Elastic Problem in Half-space

1. Direct Elastic Wave Scattering and fundamental solution

The elastic wave propagates in an isotropic homogeneous medium with *Lamé* constant λ and μ and constant density ρ, governed by time harmonic elastic wave equation (Navier's equation).

$$\nabla \cdot \sigma(u) + \rho\omega^2 u = -f \tag{1.1}$$

Equivalently, let's use *Lamé* operator Δ_e

$$\Delta_e u = (\lambda + 2\mu)\nabla\nabla \cdot u - \mu\nabla \times \nabla \times u$$

together with the constitutive relation (Hookes' law)

$$\sigma_{ij} = \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) + \lambda\frac{\partial u_k}{\partial x_k}\delta_{ij}$$

where ω is the circular frequency, and $u(x)$ denotes the displacement fields and σ is the stress tensor. In the following, we assume that the density $\rho = 1$.

Define the surface traction Tu on the normal direction n,

$$T_x u(x) := \sigma \cdot n = 2\mu\frac{\partial u}{\partial n} + \lambda n \mathrm{div} u + \mu n \times \mathrm{curl} u$$

Let's introduce the outgoing Green tensor $\Phi(x, y)$ in a homogeneous medium due to the point source y which is the fundamental solution of elastic equation,

$$\Delta_e\Phi(x, y) + \omega^2\Phi(x, y) = -\delta_y(x)\mathbb{I} \quad \text{in} \quad \mathbb{R}^2 \tag{1.2}$$

where $\mathbb{I}$ is identity matrix and Φ is a $\mathbb{C}^{2\times 2}$ matrix defined by

$$\Phi(x, y) = \frac{1}{\omega^2}(\nabla \times \nabla \cdot (g_s(x, y)\mathbb{I}) - \nabla\nabla g_p(x, y)) \tag{1.3}$$

where $g_p(x, y)$ or $g_s(x, y)$ is the fundamental solution of the scalar Helmhotlz equation with wavenumbers k_p or k_s.

$$g_\alpha = \frac{\mathrm{i}}{4}H_0^{(1)}(k_\alpha|x - y|)$$

where $H_0^{(1)}(t)$ is the Hankel function of the first type and order zero and the wavenumbers of compressional and shear waves k_p and k_s are given by

$$k_p = \frac{\omega}{\sqrt{\lambda + 2\mu}} \quad \text{and} \quad k_s = \frac{\omega}{\sqrt{\mu}}$$

2. Green Tensor with free boundary

In this section we will study the elastic Green Tensor in the half-space with free boundary [8].

$$\Delta_e G(x; y) + \omega^2 G(x, y) = -\delta_y(x)\mathbb{I} \quad \text{in} \quad \mathbb{R}^2_+, \tag{2.1}$$

$$\sigma(G(x, y))e_2 = 0 \qquad \text{on} \quad x_2 = 0 \tag{2.2}$$

where $\delta_y(x)$ is the Dirac source at $y \in R^2_+$ and G is a $\mathbb{C}^{2\times 2}$ matrix. We will first use Fourier transform to derive the formula of Green Tensor in frequency domain. Let

$$\hat{G}(\xi, x_2; y_2) = \int_{-\infty}^{+\infty} G(x_1, x_2; y) e^{-\mathbf{i}(x_1-y_1)\xi} dx_1 \tag{2.3}$$

By taking the Fourier transform of (2.1-2.2), we obtain ODEs for x_2 in R_+

$$\mu \frac{d^2\hat{G}_1}{dx_2^2} + \mathbf{i}(\lambda+\mu)\xi \frac{d\hat{G}_2}{dx_2} + (\omega^2 - (\lambda+2\mu)\xi^2)\hat{G}_1 = [-\delta_{y_2}(x_2), 0] \tag{2.4}$$

$$(\lambda+2\mu) \frac{d^2\hat{G}_2}{dx_2^2} + \mathbf{i}(\lambda+\mu)\xi \frac{d\hat{G}_1}{dx_2} + (\omega^2 - \mu\xi^2)\hat{G}_2 = [0, -\delta_{y_2}(x_2)] \tag{2.5}$$

where $\hat{G}_i = e_i^T \hat{G}$ and the boundary coditions when $x_2 = 0$ are

$$\mu \frac{d\hat{G}_1}{dx_2} + \mathbf{i}\mu\xi\hat{G}_2 = [0, 0] \tag{2.6}$$

$$(\lambda+\mu) \frac{d\hat{G}_2}{dx_2} + \mathbf{i}\lambda\xi\hat{G}_1 = [0, 0] \tag{2.7}$$

For simplicity, the ordinary differential operator and boundary condition in (2.4-2.7) are denoted as $\mathbb{A}_\xi$ and $\mathbb{B}_\xi$. Now, we recall that

$$\hat{\Phi}(\xi, x_2; y_2) = \frac{\mathbf{i}}{2\omega^2}\left[\begin{pmatrix} \mu_s & -\xi\frac{x_2-y_2}{|x_2-y_2|} \\ -\xi\frac{x_2-y_2}{|x_2-y_2|} & \frac{\xi^2}{\mu_s} \end{pmatrix} e^{\mathbf{i}\mu_s|x_2-y_2|} + \begin{pmatrix} \frac{\xi^2}{\mu_p} & \xi\frac{x_2-y_2}{|x_2-y_2|} \\ \xi\frac{x_2-y_2}{|x_2-y_2|} & \mu_p \end{pmatrix} e^{\mathbf{i}\mu_p|x_2-y_2|}\right]$$

Denote $U = \hat{G} - \hat{\Phi}$ and it satisifies the following 2-order homogeneous ordinary differential equations with constant coefficients

$$\mathbb{A}_\xi U^i = 0 \qquad \text{in } \mathbb{R}^2_+ \tag{2.8}$$

$$\mathbb{B}_\xi U^i = \mathbb{B}_\xi \hat{\Phi} \qquad \text{on } x_2 = 0 \tag{2.9}$$

where $U^i = Ue_i$.

Throughout the paper, we will assume that for $z \in \mathbb{C}$, $z^{1/2}$ is the analytic branch of $\sqrt{z}$ such that $\operatorname{Im}(z^{1/2}) \geq 0$. This corresponds to the rigt half real axis as the branch cut in the complex plane. For $z = z_1 + \mathrm{i}z_2$, $z_1, z_2 \in \mathbb{R}$, we have

$$z^{1/2} = sgn(z_2)\sqrt{\frac{|z| + z_1}{2}} + \mathrm{i}\sqrt{\frac{|z| - z_1}{2}} \tag{2.10}$$

For z on the right half real axis, we take $z^{1/2}$ as the limit of $(z + \mathbf{i}\varepsilon)^{1/2}$ as $\varepsilon \to 0^+$.

By the standard arguement in ODEs, solutions of (2.8) are linear combinations of vectors of the form

$$\mathbf{r}(x_2) = \mathbf{v}e^{\mathbf{i}\nu x_2} \tag{2.11}$$

where $\mathbf{v} = (v_1, v_2)^T \in \mathbb{C}^2$ and $\nu \in \mathbb{C}$. It is well kown that the admissible values of ν are in $\Lambda := \{\pm\mu_p, \pm\mu_s\}$ and where

$$\mu_\alpha = (k_\alpha^2 - \xi^2)^{1/2} \qquad \text{for} \quad \alpha = s, p \tag{2.12}$$

Associated with each value of admissible $\nu \in \Lambda$ there is an eigenvector $\mathbf{v}$. The respective eigenvectors are

$$\mathbf{v}_s^+ = \begin{bmatrix} \mathbf{i}\mu_s \\ -\mathbf{i}\xi \end{bmatrix}, \quad \mathbf{v}_p^+ = \begin{bmatrix} \mathbf{i}\xi \\ \mathbf{i}\mu_p \end{bmatrix}, \quad \mathbf{v}_s^- = \begin{bmatrix} \mathbf{i}\mu_s \\ \mathbf{i}\xi \end{bmatrix}, \quad \mathbf{v}_p^- = \begin{bmatrix} -\mathbf{i}\xi \\ \mathbf{i}\mu_p \end{bmatrix}$$

We denote by $\hat{\sigma}(\mathbf{r})e_2 = \mathbf{u}e^{\mathbf{i}\nu x_2}$ the traction of such a vector in the Fourier domain, then we obtain the respective expression for $\mathbf{v}$

$$\mathbf{u}_s^+ = \begin{bmatrix} -\mu\beta \\ 2\mu\xi\mu_s \end{bmatrix}, \ \mathbf{u}_p^+ = \begin{bmatrix} -2\mu\xi\mu_p \\ -\mu\beta \end{bmatrix}, \ \mathbf{u}_s^- = \begin{bmatrix} \mu\beta \\ 2\mu\xi\mu_s \end{bmatrix}, \ \mathbf{u}_p^- = \begin{bmatrix} -2\mu\xi\mu_p \\ \mu\beta \end{bmatrix}$$

where $\beta = k_s^2 - 2\xi^2$. By allowing only bounded Fourier modes for the Green function, we must choose $V_S = \mathbf{v}_s^+ e^{\mathbf{i}\mu_s x_2}$ and $V_p = \mathbf{v}_p^+ e^{\mathbf{i}\mu_p x_2}$. Thus U_i must be written as the linear combination,

$$U_i = \alpha_i V_s + \beta_i V_p \quad \text{ for } \ i = 1, 2 \tag{2.13}$$

Combining (2.9) with the linear independence of V_s, V_p, then we can obtain α_i, β_i.

Therefore, the Green Tensor in half-space can be deduced as

$$\hat{G}(\xi, x_2; y_2) = \hat{\Phi}(\xi, x_2; y_2) - \hat{\Phi}(\xi, x_2; -y_2) + \hat{G}_c(\xi, x_2; y_2) \tag{2.14}$$

$$\begin{aligned} \hat{G}_c(\xi, x_2; y_2) = \ = \frac{\mathrm{i}}{\omega^2\delta(\xi)} \Big\{ & A(\xi)e^{\mathrm{i}\mu_s(x_2+y_2)} + B(\xi)e^{\mathrm{i}\mu_p(x_2+y_2)} \\ & + C(\xi)e^{\mathrm{i}\mu_s x_2 + \mu_p y_2} + D(\xi)e^{\mathrm{i}\mu_p x_2 + \mu_s y_2} \Big\} \end{aligned} \tag{2.15}$$

where

$$A(\xi) = \begin{pmatrix} \mu_s\beta^2 & -4\xi^3\mu_s\mu_p \\ -\xi\beta^2 & 4\xi_4\mu_p \end{pmatrix} \qquad B(\xi) = \begin{pmatrix} 4\xi^4\mu_s & \xi\beta^2 \\ 4\xi^3\mu_s\mu_p & \mu_p\beta^2 \end{pmatrix}$$

$$C(\xi) = \begin{pmatrix} 2\xi^2\mu_s\beta & -2\xi\mu_s\mu_p\beta \\ -2\xi^3\beta & 2\xi^2\mu_p\beta \end{pmatrix} \quad D(\xi) = \begin{pmatrix} 2\xi^2\mu_s\beta & 2\xi^3\beta \\ 2\xi\mu_s\mu_p\beta & 2\xi^2\mu_p\beta \end{pmatrix}$$

and $\beta(\xi) = k_s^2 - 2\xi^2$, $\delta(\xi) = \beta^2 + 4\xi^2\mu_s\mu_p$.

The desired Green function should be obtained by taking the inverse Fourier transform of $\hat{G}(\xi, x_2; y_2)$. Unfortunately, one cannot simply take the inverse Fourier transform in the above formula because $\delta(\xi)$ have zero points in the real axis by lemma 2.1 [1][10].

Lemma 2.1 *Let* Lamé *constant* $\lambda, \mu \in \mathbb{R}^+$, *then the Rayleigh equation* $\delta(\xi) = 0$ *has only two roots denoted by* $\pm k_R$ *in complex plane. Morever,* $k_R > k_s > k_p$, $k_R \in \mathbb{R}$ *and* k_R *is called Rayleigh wave number.*

Proof. For the sake of completeness, we include a proof here. It is well known that

$$\delta(\xi) = (k_s^2 - 2\xi^2)^2 + 4\xi^2(k_s^2 - \xi^2)^{1/2}(k_p^2 - \xi^2)^{1/2} \tag{2.16}$$

However, $\delta(\xi)$ is rendered single-valued by selecting branch cuts along $k_p < \mathrm{Re}\,(\xi) < k_s, \mathrm{Im}\,(\xi) = 0$ which is consistent with the convention (2.10). A simple computation show that $\delta(\pm k_s) > 0$ and $\delta(\pm\infty + 0\mathbf{i}) < 0$. By the continuity of $\delta(\xi)$, we can obtain that it has at least two real zero points which denoted by $\pm k_R$.

