CSE215
Foundations of Computer Science

State University of New York, Korea

Agenda

- Functions
- One-on-one functions
- Onto functions

Level test

Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. Define a function f from X to Y by the arrow diagram in Figure 7.1.3.

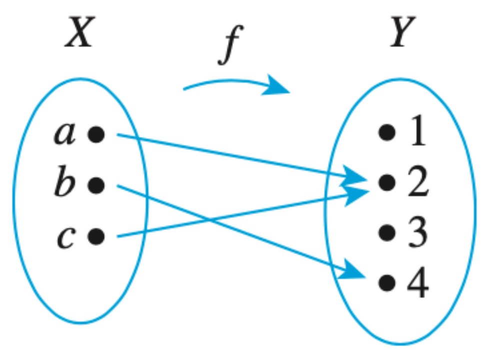


Figure 7.1.1

a. Write the domain and co-domain of f.

b. Find $f(a)$, $f(b)$, and $f(c)$.

c. What is the range of f?

d. Is c an inverse image of 2? Is b an inverse image of 3?

e. Find the inverse images of 2, 4, and 1.

f. Represent f as a set of ordered pairs.

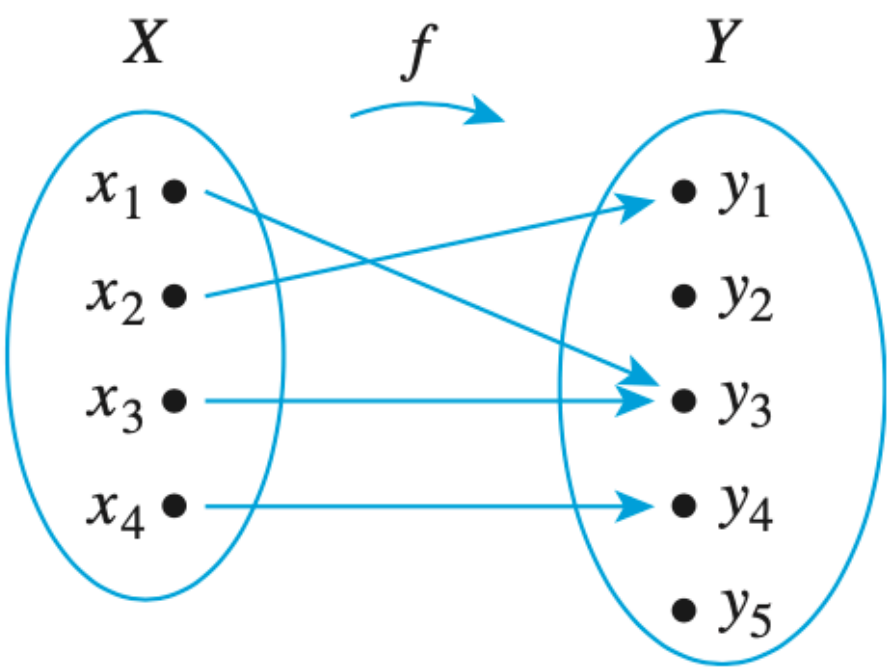
g. Is function f injective?

h. Is function f surjective?

Functions are omnipresent in CS applications

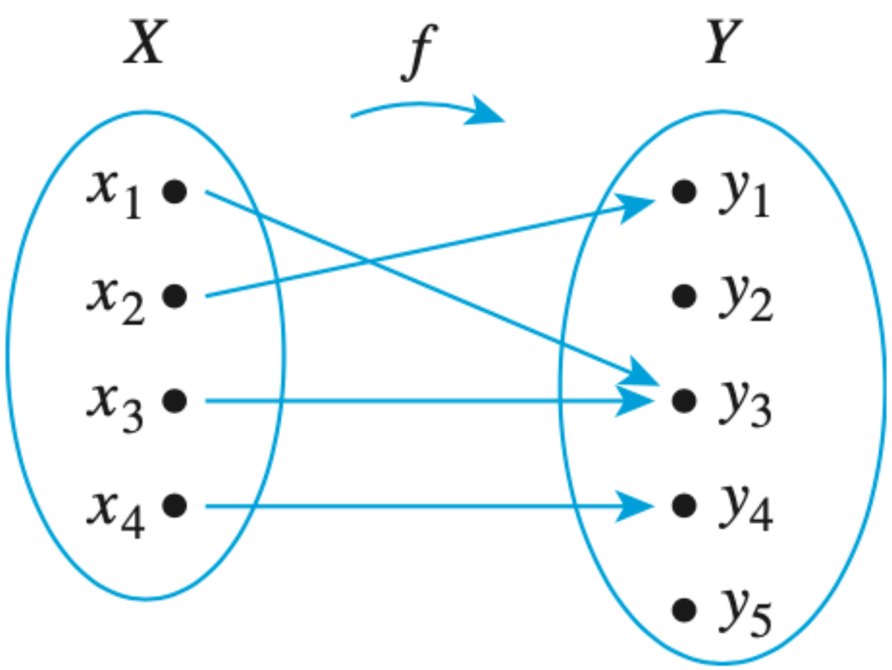
- Spam detector
- Housing price prediction
- Speech-to-text converter

