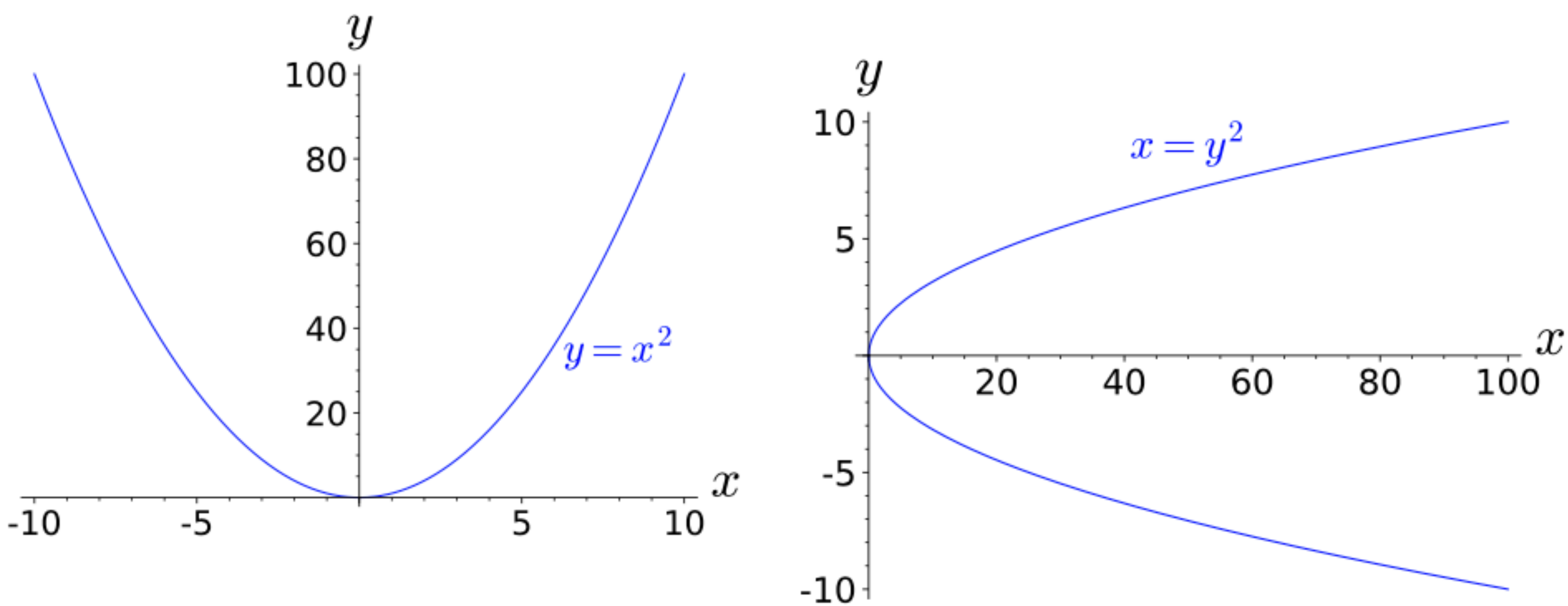
CSE215
Foundations of Computer Science

State University of New York, Korea

Relation

Functions vs. relations



	$y = x^2$	$y = \pm\sqrt{x}$
Function?	✓	✗
Relation?	✓	✓

Relation

Definition A **relation** on a set A is a subset $R \subseteq A \times A$. We often abbreviate the statement $(x, y) \in R$ as xRy. The statement $(x, y) \notin R$ is abbreviated as $x \not R y$.

$$5 < 10 \qquad 5 \le 5 \qquad 6 = \tfrac{30}{5} \qquad 5 \mid 80 \qquad 7 > 4 \qquad x \ne y \qquad 8 \nmid 3$$

$$a \equiv b \pmod{n} \qquad 6 \in \mathbb{Z} \qquad X \subseteq Y \qquad \pi \approx 3.14 \qquad 0 \ge -1 \qquad \sqrt{2} \notin \mathbb{Z} \qquad \mathbb{Z} \nsubseteq \mathbb{N}$$

Example: Less than

Problem

- A relation $L : \mathbb{R} \to \mathbb{R}$ as follows.
  For all real numbers x and y, $(x, y) \in L \Leftrightarrow x \; L \; y \Leftrightarrow x < y$.
  Draw the graph of L as a subset of the Cartesian plane $\mathbb{R} \times \mathbb{R}$.

