

CSE215

Foundations of Computer Science

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Plan ahead

- 11.14 Tu: Sets - REC on Sets; no homework review in REC
- 11.16 Review Midterm 2 No homework to be announced
- 11.21 Tu **Midterm 2** - REC on Functions
- 11.23 Function Homework to be announced
- 11.28 Tu Function -
- 11.30 Function Ungraded homework to be announced
- 12.05 Tu Final Review -
- 12.07 Final Review No homework to be announced
- 12.10 Mon Final -

- Clarification about sets 1: Membership and subsets
- Clarification about sets 2: Power sets,
- Clarification about sets 3: infinite union/intersection notation
- Clarification about sets 4: Element argument

Clarification on sets 1: Membership and Subsets

Rules for survival

- $a \in A$ if you can find the exact “a” in A’s elements
- Translate $a \notin A$ as $\sim (a \in A)$
- Translate $B \subseteq A$ to for any $b \in B$, $b \in A$
- Emptset is a subset of anything

Examples

1. $1 \in \{1, \{1\}\}$ 1 is the first element listed in $\{1, \{1\}\}$
2. $1 \notin \{1, \{1\}\}$ because 1 is not a set
3. $\{1\} \in \{1, \{1\}\}$ $\{1\}$ is the second element listed in $\{1, \{1\}\}$

Examples

4. $\{1\} \subseteq \{1, \{1\}\}$ make subset $\{1\}$ by selecting 1 from $\{1, \{1\}\}$
5. $\{\{1\}\} \notin \{1, \{1\}\}$ because $\{1, \{1\}\}$ contains only 1 and $\{1\}$, and not $\{\{1\}\}$
6. $\{\{1\}\} \subseteq \{1, \{1\}\}$ make subset $\{\{1\}\}$ by selecting $\{1\}$ from $\{1, \{1\}\}$

Examples

- 7. $\mathbb{N} \notin \mathbb{N}$ \mathbb{N} is a set (not a number) and \mathbb{N} contains only numbers
- 8. $\mathbb{N} \subseteq \mathbb{N}$ because $X \subseteq X$ for every set X
- 9. $\emptyset \notin \mathbb{N}$ because the set \mathbb{N} contains only numbers and no sets

Examples

- 10. $\emptyset \subseteq \mathbb{N}$ because \emptyset is a subset of every set
- 11. $\mathbb{N} \in \{\mathbb{N}\}$ because $\{\mathbb{N}\}$ has just one element, the set \mathbb{N}
- 12. $\mathbb{N} \not\in \{\mathbb{N}\}$ because, for instance, $1 \in \mathbb{N}$ but $1 \notin \{\mathbb{N}\}$

Examples

- 13. $\emptyset \notin \{\mathbb{N}\}$ note that the only element of $\{\mathbb{N}\}$ is \mathbb{N} , and $\mathbb{N} \neq \emptyset$
- 14. $\emptyset \subseteq \{\mathbb{N}\}$ because \emptyset is a subset of every set
- 15. $\emptyset \in \{\emptyset, \mathbb{N}\}$ \emptyset is the first element listed in $\{\emptyset, \mathbb{N}\}$

Examples

- 16. $\emptyset \subseteq \{\emptyset, \mathbb{N}\}$ because \emptyset is a subset of every set
- 17. $\{\mathbb{N}\} \subseteq \{\emptyset, \mathbb{N}\}$ make subset $\{\mathbb{N}\}$ by selecting \mathbb{N} from $\{\emptyset, \mathbb{N}\}$
- 18. $\{\mathbb{N}\} \not\subseteq \{\emptyset, \{\mathbb{N}\}\}$ because $\mathbb{N} \notin \{\emptyset, \{\mathbb{N}\}\}$
- 19. $\{\mathbb{N}\} \in \{\emptyset, \{\mathbb{N}\}\}$ $\{\mathbb{N}\}$ is the second element listed in $\{\emptyset, \{\mathbb{N}\}\}$

Quiz: True or False

$$\{(1,2), (2,2), (7,1)\} \subseteq \mathbb{N} \times \mathbb{N}$$

Clarification on sets 2:

Powerset

Definition : If A is a set, the **power set** of A is another set, denoted as $\mathcal{P}(A)$ and defined to be the set of all subsets of A . In symbols, $\mathcal{P}(A) = \{X : X \subseteq A\}$.

