CSE215
Foundations of Computer Science

State University of New York, Korea

Agenda

- Size of infinite sets
- Five classic examples

Cardinality/Size of Infinite sets

- There are as many squares as there are numbers because they are just as numerous as their roots.

    - — Galileo Galilei, 1632

Same cardinality

Definition

- Let A and B be any sets. A has the same cardinality as B if, and only if, there is a one-to-one correspondence from A to B.
- A has the same cardinality as B if, and only if, there is a function f from A to B that is both one-to-one and onto.

Properties of infinite sets

Properties

For all sets A, B, and C:

- Reflexive property.
  A has the same cardinality as A.
- Symmetric property.
  If A has the same cardinality as B,
  then B has the same cardinality as A.
- Transitive property.
  If A has the same cardinality as B
  and B has the same cardinality as C,
  then A has the same cardinality as C.

Example 1

Integers and even numbers are of the same size

$$
\begin{array}{|c|ccc|c|}
\boxed{\mathbb{Z}} & & & & \boxed{\mathbb{Z}^{\textsf{even}}} \\
\vdots & & & & \vdots \\
-4 & & \longrightarrow & & -8 \\
-3 & & \longrightarrow & & -6 \\
-2 & & \longrightarrow & & -4 \\
-1 & & \longrightarrow & & -2 \\
0 & & \longrightarrow & & 0 \\
1 & & \longrightarrow & & 2 \\
2 & & \longrightarrow & & 4 \\
3 & & \longrightarrow & & 6 \\
4 & & \longrightarrow & & 8 \\
\vdots & & & & \vdots
\end{array}
$$

Proof

Problem

- Prove that the cardinality of integers and even numbers are the same.

Solution

Problem

- Prove that the cardinality of integers and even numbers are the same.

- To prove that $|\mathbb{Z}| = |\mathbb{Z}^{\text{even}}|$, we need to prove that there is a one-to-one correspondence, say f, between $\mathbb{Z}$ and $\mathbb{Z}^{\text{even}}$. Suppose $f = 2n$ for all integers $n \in \mathbb{Z}$.
- Prove that f is one-to-one.
  Suppose $f(n_1) = f(n_2)$.
  $\implies 2n_1 = 2n_2 \quad (\because \text{Defn. of } f)$
  $\implies n_1 = n_2 \quad (\because \text{Simplify})$
- Prove that f is onto.
  Suppose $m \in \mathbb{Z}^{\text{even}}$.
  $\implies m \text{ is even} \quad (\because \text{Defn. of } \mathbb{Z}^{\text{even}})$
  $\implies m = 2k \text{ for } k \in \mathbb{Z} \quad (\because \text{Defn. of even})$
  $\implies f(k) = m \quad (\because \text{Defn. of } f)$

An infinite set and its proper subset can have the same size!

Countable sets

$\boxed{\mathbb{N}}$		$\boxed{A}$
1	$\longrightarrow$	"First" element of A
2	$\longrightarrow$	"Second" element of A
3	$\longrightarrow$	"Third" element of A
4	$\longrightarrow$	"Fourth" element of A
5	$\longrightarrow$	"Fifth" element of A
$\vdots$		$\vdots$

Definition

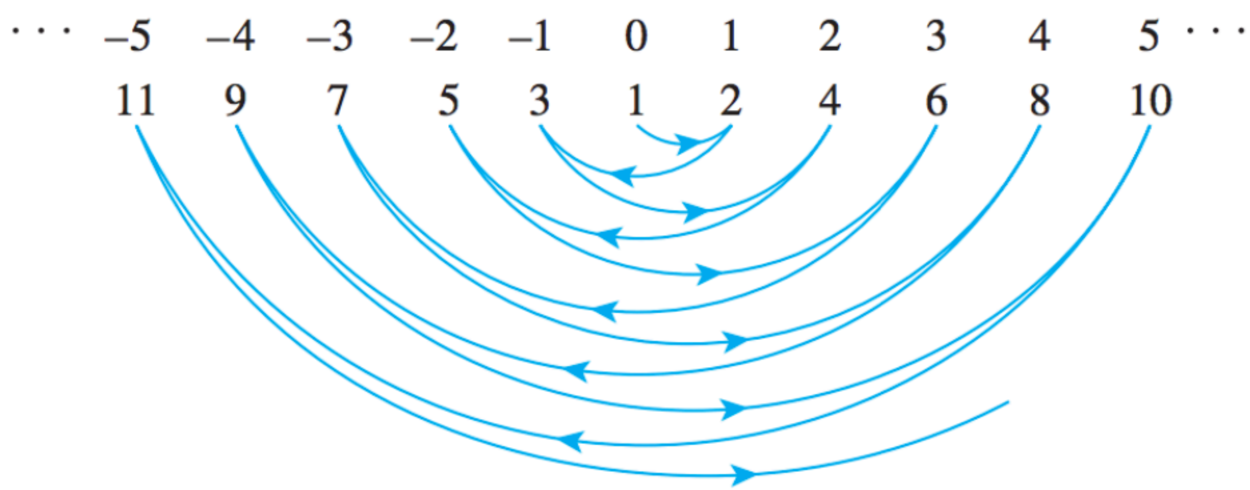
- A set is called countably infinite if, and only if, it has the same cardinality as the set of positive integers.
- A set is called countable if, and only if, it is finite or countably infinite. A set that is not countable is called uncountable.

