椭圆方程积分恒等式的应用

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摘要

本文整理了麻希南老师 2023 年 6 月 1 日-2 日于中科院数学所所做系列报告的笔记. 第一部分, 我们整理了椭圆方程解的分类问题的历史, 以及相关未解决的问题. 第二部分, 我们利用积分恒等式技巧给出 $\mathbb{R}^n$ 上方程 $\Delta u + u^\alpha = 0$ 次临界指标情形解的分类问题的新证明.

目录

第 1 节 历史与问题

简要回顾椭圆方程解的分类问题的历史.

¶ $\mathbb{R}^n$ 上的 Laplace 算子.

<u>**次临界指标.**</u>

• $\Delta u + u^q = 0$, in $\mathbb{R}^n$. Giads-Spruck[11] 得到, 当 $1 \le q < \frac{n+2}{n-2}$ 时, 方程的非负解恒为 0. 方法: 分部积分 (Obata 积分恒等式技巧 [20]).

• $\Delta u + |\nabla u|^q = 0$, in $\mathbb{R}^n$. Lions[17] 得到, 当 $q > 1$ 时, 任意的 C^2 解均为常数. 方法: 最大值原理 (Bernstein 技巧).

• $\Delta u + u^p|\nabla u|^q = 0$, in $\Omega \subset \mathbb{R}^n$. Bidaut-Marie-García-Véron[1] 得到, 当 n, p, q 满足一定条件时, 方程的正解为常数. 方法: Bernstein 技巧.

• $\Delta u + u^p + M|\nabla u|^q = 0$, in $\mathbb{R}^n$. Bidaut-Marie-García-Véron[2] 得到, 当 $n \geq 3$, $1 < p < \frac{n+2}{n-2}$, $q = \frac{2p}{p+1}$, $|M| < \varepsilon(n,p)(\varepsilon > 0)$ 时, 方程不存在非平凡的非负解. 方法: Bernstein 技巧.

• 对于更一般的方程 $\Delta u + Nu^p + M|\nabla u|^q = 0(N > 0)$, in $\mathbb{R}^n$. Ma-Wu-Zhang[19] 利用积分恒等式的方法得到, 当 $n \geq 3$, $1 < p \leq \frac{n+2}{n-2}$, $q = \frac{2p}{p+1}$, $M > 0$ 时, 方程的非负解恒为 0.

<u>**临界指标.**</u>

对于方程 $\Delta u + u^{\frac{n+2}{n-2}} = 0$, in $\mathbb{R}^n$. Caffarelli-Gidas-Spruck[4] 利用移动平面法和几何测度论方法得到方程的正解分类. 后来, Chen-Li[6] 利用移动平面法给出了方程 $\Delta u + u^q = 0$ 次临界指标和临界指标正解的完全分类. 最近, Ou[22] 利用积分恒等式的方法得到 $3 \leq n \leq 5$ 正解的分类.

Q1. 利用积分恒等式的方法给出临界指标方程 $\Delta u + u^{\frac{n+2}{n-2}} = 0$, in $\mathbb{R}^n$, 任意维数正解的分类.

¶ $\mathbb{R}^n$ 上的 p-Laplace 算子.

<u>**次临界指标.**</u>

• $\Delta_p u + u^q = 0$, in $\mathbb{R}^n$. Serrin-Zou[24] 得到, 当 $1 < p < n$, $1 < q < p^* - 1$, $p^* = \frac{np}{n-p}$ 时, 方程的非负解恒为 0. 方法: 分部积分.

• $\Delta_m u + u^p + M|\nabla u|^q = 0$, in $\Omega \subset \mathbb{R}^n$. Wu-Zhang[26] 得到, 当 $n \geq 2$, $m - 1 < p \leq \frac{(m-1)n+m}{n-m}$, $q = \frac{mp}{p+1}$ 时, 则存在 $M_1 \sim n, m, p$, 使得 $0 < M < M_1$ 时, 或存在 $M_2 \sim n, m, p$, 使得 $-M_2 < M < 0$ 时, 方程的非负解恒为 0. 方法: 积分恒等式.

• Véron 在 1992 年 [25] 提出: $\Delta_p u + u^q = 0$, in $B_1(0) \setminus \{0\}$. 当 n, p, q, 满足一定的条件时, 方程的正解为常数. Lin-Ma[15], 方法: 积分恒等式.

<u>**临界指标.**</u>

对于方程 $\Delta_p u + u^{p^*-1} = 0$ in $\mathbb{R}^n$, $p^* = \frac{np}{n-p}$. Ciraolo-Figalli-Roncoroni[7], 在附加条件 $\int_{\mathbb{R}^n} |\nabla u|^p < \infty$ 下, 利用积分估计的方法得到正解的分类. Ou[22] 在不加条件的情况下, 利用积分恒等式的方法得到 $\frac{n+1}{3} \leq p \leq n$ 正解的分类.

Q2. 在不添加条件 $\int_{\mathbb{R}^n} |\nabla u|^p < \infty$ 的情况下, 得到临界指标方程 $\Delta_p u + u^{p^*-1} = 0$, in $\mathbb{R}^n$, 任意 n 和 p 正解的分类.

¶ Heisenberg 群 $\mathbb{H}^n$ 上的 $(p\text{-})$Laplace 算子.

<u>**次临界指标.**</u>

• $\Delta_{\mathbb{H}^n} u + u^q = 0$, on $\mathbb{H}^n$. Ma-Ou[18] 得到, 当 $1 < q < \frac{Q+2}{Q-2}$, $Q = 2n + 2$ 时, 方程的

非负解恒为 0. 方法: 分部积分 (Jerison-Lee 恒等式技巧 [12])

临界指标.

