A Direct Imaging Method for Half-Space Inverse Elastic Scattering Problems

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Abstract. We propose a direct imaging method based on the reverse time migration to reconstruct extended obstacles in the half space with finite aperture elastic scattering data at a fixed frequency. We prove the resolution of the reconstruction method in terms of the aperture and the depth of the obstacle embedded in the half space. The resolution analysis is studied by virtue of the point spread function and implies that the imaginary part of the cross-correlation imaging function always peaks on the upper boundary of the obstacle. Numerical examples are included to illustrate the effectiveness of the method.

1. Introduction

section1

Inverse elastic wave scattering problems have considerable interests in diverse application fields including non-destructive testings, medical imaging, and seismic explorations. The purpose of this paper is to study a direct imaging method to find the shape and location of unknown obstacles embedded in the half-space isotropic and homogeneous elastic medium. We assume the obstacles are far away from the surface of the medium where the sources and receivers are located. The imaging method is based on the idea of the reverse time migration (RTM) and does not require the knowledge of the physical properties of the obstacles such as penetrable or non-penetrable, and for non-penetrable obstacles, the type of boundary conditions on the boundary of the obstacle.

Let $D \subset \mathbb{R}_+^2 = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 > 0$ be a bounded Lipschitz domain with the unit outer normal ν to its boundary Γ_D. We assume the incident wave is emitted by a point source at x_s on the surface $\Gamma_0 = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 = 0\}$, along the polarization direction $q \in \mathbb{R}^2$. Let $\mathbb{N}(x, x_s)q$ be the Neumann Green tensor for the half-space elastic scattering problem with free surface condition on Γ_0 (see (2.2)-(2.3) below). The measured data is $u(x_r, x_s) = u_q^s(x_r, x_s) + \mathbb{N}(x_r, x_s)q, x_r \in \Gamma_0$, with $u_q^s(x, x_s)$ satisfying the followinging equations

$$\Delta_e u_q^s(x, x_s) + \rho\,\omega^2 u_q^s(x, x_s) = 0 \quad \text{in } \mathbb{R}_+^2 \backslash \bar{D}, \tag{1.1}$$

$$u_q^s(x, x_s) = -\mathbb{N}(x, x_s)q \ \text{ on } \Gamma_D, \quad \sigma(u_q^s(x, x_s))e_2 = 0 \ \text{ on } \Gamma_0, \tag{1.2}$$

where $\Delta_e u := (\lambda + \mu)\nabla \mathrm{div} u + \mu \Delta u$ is the elastic Laplacian operator with Lamé constants λ and μ satisfying $\lambda > 0, \mu > 0$, ρ is the density, $\omega > 0$ is the circular frequency, and e_i is the unit vector along the x_i axis, $i = 1, 2$. In the following, we will always assume $\rho = 1$. In the boundary condition (1.2), $\sigma(u) \in \mathbb{C}^{2 \times 2}$ is the stress tensor, which relates the strain tensor $\varepsilon(u) \in \mathbb{C}^{2 \times 2}$ through the following constitutive law

$$\sigma(u) = 2\mu\varepsilon(u) + \lambda\mathrm{div} u \mathbb{I}, \quad \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T).$$

Here $\mathbb{I} \in \mathbb{R}^{2 \times 2}$ is the identity matrix.

The equations (1.1)-(1.2) must be complemented by appropriate boundary conditions at infinity to make the problem well-posed. In this paper we shall take the method of limiting absorption principle to define the scattering solution of the problem (1.1)-(1.2). The limiting absorption principle defines the scattering solution of (1.1)-(1.2) as the limit of the solution of the same equations with the complex frequency $\omega(1 + \mathbf{i}\varepsilon)$ when $\varepsilon \to 0^+$. The limiting absorption principle for the half-space elastic scattering problems is proved in [18], [28] for the scatterer with traction free boundary conditions. The results can be easily extended to cover penetrable scatterers or non-penetrable scatterers with Dirichlet or impedance boundary conditions. We also refer to [20], [26] and the references therein for the study of radiation conditions for half-space elastic scattering problems.

The RTM method, whose imaging function is defined as the cross-correction between the incident wave field and the back-propagated wave field using the complex

conjugated data, is nowadays widely used in exploration geophysics [15, 7, 6]. In [9, 10, 11, 12], the RTM method for reconstructing extended targets using acoustic, electromagnetic and elastic waves at a fixed frequency in the free space is proposed and studied. The resolution analysis in [9, 10, 11, 12] is achieved without using the small inclusion or geometrical optics assumption previously made in the literature (e.g. [3, 6]).

In the geophysics literature, the RTM method for elastic scattering data usually consists of three steps: 1) back-propagating the received elastic data on the surface to the medium using the full elastic wave equation; 2) decomposing the back-propagated wave field to obtain the p-wave and s-wave components by Helmholtz decomposition; and 3) cross-correlating each decomposed mode component with the corresponding mode component of the incident field to output the imaging profile [14, 17, 27]. In this paper, we study the elastic wave RTM method without using field separation, i.e., the imaging condition is defined as the cross-correlation between the back-propagated vector field with the incident vector field. More precisely, we propose and study the following imaging function (see (4.10) below):

$$\hat{I}_d(z) = \text{Im} \sum_{q=e_1,e_2} \int_{\Gamma_0^d} \int_{\Gamma_0^d} [\mathbb{T}_D(x_s, z)^T q][\mathbb{T}_D(x_r, z)^T \overline{u_q^s(x_r, x_s)}] \, ds(x_r) ds(x_s).$$

Here $\Gamma_0^d = \{x \in \Gamma_0 : x_1 \in (-d, d)\}$ is the interval where the data are collected and $\mathbb{T}_D(x, z)$ is the traction tensor on Γ_0 associated with the Dirichlet Green tensor (see (2.21) below).

Our resolution analysis, which extends the study in [12] for the half-space acoustic scattering data , indicates that the imaging function always peaks on the illuminating part of the obstacle. The important Rayleigh surface wave is considered in our resolution analysis which shows that the contribution of the surface wave decays exponentially in the imaging function. The elastic wave RTM method based on the wave separation can be studied using the techniques developed in this paper and will be considered in a forthcoming work.

The layout of the paper is as follows. In section 2 we study the Neumann and Dirichlet Green tensors for the half-space elastic scattering problem by using the method of Fourier transform. In section 3 we study the point spread function defined by the RTM method. In section 4 we study the resolution of our RTM method for locating extended targets. In section 5 the extension of our resolution results to other types of obstacles are briefly considered. In section 6 we report extensive numerical results of our RTM method for synthesized scattering data. In section 7 we prove a technical result used in the resolution analysis which may be of independent interest.

2. Elastic Green Tensors in the half-space

In this section we introduce the elastic Green tensors and study their horizontal asymptotic behavior on the surface Γ_0, which will play an crucial role in the resolution analysis for the RTM method in this paper. Throughout the paper, we will assume

that for $z \in \mathbb{C}\backslash\{0\}$, $z^{1/2}$ is the analytic branch of $\sqrt{z}$ such that $\mathrm{Im}\,(z^{1/2}) \geq 0$. This corresponds to the right half real axis as the branch cut in the complex plane. For $z = z_1 + \mathbf{i}z_2, z_1, z_2 \in \mathbb{R}$, we have

$$z^{1/2} = \mathrm{sgn}(z_2)\sqrt{\frac{|z| + z_1}{2}} + \mathbf{i}\sqrt{\frac{|z| - z_1}{2}}, \quad \forall z \in \mathbb{C}\backslash\mathbb{R}_+. \tag{2.1} \boxed{\texttt{convention\_1}}$$

For z on the right half real axis $\mathbb{R}_+ = \{z \in \mathbb{C} : \mathrm{Re}\,z > 0, \mathrm{Im}\,z = 0\}$, we take $z^{1/2}$ as the limit of $(z + \mathbf{i}\varepsilon)^{1/2}$ as $\varepsilon \to 0^+$.

We start by introducing the Neumann Green tensor $\mathbb{N}(x, y) \in \mathbb{C}^{2\times 2}, y \in \mathbb{R}^2_+$, which satisfies, for any $q \in \mathbb{R}^2$,

$$\Delta_e[\mathbb{N}(x; y)q] + \omega^2[\mathbb{N}(x, y)q] = -\delta_y(x)q \quad \text{in } \mathbb{R}^2_+, \tag{2.2} \boxed{\texttt{eq\_n1}}$$

$$\sigma(\mathbb{N}(x, y)q)e_2 = 0 \quad \text{on } \Gamma_0, \tag{2.3} \boxed{\texttt{eq\_n2}}$$

where $\delta_y(x)$ is the Dirac source at y. We use the method of Fourier transform to derive a formula of the Neumann Green tensor [20]. Let

$$\hat{\mathbb{N}}(\xi, x_2; y_2) = \int_{-\infty}^{+\infty} \mathbb{N}(x_1, x_2; y)e^{-\mathbf{i}(x_1 - y_1)\xi}dx_1, \quad \forall \xi \in \mathbb{C}, \tag{2.4} \boxed{\texttt{a1}}$$

be the spectral Neumann Green tensor. Let $\mathbb{G}(x, y)$ be the fundamental solution matrix of the elastic equation [23] whose Fourier transform is $\hat{\mathbb{G}}(\xi, x_2; y_2) = \hat{\mathbb{G}}_s(\xi, x_2; y_2) + \hat{\mathbb{G}}_p(\xi, x_2; y_2)$ with

$$\hat{\mathbb{G}}_s(\xi, x_2; y_2) = \frac{\mathbf{i}}{2\omega^2}\begin{pmatrix} \mu_s & -\xi\frac{x_2 - y_2}{|x_2 - y_2|} \\ -\xi\frac{x_2 - y_2}{|x_2 - y_2|} & \frac{\xi^2}{\mu_s} \end{pmatrix}e^{\mathbf{i}\mu_s|x_2 - y_2|},$$

$$\hat{\mathbb{G}}_p(\xi, x_2; y_2) = \frac{\mathbf{i}}{2\omega^2}\begin{pmatrix} \frac{\xi^2}{\mu_p} & \xi\frac{x_2 - y_2}{|x_2 - y_2|} \\ \xi\frac{x_2 - y_2}{|x_2 - y_2|} & \mu_p \end{pmatrix}e^{\mathbf{i}\mu_p|x_2 - y_2|}.$$