Example 2

Problem

- Prove that the set of integers is countably infinite.

Intuition



Integers are countable

Solution (continued)

$\boxed{\mathbb{N}}$		$\boxed{\mathbb{Z}}$
1	$\longrightarrow$	0
2	$\longrightarrow$	1
3	$\longrightarrow$	-1
4	$\longrightarrow$	2
5	$\longrightarrow$	-2
$\vdots$		$\vdots$
n	$\longrightarrow$	$f(n)$
$\vdots$		$\vdots$

- Define a function $f(n) : \mathbb{N} \to \mathbb{Z}$ such that

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is an even natural number,} \\ -\left(\frac{n-1}{2}\right) & \text{if } n \text{ is an odd natural number.} \end{cases}$$

- As f is a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{Z}$, the set of integers is countably infinite.

Example 3: True or False?

Set of positive rationals is uncountable

$\boxed{\mathbb{N}}$		$\boxed{\mathbb{Q}^+}$
		$\vdots$
1	$\longrightarrow$	$\frac{1}{1}$
		$\vdots$
2	$\longrightarrow$	$\frac{2}{1}$
		$\vdots$
3	$\longrightarrow$	$\frac{3}{1}$
$\vdots$		$\vdots$

$$
\begin{array}{ccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\[1em]
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \cdots \\[1em]
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \cdots \\[1em]
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \cdots \\[1em]
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \frac{5}{6} & \cdots \\[1em]
\frac{6}{1} & \frac{6}{2} & \frac{6}{3} & \frac{6}{4} & \frac{6}{5} & \frac{6}{6} & \cdots \\[1em]
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Set of positive rationals is countable

Problem

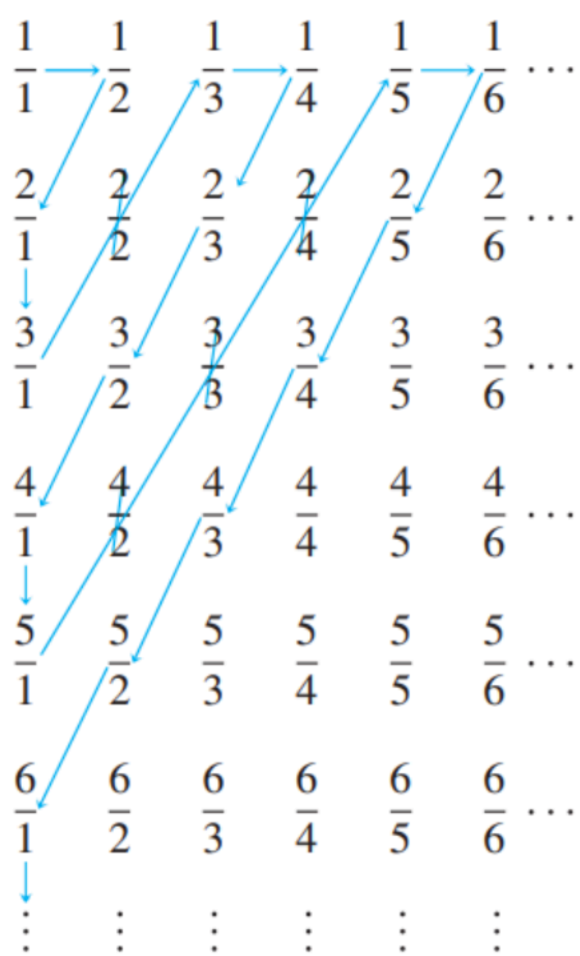
- Prove that the set of positive rational numbers are countable.

Set of positive rationals is countable

Problem

- Prove that the set of positive rational numbers are countable.

