CSE215
Foundations of Computer Science

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Some slides taken from Prof. Pramod Ganapathi (Stony Brook). Thanks!

Sequences

"Mathematics is the art of giving the same name to different things." - Henri Poincaré

What are sequences?

Types of sequences

- Finite sequence: $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$
  e.g.: $1^1, 2^2, 3^2, \ldots, 100^2$
- Infinite sequence: $a_m, a_{m+1}, a_{m+2}, \ldots$
  e.g.: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$

Term

- Closed-form formula: $a_k = f(k)$
  e.g.: $a_k = \frac{k}{k+1}$
- Recursive formula: $a_k = g(k, a_{k-1}, \ldots, a_{k-c})$
  e.g.: $a_k = a_{k-1} + (k-1)a_{k-2}$

Growth of sequences

- Increasing sequence

  e.g.: $2, 3, 5, 7, 11, 13, 17, \ldots$
- Decreasing sequence

  e.g.: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$

Sums and products of sequences

Sum

- Summation form:

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

where, $k =$ index, $m =$ lower limit, $n =$ upper limit
e.g.: $\sum_{k=m}^{n} \frac{(-1)^k}{k+1}$

Product

- Product form:

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdot \cdots \cdot a_n$$

where, $k =$ index, $m =$ lower limit, $n =$ upper limit
e.g.: $\prod_{k=m}^{n} \frac{k}{k+1}$

Properties of sums and products

- Suppose $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number

Sum

- $\sum_{k=m}^{n} a_k = \sum_{k=m}^{i} a_k + \sum_{k=i+1}^{n} a_k$ for $m \leq i < n$ where, i is between m and $n-1$ (inclusive)
- $c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n}(c \cdot a_k)$
- $\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n}(a_k + b_k)$

Product

- $(\prod_{k=m}^{n} a_k) \cdot (\prod_{k=m}^{n} b_k) = \prod_{k=m}^{n}(a_k \cdot b_k)$

Proof by
Mathematical Induction

Proof for dominos

Core idea

- A starting domino falls. From the starting domino, every successive domino falls. Then, every domino from the starting domino falls.

Mathematical Induction

- Let P(n) denote a predicate, and our objective is to prove P(n) for all integers n>=0. Mathematical induction works in two steps.

- Base step: prove P(0)

- Inductive step: prove P(k) -> P(k+1) for any k>=0

Proof by mathematical induction: Example 1

Proposition

- $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$ for all integers $n \geq 2$.

Proof by mathematical induction: Example 1

Proposition

- $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$ for all integers $n \geq 2$.

Proof

Let $P(n)$ denote $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$.

- Basis step. $P(2)$ is true. $\triangleright$ How?
- Induction step.

  Assume $P(k)$: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)=\frac{1}{k}$ for some $k \geq 2$.

  Prove $P(k+1)$: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k+1}\right)=\frac{1}{k+1}$

  LHS of $P(k+1)$

  $=\left[\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)\right]\left(1-\frac{1}{k+1}\right)$

  $=\frac{1}{k}\left(1-\frac{1}{k+1}\right)$ $\quad$ ($\because$ $P(k)$ is true)

  $=\frac{1}{k}\cdot\frac{k}{k+1}$ $\quad$ ($\because$ common denominator)

  $=\frac{1}{k+1}$ $\quad$ ($\because$ remove common factor)

  $=$ RHS of $P(k+1)$

Proof by mathematical induction: Example 2

Proposition

- Fibonacci sequence is: $F(0) = 1$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Prove that: $F(0)^2 + F(1)^2 + \cdots + F(n)^2 = F(n)F(n+1)$ for all $n \geq 0$.

Proof by Mathematical Induction — Guideline

- The problem usually looks like this:
  - Prove: For any integer n >=1, P(n) holds
- Check P(1), P(2) … before the proof
- Try to solve P(k) -> P(k+1)
- Write a proof that has:
  - A notation for predicate P(n)
  - A base step
  - An inductive step

Exercises

Use mathematical induction to prove the following identities.

(a) [5 points] For all integers $n \geq 1$**,**

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

Use mathematical induction to prove the following identities.