Here $k_p = \omega/\sqrt{\lambda + 2\mu}, k_s = \omega/\sqrt{\mu}$ are wave number of p-wave and s-wave, and $\mu_\alpha = (k_\alpha^2 - \xi^2)^{1/2}$ for $\alpha = s, p$. Using the spectral fundamental solution matrix, one can write the spectral Neumann Green tensor as

$$\hat{\mathbb{N}}(\xi, x_2; y_2) = \hat{\mathbb{G}}(\xi, x_2; y_2) - \hat{\mathbb{G}}(\xi, x_2; -y_2) + \frac{\mathbf{i}}{\omega^2\delta(\xi)}\sum_{\alpha, \beta = p, s} \mathbb{A}_{\alpha\beta}(\xi)e^{\mathbf{i}(\mu_\alpha x_2 + \mu_\beta y_2)}, \tag{2.5} \boxed{\texttt{NGT}}$$

where $\beta(\xi) = k_s^2 - 2\xi^2$, $\delta(\xi) = \beta(\xi)^2 + 4\xi^2\mu_s\mu_p$, and

$$\mathbb{A}_{ss}(\xi) = \begin{pmatrix} \beta^2\mu_s & -4\xi^3\mu_s\mu_p \\ -\xi\beta^2 & 4\xi^4\mu_p \end{pmatrix}, \quad \mathbb{A}_{sp}(\xi) = \begin{pmatrix} 2\xi^2\beta\mu_s & -2\xi\beta\mu_s\mu_p \\ -2\xi^3\beta & 2\xi^2\beta\mu_p \end{pmatrix},$$

$$\mathbb{A}_{ps}(\xi) = \begin{pmatrix} 2\xi^2\beta\mu_s & 2\xi^3\beta \\ 2\xi\beta\mu_s\mu_p & 2\xi^2\beta\mu_p \end{pmatrix}, \quad \mathbb{A}_{pp}(\xi) = \begin{pmatrix} 4\xi^4\mu_s & \xi\beta^2 \\ 4\xi^3\mu_s\mu_p & \beta^2\mu_p \end{pmatrix}.$$

The desired Neumann Green tensor should be obtained by taking the inverse Fourier transform of the spectral Green tensor $\hat{\mathbb{N}}(\xi, x_2; y_2)$. Unfortunately, one cannot simply take the inverse Fourier transform in the above formula because it is well known that $\delta(\xi)$ have zeros in the real axis [1, 22].

lem:2.1 **Lemma 2.1** *The Rayleigh equation $\delta(\xi) = 0$ has only two zeros $\pm k_R$, $k_R > k_s$, in the complex plane.*

Proof. For the sake of completeness, we include a proof here. By (2.1), It is clear that
$\delta(\xi)$ is analytic outside the branch cuts $C_l = \{\xi = \xi_1 + \mathbf{i}\xi_2 \in \mathbb{C} : \xi_1 \in [-k_s, -k_p], \xi_2 = 0\}$
and $C_r = \{\xi = \xi_1 + \mathbf{i}\xi_2 \in \mathbb{C} : \xi_1 \in [k_p, k_s], \xi_2 = 0\}$. On the branch cuts,

$$\delta(\xi) = (k_s^2 - 2\xi^2)^2 + \mathbf{i}\,[4\xi^2(k_s^2 - \xi^2)^{1/2}(\xi^2 - k_p^2)^{1/2}] \quad \forall \xi \in C_l \cup C_r.$$

Thus, $\delta(\xi)$ has no zeros in $C_l \cup C_r$ and at least two real zeros $\pm k_R$, $k_R > k_s$ since
$\delta(\pm k_s) = 0$, $\delta(\pm\infty) < 0$.

To conclude the proof, we now show that $\delta(\xi)$ has only two zeros in the complex plane by the principle of argument [2]. Let Γ be a circle with sufficiently large radius. We consider the domain $\mathcal{D}$ surrounded by the contour C consisting of Γ, the upper and lower edges of the branch cuts $C_l^\pm$ and $C_r^\pm$. Since $\delta(\xi)$ has no poles in the complex plane, we know from the principle of argument that the number of zeros in $\mathcal{D}$ is

$$Z = \frac{1}{2\pi\mathbf{i}} \int_C \frac{\delta'(\xi)}{\delta(\xi)} d\xi. \qquad (2.6) \quad \boxed{\texttt{zero}}$$

It is clear that $\delta(\xi) = \delta^\pm(\xi)$ for $\xi \in C_r^\pm$, where

$$\delta^\pm(\xi) = (k_s^2 - 2\xi^2)^2 \mp \mathbf{i}\,[4\xi^2(k_s^2 - \xi^2)^{1/2}(\xi^2 - k_p^2)^{1/2}] := f_1(\xi) \mp \mathbf{i}f_2(\xi).$$

Then we have

$$\int_{C_r^\pm} \frac{\delta'(\xi)}{\delta(\xi)} d\xi = \int_{k_p}^{k_s} \left(\frac{\delta'_+(\xi)}{\delta_+(\xi)} - \frac{\delta'_-(\xi)}{\delta_-(\xi)} \right) d\xi = 2\mathbf{i} \int_{k_p}^{k_s} \frac{f'_1(\xi)f_2(\xi) - f_1(\xi)f'_2(\xi)}{f_1^2(\xi) + f_2^2(\xi)} d\xi$$

$$= -2\mathbf{i}\arctan\frac{f_2(\xi)}{f_1(\xi)}\bigg|_{k_p}^{k_s} = 0.$$

Similarly, we have $\int_{C_l^\pm} \frac{\delta'(\xi)}{\delta(\xi)} d\xi = 0$. Moreover, for $|\xi|$ large, we have $\delta(\xi) = -2(k_p^2 + 3k_s^2)\xi^2 + O(1)$, and consequently $\int_\Gamma \frac{\delta'(\xi)}{\delta(\xi)} d\xi = 4\pi$. This yields $Z = 2$ and completes the proof. $\qquad\square$

Let $\mathbb{N}_{\omega(1+\mathbf{i}\varepsilon)}(x, y)$ be the Neumann Green tensor with complex circular frequency $\omega(1 + \mathbf{i}\varepsilon)$, that is, ω in (2.2) is replaced by $\omega(1 + \mathbf{i}\varepsilon)$. Let $\hat{\mathbb{N}}_{\omega(1+\mathbf{i}\varepsilon)}(\xi, x_2; y_2)$ be the corresponding spectral Neumann Green tensor which can be obtained by respectively replacing k_s, k_p in (2.5) by $k_s(1 + \mathbf{i}\varepsilon), k_p(1 + \mathbf{i}\varepsilon)$. The Neumann Green tensor $\mathbb{N}(x, y)$ is defined by the limit absorption principle as

$$\mathbb{N}(x, y) = \lim_{\varepsilon \to 0^+} \mathbb{N}_{\omega(1+\mathbf{i}\varepsilon)}(x, y) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\mathbb{N}}_{\omega(1+\mathbf{i}\varepsilon)}(\xi, x_2; y_2)e^{\mathbf{i}(x_1 - y_1)\xi} d\xi. \quad (2.7) \quad \boxed{\texttt{NGT1}}$$

The above limit can be computed by the following lemma on the Cauchy principal value (cf. e.g. [24, Chapter 4, Theorem 5]).

lem:2.2 **Lemma 2.2** *Let $a, b \in \mathbb{R}$, $a < b$, and $t_0 \in (a, b)$. If γ is Hölder continuous in $[a, b]$, that is, there exists a constant $\alpha \in (0, 1]$ and a constant $C > 0$ such that for any $s, t \in [a, b]$,*

$|\gamma(s) - \gamma(t)| \leq C|s - t|^\alpha$, *then*

$$\lim_{z \to t_0, \pm \text{Im } z > 0} \int_a^b \frac{\gamma(t)}{t - z} dt = \text{p.v.} \int_a^b \frac{\gamma(t)}{t - t_0} dt \pm \pi \mathbf{i} \gamma(t_0),$$

where p.v. $\int_a^b$ *denotes the Cauchy principal value of the integral.*

Lemma 2.2 and (2.7) yield the following representation formula for the Neumann Green function. For any $x, y \in \mathbb{R}_+^2$,

$$\mathbb{N}(x, y) = \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}} \hat{\mathbb{N}}(\xi, x_2; y_2) e^{\mathbf{i}(x_1 - y_1)\xi} d\xi + \frac{\mathbf{i}}{2} \left[\frac{\hat{\mathbb{N}}_c(\xi, x_2; y_2)}{\delta'(\xi)} e^{\mathbf{i}(x_1 - y_1)\xi} \right]_{-k_R}^{k_R},$$

where $[f(\xi)]_a^b := f(b) - f(a)$, *and*

$$\hat{\mathbb{N}}_c(\xi, x_2; y_2) = \frac{\mathbf{i}}{\omega^2} \sum_{\alpha, \beta = p, s} \mathbb{A}_{\alpha\beta}(\xi) e^{\mathbf{i}(\mu_\alpha x_2 + \mu_\beta y_2)}.$$

For $x_s \in \Gamma_0$, we can define $\mathbb{N}(x, x_s), x \in \mathbb{R}^2$, as the limit of $\mathbb{N}(x, y)$ as $y \in \mathbb{R}_+^2, y \to x_s$. It is also easy to check that $\mathbb{N}(x, y) = \mathbb{N}(y, x)^T$ for $x, y \in \mathbb{R}_+^2$.

