Solution of Ali Math Competition

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1. problem 1.a

We can buy earphone with another good which is worth 50. Then we should only pay 300-60-5*5= 215. For the audio, we should pay 600-5*10-60=490. So, all we should pay is 215+490=705.

2. problem 1.b

1)If we buy earphone with another good which is worth 49, we should pay 299-60-x. Then 299-60-x = 215-1 which implies x=25.
2) 299-60-x+600-60-x= 705-1 which implies x=37.5.

3. problem 1.c

1)The profit $a_i = p_i - c_i$, then the expect of profit is $\int_{p_i}^{u_i} \frac{p_i - c_i}{u_i} dt = \frac{(u_i - p_i)(p_i - c_i)}{u_i}$. It is easy to see that the best price $p_i = \frac{u_i + c_i}{2}$ and the corresponding profit expect is $\frac{(u_i - c_i)^2}{4u_i}$
2)The expect of profit is

$$\int_{p_{12} \leq t_1 + t_2} \frac{p_{12} - c_{12}}{u_1 u_2} dt_1 dt_2 = \begin{cases} \frac{(2u_1u_2 - p_{12}^2)(p_12 - c_12)}{2u_1u_2}, & p_{12} \leq \min\{u_1, u_2\} \\ \frac{(2u_1 + u_2 - 2p_{12})u_2(p_12 - c_12)}{2u_1u_2}, & \min\{u_1, u_2\} < p_{12} < \max\{u_1, u_2\} \\ \frac{(u_1 + u_2 - p_{12})^2 (p_{12} - c_{12})}{2u_1u_2}, & p_{12} > \max\{u_1, u_2\} \end{cases}$$

4. problem 2.a

The path: 2 3 4 15 14 13 12 9 5 6 7 8 11
The length: 17

5. problem 2.b

1)$\sum_{k=1}^{m} \sum_{\{i_1, \cdots, i_k\} \subset \{1, \cdots, m\}} P_{i_1} \cdots P_{i_k}$
2) $1 - (1 - P_1) \cdots (1 - P_m)$

6. problem 3.b

By the assumption of problem, it is easy to see that $HH^T = nI$. Therefore the singular value of H is $\sqrt{n}$. Let M be the submatrix $a \times b$ of H such that all elements are 1. Then the rank of MM^T is 1 which implies that MM^T has only one nonezero eigenvalue. Sine the sum of all eigenvalues is equal to the trace. Then we can obtain the singular value of M is $\sqrt{ab}$. By the definition of singular value of H that $\sigma(H) = \max_{\|x\| \leq 1, \|y\| \leq 1} x^T Hy$, we can assert that the singular value of submatrix can not be larger than the singular value of H. Therefore $ab \leq n$. This completes the proof.

7. problem 3.c

Since F has finite elements, let $F = \{h_1, h_2, \cdots h_k\}, k > 0$ satisfies $h_i^{m_i} = e,\ m_i \geq 1,\ 1 \leq i \leq k$. By choosing any $g \in G$, it is easy to see that

$$(g^{-1}h_i g)^{m_i} = g^{-1}h_i^{m_i} g = e$$

Let $F_g := \{g^{-1}h_i g,\ 1 \leq i \leq k\}$, then we have $F_g = F$. Let σ_g be the permutation on F such that $\sigma_g(h_i) = g^{-1}h_i g$. Therefore, we can find integer n, for any $g \in G$, $(\sigma_g)^n$ is a identity permutation which implies $(g^{-n}hg^n) = h$. This completes the proof.