POWER SETS

- If S is the set $\{a, b, c\}$ then $\{a, c\}$ is a subset of S . There are other subsets of S ; the complete list is as follows:
 - $\{\}$ (the empty set or null set)
 - $\{a\}$
 - $\{b\}$
 - $\{c\}$
 - $\{a, b\}$
 - $\{a, c\}$
 - $\{b, c\}$
 - $\{a, b, c\}$ or S
- So the power set of S , written $P(S)$, is the set containing all the subsets above. Written out this would be the set:
- $P(S) = \{ \{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \}$

Fact

If A is a finite set, then $|\mathcal{P}(A)| = 2^{|A|}$.

$|A|$ refers to # of elements in a set A

Rules for survival

- Know how to get the powerset of N elements, for $N \geq 0$.
- `\emptyset` is always an element of the powerset
- The set itself is also always an element of powerset

Examples

1. $\mathcal{P}(\{0,1,3\}) = \{ \emptyset, \{0\}, \{1\}, \{3\}, \{0,1\}, \{0,3\}, \{1,3\}, \{0,1,3\} \}$
2. $\mathcal{P}(\{1,2\}) = \{ \emptyset, \{1\}, \{2\}, \{1,2\} \}$
3. $\mathcal{P}(\{1\}) = \{ \emptyset, \{1\} \}$

Examples

4. $\mathcal{P}(\emptyset) = \{ \emptyset \}$

5. $\mathcal{P}(\{a\}) = \{ \emptyset, \{a\} \}$

6. $\mathcal{P}(\{\emptyset\}) = \{ \emptyset, \{\emptyset\} \}$

Examples

- 7. $\mathcal{P}(\{a\}) \times \mathcal{P}(\{\emptyset\}) = \{(\emptyset, \emptyset), (\emptyset, \{\emptyset\}), (\{a\}, \emptyset), (\{a\}, \{\emptyset\})\}$
- 8. $\mathcal{P}(\mathcal{P}(\{\emptyset\})) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
- 9. $\mathcal{P}(\{1, \{1, 2\}\}) = \{\emptyset, \{1\}, \{\{1, 2\}\}, \{1, \{1, 2\}\}\}$
- 10. $\mathcal{P}(\{\mathbb{Z}, \mathbb{N}\}) = \{\emptyset, \{\mathbb{Z}\}, \{\mathbb{N}\}, \{\mathbb{Z}, \mathbb{N}\}\}$

The following are wrong .

Explain why

11. $\mathcal{P}(1) = \{ \emptyset, \{1\} \}$
12. $\mathcal{P}(\{1, \{1, 2\}\}) = \{ \emptyset, \{1\}, \{1, 2\}, \{1, \{1, 2\}\} \}$
13. $\mathcal{P}(\{1, \{1, 2\}\}) = \{ \emptyset, \{\{1\}\}, \{\{1, 2\}\}, \{ \emptyset, \{1, 2\} \} \}$

Solution

meaningless because 1 is not a set

....wrong because $\{1,2\} \not\subseteq \{1,\{1,2\}\}$

... wrong because $\{\{1\}\} \not\subseteq \{1,\{1,2\}\}$

Clarification on sets 3: Infinite union/intersection notation

Notations

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots = \{x : x \in A_i \text{ for at least one set } A_i \text{ with } 1 \leq i\}.$$

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots = \{x : x \in A_i \text{ for every set } A_i \text{ with } 1 \leq i\}.$$

Example

$$A_1 = \{-1, 0, 1\}, \quad A_2 = \{-2, 0, 2\}, \quad A_3 = \{-3, 0, 3\}, \quad \dots, \quad A_i = \{-i, 0, i\}, \quad \dots$$

Observe that $\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}$, and $\bigcap_{i=1}^{\infty} A_i = \{0\}$.

