CSE215
Foundations of Computer Science

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Today

- Revision on proof of dividing into cases
- Disproof

Prove the following statement

If $n \in \mathbb{Z}$, then $n^2 + 3n + 4$ is even.

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Proof. Suppose $n \in \mathbb{Z}$. We consider two cases.
Case 1. Suppose n is even. Then $n = 2a$ for some $a \in \mathbb{Z}$.
Therefore $n^2 + 3n + 4 = (2a)^2 + 3(2a) + 4 = 4a^2 + 6a + 4 = 2(2a^2 + 3a + 2)$.
So $n^2 + 3n + 4 = 2b$ where $b = 2a^2 + 3a + 2 \in \mathbb{Z}$, so $n^2 + 3n + 4$ is even.
Case 2. Suppose n is odd. Then $n = 2a + 1$ for some $a \in \mathbb{Z}$.
Therefore $n^2 + 3n + 4 = (2a+1)^2 + 3(2a+1) + 4 = 4a^2 + 4a + 1 + 6a + 3 + 4 = 4a^2 + 10a + 8$
$= 2(2a^2 + 5a + 4)$. So $n^2 + 3n + 4 = 2b$ where $b = 2a^2 + 5a + 4 \in \mathbb{Z}$, so $n^2 + 3n + 4$ is even.

In either case $n^2 + 3n + 4$ is even. ■

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Proof. Suppose $x^2 + 5y = y^2 + 5x$.
Then $x^2 - y^2 = 5x - 5y$, and factoring gives $(x-y)(x+y) = 5(x-y)$.
Now consider two cases.
Case 1. If $x - y \neq 0$ we can divide both sides of $(x-y)(x+y) = 5(x-y)$ by the non-zero quantity $x - y$ to get $x + y = 5$.
Case 2. If $x - y = 0$, then $x = y$. (By adding y to both sides.)
Thus $x = y$ or $x + y = 5$. ■

Disproof

dis·proof | dis'prōōf |

noun

a set of facts that prove that something is untrue: *the theory also provides a disproof of the principle of closure*.

- the action of proving that something is untrue: *considerations that are subject to scientific verification or disproof*.

Principle of disproof

- Suppose you want to disprove a statement P. In other words you want to prove that P is false.
- The way to do this is to prove that $\sim$ P is true, for if $\sim$ P is true, it follows immediately that P has to be false.

How to disprove P**:** Prove $\sim P$.

Disproving for-all

How to disprove $\forall x \in S, P(x)$.

Produce an example of an $x \in S$ that makes $P(x)$ false.

Disproving there-exists

- How to disprove a existential statement $\exists x \in S, P(x)$?
- To disprove it, we prove its negation $\sim(\exists x \in S, P(x)) = \forall x \in S, \sim P(x)$.

Disproving universal condition

How to disprove $P(x) \Rightarrow Q(x)$.

Produce an example of an x that makes $P(x)$ true and $Q(x)$ false.

Example: Prove or disprove the following conjecture

Conjecture: For every $n \in \mathbb{Z}$, the integer $f(n) = n^2 - n + 11$ is prime.

n	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
$f(n)$	23	17	13	11	11	13	17	23	31	41	53	67	83	101

Disproof. The statement "*For every $n \in \mathbb{Z}$, the integer $f(n) = n^2 - n + 11$ is prime,*" is **false**. For a counterexample, note that for $n = 11$, the integer $f(11) = 121 = 11 \cdot 11$ is not prime. ■

Disproving by "proof by contradiction"

How to disprove P with contradiction:

Assume P is true, and deduce a contradiction.

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Conjecture: There is a real number x for which $x^4 < x < x^2$.

Disproof. Suppose for the sake of contradiction that this conjecture is true. Let x be a real number for which $x^4 < x < x^2$. Then x is positive, since it is greater than the non-negative number x^4. Dividing all parts of $x^4 < x < x^2$ by the positive number x produces $x^3 < 1 < x$. Now subtract 1 from all parts of $x^3 < 1 < x$ to obtain $x^3 - 1 < 0 < x - 1$ and reason as follows:

$$\begin{array}{rcccl} x^3 - 1 & < & 0 & < & x - 1 \\ (x-1)(x^2+x+1) & < & 0 & < & (x-1) \\ x^2+x+1 & < & 0 & < & 1 \end{array}$$

Now we have $x^2 + x + 1 < 0$, which is a contradiction because x is positive. Thus the conjecture must be false. ■

Prove or disprove

- If $x,y \in R$, then $|x+y| = |x| + |y|$.
- For every natural number n, the integer $2n^2 - 4n + 31$ is prime.
- If $a,b \in N$, then $a + b < ab$
- Every odd integer is the sum of three odd integers.
- Rational + Irrational = Irrational
- Rational * Irrational = Irrational

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Exercises

Prove or disprove

- If $x,y \in R$, then $|x+y| = |x| + |y|$.

Prove or disprove

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