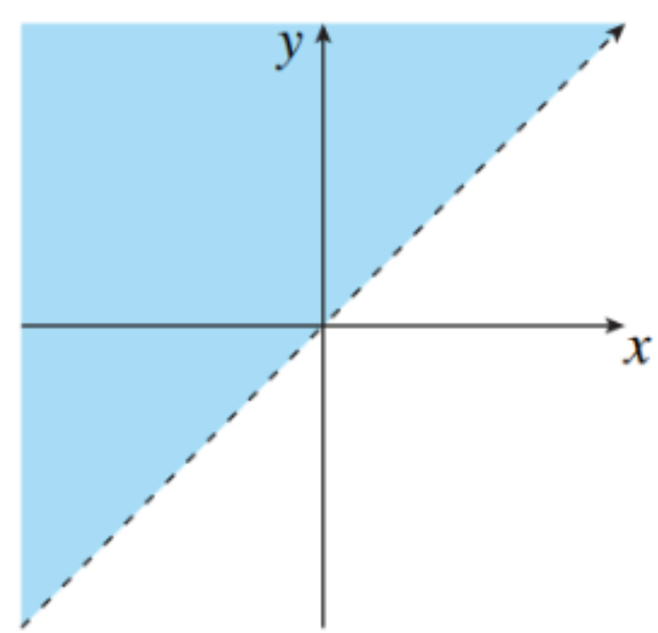
Example: Less than

Problem

- A relation $L : \mathbb{R} \to \mathbb{R}$ as follows.
  For all real numbers x and y, $(x, y) \in L \Leftrightarrow x \, L \, y \Leftrightarrow x < y$.
  Draw the graph of L as a subset of the Cartesian plane $\mathbb{R} \times \mathbb{R}$.

Solution

- $L = \{(-10.678, 30.23), (17.13, 45.98), (100/9, 200), \ldots\}$
- Graph:



Example: Congruence modulo 2

Problem

- Define a relation $C : \mathbb{Z} \to \mathbb{Z}$ as follows.
  For all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, $m\ C\ n \Leftrightarrow m - n$ is even.
- Prove that if n is any odd integer, then $n\ C\ 1$.

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- Define a relation $C : \mathbb{Z} \to \mathbb{Z}$ as follows.
  For all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, $m \, C \, n \Leftrightarrow m - n$ is even.
- Prove that if n is any odd integer, then $n \, C \, 1$.

Solution

- $A = \{(2, 4), (56, 10), (-88, -64), \ldots\}$
  $B = \{(7, 7), (57, 11), (-87, -63), \ldots\}$
  $C = A \cup B$
- Proof. $(n, 1) \in C \Leftrightarrow n \, C \, 1 \Leftrightarrow n - 1$ is even
  Suppose n is odd i.e., $n = 2k + 1$ for some integer k.
  This implies that $n - 1 = 2k$ is even.

Example: Relation on a set

Problem

- Let $A = \{3, 4, 5, 6, 7, 8\}$. Define relation R on A as follows. For all $x, y \in A$, $x \, R \, y \Leftrightarrow 2 | (x - y)$. Draw the graph of R.

Example: Relation on a set

Problem

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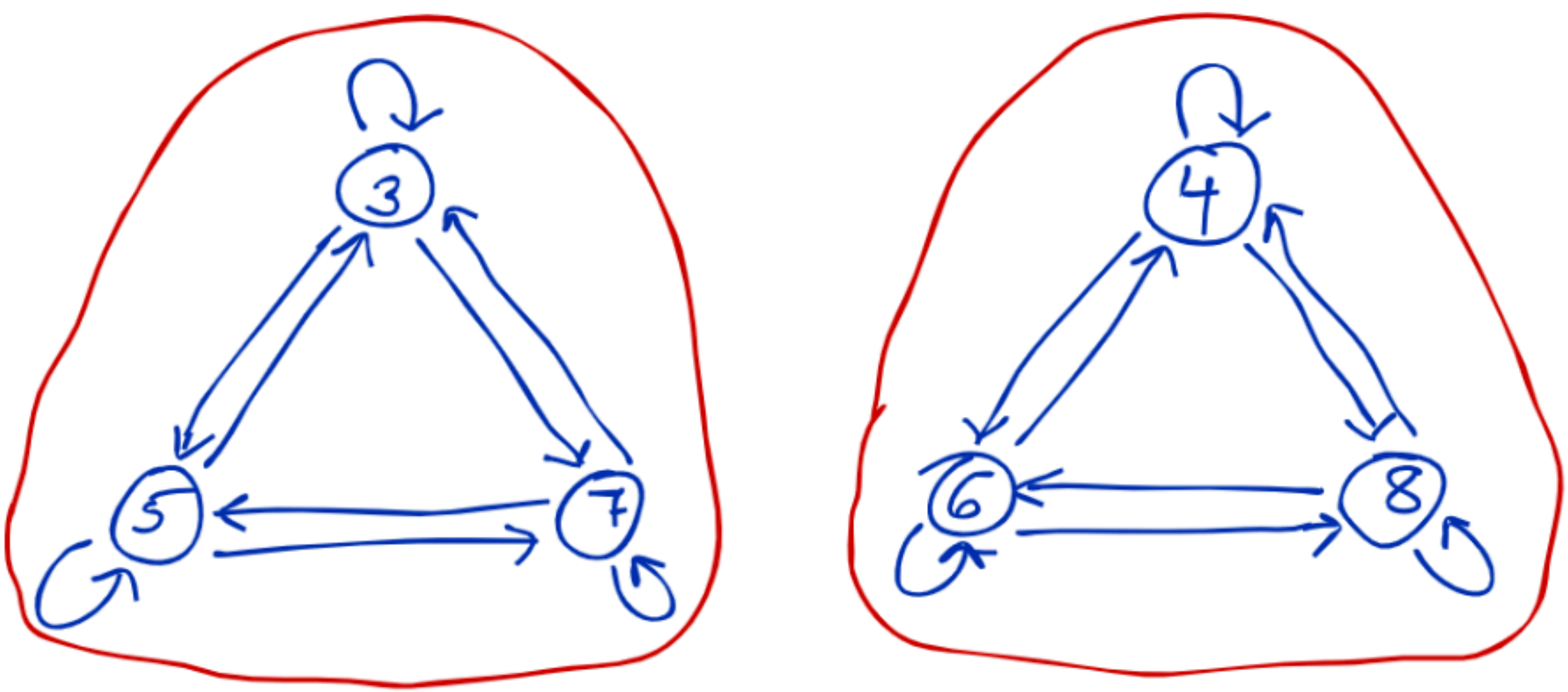