(a) [5 points] For all integers $n \geq 1$,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

- **Proof**.
  - We will prove this proposition with mathematical induction.
  - **Let P(n) denote the predicate** 1(1!) + 2(2!) + … + n(n!) = (n+1)! -1
  - **Base step**: **We prove** P(1).
    - LHS is 1, and RHS is 2! - 1=1
  - **Inductive step: We prove** P(k) -> P(k+1) for all integer k>=1
    - Let k be an arbitrary integer and k>=1.
    - Assume P(k) holds. That is 1(1!) + 2(2!) + … + k(k!) = (k+1)! -1
    - We want to prove P(k+1), namely, 1(1!) + 2(2!) + … + k(k!) + (k+1) (k+1)! = (k+2)! -1
      - LHS can be reduced to (k+1)! -1 + (k+1) (k+1)! following assumption P(k)
      - The latter can be further reduced to (k+2)! -1, namely RHS.
- **QED**.

Use mathematical induction to prove the following identities.

(b) [5 points] Consider the Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$. For all integers $n \geq 2$,

$$f_{n+2} = 1 + \sum_{i=0}^{n} f_i.$$

Use mathematical induction to prove the following identities.

(b) [5 points] Consider the Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$. For all integers $n \geq 2$,

$$f_{n+2} = 1 + \sum_{i=0}^{n} f_i.$$

- Proof.
  - We will prove this proposition with mathematical induction.
  - Let P(n) be the predicate f_{n+2} = 1+ f_0+f_1 +… + f_n
  - Base step: We first prove P(2) holds.
    - LHS = f_4 = 3. RHS = 1+ f_0 + f_1 + f_2 = 1+0+1+1=3
  - Inductive step:: Then we prove P(k) -> P(k+1) for all k>=2
    - Let k be an arbitrary integer and k>=2
    - Assume P(k) holds, namely,   f_{k+2} = 1+ f_0+f_1 +… + f_k
    - We want to prove P(k+1), that is, f_{k+3} = 1+ f_0+f_1 +… + f_k+f_{k+1}
      - LHS = f_{k+2}+f_{k+1} from Definition of Fobonacci Sequence.
      - Thus, LHS = 1+ f_0+f_1 +… + f_k + f_{k+1} following assumption P(k)
      - Thus equals RHS
- QED.=

Plan

- Review exercises on Mathematical Induction
- Recursive sequences

Use mathematical induction to prove the following identities.

(a) [5 points] For all natural numbers n,

$$1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + n^2 \times (n+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$$

Use mathematical induction to prove the following identities.

Final 2020

(a) [5 points] For all natural numbers n,

$$1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + n^2 \times (n+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$$

- Proof.
    - We will prove this proposition with mathematical induction.
    - Let P(n) be the predicate 1^2 * 2 + … + n^2 * (n+1) = n (n+1) (n+2) (3n+1)/12
    - Base step: We first prove P(1) holds.
        - LHS = 1^2 * 2=2. RHS = 1*2*3*4/12 = 2.
    - Inductive step: Then, we prove P(k) -> P(k+1) for all k>=1
        - Let k be an arbitrary integer and k>=1.
        - Assume P(k) holds. That is, 1^2 * 2 + … + k^2 * (k+1) = k (k+1) (k+2) (3k+1)/12
        - We want to prove P(k+1), that is, 1^2 * 2 + … + k^2 * (k+1) + (k+1)^2 * (k+2) = (k+1) (k+2) (k+3) (3k+4)/12
            - LHS = k (k+1) (k+2) (3k+1)/12 + + (k+1)^2 * (k+2) following assumption P(k)
            - The latter can be further reduced to (k+1)(k+2)/12 * (3k^2+k + 12k+12) = RHS
- QED.

Use mathematical induction to prove the following identities.

(b) [5 points] For all natural numbers n,

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n\times(n+1)}=\frac{n}{n+1}$$

Final 2020

Use mathematical induction to prove the following identities.

