CSE215
Foundations of Computer Science

Instructor: Zhoulai Fu

State University of New York, Korea

Agenda

- Prove set properties using
  - set identities
  - element argument

Proof using set identities (Algebraic proof)

Review example

Proposition

- Construct an algebraic proof that for all sets A, B, and C, $(A \cup B) - C = (A - C) \cup (B - C)$

Proof

- $(A \cup B) - C$
  $= (A \cup B) \cap C'$ $\quad$ ($\because$ set difference law)
  $= C' \cap (A \cup B)$ $\quad$ ($\because$ commutative law)
  $= (C' \cap A) \cup (C' \cap B)$ $\quad$ ($\because$ distributive law)
  $= (A \cap C') \cup (B \cap C')$ $\quad$ ($\because$ commutative law)
  $= (A - C) \cup (B - C)$ $\quad$ ($\because$ set difference law)

Prove set properties using element argument

Facts for element arguments

$$A \subseteq B \textit{ if and only if } \forall x, x \in A \rightarrow x \in B$$

Facts for element arguments

Let X and Y be subsets of a universal set U and suppose x and y are elements of U.

- $x \in X \cup Y \Leftrightarrow x \in X$ or $x \in Y$
- $x \in X \cap Y \Leftrightarrow x \in X$ and $x \in Y$
- $x \in X - Y \Leftrightarrow x \in X$ and $x \notin Y$
- $x \in X' \Leftrightarrow x \notin X$
- $(x, y) \in X \times Y \Leftrightarrow x \in X$ and $y \in Y$

Element argument — situation 1

Basic method for proving that a set is a subset of another

- Let sets X and Y be given. To prove that $X \subseteq Y$,
  1. suppose that x is a particular but arbitrarily chosen element of X.
  2. show that x is an element of Y.

Element argument — situation 2

Basic method for proving that two sets are equal

- Let sets X and Y be given. To prove that $X = Y$,
  1. Prove that $X \subseteq Y$.
  2. Prove that $Y \subseteq X$.

Element argument — situation 3

Basic method for proving a set equals the empty set

- To prove that a set X is equal to the empty set ϕ, prove that X has no elements.
- To do this, suppose X has an element and derive a contradiction.

Proof by element argument: Example 1

Proposition

- Prove that for all sets A, B, and C $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof

We need to prove:

1. $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
2. $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

Proof by element argument: Example 1

Proof (continued)

Proof that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Suppose $x \in A \cup (B \cap C)$.

$x \in A$ or $x \in B \cap C$ ($\because$ defn. of union)

- Case 1. $[x \in A.]$

  $x \in A \cup B$ ($\because$ defn. of union)

  $x \in A \cup C$ ($\because$ defn. of union)

  $x \in (A \cup B) \cap (A \cup C)$ ($\because$ defn. of intersection)

- Case 2. $[x \in B \cap C.]$

  $x \in B$ and $x \in C$ ($\because$ defn. of intersection)

  $x \in A \cup B$ ($\because$ defn. of union)

  $x \in A \cup C$ ($\because$ defn. of union)

  $x \in (A \cup B) \cap (A \cup C)$ ($\because$ defn. of intersection)

Proof by element argument: Example 1

Proof (continued)

Proof that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Suppose $x \in (A \cup B) \cap (A \cup C)$.

$x \in A \cup B$ and $x \in A \cup C$ $\quad$ ($\because$ defn. of intersection)

- Case 1. $[x \in A.]$

  $x \in A \cup (B \cap C)$ $\quad$ ($\because$ defn. of union)

- Case 2. $[x \notin A.]$

  $x \in A$ or $x \in B$ $\quad$ ($\because$ defn. of union)

  $x \in B$ $\quad$ ($\because x \notin A$)

  $x \in A$ or $x \in C$ $\quad$ ($\because$ defn. of union)

  $x \in C$ $\quad$ ($\because x \notin A$)

  $x \in B \cap C$ $\quad$ ($\because$ defn. of intersection)

  $x \in A \cup (B \cap C)$ $\quad$ ($\because$ defn. of union)

Proof by element argument: Example 2

Proposition

- Prove that for all sets A and B, $(A \cup B)' = A' \cap B'$.

