Unit 10: Random Walks

Instructor: Quan Zhou

10.1 Random walks on $\mathbb{R}$

In this subsection, we let $Z_1, Z_2, \ldots$ be arbitrary i.i.d. random variables, and define $X_0 = 0$ and $X_n = Z_1 + \cdots + Z_n$ for each $n \geq 1$. Let the filtration be given by $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$. The sequence $(X_n)_{n \geq 0}$ is called a random walk on $\mathbb{R}$. We first prove a result about the limiting behavior of (X_n) and then give applications of the optional sampling theorem.

Theorem 10.1. *One of the following four events happens with probability one:*

(i) $X_n = 0$ *for all* n.

(ii) $X_n \to \infty$.

(iii) $X_n \to -\infty$.

(iv) $\liminf X_n = -\infty$ *and* $\limsup X_n = \infty$.

Proof. If $Z_1 = 0$ a.s., then event (i) happens a.s. If $\mathsf{P}(Z_1 > 0) > 0$, by the continuity of measures, there exist some $\delta, \epsilon > 0$ such that $\mathsf{P}(Z_1 > \delta) > \epsilon$. Hence, it follows from Borel-Cantelli lemma that event (ii) happens a.s. if $Z_1 \geq 0$ and $\mathsf{P}(Z_1 > 0) > 0$. Similarly, if $Z_1 \leq 0$ and $\mathsf{P}(Z_1 < 0) > 0$, event (iii) happens a.s.

Now assume $\mathsf{P}(Z_1 > 0) > 0$ and $\mathsf{P}(Z_1 < 0) > 0$, which implies that there exist some $\delta, \epsilon > 0$ such that $\mathsf{P}(Z_1 > \delta) > \epsilon$ and $\mathsf{P}(Z_1 < -\delta) > \epsilon$. Let $\bar{X} = \limsup X_n$, and define $A_n = \{X_n > \bar{X} - \delta/2\}$ and $B_n = A_{n-1} \cap \{X_n > \bar{X} + \delta\}$. Since $A_{n-1} \cap \{Z_n > \delta\} = B_n$, we have $\mathsf{P}(B_n \,|\, \mathcal{F}_{n-1}) \geq \epsilon \mathbb{1}_{A_{n-1}}$. By Levy's zero-one law, whenever A_n happens infinitely often, so does B_n. An argument by contradiction yields that $\mathsf{P}(\limsup X_n \in (-\infty, \infty)) = 0$. Similarly, $\mathsf{P}(\liminf X_n \in (-\infty, \infty)) = 0$, and thus event (ii), (iii) or (iv) must happen a.s. $\square$

Remark 10.1. When Z_1 is integrable and $\mathsf{E}[Z_1] = 0$, (X_n) is a martingale. Theorem 10.1 shows that, except the trivial case where $X_n = 0$ for all 0, almost surely (X_n) does not converge to a finite limit. Theorem 5.1 (martingale convergence theorem) thus implies that $\sup \mathsf{E}|X_n| \to \infty$.

Theorem 10.2. *Suppose* $\mathsf{E}[Z_1] = 0$ *and* $\mathsf{E}[Z_1^2] = \sigma^2 < \infty$. *Let* T *be a stopping time such that* $\mathsf{E}[T] < \infty$. *Then,* $\mathsf{E}[X_T^2] = \sigma^2 \mathsf{E}[T]$.

Proof. Consider the martingale $Y_n = X_n^2 - \sigma^2 n$. Applying OST with bounded stopping times yields $\mathsf{E}[X_{T\wedge n}^2] = \sigma^2 \mathsf{E}[T \wedge n] \leq \sigma^2 \mathsf{E}[T]$. Since $\mathsf{E}[T] < \infty$, $(X_{T\wedge n})$ is a martingale bounded in L^2, and thus $X_{T\wedge n}$ converges a.s. and also in L^2. It follows that

$$\mathsf{E}[X_T^2] = \lim_{n\to\infty} \mathsf{E}[X_{T\wedge n}^2] = \lim_{n\to\infty} \mathsf{E}[T \wedge n] = \sigma^2 \mathsf{E}[T],$$

where the last step follows from MCT. $\square$

Theorem 10.3. *Suppose* $\mathsf{E}[Z_1] = 0$ *and* $\mathsf{E}[Z_1^2] = 1$. *Let*

$$T(c) = \inf\{n \geq 1 \colon |X_n| > c\sqrt{n}\}.$$

If $c < 1$, *we have* $\mathsf{E}[T(c)] < \infty$; *if* $c \geq 1$, *we have* $\mathsf{E}[T(c)] = \infty$.