Solution



$\mathbb{N}$		$\mathbb{Q}^+$
1	$\longrightarrow$	1/1
2	$\longrightarrow$	1/2
3	$\longrightarrow$	2/1
4	$\longrightarrow$	3/1
5	$\longrightarrow$	1/3
6	$\longrightarrow$	1/4
7	$\longrightarrow$	2/3
8	$\longrightarrow$	3/2
9	$\longrightarrow$	4/1
10	$\longrightarrow$	5/1
$\vdots$		$\vdots$

Set of positive rational numbers is countable

Problem

- Prove that the set of positive rational numbers are countable.

Solution (continued)

- To prove that $|\mathbb{N}| = |\mathbb{Q}^+|$, we need to prove that there is a one-to-one correspondence, say f, between $\mathbb{N}$ and $\mathbb{Q}^+$.
- Prove that f is onto.
  Every positive rational number appears somewhere in the grid.
  Every point in the grid is reached eventually.
- Prove that f is one-to-one.
  Skipping numbers that have already been counted ensures that no number is counted twice.

Example 4

Set of real numbers in $[0, 1]$ is uncountable

Problem

- Prove that the set of all real numbers between 0 and 1 is uncountable.

Set of real numbers in $[0, 1]$ is uncountable

Problem

- Prove that the set of all real numbers between 0 and 1 is uncountable.

Solution

- To prove that $|\mathbb{N}| \neq |[0..1]|$, we need to prove that there is no one-to-one correspondence between $\mathbb{N}$ and $[0..1]$.
- A powerful approach to prove the theorem is: proof by contradiction.

Set of real numbers in $[0, 1]$ is uncountable

Problem

- Prove that the set of all real numbers between 0 and 1 is uncountable.

Solution

Proof by contradiction.

- Suppose $[0..1]$ is countable.
- We will derive a contradiction by showing that there is a number in $[0..1]$ that does not appear on this list.

$$
\begin{array}{|c|c|c|}
\boxed{\mathbb{N}} & & \boxed{[0..1]} \\
 & & \\
1 & \longrightarrow & 0.a_{11}a_{12}a_{13}\ldots a_{1n}\ldots \\
2 & \longrightarrow & 0.a_{21}a_{22}a_{23}\ldots a_{2n}\ldots \\
3 & \longrightarrow & 0.a_{31}a_{32}a_{33}\ldots a_{3n}\ldots \\
\vdots & \vdots & \vdots \\
n & \longrightarrow & 0.a_{n1}a_{n2}a_{n3}\ldots a_{nn}\ldots \\
\vdots & \vdots & \vdots
\end{array}
$$

Set of real numbers in $[0,1]$ is uncountable

Solution (continued)

- Suppose the list of reals starts out as follows:

$$
\begin{array}{ccccccc}
0. & \boxed{9} & 0 & 1 & 4 & 8 & \dots \\
0. & 1 & \boxed{1} & 6 & 6 & 6 & \dots \\
0. & 0 & 3 & \boxed{3} & 5 & 3 & \dots \\
0. & 9 & 6 & 7 & \boxed{2} & 6 & \dots \\
0. & 0 & 0 & 0 & 3 & \boxed{1} & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

- Construct a new number $d = 0.d_1d_2d_3 \dots d_n \dots$ as follows:

$$
d_n = \begin{cases} 1 & a_{nn} \neq 1, \\ 2 & a_{nn} = 1. \end{cases}
$$

- We have $d = 0.12112\dots$, i.e.,

$$
\begin{array}{cccccccc}
0. & \boxed{1} & \boxed{2} & \boxed{1} & \boxed{1} & \boxed{2} & \dots
\end{array}
$$

Set of real numbers in $[0,1]$ is uncountable

Solution (continued)

- Observation:
  For each natural number n, the constructed real number d differs in the nth decimal position from the nth number on the list.

1	$\longrightarrow$	0.	$\boxed{9}$	0	1	4	8	$\ldots$
2	$\longrightarrow$	0.	1	$\boxed{1}$	6	6	6	$\ldots$
3	$\longrightarrow$	0.	0	3	$\boxed{3}$	5	3	$\ldots$
4	$\longrightarrow$	0.	9	6	7	$\boxed{2}$	6	$\ldots$
5	$\longrightarrow$	0.	0	0	0	3	$\boxed{1}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
d	$\longrightarrow$	0.	$\boxed{1}$	$\boxed{2}$	$\boxed{1}$	$\boxed{1}$	$\boxed{2}$	$\ldots$

- This implies that d is not on the list. But, $d \in [0,1]$.
- Contradiction! So, our supposition is false.
- Set of real numbers in $[0,1]$ is uncountable.