对于方程 $\Delta_{\mathbb{H}^n} u + u^{\frac{Q+2}{Q-2}} = 0$, in $\mathbb{H}^n$. Jerison-Lee[12], 在附加条件 $\displaystyle\int_{\mathbb{H}^n} u^{\frac{2Q}{Q-2}} < \infty$ 下, 得到正解的分类. Catino-Li-Monticelli-Roncoroni[5] 在不加条件的情况下, 利用积分恒等式的方法得到了 $n=1$ 正解的分类.

Q3. 在不添加条件 $\displaystyle\int_{\mathbb{H}^n} u^{\frac{2Q}{Q-2}} < \infty$ 的情况下, 得到临界指标方程任意维数正解的分类.

Q4. 对于 $\mathbb{H}^n$ 上 p-Laplace 方程 (次) 临界指标正解的分类. 即对于方程 $\Delta_p u + u^q = 0$, in $\mathbb{H}^n$, $1 < q < p^* - 1$, $p^* = \frac{Q+p}{Q-p}$, 正解的分类, 以及, 方程 $\Delta_p u + u^{p^*-1} = 0$, in $\mathbb{H}^n$, 正解的分类 (由此可以得到 Heisenberg 群上的 Sobolev 最佳常数).

¶ σ_k 算子.

• $\sigma_k(\lambda(D^2u)) = u^q$, in $\mathbb{R}^n$. Phuc-Verbitsky[23] 得到, 当 $q < \frac{nk}{n-2k}$ 时, 方程的非负解恒为 0. 方法: 位势理论. 后来, Ou[21] 利用积分估计的方法得到相同的结果.

Q5. 对于方程 $\sigma_k(\lambda(D^2u)) = u^q$, in $\mathbb{R}^n$. 当 $1 < q < \frac{n(k+1)}{n-2k} - 1$ 时, 能否得到方程的非负解恒为 0. 当 $q = \frac{n(k+1)}{n-2k} - 1$ 时, 给出方程的正解分类.

• $\sigma_k(\lambda(A)) = u^q$, 其中 A 为 Schouten tensor. Li-Li[13](in $\mathbb{R}^n$),[14](在有界区域上) 利用移动平面法得到临界指标 ($q = \frac{n+2}{n-2}$) 解的分类.

¶ 流形上的椭圆方程 ($\mathbb{S}^n$ 等).

设 $(\mathbb{S}^n, g)$ 为带有标准度量 g 的 n 维单位球, 考虑 $(\mathbb{S}^n, g)$ 的共形类中度量 $\bar{g} = u^{\frac{4}{n-2}} g$, 若共形度量 $\bar{g}$ 具有常数量曲率 $n(n-1)$, 则 u 满足如下方程

$$\Delta u - \frac{n(n-2)}{4} u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0, \quad \text{on } \mathbb{S}^n. \tag{1.1}$$

1971 年, Obata[20] 得到了如下结果.

定理 1.1 *若 $u(x)$ 为方程 (1.1) 的正解 (C^2), 则*

$$u(x) = c(\cosh t + (\sinh t) x \cdot a)^{\frac{-(n-2)}{2}},$$

其中 $c > 0$, $t \geq 0$, $a \in \mathbb{S}^n$.

Lin-Ma-Ou[16] 利用旧度量 ($\mathbb{S}^n$ 上的标准度量) 重新给出了新证明, 我们给出简要的证明思路.

证明. 令 $u=v^{-a}$, 其中 a 待定. 则

$$\Delta u=a(a+1)v^{-a-2}|\nabla v|^2-av^{-a-1}\Delta v,$$

从而方程 (1.1) 可写为

$$-av^{-a-1}\Delta v+a(a+1)v^{-a-2}|\nabla v|^2-\frac{n(n-2)}{4}v^{-a}+\frac{n(n-2)}{4}v^{-\frac{n+2}{n-2}a}=0.$$

对上述方程两边同时乘以 $-\frac{1}{\mathbf{a}}\mathbf{v}^{\mathbf{1+a}}$, 则

$$\Delta v-(a+1)\frac{|\nabla v|^2}{v}+\frac{n(n-2)}{4a}v-\frac{1}{a}v^{1+a-\frac{n+2}{n-2}a}=0,\quad \text{on } \mathbb{S}^n.$$

引入 $E_{ij}=v_{ij}-\frac{\Delta v}{n}\delta_{ij}$, 则

$$(v^{1-n}E_{ij}v_i)_j=v^{1-n}|E_{ij}|^2.$$

两边同时积分, 则有

$$0=\int_{\mathbb{S}^n}v^{1-n}|E_{ij}|^2,$$

从而 $|E_{ij}|\equiv 0$, 即

$$v(x)=c(\cosh t+(\sinh t)x\cdot a),$$

□

- $\Delta u-\lambda u+u^p=0$, on $\mathbb{S}^n$. Bidaut-Marie-Véron[3] 得到, 当 $1<p\leq\frac{n+2}{n-2}$, $0<\lambda\leq\frac{n}{p-1}$ 时 (由三个方程和两个条件得到此范围), 方程的正解为常数.(他们的结果实际上做到了紧的 Riemann 流形上).
- $\Delta u-\lambda u+u^p=0$, on $\mathbb{S}^n$. Dolbeault-Esteban- Loss[8] 及 [9] 得到相同的结果 (由两个不等式条件得到 p 和 λ 的范围). Dolbeault-Esteban- Loss[9] 考虑了与 Caffarelli-Kohn-Nirenberg 不等式相关的方程 $\nabla(|x|^{-a}\nabla u)+|x|^{-b}u^{2^*}=0$, in $\mathbb{R}^n$.

