# CSE215
# Foundations of Computer Science

State University of New York, Korea

# Previous lectures

**Truth table**

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

**Equivalence laws**

| Laws | Formula | Formula |
|---|---|---|
| Commutative laws | $p \wedge q \equiv q \wedge p$ | $p \vee q \equiv q \vee p$ |
| Associative laws | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | $(p \vee q) \vee r \equiv p \vee (q \vee r)$ |
| Distributive laws | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ |

## Propositional logic: A formal language to express facts and argue about them

**Valid arguments**

Premise$_1$
Premise$_2$
$\vdots$
Premise$_m$
$\therefore$ Conclusion

**Inference**

| Name | Rule | Name | Rule | |
|---|---|---|---|---|
| Modus Ponens | $p \to q$<br>$p$<br>$\therefore q$ | Elimination | $p \vee q$<br>$\sim q$<br>$\therefore p$ | $p \vee q$<br>$\sim p$<br>$\therefore q$ |
| Modus Tollens | $p \to q$<br>$\sim q$<br>$\therefore \sim p$ | Transitivity | $p \to q$<br>$q \to r$<br>$\therefore p \to r$ | |

# Plan

- Revision on using Inference rules for validity
- Predicate Logic, or propositions with Quantifiers
- Negation

# Revision on Using Inference Rules for Validity

In 41–44 a set of premises and a conclusion are given. Use the valid argument forms listed in Table 2.3.1 to deduce the conclusion from the premises, giving a reason for each step as in Example 2.3.8. Assume all variables are statement variables.

**41.** a. $\sim p \vee q \rightarrow r$

b. $s \vee \sim q$

c. $\sim t$

d. $p \rightarrow t$

e. $\sim p \wedge r \rightarrow \sim s$

f. $\therefore \sim q$

42. a. $p \vee q$

b. $q \to r$

c. $p \wedge s \to t$

d. $\sim r$

e. $\sim q \to u \wedge s$

f. $\therefore t$

# Propositions with Quantifiers

# Why quantifiers? To express "for all" and "there exists"

- Everyone can make mistake
- Nobody is perfect
- Every lock has a key
- There is a key for every lock
- Every nonzero real number has a reciprocal

# CS example 1: Software security

**Question: Could the program print “bad”?**

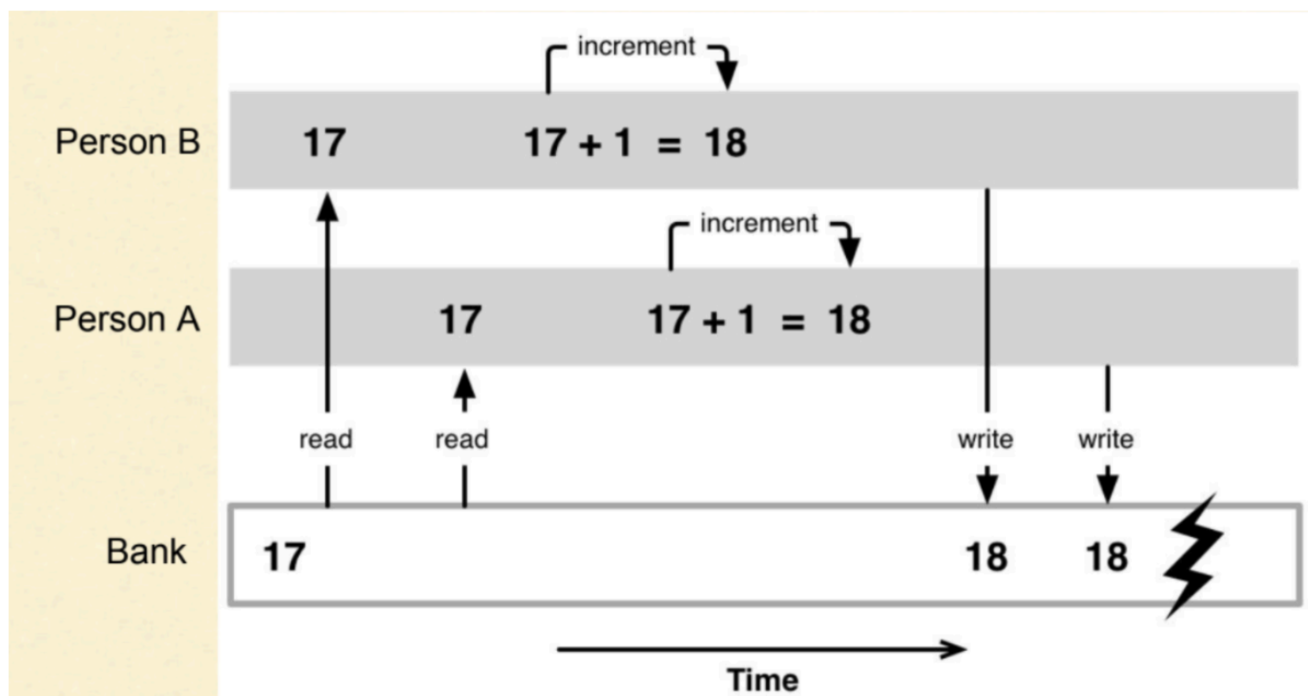
```c
1  #include <stdio.h>
2
3  void f(int input) {
4      char a[8];
5      int b = 0;
6      a[input] = 1;
7
8      if (b == 0)
9          printf("good\n");
10     else
11         printf("bad\n");
12 }
```

