CSE215
Foundations of Computer Science

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Plan ahead

- 11.14 Tu: Sets - REC on Sets; no homework review in REC
- 11.16 Review Midterm 2 No homework to be announced
- 11.21 Tu **Midterm 2** - REC on Functions
- 11.23 Function Homework to be announced
- 11.28 Tu Function -
- 11.30 Function Ungraded homework to be announced
- 12.05 Tu Final Review -
- 12.07 Final Review No homework to be announced
- 12.10 Mon Final -

- Clarification about sets 1: Membership and subsets
- Clarification about sets 2: Power sets
- Clarification about sets 3: infinite union/intersection notation
- Clarification about sets 4: Element argument

Clarification on sets 1: Membership and Subsets

Basic Rules

- a $\in$ A if you can find the exact “a” in A’s elements
- Translate $a \notin A$ as ~ (a $\in$ A)
- Translate B $\subseteq$ A to for any b, b$\in$ B -> b$\in$ A
- Emptyset is a subset of any set

Examples

1. $1 \in \{1, \{1\}\}$ .... 1 is the first element listed in $\{1, \{1\}\}$
2. $1 \not\subseteq \{1, \{1\}\}$ .... because 1 is not a set
3. $\{1\} \in \{1, \{1\}\}$ .... $\{1\}$ is the second element listed in $\{1, \{1\}\}$

Examples

4. $\{1\} \subseteq \{1,\{1\}\}$ .... make subset $\{1\}$ by selecting 1 from $\{1,\{1\}\}$
5. $\{\{1\}\} \notin \{1,\{1\}\}$ .... because $\{1,\{1\}\}$ contains only 1 and $\{1\}$, and not $\{\{1\}\}$
6. $\{\{1\}\} \subseteq \{1,\{1\}\}$ .... make subset $\{\{1\}\}$ by selecting $\{1\}$ from $\{1,\{1\}\}$

Examples

7. $\mathbb{N} \notin \mathbb{N}$ .... $\mathbb{N}$ is a set (not a number) and $\mathbb{N}$ contains only numbers
8. $\mathbb{N} \subseteq \mathbb{N}$ .... because $X \subseteq X$ for every set X
9. $\emptyset \notin \mathbb{N}$ .... because the set $\mathbb{N}$ contains only numbers and no sets

Examples

10. $\emptyset \subseteq \mathbb{N}$ .... **because $\emptyset$ is a subset of every set**
11. $\mathbb{N} \in \{\mathbb{N}\}$ .... **because $\{\mathbb{N}\}$ has just one element, the set $\mathbb{N}$**
12. $\mathbb{N} \not\subseteq \{\mathbb{N}\}$ .... **because, for instance, $1 \in \mathbb{N}$ but $1 \notin \{\mathbb{N}\}$**

Examples

13. $\emptyset \notin \{\mathbb{N}\}$ .... note that the only element of $\{\mathbb{N}\}$ is $\mathbb{N}$, and $\mathbb{N} \neq \emptyset$
14. $\emptyset \subseteq \{\mathbb{N}\}$ .... because $\emptyset$ is a subset of every set
15. $\emptyset \in \{\emptyset, \mathbb{N}\}$ .... $\emptyset$ is the first element listed in $\{\emptyset, \mathbb{N}\}$

Examples

16. $\varnothing \subseteq \{\varnothing, \mathbb{N}\}$ .... because $\varnothing$ is a subset of every set
17. $\{\mathbb{N}\} \subseteq \{\varnothing, \mathbb{N}\}$ .... make subset $\{\mathbb{N}\}$ by selecting $\mathbb{N}$ from $\{\varnothing, \mathbb{N}\}$
18. $\{\mathbb{N}\} \not\subseteq \{\varnothing, \{\mathbb{N}\}\}$ .... because $\mathbb{N} \notin \{\varnothing, \{\mathbb{N}\}\}$
19. $\{\mathbb{N}\} \in \{\varnothing, \{\mathbb{N}\}\}$ .... $\{\mathbb{N}\}$ is the second element listed in $\{\varnothing, \{\mathbb{N}\}\}$

Quiz: True or False

$$\{(1,2),(2,2),(7,1)\} \subseteq \mathbb{N} \times \mathbb{N}$$

Clarification on sets 2: Powerset

Definition : If A is a set, the **power set** of A is another set, denoted as $\mathscr{P}(A)$ and defined to be the set of all subsets of A. In symbols, $\mathscr{P}(A) = \{X : X \subseteq A\}$.