Quiz

The **congruence modulo 3** relation, T, is defined from $\mathbf{Z}$ to $\mathbf{Z}$ as follows: For all integers m and n,

$$m\ T\ n \quad \Leftrightarrow \quad 3 \mid (m - n).$$

a. Is $10\ T\ 1$? Is $1\ T\ 10$? Is $(2, 2) \in T$? Is $(8, 1) \in T$?
b. List five integers n such that $n\ T\ 0$.
c. List five integers n such that $n\ T\ 1$.
d. List five integers n such that $n\ T\ 2$.

Inverse of a relation

Definition

- Let R be a relation from A to B.
  Then inverse relation R^{-1} from B to A is:
  $\boxed{R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}}$.
- For all $x \in A$ and $y \in B$,
  $\boxed{(x, y) \in R \Leftrightarrow (y, x) \in R^{-1}}$.

Example: Inverse of a finite relation

Problem

- Let $A = \{2, 3, 4\}$ and $B = \{2, 6, 8\}$.
  Let $R : A$ to B. For all $(a, b) \in A \times B$, $a \, R \, b \Leftrightarrow a \mid b$
- Determine R and R^{-1}. Draw arrow diagrams for both.
  Describe R^{-1} in words.

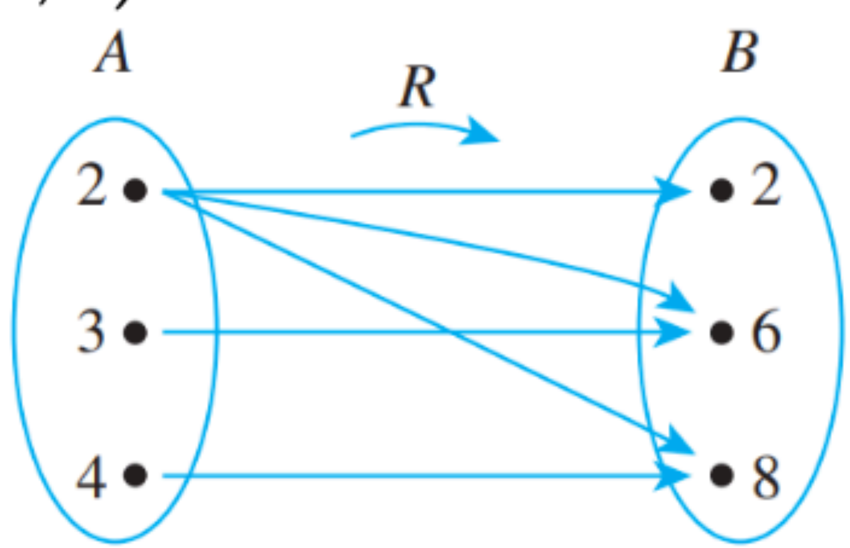
Example: Inverse of a finite relation

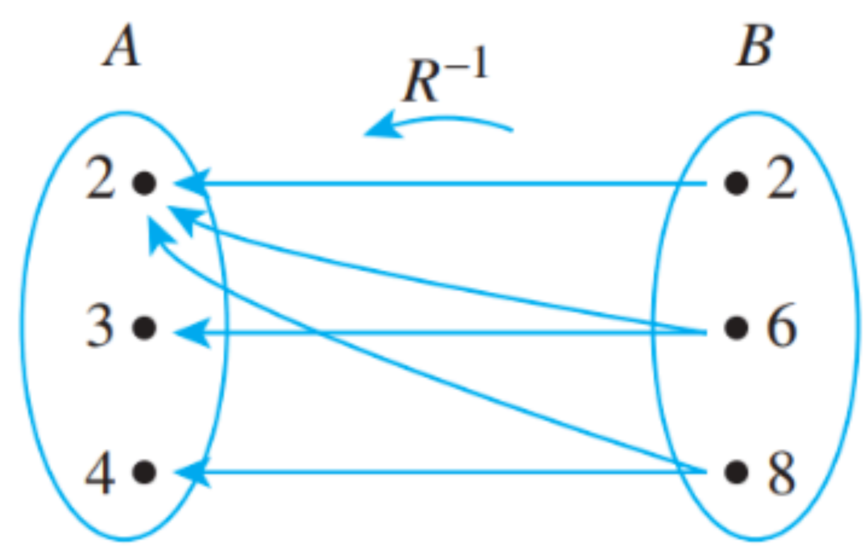
Problem

- Let $A = \{2, 3, 4\}$ and $B = \{2, 6, 8\}$.
  Let $R : A$ to B. For all $(a, b) \in A \times B$, $a \, R \, b \Leftrightarrow a \mid b$
- Determine R and R^{-1}. Draw arrow diagrams for both.
  Describe R^{-1} in words.