**Exisitential Statement**

**: there exists an integer n, such that b !=0 at line 8 when executing f(n)**

# CS example 2: Concurrency

Two persons are trying to deposit 1 dollar online into the same bank account.




**Universal Statement:**
**For all CPU schedule s, A should not read while B intends to write.**

# Predicate

- A propositional function or predicate is a sentence that contains one or more variables
- A predicate is neither true nor false
- A predicate becomes a proposition when the variables are substituted with specific values
- The domain of a predicate variable is the set of all values that may be substituted for the variable

## Examples

| Symbol | Predicate | Domain | Propositions |
|---|---|---|---|
| $p(x)$ | $x > 5$ | $x \in \mathbb{R}$ | $p(6), p(-3.6), p(0), \ldots$ |
| $p(x, y)$ | $x + y$ is odd | $x \in \mathbb{Z}, y \in \mathbb{Z}$ | $p(4, 5), p(-4, -4), \ldots$ |

# Universal statement

- Let $p(x)$ be a predicate and $D$ be the domain of $x$
- A universal statement is a statement of the form

$$\forall x \in D, p(x)$$

- Forms:
  - "$p(x)$ is true for all values of $x$"
  - "For all $x$, $p(x)$"
  - "For each $x$, $p(x)$"
  - "For every $x$, $p(x)$"
  - "Given any $x$, $p(x)$"

# Existential statement

- Let $p(x)$ be a predicate and $D$ be the domain of $x$
- An existential statement is a statement of the form

$$\exists x \in D, p(x)$$

- Forms:
  - "There exists an $x$ such that $p(x)$"
  - "For some $x$, $p(x)$"
  - "We can find an $x$, such that $p(x)$"
  - "There is some $x$ such that $p(x)$"
  - "There is at least one $x$ such that $p(x)$"

**Examples**

| Universal st.s | Domain |
|---|---|
| $\exists x \in D, x^2 \geq x$ | $D = \{1, 2, 3\}$ |
| $\exists x \in \mathbb{R}, x^2 \geq x$ | $\mathbb{R}$ |

# Propositions with multiple quantifiers

$$\forall x \in D, \exists y \in E, \text{such that } p(x, y)$$

Every lock has a key.

# Propositions with multiple quantifiers

$$\exists x \in D, \forall y \in E, \text{such that } p(x, y)$$

There is a key for every lock.

# Propositions with multiple quantifiers

$$\forall x \in D, \forall y \in E, \text{such that } p(x, y)$$

"Give me a place to stand, and a lever long enough, and I will move the Earth."

# Propositions with multiple quantifiers

$$\exists x \in D, \exists y \in E, \text{such that } p(x, y)$$

“There is someone in a park sitting on a bench”.

# Four Notes

# Note 1:
# the order of quantifiers matters

- Every lock has a key
- For any lock L, there exists a key K, such that K can unlock L.
- There is a key for every lock
- There exists a key K, such that for any lock L, K can unlock L.

