CSE215
Foundations of Computer Science

State University of New York, Korea

Some other proof tech.

If-and-only-if

- Prove: The integer n is odd if and only if n^2 is odd.
- To prove A if and only if B, first prove A -> B, then prove B ->A

Solution

Proposition The integer n is odd if and only if n^2 is odd.

Proof. First we show that n being odd implies that n^2 is odd. Suppose n is odd. Then, by definition of an odd number, $n = 2a+1$ for some integer a. Thus $n^2 = (2a+1)^2 = 4a^2+4a+1 = 2(2a^2+2a)+1$. This expresses n^2 as twice an integer, plus 1, so n^2 is odd.

Conversely, we need to prove that n^2 being odd implies that n is odd. We use contrapositive proof. Suppose n is not odd. Then n is even, so $n = 2a$ for some integer a (by definition of an even number). Thus $n^2 = (2a)^2 = 2(2a^2)$, so n^2 is even because it's twice an integer. Thus n^2 is not odd. We've now proved that if n is not odd, then n^2 is not odd, and this is a contrapositive proof that if n^2 is odd then n is odd. ■

Equivalent statements

Theorem Suppose A is an $n \times n$ matrix. The following statements are equivalent:

(a) The matrix A is invertible.
(b) The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$.
(c) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
(d) The reduced row echelon form of A is I_n.
(e) $\det(A) \neq 0$.
(f) The matrix A does not have 0 as an eigenvalue.

(How to actually prove this in details is not a part of our lecture)

$$\begin{array}{ccccc} (a) & \Longrightarrow & (b) & \Longrightarrow & (c) \\ \Uparrow & & & & \Downarrow \\ (f) & \Longleftarrow & (e) & \Longleftarrow & (d) \end{array}$$

$$\begin{array}{ccccc} (a) & \Longrightarrow & (b) & \Longleftrightarrow & (c) \\ \Uparrow & & \Downarrow & & \\ (f) & \Longleftarrow & (e) & \Longleftrightarrow & (d) \end{array}$$

$$\begin{array}{ccccc} (a) & \Longleftrightarrow & (b) & \Longleftrightarrow & (c) \\ & & \Updownarrow & & \\ (f) & \Longleftrightarrow & (e) & \Longleftrightarrow & (d) \end{array}$$

Uniqueness Proof

- Prove: there is a unique function f defined over R such that $f'(x)=2x$ and $f(0)=3$
- To prove there is a unique x such that P(x):
  - first prove there exist an x,
  - then prove “x and y both satisfy P, then x = y”

Solution

Proof. *Existence*: $f(x) = x^2 + 3$ works.

Uniqueness: If $f_0(x)$ and $f_1(x)$ both satisfy these conditions, then $f_0'(x) = 2x = f_1'(x)$, so they differ by a constant, i.e., there is a C such that $f_0(x) = f_1(x) + C$. Hence, $3 = f_0(0) = f_1(0) + C = 3 + C$. This gives $C = 0$ and so $f_0(x) = f_1(x)$. ■

Non constructive proof

Proposition There exist irrational numbers x, y for which x^y is rational.

Solution

Proposition There exist irrational numbers x, y for which x^y is rational.

Proof. Let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$. We know y is irrational, but it is not clear whether x is rational or irrational. On one hand, if x is irrational, then we have an irrational number to an irrational power that is rational:

$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2.$$

On the other hand, if x is rational, then $y^y = \sqrt{2}^{\sqrt{2}} = x$ is rational. Either way, we have a irrational number to an irrational power that is rational. ∎

Review Exercises

SBU 2020 Midterm

Problem 8. [5 points]

Prove that for all integers a, if a^3 is even, then a is even.

- Proof.
  - We want to prove ________________
  - We use **proof by contraposition** to prove the statement above.
  - That is, we want to prove __________
  - Assume ______
  - _______
  - Therefore _______
- QED.

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Problem 8. [5 points]

Prove that for any two integers a and b, if ab is odd, then a and b are both odd.

- Proof.
  - We want to prove ________________
  - We use **proof by contradiction** to prove the statement above.
  - That is, we assume __________ (A)
  - From this assumption, ______
  - So, we get a contradiction with (A)
- QED.

SBU 2020 Midterm

Problem 6. [5 points]
Prove that the product of any four consecutive integers is a multiple of 4.

- Proof.
  - Suppose ____. We want to prove ________________ (Q)
  - We prove the statement by **division into cases.**
  - Case 1: _____. In this case, we have ____.
  - Case 2: _____. In this case, we have ____.
  - Thus, (Q) holds in either case.
- QED.

SBU 2021 Midterm

Problem 8. [5 points]

Prove by contradiction that there are no integers x and y such that $x^2 = 4y + 2$.

- Proof.
  - (formal statement of proof objectives)
  - (our proof strategy, derived proof objectives, assumptions)
  - (core proof)
- QED.

Three SBU exam problems

- Prove: Given an integer a, then $a^3 + a^2 + a$ is even if and only if a is even.

SBU 2022 Midterm

Problem 6. [5 points]

Let $a_1, a_2, \ldots, a_n$ be real numbers for $n \geq 1$. Prove that at least one of these numbers is greater or equal to the average of the numbers.

SBU 2022 Midterm

Problem 7. [5 points]

Prove or disprove the following statement. If x and y are rational, then x^y is rational.