Concepts for a function f: X->Y



- Domain = X
- Co-domain = Y
- Range = Image of X = {f(x) | x \in X}
- Inverse image of an y = {x \in X| f(x) = y}
- Written as {(x1,y3), (x2, y1), …}

What about spam detection?



- Domain
- Co-domain
- Range
- Inverse image of an y
- Written as {(x1,y3), (x2, y1), …}

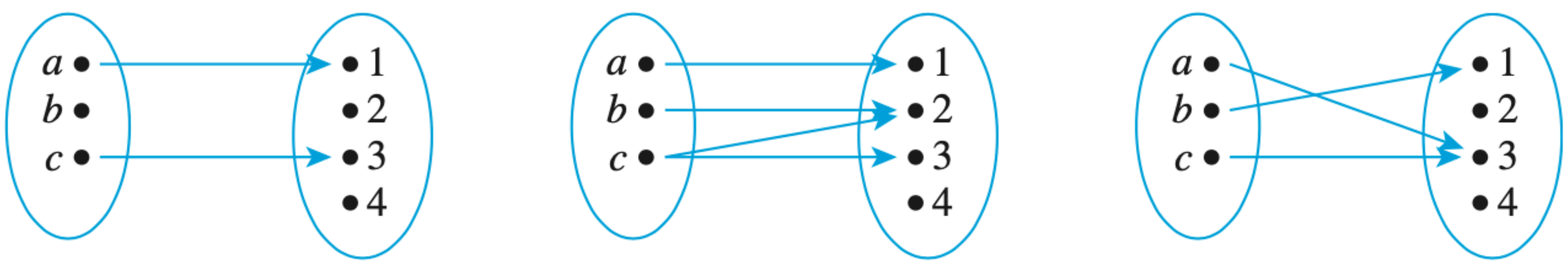
Spam detection

- End-product is a function, or called model of type
    - Emails-> {true, false}
- Machine learning process is a function of type
    - Dataset -> Functions of type (Emails ->{true, false})

Requirement to be a function

- **Each** element in X maps to a **single element** in Y

Exercise2:
Functions or non-functions



(a) (b) (c)

One-to-one functions

Definition

- A function $F : X \to Y$ is one-to-one (or injective) if and only if for all elements x_1 and x_2 in X,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2, \text{ or}$$
$$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

- A function $F : X \to Y$ is one-to-one $\Leftrightarrow$
  $\forall x_1, x_2 \in X$, if $F(x_1) = F(x_2)$ then $x_1 = x_2$.
  A function $F : X \to Y$ is not one-to-one $\Leftrightarrow$
  $\exists x_1, x_2 \in X, \quad F(x_1) = F(x_2)$ and $x_1 \neq x_2$.

One-to-one functions: Proof technique

Problem

- Prove that a function f is one-to-one.

Proof

Direct proof.

- Suppose x_1 and x_2 are elements of X such that $f(x_1) = f(x_2)$.
- Show that $x_1 = x_2$.

Problem

- Prove that a function f is not one-to-one.

Proof

Counterexample.

- Find elements x_1 and x_2 in X so that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

One-to-one functions: Example 1

Problem

- Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$. Is f one-to-one? Prove or give a counterexample.

One-to-one functions: Example 1

Problem

- Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$. Is f one-to-one? Prove or give a counterexample.

Proof

Direct proof.

- Suppose x_1 and x_2 are elements of X such that $f(x_1) = f(x_2)$.
  $\implies 4x_1 - 1 = 4x_2 - 1 \qquad (\because \text{Defn. of } f)$
  $\implies 4x_1 = 4x_2 \qquad (\because \text{Add 1 on both sides})$
  $\implies x_1 = x_2 \qquad (\because \text{Divide by 4 on both sides})$
- Hence, f is one-to-one.

