CSE215
Foundations of Computer Science

State University of New York, Korea

Previous lectures

Truth table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Equivalence laws

Laws	Formula	Formula
Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Propositional logic: A formal language to express facts and argue about them

Valid arguments

Premise$_1$
Premise$_2$
$\vdots$
Premise$_m$
$\therefore$ Conclusion

Inference

Name	Rule	Name	Rule	
Modus Ponens	$p \to q$ p $\therefore q$	Elimination	$p \vee q$ $\sim q$ $\therefore p$	$p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	Transitivity	$p \to q$ $q \to r$ $\therefore p \to r$	

Plan

- Revision on using Inference rules for validity
- Predicate Logic, or propositions with Quantifiers
- Negation

Revision on Using Inference Rules for Validity

In 41–44 a set of premises and a conclusion are given. Use the valid argument forms listed in Table 2.3.1 to deduce the conclusion from the premises, giving a reason for each step as in Example 2.3.8. Assume all variables are statement variables.

41. a. $\sim p \vee q \to r$

b. $s \vee \sim q$

c. $\sim t$

d. $p \to t$

e. $\sim p \wedge r \to \sim s$

f. $\therefore \sim q$

42. a. $p \vee q$

b. $q \to r$

c. $p \wedge s \to t$

d. $\sim r$

e. $\sim q \to u \wedge s$

f. $\therefore t$

Propositions with Quantifiers

Why quantifiers? To express "for all" and "there exists"

- Everyone can make mistake
- Nobody is perfect
- Every lock has a key
- There is a key for every lock
- Every nonzero real number has a reciprocal

CS example 1:
Software security

Question: Could the program print “bad”?

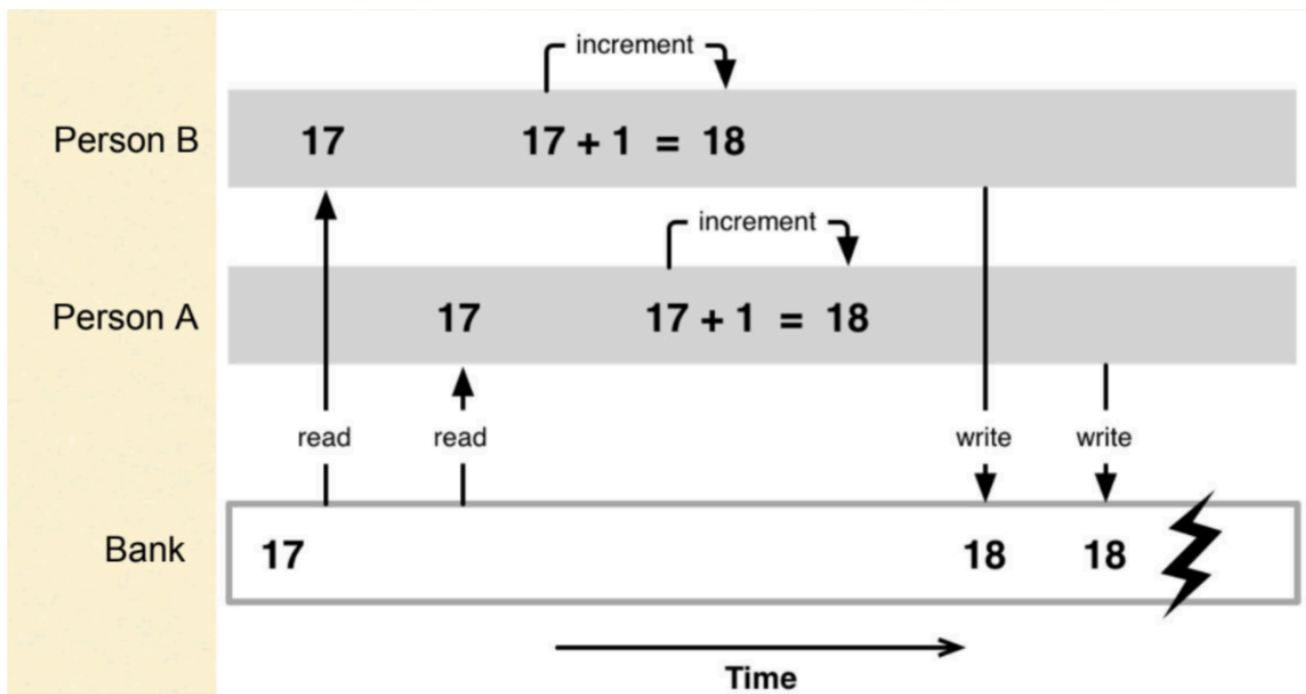
```c
1  #include <stdio.h>
2  
3  void f(int input) {
4      char a[8];
5      int b = 0;
6      a[input] = 1;
7  
8      if (b == 0)
9          printf("good\n");
10     else
11         printf("bad\n");
12 }
```

