# POWER CONVEXITY OF SOLUTION TO SPECIAL LAGRANGIAN EQUATION IN DIMENSION TWO

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Abstract. In this paper, we prove power convexity result of solution to Dirichlet problem of special Lagrangian equation in dimension two. This provides new example of fully nonlinear elliptic boundary value problem whose solution shares power convexity property previously only knew for 2-Hessian equation in dimension three. The key ingredients consist of microscopic convexity principles and deformation methods.

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## 1. Introduction

Given an elliptic boundary value problem on a bounded convex domain in $\mathbb{R}^n$, does the solution inherit some convexity properties from the boundary? This is a classical topics in elliptic PDEs which was studied by many authors since a century ago. There are two folds of this problem, i.e. convexity of solution (or composed by a strictly monotonicity function of one variable) and convexity of level sets of solution. As far as we know, the first result in this direction may be dated back to Caratheodory [2] who proved that the level curves of the Green function of a bounded convex domain in $\mathbb{R}^2$ are strictly convex. In 1957, Gabriel [14] obtained the analogous result in three dimensions. Later, these results were extended to high dimensions and more general elliptic PDEs by Lewis [27], Caffarelli and Spruck [11], Korevaar [23]. For further discussions on convexity issue of level sets of solution, we refer the readers to [6, 30, 16, 42, 17, 33], and the references therein.

On the other hand, about half century ago, Makar-Limanov [34] initiated the study of convexity of the torsion function which satisfies

$$\begin{cases} \Delta u = -2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^2$ is a bounded, smooth, strictly convex domain. By an ingenious argument involving the maximum principle, he proved that the function $\sqrt{u}$ is concave in $\Omega$. Along

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the same route as Makar-Limanov, in 1981, Acker, Payne and Philippin [1] proved the log-concavity of the first Dirichlet eigenfunction of Laplace operator on planar convex domain. Note that this result was first established by Brascamp and Lieb [8]. Recently, Ma, Shi and Ye [31], Jia, Ma and Shi [19] could generalize this technique to torsion function and the first Dirichlet eigenfunction of high dimensions.

At the beginning of 1980s, Korevaar [21, 22] made breakthrough on convexity of solution of some quasilinear elliptic equation under certain boundary value condition. More precisely, let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain and $u$ be the solution, he introduced the convexity test function

$$C(x, y, \lambda) = u((1-\lambda)x + \lambda y) - (1-\lambda)u(x) - \lambda u(y)$$

for $(x, y, \lambda) \in \bar{\Omega} \times \bar{\Omega} \times [0, 1]$. Obviously, the nonnegativity of $C(x, y, \lambda)$ is equivalent to the concavity of $u$. This method usually called Korevaar's concavity (or convexity) maximum principle. Later, Kennington [20] presented an improved version which was used to show power concavity of various categories of boundary value problems. In particular, Kennington pointed out that the concavity number $1/2$ of $u$ is sharp in (1.1) in higher dimensional case. For problems involving $p$-Laplace operator, since lack of regularity of the solution, one can not use the above concavity maximum principle directly. But by suitable approximation arguments, power concavity results are also derived in [36, 7]. In 1997, Alvarez, Lasry and Lions [3] developed a new method, i.e. convex envelope method, to establish the convexity of viscosity solution of fully nonlinear degenerate second order equation

$$F(D^2u, \nabla u, u, x) = 0$$

in a bounded convex domain $\Omega$ of $\mathbb{R}^n$. In particular, they gave alternative proofs of results in [21, 22, 20]. Recently, one more application of convex envelope method was ascribed by Crasta and Fragala [13], who proved that the unique solution $u$ to Dirichlet problem

$$\begin{cases} \Delta_\infty u = -1 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

is $3/4$-concave and in $C^1(\Omega)$.

Another powerful tool to produce convex solution is the so-called microscopic convexity principle (also called constant rank theorem). This approach was first discovered by Caffarelli and Friedman [9] for semilinear elliptic equations in two dimensional case. Later, Korevaar and Lewis [24] proved the analogous results in high dimensions. Combining the deformation process, they reproved $1/2$-concavity of the torsion function and log-concavity of the first Dirichlet eigenfunction. For historic developments on various constant rank theorems and their geometric applications, one can see [15, 5, 39, 32, 28, 12]. Among these results, we particularly mention that in [32] and [28], the authors studied the power convexity of solutions to Dirichlet problems involving 2-Hessian operator in three dimension.