In the following we are mostly interested in the Neumann Green tensor $\mathbb{N}(x, y)$ when $x \in \Gamma_0, y \in \mathbb{R}_+^2$. In this case, (2.5) simplifies to

$$\hat{\mathbb{N}}(\xi, 0; y_2) = \frac{\mathbf{i}}{\mu\delta(\xi)} \left[\begin{pmatrix} 2\xi^2 \mu_s & -2\xi\mu_s\mu_p \\ -\xi\beta & \mu_p\beta \end{pmatrix} e^{\mathbf{i}\mu_p y_2} + \begin{pmatrix} \mu_s\beta & \xi\beta \\ 2\xi\mu_s\mu_p & 2\xi^2\mu_p \end{pmatrix} e^{\mathbf{i}\mu_s y_2} \right]$$

$$:= \frac{1}{\delta(\xi)} (\mathcal{N}_p(\xi) e^{\mathbf{i}\mu_p y_2} + \mathcal{N}_s(\xi) e^{\mathbf{i}\mu_s y_2}), \tag{2.8} \quad \boxed{\text{d2}}$$

and consequently, for $x \in \Gamma_0, y \in \mathbb{R}_+^2$,

$$\mathbb{N}(x, y) = \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}} \hat{\mathbb{N}}(\xi, 0; y_2) e^{\mathbf{i}(x_1 - y_1)\xi} d\xi + \frac{\mathbf{i}}{2} \left[\sum_{\alpha = p, s} \frac{\mathcal{N}_\alpha(\xi)}{\delta'(\xi)} e^{\mathbf{i}(y_2\mu_\alpha + (x_1 - y_1)\xi)} \right]_{-k_R}^{k_R}. \tag{2.9} \quad \boxed{\text{c8}}$$

The following representation is useful in studying the asymptotic behavior of the for the Neumann Green tensor.

$\boxed{\text{lem:2.3}}$ **Lemma 2.3** *Let* $x \in \Gamma_0, y \in \mathbb{R}_+^2$ *and* $\phi \in (-\pi/2, \pi/2)$ *such that* $y_2 = |x - y| \cos \phi, x_1 - y_1 = \sin \phi$. *Assume that* $x_1 \neq y_1$, *then we have*

$$\mathbb{N}(x, y) = \frac{1}{2\pi} \int_L \mathcal{N}(t) \cos(t + \phi) e^{\mathbf{i}\lambda \cos t} dt \pm \mathbf{i} \left[\sum_{\alpha = p, s} \frac{\mathcal{N}_\alpha(\xi)}{\delta'(\xi)} e^{\mathbf{i}(y_2\mu_\alpha + (x_1 - y_1)\xi)} \right]_{\xi = \pm k_R} \tag{2.10} \quad \boxed{\text{h3}}$$

where the sign $\pm$ *is taken according to* $\text{sgn}(x_1 - y_1) = \pm 1$ *and*

$$\mathcal{N}(t) = \sum_{\alpha = p, s} k_s \frac{\mathcal{N}_\alpha(k_s \sin(t + \phi))}{\delta(k_s(\sin(t + \phi)))}. \tag{2.11} \quad \boxed{\text{h2}}$$

Here $\lambda = k_s|x - y|$ *and* L *is the integral path* $-\pi/2 + \mathbf{i}\infty$ *to* $-\pi/2$, $-\pi/2$ *to* $\pi/2$, *and* $\pi/2$ *to* $\pi/2 - \mathbf{i}\infty$ *in the complex plan in the complex plan (see Figure 1).*

Proof. We only consider the case $x_1 > y_1$ and so $\text{sgn}(x_1 - y_1) = 1$. The other case can be considered similarly and thus omitted. Notice that $\hat{N}(\xi, 0; y_2) = \sum_{\alpha=p,s} \frac{\mathcal{N}_\alpha(\xi)}{\delta(\xi)} e^{i(y_2\mu_\alpha + (x_1-y_1)\xi)}$. For $\alpha = p, s$, we use the classical transform $\xi = k_\alpha \sin t$ to obtain

$$\frac{1}{2\pi} \text{ p.v.} \int_{\mathbb{R}} \hat{N}(\xi, 0; y_2) e^{i(x_1-y_1)\xi} d\xi = \frac{1}{2\pi} \text{ p.v.} \int_L \sum_{\alpha=p,s} k_s \frac{\mathcal{N}_\alpha(k_s \sin t)}{\delta(k_s \sin t)} \cos t\, e^{i\lambda \cos(t-\phi)} dt.$$

Let $L_{-\phi}$ be the integral path which is the shift of L by $-\phi$, we know that

$$\frac{1}{2\pi} \text{ p.v.} \int_{\mathbb{R}} \hat{N}(\xi, 0; y_2) e^{i(x_1-y_1)\xi} d\xi = \frac{1}{2\pi} \text{ p.v.} \int_{L_{-\phi}} \mathcal{N}(t) \cos t\, e^{i\lambda \cos t} dt, \qquad (2.12) \quad \boxed{\text{h1}}$$

where $\mathcal{N}(t)$ is defined in ($\overset{\boxed{\text{h1}}}{2}$.12).

Let $t_R = \pi/2 - is_R \in L$, $s_R > 0$, be the image in the complex t-plane of k_R under the integral transform $\xi = k_s \sin t$, that is, $k_R = k_s \sin t_R$. For any $\varepsilon > 0$, let L^ε be the integral path from $-\pi/2 + i\infty \to -\pi/2 + i(s_0 + \varepsilon) \cup \partial B_\varepsilon^+(-t_R) \to -\pi/2 + i(s_0 - \varepsilon) \to -\pi/2 \to \pi/2 \to \pi/2 - i(s_0 - \varepsilon) \to \partial B_\varepsilon^+(t_R) \to \pi/2 - i(s_0 + \varepsilon) \to \pi/2 - i\infty$, where $\partial B_\varepsilon^+(\pm t_R)$ is the right half circle of radius ε centered at $\pm t_R$ (see Figure 1). Let $L_{-\phi}^\varepsilon$ be the shift L^ε by $-\phi$. Then by the definition of Cauchy principle value, we know that

$$\frac{1}{2\pi} \text{ p.v.} \int_{L_{-\phi}} \mathcal{N}(t) \cos t\, e^{i\lambda \cos t} dt = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{L_{-\phi}^\varepsilon} \mathcal{N}(t) \cos t\, e^{i\lambda \cos t} dt$$

$$+ \frac{i}{2} \sum_{t'=\pm t_R} \text{Res}(\mathcal{N}(t) \cos t\, e^{i\lambda \cos t}, t').$$

It is easy to see that the residue

$$\sum_{t'=\pm t_R} \text{Res}(\mathcal{N}(t) \cos t\, e^{i\lambda \cos t}, t') = \frac{i}{2} \sum_{\xi=\pm k_R} \sum_{\alpha=p,s} \frac{\mathcal{N}_\alpha(\xi)}{\delta'(\xi)} e^{i(y_2\mu_\alpha + (x_1-y_1)\xi)}.$$

By Cauchy integral theorem we have

$$\frac{1}{2\pi} \int_{L_{-\phi}^\varepsilon} \mathcal{N}(t) \cos t\, e^{i\lambda \cos t} dt = \frac{1}{2\pi} \int_L \mathcal{N}(t) \cos t\, e^{i\lambda \cos t} dt.$$

This completes the proof of the lemma by ($\overset{\boxed{\text{c8}}}{2}$.9) and ($\overset{\boxed{\text{h1}}}{2}$.12). $\qquad \square$

This lemma is the starting point of our estimate of the decay behavior of $N(x, y)$ as $x \to \infty$ on Γ_0. We recall first the following Van der Corput lemma for the oscillatory integral $\overset{\boxed{\text{grafakos}}}{[21}, \text{P.152}]$.