Q6. 方程 $\nabla(|x|^{-a}\nabla u)+|x|^{-b}u^{2^*}=0$ 的 p-Laplace 版本.

- $\Delta u-\lambda u+u^p=0$, on $\mathbb{S}^{2n+1}$(CR 流形中的单位球). Frank-Lieb[10] 得到, 当 $p<\frac{Q+2}{Q-2}$, $Q=2n+2$, $\lambda=0$ 时, 方程的正解为常数. Jerison-Lee[12] 得到, 当 $p=\frac{Q+2}{Q-2}$, $\lambda=\frac{Q(Q-2)}{4}$ 时, 正解的分类.
- $\Delta^2_{\mathbb{H}^n}u+u^p=0$, in $\mathbb{H}^n$. Frank-Lieb[10] 得到, 当 $p=\frac{Q+4}{Q-4}$, 满足条件 $u(x)\sim|x|^{2-Q}$, $|x|\to+\infty$ 时, 正解的分类.

Q7. 方程 $\Delta^2_{\mathbb{H}^n}u=u^p=0$ 的次临界指标 $p<\frac{Q+4}{Q-4}$, 以及临界指标 $p=\frac{Q+4}{Q-4}$ 去掉条件时

正解的分类.

Q8. (Lane-Emden 猜想) 方程组 $\begin{cases} -\Delta u = v^p, \\ -\Delta v = u^q, \end{cases}$, in $\mathbb{R}^n$, $u, v \geq 0$, 当 $\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{n}$ 时, $(u, v) = (0, 0)$.

第 2 节 $\Delta u + u^\alpha = 0$ 在 $\mathbb{R}^n$ 上解的分类

定理 2.1 令 $u(x)$ 为方程

$$\Delta u + u^\alpha = 0, \quad \text{on } \mathbb{R}^n, \tag{2.1}$$

的非负解, 若 $\alpha \in (-\infty, \frac{n}{n-2})$, 则 $u \equiv 0$.

证明. 对方程 (2.1) 两边同时乘以 $u^a \eta^\beta$, 其中 $\eta \in C_0^\infty(B_R)$ 为截断函数, 且 η 满足 $\eta = 0$ on $B_{R/2}$, 且 $|\nabla^a \eta| \leq \frac{C(a,n)}{R^a}$, 则有

$$\int u^{\alpha+a} \eta^\beta + \int u^a \Delta u \eta^\beta = 0.$$

利用分部积分, 则有

$$\begin{aligned} 0 &= \int u^{\alpha+a} \eta^\beta - \int (u^a \eta^\beta)_i u_i \\ &= \int u^{\alpha+a} \eta^\beta - a \int u^{a-1} |\nabla u|^2 \eta^\beta - \int u^a (\eta^\beta)_i u_i \\ &\geq \int u^{\alpha+a} \eta^\beta - a \int u^{a-1} |\nabla u|^2 \eta^\beta - \beta \int u^a \eta^{\beta-1} |\nabla \eta| |\nabla u|. \end{aligned}$$

利用 Young 不等式, 从而

$$0 \geq \int u^{\alpha+a} \eta^\beta - a \int u^{a-1} |\nabla u|^2 \eta^\beta - \varepsilon \beta \int u^{a-1} |\nabla u|^2 \eta^\beta - C(\varepsilon) \beta R^{-2} \int u^{a+1} \eta^{\beta-2},$$

即

$$(-a - \varepsilon\beta) \int u^{a-1} |\nabla u|^2 \eta^\beta + \int u^{\alpha+a} \eta^\beta \leq C(\varepsilon) \beta R^{-2} \int u^{a+1} \eta^{\beta-2}. \tag{2.2}$$

再次利用 Young 不等式, 其中指标为 $(\frac{\alpha+a}{a+1}, \frac{\alpha+a}{\alpha-1})$, 则有

$$(-a - \varepsilon\beta) \int u^{a-1} |\nabla u|^2 \eta^\beta + \int u^{\alpha+a} \eta^\beta \leq C(\varepsilon) \beta \varepsilon_1 \int u^{\alpha+a} \eta^\beta + C(\varepsilon, \varepsilon_1) \beta R^{-2\frac{\alpha+a}{\alpha-1}+n},$$

即

$$(-a-\varepsilon\beta)\int u^{a-1}|\nabla u|^2\eta^{\beta} + (1-C(\varepsilon)\beta\varepsilon_1)\int u^{\alpha+a}\eta^{\beta} \le C(\varepsilon,\varepsilon_1)\beta R^{-2\frac{\alpha+a}{\alpha-1}+n}.$$

现在要求: (1) $a<0$; (2) $\frac{\alpha+a}{a+1}>1$; (3) $n-2\frac{\alpha+a}{\alpha-1}<0$.