(b) [5 points] For all natural numbers n,

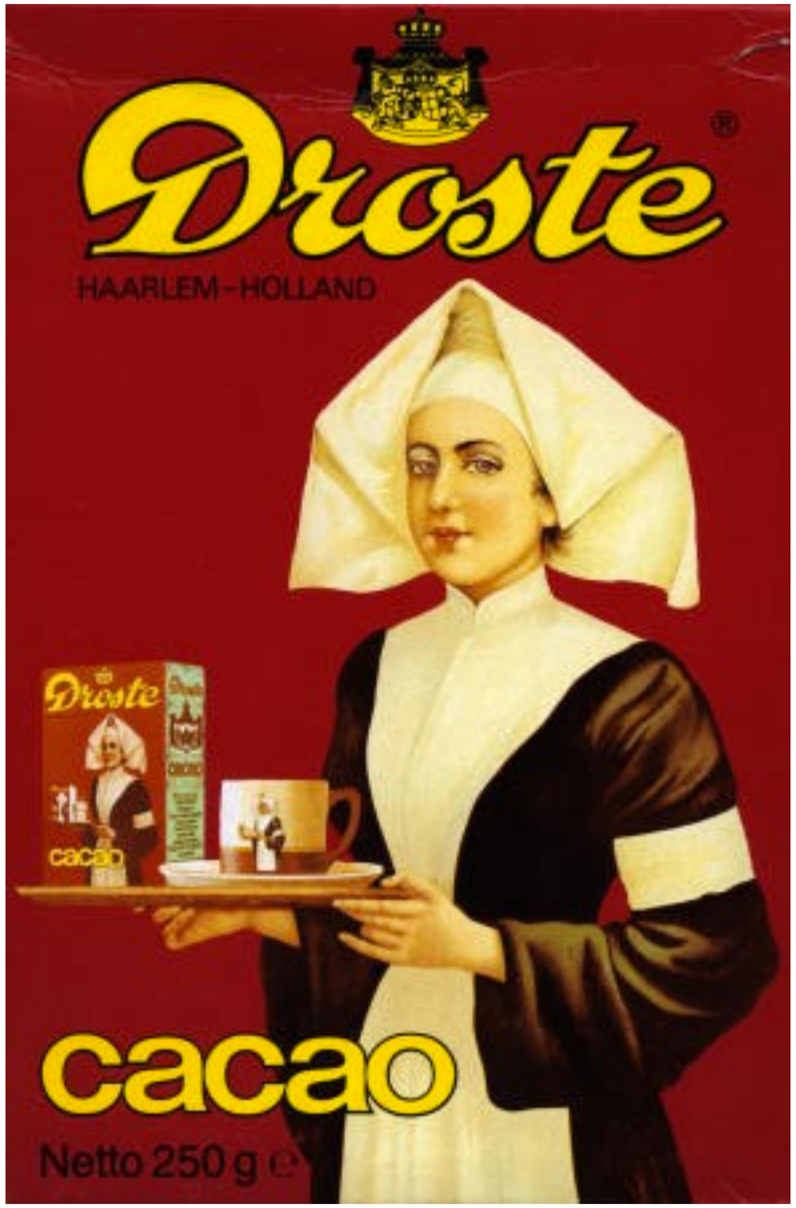
$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n\times(n+1)}=\frac{n}{n+1}$$

- Proof.
  - We will prove this proposition with mathematical induction.
  - Let P(n) be the predicate 1/(1*2) + 1(2*3) + … + 1/(n *(n+1)) = n/(n+1)
  - Base step: We first prove P(1) holds.
    - LHS is 1/2, and RHS is 1/(1+1) = 1/2
  - Inductive step: Then, we prove P(k) -> P(k+1) for all integer k>=1
    - Let k be an arbitrary integer and k>=1.
    - Assume P(k) holds. That is 1/(1*2) + 1(2*3) + … + 1/(k*(k+1)) = k/(k+1)
    - We want to prove P(k+1), namely,1/(1*2) + 1(2*3) + … + 1/(k*(k+1)) + 1/((k+1)*(k+2))= (k+1)/(k+2)
      - LHS can be reduced to k/(k+1) + 1/((k+1)*(k+2)) following the assumption P(k)
      - The latter can be further reduced to (k^2+2k+1)/((k+2)(k+1)), which equals to RHS.
- QED.