$\boxed{\text{van}}$ **Lemma 2.4** *Let $-\infty < a < b < \infty$, and u is a C^k function u in (a, b).*
1. If $|u'(t)| \geq 1$ for $t \in (a, b)$ and u' is monotone in (a,b), then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left| \int_a^b e^{i\lambda u(t)} \phi(t) dt \right| \leq 3\lambda^{-1} \left[|\phi(b)| + \int_a^b |\phi'(t)| dt \right].$$
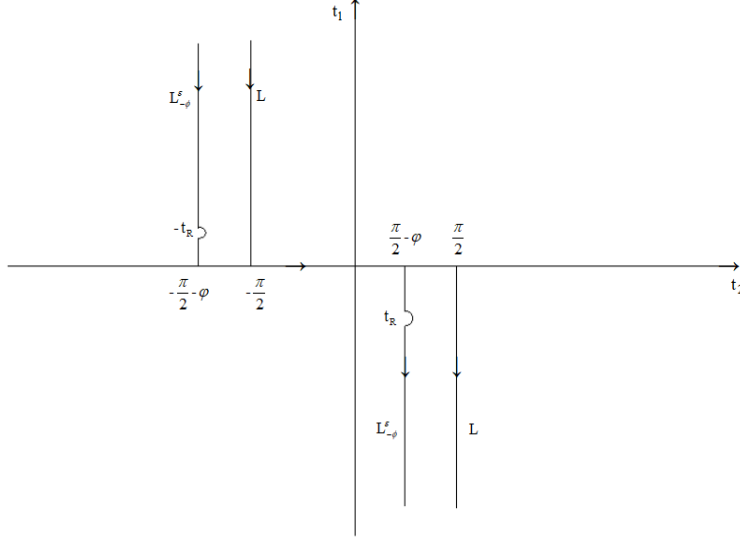
2. For all $k \geq 2$, if $|u^{(k)}(t)| \geq 1$ for $t \in (a, b)$, then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left| \int_a^b e^{i\lambda u(t)} \phi(t) dt \right| \leq Ck\lambda^{-1/k} \left[|\phi(b)| + \int_a^b |\phi'(t)| dt \right].$$

Figure 1. The integral path L and $L_{-\phi}^{\varepsilon}$.

The following lemma, which will be useful in the subsequent analysis, shows that the Van der Corput lemma is still valid when the singular points of the integrand $\phi(t)$ have a gap with the stationary phase points.

lem:2.5 **Lemma 2.5** *Assume that $\lambda \geq 1$ and $\phi \in (-\pi/2, \pi/2)$, Let $f(t)$ be continuous and have absolutely integrable derivative in $(-\pi/2, \pi/2)$. Then for any $(a, b) \subset (-\pi/2, \pi/2)$, we have*

$$\left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t) e^{i\lambda \cos t} dt \right| \leq C\lambda^{-1/2} \left[|f(0)| + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f'(t)| dt \right], \qquad (2.13) \quad \boxed{\text{c1}}$$

where the constant C is independent of a, b, ϕ, λ and the integrand f. Moreover, let $\kappa \in (0, 1)$ and $\phi \geq \phi^* > \arcsin \kappa := \phi_\kappa$, we have

$$\left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)(\kappa^2 - \sin^2(t + \phi))^{-1/2} e^{i\lambda \cos t} dt \right| \leq C\lambda^{-1/2} \left[|f(0)| + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f'(t)| dt \right], \qquad (2.14) \quad \boxed{\text{c3}}$$

where C depends only on ϕ^* and κ.

Proof. The estimate ($\overset{\text{c1}}{2.13}$) follows directly from Lemma $\overset{\text{van}}{2.4}$ since the interval (a, b) can be divided into several subintervals such that in each subinterval, either $|\sin t|$ or $|\cos t|$ is bounded below by $1/\sqrt{2}$.

Let $g(t) = \kappa^2 - \sin^2(t + \phi)$. It is easy to see that $g(t)$ has two zeros t_1, t_2 in $(-\pi/2, \pi/2)$, where $t_1 = \phi_\kappa - \phi$ and $t_2 = -\phi_\kappa - \phi$ or $t_2 = \pi - \phi_\kappa - \phi$ depending on whether $\phi + \phi_\kappa < \pi/2$ or $\phi + \phi_\kappa \geq \pi/2$. Without loss of generality, we assume the later case and thus $t_2 = \pi - \phi_\kappa - \phi$.

Let $\varepsilon_0 = \min(\frac{\phi^* - \phi_\kappa}{2}, \frac{\phi_\kappa}{2}) > 0$. Obviously, $t_1 - \varepsilon_0 > -\pi/2, t_1 + \varepsilon_0 \leq -(\phi^* - \phi_\kappa)/2$ and $t_2 - \varepsilon_0 > (\phi^* - \phi_\kappa)/2$. We divide $(-\pi/2, \pi/2)$ into five intervals: $I_1 = (-\pi/2, t_1 - \varepsilon_0), I_2 =$

$(t_1 - \varepsilon_0, t_1 + \varepsilon_0), I_3 = (t_1 + \varepsilon_0, t_2 - \varepsilon_0), I_4 = (t_2 - \varepsilon_0, t_2 + \min(t_2 + \varepsilon_0, \pi/2))$ and $I_5 = (\min(t_2 + \varepsilon_0, \pi/2), \pi/2)$. By (2.13) we have

$$\left| \int_{I_1 \cup I_3 \cup I_5} f(t)(\kappa^2 - \sin^2(t+\phi))^{-1/2} e^{i\lambda \cos t} dt \right| \leq C\lambda^{-1/2} \left[|f(0)| + \int_{-d}^0 |f'(t)| dt \right], \quad (2.15) \quad \boxed{\text{c4}}$$

where the constant C depends only on ϕ^* and κ.

Now we estimate the integration in I_2, I_4. We first observe that $|\sin t| \geq \sin((\phi^* - \phi_\kappa)/2)$ in $I_2 \cup I_4$. Moreover, $|g'(t)| = |\sin(2(t+\phi))| \geq \min(\sin \phi_\kappa, \sin(\phi^* + \phi_\kappa))$ in $I_2 \cup I_4$. Let $\delta \in (0, \varepsilon_0)$ be sufficiently small. By the mean value theorem, we have

$$|g(t)| \geq \min(\sin \phi_\kappa, \sin(\phi^* + \phi_\kappa))\delta, \quad \forall \delta \leq |t - t_j| \leq \varepsilon_0, j = 1, 2.$$

By integration by parts we then obtain

$$\left| \int_{t_1 - \varepsilon_0}^{t_1 - \delta} f(t)g(t)^{-1/2} e^{i\rho \cos t} dt \right| \leq C\delta^{-1/2}\lambda^{-1} \left[|f(0)| + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f'(t)| dt \right].$$

Similarly,

$$\left| \int_{t_1 + \delta}^{t_1 + \varepsilon_0} f(t)g(t)^{-1/2} e^{i\lambda \cos t} dt \right| \leq C\delta^{-1/2}\lambda^{-1} \left[|f(0)| + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f'(t)| dt \right].$$

Finally,

$$\left| \int_{t_1 - \delta}^{t_1 + \delta} f(t)g(t)^{-1/2} e^{i\lambda \cos t} dt \right| \leq C \max_{t \in (-\pi/2, \pi/2)} |f(t)| \int_{-\delta}^{\delta} |\kappa - \sin(\phi_\kappa + t)|^{-1/2} dt$$

$$\leq C\delta^{1/2} \max_{t \in (-\pi/2, \pi/2)} |f(t)|.$$

In conclusion, by taking $\delta = \lambda^{-1}$, we obtain

$$\left| \int_{I_2} f(t)(\kappa^2 - \sin^2(t+\phi))^{-1/2} e^{i\rho \cos t} dt \right| \leq C\lambda^{-1/2} \left[|f(0)| + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f'(t)| dt \right]. \quad (2.16) \quad \boxed{\text{c6}}$$

The integral in I_4 can be estimated similarly. This completes the proof by combining with the estimate in (2.15). $\qquad \square$

The following lemma collects some facts about the Rayleigh function $\delta(\xi)$.

$\boxed{\text{delta}}$ **Lemma 2.6** *Let $d_R = (k_R - k_s)/2$, where k_R is the wave number of the Rayleigh surface wave. There exists constant C_1, C_2 depending only on κ such that $|\delta^{(k)}(\xi)| \leq C_2 k_s^{4-k}, k = 0, 1, 2,$ for any $\xi \in \mathbb{R}$, $|\delta(\xi)| \geq C_1 k_s^4$ for any $|\xi| \leq k_s + d_R$, and $|\delta(\xi)| \geq 2k_s^2(|\xi|^2 - k_R^2)$ for any $|\xi| \geq k_R$. Moreover, let $\delta(\xi) = \delta_1(\xi)(\xi - k_R)$, then $|\delta_1(\xi)| \geq C_1 k_s^3$ for any $k_R - d_R \leq |\xi| \leq k_R + d_R$.*

Proof. By definition, we know that for $|\xi| \geq k_s$, $\delta(\xi) = k_s^4 f(\xi^2/k_s^2)$, where

$$f(t) = (2t - 1)^2 - 4t\sqrt{t-1}\sqrt{t - \kappa^2}, \quad \forall t \geq 1.$$

It is easy to see that $f'(t) \leq -2$ for $t \geq 1$, $f(1) = 1$ and $f((2 - \kappa^2)/(1 - \kappa^2)) < 0$. Thus $k_R^2 \leq \frac{2-\kappa^2}{1-\kappa^2} k_s^2$. The estimates for $\delta(\xi)$ follows easily.