Now it turn to proof that $\delta(\xi)$ has only two roots in the complex plane by the principle of arguement which follows as a theorem of the theory of complex variables[2]. Now consider the contour C consisting of Γ, and C_l and C_r where $C_r = [k_p + \mathbf{i}0^+, k_s + \mathbf{i}0^+] \cup [k_p + \mathbf{i}0^-, k_s + \mathbf{i}0^-]$ that surround $[k_p, k_s]$, $C_l = [-k_s + \mathbf{i}0^+, -k_p + \mathbf{i}0^+] \cup [-k_s + \mathbf{i}0^-, -k_p + \mathbf{i}0^-]$ that surround $[-k_s, -k_p]$ and Γ denotes a circle with enough large radius. Since the function $\delta(\xi)$ clears does not have poles in the complex ξ-plane and we find that within the contour $C = \Gamma \cup C_r \cup C_l$ the number of zeros is given by

$$Z = \frac{1}{2\pi\mathbf{i}} \int_C \frac{d\delta}{d\xi} \frac{d\xi}{\delta(\xi)} \tag{2.17}$$

The counting of the number of zeros is carried out by mapping the ξ-plane on the η-plane through the relation $\eta := \delta(\xi)$. If C_η is the mapping of C in the η-plane, the integral (2.17) in the η-plane becomes

$$Z = \frac{1}{2\pi\mathbf{i}} \int_{C_\eta} \frac{d\eta}{\eta} \tag{2.18}$$

The latter integral has a simple pole at $\eta = 0$, and Z is simply the number of times the image contour C_η encircles the origin in the η-plane in the counter-clockwise direction. To determine the number of zeros in the ξ-plane we thus carefully trace the mapping of the contour C into th η-plane.

Since $\delta(\xi) = \delta(-\xi)$ the images of C_r and C_l are the same, and one of them, say C_r, needs to be considered. We have $\delta(k_p) = (k_s^2 - 2k_p^2)^2$ and along C_r: $\delta(\xi) = (k_s^2 - \xi^2)^2 \mp \mathbf{i}4\xi^2\sqrt{k_s^2 - \xi^2}\sqrt{\xi^2 - k_p^2}$, and $\delta(k_s) = k_s^4$ where the minus sign applies above the cut, and the plus sign applies below the cut. Note that along C_r we have $\mathrm{Re}\,(\delta(\xi)) > 0$ and the mapping into the η-plane do not surround the origin. For $|\xi|$ large, we find $\delta(\xi) = A\xi^2 + O(1)$, thus the mapping of Γ encircles the origin twice. Then we obtain $Z = 2$. This completes the proof. $\square$

In order to overcome the ambiguity above, loss is assumed in the medium so that $k_{\alpha,\varepsilon} := k_\alpha(1+\mathbf{i}\varepsilon)$. When $\varepsilon > 0$, the branch point of $\mu_{\alpha,\varepsilon}$ are $\pm k_{\alpha,\varepsilon}$ and the branch cut are denoted by the equation $\xi_1\xi_2 = k_\alpha\varepsilon, -k_\alpha \le \xi \le k_\alpha$. In this case, the poles singularities are now located off the real axis and the Fouerier inverse transform becomes meaningful. In order to express lemma 2.2 concisely, we define

$$\Omega := \{\xi \in \mathbb{C} \mid k_p\varepsilon < \xi_1\xi_2 < k_s\varepsilon\ ,\quad \xi_2 > \xi_1\varepsilon\} \tag{2.19}$$

Lemma 2.2 *If the elastic medium has loss that $k_{\alpha,\varepsilon} := k_\alpha(1+\mathbf{i}\varepsilon), 0 < \varepsilon < 1$ for $\alpha = p, s$, we assert that $\delta_\varepsilon(\xi) = 0$ has only two roots in domain $\Omega^c \subset \mathbb{C}$ and exactly they are $\pm k_{R,\varepsilon}$.*

Lemma 2.3 *Let $0 < \varepsilon < 1$ and $z = Re^{\mathbf{i}\phi}$, $(1 + \mathbf{i}\varepsilon) = re^{\mathbf{i}\psi}$, where $0 \le \phi < 2\pi$, $0 < \psi < \pi/2$ and $R, r > 0$. Then the equality*

$$z^{1/2} = (1 + \mathbf{i}\varepsilon)(\frac{z}{1 + \mathbf{i}\varepsilon^2})^{1/2} \tag{2.20}$$

holds only when $2\psi \leq \phi < 2\pi$

Proof. Let $z_\varepsilon = z/(1+\mathbf{i}\varepsilon)^2 := r_\varepsilon e^{\mathbf{i}\phi_\varepsilon}$, then $\phi_\varepsilon = \phi - 2\psi$ when $2\psi \leq \phi < 2\pi$.So it is easy to see that

$$z^{1/2} = \sqrt{R}e^{\mathbf{i}\phi/2} = \sqrt{R/r}\sqrt{r}e^{\mathbf{i}(\phi/2-\psi)+\mathbf{i}\psi} = (1+\mathbf{i}\varepsilon)z_\varepsilon^{1/2}$$

Similarly, when $0 \leq \phi < 2\psi$, we have $\phi_\varepsilon = \phi - 2\psi + 2\pi$ and then $z^{1/2} = -(1+\mathbf{i}\varepsilon)z_\varepsilon^{1/2}$. This completes the proof. □

Proof of lemma 2.2. The lemma now follow lemma2.1 and lemma2.3 easily. Denote by $\mu_\varepsilon = (k^2(1+\mathbf{i}\varepsilon)^2 - \xi^2)^{1/2}, k \in \mathbb{R}^+$ and write $\xi = \xi_1 + \mathbf{i}\xi_2, \xi_1, \xi_2 \in \mathbb{R}$ and $(1+\mathbf{i}\varepsilon) = re^{\mathbf{i}\psi}$. It is easy to see that

$$\mu_\varepsilon^2 = k^2(1-\varepsilon^2) - \xi_1^2 + \xi_2^2 + \mathbf{i}(2k^2\varepsilon - 2\xi_1\xi_2) := Re^{\mathbf{i}\Theta} := a_1 + \mathbf{i}a_2 \tag{2.21}$$

Let $\Delta := \{\xi | 2\psi \leq \Theta < 2\pi\}$, then we have $\mu_\varepsilon = (k^2 - \xi_\varepsilon^2)^{1/2}(1+\mathbf{i}\varepsilon)$ when $\xi \in \Delta$ and $\mu_\varepsilon = -(k^2 - \xi_\varepsilon^2)^{1/2}(1+\mathbf{i}\varepsilon)$ when $\xi \notin \Delta$ by lemma2.3. Now we divide the set Δ into three parts

$$\begin{aligned}\Delta &= \{\xi | a_1 \leq 0\} \cup \{\xi | a_2 \leq 0\} \cup \{\xi | a_1 > 0, a_2 > 0 \text{ and } \tan\Theta \geq \tan(2\psi)\} \\ &:= \Delta_1 \cup \Delta_2 \cup \Delta_3\end{aligned} \tag{2.22}$$

Our goal now is to show where domain Δ occupies. A simple computation show that

$$\Delta_1 = \{\xi | \xi_1^2 - \xi_2^2 \geq k^2(1-\varepsilon^2)\} \tag{2.23}$$

$$\Delta_2 = \{\xi | \xi_1\xi_2 \geq k^2\varepsilon\} \tag{2.24}$$

and

$$\Delta_3 = \{\xi | \xi_1^2 - \xi_2^2 \leq k^2(1-\varepsilon^2), \xi_1\xi_2 \leq k^2\varepsilon, \frac{k^2\varepsilon - \xi_1\xi_2}{k^2(1-\varepsilon^2) - (\xi_1^2 - \xi_2^2)} \geq \frac{\varepsilon}{1-\varepsilon^2}\} \tag{2.25}$$

The domains denote by Δ_1, Δ_2 are obvious in complex plane. To locate Δ_3 in complex plane, we divide Δ_3 into tree parts $\Delta_3 = \Delta_{31} \cup \Delta_{32} \cup \Delta_{33}$ where

$$\begin{aligned}\Delta_{31} &= \{\xi_1\xi_2 \leq 0, 0 \leq \xi_1^2 - \xi_2^2 \leq k^2(1-\varepsilon^2)\} \\ \Delta_{32} &= \{0 \leq \xi_1\xi_2 \leq k^2\varepsilon, 0 \leq \xi_1^2 - \xi_2^2 \leq k^2(1-\varepsilon^2), \frac{\xi_1\xi_2}{\xi_1^2 - \xi_2^2} \leq \frac{\varepsilon}{1-\varepsilon^2}\} \\ &= \{0 \leq \xi_1\xi_2 \leq k^2\varepsilon, 0 \leq \xi_1^2 - \xi_2^2 \leq k^2(1-\varepsilon^2), \frac{\xi_2}{\xi_1} \leq \varepsilon\} \\ \Delta_{33} &= \{\xi_1\xi_2 \leq 0, \xi_1^2 - \xi_2^2 \leq 0, \frac{\xi_1\xi_2}{\xi_1^2 - \xi_2^2} \geq \frac{\varepsilon}{1-\varepsilon^2}\} \\ &= \{\xi_1\xi_2 \leq 0, \xi_1^2 - \xi_2^2 \leq 0, -\frac{\xi_1}{\xi_2} \geq \varepsilon\}\end{aligned}$$

Substituting k_s, k_p into μ_ε and let Δ_s, Δ_p denote their corresponding areas, we have

$$\mathbb{C}\backslash\Omega = (\Delta_s \cap \Delta_p) \cup (\mathbb{C}\backslash(\Delta_s \cup \Delta_p)) \tag{2.26}$$

Moreover, when $\xi \in \Omega$ it is easy to see $\delta_\varepsilon(\xi) = \delta(\xi)(1-\mathbf{i}\varepsilon)^4$. This complete the proof by lemma 2.1 □

Let $\xi = \xi_1 + \mathbf{i}\xi_2 \in \mathbb{C}$, $\xi_1, \xi_2 \in \mathbb{R}$, and the hyperbolic curve Γ defined by the equation $\xi_1^2 - \xi_2^2 = k_s^2$. Denote Γ_r^+, Γ_r^- respectively the parts of right branch of Γ in the upper-half complex plane and the lower-half complex plane. Similarily, we can define Γ_l^-, Γ_l^-.Now, we can define a new integral path in the complex plane

$$NP = \begin{cases} \Gamma_l^+ \cup \Gamma_r^+ \cup [-k_s, k_s] & \text{when} \quad x_1 - y_1 > 0 \\ \Gamma_l^- \cup \Gamma_r^- \cup [-k_s, k_s] & \text{when} \quad x_1 - y_1 < 0 \end{cases} \tag{2.27}$$

Thus, by using Cauchy integral theorem and lemma 2.2, we have

$$G_\varepsilon(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{G}_\varepsilon(\xi, x_2; y_2) e^{\mathbf{i}(x_1 - y_1)\xi} d\xi \tag{2.28}$$

$$= \frac{1}{2\pi} \int_{NP} \hat{G}_\varepsilon(\xi, x_2; y_2) e^{\mathbf{i}(x_1 - y_1)\xi} d\xi \pm \mathbf{i} Res_{\xi = \pm k_R^\varepsilon} G_\varepsilon(\xi, x_2; y_2) e^{\mathbf{i}(x_1 - y_1)\xi} \tag{2.29}$$

As the perturbation ε have nothing to do with the integration path NP, so we could take the limitation $\varepsilon \to 0$. Thus, we have the representation of Green Tensor

$$G(x, y) = \Phi(x, y) - \Phi(x, y') + \frac{1}{2\pi} \int_{NP} \hat{G}_c(\xi, x_2; y_2) e^{\mathbf{i}(x_1 - y_1)\xi} d\xi \tag{2.30}$$

$$\pm \mathbf{i} Res_{\xi = \pm\kappa_r} \hat{G}_c(\xi, x_2; y_2) e^{\mathbf{i}(x_1 - y_1)\xi}$$

where $\pm$ are corresponding $sgn(x_1 - y_1)$. Specially, $G(x, y)$ has a simple form when $x_2 = 0$ that

$$G(x, y) = \frac{1}{2\pi} \int_{NP} \hat{G}(\xi, 0; y) e^{\mathbf{i}(x_1 - y_1)\xi} d\xi \pm \mathbf{i} Res_{\xi = \pm\kappa_r} \hat{G}(\xi, x_2; y) e^{\mathbf{i}(x_1 - y_1)\xi} \tag{2.31}$$

where

$$\hat{G}(\xi, 0; y_2) = \frac{\mathrm{i}}{\mu\delta(\xi)} \left[\begin{pmatrix} 2\xi^2\mu_s & -2\xi\mu_s\mu_p \\ -\xi\beta & \mu_p\beta \end{pmatrix} e^{\mathrm{i}\mu_p y_2} + \begin{pmatrix} \mu_s\beta & \xi\beta \\ 2\xi\mu_s\mu_p & 2\xi^2\mu_p \end{pmatrix} e^{\mathrm{i}\mu_s y_2} \right] \tag{2.32}$$

and let $G_r(x_1; y_1, y_2)$ denote the first part of G and $G_s(x_1; y_1, y_2)$ denote the second part of G in (2.31). We has following estimate for G_r

Lemma 2.4 *For $y \in \mathbb{R}_+^2$, the estimate*

$$|G_r(x_1; y_1, y_2)| \leq \frac{h(y_2)}{|x_1 - y_1|} \tag{2.33}$$

holds, where $h \in C(\mathbb{R}_+)$

Proof. Since there are no stationary points in the phases of $e^{\mathbf{i}(x_1 - y_1)\xi}$, the estimate above can be easy obtained via integration by part. Here we omit the details. □

Lemma 2.5 *For $y \in \mathbb{R}_+^2$, $x \in \Gamma_0$, we have*

$$|G_r(x; y)| \leq \frac{C}{\mu|x - y|^{1/2}} \tag{2.34}$$

$$|G_s(x; y)| \leq \frac{C}{\mu}(e^{-\sqrt{k_r^2 - k_s^2} y_2} + e^{-\sqrt{k_r^2 - k_p^2} y_2}) \tag{2.35}$$

where the constant C is only dependent on κ.

