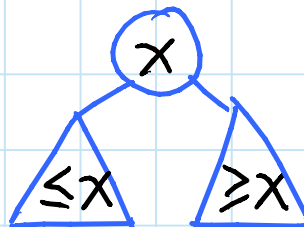
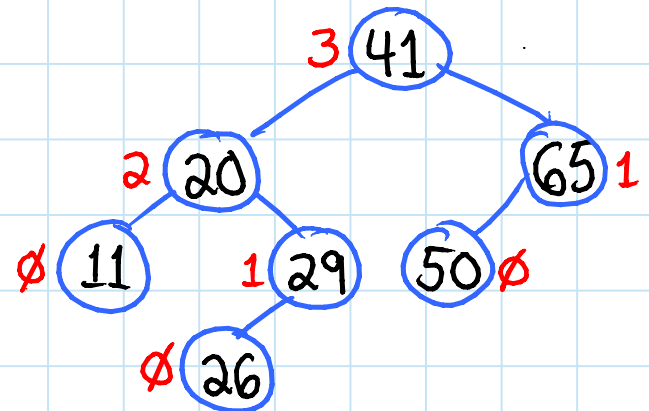


TODAY: Balanced BSTs

- The importance of being balanced
- AVL trees
 - definition & balance
 - rotations
 - insert
- Other balanced trees
- Data structures in general
- Lower bounds

Recall: Binary Search Trees (BSTs)

- rooted binary tree
- each node has
 - key
 - left pointer
 - right pointer
 - parent pointer
- BST property:

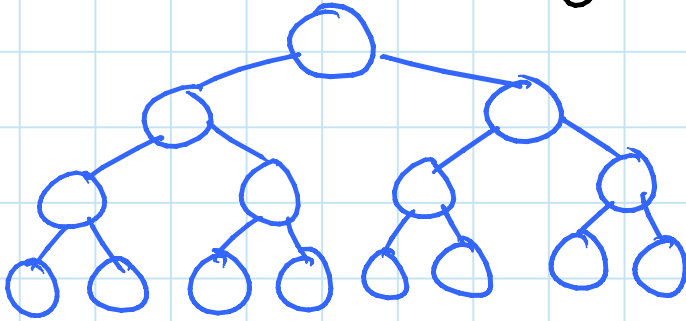


CLRS B.5

- height of node = length (# edges) of longest downward path to a leaf

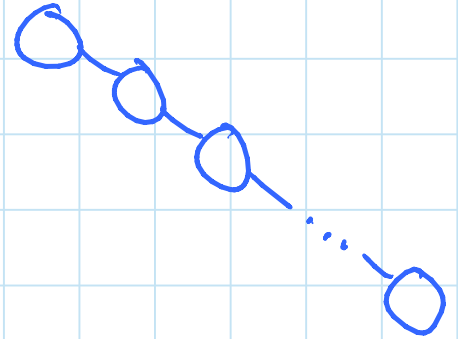
The importance of being balanced:

- BSTs support insert, delete, min, max, next-larger, next-smaller, etc, in $O(h)$ time, where h = height of tree (= height of root)
- h is between $\lg n$ and n :



perfectly balanced

vs.



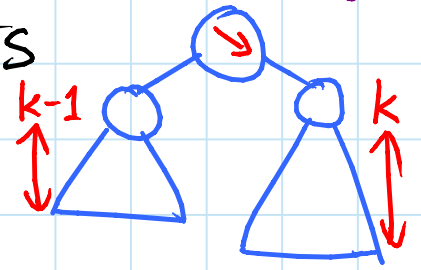
path

- balanced BST maintains $h = O(\lg n)$
 \Rightarrow all operations run in $O(\lg n)$ time

AVL trees: [Adel'son-Vel'skiĭ & Landis 1962]

for every node, require heights of left & right children to differ by at most ± 1

- treat nil tree as height -1
- each node stores its height



(DATA STRUCTURE AUGMENTATION) (like subtree size)
(alternatively, can just store difference in heights)

Balance: worst when every node differs by 1

- let $N_h = (\text{min.}) \# \text{ nodes in height-}h \text{ AVL tree}$

$$\Rightarrow N_h = N_{h-1} + N_{h-2} + 1$$
$$> 2 N_{h-2}$$

$$\Rightarrow N_h > 2^{h/2}$$

$$\Rightarrow h < 2 \lg N_h$$

Alternatively: $N_h > F_h$ (nth Fibonacci number)

- in fact $N_h = F_{h+2} - 1$ (simple induction)

- $F_h = \varphi^h / \sqrt{5}$ rounded to nearest integer
where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ (golden ratio)