Recursive sequences

Recursion = Repeating itself

- recursive sequences
- recursive functions
- recursive data structures



Example

Examples

- Suppose $f(n) = n!$, where $n \in \mathbb{W}$. Then,
  $$f(n) = \begin{cases} 1 & \text{if } n = 0, \\ n \cdot f(n-1) & \text{if } n \geq 1. \end{cases}$$
  Closed-form formula: $f(n) = n \cdot (n-1) \cdot \cdots \cdot 1$

Examples

Examples

- Suppose $f(n) = n!$, where $n \in \mathbb{W}$. Then,
$$f(n) = \begin{cases} 1 & \text{if } \; n = 0, \\ n \cdot f(n-1) & \text{if } \; n \geq 1. \end{cases}$$
Closed-form formula: $f(n) = n \cdot (n-1) \cdot \cdots \cdot 1$
- Suppose $F(n) = n$th Fibonacci number. Then,
$$F(n) = \begin{cases} 1 & \text{if } \; n = 0 \text{ or } 1, \\ F(n-1) + F(n-2) & \text{if } \; n \geq 2. \end{cases}$$
Closed-form formula: $F(n) = ?$

Example: Arithmetic sequence

Let $a_0, a_1, a_2, \ldots$ be the sequence defined recursively as follows: For all integers $k \geq 1$,

$$\text{(1)} \quad a_k = a_{k-1} + 2 \quad \text{recurrence relation}$$

$$\text{(2)} \quad a_0 = 1 \quad \text{initial condition.}$$

- Write out a_1, a_2, a_3, a_4, and a_5
- Derive an explicit formula of the sequence
- Confirm the explicit formula satisfies the recursive definition

Example: Geometric sequence

You deposit $100,000 in a bank account for a 3% interest compounded annually. How much will you get after 21 years?

Solution

- Suppose A_k = Amount in your account after k years. Then,
$$A_k = \begin{cases} 100,000 & \text{if } k = 0, \\ (1 + 3\%) \times A_{k-1} & \text{if } k \geq 1. \end{cases}$$
- Solving the recurrence by the method of iteration, we get
$$\boxed{A_k = ((1.03)^k \cdot 100,000) \text{ dollars}}$$
- Homework:

Confirm the explicit formula satisfies the recursive definition

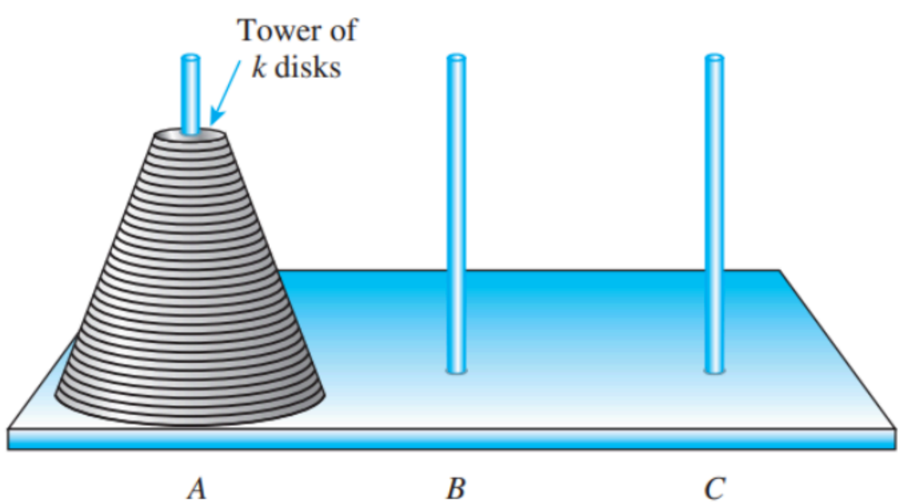
- $$A_{21} = ((1.03)^{21} \cdot 100,000) \approx 186,029.46 \text{ dollars}$$

