CSE215
Foundations of Computer Science

State University of New York, Korea

Agenda

- Functions

- One-on-one functions

- Onto functions

Level test

Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. Define a function f from X to Y by the arrow diagram in Figure 7.1.3.
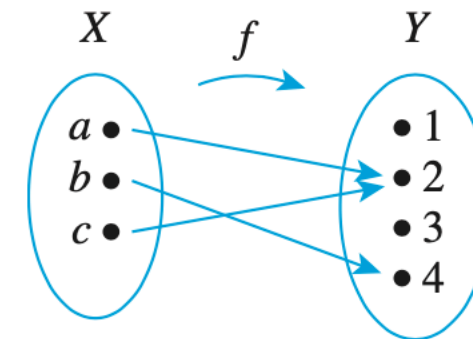
a. Write the domain and co-domain of f.

b. Find $f(a)$, $f(b)$, and $f(c)$.

c. What is the range of f?

d. Is c an inverse image of 2? Is b an inverse image of 3?

e. Find the inverse images of 2, 4, and 1.

f. Represent f as a set of ordered pairs.

g. Is function f injective?

h. Is function f surjective?

Figure 7.1.1

Functions are omnipresent in CS applications

- Spam detector

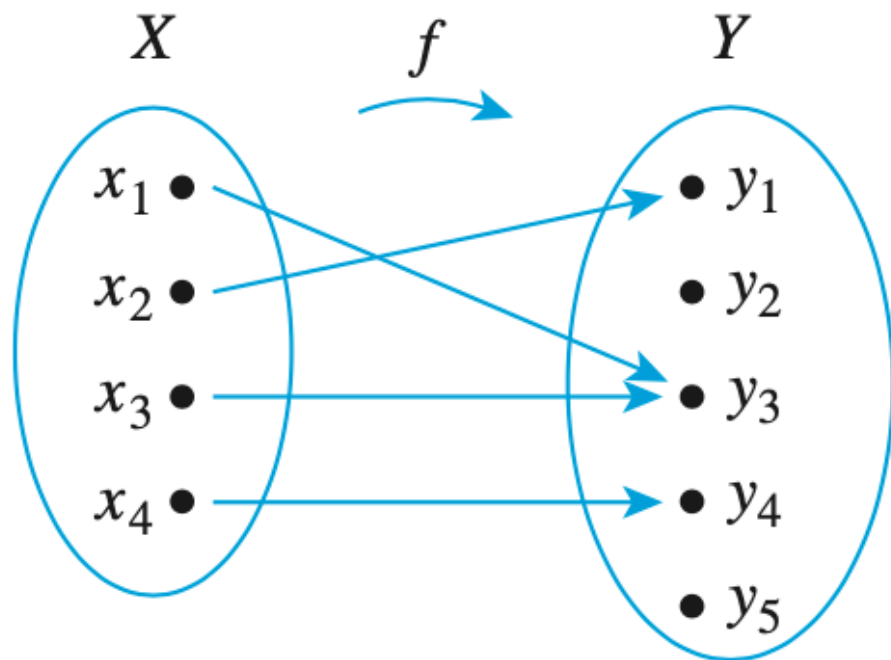- Housing price prediction

- Speech-to-text converter

Concepts for a function f: X->Y
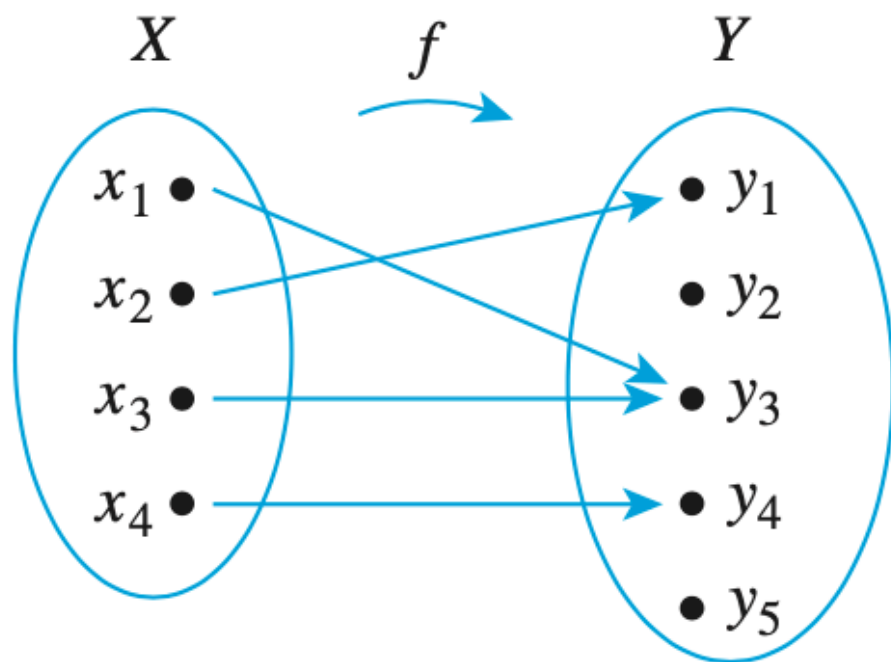
- Domain = X