One-to-one functions: Example 2

Problem

- Define $g : \mathbb{Z} \to \mathbb{Z}$ by the rule $g(n) = n^2$ for all $n \in \mathbb{Z}$. Is g one-to-one? Prove or give a counterexample.

One-to-one functions: Example 2

Problem

- Define $g : \mathbb{Z} \to \mathbb{Z}$ by the rule $g(n) = n^2$ for all $n \in \mathbb{Z}$. Is g one-to-one? Prove or give a counterexample.

Proof

Counterexample.

- Let $n_1 = -1$ and $n_2 = 1$.
  $\implies g(n_1) = (-1)^2 = 1$ and $g(n_2) = 1^2 = 1$
  $\implies g(n_1) = g(n_2)$ but, $n_1 \neq n_2$
- Hence, g is not one-to-one.

Onto functions

Definition

- A function $F : X \to Y$ is onto (or surjective) if and only if given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.
- A function $F : X \to Y$ is onto $\Leftrightarrow$
  $\forall y \in Y, \exists x \in X$ such that $F(x) = y$.
  A function $F : X \to Y$ is not onto $\Leftrightarrow$
  $\exists y \in Y, \forall x \in X$ such that $F(x) \neq y$.

Onto functions: Proof technique

Problem

- Prove that a function f is onto.

Proof

Direct proof.

- Suppose that y is any element of Y
- Show that there is an element x of X with $F(x) = y$

Problem

- Prove that a function f is not onto.

Proof

Counterexample.

- Find an element y of Y such that $y \neq F(x)$ for any x in X.

Onto functions: Example 1

Problem

- Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$. Is f onto? Prove or give a counterexample.

Proof

Direct proof.

- Let $y \in \mathbb{R}$. We need to show that $\exists x$ such that $f(x) = y$.
  Let $x = \frac{y+1}{4}$. Then
  $f\left(\frac{y+1}{4}\right) = 4\left(\frac{y+1}{4}\right) - 1 \qquad (\because \text{Defn. of } f)$
  $= y \qquad (\because \text{Simplify})$
- Hence, f is onto.

Onto functions: Example 1

Problem

- Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$. Is f onto? Prove or give a counterexample.

Onto functions: Example 2

Problem

- Define $g : \mathbb{Z} \to \mathbb{Z}$ by the rule $g(n) = 4n - 1$ for all $n \in \mathbb{Z}$. Is g onto? Prove or give a counterexample.

Proof

Counterexample.

- We know that $0 \in \mathbb{Z}$.
- Let $g(n) = 0$ for some integer n.
  $\implies 4n - 1 = 0 \qquad (\because \text{Defn. of } g)$
  $\implies n = \frac{1}{4} \qquad (\because \text{Simplify})$
  But $\frac{1}{4} \notin \mathbb{Z}$.
  So, $g(n) \neq 0$ for any integer n.
- Hence, g is not onto.

Exercises

Consider four functions defined below:

1. The function $f_1 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by the as

$$f_1(m,n) = (m+n, m+2n).$$

2. The function $f_2 : \mathbb{R} - \{0\} \to \mathbb{R}$ defined as

$$f_2(x) = \frac{1}{x} + 1.$$

Fill in the following table with "T" (for true) or "F" (for false). Be careful about the domain and co-domain of each function.

Functions	One-to-one?	Onto?
f_1		
f_2		
f_3		
f_4		

3. The function $f_3 : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f_3(x, y) = (xy, x^3).$$

4. The function $f_4 : \mathbb{R} \to \mathbb{R}$ defined as

$$f_4(n) = 2n + 1.$$

Fill in the following table with "T" (for true) or "F" (for false). Be careful about the domain and co-domain of each function.

Functions	One-to-one?	Onto?
f_1		
f_2		
f_3		
f_4		

Review on injective, surjective, bijective functions

Determine Injectivity

- Def: $x \neq y \rightarrow f(x) \neq f(y)$
- Equivalently: $f(x) = f(y) \rightarrow x = y$
- What is a non-injective function?

Determine surjectivity

- Def: for any y, there exists x such that f(x) = y -> f(x) ≠ f(y)
- What is a non-surjective function?