Solution

- $R = \{(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)\}$
  $R^{-1} = \{(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)\}$
- For all $(b, a) \in B \times A$,
  $(b, a) \in R^{-1} \Leftrightarrow b$ is a multiple of a





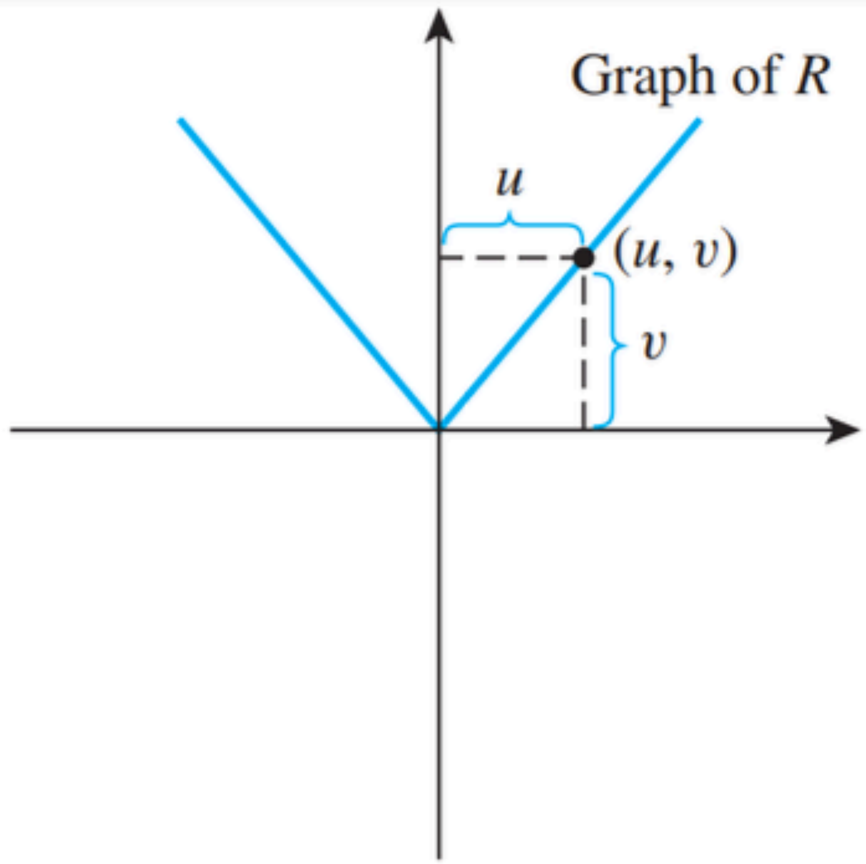
Example: Inverse of an infinite relation

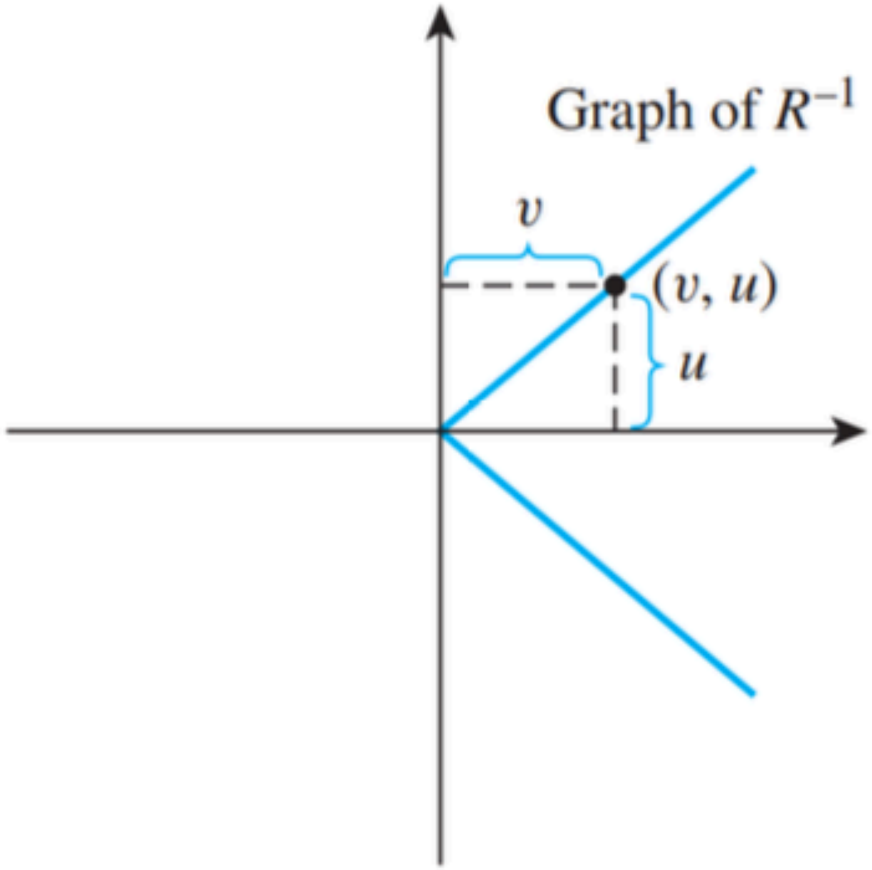
Problem

- Define a relation R from $\mathbb{R}$ to $\mathbb{R}$ as follows:
  For all $(u, v) \in \mathbb{R} \times \mathbb{R}$, $u \; R \; v \Leftrightarrow v = 2|u|$.
- Draw the graphs of R and R^{-1} in the Cartesian plane.
  Is R^{-1} a function?