Exisitential Statement

: there exists an integer n, such that b !=0 at line 8 when executing f(n)

CS example 2: Concurrency

Two persons are trying to deposit 1 dollar online into the same bank account.



Universal Statement:
For all CPU schedule s, A should not read not read while B intends to write.

Predicate

- A propositional function or predicate is a sentence that contains one or more variables
- A predicate is neither true nor false
- A predicate becomes a proposition when the variables are substituted with specific values
- The domain of a predicate variable is the set of all values that may be substituted for the variable

Examples

Symbol	Predicate	Domain	Propositions
$p(x)$	$x > 5$	$x \in \mathbb{R}$	$p(6), p(-3.6), p(0), \ldots$
$p(x, y)$	$x + y$ is odd	$x \in \mathbb{Z}, y \in \mathbb{Z}$	$p(4, 5), p(-4, -4), \ldots$

Universal statement

- Let $p(x)$ be a predicate and D be the domain of x
- A universal statement is a statement of the form

$$\forall x \in D, p(x)$$

- Forms:
  - "$p(x)$ is true for all values of x"
  - "For all x, $p(x)$"
  - "For each x, $p(x)$"
  - "For every x, $p(x)$"
  - "Given any x, $p(x)$"

Existential statement

- Let $p(x)$ be a predicate and D be the domain of x
- An existential statement is a statement of the form

$$\exists x \in D, p(x)$$

- Forms:
  - "There exists an x such that $p(x)$"
  - "For some x, $p(x)$"
  - "We can find an x, such that $p(x)$"
  - "There is some x such that $p(x)$"
  - "There is at least one x such that $p(x)$"

Examples

Universal st.s	Domain
$\exists x \in D, x^2 \geq x$	$D = \{1, 2, 3\}$
$\exists x \in \mathbb{R}, x^2 \geq x$	$\mathbb{R}$

Propositions with multiple quantifiers

$$\forall x \in D, \exists y \in E, \text{such that } p(x, y)$$

Every lock has a key.

Propositions with multiple quantifiers

$$\exists x \in D, \forall y \in E, \text{such that } p(x, y)$$

There is a key for every lock.

Propositions with multiple quantifiers

$$\forall x \in D, \forall y \in E, \text{such that } p(x, y)$$

"Give me a place to stand, and a lever long enough, and I will move the Earth."

Propositions with multiple quantifiers

$$\exists x \in D, \exists y \in E, \text{such that } p(x, y)$$

"There is someone in a park sitting on a bench".

Four Notes

Note 1:
the order of quantifiers matter

- Every lock has a key
- For any lock L, there exists a key K, such that K can unlock L.
- There is a key for every lock
- There exists a key K, such that for any lock L, K can unlock L.

Note 2: A commonly used notational equivalence

$$\forall x \in D, p(x)$$

Equivalen to

$$\forall x, x \in D \rightarrow p(x)$$

Example: All doctors wear glasses

- for all d, if d is a doctor, then d wears glasses
- Formally, if we define
  - D to be the set of doctors,
  - wear_glass to be a function that takes a person x as an input, and returns true if x wears glasses
- then the following two statements are considered the same

$$\forall d \in D, \texttt{wear\_glass}(d).$$

$$\forall d, d \in D \to \texttt{wear\_glass}(d)$$

Note 3: Universal conditional statement

- A universal conditional statement is of the form

$$\forall x, \text{ if } p(x) \text{ then } q(x)$$

Examples

- $\forall x \in \mathbb{R}$, if $x > 2$ then $x^2 > 4$
- $\forall$ real number x, if x is an integer then x is rational
  $\forall$ integer x, x is rational
- $\forall x$, if x is a square then x is a rectangle
  $\forall$ square x, x is a rectangle