More precisely, for Dirichlet problems

$$\begin{cases} \sigma_2(D^2u) = 1 & \text{in } \Omega \subset \mathbb{R}^3, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

and

$$\begin{cases} \sigma_2(D^2u) = \Lambda(-u)^2 & \text{in } \Omega \subset \mathbb{R}^3, \\ u < 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

they proved that the functions $-\sqrt{-u}$ and $-\log(-u)$ are strictly convex in $\Omega$, respectively. For new proofs and discussions on Brunn-Minkowski inequalities of 2-torsion and 2-Hessian eigenvalue, please see the well-written paper by Salani [37].

Note that many other interesting convexity results of this kind are proved in literature, such as Kulczycki [25], Langford and Scheuer [26], etc.

In this paper, we find a new class of fully nonlinear elliptic PDEs, i.e. the two dimensional special Lagrangian equations, whose solutions share the similar power convexity. In the area of power convexity of solutions, this is the second concrete example of fully nonlinear elliptic operator after the 2-Hessian operator in three dimension.

Let us briefly review the history of special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i = \theta \tag{1.3}$$

with phase $|\theta| < n\pi/2$ in all dimensions $n \geq 2$, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ are eigenvalues of the Hessian $D^2u$. Equation (1.3) originates in the special Lagrangian geometry by Harvey and Lawson [18]. The Lagrangian graph $(x, \nabla u(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ is called special when the argument of the complex number $(1+\sqrt{-1}\lambda_1)(1+\sqrt{-1}\lambda_2)\cdots(1+\sqrt{-1}\lambda_n)$ is constant or the phase is constant, and it is special if and only if the gradient graph $(x, \nabla u(x))$ is a (volume minimizing) minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$. The phase $(n-2)\pi/2$ is said critical because the level sets $\{\lambda \in \mathbb{R}^n\colon\ \lambda \text{ satisfying equation (1.3)}\}$ are convex only when $|\theta| \geq (n-2)\pi/2$. For critical and supercritical phase $|\theta| \geq (n-2)\pi/2$, interior Hessian estimates of equation (1.3) were derived in series papers [43, 44, 41]. For subcritical phase $|\theta| < (n-2)\pi/2$, singular solutions were constructed by Nadirashvili and Vlăduţ [35], Wang and Yuan [40].

With critical and supercritical phase, the Dirichlet problem

$$\begin{cases} \sum_{i=1}^{n} \arctan \lambda_i = \psi & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega \end{cases}$$

was solved by Lu [29] very recently under some general conditions on $\psi$, $\phi$ and $\Omega$, see also [10, 4] for related results. In the following, We state a special case of their existence theorems.

**Theorem 1.1** (Lu [29], Bhattacharya [4]). *Let $\Omega$ be a bounded, smooth, strictly convex domain in $\mathbb{R}^n$ for $n \geq 2$. Let $\theta \in [(n-2)\pi/2, (n-1)\pi/2)$ be a constant. Then Dirichlet problem*

$$\begin{cases} \sum_{i=1}^{n} \arctan \lambda_i = \theta & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

*admits a unique solution $u \in C^\infty(\bar{\Omega})$.*

The following power convexity theorem is our main result.

**Theorem 1.2.** *Let $\Omega$ be a bounded, smooth, strictly convex domain in $\mathbb{R}^2$. Let $\theta \in (0, \pi/2)$ be a constant. Assume that $u \in C^\infty(\bar{\Omega})$ be the unique solution of Dirichlet problem*

$$\begin{cases} \arctan \lambda_1 + \arctan \lambda_2 = \theta & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

*where $\lambda = (\lambda_1, \lambda_2)$ are the eigenvalues of $D^2u$. Then the function $v = -\sqrt{-2u}$ is strictly convex in $\Omega$.*

**Remark 1.3.** *Assume that $n \geq 3$. Under conditions of Theorem 1.1, whether the unique solution $u$ of Dirichlet problem (1.4) possesses power convexity property remains open to us. Note that sophisticated examples related to nonconvex level sets of solution to elliptic boundary value problem are constructed by Wang [42], Hamel, Nadirashvili and Sire [17].*

The paper is organized as follows. In Section 2, we prove that $D^2v$ has constant rank if the function $v$ in Theorem 1.2 is convex. Section 3 is devoted to the proof of Theorem 1.2 and the sharpness of the convexity number $1/2$.

