LECTURE NOTES ON MINIMAL SURFACES

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Abstract. This lecture note was taken for the class on minimal surfaces given by Professor Chao Xia (Xiamen University) in the summer school on Differential Geometry in the Hangzhou Dianzi University, July 2021.

The class includes three parts, geometry of submanifolds, Bernstein theorem and Plateau problem. However, this note contains only the first two parts (The Plateau problem is just an introduction in the class).

The main references for this note are [1] and [2]. We can refer to the Chapter 4 of [3] and the note written by Professor Xin Zhou (Cornell University).

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1. Geometry of Submanifolds

Suppose M is Riemannian manifold with metric g and Levi-Civita connection ∇. $\Sigma \subset M$ is $k-$dim submanifold with induced metric g_Σ and Levi-Civita connection ∇_Σ. For any $p \in \Sigma$, we have

$$T_pM = T_p\Sigma \oplus N_p\Sigma.$$

If X is a vector field on $\Sigma \subset M$, then we let $X^\top$ and $X^\perp$ denote the tangential and normal components, respectively.

For any $X, Y \in T\Sigma$, we have

$$\nabla_X Y = (\nabla_X Y)^\top + (\nabla_X Y)^\perp.$$

Proposition 1.1.

$$(\nabla_X Y)^\top = \nabla_X^\Sigma Y.$$

Definition 1.1 (Second Fundamental Form)**.** *For any $X, Y \in T\Sigma$,*

$$A(X,Y) = (\nabla_X Y)^\perp = \nabla_X Y - \nabla_X^\Sigma Y$$

is called ***second fundamental form****.*

Proposition 1.2. *A is symmetric and bilinear, i.e.,*

$$A(X,Y) = A(Y,X)$$

and

$$A(f_1X_1 + f_2X_2, Y) = f_1A(X_1,Y) + f_2A(X_2,Y), \quad \forall f_1, f_2 \in C^\infty(\Sigma).$$

Definition 1.2 (Shape Operator)**.** *Take $\nu \in N\Sigma$,*

$$\begin{aligned} S^\nu: \quad T\Sigma &\longrightarrow T\Sigma \\ X &\longmapsto (\nabla_X \nu)^\top \end{aligned}$$

is called ***shape operator****.*

Proposition 1.3. *Suppose $\nu \in N\Sigma$ and $X, Y \in T\Sigma$, then*

$$\langle A(X,Y), \nu \rangle = -\langle S^\nu(X), Y \rangle.$$

Remark 1.1. If $\Sigma^{n-1} \subset M^n$ is a hypersurface and choose $\{e_i\}$ is an orthonormal basis for $T_x\Sigma$, then

$$\begin{aligned} \langle A(e_i,e_i), \nu \rangle &= -\langle S^\nu(e_i), e_i \rangle \\ \Longrightarrow A(e_i,e_i)e_i &= -S^\nu(e_i), \end{aligned}$$

i.e.,$-A(e_i,e_i)$ are eigenvalues of S^ν and denote $(\kappa_i)_{i=1,\cdots,n-1}$ are called the **principal curvatures**.

Definition 1.3 (Mean Curvature Vector)**.** *The* ***mean curvature vector*** *$\overrightarrow{H}$ at x is by definition*

$$\overrightarrow{H} = \sum_{i=1}^k A(e_i,e_i),$$

where $\{e_i\}$ is an orthonormal basis for $T_x\Sigma$.

Remark 1.2. If $\Sigma^{n-1} \subset M^n$ is a hypersurface, then **mean curvature** of Σ is

$$H = \left\langle \overrightarrow{H}, N \right\rangle = -\sum_{i=1}^{n-1} \kappa_i.$$

Definition 1.4 (Minimal Submanifold)**.** *An immersed submanifold $\Sigma^k \subset M^n$ is said to be **minimal** if the mean curvature H vanishes identically.*

Definition 1.5 (Divergence)**.** *Suppose $X \in T\Sigma$, then the **divergence** of X at $x \in \Sigma$ is*

$$\mathrm{div}_\Sigma X = \sum_{i=1}^{k} \langle \nabla_{e_i} X, e_i \rangle,$$

where $\{e_i\}$ is an orthonormal basis for $T_x\Sigma$.

Remark 1.3. div_Σ satisfies the Leibniz rule

$$\mathrm{div}_\Sigma(fX) = \langle \nabla_\Sigma f, X \rangle + f \mathrm{div}_\Sigma(X).$$

Remark 1.4.

$$\begin{aligned}\mathrm{div}_\Sigma X^\perp &= \sum_{i=1}^{k} \left\langle \nabla_{e_1} X^\perp, e_i \right\rangle = -\sum_{i=1}^{k} \left\langle X^\perp, \nabla_{e_i} e_i \right\rangle \\ &= -\left\langle X^\perp, \overrightarrow{H} + (\nabla_{e_i} e_i)^\top \right\rangle = -\left\langle \overrightarrow{H}, X^\perp \right\rangle \left(= -\left\langle \overrightarrow{H}, X \right\rangle \right).\end{aligned}$$

Then,

$$\mathrm{div}_\Sigma X = \mathrm{div}_\Sigma X^\top + \mathrm{div}_\Sigma X^\perp = \mathrm{div}_\Sigma X^\top - \left\langle \overrightarrow{H}, X^\perp \right\rangle \left(= \mathrm{div}_\Sigma X^\top - \left\langle \overrightarrow{H}, X \right\rangle \right).$$

Definition 1.6 (Laplace Operator)**.** *For any $f \in C^\infty(\Sigma)$, define*

$$\Delta_\Sigma f = \mathrm{div}_\Sigma(\nabla_\Sigma f),$$

where

$$\langle \nabla_\Sigma f, X \rangle = X(f).$$

Remark 1.5. A function f is said to be **harmonic** on Σ if $\Delta_\Sigma f = 0$.

Definition 1.7 (Gauss Equation)**.** *For any $X, Y, Z, W \in T\Sigma$,*

$$\left\langle R^\Sigma(X,Y)Z, W \right\rangle = \left\langle R^M(X,Y)Z, W \right\rangle + A(X,Z)A(Y,W) - A(X,W)A(Y,Z).$$

Definition 1.8 (Codazzi Equation)**.** *For any $X, Y, Z \in T\Sigma$,*

$$(\nabla_X A)(Y,Z) - (\nabla_Y A)(X,Z) = -(R^M(X,Y)Z)^\perp.$$

2. Minimal Graph in $\mathbb{R}^n$

Suppose that $u : \Omega \subset \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ is a C^2 function and consider the graph of the function u

$$\Sigma_u = \{(x, u(x)) \in \mathbb{R}^n \mid x \in \Omega\}.$$

By calculus, we have

$$\mathrm{Area}(\Sigma_u) = |\Sigma_u| = \int_\Omega \sqrt{1 + |\nabla u|^2} dx,$$

and unit normal

$$N = \frac{(-\nabla u, 1)}{\sqrt{1+|\nabla u|^2}}.$$

Consider function $u + t\eta$, where $\eta \mid_{\partial\Omega} = 0$ (imply $\partial\Sigma_{u+t\eta} = \partial\Sigma_u$), we have

$$|\Sigma_{u+t\eta}| = \int_\Omega \sqrt{1+|\nabla u + t\nabla\eta|^2} dx.$$

Hence, the directional derivative of the area functional on graphs at u in the direction η is

$$\frac{d}{dt}\bigg|_{t=0} |\Sigma_{u+t\eta}| = \int_\Omega \frac{\langle \nabla u, \nabla\eta\rangle}{\sqrt{1+|\nabla u|^2}} dx = -\int_\Omega \eta \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) dx.$$

Therefore, the graph of u is a critical point for the area functional of u satisfies the divergence form equation

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$

Definition 2.1 (Minimal Surface Equation)**.**

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$

Proposition 2.1.

$$H = \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right),$$

where H is the mean curvature of Σ_u.

Theorem 2.1. *A minimal graph $\Sigma_u \subset \mathbb{R}^n$ is area-minimizing in $\Omega \times \mathbb{R}$, i.e., for any $\Sigma \subset \Omega \times \mathbb{R}$ and $\partial\Sigma = \partial\Sigma_u$, then*

$$|\Sigma_u| \leq |\Sigma|.$$

Proof. Consider

$$X(x_1, \cdots, x_{n-1}, x_n) = \frac{(-\nabla u, 1)}{\sqrt{1+|\nabla u|^2}},$$

then

$$\mathrm{div}_{\mathbb{R}^n} X = \mathrm{div}_{\mathbb{R}^n}\left(\frac{-\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + \frac{\partial}{\partial x_n}\left(\frac{1}{\sqrt{1+|\nabla u|^2}}\right) = 0,$$

where the first term by Σ_u is minimal graph.
By divergence theorem, we have

$$0 = \int_{\tilde{\Omega}} \mathrm{div}_{\mathbb{R}^n} X dx = \int_{\Sigma_u} \langle X, N\rangle + \int_\Sigma \langle X, N_\Sigma\rangle = |\Sigma_u| + \int_\Sigma \langle X, N_\Sigma\rangle,$$

where $\tilde{\Omega}$ is the region bounded by Σ and Σ_u and $N = \dfrac{(-\nabla u, 1)}{\sqrt{1+|\nabla u|^2}}$.
By Cauchy-Schwarz inequality, we obtain

$$|\Sigma_u| \leq -\int_\Sigma \langle X, N_\Sigma\rangle \leq \int_\Sigma |X||N_\Sigma| = |\Sigma|.$$

□

Corollary 2.1. *Suppose Σ_u is minimal graph in $\mathbb{R}^n$ and $\partial\Sigma_u \cap B_R = \emptyset$, then*

$$|\Sigma_u \cap B_R| \leq \frac{1}{2}\omega_{n-1}R^{n-1},$$

where $\omega_{n-1} = |S^{n-1}|$.

Proof. Suppose ∂B_R is divided by Σ_u into two parts A and B, by **Theorem2.1**, we have

$$|\Sigma_u \cap B_R| \leq \min\{|A|, |B|\} \leq \frac{1}{2}(|A|+|B|) = \frac{1}{2}|\partial B_R| = \frac{1}{2}\omega_{n-1}R^{n-1}.$$

□

3. Basic Properties of Minimal Submanifold in $\mathbb{R}^n$

Proposition 3.1. *$\Sigma^k \subset \mathbb{R}^n$ is minimal if and only if the restrictions of the coordinate functions of $\mathbb{R}^n$ to Σ are harmonic functions.*

Proof. Take $x = (x^1, \cdots, x^n) \in \Sigma$ and $x^i = \langle x, E_i\rangle$, where $\{E_i\}$ is an orthonormal basis for $\mathbb{R}^n$. Take the orthonormal basis for $T_x\Sigma$, $e_\alpha = (e_\alpha^1, \cdots, e_\alpha^n) = \sum_i e_\alpha^i E_i (1 \leq \alpha \leq k)$. Consider

$$\begin{aligned}
\nabla_\Sigma x^i &= \nabla_{e_\alpha} \langle x, E_i\rangle e_\alpha = \nabla_{e_\alpha^j E_j} \langle x, E_i\rangle e_\alpha \\
&= e_\alpha^j \nabla_{E_j} \langle x, E_i\rangle e_\alpha = e_\alpha^j \left(\langle E_j(x), E_i\rangle e_\alpha + \left\langle x, \nabla_{E_j} E_i\right\rangle e_\alpha\right) \\
&= e_\alpha^j \langle E_j, E_i\rangle e_\alpha = e_\alpha^j \delta_{ij} e_\alpha = e_\alpha^i e_\alpha \\
&= \langle e_\alpha, E_i\rangle e_\alpha = E_i - E_i^N.
\end{aligned}$$

then

$$\Delta_\Sigma x^i = \mathrm{div}_\Sigma \nabla_\Sigma x^i = \mathrm{div}_\Sigma(E_i - E_i^N) = \left\langle \overrightarrow{H}, E_i \right\rangle = 0.$$

Indeed,

$$\begin{aligned}
\mathrm{div}_\Sigma(E_i - E_i^N) &= \mathrm{div}_\Sigma(E_i) - \mathrm{div}_\Sigma E_i^N = \langle \nabla_{e_\alpha} E_i, e_\alpha\rangle - \mathrm{div}_\Sigma E_i^N \\
&= \left\langle \nabla_{e_\alpha^j E_j} E_i, e_\alpha \right\rangle + \left\langle \overrightarrow{H}, E_i \right\rangle = e_\alpha^j \left\langle \nabla_{E_j} E_i, e_\alpha\right\rangle + \left\langle \overrightarrow{H}, E_i \right\rangle \\
&= \left\langle \overrightarrow{H}, E_i \right\rangle.
\end{aligned}$$

This third equality holds by **Remark1.4**. □

Proposition 3.2. *Let $\Sigma^k \subset \mathbb{R}^n$ is minimal surface, then*

$$\Delta_\Sigma |x|^2 = 2k.$$

Proof. Take $x = (x^1, \cdots, x^n) \in \Sigma$. For any vector v, we have

$$\nabla_v x^i = \langle v, e_i \rangle .$$

Then,

$$\mathrm{div}_\Sigma (x^1, \cdots, x^n) = \sum_{i=1}^k \left\langle \nabla_{e_i} (x^1, \cdots, x^n), e_i \right\rangle = \sum_{i=1}^k \langle e_i, e_i \rangle = k.$$

Since Σ is minimal surface, then $\mathrm{div}_\Sigma Y^\perp = 0$ for any Y (by **Remark1.4**). Therefore,

$$\begin{aligned}
\Delta_\Sigma |x|^2 &= \mathrm{div}(\nabla_\Sigma |x|^2) = \mathrm{div}(\nabla |x|^2 - (\nabla |x|^2)^\perp) \\
&= \mathrm{div}_\Sigma (\nabla |x|^2) = 2\mathrm{div}(x^1, \cdots, x^n)' = 2k.
\end{aligned}$$

□

Proposition 3.3. *Let $\Sigma^{n-1} \subset \mathbb{R}^n$ be minimal hypersurface, then*

$$\Delta_\Sigma \langle N, E_i \rangle = -|A|^2 \langle N, E_i \rangle ,$$

where $|A|^2 = \sum_{\alpha,\beta} \langle A(e_\alpha, e_\beta), N \rangle^2$.