3. Green Tensor with Dirichlet boundary

In this section we study the elastic Green Tensor in the half-space with Dirichlet boundary [4].

$$\Delta_e D(x,y) + \omega^2 D(x,y) = -\delta_y(x)\mathbb{I} \quad \text{in} \quad \mathbb{R}^2_+, \tag{3.1}$$

$$D(x,y) = 0 \qquad \text{on} \quad x_2 = 0 \tag{3.2}$$

where $\delta_y(x)$ is the Dirac source at $y \in R^2_+$ and G is a $\mathbb{C}^{2\times 2}$ matrix. We will first use Fourier transform to derive the formula of Green Tensor in frequency domain. Then we can obtain $D(x,y)$ similar to $G(x,y)$. An alternative representation for $D(x,y)$ is seen that

$$\hat{D}(\xi, x_2; y_2) = \hat{\Phi}(\xi, x_2; y_2) - \hat{\Phi}(\xi, x_2; -y_2) + \hat{M}(\xi, x_2; y_2) \tag{3.3}$$

$$\begin{aligned}\hat{M}(\xi, x_2; y_2) = \frac{\mathrm{i}}{\omega^2\gamma(\xi)}\Big\{ & A(\xi)e^{\mathrm{i}\mu_s(x_2+y_2)} + B(\xi)e^{\mathrm{i}\mu_p(x_2+y_2)} \\ & - A(\xi)e^{\mathrm{i}\mu_s x_2 + \mu_p y_2} - B(\xi)e^{\mathrm{i}\mu_p x_2 + \mu_s y_2}\Big\}\end{aligned} \tag{3.4}$$

where

$$A(\xi) = \begin{pmatrix} \xi^2\mu_s & -\xi\mu_s\mu_p \\ -\xi^3 & \xi^2\mu_p \end{pmatrix} \qquad B(\xi) = \begin{pmatrix} \xi^2\mu_s & \xi^3 \\ \xi\mu_s\mu_p & \xi^2\mu_p \end{pmatrix}$$

and $\gamma(\xi) = \xi^2 + \mu_s\mu_p$.

Lemma 3.1 *Let* Lamé *constant* $\lambda, \mu \in \mathbb{C}$ *and* $\mathrm{Im}\,(k_s) \geq 0, \mathrm{Im}\,(k_p) \geq 0$, *then equation* $\gamma(\xi) = 0$ *has no root in complex plane.*

Proof. Let $F(\xi) = \gamma(\xi) * (\xi^2 - \mu_s\mu_p)$ and it is easy to see that the root of $\gamma(\xi) = 0$ is also of $F(\xi) = 0$. A simple computation show that $F(\xi) = (k_s^2 + k_p^2)\xi^2 - k_p^2 k_s^2$. However, only when $\xi^2 = k_p^2 k_s^2/(K_s^2 + k_p^2)$, $F(\xi) = 0$ but $\gamma(\xi) = 2k_p^2 k_s^2/(K_s^2 + k_p^2)$. This completes the proof. □

Thus, we get the representation of Green Tensor by inverse Fourier transform

$$D(x,y) = \Phi(x,y) - \Phi(x,y') + \frac{1}{2\pi}\int_{\mathbb{R}} \hat{M}(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \tag{3.5}$$

Let $T_D(x,y)$ denote the traction of $D(x,y)$ in direction e_2 to variable x that $T_D(x,y)e_i = T_x^{e_2}(D(x,y))e_i = T_x^{e_2}(D(x,y)e_i)$. Then we can get the representation of $T_D(x,y)$ by a trivial calculation.

$$T_D(x,y) = T(x,y) - T(x,y') + \frac{1}{2\pi}\int_{\mathbb{R}} \hat{T}_M(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \tag{3.6}$$

and

$$\begin{aligned}\hat{T}_M(\xi, x_2; y_2) = \frac{\mu}{\omega^2\gamma(\xi)}\Big\{ & E(\xi)e^{\mathrm{i}\mu_s(x_2+y_2)} + F(\xi)e^{\mathrm{i}\mu_p(x_2+y_2)} \\ & - E(\xi)e^{\mathrm{i}\mu_s x_2 + \mu_p y_2} - F(\xi)e^{\mathrm{i}\mu_p x_2 + \mu_s y_2}\Big\}\end{aligned} \tag{3.7}$$

where

$$E(\xi) = \begin{pmatrix} -\xi^2\beta & \xi\mu_p\beta \\ 2\xi^3\mu_s & -2\xi^2\mu_s\mu_p \end{pmatrix} \qquad F(\xi) = \begin{pmatrix} -2\xi^2\mu_s\mu_p & -2\xi^3\mu_p \\ -\xi\mu_s\beta & -\xi^2\beta \end{pmatrix}$$

Specially, $T_D(x, y)$ has a simple form when $x_2 = 0$ that

$$\hat{T}_D(\xi, 0; y_2) = \frac{1}{\gamma(\xi)}\left[\begin{pmatrix} \mu_s\mu_p & \xi\mu_p \\ \xi\mu_s & \xi^2 \end{pmatrix} e^{\mathrm{i}\mu_s y_2} + \begin{pmatrix} \xi^2 & -\xi\mu_p \\ -\xi\mu_s & \mu_p\mu_s \end{pmatrix} e^{\mathrm{i}\mu_p y_2}\right] \quad (3.8)$$

and

$$T_D(x_1, 0; y_1, y_2) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{T}_D(\xi, 0; y_2) e^{\mathbf{i}(x_1-y_1)\xi} d\xi \quad (3.9)$$

We has following estimate for the T_D [4]

Lemma 3.2 *For $y \in \mathbb{R}^2_+$, $|x_1 - y_1| \le e$ the estimate*

$$|T_D(x_1, 0; y_1, y_2)| \le \frac{h(y_2)}{|x_1 - y_1|^{3/2}} \quad (3.10)$$

holds, where $h \in C(\mathbb{R}_+)$

We need the following slight generalization of Van der Corput lemma for the oscillatory integral [9, P.152].

Lemma 3.3 *Let $-\infty < a < b < \infty$, and u is a C^k function u in (a, b).*
1. If $|u'(t)| \ge 1$ for $t \in (a, b)$ and u' is monotone in (a,b), then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \le 3\lambda^{-1}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

2. For all $k \ge 2$, if $|u^{(k)}(t)| \ge 1$ for $t \in (a, b)$, then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \le 12k\lambda^{-1/k}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

Proof. The assertion can be proved by extending the Van der Corptut lemma in [9]. Here we omit the details. □

We also have following more explicit estimation:

Lemma 3.4 *Let $f(\xi, \mu_s, \mu_p) = g(\xi, \mu_s, \mu_p)/\gamma(\xi, \mu_s, \mu_p)$ where g(x,y,z) is a homogeneous quadratic polynomial with respect to x,y,z. Let $a, b > 0$ and $\rho = \sqrt{a^2 + b^2}$. Assume $a/\rho > (1 + \kappa)/2$, $b/\rho < \kappa/2$, $\kappa = k_p/k_s$ and $k_s\rho > 1$, then we have*

$$\left|\int_{\mathbb{R}} f(\xi, \mu_s, \mu_p) e^{\mathbf{i}(\mu_s b+\xi a)} d\xi\right| \le C\left(\frac{k_s b}{\rho(k_s\rho)^{1/2}} + \frac{k_s a}{\rho(k_s\rho)^{3/2}}\right) \quad (3.11)$$

and

$$\left|\int_{\mathbb{R}} f(\xi, \mu_s, \mu_p) e^{\mathbf{i}(\mu_p b+\xi a)} d\xi\right| \le C\left(\frac{k_s b}{\rho(k_s\rho)^{1/2}} + \frac{k_s a}{\rho(k_s\rho)^{3/2}}\right) \quad (3.12)$$

where C is only dependent on κ.

Proof. Let I(a,b) denote the integral in the left side of inequality (3.11). To simplify the integral, the standard substitution $\xi = k_s \sin t$ is made, taking the ξ-plane to a strip $-\pi/2 < \mathrm{Re}\, t < \pi/2$ in the t-plane, and the real axis in the ξ-plane onto the path L from $-\pi/2 + \mathbf{i}\infty \to -\pi/2 \to \pi/2 \to \pi/2 - \mathbf{i}\infty$ in the t-plane. The integral I(a,b) then becomes(Let a=$\rho \sin\phi$ and b=$\rho\cos\phi$, $0 < \phi < \pi/4$)

$$k_s \int_L f(\sin t, \cos t, (\kappa^2 - \sin^2 t)^{1/2}) \cos t\; e^{\mathbf{i}k_s\rho(\cos(t-\phi))} dt \tag{3.13}$$

Taking the shift transformation of t and using cauchy integral theorem, we can obtain the representation of I(a,b):

$$\begin{aligned}
&k_s \int_L f(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2}) \cos(t+\phi) e^{\mathbf{i}k_s\rho(\cos t)} dt \\
&= k_s \cos\phi \int_L f(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2}) \cos t\; e^{\mathbf{i}k_s\rho(\cos t)} dt \\
&- k_s \sin\phi \int_L f(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2}) \sin t\; e^{\mathbf{i}k_s\rho(\cos t)} dt \\
&:= k_s(\cos\phi\; I_1 + \sin\phi\; I_2)
\end{aligned}$$

The lemma will be proved if we can show $|I_1| \le C/(k_s\rho)^{1/2}$ and $|I_2| \le C/(k_s\rho)^{3/2}$.

For I_1, we split the integral path L into $L_1 = (-\pi/2, \pi/2)$ and $L_2 = (-\pi/2 + \mathbf{i}\infty, -\pi/2) \cup (\pi/2, \pi/2 - \mathbf{i}\infty)$, then we have corresponding representation: $I_1 = I_{11} + I_{12}$. A simple calculation gives that f and $\partial f/\partial t$ are both integrable on path L_1. Further more, $\cos t > 1/2$ for any $t \in (-\pi/4, \pi/4)$ while $|\sin t| > 1/2$ for any $t \in (-\pi/2, -\pi/4) \cup (\pi/4, \pi/2)$. Then we have $|I_{11}| \le C/(k_s\rho)^{1/2}$ following the lemma 3.3.

Moreover, because f and $\partial f/\partial t$ has no singularity on L_2 and $\mathbf{i}\cos t \to -\infty$ as $t \to \infty$ along L_2, it is easy to see that $|I_{12}| \le C/(k_s\rho)$ via integration by parts.