## 2. Constant rank theorem

In this section, we restrict ourselves to the two dimensional case. It is clear that the special Lagrangian equation

$$\arctan \lambda_1 + \arctan \lambda_2 = \theta \in (0, \pi/2)$$

is equivalent to

$$\det D^2u + \cot\theta \, \Delta u - 1 = 0. \tag{2.1}$$

Let $v = -\sqrt{-2u}$. Then we have

$$u_i = -vv_i, \quad u_{ij} = -vv_{ij} - v_iv_j, \quad \text{for } i, j = 1, 2$$

and

$$\begin{aligned}\Delta u &= -\, v\Delta v - |\nabla v|^2,\\ \det D^2 u &= v^2 \det D^2 v + v|\nabla v|^2 \Delta v - v \sum_{ij} v_i v_j v_{ij}.\end{aligned}$$

Thus, equation (2.1) can be written as

$$v^2 \det D^2 v + v|\nabla v|^2 \Delta v - v \sum_{ij} v_i v_j v_{ij} - \cot\theta \; v\Delta v - \cot\theta \; |\nabla v|^2 - 1 = 0. \tag{2.2}$$

Hence $v$ is the unique solution of Dirichlet problem

$$\begin{cases} F(D^2 v, \nabla v, v) = 0 & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where

$$F(D^2 v, \nabla v, v) = v^2 \det D^2 v + v|\nabla v|^2 \Delta v - v \sum_{ij} v_i v_j v_{ij} - \cot\theta \; v\Delta v - \cot\theta \; |\nabla v|^2 - 1.$$

For the sake of convenience, we denote

$$\begin{aligned} F^{ij} &= \frac{\partial F}{\partial v_{ij}}, & F^{v_p} &= \frac{\partial F}{\partial v_p}, & F^{v} &= \frac{\partial F}{\partial v},\\ F^{ij,kl} &= \frac{\partial^2 F}{\partial v_{ij}\partial v_{kl}}, & F^{ij,v_p} &= \frac{\partial^2 F}{\partial v_{ij}\partial v_p}, & F^{ij,v} &= \frac{\partial^2 F}{\partial v_{ij}\partial v},\\ F^{v_p,v_q} &= \frac{\partial^2 F}{\partial v_p \partial v_q}, & F^{v_p,v} &= \frac{\partial^2 F}{\partial v_p\partial v}, & F^{v,v} &= \frac{\partial^2 F}{\partial v^2}. \end{aligned}$$

The following deformation lemma plays crucial role in the proof of our constant rank theorem.

**Lemma 2.1** (Deformation lemma). *Let $\Omega \subset \mathbb{R}^2$ be any domain, and $v \in C^4(\Omega)$ be a convex solution of equation (2.2). For each $x \in \Omega$, set $\phi(x) = \det D^2 v(x)$. If the Hessian $D^2 v(x)$ attains its minimal rank 1 at some point $x_0 \in \Omega$, then there exist a neighborhood $\mathcal{O}$ of $x_0$ and positive constants $C$, $c$ independent of $\phi$, such that*

$$\sum_{ij} F^{ij}\phi_{ij} \le C(|\nabla \phi|(x) + \phi(x)), \qquad x \in \mathcal{O},$$

*provided $\Delta v(x) \ge c/2$ and $x \in \mathcal{O}$.*

*Proof.* Note that $D^2 v(x)$ attains its minimal rank 1 at $x_0 \in \Omega$. We may assume that $\Delta v(x_0) \ge c$ for some positive constant $c$. By continuity, there exists a neighborhood $\mathcal{O}$ of $x_0$ such that $\Delta v(x) \ge c/2$ for each $x \in \mathcal{O}$. To simplify our calculation, we adopt the following notation.

For two functions $h$ and $k$ defined in $\mathcal{O}$, we say that $h(x) \lesssim k(x)$ if there exists a positive constant $C_1$ such that

$$h(x) - k(x) \leq C_1(|\nabla\phi(x)| + \phi(x)).$$

We also write $h(x) \sim k(x)$ if $h(x) \lesssim k(x)$ and $k(x) \lesssim h(x)$. Next, we write $h \lesssim k$ if the above inequality holds for any $x \in \mathcal{O}$, with constant $C_1$ depends only on $|v|_{C^3(\mathcal{O})}$, $\inf_{x\in\mathcal{O}} \Delta v(x)$ and $\theta$ (independent of $x$). Finally, $h \sim k$ means $h \lesssim k$ and $k \lesssim h$.

For each $x \in \mathcal{O}$ fixed, by choosing orthonormal coordinates, we suppose that $D^2v(x)$ is diagonal and $v_{11}(x) \geq v_{22}(x) \geq 0$. Recalling that $\Delta v(x) \geq c/2$ for each $x \in \mathcal{O}$, we get $v_{11}(x) \geq c/4$.