- Co-domain = Y

- Range = Image of X = {f(x) | x \in X}

- Inverse image of an y = {x \in X| f(x) = y}

- Written as {(x1,y3), (x2, y1), …}

What about spam detection?

- Domain

- Co-domain

- Range

- Inverse image of an y

- Written as {(x1,y3), (x2, y1), …}

Spam detection
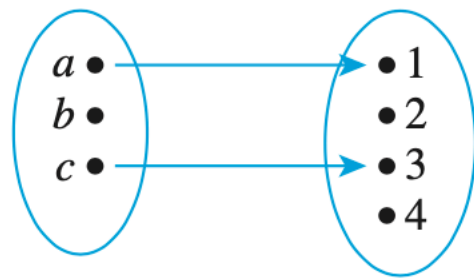
- End-product is a function, or called model of type

  - Emails-> {true, false}

- Machine learning process is a function of type

  - Dataset -> Functions of type (Emails ->{true, false})
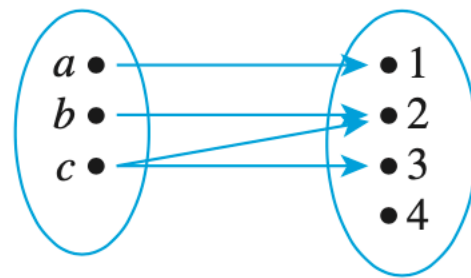
Requirement to be a function

- **Each** element in X maps to a **single element** in Y

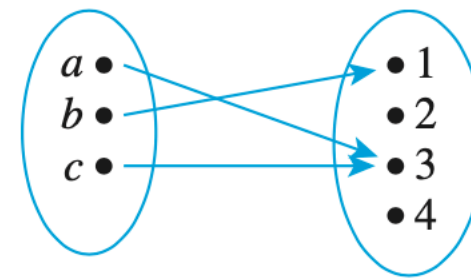# Exercise2:
Functions or non-functions

(a)   (b)   (c)

One-to-one functions

Definition

- A function $F : X \to Y$ is one-to-one (or injective) if and only if for all elements x_1 and x_2 in X,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2, \text{ or}$$
$$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

- A function $F : X \to Y$ is one-to-one $\Leftrightarrow$
$\forall x_1, x_2 \in X$, if $F(x_1) = F(x_2)$ then $x_1 = x_2$.
A function $F : X \to Y$ is not one-to-one $\Leftrightarrow$
$\exists x_1, x_2 \in X$, if $F(x_1) = F(x_2)$ then $x_1 \neq x_2$.

One-to-one functions: Proof technique

Problem

- Prove that a function f is one-to-one.

Proof

Direct proof.
- Suppose x_1 and x_2 are elements of X such that $f(x_1) = f(x_2)$.
- Show that $x_1 = x_2$.

Problem

- Prove that a function f is not one-to-one.

Proof

Counterexample.
- Find elements x_1 and x_2 in X so that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

One-to-one functions: Example 1

Problem

- Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$. Is f one-to-one? Prove or give a counterexample.

One-to-one functions: Example 1

Problem

- Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$. Is f one-to-one? Prove or give a counterexample.

Proof

Direct proof.