Proof. Write $T = T(c)$ and consider $c \geq 1$ first. If $\mathsf{E}[T] < \infty$, by Theorem 10.2, we have $\mathsf{E}[X_T^2] = \mathsf{E}[T]$. But by the definition of T_c, $X_T^2 > c^2 T \geq T$, and thus $\mathsf{E}[X_T^2] > \mathsf{E}[T]$. This yields the contradiction. The proof for the case $c < 1$ is more involved and omitted here; see [1]. $\square$

Remark 10.2. It is interesting to compare Theorem 10.3 with the law of iterated logarithm. The latter tells us that $\limsup |X_n| / \sqrt{2n \log(\log n)} = 1$, a.s., which implies that for any $c < \infty$, $|X_n| > c\sqrt{n}$ infinitely many times.

Exercise 10.1. Let $\varphi(\theta) = \mathsf{E}e^{\theta Z_1}$. Assume Z_1 is not a constant, which can be shown to imply that $\theta \mapsto \log \varphi(\theta)$ is strictly convex whenever $\varphi(\theta) < \infty$. Fix some $\theta \neq 0$ and assume $\varphi(\theta) < \infty$. Define

$$Y_n = \exp\left(\theta X_n - n \log \varphi(\theta)\right).$$

Show that (i) (Y_n) is a martingale, (ii) $\lim_{n\to 0} \mathsf{E}\sqrt{Y_n} = 0$, and (iii) $Y_n \overset{\text{a.s.}}{\to} 0$.

Exercise 10.2. Suppose $\mathsf{E}e^{\theta Z_1} = 1$ for some $\theta < 0$, and Z_1 is not a constant. Let a, b be such that $a < 0 < b$, and define

$$T = \min\{n \geq 1 \colon X_n \leq a \text{ or } X_n \geq b\}.$$

Show that (i) $\mathsf{E}[T] < \infty$, and (ii) $\mathsf{P}(X_T \leq a) \leq e^{-\theta a}$.

10.2 Simple random walks

In this subsection, we let $Z_1, Z_2, \dots$ be i.i.d. such that $\mathsf{P}(Z_1 = 1) = 1 - \mathsf{P}(Z_1 = -1) = p$. Define (X_n) as in the last subsection. We say (X_n) is a simple random walk, or a random walk on $\mathbb{Z}$. When $p = 1/2$, we say the random walk is symmetric; when $p \neq 1/2$, we say it is asymmetric.

Theorem 10.4. *Let $(X_n)_{n\geq 0}$ be a symmetric simple random walk. Then*

$$\mathsf{E}|X_n| = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{2j}{j} 4^{-j}.$$

Proof. We proved this in Example 7.1 using Doob's decomposition. □

Theorem 10.5. *Let $(X_n)_{n\geq 0}$ be a symmetric simple random walk. Let a, b be integers such that $a < 0 < b$, and define*

$$T = \min\{n \geq 1 \colon X_n \leq a \text{ or } X_n \geq b\}.$$

Then $\mathsf{P}(X_T = a) = b/(b-a)$, and $\mathsf{E}[T] = -ab$.

Proof. It is easy to show that there exists some constant $\epsilon > 0$ such that $\mathsf{E}[T \leq n + b - a \,|\, \mathcal{F}_n] \geq \epsilon$, from which we get $\mathsf{E}T < \infty$. The optional sampling theorem then yields $\mathsf{E}[X_T] = 0$, from which the first result follows.

To find $\mathsf{E}[T]$, consider the martingale $Y_n = X_n^2 - n$. The optional sampling theorem yields that

$$0 = \mathsf{E}[Y_T] = \mathsf{E}[X_T^2] - \mathsf{E}[T] = \frac{a^2 b}{b-a} - \frac{b^2 a}{b-a} - \mathsf{E}[T] = -ab - \mathsf{E}[T].$$

The proof is complete. □

Theorem 10.6. *Let $(X_n)_{n\geq 0}$ be a symmetric simple random walk. For any $b > 0$, $\mathsf{P}(T_b < \infty) = 1$ and $\mathsf{E}[T_b] = \infty$ where $T_b = \min\{n \geq 1 \colon X_n = b\}$.*

Proof. For any $a < 0$, Theorem 10.5 implies that $\min(T_a, T_b) < 1$ a.s. and $\mathsf{P}(T_b < T_a) = -a/(b-a)$. It follows that

$$\lim_{n\to\infty} \mathsf{P}(T_b < T_{-n}) = \lim_{n\to\infty} n/(b+n) = 1.$$

Define $E_n = \{T_b < T_{-n}\}$. Clearly, $E_n \subset E_{n+1}$, and thus the continuity of probability measures yields that $\lim_{n\to\infty} \mathsf{P}(T_b < T_{-n}) = \mathsf{P}(\bigcup_{n\to\infty} E_n)$. Since $\bigcup_{n\to\infty} E_n = \{T_b < \infty\}$, we get $\mathsf{P}(T_b < \infty) = 1$.