From now on, all the analyses are performed at the fixed point $x$. When we use the relation "$\lesssim$", all the constants are under controlled. Note that $D^2v(x)$ is diagonal. It is easy to see that

$$0 \sim \phi = v_{11}v_{22}.$$

Since $v_{11} \geq c/4$, we have

$$v_{22} \sim 0. \tag{2.4}$$

Taking first derivatives of $\phi$, by (2.4), we get

$$0 \sim \phi_i = v_{11i}v_{22} + v_{11}v_{22i} \sim v_{11}v_{22i}, \quad i = 1, 2.$$

Equivalently, we have

$$v_{22i} \sim 0, \quad i = 1, 2. \tag{2.5}$$

For $i, j = 1, 2$, taking second derivatives of $\phi$, we obtain

$$\begin{aligned} \phi_{ij} &= v_{11ij}v_{22} + v_{11i}v_{22j} + v_{11j}v_{22i} + v_{11}v_{22ij} - 2v_{12i}v_{12j} \\ &\sim v_{11}v_{22ij} - 2v_{12i}v_{12j}. \end{aligned}$$

Here we have used (2.4)–(2.5). It follows that

$$\begin{aligned} \frac{1}{v_{11}} \sum_{ij} F^{ij}\phi_{ij} &\sim \sum_{ij} F^{ij}v_{22ij} - \frac{2}{v_{11}} \sum_{ij} F^{ij}v_{12i}v_{12j} \\ &\triangleq \mathrm{I}_1 + \mathrm{I}_2. \end{aligned} \tag{2.6}$$

In the following, we will compute the terms $\mathrm{I}_1$ and $\mathrm{I}_2$, respectively.

For the term $\mathrm{I}_1$, differentiating the equation $F(D^2v, \nabla v, v) = 0$ twice with respect to the variable $x_2$ on both sides, we obtain

$$0 = \sum_{ij} F^{ij}v_{ij2} + \sum_p F^{v_p}v_{p2} + F^v v_2, \tag{2.7}$$

and

$$\begin{aligned} 0 = & \sum_{ijkl} F^{ij,kl} v_{ij2} v_{kl2} + \sum_{ijp} F^{ij,v_p} v_{ij2} v_{p2} + \sum_{ij} F^{ij,v} v_{ij2} v_2 + \sum_{ij} F^{ij} v_{ij22} \\ & + \sum_{pij} F^{v_p,ij} v_{p2} v_{ij2} + \sum_{pq} F^{v_p,v_q} v_{p2} v_{q2} + \sum_{p} F^{v_p,v} v_{p2} v_2 + \sum_{p} F^{v_p} v_{p22} \\ & + \sum_{ij} F^{v,ij} v_2 v_{ij2} + \sum_{p} F^{v,v_p} v_2 v_{p2} + F^{v,v} v_2^2 + F^v v_{22}. \end{aligned} \tag{2.8}$$

By (2.4)–(2.5), equations (2.7)–(2.8) can be simplified as

$$\begin{aligned} & 0 \sim F^{11} v_{112} + F^v v_2, \\ & 0 \sim F^{11,11} v_{112}^2 + 2F^{11,v} v_{112} v_2 + \sum_{ij} F^{ij} v_{ij22} + F^{v,v} v_2^2. \end{aligned}$$

It follows that

$$v_2 \sim -\frac{1}{F^v} F^{11} v_{112}, \tag{2.9}$$

$$\sum_{ij} F^{ij} v_{ij22} \sim -F^{11,11} v_{112}^2 - 2F^{11,v} v_{112} v_2 - F^{v,v} v_2^2. \tag{2.10}$$

Putting (2.9) into (2.10), we obtain

$$\mathrm{I}_1 = \sum_{ij} F^{ij} v_{ij22} \sim \left[ -F^{11,11} + \frac{2}{F^v} F^{11,v} F^{11} - \frac{1}{(F^v)^2} F^{v,v} (F^{11})^2 \right] v_{112}^2. \tag{2.11}$$

For the term $\mathrm{I}_2$, we simply have

$$\mathrm{I}_2 \sim -\frac{2}{v_{11}} F^{11} v_{112}^2. \tag{2.12}$$

Inserting (2.11)–(2.12) into (2.6), we then obtain

$$\frac{1}{v_{11}} \sum_{ij} F^{ij} \phi_{ij} \sim \left[ -F^{11,11} + \frac{2}{F^v} F^{11,v} F^{11} - \frac{1}{(F^v)^2} F^{v,v} (F^{11})^2 - \frac{2}{v_{11}} F^{11} \right] v_{112}^2. \tag{2.13}$$