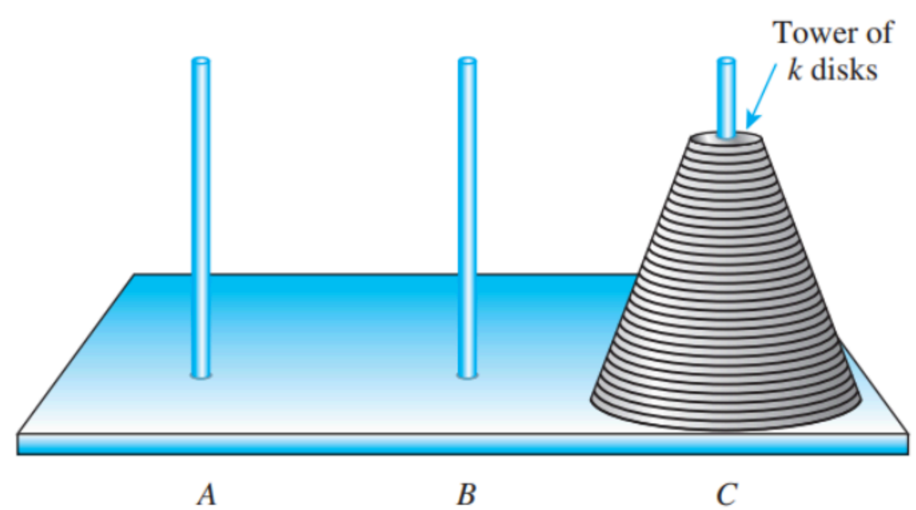
Application

Example: Towers of Hanoi

Problem

- There are k disks on peg 1. Your aim is to move all k disks from peg 1 to peg 3 with the minimum number of moves. You can use peg 2 as an auxiliary peg. The constraint of the puzzle is that at any time, you cannot place a larger disk on a smaller disk.
  What is the minimum number of moves required to transfer all k disks from peg 1 to peg 3?



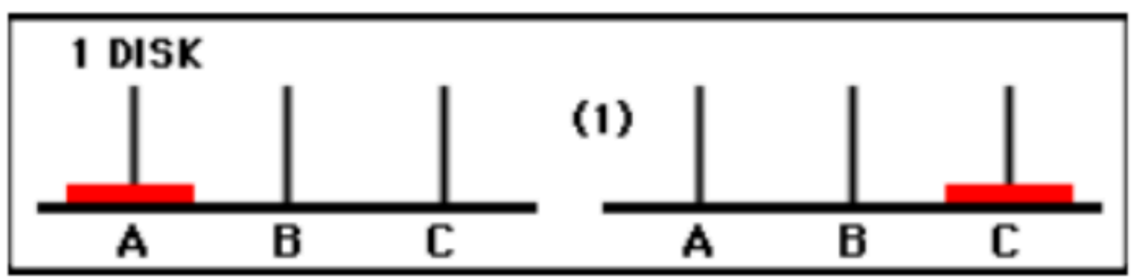


Demo: https://www.mathsisfun.com/games/towerofhanoi.html

Example: Towers of Hanoi

Solution

Suppose $k = 1$. Then, the 1-step solution is:
1. Move disk 1 from peg A to peg C.