- 若 $a+1>0$, 则 $\alpha>1$, 从而取 $\frac{n}{2}(\alpha-1)-\alpha<a<0$, 即有$\underline{1<\alpha<\frac{n}{n-2}}$, 则 $n-2\frac{\alpha+a}{\alpha-1}<0$.
- 若 $a+1<0$, 则$\underline{\alpha<1}$, 从而取 $a>\frac{n}{2}(\alpha-1)-\alpha$, 则 $n-2\frac{\alpha+a}{\alpha-1}<0$.
- 若 $a=-1$, 取$\underline{\alpha=1}$, 则 (2.2) 可写为

$$(1-\varepsilon\beta)\int u^{-2}|\nabla u|^2\eta^{\beta} + \int \eta^{\beta} \le C(\varepsilon)\beta R^{-2}\int \eta^{\beta-2} \le C(\varepsilon)\beta R^{n-2}.$$

设 $u>0$, 取 ε 足够小, 令 $R\to\infty$, 则矛盾, 即 $u=0$. □

定理 2.2 令 $u(x)$ 为方程

$$\Delta u + u^{\alpha} = 0, \quad \text{on } \mathbb{R}^n, \tag{2.3}$$

的非负解, 若 $1<\alpha<\frac{n+2}{n-2}$, 则 $u\equiv 0$.

证明. 令 $u=v^{-\beta}$, 则

$$\Delta u = \beta(\beta+1)v^{-\beta-2}|\nabla v|^2 - \beta v^{-\beta-1}\Delta v,$$

从而

$$-\beta v^{-\beta-1}\Delta v + \beta(1+\beta)v^{-\beta-2}|\nabla v|^2 + v^{-\alpha\beta} = 0, \quad \beta>0.$$

因此, 方程 (2.3) 可写为

$$\Delta v - (1+\beta)\frac{|\nabla v|^2}{v} - \frac{v^{1+\beta-\alpha\beta}}{\beta} = 0;$$

对上述方程两边同时乘以 $v^a|\nabla v|^b(\Delta v)^c$.

对于方程 (2.3), 考虑 $c=1, b=0$.

对于方程

$$\Delta v - (1+\beta)\frac{|\nabla v|^2}{v} - \frac{v^k}{\beta} = 0, \quad 其中\ k = 1+\beta-\alpha\beta \tag{2.4}$$

两边同时乘以 $v^a \Delta v$, 则

$$0 = v^a(\Delta v)^2 - (1+\beta)v^{a-1}|\nabla v|^2 \Delta v - \frac{v^{a+k}}{\beta}\Delta v := (1) + (2) + (3).$$

对于 (1), 有

$$\begin{aligned}(1) = v^a v_{ii} v_{jj} &= (v^a \Delta v v_j)_j - (v^a \Delta v)_j v_j \\ &= (v^a \Delta v v_j)_j - a v^{a-1} \Delta v |\nabla v|^2 - v^a v_{iji} v_j.\end{aligned}$$

对于最后一项,

$$v^a v_{iji} v_j = (v^a v_{ij} v_j)_i - v^a \sum_{i,j} v_{ij}^2 - a v^{a-1} v_i v_j v_{ij}.$$

令 $E_{ij} = v_{ij} - \frac{\Delta v}{n}\delta_{ij}$ $(\mathrm{tr} E_{ij} = 0)$, 及 $v_{ij} = E_{ij} + \frac{\Delta v}{n}\delta_{ij}$, 从而

$$\sum_{i,j} v_{ij}^2 = |E_{ij}|^2 + \frac{(\Delta v)^2}{n},$$

因此,

$$\begin{aligned}(1) =&(v^a \Delta v v_j - v^a v_{ij} v_i)_j - a v^{a-1} |\nabla v|^2 \Delta v \\ &+ v^a |E_{ij}|^2 + \frac{v^a}{n}(\Delta v)^2 + a v^{a-1} v_i v_j E_{ij} + \frac{a}{n} v^{a-1} |\nabla v|^2 \Delta v,\end{aligned}$$

即

$$\begin{aligned}\frac{n-1}{n} v^a (\Delta v)^2 =&(v^a \Delta v v_j - v^a v_{ij} v_i)_j + \frac{1-n}{n} a v^{a-1} |\nabla v|^2 \Delta v \\ &+ v^a |E_{ij}|^2 + a v^{a-1} v_i v_j E_{ij},\end{aligned}$$

故

$$(1) = \frac{n}{n-1}(v^a \Delta v v_j - v^a v_{ij} v_i)_j - a v^{a-1} |\nabla v|^2 \Delta v + \frac{n}{n-1} v^a |E_{ij}|^2 + \frac{n}{n-1} a v^{a-1} v_i v_j E_{ij}.$$

对于 (2), 有

$$(2) = -(1+\beta) v^{a-1} |\nabla v|^2 \Delta v.$$

对于 (3), 由 (2.4), 则有

$$\begin{aligned}(3)&=-\frac{1}{\beta}v^{a+k}\Delta v=-\frac{1}{\beta}(v^{a+k}v_i)_i+\frac{a+k}{\beta}v^{a+k-1}|\nabla v|^2\\&=-\frac{1}{\beta}(v^{a+k}v_i)_i+(a+k)v^{a-1}|\nabla v|^2\left(\Delta v-(1+\beta)\frac{|\nabla v|^2}{v}\right)\\&=-\frac{1}{\beta}(v^{a+k}v_i)_i+(a+k)v^{a-1}|\nabla v|^2\Delta v-(1+\beta)(a+k)v^{a-2}|\nabla v|^4.\end{aligned}$$