Quiz

- Determine Injectivity and subjectivity for:
  - sin : R → R
  - sin: R → [-1,1]

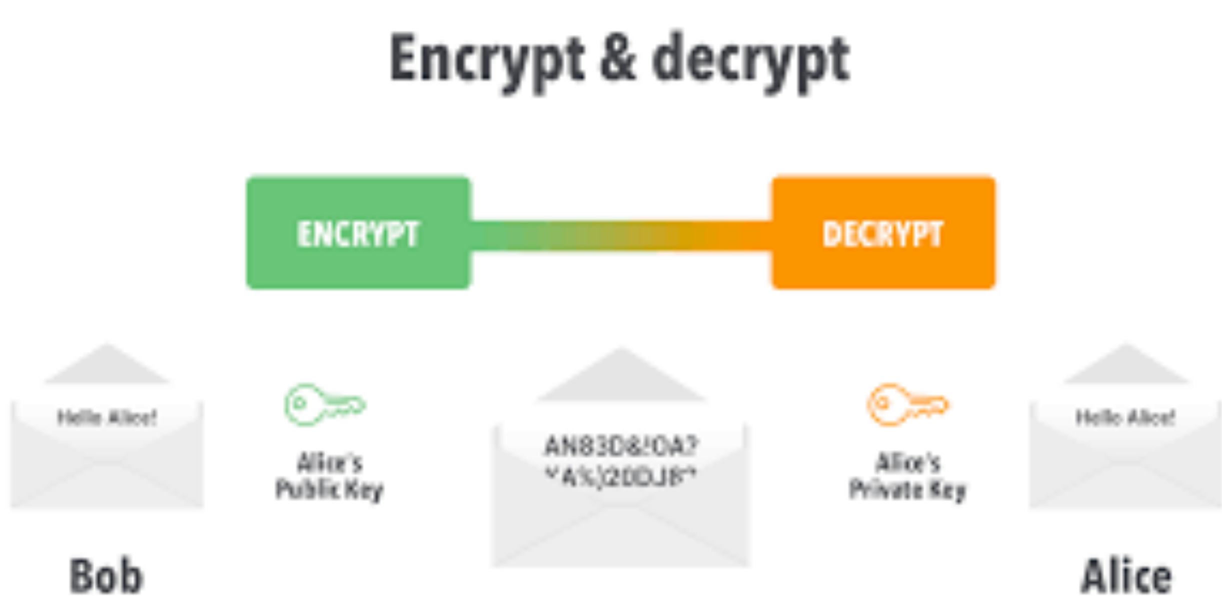
Inverse functions

Inverse functions

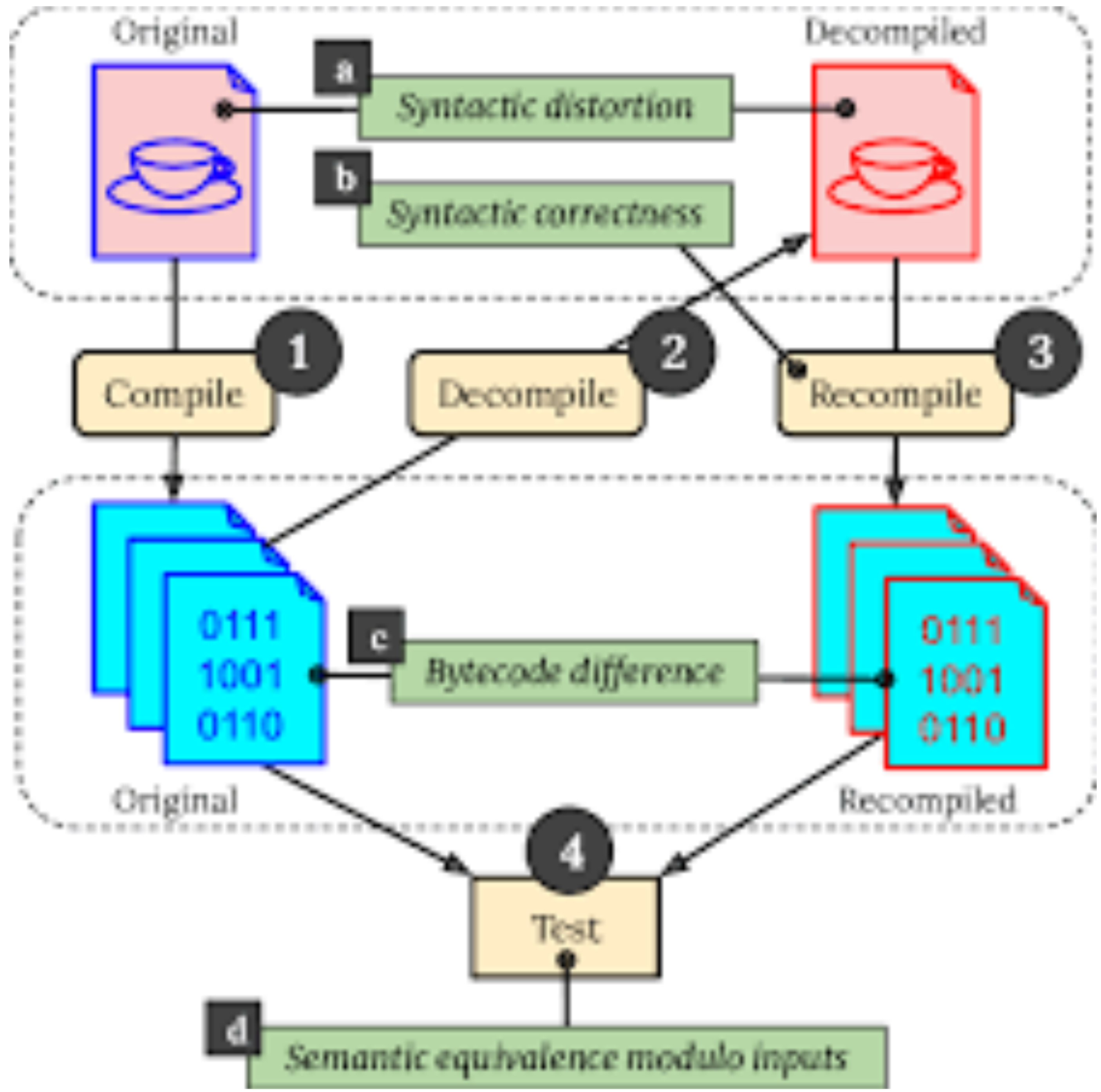
Definition

- Suppose $F : X \to Y$ is a one-to-one correspondence.
  Then, the inverse function $F^{-1} : Y \to X$ is defined as follows:
  Given any element y in Y,
  $F^{-1}(y) =$ that unique element x in X such that $F(x) = y$.

Does encryption have an inverse function?



Does Java compilation have an inverse function?



Example

Problem

- Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$. Find its inverse function.

