CSE215
Foundations of Computer Science

State University of New York, Korea

Agenda

- Size of infinite sets
- Five classic examples

Cardinality/Size of Infinite sets

- There are as many squares as there are numbers because they are just as numerous as their roots.

    - — Galileo Galilei, 1632

Same cardinality

Definition

- Let A and B be any sets. A has the same cardinality as B if, and only if, there is a one-to-one correspondence from A to B.
- A has the same cardinality as B if, and only if, there is a function f from A to B that is both one-to-one and onto.

Example 1

Integers and even numbers are of the same size

$$\begin{array}{|c|ccc|c|}
\boxed{\mathbb{Z}} & & & \boxed{\mathbb{Z}^{\text{even}}} \\
\vdots & & & \vdots \\
-4 & \longrightarrow & & -8 \\
-3 & \longrightarrow & & -6 \\
-2 & \longrightarrow & & -4 \\
-1 & \longrightarrow & & -2 \\
0 & \longrightarrow & & 0 \\
1 & \longrightarrow & & 2 \\
2 & \longrightarrow & & 4 \\
3 & \longrightarrow & & 6 \\
4 & \longrightarrow & & 8 \\
\vdots & & & \vdots
\end{array}$$

Proof

Problem

- Prove that the cardinality of integers and even numbers are the same.

Solution

Problem

- Prove that the cardinality of integers and even numbers are the same.

- To prove that $|\mathbb{Z}| = |\mathbb{Z}^{\text{even}}|$, we need to prove that there is a one-to-one correspondence, say f, between $\mathbb{Z}$ and $\mathbb{Z}^{\text{even}}$. Suppose $f = 2n$ for all integers $n \in \mathbb{Z}$.
- Prove that f is one-to-one.
  Suppose $f(n_1) = f(n_2)$.
  $\implies 2n_1 = 2n_2 \qquad (\because \text{Defn. of } f)$
  $\implies n_1 = n_2 \qquad (\because \text{Simplify})$
- Prove that f is onto.
  Suppose $m \in \mathbb{Z}^{\text{even}}$.
  $\implies m \text{ is even} \qquad (\because \text{Defn. of } \mathbb{Z}^{\text{even}})$
  $\implies m = 2k \text{ for } k \in \mathbb{Z} \qquad (\because \text{Defn. of even})$
  $\implies f(k) = m \qquad (\because \text{Defn. of } f)$

An infinite set and its proper subset can have the same size!

Countable sets

$\mathbb{N}$		A
1	$\longrightarrow$	"First" element of A
2	$\longrightarrow$	"Second" element of A
3	$\longrightarrow$	"Third" element of A
4	$\longrightarrow$	"Fourth" element of A
5	$\longrightarrow$	"Fifth" element of A
$\vdots$		$\vdots$

Definition

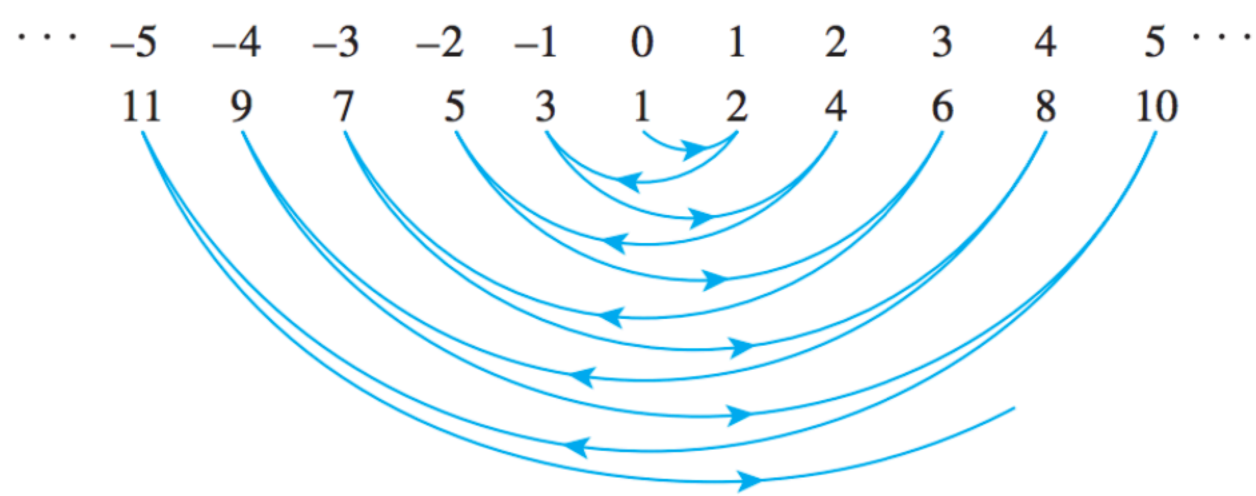
- A set is called countably infinite if, and only if, it has the same cardinality as the set of positive integers.
- A set is called countable if, and only if, it is finite or countably infinite. A set that is not countable is called uncountable.