因此,

$$\begin{aligned}0=&(1)+(2)+(3)\\=&\frac{n}{n-1}(v^a\Delta vv_j-v^av_{ij}v_i)_j-a\underline{v^{a-1}|\nabla v|^2\Delta v}\\&+\frac{n}{n-1}v^a|E_{ij}|^2+\frac{n}{n-1}av^{a-1}v_iv_jE_{ij}-(1+\beta)\underline{v^{a-1}|\nabla v|^2\Delta v}\\&-\frac{1}{\beta}(v^{a+k}v_i)_i+(a+k)\underline{v^{a-1}|\nabla v|^2\Delta v}-(1+\beta)(a+k)v^{a-2}|\nabla v|^4.\end{aligned}\tag{2.5}$$

对于 (2.5) 中下划线项, 有

$$\begin{aligned}v^{a-1}|\nabla v|^2\Delta v=&(v^{a-1}|\nabla v|^2v_i)_i-(v^{a-1}|\nabla v|^2)_iv_i\\=&(v^{a-1}|\nabla v|^2v_i)_i-(a-1)v^{a-2}|\nabla v|^4-2v^{a-1}v_iv_jv_{ij}\\=&(v^{a-1}|\nabla v|^2v_i)_i-(a-1)v^{a-2}|\nabla v|^4-2v^{a-1}v_iv_j\left(E_{ij}+\frac{\Delta v}{n}\delta_{ij}\right)\\=&(v^{a-1}|\nabla v|^2v_i)_i-(a-1)v^{a-2}|\nabla v|^4-2v^{a-1}v_iv_jE_{ij}-\frac{2}{n}v^{a-1}|\nabla v|^2\Delta v,\end{aligned}$$

即

$$\frac{n+2}{n}v^{a-1}|\nabla v|^2\Delta v=(v^{a-1}|\nabla v|^2v_i)_i-(a-1)v^{a-2}|\nabla v|^4-2v^{a-1}v_iv_jE_{ij},$$

从而

$$\begin{aligned}&(a+k-(1+\beta)-a)v^{a-1}|\nabla v|^2\Delta v\\=&-\alpha\beta v^{a-1}|\nabla v|^2\Delta v\\=&-\alpha\beta\cdot\frac{n}{n+2}\left((v^{a-1}|\nabla v|^2v_i)_i-(a-1)v^{a-2}|\nabla v|^4-2v^{a-1}v_iv_jE_{ij}\right).\end{aligned}$$

因此, 由 (2.5) 得到

$$\begin{aligned}
0 =&(1)+(2)+(3)\\
=&\frac{n}{n-1}(v^a\Delta v v_j - v^a v_{ij}v_i)_j - \frac{1}{\beta}(v^{a+k}v_i)_i - \frac{n\alpha\beta}{n+2}(v^{a-1}|\nabla v|^2 v_i)_i\\
&+\underbrace{\frac{n}{n-1}v^a|E_{ij}|^2}_{(A_1)}\\
&+\underbrace{v^{a-1}v_iv_jE_{ij}\left(\frac{na}{n-1}+\frac{2\alpha\beta n}{n+2}\right)}_{(A_2)}\\
&+\underbrace{v^{a-2}|\nabla v|^4\left(-(1+\beta)(a+k)+(a-1)\alpha\beta\frac{n}{n+2}\right)}_{(A_3)}.
\end{aligned}\tag{2.6}$$

对于 (A_1) 和 (A_2) 应用不等式,

$$a|E_{ij}|^2 + bv_iv_jE_{ij} \geq -\frac{b^2}{4a}|\nabla v|^4. \tag{2.7}$$

令 $L_{ij} = v_iv_j - \frac{|\nabla v|^2}{n}\delta_{ij}$, 显然 $\mathrm{tr}L_{ij} = 0$, 且有

$$|L_{ij}|^2 = |\nabla v|^4 - 2\frac{|\nabla v|^4}{n} + \frac{|\nabla v|^4}{n} = \frac{n-1}{n}|\nabla v|^4.$$

由于 $v_iv_j = L_{ij} + \frac{|\nabla v|^2}{n}\delta_{ij}$, 则

$$E_{ij}v_iv_j = E_{ij}L_{ij} + \frac{|\nabla v|^2}{n}\mathrm{tr}E_{ij} = E_{ij}L_{ij},$$

从而应用不等式 (2.7), 有

$$\begin{aligned}
(A_1)+(A_2) =&\frac{n}{n-1}v^a|E_{ij}|^2 + \left(\frac{a}{n-1}+\frac{2\alpha\beta}{n+2}\right)nv^{a-1}E_{ij}L_{ij}\\
\geq& -\frac{1}{4\frac{n}{n-1}v^a}\left(\frac{a}{n-1}+\frac{2\alpha\beta}{n+2}\right)^2 n^2|L_{ij}|^2v^{2a-2}\\
=& -\frac{n-1}{4n}\left(\frac{a}{n-1}+\frac{2\alpha\beta}{n+2}\right)^2 n^2\frac{n-1}{n}|\nabla v|^4v^{a-2}\\
=& -\frac{(n-1)^2}{4}\left(\frac{a}{n-1}+\frac{2\alpha\beta}{n+2}\right)^2 v^{a-2}|\nabla v|^4.
\end{aligned}$$

因此,

$$
\begin{aligned}
&(A_1)+(A_2)+(A_3)\\
\geq&\underbrace{\left[-(1+\beta)(a+k)+(a-1)\alpha\beta\frac{n}{n+2}-\frac{(n-1)^2}{4}\left(\frac{a}{n-1}+\frac{2\alpha\beta}{n+2}\right)^2\right]}_{(A)}v^{a-2}|\nabla v|^4,
\end{aligned}
$$

其中 $k=1+\beta-\alpha\beta$.