Proof. Fix $p \in \Sigma$, choose normal coordinate at p, such that $g_{\alpha\beta}(p) = \delta_{\alpha\beta}$ and $\nabla^\Sigma_{e_\alpha} e_\beta (p) = 0$.
Direct calculate, we have

$$\begin{aligned}
\Delta_\Sigma f(p) = \mathrm{div}_\Sigma (\nabla_\Sigma f) &= \sum_{\alpha=1}^k \left\langle \nabla^\Sigma_{e_\alpha} \nabla_\Sigma f, e_\alpha \right\rangle \\
&= \sum_{\alpha=1}^k \left(e_\alpha \langle \nabla_\Sigma f, e_\alpha \rangle - \left\langle \nabla_\Sigma f, \nabla^\Sigma_{e_\alpha} e_\alpha (p) \right\rangle \right) \\
&= e_\alpha e_\alpha f.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\Delta_\Sigma \langle N, E_i \rangle &= e_\alpha e_\alpha \langle N, E_i \rangle = e_\alpha \langle \nabla_{e_\alpha} N, E_i \rangle \\
&= e_\alpha \langle -A(e_\alpha, e_\beta) e_\beta, E_i \rangle \\
&= \langle -(\nabla_{e_\alpha} A(e_\alpha, e_\beta)) e_\beta - A(e_\alpha, e_\beta) \nabla_{e_\alpha} e_\beta, E_i \rangle \\
&= \left\langle -(\nabla_{e_\beta} A(e_\alpha, e_\alpha)) e_\beta - A(e_\alpha, e_\beta)(A(e_\alpha, e_\beta) N + \nabla^\Sigma_{e_\alpha} e_\beta), E_i \right\rangle \\
&= -A_{\alpha\beta} A_{\alpha\beta} \langle N, E_i \rangle \\
&= -|A|^2 \langle N, E_i \rangle .
\end{aligned}$$

The above calculation uses the following facts.
The third identity:

$$A(e_\alpha, e_\beta) = \langle \nabla_{e_\alpha} e_\beta, N\rangle = e_\alpha \langle e_\beta, N\rangle - \langle e_\beta, \nabla_{e_\alpha} N\rangle$$

$\Longrightarrow$

$$\nabla_{e_\alpha} N = A(e_\alpha, e_\beta) e_\beta.$$

The fifth identity uses the Codazzi formula and the definition of the second fundamental form, that is,

$$A(e_\alpha, e_\beta) N = \nabla_{e_\alpha} e_\beta - \nabla^\Sigma_{e_\alpha} e_\beta.$$

The sixth identity holds, since $H = \sum_{\alpha=1}^{k} A(e_\alpha, e_\alpha)$ and Σ is minimal surface, i.e., $H = 0$. □

Corollary 3.1. *There exists no closed(compact and without boundary) minimal submanifold in $\mathbb{R}^n$.*

Proof. Suppose $\Sigma \subset \mathbb{R}^n$ is closed minimal submanifold. Since

$$\Delta_\Sigma x^i = 0,$$

by the strong maximum principle, then x^i are constants. It is contradiction. □

If $\Omega \subset \mathbb{R}^n$ is a compact subset, then the smallest convex set containing Ω (convex hull) is the intersection of all half-spaces containing Ω.

Proposition 3.4 (Convex Hull Property)**.** *If $\Sigma^k \subset \mathbb{R}^n$ is a compact minimal submanifold, then*

$$\Sigma \subset \mathrm{Conv}(\partial\Sigma).$$

Proof. A half-space $H \subset \mathbb{R}^n$ can be written as

$$H = \{x \in \mathbb{R}^n \mid \langle x, e\rangle \leq a\},$$

for a vector $e \in \mathbb{S}^{n-1}$ and constant $a \in \mathbb{R}$.
By **Proposition3.1**, we have

$$\Delta_\Sigma \langle x, e\rangle = 0.$$

By the maximum principle, we obtain

$$\max_{\Sigma} \langle x, e\rangle \leq \max_{\partial\Sigma} \langle x, e\rangle = a,$$

and for any $x \in \Sigma$, $\langle x, e\rangle \leq a$.
Therefore, for any $e \in \mathbb{S}^{n-1}$, we have $\Sigma \subset H$, and

$$\Sigma \subset \bigcap_{e \in \mathbb{S}^{n-1}} H = \mathrm{Conv}(\partial\Sigma).$$

$\square$

Corollary 3.2 (Monotonicity of Topology). *If $\Sigma^2 \subset \mathbb{R}^n$ is a compact minimal disk and K is a compact convex set with $K \cap \partial\Sigma = \emptyset$, then $K \cap \Sigma$ is simply connected.*

Proof. Suppose instead that there is a closed curve $\gamma \subset K \cap \Sigma$ that does not bound a disk in $K \cap \Sigma$.
Since Σ is simply connected, γ bounds a disk $\Gamma \subset \Sigma$.
Since $\partial\Gamma = \gamma \subset K$, by **convex hull property**3.4, we have

$$\Gamma \subset \mathrm{Conv}(\partial\Sigma) \subset K.$$

It is contradiction ($\Gamma \subset K \cap \Sigma$)! $\square$

4. The First and Second Variation Formula

Suppose Σ^k is a minimal submanifold and let $F : \Sigma^k \times (-\varepsilon, \varepsilon) \longrightarrow M$ be a variation of Σ with compact support and fixed boundary. That is, $F = \mathrm{Id}$ outside a compact set,

$$F(x, 0) = x,$$

and for all $x \in \partial\Sigma$,

$$F(x, t) = x.$$

Denote $F(\Sigma, t) = \Sigma_t$, $\left.\frac{\partial}{\partial t}\right|_{t=0} F(\cdot, t) = X(\cdot, t) \in TM$ and

$$g_{t,ij} = \langle \partial_i F(x,t), \partial_j F(x,t) \rangle .$$

Theorem 4.1 (The First Variation Formula).

$$\left.\frac{d}{dt}\right|_{t=0} |\Sigma_t| = \int_\Sigma \mathrm{div}_\Sigma X \, d\mu.$$

Proof. It is obvious that

$$|\Sigma_t| = \int_\Sigma d\mu_t = \int_\Sigma \frac{\sqrt{\det(g_{t,ij})}}{\sqrt{\det(g_{ij})}} d\mu.$$

We denote by $A_t = (A_t^{ij})$ the cofactor matrix of $G_t = (g_{t,ij})$, i.e., $A_t = \det G_t \cdot G_t^{-1}$. Direct calculation, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\Big|_{t=0} \sqrt{\det g_{t,ij}} &= \frac{1}{2} \cdot \frac{1}{\sqrt{\det g_{t,ij}}}\Bigg|_{t=0} \cdot \frac{\partial}{\partial t}\Big|_{t=0} \det g_{t,ij} \\
&= \frac{1}{2} \cdot \frac{1}{\sqrt{\det g_{ij}}} \cdot \frac{\partial}{\partial t}\Big|_{t=0} (A_t^{i1} g_{t,i1} + \cdots + A_t^{in} g_{t,in}) \\
&= \frac{1}{2} \cdot \frac{1}{\sqrt{\det g_{ij}}} \cdot \frac{\partial}{\partial g_{t,ij}}\Big|_{t=0} (A_t^{i1} g_{t,i1} + \cdots + A_t^{in} g_{t,in}) \cdot \frac{\partial g_{t,ij}}{\partial t}\Big|_{t=0} \\
&= \frac{1}{2} \cdot \frac{1}{\sqrt{\det g_{ij}}} \cdot A^{ij} \cdot \frac{\partial}{\partial t}\Big|_{t=0} g_{t,ij} \\
&= \frac{1}{2} \cdot \sqrt{\det g_{ij}} \cdot g^{ij} \cdot \frac{\partial}{\partial t}\Big|_{t=0} g_{t,ij}.
\end{aligned}$$

We will calculate $\frac{\partial}{\partial t}\Big|_{t=0} g_{t,ij}$. By denotation of $g_{t,ij}$, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\Big|_{t=0} g_{t,ij} &= \frac{\partial}{\partial t}\Big|_{t=0} \left\langle \frac{\partial}{\partial x_i} F(x,t), \frac{\partial}{\partial x_j} F(x,t) \right\rangle \\
&= \left\langle \frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial x_i} F(x,t), \frac{\partial}{\partial x_j} F(x,t) \right\rangle + \left\langle \frac{\partial}{\partial x_i} F(x,t), \frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial x_j} F(x,t) \right\rangle \\
&= \left\langle \frac{\partial}{\partial x_i} \frac{\partial}{\partial t}\Big|_{t=0} F(x,t), \frac{\partial}{\partial x_j} F(x,t) \right\rangle + \left\langle \frac{\partial}{\partial x_i} F(x,t), \frac{\partial}{\partial x_j} \frac{\partial}{\partial t}\Big|_{t=0} F(x,t) \right\rangle \\
&= \left\langle \nabla_{\frac{\partial}{\partial x_i} F} X, \frac{\partial}{\partial x_j} F(x,t) \right\rangle + \left\langle \nabla_{\frac{\partial}{\partial x_j} F} X, \frac{\partial}{\partial x_i} F(x,t) \right\rangle.
\end{aligned}$$

Since,

$$g^{ij} \left\langle \nabla_{\frac{\partial}{\partial x_i} F} X, \frac{\partial}{\partial x_j} F(x,t) \right\rangle = \mathrm{div}_\Sigma X,$$

then

$$\frac{d}{dt}\Big|_{t=0} |\Sigma_t| = \int_\Sigma \mathrm{div}_\Sigma X \, d\mu.$$

□

Assume $X = \frac{\partial}{\partial t}\Big|_{t=0} F(\cdot, t) \in N\Sigma$, i.e., $X^\top = 0$ on Σ.