POWER SETS

- If *S* is the set {a, b, c} then {a,c} is a subset of *S*. There are other subsets of *S*; the complete list is as follows:
  - {} (the empty set or null set)
  - {a}
  - {b}
  - {c}
  - {a, b}
  - {a, c}
  - {b, c}
  - {a, b, c} or S
- So the power set of *S*, written *P*(*S*), is the set containing all the subsets above. Written out this would be the set:
- *P*(*S*) = { {}, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c} }

Fact

If A is a finite set, then $|\mathscr{P}(A)| = 2^{|A|}$.

|A| refers to # of elements in a set A

Rules for survival

- Know how to get the powerset of N elements, for N>=0.
- \emptyset is always an element of the powerset
- The set itself is also always an element of powerset

Examples

1. $\mathscr{P}(\{0,1,3\}) = \{\, \varnothing,\ \{0\},\ \{1\},\ \{3\},\ \{0,1\},\ \{0,3\},\ \{1,3\},\ \{0,1,3\}\,\}$
2. $\mathscr{P}(\{1,2\}) = \{\, \varnothing,\ \{1\},\ \{2\},\ \{1,2\}\,\}$
3. $\mathscr{P}(\{1\}) = \{\, \varnothing,\ \{1\}\,\}$

Examples

4. $\mathscr{P}(\varnothing) = \{\varnothing\}$
5. $\mathscr{P}(\{a\}) = \{\varnothing, \{a\}\}$
6. $\mathscr{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$

Examples

7. $\mathscr{P}(\{a\}) \times \mathscr{P}(\{\emptyset\}) = \{ (\emptyset,\emptyset),\ (\emptyset,\{\emptyset\}),\ (\{a\},\emptyset),\ (\{a\},\{\emptyset\})\ \}$
8. $\mathscr{P}(\mathscr{P}(\{\emptyset\})) = \{ \emptyset,\ \{\emptyset\},\ \{\{\emptyset\}\},\ \{\emptyset,\{\emptyset\}\} \}$
9. $\mathscr{P}(\{1,\{1,2\}\}) = \{ \emptyset,\ \{1\},\ \{\{1,2\}\},\ \{1,\{1,2\}\} \}$
10. $\mathscr{P}(\{\mathbb{Z},\mathbb{N}\}) = \{ \emptyset,\ \{\mathbb{Z}\},\ \{\mathbb{N}\},\ \{\mathbb{Z},\mathbb{N}\} \}$

The following are wrong . Explain why

11. $\mathscr{P}(1) = \{\, \emptyset,\ \{1\}\,\}$ ....

12. $\mathscr{P}\left(\{1,\{1,2\}\}\right) = \{\emptyset,\{1\},\{1,2\},\{1,\{1,2\}\}\}$ ....

13. $\mathscr{P}\left(\{1,\{1,2\}\}\right) = \{\emptyset,\{\{1\}\},\{\{1,2\}\},\{\emptyset,\{1,2\}\}\}$

Solution

meaningless because 1 is not a set

$\ldots\ldots$wrong because $\{1,2\} \not\subseteq \{1,\{1,2\}\}$

$\ldots$ wrong because $\{\{1\}\} \not\subseteq \{1,\{1,2\}\}$

Clarification on sets 3:
Infinite union/intersection notation

Notations

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots = \{x : x \in A_i \text{ for at least one set } A_i \text{ with } 1 \leq i\}.$$

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots = \{x : x \in A_i \text{ for every set } A_i \text{ with } 1 \leq i\}.$$

Example

$$A_1 = \{-1,0,1\}, \quad A_2 = \{-2,0,2\}, \quad A_3 = \{-3,0,3\}, \quad \cdots, \quad A_i = \{-i,0,i\}, \quad \cdots$$

Observe that $\displaystyle\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}$, and $\displaystyle\bigcap_{i=1}^{\infty} A_i = \{0\}$.