Example 2

Problem

- Prove that the set of integers is countably infinite.

Intuition



Integers are countable

Solution (continued)

$$
\begin{array}{|c|ccc|c|}
\boxed{\mathbb{N}} & & & & \boxed{\mathbb{Z}} \\
1 & \longrightarrow & 0 \\
2 & \longrightarrow & 1 \\
3 & \longrightarrow & -1 \\
4 & \longrightarrow & 2 \\
5 & \longrightarrow & -2 \\
\vdots & & \vdots \\
n & \longrightarrow & f(n) \\
\vdots & & \vdots
\end{array}
$$

- Define a function $f(n) : \mathbb{N} \to \mathbb{Z}$ such that

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is an even natural number,} \\ -\left(\frac{n-1}{2}\right) & \text{if } n \text{ is an odd natural number.} \end{cases}$$

- As f is a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{Z}$, the set of integers is countably infinite.

Example 3: True or False?

Set of positive rationals is uncountable

$\boxed{\mathbb{N}}$		$\boxed{\mathbb{Q}^+}$
		$\vdots$
1	$\longrightarrow$	$\frac{1}{1}$
		$\vdots$
2	$\longrightarrow$	$\frac{2}{1}$
		$\vdots$
3	$\longrightarrow$	$\frac{3}{1}$
$\vdots$		$\vdots$

$$
\begin{array}{ccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\[2ex]
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \cdots \\[2ex]
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \cdots \\[2ex]
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \cdots \\[2ex]
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \frac{5}{6} & \cdots \\[2ex]
\frac{6}{1} & \frac{6}{2} & \frac{6}{3} & \frac{6}{4} & \frac{6}{5} & \frac{6}{6} & \cdots \\[2ex]
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Set of positive rationals is countable

Problem

- Prove that the set of positive rational numbers are countable.

Set of positive rationals is countable

Problem

- Prove that the set of positive rational numbers are countable.

Solution

$$
\begin{array}{ccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \cdots \\
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \frac{5}{6} & \cdots \\
\frac{6}{1} & \frac{6}{2} & \frac{6}{3} & \frac{6}{4} & \frac{6}{5} & \frac{6}{6} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

$\mathbb{N}$		$\mathbb{Q}^+$
1	$\longrightarrow$	$1/1$
2	$\longrightarrow$	$1/2$
3	$\longrightarrow$	$2/1$
4	$\longrightarrow$	$3/1$
5	$\longrightarrow$	$1/3$
6	$\longrightarrow$	$1/4$
7	$\longrightarrow$	$2/3$
8	$\longrightarrow$	$3/2$
9	$\longrightarrow$	$4/1$
10	$\longrightarrow$	$5/1$
$\vdots$		$\vdots$

Set of positive rational numbers is countable

Problem

- Prove that the set of positive rational numbers are countable.

Solution (continued)

- To prove that $|\mathbb{N}| = |\mathbb{Q}^+|$, we need to prove that there is a one-to-one correspondence, say f, between $\mathbb{N}$ and $\mathbb{Q}^+$.
- Prove that f is onto.
  Every positive rational number appears somewhere in the grid.
  Every point in the grid is reached eventually.
- Prove that f is one-to-one.
  Skipping numbers that have already been counted ensures that no number is counted twice.

Example 4

Set of real numbers in $[0, 1]$ is uncountable

Problem

- Prove that the set of all real numbers between 0 and 1 is uncountable.

Set of real numbers in $[0, 1]$ is uncountable

Problem

- Prove that the set of all real numbers between 0 and 1 is uncountable.

Solution

- To prove that $|\mathbb{N}| \neq |[0..1]|$, we need to prove that there is no one-to-one correspondence between $\mathbb{N}$ and $[0..1]$.
- A powerful approach to prove the theorem is:
  proof by contradiction.

Set of real numbers in $[0, 1]$ is uncountable

Problem

- Prove that the set of all real numbers between 0 and 1 is uncountable.

Solution

Proof by contradiction.