There are different types of ∞!

Example 5

$\mathbb{R}$ and $[0, 1]$ have the same size

Problem

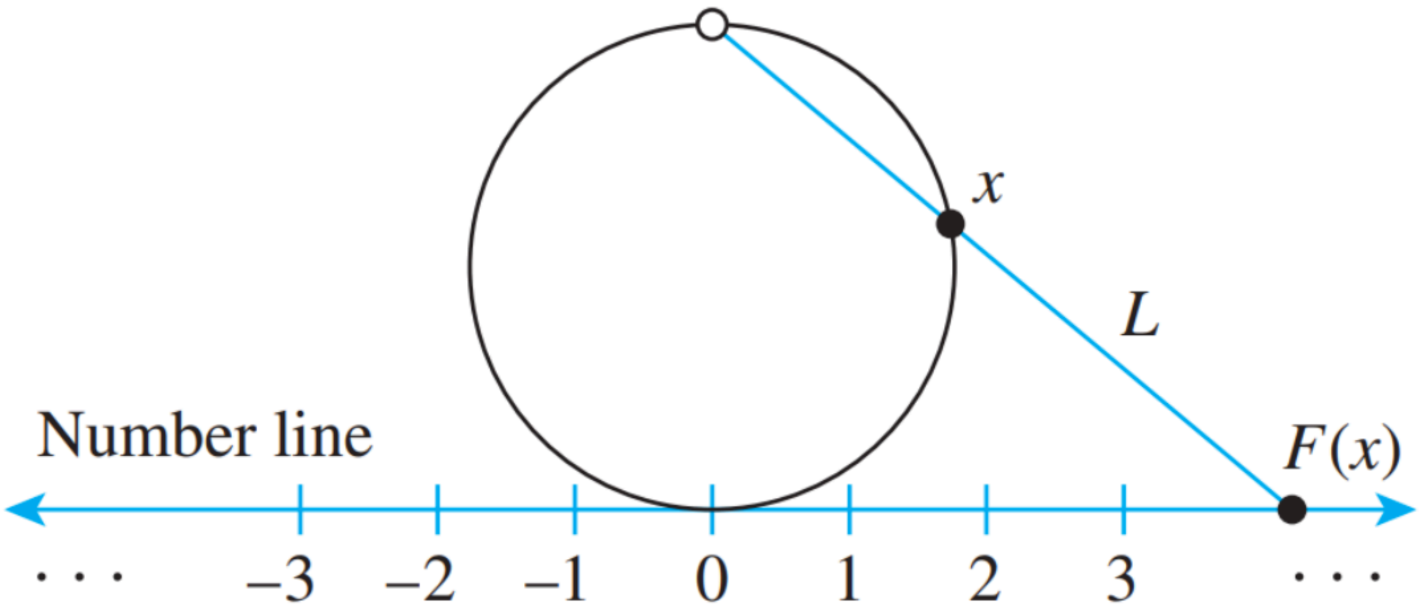
- Prove that the set of all real numbers has the same cardinality as the set of real numbers between 0 and 1.

Related fact: Open and closed Interval have same cardinality.
Proof here: https://planetmath.org/openandclosedintervalshavethesamecardinality

Proof

Solution

- Let $S = \{x \in \mathbb{R} \mid 0 < x < 1\}$
- Bend S to create a circle as shown in the diagram.
- Define $F : S \to \mathbb{R}$ as follows.
- $F(x)$ is called the projection of x onto the number line.