Proof

We need to prove:
1. $(A \cup B)' \subseteq A' \cap B'$
2. $A' \cap B' \subseteq (A \cup B)'$

Proof by element argument: Example 2

Proof (continued)

- Proof that $(A \cup B)' \subseteq A' \cap B'$.
  Suppose $x \in (A \cup B)'$.
  $x \notin A \cup B$ ($\because$ defn. of complement)
  It is false that (x is in A or x is in B).
  x is not in A and x is not in B ($\because$ De Morgan's law of logic)
  $x \notin A$ and $x \notin B$
  $x \in A'$ and $x \in B'$ ($\because$ defn. of complement)
  $x \in (A' \cap B')$ ($\because$ defn. of intersection)
  Hence, $(A \cup B)' \subseteq A' \cap B'$ ($\because$ defn. of subset)

Proof by element argument: Example 2

Proof (continued)

- Proof that $A' \cap B' \subseteq (A \cup B)'$.
  Suppose $x \in A' \cap B'$.
  $x \in A'$ and $x \in B'$ $\quad$ ($\because$ defn. of intersection)
  $x \notin A$ and $x \notin B$ $\quad$ ($\because$ defn. of complement)
  x is not in A and x is not in B
  It is false that (x is in A or x is in B)
  ($\because$ De Morgan's law of logic)
  $x \notin A \cup B$
  $x \in (A \cup B)'$ $\quad$ ($\because$ defn. of complement)
  Hence, $A' \cap B' \subseteq (A \cup B)'$ $\quad$ ($\because$ defn. of subset)

Proof by element argument: Example 3

Proposition

- For any sets A and B, if $A \subseteq B$, then (a) $A \cap B = A$ and (b) $A \cup B = B$.

Proof

Part (a): We need to prove:

1. $A \cap B \subseteq A$
2. $A \subseteq A \cap B$

Part (b): We need to prove:

1. $A \cup B \subseteq B$
2. $B \subseteq A \cup B$

Proof by element argument: Example 3

Proof (continued)

Part (a).

1. Proof that $A \cap B \subseteq A$.

   $A \cap B \subseteq A \qquad (\because \text{inclusion of intersection})$

2. Proof that $A \subseteq A \cap B$.

   Suppose $x \in A$

   $x \in B \qquad (\because A \subseteq B)$

   $x \in A$ and $x \in B$

   $x \in A \cap B \qquad (\because \text{defn. of intersection})$

Proof by element argument: Example 3

Proof (continued)

Part (b).

1. Proof that $A \cup B \subseteq B$.

   Suppose $x \in A \cup B$

   $x \in A$ or $x \in B$ ($\because$ defn. of union)

   If $x \in A$, then $x \in B$ ($\because A \subseteq B$)

   $x \in B$ ($\because$ Modus Ponens and division into cases)

2. Proof that $B \subseteq A \cup B$.

   $B \subseteq A \cup B$ ($\because$ inclusion in union)

Proof by element argument: Example 4

Proposition

- If E is a set with no elements and A is any set, then $E \subseteq A$.

Proof

Proof by contradiction.

- Suppose there exists a set E with no elements and a set A such that $E \not\subseteq A$.
- $\exists x$ such that $x \in E$ and $x \notin A$ $\quad$ ($\because$ defn. of a subset)
- But there can be no such element since E has no elements.
- Contradiction!
- Hence, if E is a set with no elements and A is any set, then $E \subseteq A$.

Proof by element argument: Example 5

Proposition

- There is only one set with no elements.

Proof

- Suppose E_1 and E_2 are both sets with no elements.
- $E_1 \subseteq E_2 \qquad (\because \text{previous proposition})$
- $E_2 \subseteq E_1 \qquad (\because \text{previous proposition})$
- Thus, $E_1 = E_2$

Proof by element argument: Example 6

Proposition

- Prove that for any set A, $A \cap \phi = \phi$

Proof

Proof by contradiction.

- Suppose there is an element x such that $x \in A \cap \phi$
- $x \in A$ and $x \in \phi$ $\qquad$ ($\because$ defn. of intersection)
- $x \in \phi$
- Impossible because ϕ cannot have any elements
- Hence, the supposition is incorrect.
- So, $A \cap \phi = \phi$

Proof by element argument: Example 7

Proposition

- For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C'$, then $A \cap C = \phi$.

Proof

Proof by contradiction.

- Suppose there is an element x such that $x \in A \cap C$
- $x \in A$ and $x \in C$ $\quad$ ($\because$ defn. of intersection)
- $x \in A$
- $x \in B$ $\quad$ ($\because x \in A$ and $A \subseteq B$)
- $x \in C'$ $\quad$ ($\because x \in B$ and $B \subseteq C'$)
- $x \notin C$ $\quad$ ($\because$ defn. of complement)
- $x \in C$ and $x \notin C$
- Contradiction!
- Hence, the supposition is incorrect.
- So, $A \cap C = \phi$