Solution

- R^{-1} is not a function. Why?



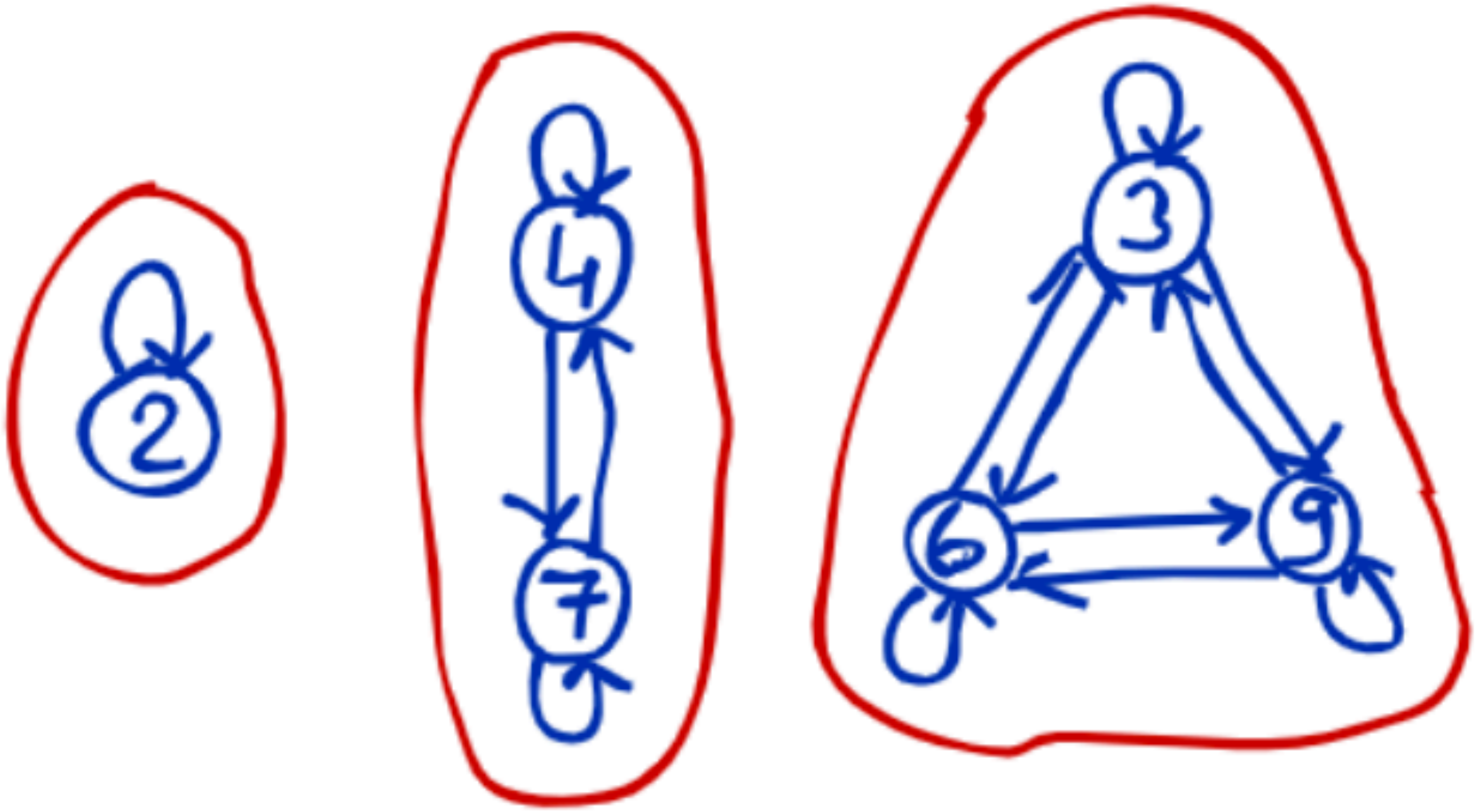


Equivalence Relation

Reflexivity, symmetry, and transitivity

Properties

- Set $A = \{2, 3, 4, 6, 7, 9\}$
  Relation R on set A is: $\forall x, y \in A,\ x\ R\ y \Leftrightarrow 3 \mid (x - y)$



- Reflexivity. $\forall x \in A, (x, x) \in R$.
- Symmetry. $\forall x, y \in A$, if $(x, y) \in R$, then $(y, x) \in R$.
- Transitivity.
  $\forall x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Example