Problem

- Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$. Find its inverse function.

Proof

For any y in R, by definition of f^{-1}

- f^{-1} = unique number x such that $f(x) = y$

  Consider $f(x) = y$

  $\implies 4x - 1 = y \qquad (\because \text{Defn. of } f)$

  $\implies x = \frac{y+1}{4} \qquad (\because \text{Simplify})$
- Hence, $f^{-1}(y) = \frac{y+1}{4}$ is the inverse function.

Quiz

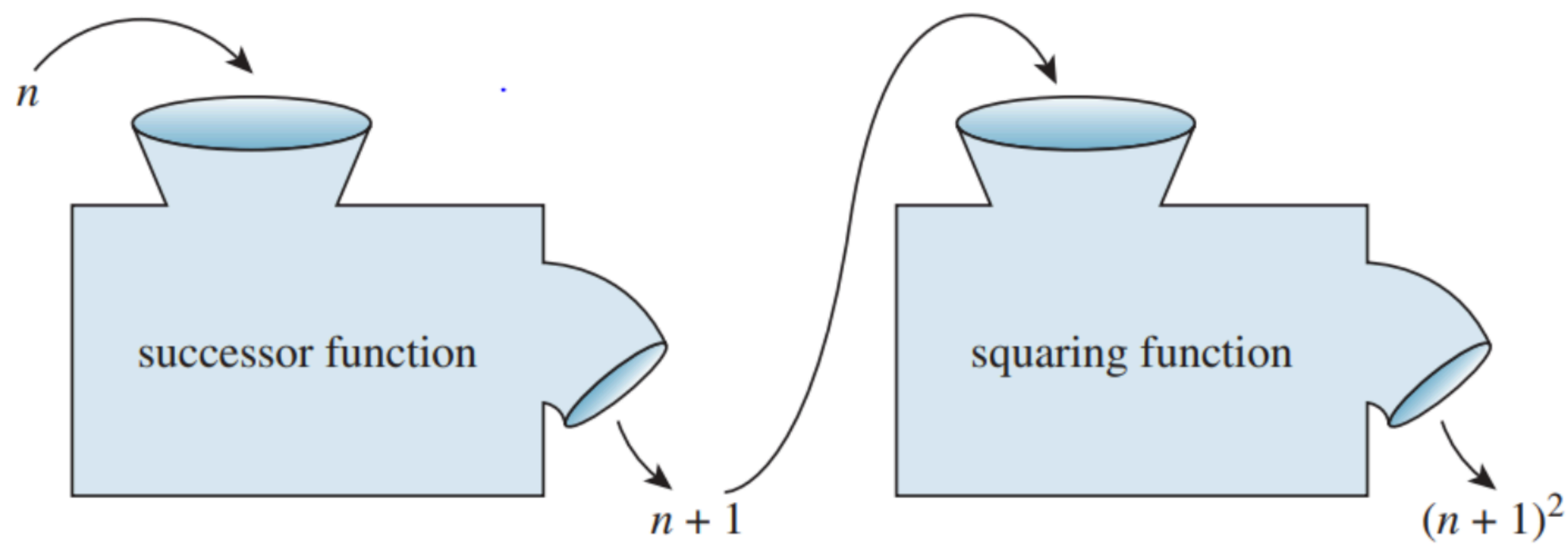
- The function $g: Z \times Z \rightarrow Z \times Z$ defined by the formula $g(m,n) = (m+n,\ m+2n)$ is a one-to-one correspondence. Find its inverse.