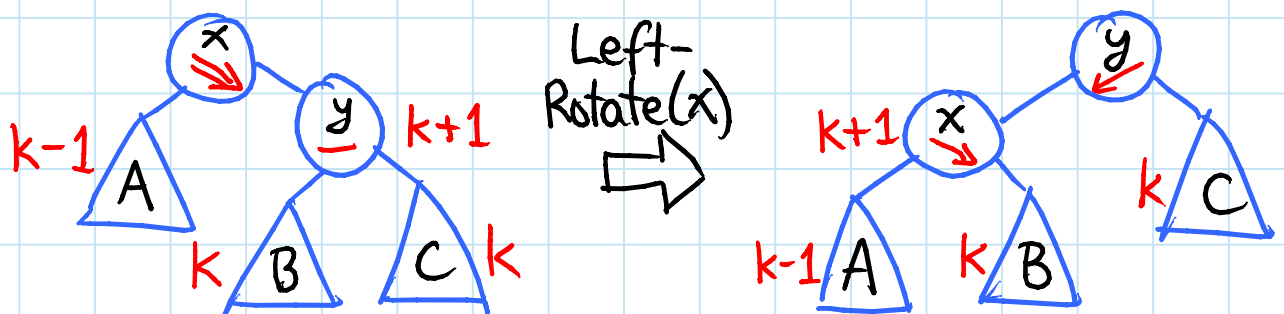
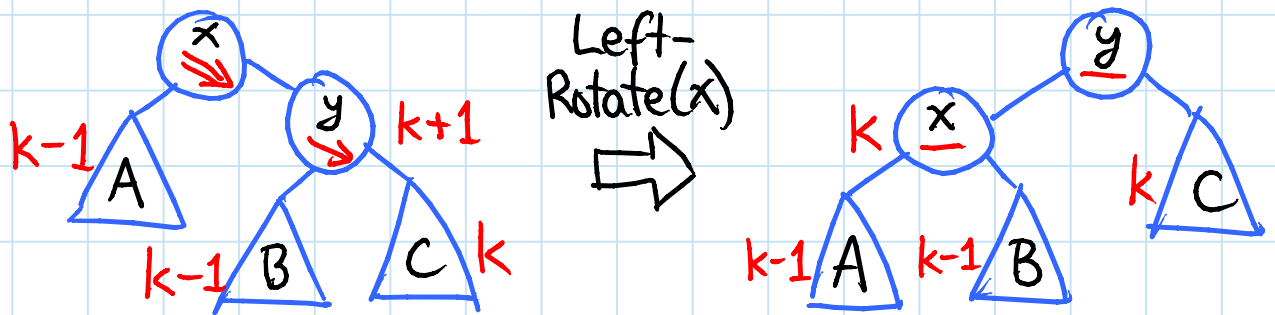
$$\Rightarrow \text{max. } h \approx \log_{\varphi} n \approx 1.440 \lg n$$

AVL insert:

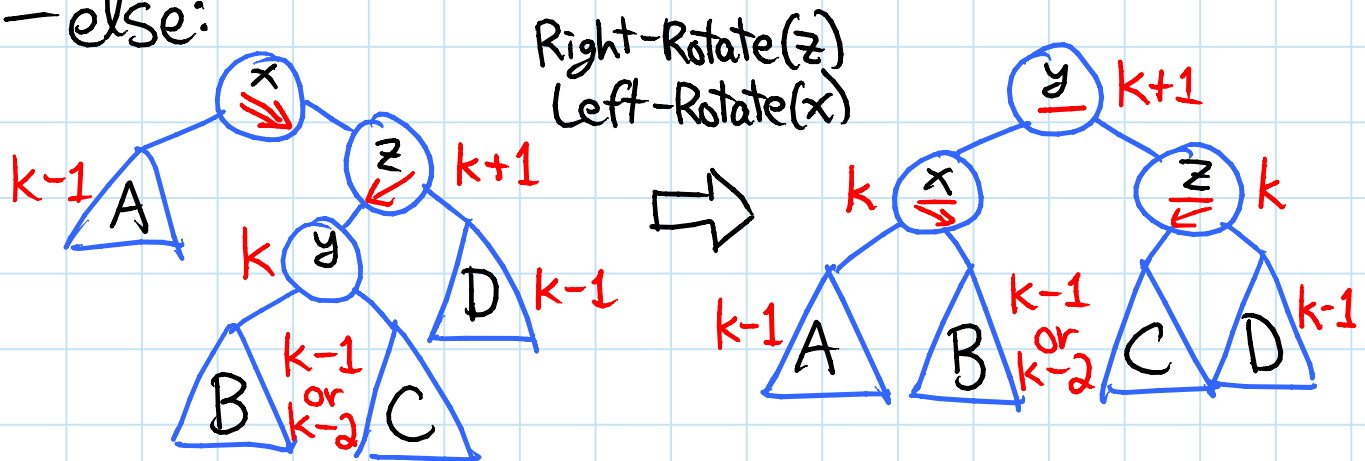
- ① insert as in simple BST
- ② work your way up tree, restoring AVL property (and updating heights as you go)