Note 4: Implicit quantifiers

Examples

- If a number is an integer, then it is a rational number
  Implicit meaning: $\forall$ number x, if x is an integer, x is rational
- The number 10 can be written as a sum of two prime numbers
  Implicit meaning: $\exists$ prime numbers p and q such that $10 = p+q$
- If $x > 2$, then $x^2 > 4$
  Implicit meaning: $\forall$ real x, if $x > 2$, then $x^2 > 4$

Definition

- Let $p(x)$ and $q(x)$ be predicates and D be the common domain of x. Then implicit quant. symbols $\Rightarrow, \Leftrightarrow$ are defined as:

$$p(x) \Rightarrow q(x) \equiv \forall x, p(x) \rightarrow q(x)$$
$$p(x) \Leftrightarrow q(x) \equiv \forall x, p(x) \leftrightarrow q(x)$$

Exercises

Exercise 1
translate to formal logic

Everyone can make mistake

Exercise 2
translate to formal logic

Nobody is perfect

Exercise 3
translate to formal logic

Every lock has a key

Exercise 4
translate to formal logic

There is a key for every lock

Exercise 5
translate to formal logic

Every nonzero real number has a reciprocal

The reciprocal of 4 is 1/4 (namely 0.25)

Solution

- Everyone can make mistake
  - (ok) for any person p, p can make mistake
  - (better) for any person p, there exists t in p’s life time, make_mistake(p, t)
- Nobody is perfect
  - (ok) there does not exist p, such that p is perfect
  - (better) for every person p, p is not perfect
- Every lock has a key
  - for every lock l, there is a key k such that k can unlock l
- There is a key for every lock
  - there is a key k, such that for every lock l, k can unlock l
- Every nonzero real number has a reciprocal
  - for any nonzero real number r, there exists a real number s, such that r * s = 1

Exercise: Translate to formal logic

- Given:
  - S: set of students
  - P(s): s passed the exam.
  - W(s): s worked hard.
  - C(s): s is in Professor Cho’s class.
- **Every student in Professor Cho’s class passed the exam**

Exercise: Translate to formal logic

- Given:
    - S: set of students
    - P(s): s passed the exam.
    - W(s): s worked hard.
    - C(s): s is in Professor Cho's class.
- **Some students studied hard but did not pass the exam**

Exercise: Translate to formal logic

- Given:
  - S: set of students
  - P(s): s passed the exam.
  - W(s): s worked hard.
  - C(s): s is in Professor Cho’s class.
- **There are students who did not study hard but passed the exam**

Exercise: Translate to formal logic

- Given:
  - S: set of students
  - P(s): s passed the exam.
  - W(s): s worked hard.
  - C(s): s is in Professor Cho’s class.
- **All students who studied hard passed the exam.**

Exercise: Translate to formal logic

- Given:
  - S: set of students
  - P(s): s passed the exam.
  - W(s): s worked hard.
  - C(s): s is in Professor Cho’s class.
- **No student in Professor Cho’s class failed the exam.**

Exercise: Translate to formal logic

- Given:
    - S: set of students
    - P(s): s passed the exam.
    - W(s): s worked hard.
    - C(s): s is in Professor Cho’s class.
- **There are no students in Professor Cho’s class who did not study hard but still passed the exam.**

Exercise: Translate to formal logic

- Given:
  - L: set of locks
  - K set of keys
  - unlock (k,l): k can unlock l
- **Some keys cannot unlock any lock.**

Negation

Negation of a universal statement

$$\sim (\forall x \in D, p(x)) \equiv \exists x \in D, \sim p(x)$$

Example: ”All swans are white.”

Negation: "There exists at least one swan that is not white."

Exercise

- Negate "Every phone on the table is turned off."

Negation of an existential statement

$$\sim (\exists x \in D, p(x)) \equiv \forall x \in D, \sim p(x)$$

Example: ”There is a car in the parking lot that is electric.”

Negation: “No car in the parking lot is electric”, or “For every car c in the parking lot, c is not electric”

Exercise

- Negate "There is a person in the village who speaks Italian."