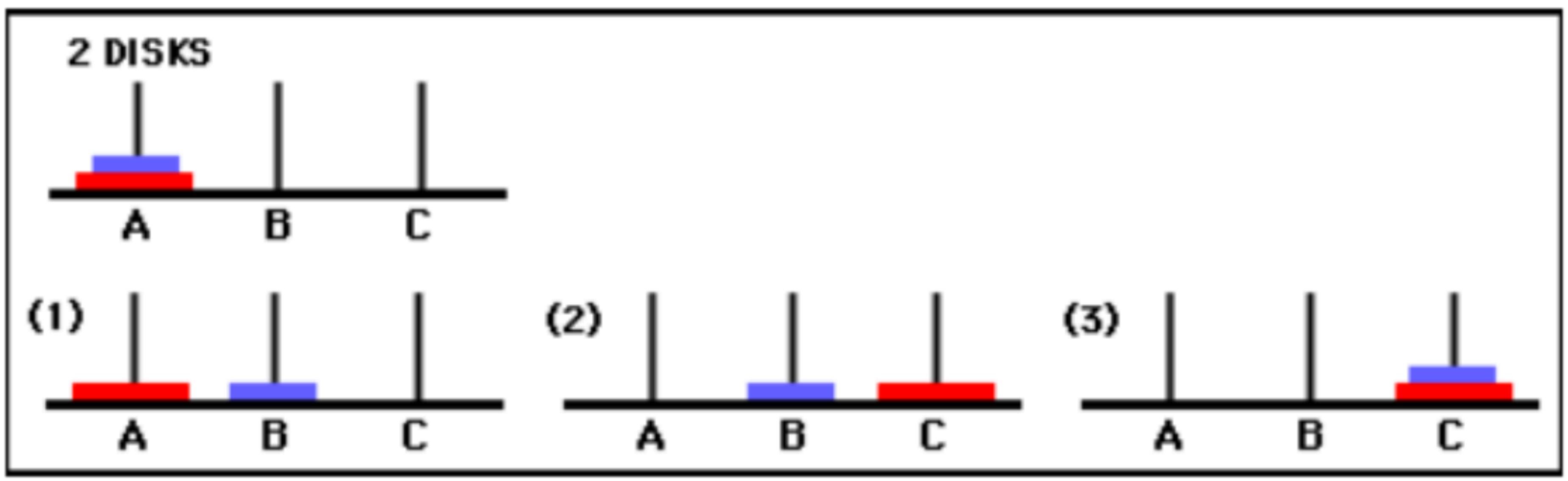


Example: Towers of Hanoi

Solution

Suppose $k = 2$. Then, the 3-step solution is:

1. Move disk 1 from peg A to peg B.
2. Move disk 2 from peg A to peg C.
3. Move disk 1 from peg B to peg C.

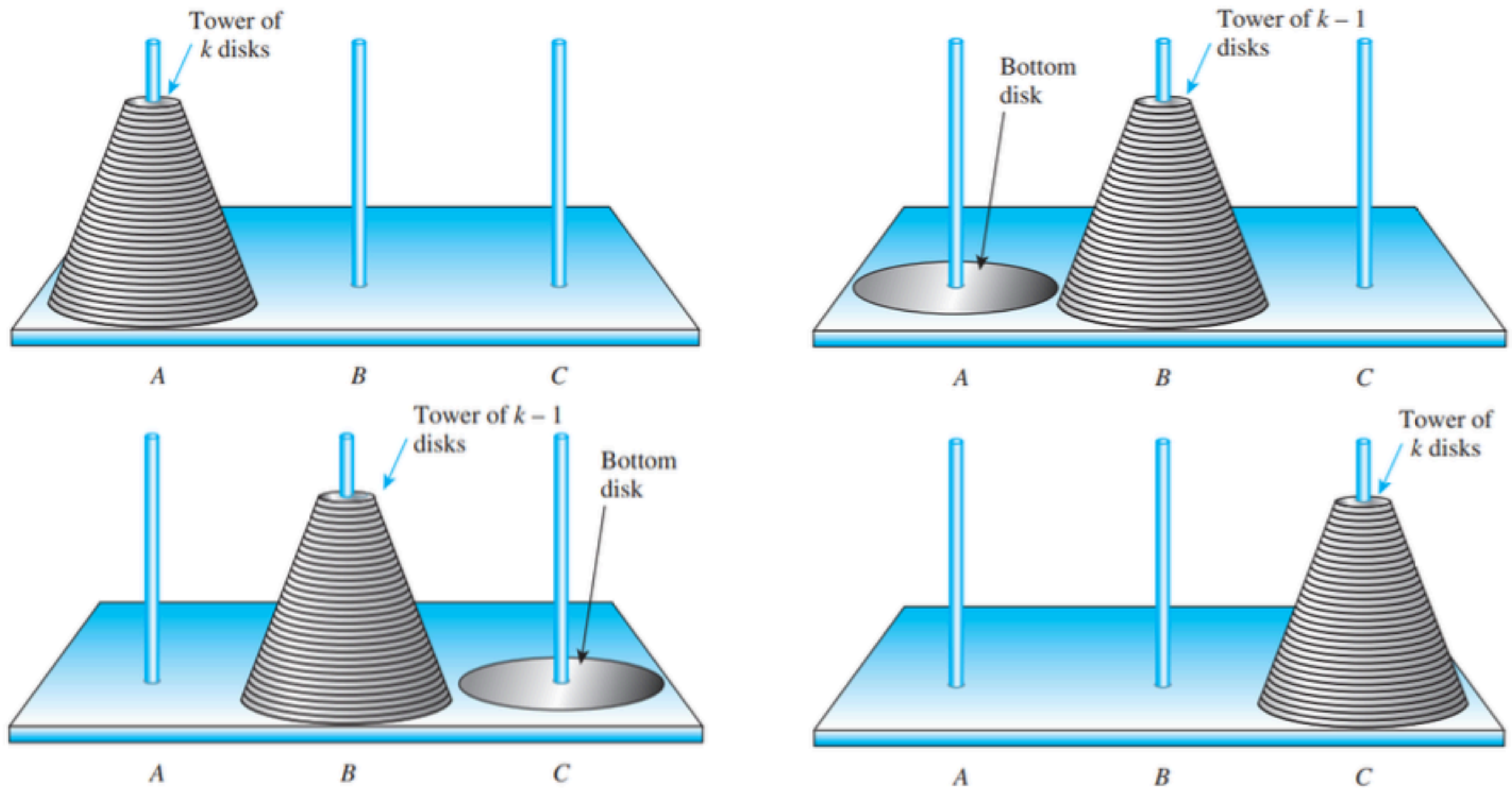


Example: Towers of Hanoi

Solution

For any $k \geq 2$, the recursive solution is:

1. Transfer the top $k-1$ disks from peg A to peg B.
2. Move the bottom disk from peg A to peg C.
3. Transfer the top $k-1$ disks from peg B to peg C.