Theorem 4.2 (The Second Variation Formula)**.**

$$\frac{d^2}{dt^2}\Big|_{t=0} |\Sigma_t| = \int_\Sigma |\nabla^\perp X|^2 - |\langle A, X\rangle|^2 - \mathrm{tr}\left\langle R^M(\cdot, X)\cdot, X\right\rangle.$$

Proof. Note that
$\bullet |\nabla^\perp X|^2 = |(\nabla_{e_i} X)^\perp|^2$,

•$|\langle A, X\rangle| = \sum_{i,j} \langle A(e_i,e_j),X\rangle^2$,
•$\mathrm{tr}(R^M(\cdot,X)\cdot,X) = \sum_i \left\langle R^M(e_i,X)e_i,X\right\rangle$.
Similar to the above proof, we have

$$|\Sigma_t| = \int_\Sigma d\mu_t = \int_\Sigma \frac{\sqrt{\det(g_{t,ij})}}{\sqrt{\det(g_{ij})}} d\mu.$$

Direct calculation, we have

$$\frac{\partial}{\partial t}\sqrt{\det g_{t,ij}} = \frac{1}{2}\cdot\sqrt{\det g_{t,ij}}\cdot g_t^{ij}\cdot\frac{\partial}{\partial t}g_{t,ij},$$

and

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\sqrt{\det g_{t,ij}} =& \frac{1}{2}\frac{\partial}{\partial t}(\sqrt{\det g_{t,ij}})\cdot g_t^{ij}\cdot\frac{\partial g_{t,ij}}{\partial t} + \frac{1}{2}\sqrt{\det g_{t,ij}}\cdot\frac{\partial}{\partial t}g_t^{ij}\cdot\frac{\partial g_{t,ij}}{\partial t}\\ &+\frac{1}{2}\sqrt{\det g_{t,ij}}\cdot g_t^{ij}\cdot\frac{\partial^2}{\partial t^2}g_{t,ij}\\ =&\frac{1}{4}\sqrt{\det g_{t,ij}}g_t^{kl}\frac{\partial g_{tkl}}{\partial t}g_t^{ij}\frac{\partial g_{t,ij}}{\partial t} - \frac{1}{2}\sqrt{\det g_{t,ij}}g_t^{ik}\frac{\partial g_{t,kl}}{\partial t}g_t^{lj}\frac{\partial g_{t,ij}}{\partial t}\\ &+\frac{1}{2}\sqrt{\det g_{t,ij}}g_t^{ij}\frac{\partial^2 g_{t,ij}}{\partial t^2}\end{aligned}$$

Thus, at $t=0$, (notation: $\dfrac{\partial}{\partial x_i}F = e_i$)

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\bigg|_{t=0}\sqrt{\det g_{t,ij}} =&\frac{1}{4}\sqrt{\det g_{ij}}\left(g^{ij}(\langle\nabla_{e_i}X,e_j\rangle + \left\langle\nabla_{e_j}X,e_i\right\rangle)\right)^2\\ &-\frac{1}{2}\sqrt{\det g_{ij}}\left(\langle\nabla_{e_i}X,e_j\rangle + \left\langle\nabla_{e_j}X,e_i\right\rangle\right)^2\\ &+\frac{1}{2}\sqrt{\det g_{ij}}\ \frac{\partial^2}{\partial t^2}\bigg|_{t=0} g_{t,ii}.\end{aligned}$$

Since

$$
\begin{aligned}
\frac{1}{2}\left.\frac{\partial^2}{\partial t^2}\right|_{t=0} g_{t,ii} =& \frac{1}{2}\left.\frac{\partial^2}{\partial t^2}\right|_{t=0}\left\langle \frac{\partial}{\partial x_i}F, \frac{\partial}{\partial x_i}F\right\rangle \\
&= \left\langle \frac{\partial^2}{\partial t^2}\frac{\partial}{\partial x_i}F, \frac{\partial}{\partial x_i}F\right\rangle + \left.\left\langle \frac{\partial}{\partial t}\frac{\partial}{\partial x_i}F, \frac{\partial}{\partial t}\frac{\partial}{\partial x_i}F\right\rangle\right|_{t=0} \\
&= \left.\left\langle \nabla_{\frac{\partial}{\partial t}F}\nabla_{\frac{\partial}{\partial t}F}\nabla_{e_i}F, \nabla_{e_i}F\right\rangle\right|_{t=0} + \sum_i |\nabla_{e_i}X|^2 \\
&= \left.\left\langle \nabla_{\frac{\partial}{\partial t}F}\nabla_{e_i}F\nabla_{\frac{\partial}{\partial t}F}F, \nabla_{e_i}F\right\rangle\right|_{t=0} \\
&\quad + \sum_i |(\nabla_{e_i}X)^{\perp}|^2 + \sum_i \left(\sum_j \langle \nabla_{e_i}X, e_j\rangle e_j\right)^2 \\
&= \left.\left\langle \nabla_{e_i}\nabla_{\frac{\partial}{\partial t}F}\nabla_{\frac{\partial}{\partial t}F}F, \nabla_{e_i}F\right\rangle\right|_{t=0} + \left.\left\langle R^M\left(e_i, \frac{\partial}{\partial t}F\right)\frac{\partial}{\partial t}F, e_i\right\rangle\right|_{t=0} \\
&\quad + \sum_i |(\nabla_{e_i}X)^{\perp}|^2 + \sum_{i,j} |\langle A(e_i,e_j), X\rangle|^2 \\
&\quad = \operatorname{div}(\nabla_X X) + \sum_i \left\langle R^M(e_i,X)e_i, X\right\rangle + |(\nabla_{e_i}X)^{\perp}|^2 + \sum_{i,j}\langle A(e_i,e_j),X\rangle^2,
\end{aligned}
$$

$$
\frac{1}{4}\left(g^{ij}(\langle \nabla_{e_i}X, e_j\rangle + \langle \nabla_{e_j}X, e_i\rangle)\right)^2 = (\operatorname{div}X)^2 = -\langle H, X\rangle = 0,
$$

and

$$
-\frac{1}{2}\left(\langle \nabla_{e_i}X, e_j\rangle + \langle \nabla_{e_j}X, e_i\rangle\right)^2 = -2\langle A(e_i,e_j),X\rangle^2.
$$

We obtain the second variation formula. □

5. The Monotonicity Formula

Recall **coarea formula**.

Lemma 5.1 (Coarea Formula). *If Σ is a submanifold and*

$$
h : \Sigma \longrightarrow \mathbb{R}
$$

is a proper (i.e., $h^{-1}((-\infty,t])$ is compact for all $t \in \mathbb{R}$) Lipschitz function on Σ, then for all locally integrable functions f on Σ and $t \in \mathbb{R}$,

$$
\int_{\{h\le t\}} f|\nabla_\Sigma h| = \int_{-\infty}^{t}\int_{h=\tau} f d\tau.
$$

Remark 5.1. Consider the special case that $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold and $x_0 \in \mathbb{R}^n$, then

$$
\int_{B_r\cap\Sigma} f|\nabla_\Sigma h| = \int_0^r \int_{\partial B_\tau\cap\Sigma} f,
$$

where $h = |x - x_0|$ for any $x \in \Sigma$.

Theorem 5.1 (The Monotonicity Formula). *Suppose that $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold and $x_0 \in \mathbb{R}^n$, then for all $0 < s < t$,*

$$\frac{|B_t(x_0) \cap \Sigma|}{t^k} - \frac{|B_s(x_0) \cap \Sigma|}{s^k} = \int_{(B_t(x_0) \backslash B_s(x_0)) \cap \Sigma} \frac{|(x - x_0)^{\perp}|^2}{|x - x_0|^{k+2}} dx.$$

Proof. Set $r = |x - x_0|$, then $\nabla_\Sigma r = \dfrac{(x - x_0)^\top}{|x - x_0|}$ and $\nabla_\Sigma r^2 = 2(x - x_0)^\top$.
Since Σ is minimal submanifold, by **Proposition3.2**, we have

$$\Delta_\Sigma r^2 = \mathrm{div}_\Sigma \nabla_\Sigma r^2 = 2\mathrm{div}_\Sigma((x - x_0) - (x - x_0)^\perp) = 2\mathrm{div}_\Sigma(x - x_0) = 2k.$$

By **Green formula**, then

$$\begin{aligned} k|\Sigma \cap B_\tau(x_0)| &= \int_{\Sigma \cap B_\tau(x_0)} \Delta_\Sigma \frac{1}{2} r^2 = \int_{\Sigma \cap \partial B_\tau(x_0)} \nabla_\Sigma \frac{1}{2} r^2 \cdot \mu \\ &= \int_{\Sigma \cap \partial B_\tau(x_0)} |(x - x_0)^\top| = \tau \int_{\Sigma \cap \partial B_\tau(x_0)} |\nabla_\Sigma r|, \end{aligned}$$

where $\mu = \dfrac{(x - x_0)^\top}{|(x - x_0)^\top|}$.
By **coarea formula5.1**, then

$$\frac{d}{d\tau} \int_{\Sigma \cap B_\tau(x_0)} |\nabla_\Sigma r|^2 = \frac{d}{d\tau} \int_{\Sigma \cap B_\tau(x_0)} |\nabla_\Sigma r| \cdot |\nabla_\Sigma r| = \int_{\Sigma \cap \partial B_\tau(x_0)} |\nabla_\Sigma r|.$$

Therefore,

$$\begin{aligned} k|\Sigma \cap B_\tau(x_0)| &= \tau \frac{d}{d\tau} \int_{\Sigma \cap B_\tau(x_0)} |\nabla_\Sigma r|^2 \\ &= \tau \frac{d}{d\tau} \int_{\Sigma \cap B_\tau(x_0)} 1 - |(\nabla r)^\perp|^2 \\ &= \tau \frac{d}{d\tau} |\Sigma \cap B_\tau(x_0)| - \tau \frac{d}{ds} \int_{\Sigma \cap B_\tau(x_0)} |(\nabla r)^\perp|^2. \end{aligned}$$

We obtain

$$\frac{d}{d\tau}(\tau^{-k} |\Sigma \cap B_\tau(x_0)|) = \frac{d}{d\tau} \int_{B_\tau(x_0) \cap \Sigma} \frac{|(\nabla r)^\perp|^2}{r^k}.$$

Thus,

$$\int_s^t \frac{d}{d\tau}(\tau^{-k} |\Sigma \cap B_\tau(x_0)|) d\tau = \int_s^t \frac{d}{d\tau} \int_{B_\tau(x_0) \cap \Sigma} \frac{|(\nabla r)^\perp|^2}{r^k} d\tau,$$

or

$$\frac{|B_t(x_0) \cap \Sigma|}{t^k} - \frac{|B_s(x_0) \cap \Sigma|}{s^k} = \int_{(B_t(x_0) \backslash B_s(x_0)) \cap \Sigma} \frac{|(x - x_0)^{\perp}|^2}{|x - x_0|^{k+2}} dx.$$

□

Suppose $\Sigma^k \subset \mathbb{R}^{n+1}$, $\mathcal{C}$ is the **cone** with vertex at 0 given by

$$\mathcal{C}_\Sigma = \{tx \in \mathbb{R}^{n+1} \mid t \in [0, +\infty), x \in \Sigma\}.$$

Give an important lemma that

Lemma 5.2. *$\Sigma \subset \mathbb{R}^{n+1}$ is a cone with vertex at 0 if and only if $\langle x, N(x)\rangle = 0$ for any $x \in \Sigma$.*

Proof. Consider graph that $(x, u(x))(x = (x_1, \cdots, x_n))$ and

$$N = \frac{(-\nabla u, 1)}{\sqrt{1 + |\nabla u|^2}}.$$

Since $0 = \langle x, N(x)\rangle = -\langle x, \nabla u(x)\rangle + u(x)$, then

$$u(x) = \langle x, \nabla u(x)\rangle .$$

Consider $v(t) = tu(x) - u(tx)$, we will prove $v(t) = 0$ for any $t \in \mathbb{R}$.
Since

$$\frac{dv}{dt} = u(x) - \langle x, \nabla u(tx)\rangle = \frac{tu(x)}{t} - \frac{u(tx)}{t} = \frac{v(t)}{t},$$

where $u(tx) = \langle tx, \nabla u(tx)\rangle$.
Then, we have the following ODE(Cauchy problem)

$$\begin{cases} \dfrac{dv}{dt} = \dfrac{v(t)}{t} \\ v(1) = 0 \end{cases}$$

and $v(t) = 0$ for any $t \in \mathbb{R}$.
In fact, we have

$$\int \frac{1}{v(t)} dv(t) = \int \frac{1}{t} dt \Longrightarrow \ln v(t) = \ln t + C \Longrightarrow v(t) = e^c t,$$

since $v(1) = 0$, then $v(t) = 0$ for any $t \in \mathbb{R}$. □

Suppose that $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold and $x_0 \in \mathbb{R}^n$. Consider the function

$$\Theta_{x_0}(s) = \frac{|B_s(x_0) \cap \Sigma|}{|B_s \subset \mathbb{R}^k|}.$$

By **Theorem5.1**,$\Theta_{x_0}(s)$ is monotone nondecreasing. Assume $x_0 \in \Sigma$, It is obvious that

$$\lim_{s \to 0} \Theta_{x_0}(s) \geq 1,$$

and $\Theta_{x_0}(s) \geq 1$ by monotonicity of $\Theta_{x_0}(s)$. Thus, define the **density** at x_0 by

$$\Theta_{x_0} = \lim_{s \to 0} \Theta_{x_0}(s).$$

This limit is well-defined (bounded and monotonicity). In fact, so long as Σ is smooth, Θ_{x_0} is a nonnegative integer equal to the multiplicity of Σ at x_0. Note that if Σ is not embedded, then this multiplicity can be greater than one.

