General Grading Criteria

- Grading will be *highly strict*, with little tolerance for mistakes.
- Factual errors in grading will be corrected upon review, but partial grading will not be reconsidered.
- If a submission is found to be plagiarized or appears to be AI-generated, the issue will be reported to the instructor for further investigation.

Grading Criteria for Proof Questions

Critical errors that will result in point deductions include:

- Unclear reasoning due to ambiguous language, broken English, or lack of explanation.
- Missing assumptions necessary for a valid proof.
- Logical errors or invalid arguments that do not follow from the premises.
- Confusion in the proof structure, such as misidentifying the proof target, assumptions, or key facts.

Exercise 1 (10 points)

An integer is a perfect square if it can be expressed as the square of another integer. For example, 81 is a perfect square, while 80 is not.

Prove that there exists a perfect square that can be written as the sum of two other perfect squares.

Exercise 2 (60 points)

For each statement below, determine whether it is true or false. If you believe the statement is true, provide a formal proof. If you believe the statement is false, negate the statement and prove the negated version is true.

1. For any rational number r, and irrational number ir, $\frac{r}{ir}$ is irrational.
2. For any two irrational numbers ir_1 and ir_2, their product $ir_1 \times ir_2$ is irrational.
3. The sum of any two positive irrational numbers is irrational.
4. The square root of any rational number is irrational.

Example: Consider the statement "For any two odd numbers m_1 and m_2, $\frac{m_1+m_2}{2}$ is odd." This is false. The negated statement would be: "There exist two odd numbers m_1 and m_2 such that $\frac{m_1+m_2}{2}$ is even." To prove this negated statement, you could write Proof. we let $m_1 = 7$ and $m_2 = 9$ (both odd), then $\frac{7+9}{2} = 8$, which is even. QED.

Exercise 3 (10 points)

Below is a proof of a mathematical statement. For each of the numbered lines, indicate whether it is a proof target, assumption, or fact.

Proof.

(1) We want to prove: for any even number m, and integer n, $m \times n$ is even.

(2) Assume m is even, and n is an integer.

(3) $m = 2 \times k$ for some integer k.

(4) $m \times n = 2 \times k \times n$.

(5) Therefore, $m \times n$ is even.

QED.

Exercise 4 (20 points)

Suppose x is a real number.

1. Use direct proof to prove that if $x^3 - x > 0$, then $x > -1$.
2. Use proof by contraposition to prove the same statement.