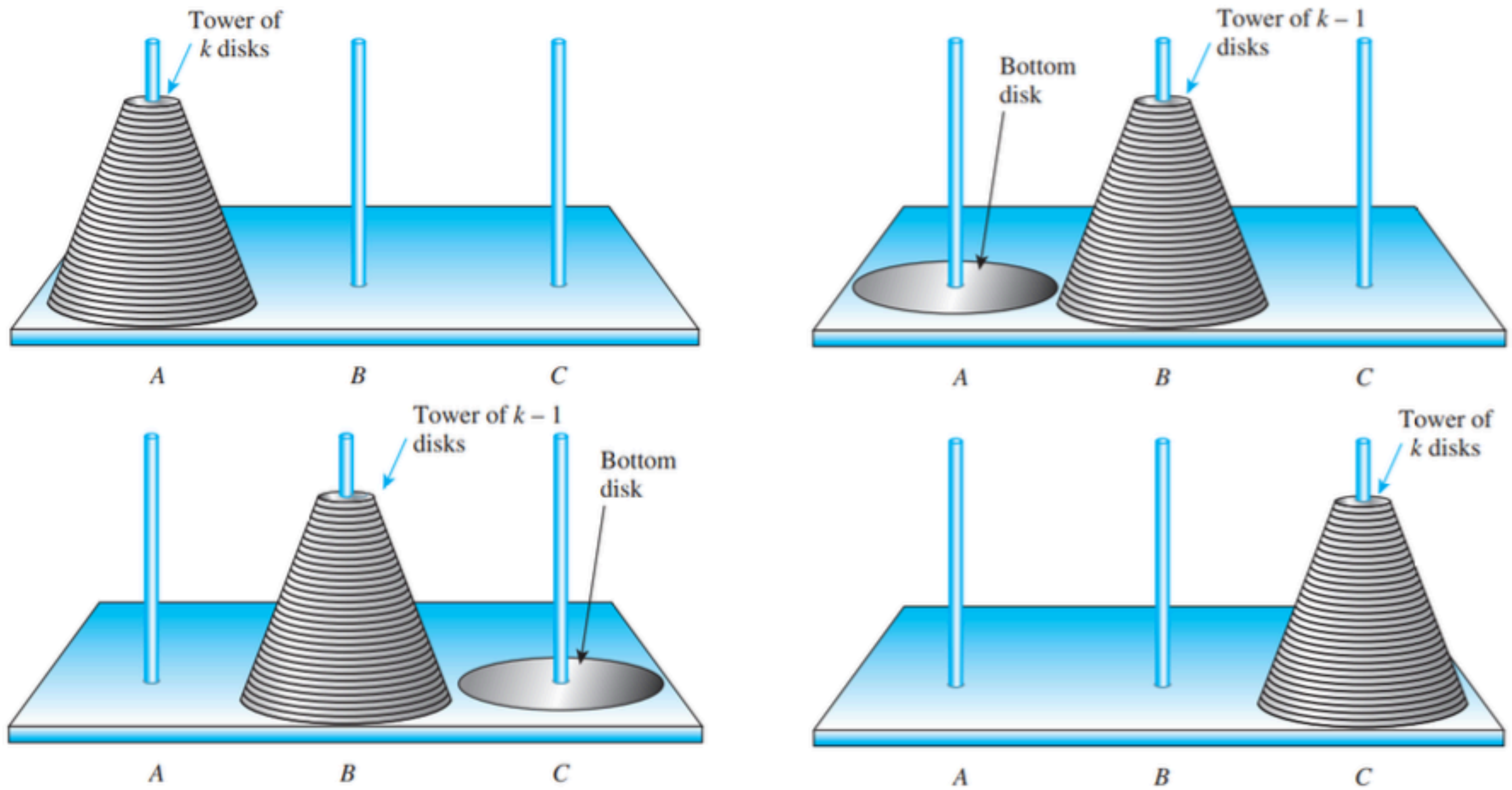


Example: Towers of Hanoi

Code Demo

Towers-of-Hanoi(k, A, C, B)

1. if $k = 1$ then
2. Move disk k from A to C.
3. elseif $k \geq 2$ then
4. Towers-of-Hanoi($k-1, A, B, C$)
5. Move disk k from A to C.
6. Towers-of-Hanoi($k-1, B, C, A$)



Example: Towers of Hanoi

Solution (continued)

- Let $M(k)$ denote the minimum number of moves required to move k disks from one peg to another peg. Then
$$M(k) = \begin{cases} 1 & \text{if } k = 1, \\ 2 \cdot M(k-1) + 1 & \text{if } k \geq 2. \end{cases}$$
- Solving the recurrence by the method of iteration, we get
$$\boxed{M(k) = 2^k - 1}$$
▷ How?

Why minimum?
https://proofwiki.org/wiki/Tower_of_Hanoi

Exercises

Exercise 1

- Consider the recursive sequence below. Find its explicit form and prove your answer, namely Confirm the explicit formula satisfies the recursive definition

$$a_k = ka_{k-1}, \text{ for all integers } k \geq 1$$
$$a_0 = 1$$

Solution

We have: $a_k = k \cdot a_{k-1} = k \cdot (k-1) \cdot a_{k-2} = k(k-1)(k-2) \cdots 1 \cdot a_0 = k!$

- To prove it indeed satisfies the recursive definition, we plug it to the recursive definition:
  - The explicit form clearly satisfies a_0=1
  - The explicit form also satisfies a_k = k a_{k-1}, since the left is k!, the right is k * (k-1)! which is also k!

Exercise 2

- Consider the recursively defined sequence below. Find its explicit form and prove your answer.

$$b_k = \frac{b_{k-1}}{1 + b_{k-1}}, \text{ for all integers } k \geq 1$$
$$b_0 = 1$$

Solution

- We have $b_0=1$, $b_1= 1/2$, $b_2=1/3$, $b3=1/4$… So a possible explicit form is $b_n=1/(n+1)$
- To prove it indeed satisfies the recursive definition, we plug it to the recursive definition:
    - The explicit form clearly satisfies $b_0=1$
    - The explicit form also satisfies $b_k = b_{k-1} / (1+b_{k-1})$, since LHS=$1/(k+1)$, RHS = $1/(k+1) / (1 + 1/(k+1)) = 1/(k+1)$

Exercise 3

- Consider the recursively defined sequence below. Find its explicit form and prove your answer.

$$c_k = 3c_{k-1} + 1, \text{ for all integers } k \geq 2$$
$$c_1 = 1$$

Solution

- We have $c_1=1$, $c_2= 4$, $c_3=13$, b4=40… So a possible explicit form is $c_n=(3^n-1)/2$
- To prove it indeed satisfies the recursive definition, we plug it to the recursive definition:
    - The explicit form clearly satisfies $c_1=1$
    - The explicit form also satisfies_____, since LHS= ___ RHS = ___ = ____