Corollary 5.1. *The function $\Theta_{x_0}(s)$ is constant in s if and only if Σ is cone about x_0. In particular, if for some $s > 0$, $\Theta_{x_0}(s) = 1$, then $B_s \cap \Sigma$ is a ball in some k-dimensional plane.*

Proof. By **Theorem5.1**, we have

$$(x - x_0)^{\perp} = 0,$$

and Σ is cone by **Lemma5.2**.
Since

$$1 = \Theta_{x_0}(s) \geq \Theta_{x_0} \geq 1,$$

then $\Theta_{x_0}(s) \equiv 1$ or Σ is cone and is contained in a k-plane, since Σ is smooth. Thus, $B_s \cap \Sigma$ is a ball in some k-dimensional plane. □

Corollary 5.2. *If $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold, then Θ_x is an upper semicontinuous function on $\mathbb{R}^n$, i.e., $x_i \longrightarrow x$, then*

$$\limsup_{x_i \to x} \Theta_{x_i} \leq \Theta_x.$$

Consequently, for any $\Lambda \geq 0$, the set

$$\{x \in \Sigma \mid \Theta_x \geq \Lambda\}$$

is closed.

Proof. Given any $\delta > 0$, there exists an $s > 0$ such that

$$\Theta_x \geq \Theta_x(2s) - \delta,$$

and we can choose $0 < \varepsilon < s$ so that

$$\Theta_x \geq (1 + s^{-1}\varepsilon)^k \Theta_x(2s) - 2\delta.$$

For any x_i with $|x - x_i| < \varepsilon$, we have

$$\begin{aligned}
\Theta_{x_i} \leq \Theta_{x_i}(s) = \frac{|B_s(x_i) \cap \Sigma|}{|B_s \subset \mathbb{R}^k|} &\leq \frac{|B_{s+\varepsilon}(x) \cap \Sigma|}{|B_s \subset \mathbb{R}^k|} \\
&= \frac{|B_{s+\varepsilon}(x) \cap \Sigma|}{|B_{s+\varepsilon} \subset \mathbb{R}^k|} \cdot \frac{|B_{s+\varepsilon} \subset \mathbb{R}^k|}{|B_s \subset \mathbb{R}^k|} \\
&= \Theta_x(s+\varepsilon) \cdot \frac{(s+\varepsilon)^k}{s^k} = (1 + s^{-1}\varepsilon)^k \Theta_x(s+\varepsilon) \\
&\leq 2\delta + \Theta_x.
\end{aligned}$$

Since δ was arbitrary small, then

$$\Theta_x \geq \limsup_{x_i \to x} \Theta_{x_i}.$$

It follows immediately that the set

$$\{x \in \Sigma \mid \Theta_x \geq \Lambda\}$$

is closed. $\square$

Theorem 5.2 (The General Monotonicity Formula). *If $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold and f is a function on Σ, then*

$$t^{-k}\int_{B_t\cap\Sigma} f - s^{-k}\int_{B_s\cap\Sigma} f = \int_{(B_t\setminus B_s)\cap\Sigma} f\frac{|x^{\perp}|^2}{|x|^{k+2}} + \frac{1}{2}\int_s^t \tau^{-k-1}\int_{B_\tau\cap\Sigma}(\tau^2 - |x|^2)\Delta_\Sigma f d\tau.$$

Remark 5.2. Take $f = 1$, we obtain **Theorem5.1**.

Proof. Since Σ is minimal submanifold, then $2k = \Delta_\Sigma |x|^2$ and $\nabla_\Sigma |x|^2 = 2|x^\top|$. By integration by parts, we have

$$\begin{aligned} 2k\int_{B_s\cap\Sigma} f &= \int_{B_s\cap\Sigma} f\Delta_\Sigma|x|^2 \\ &= \int_{B_s\cap\Sigma}|x|^2\Delta_\Sigma f + \int_{\partial B_s\cap\Sigma} f\nabla_\Sigma|x|^2\cdot\mu - \int_{\partial B_s\cap\Sigma}|x|^2\nabla_\Sigma f\cdot\mu \\ &= \int_{B_s\cap\Sigma}|x|^2\Delta_\Sigma f + 2\int_{\partial B_s\cap\Sigma} f|x^\top| - s^2\int_{B_s\cap\Sigma}\Delta_\Sigma f. \end{aligned}$$

By **coarea formula5.1**, then

$$\begin{aligned} \frac{d}{ds}\int_{B_s\cap\Sigma} f &= \frac{d}{ds}\int_{B_s\cap\Sigma} f|\nabla_\Sigma|x||\cdot|\nabla_\Sigma|x||^{-1} \\ &= \frac{d}{ds}\int_0^s\int_{\partial B_\tau\cap\Sigma} f|\nabla_\Sigma|x||^{-1} \\ &= \int_{\partial B_s\cap\Sigma} f\frac{|x|}{|x^\top|}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d}{ds}\left(s^{-k}\int_{B_s\cap\Sigma} f\right) =& -ks^{-k-1}\int_{B_s\cap\Sigma} f + s^{-k}\int_{\partial B_s\cap\Sigma} f\frac{|x|}{|x^\top|}\\
=& -\frac{1}{2}s^{-k-1}\left(\int_{B_s\cap\Sigma}|x|^2\Delta_\Sigma f + 2\int_{\partial B_s\cap\Sigma} f|x^\top| - s^2\int_{B_s\cap\Sigma}\Delta_\Sigma f\right)\\
&+ s^{-k}\int_{\partial B_s\cap\Sigma} f\frac{|x|}{|x^\top|}\\
=& -\frac{1}{2}s^{-k-1}\int_{B_s\cap\Sigma}|x|^2\Delta_\Sigma f - s^{-k-1}\int_{\partial B_s\cap\Sigma} f|x^\top| + \frac{1}{2}s^{-k-1}\int_{B_s\cap\Sigma} s^2\Delta_\Sigma f\\
&+ s^{-k}\int_{\partial B_s\cap\Sigma} f\frac{|x|}{|x^\top|}\\
=& s^{-k-1}\int_{\partial B_s\cap\Sigma} f\left(\frac{|x|}{|x^\top|}\cdot\frac{1}{|x|} - |x^\top|\right) + \frac{1}{2}s^{-k-1}\int_{B_s\cap\Sigma}(s^2-|x|^2)\Delta_\Sigma f\\
=& s^{-k-1}\int_{\partial B_s\cap\Sigma} f\frac{|x^\perp|^2}{|x^\top|} + \frac{1}{2}s^{-k-1}\int_{B_s\cap\Sigma}(s^2-|x|^2)\Delta_\Sigma f,
\end{aligned}$$

where $1-|x^\top|^2 = |x^\perp|^2$.

Thus,

$$\int_s^t \frac{d}{d\tau}\left(\tau^{-k}\int_{B_\tau\cap\Sigma} f\right)d\tau = \int_s^t\int_{\partial B_\tau\cap\Sigma} f\frac{|x^\perp|^2}{|x^\top|}\cdot\frac{1}{|x|^{k+1}} + \frac{1}{2}\int_s^t \tau^{-k-1}\int_{B_\tau\cap\Sigma}(\tau^2-|x|^2)\Delta_\Sigma f d\tau.$$

By **coarea formula**5.1 again, we obtain

$$t^{-k}\int_{B_t\cap\Sigma} f - s^{-k}\int_{B_s\cap\Sigma} f = \int_{(B_t\setminus B_s)\cap\Sigma} f\frac{1}{|x|^{k+1}}\frac{|x^\perp|^2}{|x^\top|}\cdot\frac{|x^\top|}{|x|} + \frac{1}{2}\int_s^t \tau^{-k-1}\int_{B_\tau\cap\Sigma}(\tau^2-|x|^2)\Delta_\Sigma f d\tau$$

or

$$t^{-k}\int_{B_t\cap\Sigma} f - s^{-k}\int_{B_s\cap\Sigma} f = \int_{(B_t\setminus B_s)\cap\Sigma} f\frac{|x^\perp|^2}{|x|^{k+2}} + \frac{1}{2}\int_s^t \tau^{-k-1}\int_{B_\tau\cap\Sigma}(\tau^2-|x|^2)\Delta_\Sigma f d\tau.$$

□

Corollary 5.3 (The Mean Value Inequality)**.** *Suppose that $\Sigma^k\subset\mathbb{R}^n$ is a minimal submanifold, $x_0\in\Sigma$, and $s>0$ satisfy $B_s(x_0)\cap\partial\Sigma=\emptyset$. If f is a nonnegative function on Σ with $\Delta_\Sigma f\geq -\lambda s^{-2}f$, then*

$$f(x_0)\leq e^{\frac{\lambda}{2}}\frac{1}{|B_s\subset\mathbb{R}^k|}\int_{B_s(x_0)\cap\Sigma} f.$$

Proof. Define

$$g(t) = t^{-k}\int_{B_t(x_0)\cap\Sigma} f,$$

then **Theorem5.2** implies that

$$\begin{aligned} g'(t) &\geq \frac{1}{2}t^{-k-1}\int_{B_t(x_0)\cap\Sigma}(t^2-r^2)\Delta_\Sigma f \\ &\geq -\frac{1}{2}t^{-k-1}\int_{B_t(x_0)\cap\Sigma}t^2(\lambda s^{-2}f) \\ &= -\frac{\lambda}{2}s^{-2}t^{1-k}\int_{B_t(x_0)\cap\Sigma}f \\ &= -\frac{\lambda}{2}s^{-2}tg(t) \end{aligned}$$

or

$$\frac{g'(t)}{g(t)} \geq -\frac{\lambda}{2}s^{-2}t \geq -\frac{\lambda}{2s}.$$

It is obvious that $e^{\lambda t/2s}g(t)$ is monotone nondecreasing. Then,

$$\lim_{t\to 0} e^{\frac{\lambda t}{2s}}g(t) = \lim_{t\to 0}\frac{1}{t^k}\int_{B_t(x_0)\cap\Sigma} f = \omega^k f(x_0).$$

Therefore,

$$\omega^k f(x_0) \leq e^{\frac{s\lambda}{2s}}g(s) = e^{\frac{\lambda}{2}}s^{-k}\int_{B_s(x_0)\cap\Sigma} f$$

and

$$f(x_0) \leq e^{\frac{\lambda}{2}}\frac{1}{|B_s\subset\mathbb{R}^k|}\int_{B_s(x_0)\cap\Sigma} f.$$

□

A function f is said to be subharmonic on Σ if $\Delta_\Sigma f \geq 0$. We get immediately the following mean value inequality for the special case of nonnegative subharmonic functions:

Corollary 5.4. *Suppose that $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold, $x_0 \in \mathbb{R}^n$, and f is a nonnegative subharmonic function on Σ, then*

$$s^{-k}\int_{B_s(x_0)\cap\Sigma} f$$

is a nondecreasing function of s. In particular, if $x_0 \in \Sigma$, then for all $s > 0$,

$$f(x_0) \leq \frac{1}{|B_s\subset\mathbb{R}^k|}\int_{B_s(x_0)\cap\Sigma} f.$$

6. The Strong Maximum Principle

First note that the difference of two solutions of the minimal surface equation satisfies a uniformly elliptic form equation.

Proposition 6.1. *If u_1 and u_2 are solutions of the minimal surface equation on a domain $\Omega \subset \mathbb{R}^n$, then $v = u_1 - u_2$ satisfies an equation of the form*

$$\operatorname{div}(a_{ij}\nabla v) = 0,$$

where the eigenvalues of matrix $a_{ij} = a_{ij}(x)$ satisfy

$$0 < \mu \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \frac{1}{\mu}.$$

The constant μ depends only on the upper bounds for the gradients of $|\nabla u_i|$.