Next by the mean value theorem,

$$\max_{k_R-d_R\leq|\xi|\leq k_R+d_R}|\delta_1(\xi)| = \max_{k_R-d_R\leq|\xi|\leq k_R+d_R}|\delta'(\xi)| \geq C_1 k_s^3,$$

where we have used the fact $f'(t) \leq -2$ for $t \geq 1$. This completes the proof. □

Lemma 2.7 Let $\phi \in (0, \pi/2)$ and $H = \{\xi = \xi_1 + \mathbf{i}\xi_2 \in \mathbb{C} : (\xi_1/(k_s\cos\phi))^2 - (\xi_2/(k_s\sin\phi))^2 = 1\}$ be the hyperbola. Let $f(\xi)$ be analytic on H. Then there exists a constant C depending only κ such that

$$\left|\int_{L\setminus[-\pi/2,\pi/2]} f(k_s\sin(t+\phi))e^{\mathbf{i}\lambda\cos t}dt\right| + \left|\int_{L\setminus[-\pi/2,\pi/2]} f(k_s\sin(t+\phi))\cos t e^{\mathbf{i}\lambda\cos t}dt\right|$$
$$\leq C\lambda^{-1}\lambda^{-1}\max_{\xi\in H}(|f(\xi)| + k_s|f'(\xi)|).$$

Proof. Notice that for $t = -\pi/2 + \mathbf{i}s$, $s > 0$, $k_s\sin(t+\phi) = -\cosh(s)\cos\phi + \mathbf{i}\sinh(s)\sin\phi \in H$. Thus

$$\int_{-\pi/2}^{-\pi/2+\mathbf{i}\infty} f(k_s\sin(t+\phi))e^{\mathbf{i}\lambda\cos t}dt$$
$$= \mathbf{i}\int_0^\infty f(-\cosh(s)\cos\phi + \mathbf{i}\sinh(s)\sin\phi)e^{-\lambda\sinh(s)}ds.$$

This implies the estimate on the integral path $-\pi/2 + \mathbf{i}\infty \to -\pi/2$ by integration by parts. The estimate on the path $\pi/2 \to \pi/2 - \mathbf{i}\infty$ is similar. Thus the estimate of the first term of the lemma follows. The second term can be proved similarly. Here we omit the details. This completes the proof. □

The following theorem is the first main result in this section.

Theorem 2.1 Let $x \in \Gamma_0$, $y \in \mathbb{R}_+^2$ satisfy $|x_1 - y_1|/|x - y| > (1 + \kappa)/2$ and $k_s y_2 \geq 1$. There exists a constant C depending only on κ such that

$$|\mathbb{N}(x, y)| \leq \frac{C}{\mu}\left(\frac{k_s y_2}{(k_s|x-y|)^{3/2}} + e^{-\sqrt{k_R^2-k_s^2}y_2}\right).$$

Proof. The starting point is (2.10) in Lemma 2.3. Without loss of generality, we assume $x_1 > y_1$ and thus $\phi \in (0, \pi/2)$. It is easy to see from Lemma ?? that the second term in (2.10) is bounded by $C\mu^{-1}e^{-\sqrt{k_R^2-k_s^2}y_2}$.

For the first term in (2.10) we first note that

$$\frac{1}{2\pi}\int_L \mathcal{N}(t)\cos(t+\phi)e^{\mathbf{i}\lambda\cos t}dt = \frac{1}{2\pi}\int_L\sum_{\alpha=p,s}k_s\frac{\mathcal{N}_\alpha(k_s\sin(t+\phi))}{\delta(k_s\sin(t+\phi))}\cos(t+\phi)e^{\mathbf{i}\lambda\cos t}dt.$$

We shall only estimate the term involving $[\mathcal{N}_p(k_s\sin(t+\phi))]_{22} = \mu^{-1}[\beta\mu_p](k_s\sin(t+\phi))$. The other elements can be proved similarly. Thus denote by

$$g(t) = k_s\frac{[\mathcal{N}_p(k_s\sin(t+\phi))]_{22}}{\delta(k_s\sin(t+\phi))} := f(t)(\kappa^2 - \sin^2(t+\phi))^{1/2}, \quad f(t) = \frac{k_s^2}{\mu}\frac{\beta(k_s\sin(t+\phi))}{\delta(k_s\sin(t+\phi))}.$$

Note first by integration by parts that

$$\int_L g(t)\cos(t+\phi)e^{\mathbf{i}\lambda\cos t}dt = \cos\phi\int_L g(t)\cos t e^{\mathbf{i}\lambda\cos t}dt - \sin\phi\int_L g(t)\sin t e^{\mathbf{i}\lambda\cos t}dt$$

$$= \cos\phi\int_L g(t)\cos t e^{\mathbf{i}\lambda\cos t}dt - \frac{\sin\phi}{\mathbf{i}\lambda}\int_L g'(t)e^{\mathbf{i}\lambda\cos t}dt$$

$$= \mathrm{I}_1 + \mathrm{I}_2.$$

By Lemma 2.7 and (2.13) in Lemma 2.5, we know that

$$\left|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(t)\cos t e^{\mathbf{i}\lambda\cos t}dt\right| \le C\lambda^{-1/2}\left[|g(0)| + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |g'(t)|dt\right] \le C\mu^{-1}\lambda^{-1/2}.$$

Since $\delta(\xi)$ is bounded below by $Ck_s^2|\xi|^2$ in then hyperbola H, we conclude from Lemma 2.7 that

$$\left|\int_{L\setminus[-\frac{\pi}{2},\frac{\pi}{2}]} g(t)\cos t e^{\mathbf{i}\lambda\cos t}dt\right| \le C\mu^{-1}\lambda^{-1}.$$

Thus $|\mathrm{I}_1| \le C\mu^{-1}\lambda^{-1/2}\cos\phi$. Similarly, we can show $|\mathrm{I}_2| \le C\mu^{-1}\lambda^{-3/2}$. The only difference is that since $g'(t) = f'(t)(\kappa^2-\sin^2(t+\phi))^{1/2} - f(t)(\kappa^2-\sin^2(t+\phi))^{-1/2}\sin(t+\phi)\cos(t+\phi)$, one has to use both (2.13) and (2.14) in Lemma 2.5 to obtain

$$\left|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g'(t)e^{\mathbf{i}\lambda\cos t}dt\right| \le C\mu^{-1}\lambda^{-1/2}.$$

Thus $|\mathrm{I}_1 + \mathrm{I}_2| \le C\mu^{-1}\lambda^{-1/2}\cos\phi \le C\mu^{-1}(k_s y_2)/(k_s|x-y|)^{3/2}$, where have used the condition $k_s y_2 \ge 1$. This completes the proof. $\square$

Now we introduce the Dirichlet Green Tensor $\mathbb{D}(x,y), y\in\mathbb{R}_+^2$ which satisfies [4]

$$\Delta_e[\mathbb{D}(x,y)q] + \omega^2[\mathbb{D}(x,y)q] = -\delta_y(x)q\mathbb{I} \quad \text{in} \quad \mathbb{R}_+^2, \qquad (2.17)$$

$$\mathbb{D}(x,y)q = 0 \quad \text{on} \quad \Gamma_0. \qquad (2.18)$$

The spectral Dirichlet Green Tensor $\hat{\mathbb{D}}(\xi,x_2;y_2)$ is defined similar to the spectral Neumann Green tensor in (2.4) and it follows that

$$\hat{\mathbb{D}}(\xi,x_2;y_2) = \hat{\mathbb{G}}(\xi,x_2;y_2) - \hat{\mathbb{G}}(\xi,x_2;-y_2) + \frac{\mathbf{i}}{\omega^2\gamma(\xi)}\sum_{\alpha,\beta=s,p}\mathbb{B}_{\alpha\beta}(\xi)e^{\mathbf{i}(x_2\mu_\alpha+y_2\mu_\beta)}, \quad (2.19)$$

where $\gamma(\xi) = \xi^2 + \mu_s\mu_p$, $\mathbb{B}_{sp}(\xi) = -\mathbb{B}_{ss}(\xi)$, $\mathbb{B}_{ps}(\xi) = -\mathbb{B}_{pp}(\xi)$, and

$$\mathbb{B}_{ss}(\xi) = \begin{pmatrix} \xi^2\mu_s & -\xi\mu_s\mu_p \\ -\xi^3 & \xi^2\mu_p \end{pmatrix}, \qquad \mathbb{B}_{pp}(\xi) = \begin{pmatrix} \xi^2\mu_s & \xi^3 \\ \xi\mu_s\mu_p & \xi^2\mu_p \end{pmatrix}.$$

The Dirichlet Green tensor $\mathbb{D}(x,y)$ is obtained as the limit of $\mathbb{D}_{\omega(1+\mathbf{i}\varepsilon)}(x,y)$ when $\varepsilon\to 0^+$, where $\mathbb{D}_{\omega(1+\mathbf{i}\varepsilon)}(x,y)$ is Dirichlet Green tensor with the complex circular frequency $\omega(1+\mathbf{i}\varepsilon)$, that is ω in (2.17) is replaced by $\omega(1+\mathbf{i}\varepsilon)$. The corresponding spectral Dirichle Green function $\mathbb{D}_{\omega(1+\mathbf{i}\varepsilon)}(x,y)$ can be obtained by respectively replacing k_s, k_p in (2.19) by $k_s(1+\mathbf{i}\varepsilon), k_p(1+\mathbf{i}\varepsilon)$. Thus

$$\mathbb{D}(x,y) = \lim_{\varepsilon\to 0^+}\mathbb{D}_{\omega(1+\mathbf{i}\varepsilon)}(x,y) = \lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{\mathbb{D}}_{\omega(1+\mathbf{i}\varepsilon)}(\xi,x_2;y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi.$$