Function composition

Concept about function equality

- Let f and g two functions A -> B
- We say f = g if for any a in A, f(a) = g(a)
- We say f!=g if ~(f = g), namely, there exists a in A, f(a) != g(a)
- sin(2x) = 2 sin(x) cos(x)
- (x+1)^2 != x^2 + 1

Composition of functions



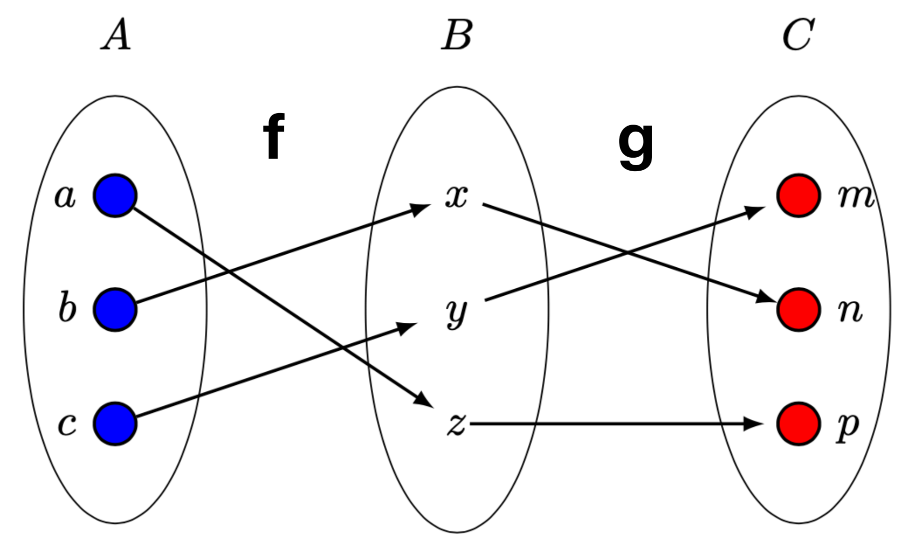
Composition of functions

Definition

- Let $f : X \to Y$ and $g : Y \to Z$. Let the range of f is a subset of the domain of g.
- Define a new composition function $g \circ f : X \to Z$ as follows:

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in X,$$

Quiz



- Write f as a pair of pairs
- Write g as a pair of pairs
- Write g of f as a pair of pairs

Quiz

Problem

- Let $f : \mathbb{Z} \to \mathbb{Z}$ be the successor function and let $g : \mathbb{Z} \to \mathbb{Z}$ be the squaring function. Then $f(n) = n + 1$ for all $n \in \mathbb{Z}$ and $g(n) = n^2$ for all $n \in \mathbb{Z}$. Find $g \circ f$. Find $f \circ g$. Is $g \circ f = f \circ g$?

Quiz

. Define $L: \mathbf{Z} \to \mathbf{Z}$ and $M: \mathbf{Z} \to \mathbf{Z}$ by the rules $L(a) = a^2$ and $M(a) = a \, mod \, 5$ for all integers a.

a. Find $(L \circ M)(12)$, $(M \circ L)(12)$, $(L \circ M)(9)$, and $(M \circ L)(9)$.

b. Is $L \circ M = M \circ L$?