Proof. Define the mapping $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by

$$F(X) = \frac{X}{(1+|X|^2)^{1/2}}.$$

By the **fundamental theorem of calculus** and the **chain rule**, we have

$$\begin{aligned} F(\nabla u_2) - F(\nabla u_1) &= \int_0^1 \frac{d}{dt}(F(\nabla u_1 + t(\nabla u_2 - \nabla u_1)))dt \\ &= \int_0^1 dF(\nabla u_1 + t(\nabla u_2 - \nabla u_1))\nabla(u_2 - u_1)dt \\ &= \left(\int_0^1 dF(\nabla u_1 + t(\nabla u_2 - \nabla u_1))dt\right)\nabla(u_2 - u_1). \end{aligned}$$

Since u_1 and u_2 are solutions of the minimal surface equation on a domain $\Omega \subset \mathbb{R}^n$, i.e.,

$$\operatorname{div}(F(\nabla u_2) - F(\nabla u_1)) = 0.$$

Then,

$$\operatorname{div}\left(\left(\int_0^1 dF(\nabla u_1 + t(\nabla u_2 - \nabla u_1))dt\right)\nabla(u_2 - u_1)\right) = 0,$$

i.e., $v = u_1 - u_2$ satisfies an equation of the form

$$\operatorname{div}(a_{ij}\nabla v) = 0.$$

Given a unit vector $V \in \mathbb{S}^{n-1}$ and $X \in \mathbb{R}^n$, we see that

$$dF(X)V = \frac{V}{(1+|X|^2)^{1/2}} - \frac{\langle X, V\rangle}{(1+|X|^2)^{3/2}}X,$$

and

$$dF(X)\langle V, V\rangle = \frac{|V|^2}{(1+|X|^2)^{1/2}} - \frac{\langle X, V\rangle^2}{(1+|X|^2)^{3/2}}.$$

It is obvious that

$$dF(X)\langle V, V\rangle \leq \frac{1}{(1+|X|^2)^{1/2}},$$

($|V|^2 = 1$)and

$$dF(X)\langle V,V\rangle = \frac{1+|X|^2}{(1+|X|^2)^{3/2}} - \frac{\langle X.V\rangle^2}{(1+|X|^2)^{3/2}} \geq \frac{1}{(1+|X|^2)^{3/2}}.$$

Therefore, the difference of two solutions of the minimal surface equation satisfies a uniformly elliptic divergence form equation. □

The following corollary is the local version of the strong maximum principle for minimal hypersurface.

Corollary 6.1. *Let $\Omega \subset \mathbb{R}^n$ be an open connected neighborhood of the origin. If $u_1, u_2 : \Omega \longrightarrow \mathbb{R}$ are solutions of the minimal surface equation with $u_1 \leq u_2$ and $u_1(0) = u_2(0)$, then $u_1 \equiv u_2$.*

Proof. Set $v = u_1 - u_2$, since $v(0) = 0$ and $v(x) \leq 0$ for any $x \in \Omega$, by **proposition6.1** and apply the **maximum principle** of uniformly elliptic equation, we obtain

$$v(x) \equiv 0, \quad \forall x \in \Omega,$$

i.e., $u_1 \equiv u_2$. □

Corollary 6.2 (The Strong Maximum Principle)**.** *If $\Sigma_1, \Sigma_2 \subset \mathbb{R}^n$ are completed connected minimal hypersurfaces (without boundaries), $\Sigma_1 \cap \Sigma_2 \neq \emptyset$, and Σ_2 lies on one side of Σ_1, then $\Sigma_1 = \Sigma_2$.*

Proof. Since Σ_1, Σ_2 locally be written as graph near some points, then we obtain the result by **Corollary6.1**. □

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^2$ be strictly convex and $\sigma \subset \mathbb{R}^3$ a simple closed curve which is a graph over $\partial\Omega$ with bounded slope. Then any minimal surface $\Sigma \subset \mathbb{R}^3$ with $\partial\Sigma = \sigma$ must be graphical over Ω and hence unique.*

Proof. By the maximum principle, the interior of Σ is contained in the interior of the cylinder $\Omega \times \mathbb{R}$.
Given a plane $\{x_3 = t\}$, we divide Σ into the portions Σ_t^+ above and Σ_t^- below the plane. Reflecting Σ_t^+ below the plane gives a new minimal surface then $\widetilde{\Sigma}_t^+$ below the plane.
By the maximum principle, there cannot be a first t where Σ_t^- and $\widetilde{\Sigma}_t^+$ have an interior point of contact.
Since $\partial\Sigma$ is a graph, there cannot be a boundary point of contact.
It follows then that the projection of Σ to the plane $\{x_3 = 0\}$ must be one to one, as desired. □

Remark 6.1. The higher dimensional case.
Let $\Omega \subset \mathbb{R}^{n-1}$ be strictly convex and $\sigma \subset \mathbb{R}^n$ a co-dimensional submanifold which is a

graph over $\partial\Omega$. Then any minimal surface $\Sigma \subset \mathbb{R}^n$ with $\partial\Sigma = \sigma$ must be graphical over Ω and hence unique.

7. Bernstein Theorem—2 Dimensional

Lemma 7.1. *If $u : \Omega \subset \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ is solution to the minimal surface equation (Σ_u is minimal graph), then for all nonnegative smooth function φ with support contained in $\Omega \times \mathbb{R}$,*

$$\int_{\Sigma_u} |\nabla_{\Sigma_u}\varphi|^2 - |A|^2\varphi^2 \geq 0.$$

Proof. Set $u = \langle N, E_N\rangle$, by **Proposition3.3**, we have

$$\Delta_\Sigma u = -|A|^2 u.$$

Set $w = \log u$, then

$$\Delta_\Sigma w = -|A|^2 - |\nabla_\Sigma w|^2,$$

and

$$-\int_{\Sigma_u} \varphi^2 \Delta_{\Sigma_u} w = \int_{\Sigma_u} \varphi^2 |\nabla_{\Sigma_u} w|^2 + |A|^2\varphi^2.$$

Apply integration by parts, we obtain

$$-\int_{\Sigma_u} \varphi^2 \Delta_{\Sigma_u} w = \int_{\Sigma_u} 2\varphi\nabla_{\Sigma_u}\varphi\nabla_{\Sigma_u} w \leq \int_{\Sigma_u} |\nabla_{\Sigma_u}\varphi|^2 + \varphi^2|\nabla_{\Sigma_u} w|^2 \quad (\varphi \in C_c^\infty(\Sigma_u)).$$

Then

$$\int_{\Sigma_u} |\nabla_{\Sigma_u}\varphi|^2 - |A|^2\varphi^2 \geq 0.$$

□

Remark 7.1. The lemma shows that we can bound the total curvature in terms of the energy of a cutoff function φ.

Apply the lemma, we get a total curvature bound for a minimal graph.

Theorem 7.1. *If $u : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a solution to the minimal surface equation (Σ is minimal graph), $\kappa > 1$, and Ω contains a ball of radius κR centered at the origin, then*

$$\int_{B_{\sqrt{\kappa}R}\cap|\Sigma|} |A|^2 \leq \frac{C}{\log\kappa}.$$

Proof. Define the cutoff function η on all of $\mathbb{R}^3$ and then restrict it to the graph of u as follows: Let r denotes the distance to the origin in $\mathbb{R}^3$ and define η by

$$\eta = \begin{cases} 1 & r^2 \leq \kappa R^2 \\ 2 - 2\frac{\log(rR^{-1})}{\log\kappa} & \kappa R^2 < r^2 \leq \kappa^2 R^2 \\ 0 & r^2 > \kappa^2 R^2. \end{cases}$$

Since $|\nabla_\Sigma r| \le |\nabla r| = 1$, we have

$$|\nabla_\Sigma \eta| = |\frac{2}{r\log\kappa}\nabla_\Sigma r| \le \frac{2}{r\log\kappa}.$$

Apply **Lemma7.1** with this cutoff function η and using the area bound **Corollary2.1**, we have

$$\begin{aligned}
\int_{B_{\sqrt{\kappa}R}\cap\Sigma} |A|^2 &\le \int_{B_{\kappa R}\cap\Sigma} \eta^2|A|^2 \le \int_{B_{\kappa R}\cap\Sigma} |\nabla_\Sigma\eta|^2 \\
&= \int_{(B_{\kappa R}\setminus B_{\sqrt{\kappa}R})\cap\Sigma} |\nabla_\Sigma\eta|^2 \\
&\le \frac{4}{(\log\kappa)^2}\int_{(B_{\kappa R}\setminus B_{\sqrt{\kappa}R})\cap\Sigma} r^{-2} \\
&\le \frac{4}{(\log\kappa)^2}\sum_{l=\log\sqrt{\kappa}}^{\log\kappa}\int_{(B_{e^l R}\setminus B_{e^{l-1}R})\cap\Sigma} r^{-2} \\
&\le \frac{4}{(\log\kappa)^2}\sum_{l=\log\sqrt{\kappa}}^{\log\kappa} 2\pi(e^l R)^2\cdot\frac{1}{e^{2(l-1)}R^2} \\
&= \frac{4}{(\log\kappa)^2}\sum_{l=\log\sqrt{\kappa}}^{\log\kappa} 2\pi e^2 \le \frac{C}{\log\kappa}.
\end{aligned}$$

Here, for simplicity, we assumed that $\dfrac{\log\kappa}{2}$ is an integer. $\square$

Theorem 7.2 (Bernstein Theorem(2-dimensional))**.** *If $u:\mathbb{R}^2 \longrightarrow \mathbb{R}$ is an entire solution to the minimal surface equation, then $u(x,y) = ax+by+c$ for some constants $a,b,c\in\mathbb{R}$.*

Proof. By the **Theorem7.1**, take $\kappa = R$ and $R = 1$, we get for all $R > 1$,

$$\int_{B_{\sqrt{R}}} |A|^2 \le \frac{C}{\log R}.$$

Letting $R \longrightarrow \infty$, we conclude that $|A|^2 = 0$. By definition of the second fundamental form on $\mathbb{R}^3$, we have

$$0 = u_{xx} = u_{yy} = u_{xy}.$$

Therefore $u = ax+by+c$ for some constants $a,b,c\in\mathbb{R}$. $\square$

8. Bernstein Theorem—$n \le 6$ Dimensional

First, we give some calculations about Riemannian Geometry.

Take E_i is an orthonormal frame for a minimal hypersurface (minimal submanifold) Σ

in $\mathbb{R}^n$ with metric g and connection ∇^Σ, then the curvature R is given by

$$R_{ijkl} = g(R(E_i, E_j)E_k, E_l) = g(\nabla^\Sigma_{E_j}\nabla^\Sigma_{E_i}E_k - \nabla^\Sigma_{E_i}\nabla^\Sigma_{E_j}E_k + \nabla^\Sigma_{\nabla^\Sigma_{E_i}E_j - \nabla^\Sigma_{E_j}E_i}E_k, E_l).$$

The **Gauss equation** expresses the curvature R in terms of the second fundamental form of Σ:

$$R_{ijkl} = a_{ik}a_{jl} - a_{jk}a_{il},$$

where $a_{jk} = \langle A(E_j, E_k), E_n\rangle$ and E_n is a unit normal.
Let a be the symmetric two-tensor on Σ given by

$$a(X, Y) = \langle A(X, Y), E_n\rangle = \left\langle \nabla_X Y - (\nabla_X Y)^\top, E_n \right\rangle = \langle \nabla_X Y, E_n\rangle = -\langle \nabla_X E_n, Y\rangle\,,$$

and set $a_{ij} = -\langle \nabla_{E_i}E_n, E_j\rangle$. That is,

$$a = \sum_{i,j=1}^{n-1} a_{ij}\Theta_i \otimes \Theta_j,$$

where Θ_i is the dual (to E_i) orthonormal frame.
Let $a_{\cdot\cdot,k}$ and $a_{ij,k}$ be defined by

$$a_{\cdot\cdot,k} = \sum_{i,j=1}^{n-1} a_{ij,k}\Theta_i \otimes \Theta_j = \nabla_{E_k} a.$$

Since a is a symmetric two-tensor, it follows that $a_{\cdot\cdot,k}$ is also a symmetric two-tensor.
Since the curvature R of $\mathbb{R}^n$ vanishes, we have the **Codazzi equation**

$$\begin{aligned}
a_{ij,k} &= (\nabla^\top_{E_k} a)(E_i, E_j) \\
&= E_k a(E_i, E_j) - a(\nabla^\top_{E_k}E_i, E_j) - a(E_i, \nabla^\top_{E_k}E_j) \\
&= -E_k\langle \nabla_{E_i}E_n, E_j\rangle + \left\langle \nabla_{\nabla^\top_{E_k}E_i}E_n, E_j\right\rangle + \langle \nabla_{E_i}E_n, \nabla_{E_k}E_j\rangle \\
&= -\langle \nabla_{E_k}\nabla_{E_i}E_n, E_j\rangle - \langle \nabla_{E_i}E_n, \nabla_{E_k}E_j\rangle + \left\langle \nabla_{\nabla^\top_{E_k}E_i}E_n, E_j\right\rangle + \langle \nabla_{E_i}E_n, \nabla_{E_k}E_j\rangle \\
&= -\langle \nabla_{E_k}\nabla_{E_i}E_n, E_j\rangle + \left\langle \nabla_{\nabla^\top_{E_k}E_i}E_n, E_j\right\rangle \\
&= -\langle \nabla_{E_i}\nabla_{E_k}E_n, E_j\rangle + \left\langle \nabla_{\nabla^\top_{E_i}E_k}E_n, E_j\right\rangle \\
&= a_{kj,i}.
\end{aligned}$$