We have the following representation of Dirichlet Green tensor

$$\mathbb{D}(x,y) = \mathbb{G}(x,y) - \mathbb{G}(x,y') + \frac{1}{2\pi}\int_{\mathbb{R}}\hat{\mathbb{D}}_c(\xi,x_2;y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi, \qquad (2.20) \quad \boxed{\texttt{DGT1}}$$

where

$$\hat{\mathbb{D}}_c(\xi,x_2;y_2) = \frac{\mathbf{i}}{\omega^2\gamma(\xi)}\sum_{\alpha,\beta=s,p}\mathbb{B}_{\alpha\beta}(\xi)e^{\mathbf{i}(x_2\mu_\alpha+y_2\mu_\beta)}.$$

Let $\mathbb{T}_D(x,y) \in \mathbb{C}^{2\times 2}$ denote the traction tensor of $\mathbb{D}(x,y)$ in the direction e_2 with respect to x, that is, $\mathbb{T}_D(x,y)q = \sigma(\mathbb{D}(x,y)q)e_2, \forall q \in \mathbb{R}^2$. We are particularly interested in $\mathbb{T}_D(x,y)$ on Γ_0. By $\overset{\texttt{DGT1}}{(2.20)}$ we have

$$\mathbb{T}_D(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{\mathbb{T}}_D(\xi,0;y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi, \qquad \forall x \in \Gamma_0, \qquad (2.21) \quad \boxed{\texttt{DGT2}}$$

where

$$\hat{\mathbb{T}}_D(\xi,0;y_2) = \frac{1}{\gamma(\xi)}\left[\begin{pmatrix} \xi^2 & -\xi\mu_p \\ -\xi\mu_s & \mu_p\mu_s \end{pmatrix}e^{\mathbf{i}\mu_p y_2} + \begin{pmatrix} \mu_s\mu_p & \xi\mu_p \\ \xi\mu_s & \xi^2 \end{pmatrix}e^{\mathbf{i}\mu_s y_2}\right]$$

$$:= \mathcal{T}_p(\xi)e^{\mathbf{i}\mu_p y_2} + \mathcal{T}_s(\xi)e^{\mathbf{i}\mu_s y_2}. \qquad (2.22) \quad \boxed{\texttt{d1}}$$

The Dirichlet Green tensor $\mathbb{D}(x,x_s)$ for $x_s \in \Gamma_0, x \in \mathbb{R}^2_+$ can be defined in a similar way as the limit of $\mathbb{D}(x,y)$ as $y \in \mathbb{R}^2_+, y \to x_s$. One also has $\mathbb{D}(x,y) = \mathbb{D}(y,x)^T$ for $x,y \in \mathbb{R}^2_+$.

The following theorem on the decay behavior of the traction tensor on Γ_0 improves $\overset{\texttt{arens1999}}{[4, \text{Lemma } 2.2]}$ in the sense that we provide exact dependence on y_2 in the numerator. It can be proved by the same (and simpler) argument as that in the proof of Theorem $\overset{\texttt{thm:2.1}}{2.1}$. We omit the details.

$\boxed{\texttt{thm:2.2}}$ **Theorem 2.2** Let $x \in \Gamma_0$, $y \in \mathbb{R}^2_+$ satisfy $|x_1 - y_1|/|x - y| \geq (1 + \kappa)/2$ and $k_s y_2 \geq 1$. There exists a constant C depending only on κ such that $|\mathbb{T}_D(x,y)| \leq C(k_s y_2)/(k_s|x - y|)^{3/2}$.

3. The point spread function

In this section we introduce the point spread function for imaging a point source embedded in the half-space elastic medium, which extends the study in $\overset{\texttt{RTMhalf\_aco}}{[12]}$ for acoustic waves. Let $\mathbb{N}(x,y)$ be the Neumann Green tensor which is the data collected on the surface Γ_0 with a point source $y \in \mathbb{R}^2_+$. The finite aperture point spread function $\mathbb{J}_d(x,y)$, $x,y \in \mathbb{R}^2_+$, is a $\mathbb{C}^{2\times 2}$ matrix, which is the back-propagated field with $\mathbb{N}(x,y)\chi_{(-d,d)}$ as the Dirichlet boundary condition, where $\chi_{(-d,d)}$ is the characteristic function of the interval $(-d,d)$. More precisely, $\mathbb{J}_d(x,y)e_j, j = 1,2$, is the scattering solution of the following problem

$$\Delta_e[\mathbb{J}_d(x,y)e_j] + \omega^2[\mathbb{J}_d(x,y)e_j] = 0 \text{ in } \mathbb{R}^2_+,$$
$$\mathbb{J}_d(x,y)e_j = \overline{\mathbb{N}(x,y)}e_j\chi_{(-d,d)} \text{ on } \Gamma_0.$$

By the integral representaion formula, for any $x,y \in \mathbb{R}^2_+$,

$$[\mathbb{J}_d(x,y)]_{ij} = e_i \cdot [\mathbb{J}_d(x,y)e_j] = \int_{\Gamma_0^d}\mathbb{T}_D(x,z)e_i \cdot \overline{\mathbb{N}(x,y)}e_j ds(x), \quad i,j = 1,2. \qquad (3.1) \quad \boxed{\texttt{jd}}$$

By Lemmas 2.2 and 2.1, we know that the integral converges as $d \to \infty$. Thus we can define the half-space elastic point spread function $\mathbb{J}(x, y) \in C^{2 \times 2}$, $x, y \in \mathbb{R}^2_+$, as

$$[\mathbb{J}(x, y)]_{ij} := e_i \cdot [\mathbb{J}(x, y)e_j] = \int_{\Gamma_0} \mathbb{T}_D(x, z)e_i \cdot \overline{\mathrm{N}(x, y)}e_j ds(x), \quad i, j = 1, 2. \qquad (3.2) \quad \boxed{\text{j}}$$

By the limiting absorption principle, we know that

$$[\mathbb{J}(z, y)]_{ij} = \lim_{\varepsilon \to 0^+} \int_{\Gamma_0} [\sigma(\mathbb{D}_{\omega(1+\mathbf{i}\varepsilon)}(x, z)e_i)e_2] \cdot [\overline{\mathrm{N}_{\omega(1+\mathbf{i}\varepsilon)}(x, y)}e_j] ds(x).$$

By using Parserval identity, Lemma 2.2, (2.22) and (2.8), we obtain

$$\mathbb{J}(z, y) = \frac{1}{2\pi} \sum_{\alpha, \beta = p, s} \text{p.v.} \int_{\mathbb{R}} \frac{\mathcal{T}_\alpha(\xi)^T \overline{\mathcal{N}_\beta}(\xi)}{\delta(\xi)} e^{\mathbf{i}(\mu_\alpha z_2 - \overline{\mu}_\beta y_2) + \mathbf{i}(y_1 - z_1)\xi} d\xi$$

$$- \frac{\mathbf{i}}{2} \sum_{\alpha, \beta = p, s} \left[\frac{\mathcal{T}_\alpha(\xi)^T \overline{\mathcal{N}_\beta}(\xi)}{\delta'(\xi)} e^{\mathbf{i}(\mu_\alpha z_2 - \overline{\mu}_\beta y_2) + \mathbf{i}(y_1 - z_1)\xi} \right]_{-k_R}^{k_R}. \qquad (3.3) \quad \boxed{\text{d3}}$$

To proceed, we define

$$\mathbb{F}(z, y) = \frac{1}{2\pi} \int_{-k_p}^{k_p} \frac{\mathcal{T}_p(\xi)^T \overline{\mathcal{N}_p}(\xi)}{\delta(\xi)} e^{\mathbf{i}\mu_p(z_2 - y_2) + \mathbf{i}(y_1 - z_1)\xi} d\xi$$

$$+ \frac{1}{2\pi} \int_{-k_s}^{k_s} \frac{\mathcal{T}_s(\xi)^T \overline{\mathcal{N}_s}(\xi)}{\delta(\xi)} e^{\mathbf{i}\mu_s(z_2 - y_2) + \mathbf{i}(y_1 - z_1)\xi} d\xi. \qquad (3.4) \quad \boxed{\text{d4}}$$

Let the obstacle D is included in the imaging domain Ω and $h = \text{dist}(\Omega, \Gamma_0)$ be the distance between Ω and Γ_0. We assume there exist constants $0 < c_1 < 1, c_2 > 0, c_3 > 0$ such that

$$|x_1| \leq c_1 d, \quad |x_1 - y_1| \leq c_2 h, \quad |x_2| \leq c_3 h, \qquad \forall x, y \in \Omega. \qquad (3.5) \quad \boxed{\text{d0}}$$

We remark that this assumption is rather mild in practical applications. The aim of this section is to show that for $z, y \in \Omega$, $\mathbb{F}(z, y)$ is the main contribution in $\mathbb{J}_d(z, y)$ and $\mathbb{F}(z, y)$ decays as $|z - y| \to \infty$ and the imaginary part of the function $|\text{Im}\,\mathbb{J}_{ii}(z, y)|, i = 1, 2$, peaks when $z = y$.

We start with the following lemma.