For I_2, using integration by parts on path L first, we have

$$\begin{aligned}
I_2 &= \frac{1}{\mathbf{i}k_s\rho} \int_L f(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2}) d\; e^{\mathbf{i}(k_s\rho\cos t)} && (3.14)\\
&= -\frac{1}{\mathbf{i}k_s\rho} \int_{L_1\cup L_2} \frac{\partial f(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2})}{\partial t}\; e^{\mathbf{i}(k_s\rho\cos t)} dt && (3.15)\\
&= -\frac{1}{\mathbf{i}k_s\rho}(I_{21} + I_{22}) && (3.16)
\end{aligned}$$

Then $|I_{22}| \le C/(k_s\rho)$ can be proved by the same method used above. Following a tedious computation, we obtain a simple form of $\partial f/\partial t$:

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{(\gamma\partial_t g - g\partial_t\gamma)(\kappa^2 - \sin^2 t)^{1/2}}{(\sin^2 t + \cos t(\kappa^2 - \sin^2 t)^{1/2})^2} \frac{1}{(\kappa^2 - \sin^2 t)^{1/2}} && (3.17)\\
&:= \frac{h(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2})}{(\kappa^2 - \sin^2 t)^{1/2}} && (3.18)
\end{aligned}$$

where h and $\partial h/\partial t$ are integrable on path L_1. By the assumption above, there exist $0 < \delta < \pi/4$ only dependent on κ such that $|\sin(t+\phi)| > (1+\kappa)/2, |\cos(t+\phi)| < \kappa/2$ for any $t \in (-\delta, \delta)$ while $|\cos(t+\phi)| > (1+\kappa)/2, |\sin(t+\phi)| < \kappa/2$ for any $t \in (-\pi/2, -\pi/2+\delta) \cup (\pi/2-\delta, \pi/2)$. Let define $t_1, t_2 \in \chi_1 = (-\pi/2+\delta, -\delta) \cup (\delta, \pi/2-\delta)$

which satisfy $\kappa^2 = \sin^2(t_i+\phi)$, $i = 1, 2$. Moreover, for any $0 < \lambda_1 < 1$ and $1 < \lambda_2 < 1/\kappa$, there exists $\sigma > 0$, which satisfy that $\chi_2 = (t_1 - \sigma, t_1 + \sigma) \cup (t_2 - \sigma, t_2 + \sigma) \subset \chi_1$ and is only dependent on $\lambda_1, \lambda_2, \kappa$, such that

$$\lambda_1 \kappa < |\sin(t + \phi)| < \lambda_2 \kappa. \tag{3.19}$$

for any $t \in \chi_2$. We are now in a position to estimate I_{21}. Similarly, we split the path L_1 into χ_2 and $L_1 \backslash \chi_2$, then we have the corresponding representation: $I_{21} = I_{\chi_2} + I_{L_1 \backslash \chi_2}$.

For I_{χ_2}, we only analysis the integral on $\chi_{21} = (t_1 - \sigma, t_1 + \sigma)$ denoted by $I^1_{\chi_2}$, the procedure of the another is same. Without loss of generality, we assume that $\sin(t_1 - \sigma + \phi) < \kappa < \sin(t_1 + \sigma + \phi)$. It is easy to see that $\sin(t+\phi)$ is monotonic increasing in χ_{21}. Let $\sin(t + \phi) = \kappa \sin\theta$ and the implicit mapping from θ to t is denoted by $t(\theta)$ while the inverse mapping by $\theta(t)$, taking the interval χ_{21} onto $L_\theta : \theta_1 \to \pi/2 \to \pi/2 - \mathbf{i}\theta_2$ where $\sin(t_1 - \sigma + \phi) = \kappa \sin\theta_1, \sin(t_1 + \sigma + \phi) = \kappa \sin(\pi/2 - \mathbf{i}\theta_2)$. By substituting $t(\theta)$ into $I^1_{\chi_2}$, we have

$$I^1_{\chi_2} = \int_{L_\theta} \frac{h(\kappa\sin\theta, (1 - \kappa^2 \sin^2\theta)^{1/2}, \kappa\cos\theta)}{(1 - \kappa^2 \sin^2\theta)^{1/2}}\, e^{\mathbf{i}k_s\rho(\cos(t(\theta)))} d\theta \tag{3.20}$$

Because of inequality 3.19, we assert that h and $\partial h/\partial\theta$ are integrable on the path L_θ. A simple computation show that

$$\frac{dt(\theta)}{d\theta} = \frac{\kappa\cos\theta}{\cos(t+\phi)} \quad \frac{d^2t(\theta)}{dt^2} = \frac{\kappa^2\cos^2\theta\sin(t+\phi) - \kappa\sin\theta\cos^2(t+\phi)}{\cos^3(t+\phi)}$$

Then we can obtain

$$\begin{aligned}
\frac{d\cos t}{d\theta} &= \frac{-\kappa\sin t\cos\theta}{\cos(t+\phi)} \\
\frac{d^2\cos t}{d\theta^2} &= \frac{d^2\cos t}{dt^2}\left(\frac{dt}{d\theta}\right)^2 + \frac{d\cos t}{dt}\frac{d^2t}{d\theta^2} \\
&= \frac{-\kappa^2\cos^2\theta\cos t}{\cos^2(t+\phi)} + \frac{\kappa\sin\theta\cos^2(t+\phi)\sin t - \kappa^2\cos^2\theta\sin(t+\phi)\sin t}{\cos^3(t+\phi)} \\
&= \frac{-\kappa^2\cos^2\theta\cos\phi + \kappa\sin\theta\cos^2(t+\phi)\sin t}{\cos^3(t+\phi)} \\
&= \frac{(\sin^2(t+\phi) - \kappa^2)\cos\phi + \cos^2(t+\phi)\sin(t+\phi)\sin t}{\cos^3(t+\phi)}
\end{aligned}$$

It is simple to see that $\theta = \pi/2$ is the only stationary point of $\cos(t(\theta))$ and we can obtain

$$\left|\frac{d^2\cos t}{d\theta^2}(\pi/2)\right| = \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}|\sin t| > \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}\sin\delta \tag{3.21}$$

Therefore, we can choose appropriate λ_1, λ_2, only dependent on κ, such that $|\frac{d^2\cos t}{d\theta^2}| > \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}\sin\delta$ for any $\theta \in \theta(\chi_{21})$. Therefore, we can decompose $\theta(\chi_{21})$ into several intervals such that in each either $|\partial\cos(t(\theta))/\partial\theta|$ or $|\partial^2\cos(t(\theta))/\partial\theta^2|$ has positive lower bound and $|\partial\cos(t(\theta))/\partial\theta|$ is monotonous. Since the amplitude function of integrand in $I^1_{\chi_2}$ and its derivative with respect to θ are both integrable on L_θ, the estimation $|I^1_{\chi_2}| \leq C/(k_s\rho)^{1/2}$ can be obtained immediately by lemma 3.3. Finally, an argument similar to the estimation of I_{11} used shoes that $|I_{L_1\backslash\chi_2}| \leq C/(k_s\rho)^{1/2}$. This completes the proof.

□

Now, another more sophisticated estimation of $T_D(x_1, 0; y_1, y_2)$ is a direct consequence of lemma 3.4.

Lemma 3.5 *For every $x \in \Gamma_0$, $y \in \mathbb{R}^2_+$ that $|x_1-y_1|/|x-y| > (1+\kappa)/2$, $y_2/|x-y| < \kappa/2$ and $k_s|x-y| > 1$, we have*

$$|T_D(x,y)| \leq C(\frac{k_s y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{1/2}} + \frac{k_s|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{3/2}}) \quad (3.22)$$

where C is only dependent on κ.

Lemma 3.6 *Let $f(\xi,\mu_s,\mu_p) = g(\xi,\mu_s,\mu_p)/\gamma(\xi,\mu_s,\mu_p)$ where g(x,y,z) is a homogeneous quadratic polynomial with respect to x,y,z. Let $a, b > 0$ and $\rho = \sqrt{a^2+b^2}$. Assume $\kappa = k_p/k_s$ and $k_s\rho > 1$, then we have*

$$\Big|\int_{\mathbb{R}} f(\xi,\mu_s,\mu_p)e^{\mathbf{i}(\mu_s b+\xi a)}d\xi - f_{\xi=\frac{k_s a}{\rho}}\frac{k_s b}{\rho}(\frac{2\pi}{k_s\rho})^{1/2}e^{\mathbf{i}(k_s\rho-\frac{\pi}{4})}\Big| \quad (3.23)$$

$$\leq C(\frac{k_s b}{\rho(k_s\rho)^{3/4}} + \frac{k_s a}{\rho(k_s\rho)^{5/4}}) \quad (3.24)$$

and

$$\Big|\int_{\mathbb{R}} f(\xi,\mu_s,\mu_p)e^{\mathbf{i}(\mu_p b+\xi a)}d\xi - f_{\xi=\frac{k_p a}{\rho}}\frac{k_p b}{\rho}(\frac{2\pi}{k_p\rho})^{1/2}e^{\mathbf{i}(k_p\rho-\frac{\pi}{4})}\Big| \quad (3.25)$$

$$\leq C(\frac{k_p b}{\rho(k_p\rho)^{3/4}} + \frac{k_p a}{\rho(k_p\rho)^{5/4}}) \quad (3.26)$$

where C is only dependent on κ.

4. Reverse time migration method

In this section we introduce RTM method for inverse elastic scattering problems in the half space. Assume that there N_s sources and N_r receivers uniformly distributed on Γ_0^d, where $\Gamma_0^d = \{(x_1,x_2)^T \in \Gamma_0 : x_1 \in [-d,d]\}, d > 0$ is aperture. We denote by Ω the sampling domain in which the obstacle is sought. Let $h = dist(\Omega,\Gamma_0)$ be the distance of Ω to Γ_0. We assume the obstacle $D \subset \Omega$ and there exist constants $0 < c_1 < 1, c_2 > 0, c_3 > 0$ such that

$$|x_1| \leq c_1 d, \quad |x_1-y_1| \leq c_2 h, \quad |x_2| \leq c_3 h \quad \forall x,y \in \Omega \quad (4.1)$$

Our RTM algorithm consists of two steps[14][16]. The first step[5] is the back-propagation in which we back-propagate the complex conjugated data $\overline{u^s(x_r,x_s)}$ as the Dirichlet boundary condition into the domain. The second step is the cross-correlation in which we compute the imaginary part of the cross-correlation of the back-propagated field and the incoming wave which uses the source as the boundary codition on Γ_0.

Algorithm 4.1 *Rtm ...*

5. point spread function (backpropagate with Dirichlet Green Tensor)

The point spread function measures the resolution for finding point source[3]. In [6], the point spread function has been defined in acoustic wave case. We can define elastic point spread function $J(z,y)$, a $\mathbb{C}^{2\times 2}$ matrix, which back-propagate the conjugated data $\overline{G(x,y)}$ as the Dirichlet boundary condition. Thus, for any $z, y \in \mathbb{R}^2_+$

$$J(z,y) = \int_{\Gamma_0} (T_D(x,z))^T \overline{G(x,y)} ds(x) \tag{5.1}$$

$$= \int_{\mathbb{R}} (T_D(x_1,0;z_1,z_2))^T \overline{G(x_1,0;y_1,y_2)} dx_1 \tag{5.2}$$

The estimate in (3.2) show that the integral above exists. Now, we define functions

$$\Theta(\xi;y_1,y_2) = \frac{1}{\gamma(\xi)}\left[\begin{pmatrix} \mu_s\mu_p & -\xi\mu_p \\ -\xi\mu_s & \xi^2 \end{pmatrix} e^{\mathrm{i}\mu_s z_2} + \begin{pmatrix} \xi^2 & \xi\mu_p \\ \xi\mu_s & \mu_p\mu_s \end{pmatrix} e^{\mathrm{i}\mu_p z_2}\right] e^{\mathbf{i}\xi y_1} \tag{5.3}$$

It is easy to see that, $\Theta = \overline{\hat{T}_D}$ when $\xi \in \mathbb{R}\backslash[-k_s, k_s]$.