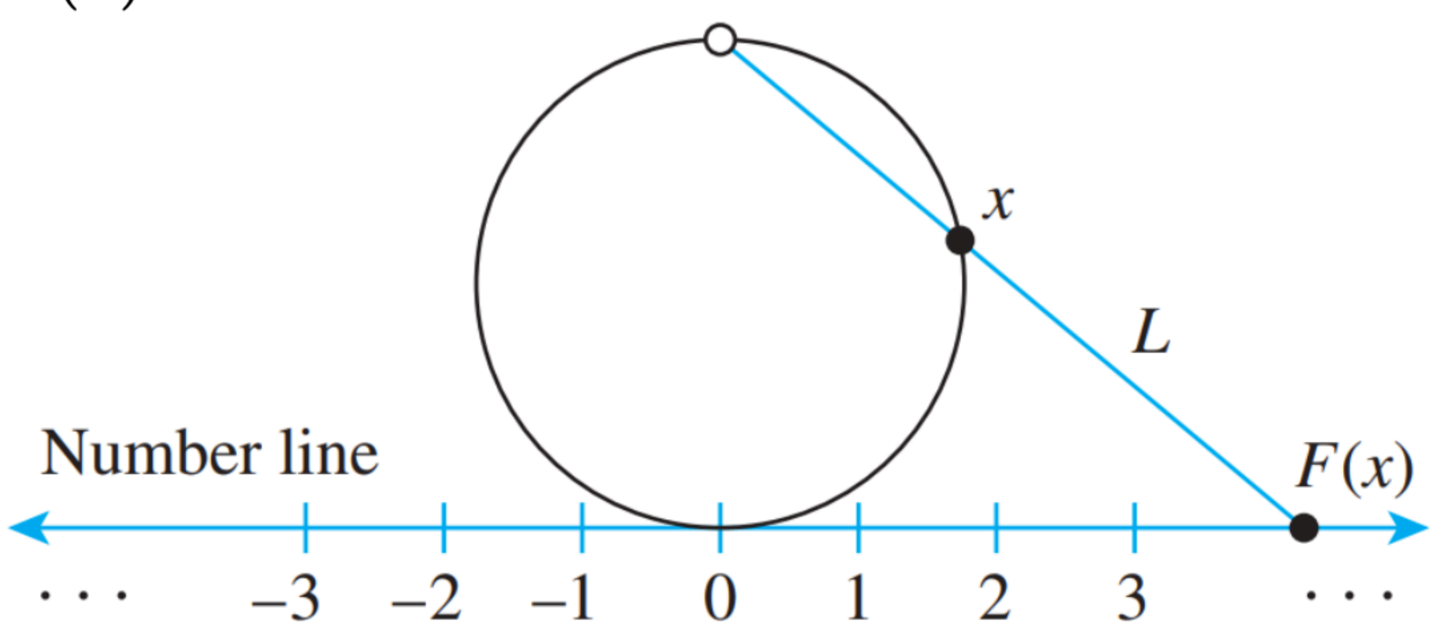


$\mathbb{R}$ and $[0, 1]$ have the same size

Solution (continued)

We show that S and $\mathbb{R}$ have the same cardinality by showing that F is a one-to-one correspondence.

- F is one-to-one. Distinct points on the circle go to distinct points on the number line.
- F is onto. Given any point y on the number line, a line can be drawn through y and the circle's topmost point. This line must intersect the circle at some point x, and, by definition, $y = F(x)$.



Exercises

- Prove real numbers and positive real number are of the same cardinality

2. Show that "there are as many squares as there are numbers" by exhibiting a one-to-one correspondence from the positive integers, $\mathbf{Z}^+$, to the set S of all squares of positive integers:

$$S = \{n \in \mathbf{Z}^+ \mid n = k^2, \text{ for some positive integer } k\}.$$

3. Let $3\mathbf{Z} = \{n \in \mathbf{Z} \mid n = 3k, \text{ for some integer } k\}$. Prove that $\mathbf{Z}$ and $3\mathbf{Z}$ have the same cardinality.

4. Let $\mathbf{O}$ be the set of all odd integers. Prove that $\mathbf{O}$ has the same cardinality as $2\mathbf{Z}$, the set of all even integers.

5. Let $25\mathbf{Z}$ be the set of all integers that are multiples of 25. Prove that $25\mathbf{Z}$ has the same cardinality as $2\mathbf{Z}$, the set of all even integers.

True or false, and proof

(h) [1 point] The size of the set of all multiples of 6 is less than the size of the set of all multiples of 3.

(i) [1 point] The size of the set of rational numbers is the same as the size of the set of real numbers in the range $[0, 0.0000001]$.

(j) [1 point] The size of the set of real numbers in the range $[1, 2]$ is the same or larger than the size of the set of real numbers in the range $[1, 4]$.

Problem 10. [5 points]

Prove that the following set is countable using a diagram and a formula for the one-to-one correspondence function.
$\{\pm 1^1, \pm 2^2, \pm 3^3, \pm 4^4, \pm 5^5, \ldots\}$

Solution

- 0 —> 1^2
- 1 —> -1^2
- 2 —> 2^2
- 3 —> -2^2
- 4 —> 3^2
- 5 —> -3^2
- 6 —> 4^2
- 7 —> -4^2
- So, f(n) = (-1) ^n (n/2+1) if n is even, or (-1) ^n (n+1)/2 if n is odd