Problem

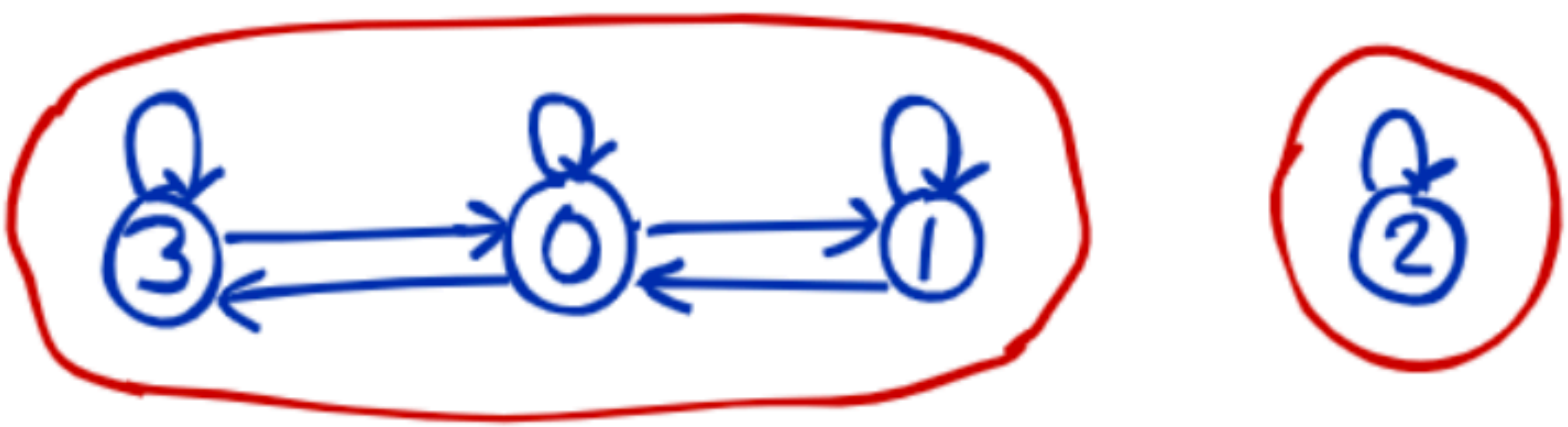
- $A = \{0, 1, 2, 3\}$.
  $R = \{(0,0), (0,1), (0,3), (1,0), (1,1), (2,2), (3,0), (3,3)\}$.
  Is R reflexive, symmetric, and transitive?

Example

Problem

- $A = \{0, 1, 2, 3\}$.
  $R = \{(0,0), (0,1), (0,3), (1,0), (1,1), (2,2), (3,0), (3,3)\}$.
  Is R reflexive, symmetric, and transitive?

Solution



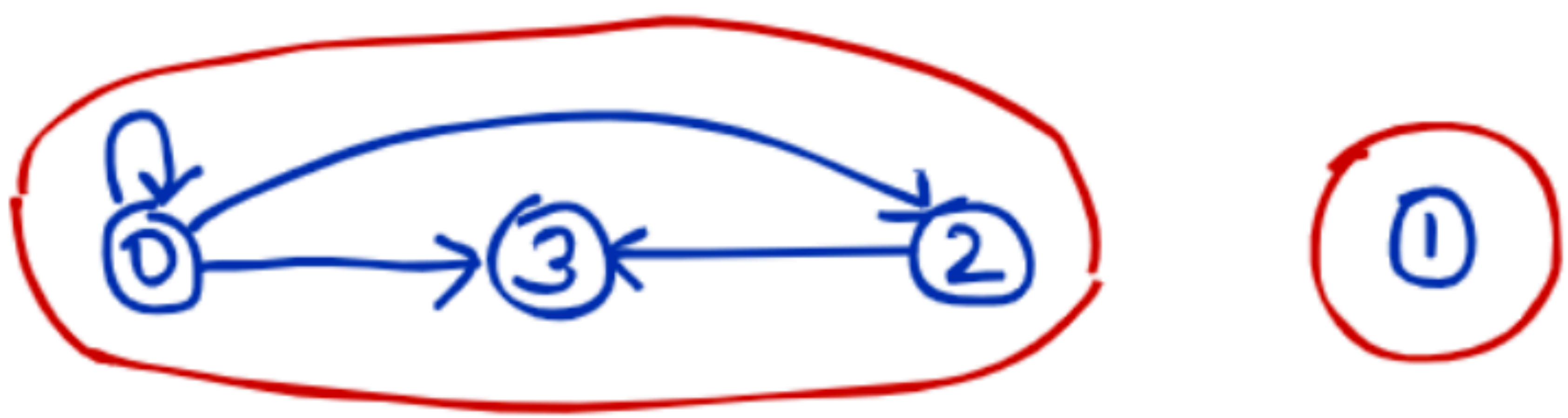
- Reflexive. $\forall x \in A, (x, x) \in R$.
- Symmetric. $\forall x, y \in A$, if $(x, y) \in R$, then $(y, x) \in R$.
- Not transitive. e.g.: $(1, 0), (0, 3) \in R$ but $(1, 3) \notin R$.
  $\exists x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \notin R$.

Exercise 1

Problem

- $A = \{0, 1, 2, 3\}$. $R = \{(0,0), (0,2), (0,3), (2,3)\}$.
  Is R reflexive, symmetric, and transitive?