We split the spectral terms into components associated with pressure and shearing waves.

$$\hat{T_D} = \hat{T^p_D} + \hat{T^s_D} \quad \hat{G} = \hat{G^p} + \hat{G^s}$$

Thus, we define

$$J^{\alpha\eta}(z,y) = \int_R (T^\alpha_D(x_1,0;z))^T \overline{G^\eta(x_1,0;y)} dx_1 \tag{5.4}$$

It's esay to see

$$J(z,y) = \sum_{\alpha=p,s}^{\eta=p,s} J^{\alpha\eta}(z,y)$$

In order to analysis the PSF, loss is assumed in the medium that $k_{\alpha,\varepsilon} := k_\alpha(1+\mathbf{i}\varepsilon)$. Then by Parseval identity and lemma 2.2, we have

$$\begin{aligned} J^{ss}(z,y) = & \lim_{\varepsilon\to 0^+} \int_R (T^s_D(x_1,0;z_1,z_2))^T \overline{G^{s,\varepsilon}(x_1,0;y_1,y_2)} dx_1 \\ = & \lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int_R (\hat{T}^s_D(\xi,0;z))^T \overline{\hat{G}^{s,\varepsilon}(\xi,0;y)} d\xi \\ = & \frac{1}{2\pi} \int_{-k_s}^{k_s} (\hat{T}^s_D(\xi,0;z))^T \overline{\hat{G}^{s,\varepsilon}(\xi,0;y)} d\xi \\ + & \lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int_{R\backslash[-k_s,k_s]} (\hat{T}_D(\xi,0;z))^T \overline{\hat{G}^{s,\varepsilon}(\xi,0;y)} d\xi \\ := & F^{ss}(z,y) + R^{ss}(z,y) \end{aligned}$$

$$\begin{aligned} J^{pp}(z,y) = & \lim_{\varepsilon\to 0^+} \int_R (T^p_D(x_1,0;z_1,z_2))^T \overline{G^{p,\varepsilon}(x_1,0;y_1,y_2)} dx_1 \\ = & \lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int_R (\hat{T}^p_D(\xi,0;z))^T \overline{\hat{G}^{p,\varepsilon}(\xi,0;y)} d\xi \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-k_p}^{k_p} (\hat{T}_D^p(\xi,0;z))^T\overline{\hat{G}^{p,\varepsilon}(\xi,0;y)}d\xi \\
&+ \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{R\backslash[-k_p,k_p]} (\hat{T}_D^p(\xi,0;z))^T\overline{\hat{G}^{p,\varepsilon}(\xi,0;y)}d\xi \\
&:= F^{pp}(z,y) + R^{pp}(z,y)
\end{aligned}
$$

$$
\begin{aligned}
J^{sp}(z,y) &= \lim_{\varepsilon\to 0^+} \int_R (T_D^s(x_1,0;z_1,z_2))^T\overline{G^{p,\varepsilon}(x_1,0;y_1,y_2)}dx_1 \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_R (\hat{T}_D^s(\xi,0;z))^T\overline{\hat{G}^{p,\varepsilon}(\xi,0;y)}d\xi \\
&= \frac{1}{2\pi}\int_{-k_p}^{k_p} (\hat{T}_D^s(\xi,0;z))^T\overline{\hat{G}^{p,\varepsilon}(\xi,0;y)}d\xi \\
&+ \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{R\backslash[-k_p,k_p]} (\hat{T}_D^s(\xi,0;z))^T\overline{\hat{G}^{p,\varepsilon}(\xi,0;y)}d\xi \\
&:= F^{sp}(z,y) + R^{sp}(z,y)
\end{aligned}
$$

$$
\begin{aligned}
J^{ps}(z,y) &= \lim_{\varepsilon\to 0^+} \int_R (T_D^p(x_1,0;z_1,z_2))^T\overline{G^{s,\varepsilon}(x_1,0;y_1,y_2)}dx_1 \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_R (\hat{T}_D^p(\xi,0;z))^T\overline{\hat{G}^{s,\varepsilon}(\xi,0;y)}d\xi \\
&= \frac{1}{2\pi}\int_{-k_p}^{k_p} (\hat{T}_D^p(\xi,0;z))^T\overline{\hat{G}^{s,\varepsilon}(\xi,0;y)}d\xi \\
&+ \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{R\backslash[-k_p,k_p]} (\hat{T}_D^p(\xi,0;z))^T\overline{\hat{G}^{s,\varepsilon}(\xi,0;y)}d\xi \\
&:= F^{ps}(z,y) + R^{ps}(z,y)
\end{aligned}
$$

and by the process of obtaining Neumann Green Tensor

$$
\begin{aligned}
\overline{R^{ss}(y,z)} &= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{R\backslash[-k_s,k_s]} \overline{(\hat{T}_D^s(\xi,0;z))^T}\hat{G}^{s,\varepsilon}(\xi,0;y)d\xi \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{R\backslash[-k_s,k_s]} (\Theta^s(\xi;z))^T\hat{G}^{s,\varepsilon}(\xi,0;y)d\xi \\
&= \frac{1}{2\pi}\int_{\Gamma_l^\pm\cup\Gamma_r^\pm} (\Theta^s(\xi;z))^T\hat{G}^{s}(\xi,0;y)d\xi + ResiduePart \\
&:= \mathrm{I}^{ss}(z,y) + \mathrm{II}^{ss}(z,y)
\end{aligned}
$$

$$
\begin{aligned}
\overline{R^{pp}(y,z)} &= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{R\backslash[-k_p,k_p]} \overline{(\hat{T}_D^p(\xi,0;z))^T}\hat{G}^{p,\varepsilon}(\xi,0;y)d\xi \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{R\backslash[-k_p,k_p]} (\Theta^p(\xi;z))^T\hat{G}^{p,\varepsilon}(\xi,0;y)d\xi \\
&= \frac{1}{2\pi}\int_{\Gamma_l^\pm\cup\Gamma_r^\pm} (\Theta^p(\xi;z))^T\hat{G}^{p}(\xi,0;y)d\xi \\
&+ \frac{1}{2\pi}\int_{(-k_s,-k_p)\cup(k_p,k_s)} \overline{(T^p(\xi;z))^T}\hat{G}^{p}(\xi,0;y)d\xi + ResiduePart \\
&:= \mathrm{I}^{pp}(z,y) + \mathrm{II}^{pp}(z,y) + \mathrm{III}^{pp}(z,y)
\end{aligned}
$$

$$\begin{aligned}
\overline{R^{sp}(y,z)} =& \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{R\backslash[-k_p,k_p]} \overline{(\hat{T}_D^s(\xi,0;z))^T}\hat{G}^{p,\varepsilon}(\xi,0;y)d\xi \\
=& \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{R\backslash[-k_p,k_p]} (\Theta^s(\xi;z))^T\hat{G}^{p,\varepsilon}(\xi,0;y)d\xi \\
=& \frac{1}{2\pi}\int_{\Gamma_l^\pm\cup\Gamma_r^\pm} (\Theta^s(\xi;z))^T\hat{G}^{p}(\xi,0;y)d\xi \\
&+ \frac{1}{2\pi}\int_{(-k_s,-k_p)\cup(k_p,k_s)} \overline{(T^s(\xi;z))^T}\hat{G}^{p}(\xi,0;y)d\xi + ResiduePart \\
:=& \mathrm{I}^{sp}(z,y)+\mathrm{II}^{sp}(z,y)+\mathrm{III}^{sp}(z,y)
\end{aligned}$$

$$\begin{aligned}
\overline{R^{ps}(y,z)} =& \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{R\backslash[-k_p,k_p]} \overline{(\hat{T}_D^p(\xi,0;z))^T}\hat{G}^{s,\varepsilon}(\xi,0;y)d\xi \\
=& \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{R\backslash[-k_p,k_p]} (\Theta^p(\xi;z))^T\hat{G}^{s,\varepsilon}(\xi,0;y)d\xi \\
=& \frac{1}{2\pi}\int_{\Gamma_l^\pm\cup\Gamma_r^\pm} (\Theta^p(\xi;z))^T\hat{G}^{s}(\xi,0;y)d\xi \\
&+ \frac{1}{2\pi}\int_{(-k_s,-k_p)\cup(k_p,k_s)} \overline{(T^p(\xi;z))^T}\hat{G}^{s}(\xi,0;y)d\xi + ResiduePart \\
:=& \mathrm{I}^{ps}(z,y)+\mathrm{II}^{ps}(z,y)+\mathrm{III}^{ps}(z,y)
\end{aligned}$$

where $\pm$ are corresponding $sgn(z_1-y_1)$. In the sequel, A^{ij} denotes the (i,j) element of a 2×2 matrix.

Lemma 5.1 *For any $z,y\in\mathbb{R}^2_+$,*

$$|\mathrm{I}_{ij}^{\alpha\beta}(x,y)| \le \frac{C}{\mu}\sum_{j=1}^{4}(k_s(y_2+z_2))^{-j},\ \ \alpha,\beta=s,p \tag{5.5}$$

where C is only dependent on $\kappa := k_p/k_s$.

Proof. Substituting (5.3) and (2.32) into I^{ss}, we have

$$\begin{aligned}
\mathrm{I}^{ss}(z,y) =& \frac{1}{2\pi}\int_{\Gamma_l^\pm\cup\Gamma_r^\pm} \frac{\mathbf{i}(k_s^2-4\xi^2)}{\mu\gamma(\xi)\delta(\xi)}\begin{pmatrix} \mu_s^2\mu_p & \xi\mu_s\mu_p \\ -\xi\mu_s\mu_p & -\xi^2\mu_p \end{pmatrix} e^{\mathbf{i}\mu_s(z_2+y_2)+\mathbf{i}\xi(z_1-y_1)} \\
=& A_l + A_r
\end{aligned}$$

where A_r, A_l are respected the integrantion on the Γ_r^+, Γ_l^+. Now, we turn to estimation of I^{ss}. The integration path Γ_r^+ can be parameterized by $\xi := k_s q(t) = k_s\sqrt{t^2+1}+\mathrm{i}k_s t$, with $0<t<+\infty$. We define $\gamma_p(t) := (\kappa^2-q(t)^2)^{1/2}$, $\gamma_s(t) = (1-q(t)^2)^{1/2}$, and $\tau(t) := 1-2t^2$. It is well know that $\mu_\alpha = k_s\gamma_\alpha(t)$, $\beta(\xi) = k_s^2\tau(t)$, $\gamma(\xi) = k_s^2(q(t)^2+\gamma_p(t)\gamma_s(t))$ and $\delta(\xi) = k_s^4(\tau(t)^2+4q(t)^2\gamma_p(t)\gamma_s(t))$ when ξ on Γ_r^+. Now, substiting the parameterized representation of ξ on Γ_r^+, we obtain

$$A_r^{11} = \int_0^{+\infty}\frac{A(t)}{\mu}e^{\mathbf{i}k_s[\gamma_s(t)(z_2+y_2)+q(t)(z_1-y_1)]}dt \tag{5.6}$$

where $A(t) = \frac{\mathbf{i}(1-4q(t)^2)\gamma_s(t)^2\gamma_p(t)(\mathbf{i}+t/\sqrt{1+t^2})}{2\pi(q(t)^2+\gamma_p(t)\gamma_s(t))(\tau(t)^2+4q(t)^2\gamma_p(t)\gamma_s(t))}$.

A simple computation show that $A(t) = O(t^3),\ t \to +\infty$. It is obvious that there exit $T, C > 0$ which only dependent on κ

$$|A(t)| \le Ct^3 \tag{5.7}$$

when $t > T$. By the convention in (2.10), it is easy to see

$$\gamma_s(t) = (1+t^2)^{1/4} t^{1/2}(-1+\mathbf{i}) \tag{5.8}$$

Cause $\delta(\xi)$ and $\gamma(\xi)$ have no zero point on Γ_r^+, we have

$$\begin{aligned}
|A_r| &\le \frac{1}{\mu}\int_0^{+\infty} |A(t)| e^{-ks\mathrm{Im}\,\gamma_s(t)(z_2+y_2)} dt \\
&\le \frac{1}{\mu}\int_0^{T} \max_{[0,T]} A(t) e^{-ks\mathrm{Im}\,\gamma_s(t)(z_2+y_2)} dt + \frac{C}{\mu}\int_T^{+\infty} t^3 e^{-ks\mathrm{Im}\,\gamma_s(t)(z_2+y_2)} dt \\
&= \frac{\max_{[0,T]} A(t)}{\mu}\int_0^{T} e^{-ks(1+t^2)^{1/4}t^{1/2}(z_2+y_2)} dt + \frac{C}{\mu}\int_T^{+\infty} t^3 e^{-ks(1+t^2)^{1/4}t^{1/2}(z_2+y_2)} dt \\
&\le \frac{\max_{[0,T]} A(t)}{\mu}\int_0^{T} e^{-kst(z_2+y_2)} dt + \frac{C}{\mu}\int_T^{+\infty} t^3 e^{-kst(z_2+y_2)} dt \\
&\le \frac{C}{\mu}\sum_{j=1}^{4} (k_s(y_2+z_2))^{-j}
\end{aligned}$$

where we use integration by parts for last inequatily. The estimate of A_l and the case of $y_1 - z_1 < 0$ can be proved similarly. Thus, we obtain (5.5) when $i = j = 1, \alpha = \beta = s$. The estimation of other term $\mathrm{I}_{\alpha\beta}^{ij}(z,y)$ can be proved similarly via integration by parts argument and the fact that $\mathrm{Im}\,\gamma_s(t) \le \mathrm{Im}\,\gamma_p(t),\ t > 0$. This completes the proof. □

Lemma 5.2 *For any $z, y \in \mathbb{R}_+^2$,*

$$|\mathrm{II}_{ij}^{pp}(x,y)| \le \frac{C}{\mu k_s(y_2+z_2)} \tag{5.9}$$

$$|\mathrm{II}_{ij}^{sp}(x,y)| \le \frac{C}{\mu k_s y_2} \tag{5.10}$$

$$|\mathrm{II}_{ij}^{ps}(x,y)| \le \frac{C}{\mu k_s z_2} \tag{5.11}$$

where C is only dependent on $\kappa := k_p/k_s$.

