CSE215
Foundations of Computer Science

Instructor: Zhoulai Fu

State University of New York, Korea

Some slides taken from Prof. Pramod Ganapathi (SBU). Thanks!

Previous lectures

Truth table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Equivalence laws

Laws	Formula	Formula
Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Propositional logic: A formal language to express facts and argue about them

Valid arguments

Premise_1
Premise_2
$\vdots$
Premise_m
$\therefore$ Conclusion

Inference

Name	Rule	Name	Rule	
Modus Ponens	$p \to q$ p $\therefore q$	Elimination	$p \vee q$ $\sim q$ $\therefore p$	$p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	Transitivity	$p \to q$ $q \to r$ $\therefore p \to r$	

Today

- Study propositions with quantifiers: for all integers a, b, and c, $(a + b) + c = a + (b + c)$
- Predicate
- Domain
- Quantifier

Quantifiers

Predicate

- A propositional function or predicate is a sentence that contains one or more variables
- A predicate is neither true nor false
- A predicate becomes a proposition when the variables are substituted with specific values
- The domain of a predicate variable is the set of all values that may be substituted for the variable

Examples

Symbol	Predicate	Domain	Propositions
$p(x)$	$x > 5$	$x \in \mathbb{R}$	$p(6), p(-3.6), p(0), \ldots$
$p(x, y)$	$x + y$ is odd	$x \in \mathbb{Z}, y \in \mathbb{Z}$	$p(4, 5), p(-4, -4), \ldots$

Predicate logic example

$$\forall x, P(x) \land Q(x)$$

- This statement asserts that for all objects x, both P(x) and Q(x) are true.

- The symbol ∀ is the **universal quantifier**, which means "for all". P(x) and Q(x) are **predicates** that are evaluated for each object x in the universe of discourse. The ∧ symbol represents the logical connective "and", which requires both P(x) and Q(x) to be true for the whole statement to be true.

Universal statement

- Let $p(x)$ be a predicate and D be the domain of x
- A universal statement is a statement of the form

$$\forall x \in D, p(x)$$

- Forms:
  - "$p(x)$ is true for all values of x"
  - "For all x, $p(x)$"
  - "For each x, $p(x)$"
  - "For every x, $p(x)$"
  - "Given any x, $p(x)$"

Existential statement

- Let $p(x)$ be a predicate and D be the domain of x
- An existential statement is a statement of the form

$$\exists x \in D, p(x)$$

- Forms:
  - "There exists an x such that $p(x)$"
  - "For some x, $p(x)$"
  - "We can find an x, such that $p(x)$"
  - "There is some x such that $p(x)$"
  - "There is at least one x such that $p(x)$"

Examples

Universal st.s	Domain
$\exists x \in D, x^2 \geq x$	$D = \{1, 2, 3\}$
$\exists x \in \mathbb{R}, x^2 \geq x$	$\mathbb{R}$

Universal conditional statement

- A universal conditional statement is of the form

$$\forall x, \text{ if } p(x) \text{ then } q(x)$$

Examples

- $\forall x \in \mathbb{R}$, if $x > 2$ then $x^2 > 4$
- $\forall$ real number x, if x is an integer then x is rational
  $\forall$ integer x, x is rational
- $\forall x$, if x is a square then x is a rectangle
  $\forall$ square x, x is a rectangle

Can be extended to existential conditional statement $(\exists, \rightarrow)$

Implicit quantifiers

Examples

- If a number is an integer, then it is a rational number
  Implicit meaning: $\forall$ number x, if x is an integer, x is rational
- The number 10 can be written as a sum of two prime numbers
  Implicit meaning: $\exists$ prime numbers p and q such that $10 = p+q$
- If $x > 2$, then $x^2 > 4$
  Implicit meaning: $\forall$ real x, if $x > 2$, then $x^2 > 4$

Definition

- Let $p(x)$ and $q(x)$ be predicates and D be the common domain of x. Then implicit quant. symbols $\Rightarrow, \Leftrightarrow$ are defined as:

$$p(x) \Rightarrow q(x) \equiv \forall x, p(x) \rightarrow q(x)$$
$$p(x) \Leftrightarrow q(x) \equiv \forall x, p(x) \leftrightarrow q(x)$$

Propositions with multiple quantifiers

$$\forall x \in D, \exists y \in E, \text{such that } p(x, y)$$

$$\exists x \in D, \forall y \in E, \text{such that } p(x, y)$$

$$\forall x \in D, \forall y \in E, \text{such that } p(x, y)$$

$$\exists x \in D, \exists y \in E, \text{such that } p(x, y)$$

Note:
the order of quantifiers matter

- Every lock has a key
- For any lock L, there exists a key K, such that K can unlock L.
- There is a key for every lock
- There exists a key K, such that for any lock L, K can unlock L.

Exercise 1
translate to formal logic

Everyone can make mistake

Exercise 2
translate to formal logic

Nobody is perfect

Exercise 3
translate to formal logic

Every lock has a key

Exercise 4
translate to formal logic

There is a key for every lock

Exercise 5
translate to formal logic

Every nonzero real number has a reciprocal

The reciprocal of 4 is 1/4 (namely 0.25)

Solution

- Everyone can make mistake
  - for any person p, p can make mistake
- Nobody is perfect
  - there does not exist p, such that p is perfect
  - = for every person p, p is not perfect
- Every lock has a key
  - for every lock l, there is a key k such that k can unlock l
- There is a key for every lock
  - there is a key k, such that for every lock l, k can unlock l
- Every nonzero real number has a reciprocal
  - for any nonzero real number r, there exists a real number s, such that r * s = 1