Solution



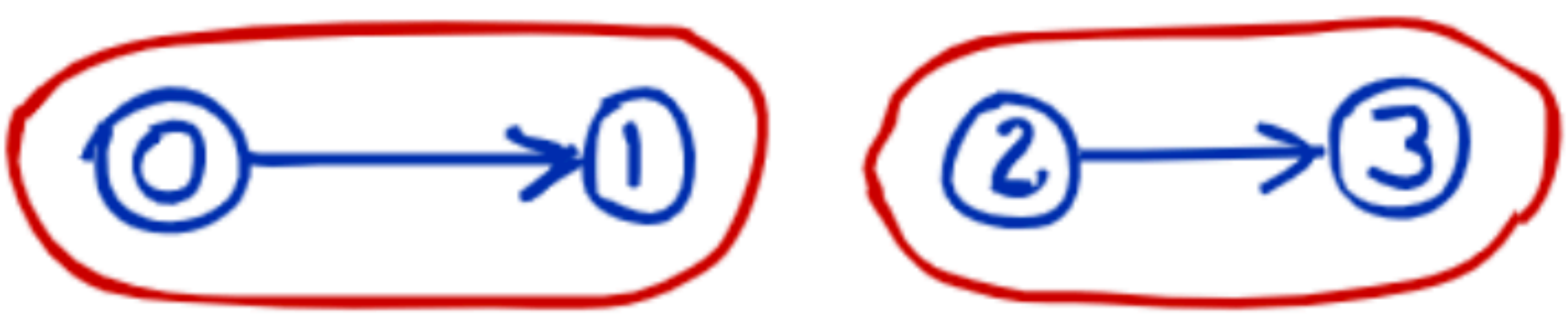
- Not reflexive. e.g.: $(1,1) \notin R$. $\exists x \in A, (x,x) \notin R$.
- Not symmetric. e.g.: $(0,3) \in R$ but $(3,0) \notin R$.
  $\exists x, y \in A$, if $(x,y) \in R$, then $(y,x) \notin R$.
- Transitive.
  $\forall x, y, z \in A$, if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$.

Exercise 2

Problem

- $A = \{0,1,2,3\}$. $R = \{(0,1),(2,3)\}$.
  Is R reflexive, symmetric, and transitive?

Solution



- Not reflexive. e.g.: $(0,0) \notin R$. $\exists x \in A, (x,x) \notin R$.
- Not symmetric. e.g.: $(0,1) \in R$ but $(1,0) \notin R$.
  $\exists x, y \in A$, if $(x,y) \in R$, then $(y,x) \notin R$.
- Transitive. Why?
  $\forall x, y, z \in A$, if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$.

Equivalence relation and equivalence class

Definition

- Relation R on set A is an equivalence relation iff
  R is reflexive, symmetric, and transitive.
- Equivalence class of element a, denoted by $[a]$, for an equivalence relation is defined as:
  $[a] = \{x \in A \mid (x, a) \in R\}$.

Example: Less than

Problem

- Suppose R is a relation on $\mathbb{R}$ such that $x \; R \; y \Leftrightarrow x < y$. Is R an equivalence relation?

Example: Less than

Problem

- Suppose R is a relation on $\mathbb{R}$ such that $x \; R \; y \Leftrightarrow x < y$. Is R an equivalence relation?

Solution

- Not reflexive. e.g.: $0 \not< 0$. $\exists x \in \mathbb{R}, x \not< x$.
- Not symmetric. e.g.: $0 < 1$ but $1 \not< 0$. $\exists x, y \in \mathbb{R}$, if $x < y$, then $y \not< x$.
- Transitive. $\forall x, y, z \in \mathbb{R}$, if $x < y$ and $y < z$, then $x < z$.

So, R is not an equivalence relation.

Example: Congruence modulo 3

Problem

- Suppose R is a relation on $\mathbb{Z}$ such that $m \, R \, n \Leftrightarrow 3 \mid (m - n)$. Is R an equivalence relation?

- If yes, what is the equivalence class [1], [2] and [3] ?

Example: Congruence modulo 3

Problem

- Suppose R is a relation on $\mathbb{Z}$ such that $m \; R \; n \Leftrightarrow 3 \mid (m - n)$. Is R an equivalence relation?

Solution

- Reflexive. $\forall m \in A, 3 \mid (m - m)$.
- Symmetric. $\forall m, n \in A$, if $3 \mid (m - n)$, then $3 \mid (n - m)$.
- Transitive.
  $\forall m, n, p \in A$, if $3 \mid (m - n)$ and $3 \mid (n - p)$, then $3 \mid (m - p)$.

So, R is an equivalence relation.

Example: Congruence modulo 3

Solution

- Equivalence classes.

  Three distinct equivalence classes are $[0]$, $[1]$, and $[2]$.

  $[0] = \{a \in \mathbb{Z} \mid a \equiv 0 \pmod{3}\} = \{0, \pm 3, \pm 6, \pm 9, \ldots\}$

  $[1] = \{a \in \mathbb{Z} \mid a \equiv 1 \pmod{3}\} = \{1, 1 \pm 3, 1 \pm 6, 1 \pm 9, \ldots\}$

  $[2] = \{a \in \mathbb{Z} \mid a \equiv 2 \pmod{3}\} = \{2, 2 \pm 3, 2 \pm 6, 2 \pm 9, \ldots\}$

  Intuition.