Each step:

- suppose x is lowest node violating AVL
- assume x is right-heavy (left case symmetric)
- if x 's right child is right-heavy or balanced:



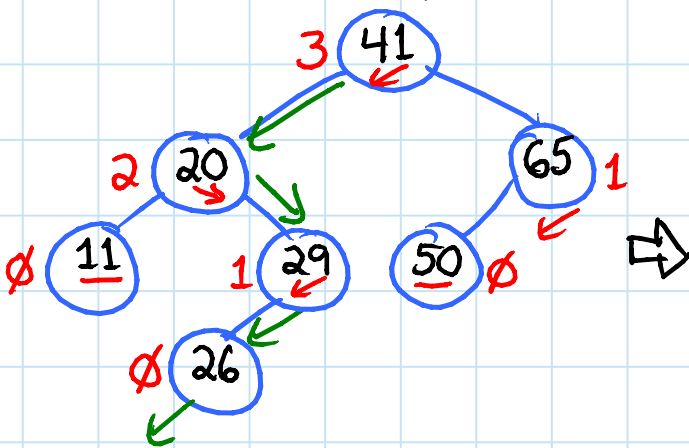
- else:



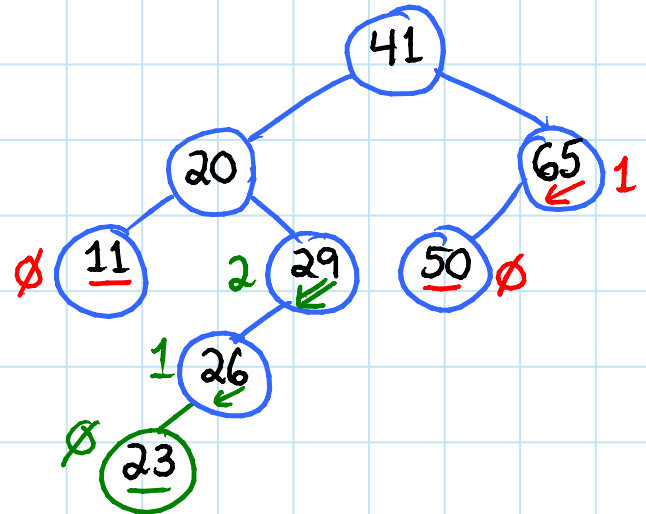
- then continue up to x 's grandparent, greatgp,...

Example:

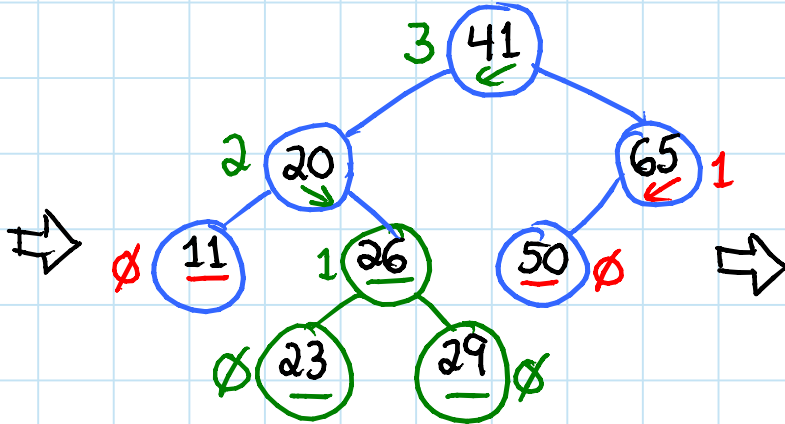
Insert(23)