Note that the second equality is the Leibniz rule and sixth equality is R_{iknj} vanishes on $\mathbb{R}^n$.
Therefore, let $a_{\cdot\cdot,\cdot}$ be the symmetric three-tensor given by

$$a_{\cdot\cdot,\cdot} = \sum_{i,j,k=1}^{n-1} a_{ij,k}\Theta_i \otimes \Theta_j \otimes \Theta_k.$$

Let $a_{\cdot\cdot,\cdot l}$ and $a_{ij,kl}$ be defined by

$$a_{\cdot\cdot,\cdot l} = \sum_{i,j,k=1}^{n-1} a_{ij,kl}\Theta_i \otimes \Theta_j \otimes \Theta_k = \nabla_{E_l} a_{\cdot\cdot,\cdot}.$$

Apply the Leibniz rule, we have

$$\begin{aligned}
a_{ij,kl} =& \nabla^{\top}_{E_l} a_{\cdot\cdot,\cdot}(E_i, E_j, E_k) \\
=& E_l a_{\cdot\cdot,\cdot}(E_i, E_j, E_k) - a_{\cdot\cdot,\cdot}(\nabla^{\top}_{E_l} E_i, E_j, E_k) \\
& - a_{\cdot\cdot,\cdot}(E_i, \nabla^{\top}_{E_l} E_j, E_k) - a_{\cdot\cdot,\cdot}(E_i, E_j, \nabla^{\top}_{E_l} E_k) \\
=& E_l a_{\cdot\cdot,k}(E_i, E_j) - a_{\cdot\cdot,k}(\nabla^{\top}_{E_l} E_i, E_j) \\
& - a_{\cdot\cdot,k}(E_i, \nabla^{\top}_{E_l} E_j) - \nabla^{\top}_{\nabla^{\top}_{E_l} E_k} a(E_i, E_j) \\
=& E_l \nabla^{\top}_{E_k} a(E_i, E_j) - \nabla^{\top}_{E_k} a(\nabla^{\top}_{E_l} E_i, E_j) - \nabla^{\top}_{E_k} a(E_i, \nabla^{\top}_{E_l} E_j) - \nabla^{\top}_{\nabla^{\top}_{E_l} E_k} a(E_i, E_j) \\
=& E_l E_k a(E_i, E_j) - E_l a(\nabla^{\top}_{E_k} E_i, E_j) - E_l a(E_i, \nabla^{\top}_{E_k} E_j) \\
& - E_k a(\nabla^{\top}_{E_l} E_i, E_j) + a(\nabla^{\top}_{E_k}\nabla^{\top}_{E_l} E_i, E_j) + a(\nabla^{\top}_{E_l} E_i, \nabla^{\top}_{E_k} E_j) \\
& - E_k a(E_i, \nabla^{\top}_{E_l} E_j) + a(\nabla^{\top}_{E_k} E_i, \nabla^{\top}_{E_k} E_i, \nabla^{\top}_{E_l} E_j) + a(E_i, \nabla^{\top}_{E_k}\nabla^{\top}_{E_l} E_j) \\
& - \nabla^{\top}_{E_l} E_k a(E_i, E_j) + a(\nabla^{\top}_{\nabla^{\top}_{E_l} E_k} E_i, E_j) + a(E_i, \nabla^{\top}_{\nabla^{\top}_{E_l} E_k} E_j) \\
=& a_{ij,lk} - a(\nabla^{\top}_{E_l}\nabla^{\top}_{E_k} E_i, E_j) - a(\nabla^{\top}_{\nabla^{\top}_{E_k} E_l} E_i, E_j) \\
& - a(E_i, \nabla^{\top}_{E_l}\nabla^{\top}_{E_k} E_j) - a(E_i, \nabla^{\top}_{\nabla^{\top}_{E_k} E_l} E_j) \\
& + a(\nabla^{\top}_{E_k}\nabla^{\top}_{E_l} E_i, E_j) + a(\nabla^{\top}_{\nabla^{\top}_{E_l} E_k} E_i, E_j) \\
& + a(E_i, \nabla^{\top}_{E_k}\nabla^{\top}_{E_l} E_j) + a(E_i, \nabla^{\top}_{\nabla^{\top}_{E_l} E_k} E_j).
\end{aligned}$$

Since

$$\begin{aligned}
& a(\nabla^{\top}_{E_k}\nabla^{\top}_{E_l} E_i, E_j) + a(\nabla^{\top}_{\nabla^{\top}_{E_l} E_k} E_i, E_j) - a(\nabla^{\top}_{E_l}\nabla^{\top}_{E_k} E_i, E_j) - a(\nabla^{\top}_{\nabla^{\top}_{E_k} E_l} E_i, E_j) \\
=& \left\langle A(\nabla^{\top}_{E_k}\nabla^{\top}_{E_l} E_i + \nabla^{\top}_{\nabla^{\top}_{E_l} E_k} E_i, E_j), E_n \right\rangle - \left\langle A(\nabla^{\top}_{E_l}\nabla^{\top}_{E_k} E_i + \nabla^{\top}_{\nabla^{\top}_{E_k} E_l} E_i, E_j), E_n \right\rangle \\
=& \left\langle A(\nabla^{\top}_{E_k}\nabla^{\top}_{E_l} E_i - \nabla^{\top}_{E_l}\nabla^{\top}_{E_k} E_i + \nabla^{\top}_{\nabla^{\top}_{E_l} E_k - \nabla^{\top}_{E_k} E_l} E_i, E_j), E_n \right\rangle \\
=& \sum_{m=1}^{n-1} \langle A(R_{lkim} E_m, E_j), E_n \rangle = \sum_{m=1}^{n-1} R_{lkim} \langle A(E_m, E_j), E_n \rangle \\
=& \sum_{m=1}^{n-1} R_{lkim} a_{mj},
\end{aligned}$$

then

$$a_{ij,kl} = a_{ij,lk} + \sum_{m=1}^{n-1} R_{lkim} a_{mj} + \sum_{m=1}^{n-1} R_{lkjm} a_{mi}.$$

Apply the **Gauss equation**, we obtain

$$a_{ik,jk} = a_{ik,kj} + \sum_{m=1}^{n-1} (a_{ki}a_{jm} - a_{ji}a_{km})a_{mk} + \sum_{m=1}^{n-1} (a_{kk}a_{jm} - a_{jk}a_{km})a_{mi}. \qquad (*)$$

We derive the Laplacian of the norm squared of the second fundamental form of minimal hypersurface Σ in $\mathbb{R}^n$.

Theorem 8.1 (Simons' Inequality)**.** *Suppose that $\Sigma^{n-1} \subset \mathbb{R}^n$ is a minimal hypersurface, then*

$$\Delta_\Sigma |A|^2 \geq -2|A|^4 + 2\left(1 + \frac{2}{n-1}\right)|\nabla_\Sigma |A||^2.$$

Note that this can equivalently be expressed as

$$|A|\Delta_\Sigma |A| + |A|^4 \geq \frac{2}{n-1}|\nabla_\Sigma |A||^2.$$

Proof. By $(*)$, we have

$$\begin{aligned}
\frac{1}{2}\Delta_\Sigma |A|^2 =& \frac{1}{2}\sum_{i,j=1}^{n-1} \Delta_\Sigma a_{ij}^2 = \frac{1}{2}\sum_{i,j,k=1}^{n-1} (a_{ij}\cdot a_{ij})_{kk} \\
=&\text{or} \left(\sum_{i,j=1}^{n-1} a_{ij}\Delta_\Sigma a_{ij} + \sum_{i,j=1}^{n-1} |\nabla_\Sigma a_{ij}|^2\right) \qquad (\textbf{The Bochner Formula}) \\
=& \sum_{i,j,k=1}^{n-1} a_{ij}a_{ij,kk} + \sum_{i,j,k=1}^{n-1} a_{ij,k}^2 = \sum_{i,j,k=1}^{n-1} a_{ij}a_{ik,jk} + \sum_{i,j,k=1}^{n-1} a_{ij,k}^2 \\
=& \sum_{i,j,k=1}^{n-1} a_{ij}a_{kk,ij} + \sum_{i,j,k,m}^{n-1} a_{ij}(a_{ki}a_{jm} - a_{ji}a_{km})a_{mk} \\
&+ \sum_{i,j,k,m=1}^{n-1} a_{ij}(a_{kk}a_{jm} - a_{jk}a_{km})a_{mi} + \sum_{i,j,k=1}^{n-1} a_{ij,k}^2 \\
=& -\sum_{i,j,k,m=1}^{n-1} a_{ij}^2 a_{km}^2 + \sum_{i,j,k=1}^{n-1} a_{ij,k}^2.
\end{aligned}$$

Note that the last equality used that Σ is minimal ,i.e., $H = \sum_{k=1}^{n-1} a_{kk} = 0$ and the fifth, sixth equality used that **Codazzi equation**, i.e., $a_{ij,k} = a_{kj,i}$ and a is a symmetric two-tensor, i.e., $a_{ij} = a_{ji}$.

Therefore,

$$\Delta_\Sigma |A|^2 = -2|A|^4 + 2\sum_{i,j,k=1}^{n-1} a_{ij,k}^2$$

which is **Simons' equation**.

Since a is symmetric, we may choose E_i, $i = 1, \cdots, n-1$, such that at x we have

$$a_{ij} = \lambda_i \delta_{ij}.$$

Since $\nabla |A|^2 = 2|A|\nabla|A|$, we have

$$\begin{aligned}
4|A|^2|\nabla|A||^2 = |\nabla|A|^2|^2 &= \sum_{k=1}^{n-1}\left(\left(\sum_{i,j=1}^{n-1} a_{ij}^2\right)_k\right)^2 \\
&= 4\sum_{k=1}^{n-1}\left(\sum_{i,j=1}^{n-1} a_{ij}a_{ij,k}\right)^2 = 4\sum_{k=1}^{n-1}\left(\sum_{i=1}^{n-1} a_{ii,k}\lambda_i\right)^2 \\
&\le 4\sum_{k=1}^{n-1}\left(\left(\sum_{i=1}^{n-1} a_{ii,k}^2\right)^{1/2}\left(\sum_{i=1}^{n-1}\lambda_i^2\right)^{1/2}\right)^2 \\
&= 4|A|^2\sum_{i,k=1}^{n-1} a_{ii,k}^2.
\end{aligned}$$

Hence,

$$|\nabla|A||^2 \le \sum_{i,k=1}^{n-1} a_{ii,k}^2.$$

Therefore,

$$\begin{aligned}
|\nabla|A||^2 \le \sum_{i,k=1}^{n-1} a_{ii,k}^2 &= \sum_{i\neq k} a_{ii,k}^2 + \sum_{i=1}^{n-1} a_{ii,i}^2 \\
&= \sum_{i\neq k} a_{ii,k}^2 + \sum_{i=1}^{n-1}\left(-\sum_{i\neq j} a_{jj,i}\right)^2 \\
&\le \sum_{i\neq k} a_{ii,k}^2 + (n-2)\sum_{i=1}^{n-1}\sum_{i\neq j} a_{jj,i}^2 = (n-1)\sum_{i\neq k} a_{ii,k}^2 \\
&= (n-1)\sum_{i\neq k} a_{ik,i}^2 = \frac{n-1}{2}\left(\sum_{i\neq k} a_{ik,i}^2 + \sum_{i\neq k} a_{ki,i}^2\right).
\end{aligned}$$

The second equality used Σ is minimal and the second inequality used the algebraic fact that

$$\left(\sum_{j=1}^{n-2} b_j\right)^2 \leq (n-2)\sum_{j=1}^{n-2} b_j^2.$$

We obtain

$$\left(1+\frac{2}{n-1}\right)|\nabla|A||^2 \leq \sum_{i,k=1}^{n-1} a_{ii,k}^2 + \sum_{i\neq k} a_{ik,i}^2 + \sum_{i\neq k} a_{ki,i}^2 \leq \sum_{i,j,k=1}^{n-1} a_{ij,k}^2.$$

By **Simons' equation**, we obtain

$$\left(1+\frac{2}{n-1}\right)|\nabla|A||^2 \leq \frac{1}{2}\Delta|A|^2 + |A|^4.$$

Thus,

$$\Delta_\Sigma|A|^2 \geq -2|A|^4 + 2\left(1+\frac{2}{n-1}\right)|\nabla_\Sigma|A||^2.$$

□

Definition 8.1 (Stable)**.** *Suppose $\Sigma^k \subset M^n$ is minimal submanifold, Σ^k is* ***stable*** *if for all variations F with boundary fixed*

$$\left.\frac{d^2}{dt^2}\right|_{t=0} |F(\Sigma,t)| \geq 0.$$

Theorem 8.2 (The Stability Inequality)**.** *Suppose that $\Sigma^{n-1} \subset M^n$ is a stable minimal hypersurface with trivial normal bundle, then for all Lipschitz functions η with compact support*

$$\int_\Sigma \left(\inf_M \mathrm{Ric}_M + |A|^2\right)\eta^2 \leq \int_\Sigma |\nabla_\Sigma \eta|^2.$$

We combine **Simons' inequality**8.1 with the **stability inequality**8.2 to show higher L^p bounds for the square of the norm of the second fundamental form for stable minimal hypersurface.