Summary

$$\sim (\forall x \in D, p(x)) \equiv \exists x \in D, \sim p(x)$$
$$\sim (\exists x \in D, p(x)) \equiv \forall x \in D, \sim p(x)$$

- Negation of a universal statement ("all are") is logically equivalent to an existential statement ("there is at least one that is not")
  Negation of an existential statement ("some are") is logically equivalent to a universal statement ("all are not")

Exercise

- $\forall$ primes p, p is odd

Exercise

- $\exists$ triangle T, sum of angles of T equals 200°

Exercise

- No child is left behind

Common mistakes

Examples

- All mathematicians wear glasses
  Negation (incorrect): No mathematician wears glasses

  Negation (correct): There is at least one mathematician who does not wear glasses
- Some snowflakes are the same
  Negation (incorrect):: Some snowflakes are different
  Negation (correct):: All snowflakes are different

Negation of universal conditional statements

Definition

- Formally,

$$\begin{aligned} \sim (\forall x, p(x) \rightarrow q(x)) &\equiv \exists x, \sim (p(x) \rightarrow q(x)) \\ &\equiv \exists x, (p(x) \wedge \sim q(x)) \end{aligned}$$

Negation of universal conditional statements

Definition

- Formally,

$$\begin{aligned}\sim(\forall x, p(x) \to q(x)) &\equiv \exists x, \sim(p(x) \to q(x)) \\ &\equiv \exists x, (p(x) \land \sim q(x))\end{aligned}$$

Examples

- $\forall$ real x, if $x > 10$, then $x^2 > 100$.
  Negation: $\exists$ real x such that $x > 10$ and $x^2 \leq 100$.
- If a computer program has more than 100,000 lines, then it contains a bug.
  Negation: There is at least one computer program that has more than 100,000 lines and does not contain a bug.

Negation of propositions with multiple quantifiers

$$\begin{aligned}
&\sim (\forall x \text{ in } D,\ \exists y \text{ in } E \text{ such that } P(x,y)) \\
&\equiv \exists x \text{ in } D \text{ such that } \sim (\exists y \text{ in } E \text{ such that } P(x,y)) \\
&\equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E,\ \sim P(x,y)
\end{aligned}$$

$$\begin{aligned}
&\sim (\exists x \text{ in } D \text{ such that } \forall y \text{ in } E,\ P(x,y)) \\
&\equiv \forall x \text{ in } D,\ \sim (\forall y \text{ in } E,\ P(x,y)) \\
&\equiv \forall x \text{ in } D,\ \exists y \text{ in } E \text{ such that } \sim P(x,y)
\end{aligned}$$

Exercise

- Do some research: Formal definition of continuity of a real-valued function f on a point x

- Give a formal definition of f being discontinuous on x

Exercise: Translate to formal logic, and then negate

- Given:
    - $P(x)$: x is a person.
    - $Q(x)$: x is busy.
    - $R(x, y)$: x likes y.
- **All people are busy.**

Exercise: Translate to formal logic, and then negate

- Given:
    - $P(x)$: x is a person.
    - $Q(x)$: x is busy.
    - $R(x, y)$: x likes y.
- **Some people are not busy.**

Exercise: Translate to formal logic, and then negate

- Given:
    - $P(x)$: x is a person.
    - $Q(x)$: x is busy.
    - $R(x, y)$: x likes y.
- **Every person likes themselves.**

Exercise: Translate to formal logic, and then negate

- Given:
    - $P(x)$: x is a person.
    - $Q(x)$: x is busy.
    - $R(x, y)$: x likes y.
- **There's someone who doesn't like themselves.**

Exercise: Translate to formal logic, and then negate

- Given:
  - P(x): x is a person.
  - Q(x): x is busy.
  - R(x, y): x likes y.
- **There's at least one person that everyone likes.**

Exercise: Translate to formal logic, and then negate

- Given:
  - $P(x)$: x is a person.
  - $Q(x)$: x is busy.
  - $R(x, y)$: x likes y.
- **Everyone likes at least one person.**

Exercise: Translate to formal logic, and then negate

- Given:
  - $P(x)$: x is a doctor.
  - $Q(x)$: x is busy.
  - $R(x, y)$: x likes y.
- **Some people don't like themselves.**