# CSE215
# Foundations of Computer Science

**State University of New York, Korea**

# Proof by contraposition

# Review Exercise

- Are P-> Q and~Q -> ~P equivalent, and why?

# Contraposition, Contrapostive

- contraposition = inference of going from a conditional statement into its logically equivalent contrapositive
- P -> Q
  - contrapositive is ~Q -> ~P
- "If it is raining, then I wear my coat”
  - contrapositive is "If I don't wear my coat, then it isn't raining.”
- (This slide is taken from Wikipedia)

# Proof by contraposition

- Proposition: P -> Q
- Proof
  - Suppose ~Q
  - ….
  - Therefore ~P
- QED.

# Example 1

- Suppose x is an arbitrary integer. Prove: If 7x + 9 is even, then x is odd.
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**Proposition** Suppose $x \in \mathbb{Z}$. If $7x+9$ is even, then $x$ is odd.

*Proof.* (Direct) Suppose $7x+9$ is even.
Thus $7x+9=2a$ for some integer $a$.
Subtracting $6x+9$ from both sides, we get $x = 2a-6x-9$.
Thus $x = 2a-6x-9 = 2a-6x-10+1 = 2(a-3x-5)+1$.
Consequently $x = 2b+1$, where $b = a-3x-5 \in \mathbb{Z}$.
Therefore $x$ is odd. ■

Here is a contrapositive proof of the same statement:

**Proposition** Suppose $x \in \mathbb{Z}$. If $7x+9$ is even, then $x$ is odd.

*Proof.* (Contrapositive) Suppose $x$ is not odd.
Thus $x$ is even, so $x = 2a$ for some integer $a$.
Then $7x+9 = 7(2a)+9 = 14a+8+1 = 2(7a+4)+1$.
Therefore $7x+9 = 2b+1$, where $b$ is the integer $7a+4$.
Consequently $7x+9$ is odd.
Therefore $7x+9$ is not even. ■

# Example 2

**Prove:** If $x^2 - 6x + 5$ is even, then $x$ is odd.

**Proposition** Suppose $x \in \mathbb{Z}$. If $x^2 - 6x + 5$ is even, then $x$ is odd.

*Proof.* (Contrapositive) Suppose $x$ is not odd.
Thus $x$ is even, so $x = 2a$ for some integer $a$.
So $x^2 - 6x + 5 = (2a)^2 - 6(2a) + 5 = 4a^2 - 12a + 5 = 4a^2 - 12a + 4 + 1 = 2(2a^2 - 6a + 2) + 1$.
Therefore $x^2 - 6x + 5 = 2b + 1$, where $b$ is the integer $2a^2 - 6a + 2$.
Consequently $x^2 - 6x + 5$ is odd.
Therefore $x^2 - 6x + 5$ is not even. ■

# Example 3

**Prove:** Suppose $x, y \in \mathbb{R}$. If $y^3 + yx^2 \leq x^3 + xy^2$, then $y \leq x$.

*Proof.* (Contrapositive) Suppose it is not true that $y \leq x$, so $y > x$. Then $y - x > 0$. Multiply both sides of $y - x > 0$ by the positive value $x^2 + y^2$.

$$
\begin{aligned}
(y-x)(x^2+y^2) &> 0(x^2+y^2) \\
yx^2 + y^3 - x^3 - xy^2 &> 0 \\
y^3 + yx^2 &> x^3 + xy^2
\end{aligned}
$$

Therefore $y^3 + yx^2 > x^3 + xy^2$, so it is not true that $y^3 + yx^2 \leq x^3 + xy^2$. ■

# Example 4

**Prove:** If $x^2 - 6x + 5$ is even, then $x$ is odd.

*Proof.* (Contrapositive) Suppose $x$ is not odd.
Thus $x$ is even, so $x = 2a$ for some integer $a$.
So $x^2-6x+5 = (2a)^2-6(2a)+5 = 4a^2-12a+5 = 4a^2-12a+4+1 = 2(2a^2-6a+2)+1$.
Therefore $x^2-6x+5 = 2b+1$, where $b$ is the integer $2a^2-6a+2$.
Consequently $x^2-6x+5$ is odd.
Therefore $x^2-6x+5$ is not even. ■

# Example 5

**Prove:** Suppose $x, y \in \mathbb{Z}$. If $5 \nmid xy$, then $5 \nmid x$ and $5 \nmid y$.

*Proof.* (Contrapositive) Suppose it is not true that $5 \nmid x$ **and** $5 \nmid y$.
By DeMorgan's law, it is not true that $5 \nmid x$ **or** it is not true that $5 \nmid y$.
Therefore $5 \mid x$ or $5 \mid y$. We consider these possibilities separately.
**Case 1.** Suppose $5 \mid x$. Then $x = 5a$ for some $a \in \mathbb{Z}$.
From this we get $xy = 5(ay)$, and that means $5 \mid xy$.
**Case 2.** Suppose $5 \mid y$. Then $y = 5a$ for some $a \in \mathbb{Z}$.
From this we get $xy = 5(ax)$, and that means $5 \mid xy$.
The above cases show that $5 \mid xy$, so it is not true that $5 \nmid xy$. ■

# Exercises

# Warm up

1. Suppose $n \in \mathbb{Z}$. If $n^2$ is even, then $n$ is even.
2. Suppose $n \in \mathbb{Z}$. If $n^2$ is odd, then $n$ is odd.
3. Suppose $a, b \in \mathbb{Z}$. If $a^2(b^2 - 2b)$ is odd, then $a$ and $b$ are odd.

**4.** Suppose $a,b,c \in \mathbb{Z}$. If $a$ does not divide $bc$, then $a$ does not divide $b$.

**5.** Suppose $x \in \mathbb{R}$. If $x^2 + 5x < 0$ then $x < 0$.

**6.** Suppose $x \in \mathbb{R}$. If $x^3 - x > 0$ then $x > -1$.

**9.** Suppose $n \in \mathbb{Z}$. If $3 \nmid n^2$, then $3 \nmid n$.

**10.** Suppose $x, y, z \in \mathbb{Z}$ and $x \neq 0$. If $x \nmid yz$, then $x \nmid y$ and $x \nmid z$.

**11.** Suppose $x, y \in \mathbb{Z}$. If $x^2(y+3)$ is even, then $x$ is even or $y$ is odd.

# $n^2$ is even $\implies$ $n$ is even

Prove:

Suppose n is an integer.  If n^2 is even, then n is even

# Exercise 2: Prove the following

Suppose $x, y \in \mathbb{Z}$. If $x^2(y+3)$ is even, then $x$ is even or $y$ is odd.

# Exercise 3: Prove the following

Suppose $x \in \mathbb{R}$. If $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 \geq 0$, then $x \geq 0$.

# Exercise 4: Prove the following

Suppose $a \in \mathbb{Z}$. If $a^2$ is not divisible by 4, then $a$ is odd.

# Exercise 5: Prove the following

Suppose $x \in \mathbb{R}$. If $x^5 + 7x^3 + 5x \geq x^4 + x^2 + 8$, then $x \geq 0$.