- Suppose x_1 and x_2 are elements of X such that $f(x_1) = f(x_2)$.
  $\implies 4x_1 - 1 = 4x_2 - 1 \qquad (\because \text{Defn. of } f)$
  $\implies 4x_1 = 4x_2 \qquad (\because \text{Add 1 on both sides})$
  $\implies x_1 = x_2 \qquad (\because \text{Divide by 4 on both sides})$
- Hence, f is one-to-one.

One-to-one functions: Example 2

Problem

- Define $g : \mathbb{Z} \to \mathbb{Z}$ by the rule $g(n) = n^2$ for all $n \in \mathbb{Z}$. Is g one-to-one? Prove or give a counterexample.

One-to-one functions: Example 2

Problem

- Define $g : \mathbb{Z} \to \mathbb{Z}$ by the rule $g(n) = n^2$ for all $n \in \mathbb{Z}$. Is g one-to-one? Prove or give a counterexample.

Proof

Counterexample.

- Let $n_1 = -1$ and $n_2 = 1$.
  $\implies g(n_1) = (-1)^2 = 1$ and $g(n_2) = 1^2 = 1$
  $\implies g(n_1) = g(n_2)$ but, $n_1 \neq n_2$
- Hence, g is not one-to-one.

Onto functions

Definition

- A function $F : X \to Y$ is **onto** (or surjective) if and only if given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.
- A function $F : X \to Y$ is **onto** $\Leftrightarrow$
  $\forall y \in Y, \exists x \in X$ such that $F(x) = y$.
  A function $F : X \to Y$ is **not onto** $\Leftrightarrow$
  $\exists y \in Y, \forall x \in X$ such that $F(x) \neq y$.

Onto functions: Proof technique

Problem

- Prove that a function f is onto.

Proof

Direct proof.
- **Suppose** that y is any element of Y
- **Show** that there is an element x of X with $F(x) = y$

Problem

- Prove that a function f is not onto.

Proof

Counterexample.
- **Find** an element y of Y such that $y \neq F(x)$ for any x in X.

Onto functions: Example 1

Problem

- Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$. Is f onto? Prove or give a counterexample.

Proof

Direct proof.
- Let $y \in \mathbb{R}$. We need to show that $\exists x$ such that $f(x) = y$. Let $x = \frac{y+1}{4}$. Then
$$f\left(\frac{y+1}{4}\right) = 4\left(\frac{y+1}{4}\right) - 1 \qquad (\because \text{Defn. of } f)$$
$$= y \qquad (\because \text{Simplify})$$
- Hence, f is onto.

Onto functions: Example 1

- Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$. Is f onto? Prove or give a counterexample.

Onto functions: Example 2

Problem

- Define $g : \mathbb{Z} \to \mathbb{Z}$ by the rule $g(n) = 4n - 1$ for all $n \in \mathbb{Z}$. Is g onto? Prove or give a counterexample.

Proof

Counterexample.

- We know that $0 \in \mathbb{Z}$.
- Let $g(n) = 0$ for some integer n.

$$\implies 4n - 1 = 0 \qquad (\because \text{Defn. of } g)$$
$$\implies n = \tfrac{1}{4} \qquad (\because \text{Simplify})$$

But $\tfrac{1}{4} \notin \mathbb{Z}$.

So, $g(n) \neq 0$ for any integer n.

- Hence, g is not onto.

Exercises

Consider four functions defined below:

1. The function $f_1 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by the as

$$f_1(m, n) = (m + n, m + 2n).$$

2. The function $f_2 : \mathbb{R} - \{0\} \to \mathbb{R}$ defined as

$$f_2(x) = \frac{1}{x} + 1.$$

Fill in the following table with "T" (for true) or "F" (for false). Be careful about the domain and co-domain of each function.

Functions	One-to-one?	Onto?
f_1		
f_2		
f_3		
f_4		

Consider four functions defined below:

3. The function $f_3 : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f_3(x, y) = (xy, x^3).$$

4. The function $f_4 : \mathbb{R} \to \mathbb{R}$ defined as

$$f_4(n) = 2n + 1.$$

Fill in the following table with "T" (for true) or "F" (for false). Be careful about the domain and co-domain of each function.

Functions	One-to-one?	Onto?
f_1		
f_2		
f_3		
f_4		