Extended Notation

Here is a useful twist on our new notation. We can write

$$\bigcup_{i=1}^3 A_i = \bigcup_{i \in \{1,2,3\}} A_i,$$

as this takes the union of the sets A_i for $i = 1, 2, 3$. Likewise:

$$\bigcap_{i=1}^3 A_i = \bigcap_{i \in \{1,2,3\}} A_i$$

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i \in \mathbb{N}} A_i$$

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i \in \mathbb{N}} A_i$$

Extended Notations

Here we are taking the union or intersection of a collection of sets A_i where i is an element of some set, be it $\{1,2,3\}$ or \mathbb{N} . In general, the way this works is that we will have a collection of sets A_i for $i \in I$, where I is the set of possible subscripts. The set I is called an **index set**.

If A_α is a set for every α in some index set I , then

$$\bigcup_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for at least one set } A_\alpha \text{ with } \alpha \in I\}$$
$$\bigcap_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for every set } A_\alpha \text{ with } \alpha \in I\}.$$

Exercise 1

(a) $\bigcup_{i \in \mathbb{N}} [0, i + 1] =$

(b) $\bigcap_{i \in \mathbb{N}} [0, i + 1] =$

Solution

- $[0, \text{infinity})$
- $[0, 2]$

Exercise 2

(a) $\bigcup_{i \in \mathbb{N}} \mathbb{R} \times [i, i+1] =$

(b) $\bigcap_{i \in \mathbb{N}} \mathbb{R} \times [i, i+1] =$

Solution

- $\mathbb{R}^* [1, \text{infinity})$
- `\emptyset`

Exercise 3

(a) $\bigcup_{\alpha \in \mathbb{R}} \{\alpha\} \times [0, 1] =$

(b) $\bigcap_{\alpha \in \mathbb{R}} \{\alpha\} \times [0, 1] =$

Solution

- $\mathbb{R}^* [0,1]$
- `\emptyset`

**Writing element arguments more
efficiently for proving some proof
properties**

Example 1

- Prove De Morgan law: Complement of A intersection B = Complement of A union Complement of B

$$\begin{aligned}(A \cap B)^c &= \{x | x \notin A \cap B\} && \text{(Def. set complement)} \\ &= \{x | \sim (x \in A \cap B)\} && \text{(Def. of } \notin \text{)} \\ &= \{x | \sim (x \in A \wedge x \in B)\} && \text{(Def. } \cap \text{)} \\ &= \{x | x \notin A \vee x \notin B\} && \text{(De Morgan in logic)} \\ &= \{x | x \in A^c \vee x \in B^c\} && \text{(Def. set complement)} \\ &= A^c \cup B^c && \text{(Def. } \cup \text{)}\end{aligned}$$

Example 2

Given sets A , B , and C , prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

- Proof

$$\begin{aligned} A \times (B \cap C) &= \{(x, y) : (x \in A) \wedge (y \in B \cap C)\} && \text{(def. of } \times) \\ &= \{(x, y) : (x \in A) \wedge (y \in B) \wedge (y \in C)\} && \text{(def. of } \cap) \\ &= \{(x, y) : (x \in A) \wedge (x \in A) \wedge (y \in B) \wedge (y \in C)\} && (P = P \wedge P) \\ &= \{(x, y) : ((x \in A) \wedge (y \in B)) \wedge ((x \in A) \wedge (y \in C))\} && \text{(rearrange)} \\ &= \{(x, y) : (x \in A) \wedge (y \in B)\} \cap \{(x, y) : (x \in A) \wedge (y \in C)\} && \text{(def. of } \cap) \\ &= (A \times B) \cap (A \times C) && \text{(def. of } \times) \end{aligned}$$

- QED.

Exercise

- Prove De Morgan law: Complement of $A \cup B$ = Complement of A intersection Complement of B

Example (This one has to be proved by the old-style element argument)

- Let Z be the set of integers
- Let A be the set of $\{7a + 8b \mid a \in Z \text{ and } b \in Z\}$
- Prove $Z = A$

Solution

- Proof.
 - A is clearly a subset of Z
 - So we only need to prove Z is a subset of A
 - Suppose n is arbitrary element of Z, n can be written as $7(-n) + 8(n)$, Therefore Z is a subset of A
- QED.

Exercise (This one has to be proved by the old-style element argument)

- Let Z be the set of integers
- Let A be the set of $\{7a + 3b \mid a \in Z \text{ and } b \in Z\}$
- Prove $Z = A$