取 $\beta=-1$, $k=\alpha$, 则

$$
\begin{aligned}
(A)&=(a-1)\frac{n}{n+2}(-\alpha)-\frac{(n-1)^2}{4}\left(\frac{a}{n-1}-\frac{2\alpha}{n+2}\right)^2\\
&=-\frac{a^2}{4}-\frac{\alpha}{n+2}a+\frac{n\alpha}{n+2}-\frac{(n-1)^2}{(n+2)^2}\alpha^2.
\end{aligned}
$$

为确定 a, 使得 $(A)\geq 0$, 则需要

$$
\begin{aligned}
0\leq\Delta&=\frac{1}{(n+2)^2}\alpha^2+\left(\frac{n\alpha}{n+2}-\frac{(n-1)^2}{(n+2)^2}\alpha^2\right)\\
&=\frac{n\alpha}{(n+2)^2}[(n+2)-(n-2)\alpha].
\end{aligned}
$$

令 $0<\alpha<\frac{n+2}{n-2}$, 从而说明存在 a, 使得 $(A)>0$. 取 $a=-\frac{2\alpha}{n+2}$, 则 $(0<\alpha<\frac{n+2}{n-2})$

$$
(A)=\frac{n\alpha}{(n+2)^2}[(n+2)-(n-2)\alpha]>0.
$$

取截断函数 $\eta\in C_0^{\infty}(B_R)$, 且 η 满足 $\eta=1$ on $B_{R/2}$, 且 $|\nabla^{\alpha}\eta|\leq\frac{C(\alpha,n)}{R^{\alpha}}$. 在 (2.6) 两边同时乘 η^{δ} (δ 待定), 并在 $\mathbb{R}^n$ 上积分, 则有 (取 $\beta=-1$)

$$
\begin{aligned}
0=&\underbrace{\int\eta^{\delta}\frac{n\alpha}{n+2}(v^{a-1}|\nabla v|^2v_i)_i+\eta^{\delta}(v^{a+\alpha}v_i)_i+\eta^{\delta}\frac{n}{n-1}(v^a\Delta vv_j-v^av_{ij}v_i)_j}_{(B)}\\
&+\int\eta^{\delta}v^{a-2}|\nabla v|^4\cdot(A)
\end{aligned}
\tag{2.8}
$$

对于 (B), 利用分部积分, 有

$$
\begin{aligned}
(B)=&-\int\delta\eta^{\delta-1}\eta_i\frac{n\alpha}{n+2}v^{a-1}|\nabla v|^2v_i-\int\delta\eta^{\delta-1}\eta_iv^{a+\alpha}v_i-\int\delta\eta^{\delta-1}\eta_i\frac{n}{n-1}v^a\Delta vv_i\\
&+\int\delta\eta^{\delta-1}\eta_j\frac{n}{n-1}v^av_{ij}v_i
\end{aligned}
$$

$$
=\underbrace{-\frac{n}{n+2}\alpha\delta\int\eta^{\delta-1}v^{a-1}|\nabla v|^2v_i\eta_i}_{(B_1)}\underbrace{-\delta\int\eta^{\delta-1}v^{a+\alpha}v_i\eta_i}_{(B_2)}\underbrace{-\frac{n}{n-1}\delta\int\eta^{\delta-1}v^a\Delta v v_i\eta_i}_{(B_3)}
$$
$$
\underbrace{+\frac{n}{n-1}\delta\int\eta^{\delta-1}v^a v_i\eta_j v_{ij}}_{(B_4)}.
$$

接下来, 我们利用 Young 不等式, 分别处理上述四项. (目标: 我们希望将 (B) 中四项拆分成出 $\int\eta^\delta v^{a-2}|\nabla v|^4$.)

对于 (B_1), 有

$$
\begin{aligned}
(B_1)=&-\frac{n}{n+2}\alpha\delta\int\eta^{\delta-1}v^{a-1}|\nabla v|^2v_i\eta_i\\
=&-\frac{n}{n+2}\alpha\delta\int\eta^{\frac34\delta+\frac14(\delta-4)}v^{\frac34(a-2)+\frac14(a+2)}|\nabla v|^2v_i\eta_i\\
=&-\frac{n}{n+2}\alpha\delta\int\eta^{\frac34\delta}v^{\frac34(a-2)}\eta^{\frac14(\delta-4)}v^{\frac14(a+2)}|\nabla v|^2v_i\eta_i\\
\geq&-\varepsilon\frac{3n}{4(n+2)}\alpha\delta\int\eta^\delta v^{a-2}|\nabla v|^4-\varepsilon^{-3}\frac{n}{4(n+2)}\alpha\delta\int\eta^{\delta-4}v^{a+2}|\nabla\eta|^4
\end{aligned}
$$

对于 (B_2), 有

$$
\begin{aligned}
(B_2)=&-\delta\int\eta^{\delta-1}v^{a+\alpha}v_i\eta_i\\
=&-\delta\int\eta^{\frac14\delta}v^{\frac14(a-2)}\eta^{\frac34(\delta-\frac43)}v^{\frac34(a+\frac43\alpha+\frac23)}\eta_iv_i\\
\geq&-\varepsilon\frac14\delta\int\eta^\delta v^{a-2}|\nabla v|^4\underbrace{-\varepsilon^{-\frac13}\frac34\delta\int\eta^{\delta-\frac43}v^{a+\frac43\alpha+\frac23}|\nabla\eta|^{\frac43}}_{(B_2')}
\end{aligned}
$$