$\boxed{\text{lem:3.1}}$ **Lemma 3.1** *For $k_s h \geq 1, d \gg h$ and $z, y \in \Omega$, we have*

$$|\mathbb{J}(z, y) - \mathbb{J}_d(z, y)| + k_s^{-1}|\nabla_y(\mathbb{J}(z, y) - \mathbb{J}_d(z, y))|$$

$$\leq \frac{C}{\mu} \left[\left(\frac{h}{d}\right)^2 + \frac{(k_s h)^{1/2}}{e^{k_s h \sqrt{\kappa_R^2 - 1}}} \left(\frac{h}{d}\right)^{1/2} \right],$$

where the constant C depends only on κ.

Proof. By Lemma 2.2 and Lemma 2.1, when $k_s h \geq 1$ and $d/h \gg 1$, we have

$$\left| \int_d^\infty [\mathbb{T}_D(x, z)^T \overline{\mathrm{N}(x, y)}]_{x_2 = 0} dx_1 \right|$$

$$\leq \frac{C}{\mu} \int_d^\infty \frac{k_s^{1/2} z_2}{|x - z|^{3/2}} \left(\frac{k_s^{-1/2} y_2}{|x - y|^{3/2}} + e^{-\sqrt{k_R^2 - k_s^2} y_2} \right) dx_1$$

$$\leq \frac{C}{\mu} \int_{(1-c_1)d/h}^{\infty} \left(\frac{1}{(1+t^2)^{3/2}} + \frac{(k_s h)^{1/2}}{(1+t^2)^{3/4}} e^{-\sqrt{k_R^2 - k_s^2}h} \right) dt$$

$$\leq \frac{C}{\mu} \left[\left(\frac{h}{d} \right)^2 + \frac{(k_s h)^{1/2}}{e^{\sqrt{k_R^2 - k_s^2}h}} \left(\frac{h}{d} \right)^{1/2} \right].$$

Here we have used the first inequality in (3.5). Similarly, we can prove that the estimate for the integral in $(-\infty, -d)$. This shows the estimate for $\mathbb{J}(z,y) - \mathbb{J}_d(z,y)$. The estimate for $\nabla_y(\mathbb{J}(z,y) - \mathbb{J}_d(z,y))$ can be proved similarly. $\qquad\square$

The following term shows the second term on the right-hand side of (3.3) is small.

Lemma 3.2 *There exists a constant C depending only on κ such that for any $z, y \in \Omega$,*

$$\left| \sum_{\alpha,\beta=p,s} \left[\frac{\mathcal{T}_\alpha(\xi)^T \overline{\mathcal{N}_\beta(\xi)}}{\delta'(\xi)} e^{\mathbf{i}(\mu_\alpha z_2 - \overline{\mu}_\beta y_2) + \mathbf{i}(y_1 - z_1)\xi} \right]_{-k_R}^{k_R} \right| \leq \frac{C}{\mu} e^{-\sqrt{k_R^2 - k_s^2}h}.$$

Proof. We know that from (2.22), (2.8) that for $\alpha = p, s$, $|\mathcal{T}_\alpha(\pm k_R)| \leq Ck_R^2/k_s^2 \leq C$, $|\mathcal{N}_\alpha(\pm k_R)| \leq Ck_R^3$. The lemma now follows easily by using Lemma 2.7. $\qquad\square$

Lemma 3.3 *Let $g \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$. There exists a constant C depending only on κ such that for any $z, y \in \Omega$,*

$$\left| \mathrm{p.v.} \int_{|\xi|>k_s} \frac{g(\xi)}{\delta(\xi)} d\xi \right| \leq Ck_s^{-4} \int_{|\xi|>k_s} |g(\xi)| d\xi + Ck_s^{-3} \max_{\xi \in (k_R - d_R, k_R + d_R)} (|g(\xi)| + k_s |g'(\xi)|).$$

where $d_R = (k_R - k_s)/2$.

Proof. Without loss of generality, we prove the lemma for the integral in (k_s, ∞). We write $\delta(\xi) = (\xi - k_R)\delta_1(\xi)$ as in Lemma 2.7, where $\delta_1(\xi) \neq 0$ for $\xi > k_s$. By the definition of the Cauchy principle value

$$\mathrm{p.v.} \int_{k_s}^{\infty} \frac{g(\xi)}{\delta(\xi)} d\xi = \int_{k_s}^{k_R - d_R} \frac{g(\xi)}{\delta(\xi)} d\xi + \int_{k_R + d_R}^{\infty} \frac{g(\xi)}{\delta(\xi)} d\xi$$

$$+ \int_{k_R - d_R}^{k_R + d_R} \frac{g(\xi)\delta_1(\xi)^{-1} - g(k_R)\delta_1(k_R)^{-1}}{\xi - k_R} d\xi. \qquad (3.6)$$

By Lemma 2.7 we obtain easily

$$\left| \int_{k_s}^{k_R - d_R} \frac{g(\xi)}{\delta(\xi)} d\xi + \int_{k_R + d_R}^{\infty} \frac{g(\xi)}{\delta(\xi)} d\xi \right| \leq Ck_s^{-4} \int_{k_s}^{\infty} |g(\xi)| d\xi.$$

The last term in (3.6) can be proved by using the mean value theorem and the bounds for $\delta(\xi), \delta_1(\xi)$ in Lemma 2.7. This completes the proof. $\qquad\square$

Lemma 3.4 *Let $k_s h \geq 1$. There exists a constant C depending only on κ such that for any $z, y \in \Omega$,*

$$\left| \sum_{\alpha,\beta=p,s} \mathrm{p.v.} \int_{|\xi|>k_s} \frac{\mathcal{T}_\alpha(\xi)^T \overline{\mathcal{N}_\beta(\xi)}}{\delta(\xi)} e^{\mathbf{i}(\mu_\alpha z_2 - \overline{\mu}_\beta y_2) + \mathbf{i}(y_1 - z_1)\xi} d\xi \right| \leq \frac{C}{\mu} (k_s h)^{-1}.$$

Proof. For $\alpha, \beta = p, s$, we denote $g_{\alpha\beta}(\xi) = \mathcal{T}_\alpha(\xi)^T \overline{\mathcal{N}_\beta(\xi)} e^{\mathbf{i}(\mu_\alpha z_2 - \bar{\mu}_\beta y_2) + \mathbf{i}(y_1 - z_1)\xi}$. By using Lemma 3.3, we obtain easily

$$
\left| \text{p.v.} \int_{|\xi| > k_s} \frac{g_{\alpha\beta}(\xi)}{\delta(\xi)} d\xi \right| \leq \frac{C}{k_s^6 \mu} \int_{k_s}^\infty |\xi|^5 e^{-\sqrt{\xi^2 - k_s^2}(y_2 + z_2)} d\xi + \frac{C}{\mu} k_s h e^{-\sqrt{(k_R - d_R)^2 - k_s^2}(y_2 + z_2)}
$$

$$
\leq \frac{C}{\mu} \int_1^\infty t^5 e^{-\sqrt{t^2 - 1}(y_2 + z_2)} dt + \frac{C}{\mu} k_s h e^{-\sqrt{(k_R - d_R)^2 - k_s^2}(y_2 + z_2)}
$$

$$
\leq \frac{C}{\mu} (k_s h)^{-1} + \frac{C}{\mu} k_s h e^{-\sqrt{(k_R - d_R)^2 - k_s^2}(y_2 + z_2)},
$$

where we have used that $y_2, z_2 \geq h$ and $d_R = (k_R - k_s)/2 \geq C_1 k_s$ for some constant $C_1 > 0$ depending only on κ. This completes the proof as the second term decays exponentially in $k_s h$. $\qquad\square$

Lemma 3.5 *Let $\phi(t) = \sqrt{1 - t^2} - \tau\sqrt{\kappa^2 - t^2} + \nu t$, where $\kappa \in (0, 1), \tau > 0, \nu \in \mathbb{R}$. There exists a constant C depending only on κ such that for any $\lambda > 0$ and $f \in C(0, \kappa)$ with absolutely integrable derivative,*

$$
\left| \int_{-\kappa}^\kappa f(t) e^{\mathbf{i}\lambda\phi(t)} dt \right| \leq C\lambda^{-\frac{1}{4}} \left(|f(\kappa)| + \int_{-\kappa}^\kappa |f'(t)| dt \right).
$$

Proof. We only prove the estimate for the integral in $(0, \kappa)$. The integral in $(-\kappa, 0)$ then follows by taking the transform $t = -t', t' \in (0, \kappa)$. It is easy to check that for $t \in (0, \kappa), m \geq 2,$, the m-the derivative $\phi^{(m)}(t) = \tau\kappa^{-1}\psi_m(t/\kappa) - \psi_m(t)$, where

$$
\psi_2(t) = (1 - t^2)^{-3/2}, \quad \psi_3(t) = 3t(1 - t^2)^{-5/2},
$$
$$
\psi_4(t) = 3(1 + 4t^2)(1 - t^2)^{-7/2}, \quad \psi_5(t) = 15t(3 + 4t^2)(1 - t^2)^{-9/2}.
$$

Obviously, $\psi_m(t), m \geq 2$, are increasing functions in $(0, \kappa)$. If $\tau\kappa^{-1} \geq \kappa$, then

$$
\phi^{(4)}(t) \geq (\tau\kappa^{-1} - 1)\psi_4(t) \geq \psi_4(0) \geq \kappa^{-1} - 1 > 0.
$$

The estimate of the oscillatory integral follows by using the Van der Corput Lemma 2.4.