  $[0]$ = Set of integers when divided by 3 leave a remainder of 0.

  $[1]$ = Set of integers when divided by 3 leave a remainder of 1.

  $[2]$ = Set of integers when divided by 3 leave a remainder of 2.

Exercises

Exercise 1

- Let $A = \{3, 4, 5\}$ and $B = \{4, 5, 6\}$ and let R be the "less than" relation. That is, for all $(x, y) \in A \times B$,

$$x \; R \; y \quad \Leftrightarrow \quad x < y.$$

State explicitly which ordered pairs are in R and R^{-1}.

Exercise 2

- Determine if the following relations is reflective, symmetric, and transitive

9. R is the "greater than or equal to" relation on the set of real numbers: For all $x, y \in \mathbf{R}$, $x\ R\ y \Leftrightarrow x \geq y$.

10. C is the circle relation on the set of real numbers: For all $x, y \in \mathbf{R}$, $x\ C\ y \Leftrightarrow x^2 + y^2 = 1$.

11. D is the relation defined on $\mathbf{R}$ as follows: For all $x, y \in \mathbf{R}$, $x\ D\ y \Leftrightarrow xy \geq 0$.

12. E is the congruence modulo 2 relation on $\mathbf{Z}$: For all $m, n \in \mathbf{Z}$, $m\ E\ n \Leftrightarrow 2 \mid (m - n)$.

13. F is the congruence modulo 5 relation on $\mathbf{Z}$: For all $m, n \in \mathbf{Z}$, $m\ F\ n \Leftrightarrow 5 \mid (m - n)$.

14. O is the relation defined on $\mathbf{Z}$ as follows: For all $m, n \in \mathbf{Z}$, $m\ O\ n \Leftrightarrow m - n$ is odd.

15. D is the "divides" relation on $\mathbf{Z}^+$: For all positive integers m and n, $m\ D\ n \Leftrightarrow m \mid n$.

Solution

- 9. R, Not S, T
- 10. Not R, S, Not T
- 11. R, S, Not T
- 12. R, S, T
- 13. R, S, T
- 14. Not R, S, Not T
- 15. R, Not S, T

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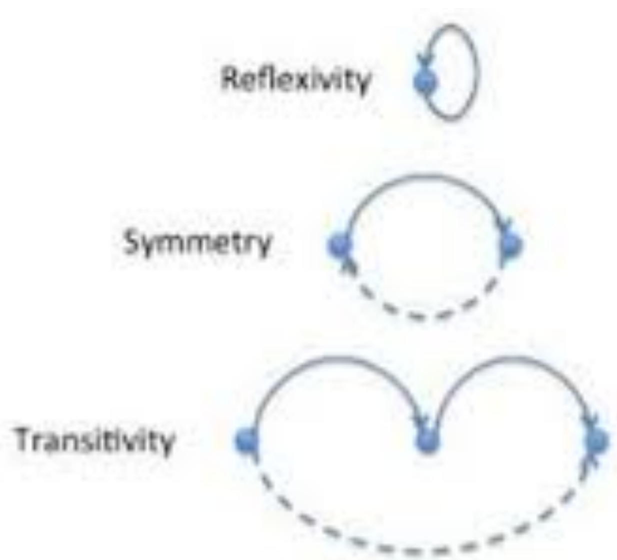
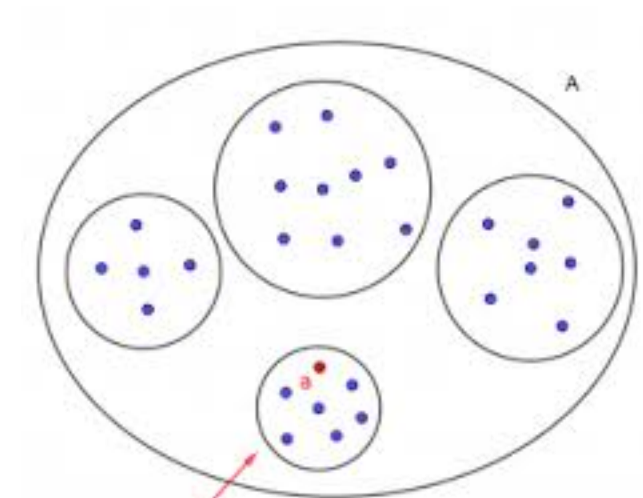
Exercise 3

2021 Final

Problem 11. [5 points]

Let A be the set of all people. Let R be the relation defined on A as follows: For persons p and q in A, we have $p\ R\ q \Leftrightarrow p$ has the same birthday as q.
Is R an equivalence relation? Prove your answer. If R is an equivalence relation, what are the distinct equivalence classes of the relation?





equivalence class

Solution

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Is R an equivalence relation? Prove your answer. If R is an equivalence relation, what are the distinct equivalence classes of the relation?

- R is an equivalence relation, as it is
  - reflective (p R p),
  - symmetric p R q <==> q R p
  - transitive p R q, q R r ==> p R r
- Equivalence classes is the set of the sets of people having the same birthday.