- Suppose $[0..1]$ is countable.
- We will derive a contradiction by showing that there is a number in $[0..1]$ that does not appear on this list.

$$
\begin{array}{|c|c|c|}
\boxed{\mathbb{N}} & & \boxed{[0..1]} \\
 & & \\
1 & \longrightarrow & 0.a_{11}a_{12}a_{13}\ldots a_{1n}\ldots \\
2 & \longrightarrow & 0.a_{21}a_{22}a_{23}\ldots a_{2n}\ldots \\
3 & \longrightarrow & 0.a_{31}a_{32}a_{33}\ldots a_{3n}\ldots \\
\vdots & \vdots & \vdots \\
n & \longrightarrow & 0.a_{n1}a_{n2}a_{n3}\ldots a_{nn}\ldots \\
\vdots & \vdots & \vdots
\end{array}
$$

Set of real numbers in $[0, 1]$ is uncountable

Solution (continued)

- Suppose the list of reals starts out as follows:

$$\begin{array}{ccccccc}
0. & \boxed{9} & 0 & 1 & 4 & 8 & \dots \\
0. & 1 & \boxed{1} & 6 & 6 & 6 & \dots \\
0. & 0 & 3 & \boxed{3} & 5 & 3 & \dots \\
0. & 9 & 6 & 7 & \boxed{2} & 6 & \dots \\
0. & 0 & 0 & 0 & 3 & \boxed{1} & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

- Construct a new number $d = 0.d_1d_2d_3\dots d_n\dots$ as follows:

$$d_n = \begin{cases} 1 & a_{nn} \neq 1, \\ 2 & a_{nn} = 1. \end{cases}$$

- We have $d = 0.12112\dots$, i.e.,

$$\begin{array}{ccccccc}
0. & \boxed{1} & \boxed{2} & \boxed{1} & \boxed{1} & \boxed{2} & \dots
\end{array}$$

Set of real numbers in $[0, 1]$ is uncountable

Solution (continued)

- Observation:
  For each natural number n, the constructed real number d differs in the nth decimal position from the nth number on the list.

$$
\begin{array}{cccccccccc}
1 & \longrightarrow & 0. & \boxed{9} & 0 & 1 & 4 & 8 & \ldots \\
2 & \longrightarrow & 0. & 1 & \boxed{1} & 6 & 6 & 6 & \ldots \\
3 & \longrightarrow & 0. & 0 & 3 & \boxed{3} & 5 & 3 & \ldots \\
4 & \longrightarrow & 0. & 9 & 6 & 7 & \boxed{2} & 6 & \ldots \\
5 & \longrightarrow & 0. & 0 & 0 & 0 & 3 & \boxed{1} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\hline
d & \longrightarrow & 0. & \boxed{1} & \boxed{2} & \boxed{1} & \boxed{1} & \boxed{2} & \ldots \\
\hline
\end{array}
$$

- This implies that d is not on the list. But, $d \in [0, 1]$.
- Contradiction! So, our supposition is false.
- Set of real numbers in $[0, 1]$ is uncountable.

There are different types of ∞!

Accepting a lemma in this course

- Given a and b, the open and closed intervals [a, b], (a, b), [a, b) and (a, b] all have the same cardinality

- Details: https://planetmath.org/openandclosedintervalshavethesamecardinality

Exercises

Exercise 1 (20 points)

The *sigmoid* function plays a pivotal role in machine learning. Particularly, it's instrumental in classification problems where we map predicted values to probabilities. The sigmoid function can squish any real-valued number into the range between 0 and 1, making it extremely useful for converting values into probabilities.

The sigmoid function S: $\mathbb{R} \rightarrow (0, 1)$, is defined as:

$$S(x) = \frac{1}{1 + e^{-x}}$$

Your task in this exercise is to check if this sigmoid function S is bijective. In other words, determine whether it's both injective (or one-to-one) and surjective (or onto).

As a reminder:

- Injectivity: Show that if S(x) = S(y), then x = y. This means that no two different inputs will yield the same output.
- Surjectivity: Show that for any number y in the range (0, 1), there is an x in the domain of real numbers such that S(x) = y. This means that every possible output is produced by some input.

- Prove real numbers and positive real number are of the same cardinality

- Find a bijection between [1,2] and [1,4] to show they are of the same cardinality

3. Let $3\mathbf{Z} = \{n \in \mathbf{Z} \mid n = 3k, \text{ for some integer } k\}$. Prove that $\mathbf{Z}$ and $3\mathbf{Z}$ have the same cardinality.

4. Let $\mathbf{O}$ be the set of all odd integers. Prove that $\mathbf{O}$ has the same cardinality as $2\mathbf{Z}$, the set of all even integers.

True or false, and proof

(h) [1 point] The size of the set of all multiples of 6 is less than the size of the set of all multiples of 3.

(i) [1 point] The size of the set of rational numbers is the same as the size of the set of real numbers in the range $[0, 0.0000001]$.

(j) [1 point] The size of the set of real numbers in the range $[1, 2]$ is the same or larger than the size of the set of real numbers in the range $[1, 4]$.