对于 (B_2) 中 (B_2'), 有

$$
\begin{aligned}
(B_2')=&-\varepsilon^{-\frac13}\frac34\delta\int\eta^{\frac23\delta}v^{\frac23(a+2\alpha)}\eta^{\frac13(\delta-4)}v^{\frac13(a+2)}|\nabla\eta|^{\frac43}\\
\geq&-\varepsilon^{-\frac13}\frac12\varepsilon_1\delta\int\eta^\delta v^{a+2\alpha}-\varepsilon^{-\frac13}\varepsilon_1^{-2}\frac14\delta\int\eta^{\delta-4}v^{a+2}|\nabla\eta|^4
\end{aligned}
$$

因此,

$$
(B_2)\geq-\varepsilon\frac14\delta\int\eta^\delta v^{a-2}|\nabla v|^4-\varepsilon^{-\frac13}\frac12\varepsilon_1\delta\int\eta^\delta v^{a+2\alpha}-\varepsilon^{-\frac13}\varepsilon_1^{-2}\frac14\delta\int\eta^{\delta-4}v^{a+2}|\nabla\eta|^4.
$$

对于 (B_3), 由于 $\beta=-1$, 则 $\Delta v+v^{\alpha}=0$, 从而

$$(B_3)=\frac{n}{n-1}\delta\int\eta^{\delta-1}v^{a+\alpha}v_i\eta_i=-\frac{n}{n-1}(B_2).$$

对于 (B_4), 有

$$\begin{aligned}(B_4)=&\frac{n}{n-1}\delta\int\eta^{\delta-1}v^{a}v_i\eta_jv_{ij}=\frac{n}{2(n-1)}\delta\int\eta^{\delta-1}v^{a}\eta_j(|\nabla v|^2)_j\\=&\frac{n}{2(n-1)}\int(\eta^{\delta})_jv^{a}(|\nabla v|^2)_j\\=&\frac{n}{2(n-1)}\left(\int(v^{a}(\eta^{\delta})_i|\nabla v|^2)_i-a\int v^{a-1}v_i(\eta^{\delta})_i|\nabla v|^2-\int v^{a}|\nabla v|^2\Delta(\eta^{\delta})\right)\\=&-\frac{na}{2(n-1)}\int v^{a-1}v_i(\eta^{\delta})_i|\nabla v|^2\underbrace{-\frac{n}{2(n-1)}\int v^{a}|\nabla v|^2\Delta(\eta^{\delta})}_{(B_4')}\end{aligned}$$

对于 (B_4) 中 (B_2'), 有

$$\begin{aligned}(B_4')=&-\frac{n}{2(n-1)}\int v^{a}|\nabla v|^2\Delta(\eta^{\delta})\\=&-\frac{n}{2(n-1)}\delta(\delta-1)\int v^{a}\eta^{\delta-2}|\nabla v|^2|\nabla\eta|^2-\frac{n}{2(n-1)}\delta\int v^{a}\eta^{\delta-1}|\nabla v|^2\Delta\eta\\\geq&-\frac{n}{2(n-1)}\delta^2\int v^{a}\eta^{\delta-2}|\nabla v|^2|\nabla\eta|^2\\=&-\frac{n}{2(n-1)}\delta^2\int v^{\frac{a-2}{2}}\eta^{\frac{\delta}{2}}v^{\frac{a+2}{2}}\eta^{\frac{\delta-4}{2}}|\nabla v|^2|\nabla\eta|^2\\\geq&-\varepsilon\frac{n}{4(n-1)}\delta^2\int v^{a-2}\eta^{\delta}|\nabla v|^4-\varepsilon^{-1}\frac{n}{4(n-1)}\delta^2\int v^{a+2}\eta^{\delta-4}|\nabla\eta|^4.\end{aligned}$$

因此,

$$\begin{aligned}(B_4)=&-\varepsilon\frac{3na}{8(n-1)}\delta\int\eta^{\delta}v^{a-2}|\nabla v|^4-\varepsilon^{-3}\frac{na}{8(n-1)}\delta\int\eta^{\delta-4}v^{a+2}|\nabla\eta|^4\\&-\varepsilon\frac{n}{4(n-1)}\delta^2\int v^{a-2}\eta^{\delta}|\nabla v|^4-\varepsilon^{-1}\frac{n}{4(n-1)}\delta^2\int v^{a+2}\eta^{\delta-4}|\nabla\eta|^4.\end{aligned}$$

故,

$$\begin{aligned}(B)=&(B_1)+(B_2)+(B_3)+(B_4)\\\geq&-\varepsilon\int\eta^{\delta}v^{a-2}|\nabla v|^4-\varepsilon_1\int\eta^{\delta}v^{a+2\alpha}-C(\varepsilon,\varepsilon_1)\int\eta^{\delta-4}v^{a+2}|\nabla\eta|^4.\end{aligned}$$

因此, (2.8) 可写为

$$0\geq -\varepsilon\int\eta^\delta v^{a-2}|\nabla v|^4-\varepsilon_1\int\eta^\delta v^{a+2\alpha}-C(\varepsilon,\varepsilon_1)\int\eta^{\delta-4}v^{a+2}|\nabla\eta|^4+\int\eta^\delta v^{a-2}|\nabla v|^4\cdot(A).$$

由于 $(A)>0$, 取 ε 足够小, 我们得到如下积分估计:

$$\int\eta^\delta v^{a-2}|\nabla v|^4\leq\varepsilon_1\int\eta^\delta v^{a+2\alpha}+C(\varepsilon_1)\int\eta^{\delta-4}v^{a+2}|\nabla\eta|^4. \tag{2.9}$$