Proof. Substituting (5.3) and (2.32) into II^{pp}, we have

$$\mathrm{II}^{pp}(z,y) = \frac{1}{2\pi}\int_{(-k_s,-k_p)\cup(k_p,k_s)} \frac{\mathbf{i}k_s^2}{\mu\overline{\gamma(\xi)}\delta(\xi)} \begin{pmatrix} \xi^2\mu_s & -\xi\mu_s\mu_p \\ \xi\mu_s\mu_p & -\mu_s\mu_p^2 \end{pmatrix} e^{\mathrm{i}\mu_p(z_2+y_2)+\mathbf{i}\xi(z_1-y_1)}$$

let $\xi = k_s t$, we have

$$\begin{aligned}
|\mathrm{II}_{11}^{pp}| &\le \int_\kappa^1 \frac{\sqrt{(1-t^2)}t^2}{\pi\mu|t^2 - \mathbf{i}\sqrt{(t^2-\kappa^2)}\sqrt{(1-t^2)}|(1-2t^2)^2 + \mathbf{i}4t^2\sqrt{(t^2-\kappa^2)}\sqrt{(1-t^2)}|} e^{-k_s\sqrt{t^2-\kappa^2}(z_2+y_2)} dt \\
&\le \frac{C}{\mu}\int_0^{1-\kappa} e^{t(z_2+y_2)} dt \le \frac{C}{\mu k_s(y_2+z_2)}
\end{aligned}$$

where we use the fact that $\gamma(\xi), \delta(\xi)$ have no roots on interval $[k_p, k_s]$, then we can get supremum of amplitude function. The method of estimating other terms are actually same, here we omit detials. This completes the proof. $\square$

Lemma 5.3 *For any $z, y \in \mathbb{R}^2_+$,*

$$|\mathrm{II}^{ss}_{ij}(x,y)| \le \frac{C}{\mu} e^{-\sqrt{k_R^2-k_s^2}(y_2+z_2)} \tag{5.12}$$

$$|\mathrm{III}^{pp}_{ij}(x,y)| \le \frac{C}{\mu} e^{-\sqrt{k_R^2-k_p^2}(y_2+z_2)} \tag{5.13}$$

$$|\mathrm{III}^{sp}_{ij}(x,y)| \le \frac{C}{\mu} e^{-\sqrt{k_R^2-k_s^2}z_2-\sqrt{k_R^2-k_p^2}y_2} \tag{5.14}$$

$$|\mathrm{III}^{ps}_{ij}(x,y)| \le \frac{C}{\mu} e^{-\sqrt{k_R^2-k_p^2}z_2-\sqrt{k_R^2-k_s^2}y_2} \tag{5.15}$$

where C is only dependent on $\kappa := k_p/k_s$.

Proof. When $z_1 - y_1 > 0$, we have

$$\begin{aligned} \mathrm{II}^{ss}_{11} &= -\frac{1}{\mu} Res_{\xi=k_R} \frac{(k_s^2-4\xi^2)\mu_s^2\mu_p}{\gamma(\xi)\delta(\xi)} e^{\mathbf{i}\mu_s(z_2+y_2)+\mathbf{i}\xi(z_1-y_1)} \\ &= -\frac{(k_s^2-4\xi^2)\mu_s^2\mu_p}{\mu(\gamma(\xi)\delta(\xi))'} e^{\mathbf{i}\mu_s(z_2+y_2)+\mathbf{i}\xi(z_1-y_1)}|_{\xi=k_R} \end{aligned}$$

Eliminating k_s in fraction, we can obtain estimation (5.12). The other terms can be estimated similarly. This completes the proof. $\square$

Now, it turn to estimate $F^{sp}(z,y)$ and $F^{ps}(z,y)$.

Lemma 5.4 *For any $z, y \in \mathbb{R}^2_+$,*

$$|F^{sp}_{ij}| \le \frac{C}{\mu}(1+k_s|z_1-y_1|)\left(\frac{1}{k_s z_2} + \frac{1}{(k_s z_2)^{1/4}} + \frac{1}{(k_s z_2)^{1/(2k_{c_3}-1)}}\right) \tag{5.16}$$

$$|F^{ps}_{ij}| \le \frac{C}{\mu}(1+k_s|z_1-y_1|)\left(\frac{1}{k_s y_2} + \frac{1}{(k_s y_2)^{1/4}} + \frac{1}{(k_s y_2)^{1/(2k_{c_3}-1)}}\right) \tag{5.17}$$

where C is only dependent on κ, c_3 and k_{c_3} is the minimum of k which satisfy $\kappa^{2k-1} < 1/c_3$.

Proof. Substituting (5.3) and (2.32) into F^{ps} and F^{sp}, we have

$$F^{ps}(z,y) = \frac{1}{2\pi}\int_{(-k_p,k_p)} \frac{-\mathbf{i}(\beta-2\mu_s\mu_p)\xi}{\mu\gamma(\xi)\delta(\xi)} \begin{pmatrix} -\xi\mu_s & \mu_s\mu_p \\ -\xi^2 & \xi\mu_p \end{pmatrix} e^{\mathbf{i}\mu_s z_2 - \mathbf{i}\mu_p y_2 - \mathbf{i}\xi(z_1-y_1)}$$

$$F^{sp}(z,y) = \frac{1}{2\pi}\int_{(-k_p,k_p)} \frac{-\mathbf{i}(\beta-2\mu_s\mu_p)\xi}{\mu\gamma(\xi)\delta(\xi)} \begin{pmatrix} \xi\mu_s & -\mu_s\mu_p \\ \xi^2 & -\xi\mu_p \end{pmatrix} e^{\mathbf{i}\mu_p z_2 - \mathbf{i}\mu_s y_2 - \mathbf{i}\xi(z_1-y_1)}$$

We only estimate $F^{ps}(z,y)$ here, the other terms can be proved in the same way. let $\xi = k_s t$, we have

$$F^{ps}_{11}(z,y) = \frac{1}{\mu}\int_{-\kappa}^{\kappa} f(t,\kappa) e^{\mathbf{i}k_s t(y_1-z_1)} e^{\mathbf{i}k_s z_2 \phi(t,\tau)} dt$$

where $\phi(t,\tau) = (\sqrt{1-t^2} - \tau\sqrt{\kappa^2 - t^2})$ and $\tau = y_2/z_2$. From the convention (4.1) we have $1/c_3 < \tau < c_3$. A simple computation show that

$$\begin{aligned}
\phi^{(1)}(t,\tau) &= -\frac{t}{\sqrt{1-t^2}} + \frac{\tau t}{\sqrt{\kappa^2 - t^2}} \\
\phi^{(2)}(t,\tau) &= -\frac{1}{(1-t^2)^{3/2}} + \frac{\tau\kappa^2}{(\kappa^2 - t^2)^{3/2}} \\
\phi^{(3)}(t,\tau) &= -\frac{3t}{(1-t^2)^{5/2}} + \frac{3\tau\kappa^2 t}{(\kappa^2 - t^2)^{5/2}} \\
\phi^{(4)}(t,\tau) &= -\frac{12t^2 + 3}{(1-t^2)^{7/2}} + \frac{\tau(12\kappa^2 t^2 + 3\kappa^4)}{(\kappa^2 - t^2)^{7/2}}
\end{aligned}$$

Moreover,

$$\phi^{(n)}(t,\tau) = \psi(t) - \tau/\kappa^{n-1}\psi(t/\kappa) \tag{5.18}$$

where $\phi^{(n)}(t,\tau) = \frac{\partial^n \phi(t,\tau)}{\partial t^n}$ and

$$\begin{aligned}
\psi(t) &= \sum_{k=0}^{n} \frac{(-1)^k 2^{2k-n} t^{2k-n} (1-t^2)^{1/2-k} (3/2-k)_k (1+2k-n)_{2(-k+n)}}{(-k+n)!} \\
&= 2^n (3/2-n)_n (-t)^n (1-t^2)^{1/2-n} \ {}_2F_1((1-n)/2, -n/2; 3/2-n; (1-1/t^2))
\end{aligned}$$

Here ${}_2F_1(a,b;c;x)$ is the hypergeometric function defined by

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} (-1)^k \frac{a_k b_k}{c_k} \frac{x^k}{k!}$$

and q_n is the (rising) Pochhammer symbol, which is defined by

$$q_n = \begin{cases} 1 & n = 0, \\ q(q+1)\cdots(q+n-1) & n > 0. \end{cases}$$

Solving $\phi'(t) = 0$, we have t=0 or $t^2 = (\tau^2 - \kappa^2)/(\tau^2 - 1)$. Because $0 < \kappa < 1$, we can assert that $\phi'(t)$ has only one root 0 in $(-\kappa,\kappa)$ when $\tau \geq \kappa$ and it has three roots, 0 and $\pm((\kappa^2 - \tau^2)/(1-\tau^2))^{1/2}$, in $(-\kappa,\kappa)$ when $\tau < \kappa$.

We are now in the position to analysis the case of $\tau \geq \kappa$. Substitute 0 for t in formula (5.18), we obtain

$$\phi^{(n)}(0,\tau) = \begin{cases} 0 & \text{n is odd}, \\ (\tau/\kappa^{n-1} - 1)(1/2 - n/2)_{n/2}(3/2 - n/2)_{n/2} 2^n & \text{n is even}. \end{cases} \tag{5.19}$$

It is trivial to see that $\phi^{(2)}(0,\tau) = \tau/\kappa - 1$, $\phi^{(3)}(0,\tau) = 0$ and $\phi^{(4)}(0,\tau) = 3(\tau/\kappa^3 - 1)$. Using the fact $\kappa < 1$ we obtain $|\phi^{(4)}(0,\tau)| \geq 3(1/\kappa^2 - 1)$. Furthermore, there exists a $0 < \delta_\kappa < \kappa$, which is independent of τ, such that $|\phi^{(4)}(t,\tau)| \geq 2(1/\kappa^2 - 1)$ when $t \in [-\delta_\kappa, \delta_\kappa]$. Let $m_\kappa = \min_{([-\kappa,\kappa]\backslash(-\delta_\kappa,\delta_\kappa))\times[0,c_3]} \phi^{(1)}(t,\tau)$. Since $\phi^{(2)}(t,\tau)$ has at most 4 zero points with respect to t, then the set $[-\kappa,\kappa]\backslash(-\delta_\kappa,\delta_\kappa)$ is the union of at most 5 intervals on each of which u' is monotone‘. Thus the estimation of $F^{ps}(z,y)$ follows immediately from lemma 3.3.