Theorem 8.3 (Schoen-Simon-Yau)**.** *Suppose that $\Sigma^{n-1} \subset \mathbb{R}^n$ is an orientable stable minimal hypersurface. For all $p \in \left[2, 2+\sqrt{\frac{2}{(n-1)}}\right)$ and each nonnegative Lipschitz function ϕ with compact support*

$$\int_\Sigma |A|^{2p}\phi^{2p} \leq C(n,p)\int_\Sigma |\nabla\phi|^{2p}.$$

Proof. If we insert $\eta = |A|^{1+q}f$ in the **stability inequality**8.2 for $0 \le q < \sqrt{\frac{2}{n-1}}$, we have

$$\begin{aligned}
\int_\Sigma |A|^2(|A|^{2+2q}f^2) &\le \int_\Sigma |\nabla(|A|^{1+q}f)|^2 \\
&= \int_\Sigma |(1+q)f|A|^2\nabla|A| + |A|^{1+q}\nabla f|^2 \\
&= \int_\Sigma f^2(1+q)^2|A|^{2q}|\nabla|A||^2 + \int_\Sigma |A|^{2+2q}|\nabla f|^2 + 2(1+q)\int_\Sigma f|A|^{1+2q}\left\langle \nabla f, \nabla\,|\,A\,|\right\rangle.
\end{aligned}$$

• Consider **Simons' inequlaity**8.1

$$|A|\Delta|A| + |A|^4 \ge \frac{2}{n-1}|\nabla|A||^2,$$

and by integration by parts, we have

$$\begin{aligned}
\frac{2}{n-1}\int_\Sigma |\nabla|A||^2|A|^{2q}f^2 &\le \int_\Sigma |A|^{4+2q}f^2 + \int_\Sigma f^2|A|^{1+2q}\Delta|A| \\
&= \int_\Sigma |A|^{4+2q}f^2 - \int_\Sigma \nabla(f^2|A|^{1+2q})\nabla|A| \\
&= \int_\Sigma |A|^{4+2q}f^2 - \int_\Sigma (2f\nabla f|A|^{1+2q})\nabla|A| - \int_\Sigma f^2(1+2q)|A|^{2q}|\nabla|A||^2 \\
&= \int_\Sigma |A|^{4+2q}f^2 - 2\int_\Sigma f|A|^{1+2q}\left\langle \nabla f, \nabla\,|\,A\,|\right\rangle - (1+2q)\int_\Sigma f^2|A|^{2q}|\nabla|A||^2.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{2}{n-1}\int_\Sigma |\nabla|A||^2|A|^{2q}f^2 \le &(1+q)^2\int_\Sigma f^2|\nabla|A||^2|A|^{2q} + \int_\Sigma |A|^{2+2q}|\nabla f|^2 \\
&+ 2(1+q)\int_\Sigma f|A|^{1+2q}\left\langle \nabla f, \nabla\,|\,A\,|\right\rangle \\
&- 2\int_\Sigma f|A|^{1+2q}\left\langle \nabla f, \nabla\,|\,A\,|\right\rangle - (1+2q)\int_\Sigma f^2|A|^{2q}|\nabla|A||^2,
\end{aligned}$$

and by **Cauchy inequality**,

$$\begin{aligned}
\left(\frac{2}{n-1} - q^2\right)\int_\Sigma |\nabla|A||^2|A|^{2q}f^2 &\le \int_\Sigma |A|^{2+2q}|\nabla f|^2 + 2q\int_\Sigma f|A|^{1+2q}\left\langle \nabla f, \nabla\,|\,A\,|\right\rangle \\
&\le \int_\Sigma |A|^{2+2q}|\nabla f|^2 + \varepsilon q\int_\Sigma f^2|A|^{2q}|\nabla|A||^2 + \frac{q}{\varepsilon}\int_\Sigma |\nabla f|^2|A|^{2+2q}.
\end{aligned}$$

Hence, we obtain

$$\left(\frac{2}{n-1} - q^2 - \varepsilon q\right)\int_\Sigma f^2|A|^{2q}|\nabla|A||^2 \le \left(1 + \frac{q}{\varepsilon}\right)\int_\Sigma |\nabla f|^2|A|^{2+2q}.$$

• Apply **Cauchy-Schwarz inequality**, we have

$$\begin{aligned}\int_\Sigma |A|^{4+2q} f^2 \leq &(1+q)^2 \int_\Sigma f^2 |\nabla |A||^2 |A|^{2q} + \int_\Sigma |A|^{2+2q} |\nabla f|^2 \\ &+ (1+q)^2 \int_\Sigma f^2 |\nabla |A||^2 |A|^{2q} + \int_\Sigma |A|^{2+2q} |\nabla f|^2 \\ =&2(1+q)^2 \int_\Sigma f^2 |A|^{2q} |\nabla |A||^2 + 2 \int_\Sigma |A|^{2+2q} |\nabla f|^2 \\ \leq& \left(\frac{2(1+q)^2 \left(1+\frac{q}{\varepsilon}\right)}{\frac{2}{n-1} - q^2 - \varepsilon q} + 2 \right) \int_\Sigma |A|^{2+2q} |\nabla f|^2.\end{aligned}$$

We set $p = 2+q$ and $f = \phi^p$, then $2 \leq p < 2 + \sqrt{\dfrac{2}{n-1}}$ and **Hölder inequality** gives

$$\begin{aligned}\int_\Sigma |A|^{2p} \phi^{2p} &\leq C(n,p) \int_\Sigma |A|^{2p-2} \phi^{2p-2} |\nabla \phi|^2 \\ &\leq C(n,p) \left(\int_\Sigma |A|^{2p} \phi^{2p} \right)^{\frac{p-1}{p}} \left(\int_\Sigma |\nabla \phi|^{2p} \right)^{\frac{1}{p}}.\end{aligned}$$

Hence, we obtain

$$\int_\Sigma |A|^{2p} \phi^{2p} \leq C(n,p) \int_\Sigma |\nabla \phi|^{2p}.$$

□

Theorem 8.4 (Schoen-Simon-Yau)**.** *If $\Sigma^{n-1} \subset \mathbb{R}^n$ is a complete two-sided stable minimal hypersurface, $n \leq 6$, and there exists $V < \infty$ so that*

$$\sup_{R>0} \frac{|B_R \cap \Sigma|}{R^{n-1}} \leq V,$$

then Σ is flat.

Proof. Consider cutoff function

$$\phi = \begin{cases} 1 & B_r \cap \Sigma \\ \phi(r) & (B_{2r} \setminus B_r) \cap \Sigma \\ 0 & \Sigma \setminus B_{2r}. \end{cases}$$

By **Theorem8.3**, choose $2p = 4 + \sqrt{\dfrac{7}{5}} < 4 + \sqrt{\dfrac{8}{5}}$ and since $|\nabla \phi| \leq \dfrac{C}{r}$, then

$$\begin{aligned}\int_{B_r \cap \Sigma} |A|^{4+\sqrt{\frac{7}{5}}} &\leq C(n,p) r^{-4-\sqrt{\frac{7}{5}}} |B_{2r} \cap \Sigma| \\ &\leq C(n,p) 2^{n-1} V r^{n-5-\sqrt{\frac{7}{5}}}.\end{aligned}$$

Since $n \leq 6$, i.e., $n - 5 - \sqrt{\dfrac{7}{5}} < 0$ and let $r \longrightarrow \infty$, we have $|A|^2 = 0$. □

Remark 8.1. From the above proof, we must consider $n \leq 6$. In fact,

$$\int_{B_r \cap \Sigma} |A|^{2p} \leq C(n,p) r^{-2p} |B_{2r} \cap \Sigma| \\ \leq C(n,p) 2^{n-1} V r^{n-1-2p}.$$

Choose p such that $n-1-2p<0$ and $p \in \left[2, 2+\sqrt{\frac{2}{n-1}}\right)$, then $n \leq 6$.

Since minimal graphs are stable and, by **Corollary2.1**, we obtain the following Bernstein theorem ($n \leq 6$-dimensional):

Theorem 8.5 (Bernstein Theorem($n \leq 6$-dimensional)). *If $u : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ ($n \leq 6$) is an entire solution to the minimal surface equation, then u is a linear function.*

Proof. It is trivial by **Theorem8.4**. □

9. Bernstein Theorem—$n \leq 8$ Dimensional

- Fleming:

  If any minimizing cone is flat in $\mathbb{R}^n$, then any minimizing hypersurface is flat in $\mathbb{R}^n$. Here, cone is defined

  $$\Sigma^{n-1} = (0, \infty) \times \hat{\Sigma}^{n-2} (\subset \mathbb{S}^{n-1}) = \{ty \mid t \in (0,\infty), y \in \hat{\Sigma}^{n-2}\}.$$

- Simon:

  Fleming's assumption holds for $n \leq 7$.

- de Giorgi:

  If any minimizing cone is flat in $\mathbb{R}^n$, then any minimal graph is flat in $\mathbb{R}^{n+1}$.

Lemma 9.1. *If $\Sigma \subset \mathbb{R}^{n+1}$ is a cone, then*

$$|\nabla A|^2 - |\nabla |A||^2 \geq \frac{2|A|^2}{|x|^2}.$$

Proof. Let $e_1, \cdots, e_{n-1}, e_n$ be orthonormal basis of Σ, s.t., $e_1, \cdots, e_{n-1} \in T\Sigma'$ and $e_n = \frac{x}{|x|}$, where $\Sigma' = \Sigma \cap \mathbb{S}^n$ and $\Sigma' \subset \mathbb{S}^n$.
Give the second fundamental form of cone.
Consider $X = (t, y) = ty$, where $t \in (0, \infty)$ and $y \in \Sigma' \subset \mathbb{S}^n$. Note that $\partial_t = e_n$ and $\partial_{y_i} = e_i$, we have

$$\partial_t X = y, \quad \partial_{y_i} X = t e_i (e_i \in T\Sigma')$$

and the first fundamental form is

$$g_\Sigma = dt^2 + t^2 g_{\Sigma'}.$$

By calculating, we have

$$\partial^2_{tt}X = 0, \quad \partial^2_{ty_i}X = e_i, \quad \partial^2_{y_iy_j} = t\nabla^{\mathbb{S}^n}_{e_j}e_i, \quad N(x) = N'(y) : \Sigma' \to \mathbb{S}^n$$

and the second fundamental form is

$$h_{tt} = 0, \quad h_{ty_i} = 0, \quad h_{y_iy_j} = t\left\langle \nabla^{\mathbb{S}^n}_{e_j}e_i, N'(y)\right\rangle = th'_{ij}.$$