# Note 2: A commonly used notational equivalence

$$\forall x \in D, p(x)$$

**Equivalen to**

$$\forall x, x \in D \rightarrow p(x)$$

# Example: All doctors wear glasses

- for all d, if d is a doctor, then d wears glasses
- Formally, if we define
  - D to be the set of doctors,
  - wear_glass to be a function that takes a person x as an input, and returns true if x wears glasses
- then the following two statements are considered the same

$$\forall d \in D, \texttt{wear\_glass}(d).$$

$$\forall d, d \in D \to \texttt{wear\_glass}(d)$$

# Note 3: Universal conditional statement

- A universal conditional statement is of the form

$$\forall x, \text{ if } p(x) \text{ then } q(x)$$

Examples

- $\forall x \in \mathbb{R}$, if $x > 2$ then $x^2 > 4$
- $\forall$ real number $x$, if $x$ is an integer then $x$ is rational
  $\forall$ integer $x$, $x$ is rational
- $\forall x$, if $x$ is a square then $x$ is a rectangle
  $\forall$ square $x$, $x$ is a rectangle

# Note 4: Implicit quantifiers

**Examples**

- If a number is an integer, then it is a rational number
  Implicit meaning: $\forall$ number $x$, if $x$ is an integer, $x$ is rational
- The number 10 can be written as a sum of two prime numbers
  Implicit meaning: $\exists$ prime numbers $p$ and $q$ such that $10 = p+q$
- If $x > 2$, then $x^2 > 4$
  Implicit meaning: $\forall$ real $x$, if $x > 2$, then $x^2 > 4$

**Definition**

- Let $p(x)$ and $q(x)$ be predicates and $D$ be the common domain of $x$. Then implicit quant. symbols $\Rightarrow, \Leftrightarrow$ are defined as:

$$p(x) \Rightarrow q(x) \equiv \forall x, p(x) \rightarrow q(x)$$
$$p(x) \Leftrightarrow q(x) \equiv \forall x, p(x) \leftrightarrow q(x)$$

# Exercises

# Exercise 1
# translate to formal logic

Everyone can make mistake

# Exercise 2
# translate to formal logic

Nobody is perfect

# Exercise 3
# translate to formal logic

Every nonzero real number has a reciprocal

The reciprocal of 4 is 1/4 (namely 0.25)

# Exercise 4: Translate to formal logic

- Given:
  - S: set of students
  - P(s): s passed the exam.
  - W(s): s worked hard.
  - C(s): s is in Professor Cho’s class.
- **Every student in Professor Cho’s class passed the exam**

# Exercise 5: Translate to formal logic

- Given:
    - S: set of students
    - P(s): s passed the exam.
    - W(s): s worked hard.
    - C(s): s is in Professor Cho’s class.
- **Some students studied hard but did not pass the exam**

# Exercise 6: Translate to formal logic

- Given:
    - S: set of students
    - P(s): s passed the exam.
    - W(s): s worked hard.
    - C(s): s is in Professor Cho's class.
- **There are students who did not study hard but passed the exam**

# Exercise 7: Translate to formal logic

- Given:
  - S: set of students
  - P(s): s passed the exam.
  - W(s): s worked hard.
  - C(s): s is in Professor Cho's class.
- **All students who studied hard passed the exam.**

# Exercise 8: Translate to formal logic

- Given:
  - S: set of students
  - P(s): s passed the exam.
  - W(s): s worked hard.
  - C(s): s is in Professor Cho's class.
- **No student in Professor Cho's class failed the exam.**

# Exercise 9: Translate to formal logic

- Given:
    - L: set of locks
    - K set of keys
    - unlock (k,l): k can unlock l
- **Some keys cannot unlock any lock.**

# Negation

# Negation of a universal statement

$$\sim (\forall x \in D, p(x)) \equiv \exists x \in D, \sim p(x)$$

Example: ”All swans are white.”

Negation: "There exists at least one swan that is not white."

# Exercise

- Negate "Every phone on the table is turned off."

# Negation of an existential statement

$$\sim (\exists x \in D, p(x)) \equiv \forall x \in D, \sim p(x)$$

Example: ”There is a car in the parking lot that is electric.”

Negation: “No car in the parking lot is electric”, or “For every car c in the parking lot, c is not electric”

# Exercise

- Negate "There is a person in the village who speaks Italian."