对于方程

$$\Delta v+v^\alpha=0,$$

两边同乘 v^γ(γ 待定), 即

$$v^\gamma\Delta v+v^{\alpha+\gamma}=0.$$

注意到 $(v^\gamma v_i)_i-\gamma v^{\gamma-1}|\nabla v|^2+v^{\alpha+\gamma}=0$. 两边同乘 η^δ, 并在 $\mathbb{R}^n$ 上积分, 则有

$$\int\eta^\delta v^{\alpha+\gamma}=\int\gamma v^{\gamma-1}\eta^\delta v^{\gamma-1}|\nabla v|^2-\int\eta^\delta(v^\gamma v_i)_i.$$

取 $\gamma=a+\alpha$, 则

$$\begin{aligned}\int\eta^\delta v^{a+2\alpha}=&(a+\alpha)\int\eta^\delta v^{a+\alpha-1}|\nabla v|^2-\int\eta^\delta(v^{a+\alpha}v_i)_i\\=&\underbrace{(a+\alpha)\int\eta^\delta v^{a+\alpha-1}|\nabla v|^2}_{(C_1)}\underbrace{+\delta\int\eta^{\delta-1}v^{a+\alpha}\eta_i v_i}_{(C_2)}\end{aligned}$$

现在, 我们利用 Young 不等式, 处理 (C_1) 和 (C_2).
对于 (C_1), 有

$$\begin{aligned}(C_1)=&(a+\alpha)\int\eta^\delta v^{a+\alpha-1}|\nabla v|^2+(a+\alpha)\int\eta^{\frac{\delta}{2}}v^{\frac{a+2\alpha}{2}}\eta^{\frac{\delta}{2}}v^{\frac{a-2}{2}}|\nabla v|^2\\\leq&\varepsilon\frac{1}{2}(a+\alpha)\int\eta^\delta v^{a+2\alpha}+\varepsilon^{-1}\frac{1}{2}(a+\alpha)\int\eta^\delta v^{a-2}|\nabla v|^4.\end{aligned}$$

对于 (C_2), 有

$$\begin{aligned}(C_2)=&\delta\int\eta^{\delta-1}v^{a+\alpha}\eta_i v_i=\delta\int\eta^{\frac{\delta}{2}}v^{\frac{a+\alpha-1}{2}}\eta^{\frac{\delta-2}{2}}v^{\frac{a+\alpha+1}{2}}\eta_i v_i\\\leq&\frac{1}{2}\delta\int\eta^\delta v^{a+\alpha-1}|\nabla v|^2\underbrace{+\frac{1}{2}\delta\int\eta^{\delta-2}v^{a+\alpha+1}|\nabla\eta|^2}_{(C_2')}.\end{aligned}$$

对于 (C_2) 中 (C_2'), 有

$$\begin{aligned}(C_2') =& \frac{1}{2}\delta\int \eta^{\frac{\delta}{2}} v^{\frac{a+2\alpha}{2}} \eta^{\frac{\delta-4}{2}} v^{\frac{a+2}{2}} |\nabla\eta|^2 \\ \leq& \varepsilon_1 \frac{1}{4}\delta \int \eta^\delta v^{a+2\alpha} + \varepsilon_1^{-1}\frac{1}{4}\delta\int \eta^{\delta-4} v^{a+2}|\nabla\eta|^4.\end{aligned}$$

故,

$$\int \eta^\delta v^{a+2\alpha} \leq \varepsilon \int \eta^\delta v^{a+2\alpha} + C(\varepsilon)\int \eta^\delta v^{a-2}|\nabla v|^4 + \varepsilon_1 \int \eta^\delta v^{a+2\alpha} + C(\varepsilon_1)\int \eta^{\delta-4}v^{a+2}|\nabla\eta|^4.$$

取 $\varepsilon, \varepsilon_1$ 足够小, 我们得到如下积分估计:

$$\int \eta^\delta v^{a+2\alpha} \leq C(\varepsilon)\int \eta^\delta v^{a-2}|\nabla v|^4 + C(\varepsilon_1)\int \eta^{\delta-4}v^{a+2}|\nabla\eta|^4. \tag{2.10}$$

将 (2.9) 代入 (2.10), 依 Young 不等式, 则有

$$\begin{aligned}\int \eta^\delta v^{a+2\alpha} \leq C\int \eta^{\delta-4}v^{a+2}|\nabla\eta|^4 \leq& \frac{C}{R^4}\int \eta^{\delta-4}v^{a+2} \\ =& C\int \eta^{\delta\frac{a+2}{a+2\alpha}} \eta^{\delta-4-\delta\frac{a+2}{a+2\alpha}} v^{(a+2\alpha)\frac{a+2}{a+2\alpha}} R^{-4} \\ \leq& \frac{1}{2}\int \eta^\delta v^{a+2\alpha} + CR^{n-4\frac{a+2\alpha}{2\alpha-2}},\end{aligned}$$

从而

$$\int \eta^\delta v^{a+2\alpha} \leq CR^{n-2\frac{a+2\alpha}{\alpha-1}}. \tag{2.11}$$

令 $n-2\frac{a+2\alpha}{\alpha-1}<0$, 则 $(\alpha>1)$

$$\alpha\cdot\frac{n^2-2n-4}{n+2}<n.$$

取 $1<\alpha<\frac{n+2}{n-2}$, 则上述不等式成立, 因此 $v=u\equiv 0$. □

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