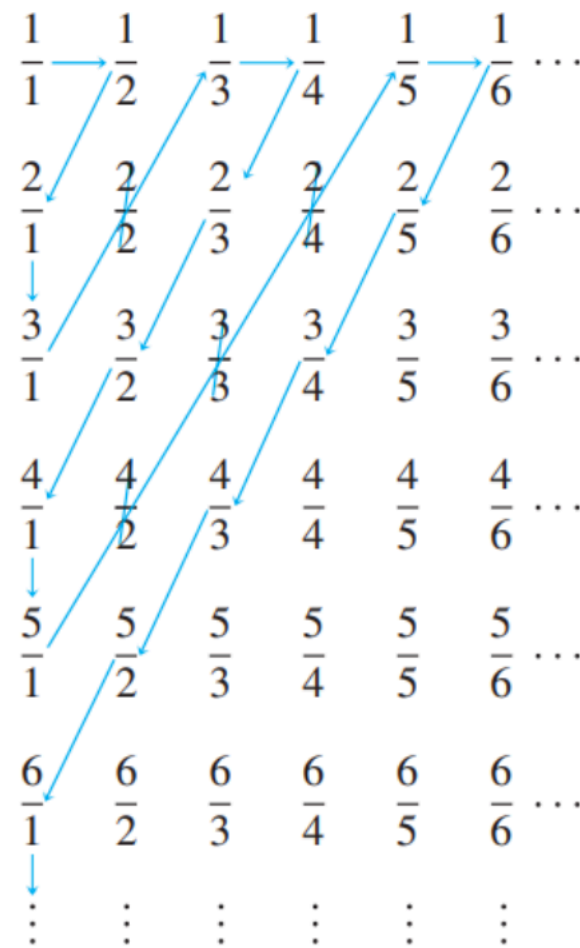
- Prove that the set of positive rational numbers are countable.

Set of positive rationals is countable

Problem

- Prove that the set of positive rational numbers are countable.

Solution



\mathbb{N}

1
2
3
4
5
6
7
8
9
10
 \vdots

→
→
→
→
→
→
→
→
→
→

\mathbb{Q}^+

1/1
1/2
2/1
3/1
1/3
1/4
2/3
3/2
4/1
5/1
 \vdots

Set of positive rational numbers is countable

Problem

- Prove that the set of positive rational numbers are countable.

Solution (continued)

- To prove that $|\mathbb{N}| = |\mathbb{Q}^+|$, we need to prove that there is a one-to-one correspondence, say f , between \mathbb{N} and \mathbb{Q}^+ .
- **Prove that f is onto.**
Every positive rational number appears somewhere in the grid.
Every point in the grid is reached eventually.
- **Prove that f is one-to-one.**
Skipping numbers that have already been counted ensures that no number is counted twice.

Example 4

Set of real numbers in $[0, 1]$ is uncountable

Problem

- Prove that the set of all real numbers between 0 and 1 is uncountable.

Set of real numbers in $[0, 1]$ is uncountable

Problem

- Prove that the set of all real numbers between 0 and 1 is uncountable.

Solution

- To prove that $|\mathbb{N}| \neq |[0..1]|$, we need to prove that there is no one-to-one correspondence between \mathbb{N} and $[0..1]$.
- A powerful approach to prove the theorem is:
proof by contradiction.

Set of real numbers in $[0, 1]$ is uncountable

Problem

- Prove that the set of all real numbers between 0 and 1 is uncountable.

Solution

Proof by contradiction.

- Suppose $[0..1]$ is countable.
- We will derive a contradiction by showing that there is a number in $[0..1]$ that does not appear on this list.

\mathbb{N}		$[0..1]$
1	→	$0.a_{11}a_{12}a_{13} \dots a_{1n} \dots$
2	→	$0.a_{21}a_{22}a_{23} \dots a_{2n} \dots$
3	→	$0.a_{31}a_{32}a_{33} \dots a_{3n} \dots$
\vdots	\vdots	\vdots
n	→	$0.a_{n1}a_{n2}a_{n3} \dots a_{nn} \dots$
\vdots	\vdots	\vdots

Set of real numbers in $[0, 1]$ is uncountable

Solution (continued)

- Suppose the list of reals starts out as follows:

0.	9	0	1	4	8	...
0.	1	1	6	6	6	...
0.	0	3	3	5	3	...
0.	9	6	7	2	6	...
0.	0	0	0	3	1	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮

- Construct a new number $d = 0.d_1d_2d_3 \dots d_n \dots$ as follows:

$$d_n = \begin{cases} 1 & a_{nn} \neq 1, \\ 2 & a_{nn} = 1. \end{cases}$$

- We have $d = 0.12112 \dots$, i.e.,

0.	1	2	1	1	2	...
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Set of real numbers in $[0, 1]$ is uncountable

Solution (continued)

- Observation:

For each natural number n , the constructed real number d differs in the n th decimal position from the n th number on the list.

1	→	0.	9	0	1	4	8	...
2	→	0.	1	1	6	6	6	...
3	→	0.	0	3	3	5	3	...
4	→	0.	9	6	7	2	6	...
5	→	0.	0	0	0	3	1	...
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮
d	→	0.	1	2	1	1	2	...

- This implies that d is not on the list. But, $d \in [0, 1]$.
- Contradiction! So, our supposition is false.
- Set of real numbers in $[0, 1]$ is uncountable.

There are different types of ∞ !



Three Exercises

Exercise 1

- Prove this: “There are as many squares as there are numbers”. (Galileo Galilei, 1632). In other words, prove
 - $\{n^2 \mid n \in \mathbb{Z}\}$ and $|\mathbb{Z}|$ are of the same cardinality.

Solution

- Let $A = \{n^2 \mid n \in \mathbb{N}\}$. We want to prove: $|A| = |\mathbb{N}|$.
- We build a function $f: A \rightarrow \mathbb{N}$ defined as $f(a) = \sqrt{a}$. We want to prove f is bijective.
- We first prove f is injective.
 - We want to prove, for any $a, b \in A$, $f(a) = f(b) \rightarrow a = b$
 - Suppose $a, b \in A$ and $f(a) = f(b)$
 - We have $\sqrt{a} = \sqrt{b}$. Therefore $a=b$
 - We have proven, for any $a, b \in A$, $f(a) = f(b) \rightarrow a = b$
- We then prove f is surjective
 - We want to prove, for any $n \in \mathbb{N}$, there exist $a \in A$, such that $f(a) = n$
 - Suppose $n \in \mathbb{N}$. We choose $a = n^2$. We have $a \in A$ and $f(a) = n$
 - We have proven, for any $n \in \mathbb{N}$, there exist $a \in A$, such that $f(a) = n$

Exercise 2

- Prove that real numbers and $(0,1)$ are of the same size.

Solution

- We only need to prove \mathbb{R} is the same size as $(0, 1)$
- Let $f: \mathbb{R} \rightarrow (0, 1)$ defined as $f(x) = 1/(1+e^{-x})$.
- We show f is bijective....

Exercise 3

- Prove the size of real number interval $[1,2]$ is the same as the size of real numbers $[1,4]$

Solution

- Let $f: [1,2] \rightarrow [1,4]$ be defined as $f(x) = 3x - 2$
- f is injective because ...
- f is surjective because ...

Expected Learning Outcomes

- Integer is countable
- Rational is countable
- Real is uncountable
- Prove simple cases that two sets of same cardinality