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1.3 (i) $P(\text{apple}) = P(r) * P(A|r) + P(b) * P(A|b) + P(g) * P(A|g)$

$$= 0.2 * \frac{3}{10} + 0.2 * \frac{1}{2} + 0.6 * \frac{3}{10}$$

$$= 0.34$$

$$P(g|O) = \frac{P(g) * P(O|g)}{P(O)} = \frac{P(g) * P(O|g)}{P(r) * P(O|r) + P(b) * P(O|b) + P(g) * P(O|g)}$$

$$= \frac{0.2 * \frac{1}{2}}{0.2 * \frac{4}{10} + 0.2 * \frac{1}{2} + 0.6 * \frac{3}{10}}$$

$$= \frac{5}{18}$$

1.4.

Consider first the way a function $f(x)$ behaves when we change to a new variable y where the two variables are related by $x=g(y)$, this defines a new function of y given by

$$\tilde{f}(y)=f(g(y)) \cdots$$

suppose $f(x)$ has a mode (such as max) at $\hat{x}$ so that $f'(\hat{x})=0$,

$$\tilde{f}'(\hat{y})=f'(g(\hat{y}))g'(\hat{y})=0$$

Assuming $g'(\hat{y})\neq 0$, then $f'(g(\hat{y}))=0$,

$\because$ $f(\hat{x})=0$

$\therefore$ the location of the mode expressed in terms of x; y are $\hat{x}=g(\hat{y})$.

$\therefore$ find a mode with respect to the variable x is completely first transforming to the variable y.

$\because$ $g'(y)=s|g'(y)|$ $s\in\{-1,+1\}$

$$p_y(y)=p_x(g(y))sg'(y)$$

$$\therefore \quad p_y'(y)=sp_x'(g(y))\{g'(y)\}^2+sp_x(g(y))g''(y)$$

$\because \quad \hat{x} = g(\hat{y})$

$\therefore$ the value of x obtained by maximizing $p_x(x)$.
will not be the value obtained by transforming to $p_y(y)$

$\because \quad x = g(y) = \ln(y) - \ln(1-y) + 5$

$$\therefore \quad y = g^{-1}(x) = \frac{1}{1+\exp(-x+5)}$$

If we simply transform $p_x(x)$ as a function of x, we obtain the green curve $p_x(g(y))$ shown, and we see that the mode of density $p_x(x)$ is transformed via the sigmoid function to the mode of the curve. However, the density over y transforms instead according to (1.27) and is shown by the magenta curve on the left side of the diagram.
To confirm this result, we take our sample of 50,000 values of x, evaluate the corresponding value of y.

1.7. $\because \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

and $x^2+y^2=r^2$.

$\mathfrak{X}\because \quad \dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$

$$\therefore I^2 = \int_0^{2\pi}\int_0^{\infty} \exp\left(-\frac{r^2}{2\sigma^2}\right) r\,dr\,d\theta$$

$$= 2\pi \int_0^{\infty} \exp\left(-\frac{u}{2\sigma^2}\right) \frac{1}{2}\,du$$

$$= \pi \left[\exp\left(-\frac{u}{2\sigma^2}\right)(-2\sigma^2)\right]_0^{\infty}$$

$$= 2\pi\sigma^2$$

$\because \quad r^2 = u$

$$\therefore I = (2\pi\sigma^2)^{1/2}$$

$\because \quad y = x - \mu$

$$\therefore \int_{-\infty}^{\infty} N(x|\mu,\sigma^2)\,dx = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy$$

$$= \frac{I}{(2\pi\sigma^2)^{\frac{1}{2}}} = 1$$

1.8. $E[x] = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\} x \, dx$

chang variable using $y = x - \mu$

$$E[x] = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left\{-\frac{1}{2\sigma^2} y^2\right\} (y+\mu) \, dy$$

$$\because \int_{-\infty}^{\infty} \left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\} dx = (2\pi\sigma^2)^{\frac{1}{2}}$$

$$\therefore \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\} (x-\mu)^2 dx = \sigma^2$$

$$\therefore E[(x-\mu)^2] = var[x] = \sigma^2$$

$$\therefore E[x^2] - 2\mu E[x] + \mu^2 = \sigma^2.$$

$$\because E[x] = \int_{-\infty}^{\infty} N(x|\mu, \sigma^2) x \, dx = \mu$$

and $E[x^2] = \int_{-\infty}^{\infty} N(x|\mu, \sigma^2) x^2 dx = \mu^2 + \sigma^2$

$$\therefore E[x^2] - E[x]^2 = (\mu^2 + \sigma^2) - \mu^2 = \sigma^2$$

1.10. $\because p(x,z) = p(x)p(z)$.

$$\therefore E[x+z] = \iint (x+z)p(x)p(z)\,dx\,dz$$
$$= \int x p(x)\,dx + \int z p(z)\,dz$$
$$= E[x] + E[z]$$

$$\because (x+z-E[x+z])^2 = (x-E[x])^2 + (z-E[z])^2 + 2(x-E[x])(z-E[z])$$

$$\therefore var(x+z) = \iint (x+z-E[x+z])^2 p(x)p(z)\,dx\,dz$$
$$= \int (x-E[x])^2 p(x)\,dx + \int (z-E[z])^2 p(z)\,dz$$
$$= var(x) + var(z)$$

1.17. $\Gamma(x+1) = \int_0^\infty u^x e^{-u} du$

$$= [-e^{-u}u^x]_0^\infty + \int_0^\infty x u^{x-1} e^{-u} du = 0 + x\Gamma(x)$$

for $x=1$:

$$\Gamma(1) = \int_0^\infty e^{-u} du = [-e^{-u}]_0^\infty = 1$$

$\because$ x is an integer.

suppose $\Gamma(x+1) = x!$

$\therefore$ $\Gamma(x+2) = x+1\,\Gamma(x+1) = (x+1)!$

$\therefore$ $\Gamma(x+1) = x!$

$\therefore$ $\Gamma(1) = 1 = 0!$