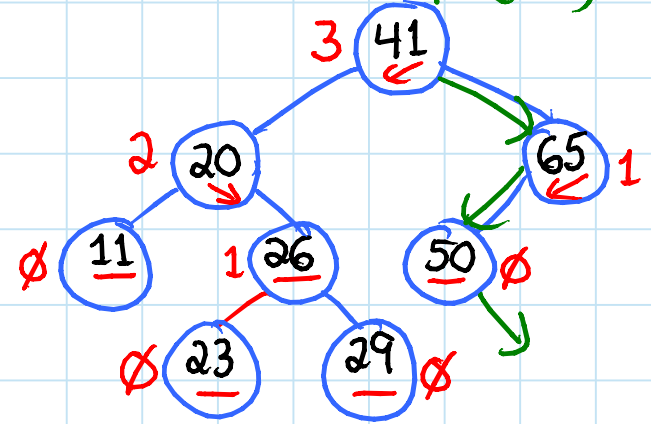
x=29: left-left case



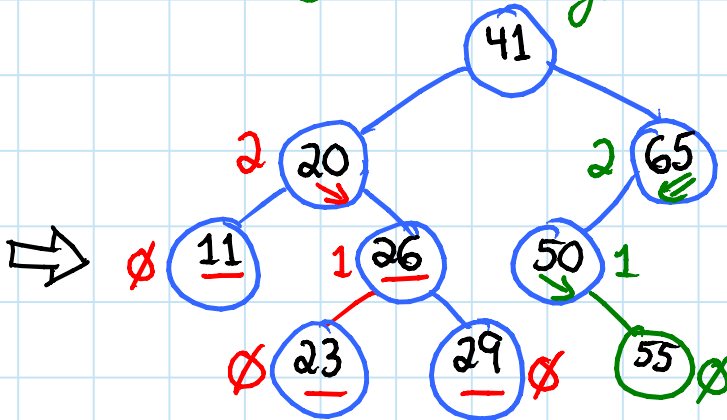
Done.



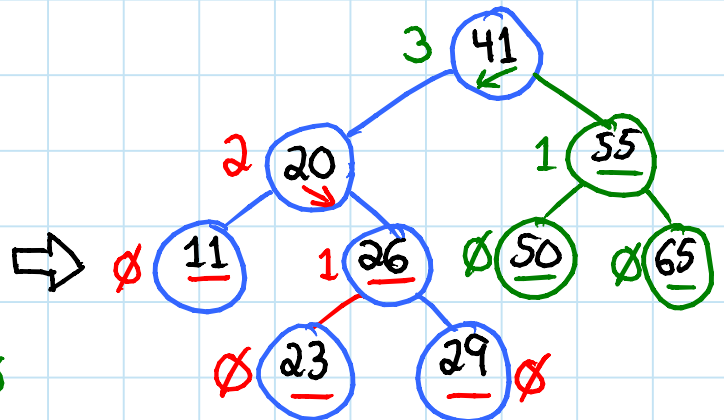
Insert(55)



x=65: left-right case



Done.



- in general may need several rotations before done with an Insert
- Delete(-min) is similar

AVL sort:

- insert each item into AVL tree
- in-order traversal

$$\begin{array}{r} \Theta(\ln n) \\ \Theta(n) \\ \hline \Theta(\ln n) \end{array}$$

Balanced search trees: there are many!

- AVL trees [Adelson-Velski & Landis 1962]
- B-trees / 2-3-4 trees [Bayer & McCreight 1972] [CLRS 18]
- BB[α] trees [Nievergelt & Reingold 1973]
- red-black trees [CLRS ch. 13]
- (A) - splay trees [Sleator & Tarjan 1985]
- (R) - skip lists [Pugh 1989]
- (A) - scapegoat trees [Galperin & Rivest 1993]
- (R) - treaps [Seidel & Aragon 1996]

(R) = use random numbers to make decisions
fast with high probability

(A) = "amortized": adding up costs for
several operations \Rightarrow fast on average

e.g. splay trees are a current research topic
- see 6.854 (Advanced Algorithms)
& 6.851 (Advanced Data Structures)

Big picture:

Abstract Data Type (ADT): interface spec.
vs. Data Structure (DS): algorithm for each op.

- many possible DSs for one ADT
e.g. much later, "heap" priority queue

Priority Queue ADT:

- $Q = \text{new-empty-queue}()$
- $Q.\text{insert}(x)$
- $x = Q.\text{deletemin}()$
- $x = Q.\text{findmin}()$

heap

$\Theta(1)$
 $\Theta(\lg n)$
 $\Theta(\lg n)$
 $\Theta(1)$

AVL tree

$\Theta(1)$
 $\Theta(\lg n)$
 $\Theta(\lg n)$
 $\Theta(\lg n)$
 $\hookrightarrow \Theta(1)$

Predecessor/Successor ADT:

- $S = \text{new-empty}()$
- $S.\text{insert}(x)$
- $S.\text{delete}(x)$
- $y = S.\text{predecessor}(x)$
 $\hookrightarrow \text{next-smaller}$
- $y = S.\text{successor}(x)$
 $\hookrightarrow \text{next-larger}$

heap

$\Theta(1)$
 $\Theta(\lg n)$
 $\Theta(\lg n)$
 $\Theta(n)$

AVL tree

$\Theta(1)$
 $\Theta(\lg n)$
 $\Theta(\lg n)$
 $\Theta(\lg n)$
 $\Theta(\lg n)$