Extended Notation

Here is a useful twist on our new notation. We can write

$$\bigcup_{i=1}^{3} A_i = \bigcup_{i\in\{1,2,3\}} A_i,$$

as this takes the union of the sets A_i for $i = 1, 2, 3$. Likewise:

$$\bigcap_{i=1}^{3} A_i = \bigcap_{i\in\{1,2,3\}} A_i$$

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i\in\mathbb{N}} A_i$$

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i\in\mathbb{N}} A_i$$

Extended Notations

Here we are taking the union or intersection of a collection of sets A_i where i is an element of some set, be it $\{1,2,3\}$ or $\mathbb{N}$. In general, the way this works is that we will have a collection of sets A_i for $i \in I$, where I is the set of possible subscripts. The set I is called an **index set**.

If A_α is a set for every α in some index set I, then

$$\begin{aligned}
\bigcup_{\alpha \in I} A_\alpha &= \{x : x \in A_\alpha \text{ for at least one set } A_\alpha \text{ with } \alpha \in I\} \\
\bigcap_{\alpha \in I} A_\alpha &= \{x : x \in A_\alpha \text{ for every set } A_\alpha \text{ with } \alpha \in I\}.
\end{aligned}$$

Exercise 1

(a) $\bigcup_{i\in\mathbb{N}}[0, i+1] =$

(b) $\bigcap_{i\in\mathbb{N}}[0, i+1] =$

Solution

- [0, infinity)
- [0, 2]

Exercise 2

(a) $\bigcup_{i \in \mathbb{N}} \mathbb{R} \times [i, i+1] =$

(b) $\bigcap_{i \in \mathbb{N}} \mathbb{R} \times [i, i+1] =$

Solution

- R * [1, infinity)
- \emptyset

Exercise 3

(a) $\displaystyle\bigcup_{\alpha\in\mathbb{R}} \{\alpha\} \times [0,1] =$

(b) $\displaystyle\bigcap_{\alpha\in\mathbb{R}} \{\alpha\} \times [0,1] =$

Solution

- R * [0,1]
- \emptyset

Writing element arguments more efficiently for proving some proof properties

Example 1

- Prove De morgen law: Complement of A intersection B = Complement of A union Complement of B

$$
\begin{aligned}
(A \cap B)^c &= \{x | x \notin A \cap B\} && \text{(Def. set complement)} \\
&= \{x | \sim (x \in A \cap B)\} && \text{(Def. of } \notin) \\
&= \{x | \sim (x \in A \wedge x \in B)\} && \text{(Def. } \cap) \\
&= \{x | x \notin A \vee x \notin B)\} && \text{(De Morgen in logic)} \\
&= \{x | x \in A^c \vee x \in B^c)\} && \text{(Def. set complement)} \\
&= A^c \cup B^c && \text{(Def. } \cup)
\end{aligned}
$$

Example 2

Given sets A, B, and C, prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

- Proof

$$
\begin{aligned}
A \times (B \cap C) &= \{(x,y) : (x \in A) \wedge (y \in B \cap C)\} && \text{(def. of } \times\text{)} \\
&= \{(x,y) : (x \in A) \wedge (y \in B) \wedge (y \in C)\} && \text{(def. of } \cap\text{)} \\
&= \{(x,y) : (x \in A) \wedge (x \in A) \wedge (y \in B) \wedge (y \in C)\} && (P = P \wedge P) \\
&= \{(x,y) : ((x \in A) \wedge (y \in B)) \wedge ((x \in A) \wedge (y \in C))\} && \text{(rearrange)} \\
&= \{(x,y) : (x \in A) \wedge (y \in B)\} \cap \{(x,y) : (x \in A) \wedge (y \in C)\} && \text{(def. of } \cap\text{)} \\
&= (A \times B) \cap (A \times C) && \text{(def. of } \times\text{)}
\end{aligned}
$$

- QED.

Exercise

- Prove De morgen law: Complement of A union B = Complement of A intersection Complement of B

Example (This one has to be proved by the old-style element argument)

- Let Z be the set of integers
- Let A be the set of {7a + 8b | a \in Z and b \in Z}
- Prove Z = A

Solution

- Proof.
    - A is clearly a subset of Z
    - So we only need to prove Z is a subset of A
    - Suppose n is arbitrary element of Z, n can be written as 7 (-n) + 8 (n), Therefore Z is a subset of A
- QED.

Exercise (This one has to be proved by the old-style element argument)

- Let Z be the set of integers
- Let A be the set of {7a + 3b | a \in Z and b \in Z}
- Prove Z = A