Recall that

$$F(D^2 v, \nabla v, v) = v^2 \det D^2 v + v|\nabla v|^2 \Delta v - v \sum_{ij} v_i v_j v_{ij} - \cot\theta \ v\Delta v - \cot\theta \ |\nabla v|^2 - 1.$$

Direct calculation gives

$$\begin{aligned} F^{11} &= v^2 v_{22} + v|\nabla v|^2 - v v_1^2 - \cot\theta \ v \sim v v_2^2 - \cot\theta \ v, \\ F^v &= 2v \det D^2 v + |\nabla v|^2 \Delta v - \sum_{ij} v_i v_j v_{ij} - \cot\theta \ \Delta v \\ &\sim v_2^2 v_{11} - \cot\theta \ v_{11}, \end{aligned}$$

and

$$\begin{aligned} F^{11,11} &= 0, \\ F^{11,v} &= 2vv_{22} + |\nabla v|^2 - v_1^2 - \cot\theta \sim v_2^2 - \cot\theta, \\ F^{v,v} &= 2\det D^2 v \sim 0. \end{aligned}$$

Via (2.13), these relations imply that

$$\begin{aligned} \frac{1}{v_{11}} \sum_{ij} F^{ij}\phi_{ij} &\sim 2\left(\frac{1}{F^v}F^{11,v} - \frac{1}{v_{11}}\right) F^{11} v_{112}^2 \\ &\sim 0. \end{aligned}$$

We complete the proof of Lemma 2.1. □

Now we state the constant rank theorem in the following.

**Theorem 2.2** (Constant rank theorem). *Let $\Omega \subset \mathbb{R}^2$ be any domain, and $v \in C^4(\Omega)$ be a convex solution of (2.2). Then the Hessian $D^2 v$ has constant rank in $\Omega$.*

*Proof.* Denote

$$l = \min_{x\in\Omega} \operatorname{rank}(D^2 v(x)).$$

For each $x \in \Omega$, since $D^2 v(x)$ is a $2 \times 2$ square matrix, there are only three possible values for $l$, i.e. $l = 0$, $l = 1$ or $l = 2$. If $l = 0$, it will contradict to equation (2.2) at these points $x$ where $D^2 v(x)$ is null matrix. If $l = 2$, there is nothing to prove.

It suffices to consider the case $l = 1$. Assume that there exists $x_0 \in \Omega$ such that $\operatorname{rank}(D^2 v(x_0)) = 1$. We will prove that $\operatorname{rank}(D^2 v(x)) = 1$ for all $x \in \Omega$. Define the set

$$\Omega^* = \{x \in \Omega : \operatorname{rank}(D^2 v(x)) = 1\}.$$

It is clear that $\Omega^*$ is relatively closed in $\Omega$. Combining the strong maximum principle and Lemma 2.1, we know that $\Omega^*$ is also open. Thus $\Omega^* = \Omega$, i.e. the rank of $D^2 v$ is always 1 in $\Omega$.

We complete the proof of Theorem 2.2. □

## 3. Proof of Theorem 1.2

In this section, we will give the proof of Theorem 1.2 by continuity method. With Theorem 2.2 (Constant rank theorem) in hand, it is well known that the proof of Theorem 1.2 is standard, see for example the earlier paper by Caffarelli and Friedman [9], or the papers [24, 32, 28].

Before we go further, we need the boundary convexity estimates for the function $v = -\sqrt{-2u}$, where $u$ solves the Dirichlet problem (1.5). We will take $\alpha = 1/2$ in the following lemma.

**Lemma 3.1** (See Korevaar [22], or Caffarelli and Friedman [9])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth, strictly convex domain. Let $u \in C^2(\bar{\Omega})$ satisfy*

$$u < 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \qquad |\nabla u| > 0 \quad \text{on } \partial\Omega.$$

*Then there exists $\varepsilon > 0$ such that the function $v = -\log(-u)$ or $v = -(-u)^\alpha$ $(0 < \alpha < 1)$ is strictly convex in boundary strip $\Omega \setminus \Omega_\varepsilon$, where*

$$\Omega_\varepsilon = \{x \in \Omega : \mathrm{d}(x, \partial\Omega) > \varepsilon\}.$$

Let us sketch the proof of Theorem 1.2.