It remains to analysis the case of $1/c_3 < \tau < \kappa$. Let $t_\tau = ((\kappa^2-\tau^2)/(1-\tau^2))^{1/2}$. By substituting $\pm t_\tau$ into $\phi^{(4)}(t,\tau)$, we obtain

$$\phi^{(4)}(\pm t_\tau, \tau) = \frac{3(5\tau^8 - \tau^6 - 4\kappa^2\tau^6 - \kappa^4\tau^2 - 4\kappa^2\tau^2 + 5\kappa^4)}{\tau^6(1-\tau^2)} \tag{5.20}$$

$$= \frac{(4\kappa^2 - 4\tau^4)(\kappa^2-\tau^2) + (\kappa^4 - \tau^6)(1-\tau^2)}{\tau^6(1-\tau^2)} \tag{5.21}$$

$$\geq \frac{\kappa^4(1-\tau^2)}{\tau^6(1-\tau^2)} \geq \frac{1}{\kappa^2} \tag{5.22}$$

It is easy to see that $\phi^{(2k_{c_3})}(0,\tau) \geq C(\frac{1}{c_3\kappa^{2k_{c_3}-1}} - 1)$ due to $\kappa^{2k_{c_3}-1} < 1/c_3$. Consequently, the estimation here can be proved by the same method as employed in the last case. This completes the proof. □

To complete the analysis of the point spread function, Let $F(z,y) = F_{ss}(z,y) + F_{pp}(z,y)$, where

$$F^{pp}(z,y) = -\frac{1}{2\pi}\int_{(-k_p,-k_p)} \frac{\mathbf{i}k_s^2\mu_s}{\mu\gamma(\xi)\delta(\xi)} \begin{pmatrix} \xi^2 & -\xi\mu_p \\ -\xi\mu_p & \mu_p^2 \end{pmatrix} e^{\mathrm{i}\mu_p(z_2-y_2)+\mathbf{i}\xi(y_1-z_1)}$$

$$\begin{aligned} F^{ss}(z,y) &= -\frac{1}{2\pi}\int_{(-k_p,-k_p)} \frac{\mathbf{i}k_s^2\mu_p}{\mu\gamma(\xi)\delta(\xi)} \begin{pmatrix} \mu_s^2 & \xi\mu_s \\ \xi\mu_s & \xi^2 \end{pmatrix} e^{\mathrm{i}\mu_p(z_2-y_2)+\mathbf{i}\xi(y_1-z_1)} \\ &\quad -\frac{1}{2\pi}\int_{(-k_s,k_s)\setminus(-k_p,k_p)} \frac{\mathbf{i}(k_s^2-4\xi^2)\mu_p}{\mu\gamma(\xi)\overline{\delta(\xi)}} \begin{pmatrix} \mu_s^2 & \xi\mu_s \\ \xi\mu_s & \xi^2 \end{pmatrix} e^{\mathrm{i}\mu_s(z_2-y_2)+\mathbf{i}\xi(y_1-z_1)} \\ &:= F^{ss1}(z,y) + F^{ss2}(z,y) \end{aligned}$$

and $R(z,y) = R^{ss}(z,y) + R^{pp}(z,y) + J^{sp}(z,y) + J^{ps}(z,y)$. Then we have $J(z,y) = F(z,y) + R(z,y)$. By the lemma 5.1-5.3 and lemma 5.4, the main contribution to the point spread function is from $F(z,y)$ when z, y far away from Γ_0. Based on the above argument, we know that $R(z,y)$ becomes small when z,y move away from Γ_0. Our goal is to show $F(z,y)$ has the similar decay to the elastic fundamental solution $\operatorname{Im}\Phi(z,y)$ as $|z-y| \to \infty$.

Lemma 5.5 *For any $z, y \in \mathbb{R}^2_+$, when $z = y$*

$$\begin{aligned} |\operatorname{Im} F_{ii}(z,y)| &\geq \frac{1}{4(\lambda+2\mu)}\ ,\ i = 1,2 \\ \operatorname{Im} F_{12}(z,y) &= \operatorname{Im} F_{21}(z,y) = 0 \end{aligned}$$

and for $z \neq y$

$$|F_{ij}(z,y)| \leq \frac{C}{\mu}[(k_s|z-y|)^{-1/2}) + (k_s|z-y|^{-1})]$$

where constant C is only dependent on κ.

Proof. We only proof the case of $i = 1$, the other ones are similar. First, we have $\gamma(\xi) \leq k_s^2$, $\delta(\xi) \leq k_s^4$ and $\mu_p \leq \mu_s$ when $\xi \in (-k_p, k_p)$. Then, if $z = y$

$$-\operatorname{Im}(F_{11}^{pp} + F_{11}^{ss1}) \geq \frac{1}{2\pi\mu}\int_{(-k_p,k_p)} \frac{\mu_p}{k_s^2} d\xi \tag{5.23}$$

$$= \frac{k_p^2}{2\pi\mu k_s^2}\int_0^\pi \sin^2(t)dt = \frac{1}{4(\lambda+2\mu)} \tag{5.24}$$

It's left to proof $-\mathrm{Im}\, F_{11}^{ss2} > 0$. If $\xi \in (-k_s, k_s)\backslash(-k_p, k_p)$, $\mu_p = \mathbf{i}\sqrt{\xi^2 - k_p^2}$. Substituting it into F^{ss2}, we have

$$F_{11}^{ss2} = \frac{1}{2\pi\mu}\int_{(-k_s,k_s)\backslash(-k_p,k_p)} \frac{\mu_s^2\sqrt{\xi^2-k_p^2}(k_s^2-4\xi^2)}{(\xi^2+\mathbf{i}\mu_s\sqrt{\xi^2-k_p^2})(\beta^2-\mathbf{i}4\xi^2\mu_s\sqrt{\xi^2-k_p^2})}d\xi \tag{5.25}$$

let $\alpha = (\xi^2+\mathbf{i}\mu_s\sqrt{\xi^2-k_p^2})(\beta^2-\mathbf{i}4\xi^2\mu_s\sqrt{\xi^2-k_p^2})$. A simple computation show that $\mathrm{Im}\,\alpha = k_s^2\mu_s\sqrt{\xi^2-k_p^2}(k_s^2-4\xi^2)$. It is easy to see that

$$-\mathrm{Im}\, F_{11}^{ss2} = \frac{k_s^2}{2\pi\mu}\int_{(-k_s,k_s)\backslash(-k_p,k_p)} \frac{\mu_s^3(\xi^2-k_p^2)(k_s^2-\xi^2)^2}{|\alpha|^2}d\xi > 0$$

For $z \neq y$, we denot $y - z = |y-z|(\cos\phi, \sin\phi)^T$ for some $0 \le \phi \le 2\pi$. Then it is easy to see that

$$F^{pp}(z,y) = \frac{1}{\mu}\int_0^\pi A(\theta,\kappa)e^{\mathbf{i}k_s|z-y|\cos(\theta-\phi)}$$

The phase function $f(\theta) = \cos(\theta-\phi)$ satisfies $f'(\theta) = -\sin(\theta-\phi), f''(\theta) = -\cos(\theta-\phi)$. For any given $0 \le \phi \le 2\pi$, we can decompose $[0,\pi]$ into several intervals such that in each either $|f''(\theta)| \ge 1/2$ or $|f'(\theta)| \ge 1/2$ and $f'(\theta)$ is monotonous. The amplitude function $A(\theta,\kappa)$ and their derirates are integrable on $[0,\pi]$. Then the estimate for $F_{pp}(z,y)$ follows by using lemma 3.3. The estimation of $F^{ss}(z,y)$ can be proved similarly. This completes the proof. □

Now we consider the finite aperture point spread function $J_d(z,y)$:

$$\int_{-d}^{d} (T_D(x_1,0;z_1,z_2))^T\overline{G(x_1,0;y_1,y_2)}dx_1 \tag{5.26}$$

Our aim is to estimate the difference $J(z,y) - J_d(z,y)$. It is easy to see that

$$\frac{(x_1-z_1)^2}{\rho^2} = \frac{1}{1+\frac{z_2^2}{(x_1-z_1)^2}} \ge \frac{1}{1+\frac{c_3^2h^2}{(1-c_1)^2d^2}} := m(h/d) \tag{5.27}$$

$$\frac{z_2^2}{\rho^2} = \frac{1}{1+\frac{(x_1-z_1)^2}{z_2^2}} \le \frac{1}{1+\frac{(1-c_1)^2d^2}{c_3^2h^2}} := M(h/d) \tag{5.28}$$

where $\rho = \sqrt{(x_1-z_1)^2+z_2^2}$ and $z \in \Omega, x \in \Gamma_0\backslash(-d,d)$.

Theorem 5.1 *Assume $m(h/d) > (1+\kappa)^2/4$, $M(h/d) < \kappa^2/4$ and $k_sh \ge 1$. Then for $z, y \in \Omega$, we have*

$$|J(z,y) - J_d(z,y)| + k_s^{-1}|\nabla_y(J(z,y) - J_d(z,y))| \le \frac{C}{\mu}(\frac{h}{d} + \frac{(k_sh)^{1/2}}{e^{\sqrt{k_r^2-k_s^2}h}}(\frac{h}{d})^{1/2}) \tag{5.29}$$

where the constant C is only dependent on κ.

Proof. By lemma 2.5, lemma 3.5 and $k_s h \geq 1$, we have

$$
\begin{aligned}
&\Big|\int_d^\infty (T_D(x_1,0;z_1,z_2))^T\overline{G(x_1,0;y_1,y_2)}dx_1\Big| \\
&\leq \frac{C}{\mu}\int_d^\infty \frac{k_s z_2}{|x-z|}\frac{1}{|x-z|^{1/2}}\Big(\frac{1}{|x-y|^{1/2}}+e^{-\sqrt{k_r^2-k_s^2}y_2}\Big)dx_1 \\
&\leq \frac{C}{\mu}\int_{(1-c_1)d/h}^\infty \frac{1}{1+t^2}+\frac{(k_s h)^{1/2}}{(1+t^2)^{3/4}}e^{-\sqrt{k_r^2-k_s^2}h}dt \\
&\leq \frac{C}{\mu}(\frac{h}{d}+\frac{(k_s h)^{1/2}}{e^{\sqrt{k_r^2-k_s^2}h}}(\frac{h}{d})^{1/2})
\end{aligned}
$$

Here we have used the first inequeality in (4.1). Similarly, we can prove that the estimate for te integral in $[-\infty,-d]$. This shows the estimate for $J(z,y)-J_d(z,y)$. The estimate for $\nabla_y(J(z,y)-J_d(z,y))$ can be proved similarly. □

6. Analysis of the forward scattering problem

In this section we introduce the following stability estimate of the forward elastic scattering problem in the half space which can be proved by the limiting absorption principle by extending the classical argument in [11, 15, 7]. Let the obstacle occupy a bounded Lipschitz domain $D\subset\mathbb{R}^2_+$.

Theorem 6.1 *Let $g\in H^{1/2}(\Gamma_D)$, then the scattering problem of elastic equation in the half space*

$$\Delta_e u+\omega^2 u=0 \qquad \text{in } \mathbb{R}^2_+\backslash\bar{D}, \tag{6.1}$$

$$u=g \quad \text{ on } \Gamma_D, \tag{6.2}$$

$$\sigma(u)e_2=0 \quad \text{on}\Gamma_0, \tag{6.3}$$

u satisfies the generalized radiation codition[12] such that

$$\lim_{r\to\infty}\int_{S_r^+}(\sigma(G(x,y)e_i)\hat{r})\cdot u(x)-(G(x,y)e_i)\cdot(\sigma(u)\hat{r})ds(x)=0 \tag{6.4}$$

where $S_r^+:=\{x\in\mathbb{R}^2_+\mid \|x\|=r^2\}$, $\hat{r}=x/r$ and $y\in\mathbb{R}^2_+$. Then the problem (6.1)-(6.4) admits a unique solution $u\in H^1_{\rm loc}(\mathbb{R}^2_+\backslash\bar{D})$. Moreover, for any bounded open set $\mathcal{O}\subset\mathbb{R}^2_+\backslash\bar{D}$ there exists a constant $C>0$ such that

$$\|u\|_{H^1(\mathcal{O})}\leq C\|g\|_{H^{-1/2}(\Gamma_D)} \tag{6.5}$$

The existence of the solution can be proved by the method of limiting absorption principle. The argument is standard and we give several lemmas below, see e.g. [11] for the consideration for Helmholtz equation. For any $z=1+\mathbf{i}\varepsilon,\varepsilon>0$, $f\in H^1(\mathbb{R}^2_+)'$ with compact support in $B_R=\{x||x|^2<R^2,x\in\mathbb{R}^2_+\}\subsetneq\mathbb{R}^2_+$ where B_R is a disk of radius R , we consider the problem

$$\Delta_e u_z+z\omega^2 u=-f \qquad \text{in } \mathbb{R}^2_+ \tag{6.6}$$

$$\sigma(u_z)e_2=0 \quad \text{ on } \quad \Gamma_0 \tag{6.7}$$

By Lax-Milgrim lemma we know that (6.6-6.7) has a unique solution $u_z \in H^1(\mathbb{R}^2_+)$. For any domain $\mathcal{D} \subset \mathbb{R}^2_+$, we define the weighted space $L^{2,s}(\mathcal{D}), s \in \mathbb{R}$, by

$$L^{2,s}(\mathcal{D}) = \{v \in L^2_{\text{loc}}(\mathcal{D}) : (1+|x|^2)^{s/2}v \in L^2(\mathcal{D})\}$$

with the norm $\|v\|_{L^{2,s}(\mathcal{D})} = (\int_{\mathcal{D}}(1+|x|^2)^s|v|^2dx)^{1/2}$. The weighted Sobolev space $H^{1,s}(\mathcal{D}), s \in \mathbb{R}$, is defined as the set of functions in $L^{2,s}(\mathcal{D})$ whose first derivative is also in $L^{2,s}(\mathcal{D})$. The norm $\|v\|_{H^{1,s}(\mathcal{D})} = (\|v\|^2_{L^{2,s}(\mathcal{D})} + \|\nabla v\|^2_{L^{2,s}(\mathcal{D})})^{1/2}$.