Here, h'_{ij} is the second fundamental form of $\Sigma' \subset \mathbb{S}^n$.
Direct calculate, we have

$$\begin{aligned}
h_{ij,n} &= \nabla^{\Sigma}_{e_n}h_{ij} = \nabla^{\Sigma}_{\partial_t}h_{ij} = \nabla^{\Sigma}_{\partial_t}h(\partial_{y_i}, \partial_{y_j})\\
&= \partial_t(th'_{ij}) - h(\nabla_{\partial_{y_t}}\partial_{y_i}, \partial_{y_j}) - h(\partial_{y_i}, \nabla_{\partial_{y_t}}\partial_{y_j})\\
&= \partial_t(th'_{ij}) - h(\Gamma^k_{ti}\partial_{y_k}, \partial_{y_j}) - h(\partial_{y_i}, \Gamma^k_{tj}\partial_{y_k})\\
&= \partial_t(th'_{ij}) - h_{kj}\Gamma^k_{ti} - h_{ik}\Gamma^k_{tj}\\
&= -\frac{h_{ij}}{t},
\end{aligned}$$

where $\Gamma^k_{ti} = \dfrac{1}{2}g^{kl}\partial_t g_{ik} = \dfrac{1}{t}\delta_{ik}$. Note that $h_{in} = 0$. We obtain

$$\begin{aligned}
|\nabla A|^2 - |\nabla|A||^2 &= \sum_{i,j,k=1}^{n} h^2_{ij,k} - \frac{\sum_{i,j,k=1}^{n}(h_{ij}h_{ij,k})^2}{\sum_{s,t=1}^{n} h^2_{st}}\\
&= \frac{1}{|A|^2}\left(\left(\sum_{i,j,k=1}^{n} h^2_{ij,k}\right)\left(\sum_{s,t=1}^{n} h^2_{st}\right) - \sum_{i,j,k=1}^{n}(h_{ij}h_{ij,k})^2\right)\\
&= \frac{1}{2|A|^2}\sum_{i,j,r,s,k=1}^{n}(h_{rs}h_{ij,k} - h_{ij}h_{rs,k})^2\\
&\geq \frac{2}{|A|^2}\left(\sum_{j,r,s,k=1}^{n}(h_{rs}h_{nj,k} - h_{nj}h_{rs,k})^2\right)\\
&= \frac{2}{|A|^2}\sum_{j,r,s,k=1}^{n}(h_{rs}h_{nj,k})^2 = \frac{2}{|A|^2}\sum_{j,r,s,k=1}^{n}(h_{rs}h_{jk,n})^2\\
&= \frac{2}{|A|^2}\sum_{j,r,s,k=1}^{n}\left(h_{rs}(-\frac{1}{r})h_{jk}\right)^2 = \frac{2|A|^2}{|x|^2}.
\end{aligned}$$

□

Theorem 9.1. *Any are minimizing cone in $\mathbb{R}^n (3 \leq n \leq 7)$ is flat.*

Proof. Since minimizing hypersurface is stable, then

$$\int_{\Sigma} |A|^2\eta^2 - |\nabla\eta|^2 \leq 0.$$

Set $\eta = \eta|A|$, then

$$\int_\Sigma |A|^4\eta^2 \le \int_\Sigma |\nabla(\eta|A|)|^2 = \int_\Sigma (\eta\nabla|A| + |A|\nabla\eta)^2$$
$$= \int_\Sigma \eta^2(\nabla|A|)^2 + 2|A|\eta\,\langle\nabla\,|\,A\,|\,,\nabla\eta\rangle + |A|^2|\nabla\eta|^2$$
$$= \int_\Sigma |\nabla\eta|^2|A|^2 + |\nabla|A||^2\eta^2 + \nabla\eta^2\nabla(\frac{1}{2}|A|^2).$$

By the **Simon equation** and the **divergence theorem**, we have

$$\int_\Sigma \nabla\eta^2\nabla(\frac{1}{2}|A|^2) = -\int_\Sigma \eta^2\nabla(\frac{1}{2}|A|^2) = -\int_\Sigma \eta^2(|\nabla A|^2 - |A|^4).$$

Lemma9.1 implies that

$$\int_\Sigma \nabla\eta^2\nabla(\frac{1}{2}|A|^2) \le -\int_\Sigma \eta^2\left(|\nabla|A||^2 + \frac{2|A|^2}{r^2} - |A|^4\right).$$

Thus,

$$2\int_\Sigma \frac{|A|^2\eta^2}{r^2} \le \int_\Sigma |\nabla\eta|^2|A|^2.$$

By multiply a suitable cutoff function, we can prove that the above inequality still holds true for $\eta \in \mathrm{Lip}(\mathbb{R}^{n+1})$ provided

$$\int_\Sigma \frac{|A|^2\eta^2}{r^2} < \infty.$$

Our next step is to find a suitable $\eta \in \mathrm{Lip}(\mathbb{R}^{n+1})$, s.t.,

$$(2-\varepsilon)\int_\Sigma \frac{|A|^2\eta^2}{r^2} \ge \int_\Sigma |A|^2|\nabla\eta|^2$$

for some $\varepsilon > 0$.
Hence, $|A| \equiv 0$, which means that Σ is actually a plane.
Since

$$\int_\Sigma \frac{|A|^2}{r^2}\eta^2 = \int_0^\infty \int_{\Sigma\cap\mathbb{S}^n(r)} \frac{|A|^2}{r^2}\eta^2 dr = \int_0^\infty r^{n-1}\int_{\Sigma\cap\mathbb{S}^n(1)} \frac{|A|^2}{r^4}\eta^2 dr.$$

If we take

$$\eta(x) = \max\{1,|x|\}^{1-\frac{n}{2}-2\varepsilon}|x|^{1+\varepsilon},$$

then

$$\int_\Sigma \frac{|A|^2\eta^2}{r^2} < \infty,$$

provided $n \ge 2$. It is obvious that

$$|\eta(x)| = \begin{cases} |x|^{2-\frac{n}{2}-\varepsilon} & \text{when } |x| \text{ is big enough,} \\ |x|^{1+\varepsilon} & \text{when } |x| \text{ is small,} \end{cases}$$

and

$$|\nabla\eta(x)| \le \begin{cases} |2-\frac{n}{2}-\varepsilon|\frac{\eta(x)}{|x|} & \text{when } |x| \text{ is big enough,} \\ (1+\varepsilon)\frac{|\eta(x)|}{|x|} & \text{when } |x| \text{ is small.} \end{cases}$$

Hence,

$$\begin{aligned} \int_\Sigma |A|^2|\nabla\eta|^2 &= \int_0^\infty r^{n-1}\int_{\Sigma\cap\mathbb{S}^n(1)} \frac{|A|^2}{r^2}|\nabla\eta|^2 \\ &\le \max\left\{1+\varepsilon, |2-\frac{n}{2}-\varepsilon|\right\}\int_0^\infty r^{n-1}\int_{\Sigma\cap\mathbb{S}^n(1)} \frac{|A|^2}{r^4}|\eta|^2 \\ &\le (2-\varepsilon)\int_\Sigma \frac{|A|^2\eta^2}{r^2}, \end{aligned}$$

provided $n \le 7$ ($|2-\dfrac{n}{2}-\varepsilon| \le 2-\varepsilon \Longrightarrow n \le 8-4\varepsilon$). ☐

First, give an important lemma.
Assume $E \subset \mathbb{R}^n$ is open set. Set

$$P(E) = \sup\left\{\int_E \operatorname{div}\varphi dx \mid \varphi \in C_c^\infty(\mathbb{R}^n,\mathbb{R}), |\varphi| \le 1\right\},$$

i.e., perimeter of E.

Lemma 9.2. *Assume $E \subset \mathbb{R}^n$ is finite perimeter ($P(E) < \infty$) and E is area minimizing. Then, exists subsequence $E_i \longrightarrow E$ ($P(E_i) \longrightarrow P(E)$), E_i is set of finite perimeter and E is area minimizing.*

Theorem 9.2. *If any minimizing cone (possibly singularity at 0) is flat, then any minimizing hypersurface $\Sigma \subset \mathbb{R}^n$ is flat.*

Proof. Fix $p \in \Sigma$, consider $\Sigma_r = \dfrac{1}{r}(\Sigma - p)$, then Σ_r is minimal for all r when Σ is minimal.
Fix $R > 0$ and $|\Sigma_r \cap B_R| \le \dfrac{1}{2}\omega_{n-1}R^{n-1}$, by **Lemma9.2**, exists subsequence Σ_{r_i}, such that $\Sigma_{r_i} \longrightarrow \mathcal{C}$ ($r_i \longrightarrow \infty$) and $\mathcal{C}$ is minimal and $P(\mathcal{C}) < \infty$.
Claim: $\mathcal{C}$ is area-minimizing cone.
By **Corollary5.1**, we will show $\dfrac{|\mathcal{C}\cap B_R|}{R^{n-1}}$ is constant.
Since $\lim\limits_{i\to\infty}\dfrac{|\Sigma_{r_i}\cap B_R|}{R^{n-1}} = \dfrac{|\mathcal{C}\cap B_R|}{R^{n-1}}$ and $\dfrac{|\Sigma_{r_i}\cap B_R|}{R^{n-1}} = \dfrac{|\Sigma\cap B_{r_iR}|}{(r_iR)^{n-1}}|$, by scaling, then we should prove

$$\lim_{i\to\infty}\frac{|\Sigma\cap B_{r_iR}|}{(r_iR)^{n-1}} = \text{constant}.$$

By monotonicity and area bound,

$$\Theta_R = \lim_{i\to\infty}\frac{|\Sigma\cap B_{r_iR}|}{(r_iR)^{n-1}}.$$

Suppose $R_1 < R_2$, $r_i \longrightarrow \infty$, exists m_i, such that $r_i R_1 > r_{i-m_i} R_2$, then

$$\frac{|\Sigma \cap B_{r_{i-m_i}R_2}|}{(r_{i-m_i}R_2)^{n-1}} \leq \frac{|\Sigma \cap B_{r_i R_1}|}{(r_i R_1)^{n-1}} \leq \frac{|\Sigma \cap B_{r_i R_2}|}{(r_i R_2)^{n-1}}.$$

As $r_i \longrightarrow \infty$ $(i \longrightarrow \infty)$, then

$$\Theta_{R_2} \leq \Theta_{R_1} \leq \Theta_{R_2}$$

and

$$\Theta_R = \text{constant}.$$

Therefore $\mathcal{C}$ is cone.
Since $\mathcal{C}$ is flat, then

$$\frac{|\mathcal{C} \cap B_R|}{R^{n-1}} = \omega_{n-1}.$$

By monotonicity,

$$\omega_{n-1} \leq \frac{|\Sigma \cap B_r|}{r^{n-1}} \leq \lim_{i \to \infty} \frac{|\Sigma \cap B_{r_i R}|}{(r_i R)^{n-1}} = \frac{|\mathcal{C} \cap B_R|}{R^{n-1}} = \omega_{n-1}.$$

Then

$$|\Sigma \cap B_r| = \omega_{n-1} r^{n-1}.$$

By **Corollary5.1**, Σ is flat. □

Combine **Theorem9.1** with **Theorem9.2**, we obtain

Theorem 9.3 (Bernstein Theorem($n \leq 7$-dimensional))**.** *If $u : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ ($n \leq 7$) is an entire solution to the minimal surface equation, then u is a linear function.*

- de Giorgi's argument:

Assume $\Sigma = (x, u(x)) : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^n$, then

$$\frac{1}{r}\Sigma = \left(\frac{x}{r}, \frac{u(x)}{r}\right) = \left(y, \frac{u(ry)}{r}\right) : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^n.$$

$\left(y, \frac{1}{r}u(ry)\right) \longrightarrow \mathcal{C}(r \longrightarrow \infty)$ area minimizing cone which is a quasi-graph.

Regularity theorem:
Let $E \subset \mathbb{R}^n$ is minimizing and $P(E) < \infty$, then

$$\begin{cases} 2 \leq n \leq 7 & \partial E \text{ smooth} \\ n = 8 & \partial E \text{ contains isolated points} \\ n \geq 9 & \mathcal{H}^s(\text{sing}(\partial E)) = 0, \forall s > n - 8. \end{cases}$$

If $\mathcal{C}^7 \cap \mathbb{S}^7$ contains singularity p, then $tp(t \in (0, \infty))$ is singularity of $\mathcal{C} \subset \mathbb{R}^8$(contradiction to regularity theorem). Thus $\mathcal{C}^7 \cap \mathbb{S}^7$ is smooth and minimal hypersurface.

Since $\langle N', e_n \rangle \geq 0$ and

$$\Delta \left\langle N', e_n \right\rangle = -|A'|^2 \left\langle N', e_n \right\rangle \leq 0,$$

where N' is outward normal vector of $\mathcal{C}$. By **strong maximum principle**, we have $\langle N', e_n \rangle$ = constant and $\mathcal{C}$ is flat. By **Theorem9.2**, we obtain Σ is flat. Then, we obtain

Theorem 9.4 (Bernstein Theorem($n \leq 8$-dimensional))**.** *If $u : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ ($n \leq 8$) is an entire solution to the minimal surface equation, then u is a linear function.*

- Bombieri-de Giorgi-Giusti.

Exist minimal graph $\Sigma : \mathbb{R}^8 \longrightarrow \Sigma$, Σ is not flat. Consider the following function

$$u(x, y) = u(|x|, |y|),$$

where $x \in \mathbb{R}^4$ and $y \in \mathbb{R}^4$.

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