*Proof of Theorem 1.2.* If $\Omega_0 = B_1$, the unit disk in $\mathbb{R}^2$, then the unique solution to Dirichlet problem (2.3) is

$$v_0(x) = -\sqrt{\tan\frac{\theta}{2}\ (1 - |x|^2)}, \quad x \in B_1.$$

Clearly, the function $v_0$ is strictly convex in $B_1$. For an arbitrary bounded, smooth, strictly convex domain $\Omega_1 = \Omega$, set

$$\Omega_t = (1-t)B_1 + t\Omega, \qquad 0 < t < 1.$$

Then from the theory of convex bodies, we can deform $B_1$ smoothly into $\Omega$ by the family $\{\Omega_t\}_{0 \le t \le 1}$, of bounded, smooth, strictly convex domain. For more details, see the excellent book by Schneider [38]. For $0 < t < 1$, assume that the function $v_t$ solves Dirichlet problem

$$\begin{cases} F(D^2v, \nabla v, v) = 0 & \text{in } \Omega_t, \\ v = 0 & \text{on } \partial\Omega_t, \end{cases}$$

where

$$F(D^2v, \nabla v, v) = v^2 \det D^2 v + v|\nabla v|^2 \Delta v - v \sum_{ij} v_i v_j v_{ij} - \cot\theta\ v\Delta v - \cot\theta\ |\nabla v|^2 - 1.$$

From the a priori estimates of special Lagrangian equations, we have uniform estimates for $\|v_t\|_{C^\infty(\Omega_t)}$ with the bound depends only on $\theta$ and the geometry of $\Omega$ (independent of $t$). On the one hand, we know that $v_t$ is strictly convex for all $t \in (0, \delta)$ ($\delta > 0$ sufficiently small) since $v_0$ is strictly convex. On the other hand, we also have the fact that if $v_t$ is strictly convex in $\Omega_t$ for all $t \in (0, t^*)$ $(0 < t^* < 1)$, then $v_{t^*}$ is convex in $\Omega_{t^*}$.

Now we suppose that $v_1$ is not strictly convex in $\Omega_1$. Then there exists $t_0 \in (0,1)$; it is the first time that the function $v_{t_0}$ is convex but not strictly convex. This contradicts to Theorem 2.2 (Constant rank theorem) and Lemma 3.1 (Boundary convexity estimates). □

**Remark 3.2.** *The following example shows that the convexity number $1/2$ is sharp. Similar examples for Dirichlet problems (1.1) and (1.2) appeared in Kennington [20] and Ma and Xu [32], respectively.*

*Define an infinite open cone* $\mathcal{C} = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < ax_2\}$, *where* $a \in (0,1)$ *is a constant such that* $\cot\theta > 2a/(1-a^2)$. *Let* $\Omega \subset \mathcal{C}$ *be a bounded, convex domain with* $y = (0,0)$ *on* $\partial\Omega$ *and* $z = (0,1)$ *in* $\Omega$. *Define* $w(x) = A(x_1^2 - a^2x_2^2)/2$, *where* $A > 0$ *is a constant satisfies the quadratic equation* $a^2A^2 - \cot\theta\ (1-a^2)A + 1 = 0$.

*Straightforward computation gives*

$$\begin{cases} \arctan\lambda_1(D^2w) + \arctan\lambda_2(D^2w) = \theta, & \text{in } \Omega, \\ w \leq 0, & \text{on } \partial\Omega. \end{cases}$$

*Note that* $u$ *solves Dirichlet problem* (1.5) *in the same domain* $\Omega$. *By comparison principle, we have* $w \leq u$ *in* $\Omega$. *Particularly, for* $t \in (0,1)$, *we have* $-Aa^2t^2/2 = w(tz) \leq u(tz) \leq 0$. *It yields* $\lim_{t\to 0+} t^{-1/\alpha}u(tz) = 0$ *provided* $\alpha > 1/2$. *This implies that* $-(-u)^\alpha$ *is not convex provided* $\alpha > 1/2$. *Otherwise, we suppose that* $-(-u)^\alpha$ *is convex. Then there holds*

$$-(-u)^\alpha((1-t)y + tz) \leq -(1-t)(-u)^\alpha(y) - t(-u)^\alpha(z) = -t(-u)^\alpha(z).$$

*Namely,* $t^{-1/\alpha}u(tz) \leq u(z) < 0$, *which contradicts to* $\lim_{t\to 0+} t^{-1/\alpha}u(tz) = 0$.

*This illustrates the sharpness of the convexity number* $1/2$.

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