We need the following sligt generalization of Rellich Theorem:

Lemma 6.1 *Let Ω be an open Lipschitz domain, then the sobolev space $H^{1,-s}(\Omega)$ is compactly embeded in $L^{2,-s'}(\Omega)$ for every $s' > s > 0$.*

Lemma 6.2 *Let $f \in L^2(\mathbb{R}^2_+)$ with compact support in B_R. For any $z = 1+\mathbf{i}\varepsilon$, $0 < \varepsilon < 1$, we have, for any $s > 1/2$, $\|u_z\|_{H^{1,-s}(\mathbb{R}^2_+)} \leq C\|f\|_{L^2(\mathbb{R}^2_+)}$ for some constant independent of ε, u_z, and f.*

Proof. Let R_z denote the map from $L^2_c(\mathbb{R}^2_+)$ to $H^{1,-s}(\mathbb{R}^2_+)$ such that $R_z(f) = u_z$ where $L^2_c(\mathbb{R}^2_+)$ is denoted by all $f \in L^2(\mathbb{R}^2_+)$ with compact support in B_R, then it is easy to see that R_z is a linear bounded operator. It follows from theorem 3.7 in [7] that R_z is a uniformly continuous operator continues valued function on $z = 1+\mathbf{i}\varepsilon$, $0 < \varepsilon < 1$ with value in $B(L^2_c(\mathbb{R}^2_+), H^{1,-s}(\mathbb{R}^2_+))$. Then, we can obtain that R_z is uniformly bounded in $B(L^2_c(\mathbb{R}^2_+), H^{1,-s}(\mathbb{R}^2_+))$. This complete the proof by the defintion of the operator norm. $\square$

We next recall the following lemma which states the absence of positive eigenvalues for the linear elasticity system in half space [13].

Lemma 6.3 *Let $u \in L^2(\mathbb{R}^2_+\backslash\bar{D})$ such that u satisfies (6.1) and (6.3), than we assert that $u = 0$ in $\mathbb{R}^2_+\backslash\bar{D}$*

Proof. The asserting above can be proved by extending [13, theorem 3.1], here we omit the details. $\square$

For any $0 < \varepsilon < 1$, we consider the problem

$$\Delta_e u_\varepsilon + (1+\mathbf{i}\varepsilon)\omega^2 u_\varepsilon = 0 \qquad \text{in } \mathbb{R}^2_+\backslash\bar{D} \tag{6.8}$$

$$u_\varepsilon = g \quad \text{on } \Gamma_D \tag{6.9}$$

$$\sigma(u_\varepsilon)e_2 = 0 \quad \text{on}\Gamma_0 \tag{6.10}$$

We know that the above problem has a unique solution $u_\varepsilon \in H^1(\mathbb{R}^2_+\backslash\bar{D})$ by the Lax-Milgram Lemma. Thus, we have next lemma

Lemma 6.4 *Let $g \in H^{1/2}(\Gamma_D)$. For any $0 < \varepsilon < 1$, we have, for any $s > 1/2$, $\|u_\varepsilon\|_{H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})} \leq C\|g\|_{H^{1/2}(\Gamma_D)}$ for some constant independent of $\varepsilon, u_\varepsilon$, and g.*

Proof. Because $h = dist(D, \Gamma_0) > 0$, we can find three concentric circles $B_{R_1}, B_{R_2}, B_{R_3}$ such that $D \subsetneq B_{R_1} \subsetneq B_{R_2} \subsetneq B_{R_3} \subsetneq \mathbb{R}^2_+$. Let $\chi \in C_0^\infty(\mathbb{R}^2_+)$ be the cut-off function such that $0 \leq \chi \leq 1$, $\chi = 0$ in B_{R_1}, and $\chi = 1$ outside of B_{R_2}. Let $v_\varepsilon = \chi u_\varepsilon$. Then v_ε satisfies (6.6) with $z = 1 + \mathbf{i}\varepsilon$ and $q = \sigma(u_\varepsilon)\nabla\chi + (\lambda+\mu)(\nabla^2\chi u_\varepsilon + \nabla u_\varepsilon \nabla\chi) + \mu\Delta\chi u_\varepsilon + \mu \mathrm{div} u_\varepsilon \nabla\chi$, where $\nabla^2\chi$ is the Hessian matrix of χ. Clearly q has compact support. By lemma 6.2 we can obtain

$$\|v_\varepsilon\|_{H^{1,-s}(\mathbb{R}^2_+)} \leq C\|u_\varepsilon\|_{H^1(B_{R_2}\backslash\bar{D})} \tag{6.11}$$

for some constant C independent of $\varepsilon > 0$. Now let $\chi_1 \in C_0^\infty(\mathbb{R}^2_+)$ be the cut-off function with that $0 \leq \chi_1 \leq 1$, $\chi_1 = 1$ in B_{R_2}, and $\chi_1 = 0$ outside of B_{R_3}. For $g \in H^{1/2}(\Gamma_D)$, let $u_g \in H^1(\mathbb{R}^2_+\backslash\bar{D})$ be the lifting function such that $u_g = g$ on Γ_D and $\|u_g\|_{H^1(\mathbb{R}^2_+\backslash\bar{D})} \leq C\|g\|_{H^{1/2}(\Gamma_D)}$. By testing 6.8 with $\chi_1^2(\overline{u_\varepsilon - u_g})$ and using the standard argument we have

$$\|u_\varepsilon\|_{H^1(B_{R_2}\backslash\bar{D})} \leq C(\|u_\varepsilon\|_{L^2(B_{R_3}\backslash\bar{D})} + \|g\|_{H^{1/2}(\Gamma_D)}). \tag{6.12}$$

A combination of (6.11) and the above estimate yields

$$\|u_\varepsilon\|_{H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})} \leq C(\|u_\varepsilon\|_{L^2(B_{R_3}\backslash\bar{D})} + \|g\|_{H^{1/2}(\Gamma_D)}). \tag{6.13}$$

Now we claim

$$\|u_\varepsilon\|_{L^2(B_{R_3}\backslash\bar{D})} \leq C\|g\|_{H^{1/2}(\Gamma_D)}, \tag{6.14}$$

for any $g \in H^{1/2}(\Gamma_D)$ and $\varepsilon > 0$. If it were false, there would exist sequences $\{g_m\} \subset H^{1/2}(\Gamma_D)$ and $\{\varepsilon_m\} \subset (0,1)$, and $\{u_{\varepsilon_m}\}$ be the corresponding solution of (6.8)-(6.10) such that

$$\|u_{\varepsilon_m}\|_{L^2(B_{R_3}\backslash\bar{D})} = 1 \text{ and } \|g_m\|_{H^{-1/2}(\Gamma_D)} \leq \frac{1}{m}. \tag{6.15}$$

Then $\|u_{\varepsilon_m}\|_{H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})} \leq C$, and thus there is a subsequence of $\{\varepsilon_m\}$, which is still denoted by $\{\varepsilon_m\}$, such that $\varepsilon_m \to \varepsilon' \in [0,1]$, and a subsequence of $\{u_{\varepsilon_m}\}$, which is still denoted by $\{u_{\varepsilon_m}\}$, such that it converges to some $u_{\varepsilon'}$ in $H^{1,-s'}(\mathbb{R}^2_+\backslash\bar{D})$ by choosing $s' > s$. This is a consequence of Korn's inequality and Rellich theorem. So $u_{\varepsilon'} \in H^{1,-s'}(\mathbb{R}^2_+\backslash\bar{D})$ satisfies (6.8-6.10) with $g = 0$ and $\varepsilon = \varepsilon'$.

By the integral representation satistied by u_{ε_m}, we know that for $y \in \mathbb{R}^2_+\backslash\bar{B}_{R_1}$ and $i = 1, 2$

$$u_{\varepsilon'}(y) \cdot e^i = \int_{\partial B_{R_1}} (\sigma(G_{\varepsilon'}(x,y)e_i)\nu) \cdot u_{\varepsilon'}(x) - (G_{\varepsilon'}(x,y)e_i) \cdot (\sigma(u_{\varepsilon'})_{\varepsilon'}\nu) ds(x) \tag{6.16}$$

If $\varepsilon' > 0$, we deduce from (6.16) that $u_{\varepsilon'}$ decays exponentially and thus $u_{\varepsilon'} \in H^1(\mathbb{R}^2_+\backslash\bar{D})$, then $u_{\varepsilon'} = 0$ by the uniqueness of the solution in $H^1(\mathbb{R}^2_+\backslash\bar{D})$ with positive absorption. If $\varepsilon' = 0$, by the [7, theorem 5.2], we have $u_{\varepsilon'} \in L^2(\mathbb{R}^2_+\backslash\bar{D})$. Then we conclude $u_{\varepsilon'} = 0$ by the lemma 6.3 Therefore, in any case $u_{\epsilon'} = 0$, which , however contradicts to 6.15. This complete the proof. □

Now we are in the position to prove the exsitence of Theorem 6.1.

Lemma 6.5 *For any $s > 1/2$, $u_\varepsilon : (0,1) \to H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})$ is a uniformly continuous operator valued function. Immediately, u_ε converges to some u_0 in $H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})$ and u_0 is a solution of (6.1-6.5).*

Proof. We also give a indirect prove here. Let $\delta_0 > 0$ and $\{\mu_n\}$ and $\{\nu_n\}$ be sequences in $(0, 1)$ such that

$$|\mu_n - \nu_n| \leq 1/n \qquad \text{and} \qquad \|u_{\mu_n} - u_{\nu_n}\|_{H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})} \geq \delta_0 \tag{6.17}$$

Thus there is a subsequence of $\{\mu_n\}$, which is still denoted by $\{\mu_n\}$, such that $\{\mu_n\} \to \epsilon \in [0, 1]$ and also $\{\nu_n\} \to \epsilon$. Then using lemma 6.4 and the procedure proving it, we get the $u_\epsilon, v_\epsilon \in H^{1,-s'}(\mathbb{R}^2_+\backslash\bar{D})$, by choosing $s' > s$, such that

$$\begin{aligned}
&\|u_{\mu_n} - u_\epsilon\|_{H^{1,-s'}(\mathbb{R}^2_+\backslash\bar{D})} \to 0 \\
&\|u_{\nu_n} - v_\epsilon\|_{H^{1,-s'}(\mathbb{R}^2_+\backslash\bar{D})} \to 0
\end{aligned}$$

and $u_\epsilon = v_\epsilon$ by the same arguement in lemma 6.4 which is contradict a contradiction. Thus we have proved u_ε is uniformly continuously for $\varepsilon \in (0, 1)$. Then it is easy to see u_ε has a limitation in $H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})$ and the estimation of u_0 can be obtained by (6.14). This completes the proof. □

It is remain to prove the uniqueness in theorem 6.1. Actually, it can be obtained following the existence of solution with any $g \in H^{1/2}(\Gamma_D)$.

prove of Theorem 6.1 By the linearity of the problem, it is sufficient to prove that any u_0 satisfies the system (6.1-6.3) with the corresponding homogeneous boundary-value vanishes identically in $\mathbb{R}^2_+\backslash\bar{D}$. For any $y \in \mathbb{R}^2_+\backslash\bar{D}$, there exists $U^s(x, y)$ sataifies (6.1-6.3) with $g(x) = -G(x, y)$ on Γ_D following the lemma 6.5 and we define $U(x, y) = G(x, y) + U^s(x, y)$. It is easy to see that $U(x, y)$ satifies the generalized radiation condition (6.4). Thus by the integral representaion of u_0, we have

$$\lim_{r\to\infty} \int_{S_r^+} (\sigma(U(x, y)e_i)\nu) \cdot u_0(x) - (U(x, y)e_i) \cdot (\sigma(u_0)\nu)ds(x) = 0$$

Finally, combining $U(x, y) = 0, u_0(x) = 0$ on Γ_D and the Green integral theorem we find that

$$\begin{aligned}
u_0(y)e_i &= \int_{\mathbb{R}^2_+\backslash\bar{D}} -(\Delta_e(G(x, y)e_i) + \omega^2 G(x, y)e_i) \cdot u_0(x)dx \\
&= \int_{\mathbb{R}^2_+\backslash\bar{D}} \Delta u_0(x) \cdot (G(x, y)e_i) - \Delta_e(G(x, y)e_i) \cdot u_0(x) \\
&= \int_{\Gamma_D} (\sigma(U(x, y)e_i)\nu) \cdot u_0(x) - (U(x, y)e_i) \cdot (\sigma(u_0)\nu)ds(x) = 0
\end{aligned}$$

Then the desired unique exsitence follows with lemma 6.5. This comlpletes the proof of theorem 6.1. □

7. The resolution analysis

In this section we study the imaging resolution of the RTM for the Direchlet boundary obstacle in the half space.

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