To prove $\mathsf{E}T_b = \infty$, note that if $\mathsf{E}T_b < \infty$, the optional sampling theorem would yield $\mathsf{E}X_{T_b} = 0$, which gives the contradiction. □

Remark 10.3. Actually it can be shown that, for a symmetric simple random walk, $\mathsf{P}(T_1 > t) \sim Ct^{-1/2}$ for some constant $C > 0$, though we do not prove the result here.

Theorem 10.7. *Let $(X_n)_{n\geq 0}$ be an asymmetric simple random walk with $\mathsf{P}(Z_1 = 1) = p \in (1/2, 1)$. Define $T_x = \min\{n \geq 1\colon X_n = x\}$, and $\psi(x) = (1-p)^x/p^x$. Then, for integers a, b such that $a < 0 < b$,*

(i) $\mathsf{P}(T_a < T_b) = \frac{\psi(b)-1}{\psi(b)-\psi(a)}$.

(ii) $\mathsf{P}(T_a < \infty) = \psi(-a)$.

(iii) $\mathsf{P}(T_b < \infty) = 1$ *and* $\mathsf{E}[T_b] = b/(2p-1)$.

Proof. Define $Y_n = \psi(X_n)$. It is easy to show that $(Y_n)_{n\geq 0}$ is a martingale. Mimicking the proof of Theorem 10.5, we find that $\mathsf{E}[T_a \wedge T_b] < \infty$ and

$$1 = \mathsf{E}[Y_{T_a \wedge T_b}] = \mathsf{P}(T_a < T_b)\psi(a) + (1 - \mathsf{P}(T_a < T_b))\psi(b).$$

A straightforward calculation proves part (i).

As in the proof of Theorem 10.6, we find that

$$\mathsf{P}(T_a < \infty) = \lim_{b\uparrow\infty} \frac{\psi(b)-1}{\psi(b)-\psi(a)} = \psi(a)^{-1} = \psi(-a),$$
$$\mathsf{P}(T_b < \infty) = \lim_{a\downarrow-\infty} \frac{1-\psi(a)}{\psi(b)-\psi(a)} = 1.$$

Finally, to find $\mathsf{E}[T_b]$, we use the martingale (Y_n) with

$$Y_n = X_n - n(2p-1).$$

Optional sampling theorem shows that $\mathsf{E}[Y_{n\wedge T_b}] = 0$ for each n, which yields $\mathsf{E}[X_{n\wedge T_b}] = (2p-1)\mathsf{E}[n \wedge T_b]$. By monotone convergence theorem, we have $\lim_{n\to\infty} \mathsf{E}[n \wedge T_b] = \mathsf{E}[T_b]$. Hence, it only remains to justify $\lim_{n\to\infty} \mathsf{E}[X_{n\wedge T_b}] = \mathsf{E}[X_{T_b}] = b$. To show this, observe that for any $a < 0$,

$$\mathsf{P}(T_a < \infty) = \mathsf{P}\left(\inf_{n\geq 0} X_n \leq a\right) = \left(\frac{1-p}{p}\right)^{-a}.$$

Since $p > 1/2$ implies $\sum_{n=1}^{\infty}((1-p)/p)^n < \infty$, we get $\mathsf{E}|\inf_{n\geq 0} X_n| < \infty$. Since $|X_{n\wedge T_b}| \leq b \vee |\inf_{n\geq 0} X_n|$, we can apply dominated convergence theorem to conclude the proof. $\square$

Exercise 10.3. Let (X_n) be an asymmetric random walk with $p \in (1/2, 1)$, and $T_b = \min\{n \geq 1 \colon X_n = b\}$. Show that

$$\mathrm{Var}(T_b) = \frac{4bp(1-p)}{(2p-1)^3}.$$

References

[1] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.

[2] Achim Klenke. *Probability theory: a comprehensive course.* Springer Science & Business Media, 2013.

[3] David Williams. *Probability with martingales.* Cambridge university press, 1991.