# Summary

$$\sim (\forall x \in D, p(x)) \equiv \exists x \in D, \sim p(x)$$
$$\sim (\exists x \in D, p(x)) \equiv \forall x \in D, \sim p(x)$$

- Negation of a universal statement ("all are") is logically equivalent to an existential statement ("there is at least one that is not")
  Negation of an existential statement ("some are") is logically equivalent to a universal statement ("all are not")

# Exercise

- $\forall$ primes $p$, $p$ is odd

# Exercise

- $\exists$ triangle $T$, sum of angles of $T$ equals $200^\circ$

# Exercise

- No child is left behind

# Common mistakes

## Examples

- All mathematicians wear glasses
  Negation (incorrect): No mathematician wears glasses

  Negation (correct): There is at least one mathematician who does not wear glasses
- Some snowflakes are the same
  Negation (incorrect):: Some snowflakes are different
  Negation (correct):: All snowflakes are different

# Negation of universal conditional statements

**Definition**

- Formally,

$$\begin{aligned} \sim (\forall x, p(x) \to q(x)) &\equiv \exists x, \sim (p(x) \to q(x)) \\ &\equiv \exists x, (p(x) \wedge \sim q(x)) \end{aligned}$$

# Negation of universal conditional statements

## Definition

- Formally,

$$\begin{aligned} \sim (\forall x, p(x) \to q(x)) &\equiv \exists x, \sim (p(x) \to q(x)) \\ &\equiv \exists x, (p(x) \wedge \sim q(x)) \end{aligned}$$

## Examples

- $\forall$ real $x$, if $x > 10$, then $x^2 > 100$.
  Negation: $\exists$ real $x$ such that $x > 10$ and $x^2 \leq 100$.
- If a computer program has more than 100,000 lines, then it contains a bug.
  Negation: There is at least one computer program that has more than 100,000 lines and does not contain a bug.

# Negation of propositions with multiple quantifiers

$$\begin{aligned}
&\sim (\forall x \text{ in } D,\ \exists y \text{ in } E \text{ such that } P(x,y)) \\
&\equiv \exists x \text{ in } D \text{ such that } \sim (\exists y \text{ in } E \text{ such that } P(x,y)) \\
&\equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E,\ \sim P(x,y)
\end{aligned}$$

$$\begin{aligned}
&\sim (\exists x \text{ in } D \text{ such that } \forall y \text{ in } E,\ P(x,y)) \\
&\equiv \forall x \text{ in } D,\ \sim (\forall y \text{ in } E,\ P(x,y)) \\
&\equiv \forall x \text{ in } D,\ \exists y \text{ in } E \text{ such that } \sim P(x,y)
\end{aligned}$$

# Exercise

- Do some research: Formal definition of continuity of a real-valued function f on a point x

- Give a formal definition of f being discontinuous on x

# Exercise: Translate to formal logic, and then negate

- Given:
  - $P(x)$: x is a person.
  - $Q(x)$: x is busy.
  - $R(x, y)$: x likes y.
- **All people are busy.**

# Exercise: Translate to formal logic, and then negate

- Given:
  - $P(x)$: x is a person.
  - $Q(x)$: x is busy.
  - $R(x, y)$: x likes y.
- **Some people are not busy.**

# Exercise: Translate to formal logic, and then negate

- Given:
  - $P(x)$: x is a person.
  - $Q(x)$: x is busy.
  - $R(x, y)$: x likes y.
- **Every person likes themselves.**

# Exercise: Translate to formal logic, and then negate

- Given:
    - $P(x)$: x is a person.
    - $Q(x)$: x is busy.
    - $R(x, y)$: x likes y.
- **There's someone who doesn't like themselves.**

# Exercise: Translate to formal logic, and then negate

- Given:
  - $P(x)$: x is a person.
  - $Q(x)$: x is busy.
  - $R(x, y)$: x likes y.
- **There's at least one person that everyone likes.**

# Exercise: Translate to formal logic, and then negate

- Given:
  - $P(x)$: x is a person.
  - $Q(x)$: x is busy.
  - $R(x, y)$: x likes y.
- **Everyone likes at least one person.**

# Exercise: Translate to formal logic, and then negate

- Given:
    - P(x): x is a doctor.
    - Q(x): x is busy.
    - R(x, y): x likes y.
- **Some people don't like themselves.**