If $\tau\kappa^{-1} \leq \kappa$, then $\tau\kappa^2 < 1$ and $\phi^{(m)}(t)$ has only one zero in $(0, \kappa)$ at $t = t_m, m \geq 2$, where $t_m = \psi(\zeta_m), \psi(t) = \left[\kappa^2 - \frac{1 - \kappa^2}{t^{-2/3} - 1} \right]^{1/2}$, and ζ_m satisfies

$$
\zeta_2 = \tau\kappa^2, \quad \zeta_3 = (\tau\kappa^2)^{3/5}, \quad \zeta_4 = \left[\tau\kappa^2 \frac{\kappa^2 + 4t^2}{1 + 4t^2} \right]^{3/7}, \quad \zeta_5 = \left[\tau\kappa^2 \frac{3\kappa^2 + 4t^2}{3 + 4t^2} \right]^{1/3}.
$$

Since $\psi(t)$ is decreasing in $(0, \kappa)$, it is easy to check that $t_4 < t_3 < t_2, t_5 < t_3 < t_2$.

Notice that $\phi^{(3)}(t) = 3t[\tau\kappa^2(\kappa^2 - t^2)^{-5/2} - (1 - t^2)^{-5/2}] := 3t f_3(t)$. We have $f_3(t)$ has only one zero at $t = t_3$ and $f_3(0) = \tau\kappa^{-3} - 1 < 0$ and thus $f_3(t) \leq 0$ in $(0, t^3)$. Thus $\phi^{(3)}(t) \leq 0$ in $(0, t_3)$ and consequently $\phi^{(2)}(t)$ is decreasing in $(0, t_3)$. This implies $\phi^{(2)}(t) \leq \phi^{(2)}(0) = \tau\kappa^{-1} - 1 < \kappa - 1 < 0$ in $(0, t_3)$. Thus $|\phi^{(2)}(t)| \geq 1 - \kappa$ in $(0, t_3)$.

Similarly, $\phi^{(5)}(t) = 15t[\tau\kappa^2(3\kappa^2 + 4t^2)(\kappa^2 - t^2)^{-9/2} - (3 + 4t^2)(1 - t^2)^{-2/9}] := 15t f_5(t)$ and $f_5(t)$ has only one zero at $t = t_5 < t_3$ and $f_5(t) \to \infty$ as $t \to \kappa$. Thus $f_5(t) \geq 0$ in (t_3, κ) and consequently, $\phi^{(4)}(t)$ is increasing (t_3, κ) which yields $\phi^{(4)}(t) \geq \phi^{(4)}(t_3)$ in (t_3, κ). On the other hand, since ψ is decreasing and $\tau\kappa^{-1} \leq \kappa, t_3 = \psi((\tau\kappa^2)^{3/5})$ attains

minimum $\psi(\kappa^{12/5})$ when $\tau = k^2$. Thus $|\phi^{(4)}(t)| \geq \phi^{(4)}(\kappa^{12/5}) > 0$ in (t_3, κ). We now split the integral

$$\int_0^\kappa f(t)e^{\mathbf{i}\lambda\phi(t)}dt = \int_0^{t_3} f(t)e^{\mathbf{i}\lambda\phi(t)}dt + \int_{t_3}^\kappa f(t)e^{\mathbf{i}\lambda\phi(t)}dt,$$

and use the van Corput Lemma $\overset{\text{van}}{2.4}$ in the corresponding interval. This completes the proof. $\qquad\square$

The following theorem is the main result of this section.

thm:3.1 **Theorem 3.1** *Let $k_s h \geq 1$. There exists a constant C depending only on κ such that for any $z, y \in \Omega$,*

$$|\mathbb{J}(z,y) - \mathbb{F}(z,y)| + k_s^{-1}|\nabla_y(\mathbb{J}(z,y) - \mathbb{F}(z,y))| \leq \frac{C}{\mu}(k_s h)^{-\frac{1}{4}}.$$

Proof. By Lemma $\overset{\text{lem:3.2}}{3.2}$, Lemma $\overset{\text{lem:3.4}}{3.4}$ and the definitions of $\mathbb{J}(z,y), \mathbb{F}(z,y)$ in $(\overset{\text{d3}}{3.3})$-$(\overset{\text{d4}}{3.4})$, we know that we are left to estimate

$$\text{II}: = \frac{1}{2\pi} \sum_{\alpha,\beta=p,s} \int_{-k_s}^{k_s} \frac{\mathcal{T}_\alpha(\xi)^T \overline{\mathcal{N}_\beta(\xi)}}{\delta(\xi)} e^{\mathbf{i}(\mu_\alpha z_2 - \overline{\mu}_\beta y_2) + \mathbf{i}(y_1 - z_1)\xi}d\xi - \mathbb{F}(z,y)$$

$$= \frac{1}{2\pi} \sum_{\substack{\alpha,\beta=p,s \\ (\alpha,\beta)\neq(s,s)}} \int_{(-k_s,k_s)\backslash[-k_p,k_p]} \frac{\mathcal{T}_\alpha(\xi)^T \overline{\mathcal{N}_\beta(\xi)}}{\delta(\xi)} e^{\mathbf{i}(\mu_\alpha z_2 - \overline{\mu}_\beta y_2) + \mathbf{i}(y_1 - z_1)\xi}d\xi$$

$$+ \frac{1}{2\pi} \int_{-k_p}^{k_p} \left[\frac{\mathcal{T}_p(\xi)\overline{\mathcal{N}_s(\xi)}}{\delta(\xi)} e^{\mathbf{i}(\mu_p y_2 - \mu_s z_2)} + \frac{\mathcal{T}_s(\xi)\overline{\mathcal{N}_p(\xi)}}{\delta(\xi)} e^{\mathbf{i}(\mu_s y_2 - \mu_p z_2)} \right] e^{\mathbf{i}(y_1 - z_1)\xi}d\xi$$

$$:= \text{II}_1 + \text{II}_2.$$

For $k_p < |\xi| < k_s$, it is easy to see that $|\delta(\xi)| \geq Ck_s^4$ and for $\alpha, \beta = p, s$, $|\mathcal{T}_\alpha(\xi)| \leq C, |\mathcal{N}_\beta(\xi)| \leq C\mu^{-1}k_s^2$. This implies

$$|\text{II}_1| \leq \frac{C}{k_s\mu} \int_{k_p}^{k_s} e^{-\sqrt{\xi^2 - k_p^2}h}d\xi \leq \frac{C}{\mu}(k_s h)^{-1}.$$

For the term II_2 we use Lemma $\overset{\text{lem:3.5}}{3.5}$. The first term in II_2 can be reduced to the integral in Lemma $\overset{\text{lem:3.5}}{3.5}$ by setting

$$f(t) = k_s \frac{\mathcal{T}_p(-k_s t)\overline{\mathcal{N}_s(-k_s t)}}{\delta(-k_s t)}, \quad \lambda = k_s z_2, \tau = \frac{y_2}{z_2}, \nu = \frac{z_1 - y_1}{z_2}.$$

By the assumption $(\overset{\text{d0}}{3.5})$, it is then straightforward by using Lemma $\overset{\text{lem:3.5}}{3.5}$ to see that

$$\left| \int_{-k_p}^{k_p} \frac{\mathcal{T}_p(\xi)\overline{\mathcal{N}_s(\xi)}}{\delta(\xi)} e^{\mathbf{i}(\mu_p y_2 - \mu_s z_2)}d\xi \right| \leq \frac{C}{\mu}(k_s h)^{-\frac{1}{4}}.$$

The second integral in II_2 can be estimated similarly. This completes the proof. $\qquad\square$

The following theorem shows that $\mathbb{F}(z,y)$ has the similar behavior as the imaginary part of the elastic fundamental solution $\text{Im}\, \mathbb{G}(z,y)$.

$\boxed{\texttt{thm:3.2}}$ **Theorem 3.2** *For any $z, y \in \mathbb{R}^2_+$, $\mathbb{F}(z,y)^T = \mathbb{F}(z,y)$. When $z = y$,*

$$-\operatorname{Im} \mathbb{F}_{ii}(z,y) \geq \frac{1}{4(\lambda + 2\mu)} \ , \quad i = 1, 2, \qquad \operatorname{Im} \mathbb{F}_{12}(z,y) = \operatorname{Im} \mathbb{F}_{21}(z,y) = 0, \quad (3.7) \quad \boxed{\texttt{d6}}$$

and for $z \neq y$,

$$|\mathbb{F}(z,y)| \leq \frac{C}{\mu}[(k_s|z-y|)^{-1/2}) + (k_s|z-y|^{-1})], \qquad (3.8) \quad \boxed{\texttt{d7}}$$

where constant C depends only on κ.

Proof. Substitute ($\overset{\texttt{d1}}{2.22}$) and ($\overset{\texttt{d2}}{2.8}$) into ($\overset{\texttt{d4}}{3.4}$), we obtain

$$
\begin{aligned}
\mathbb{F}(z,y) = \ & -\frac{1}{2\pi} \int_{-k_p}^{k_p} \frac{\mathbf{i}k_s^2\mu_s}{\mu\gamma(\xi)\delta(\xi)} \begin{pmatrix} \xi^2 & -\xi\mu_p \\ -\xi\mu_p & \mu_p^2 \end{pmatrix} e^{\mathbf{i}\mu_p(z_2-y_2)+\mathbf{i}\xi(y_1-z_1)} d\xi \\
& -\frac{1}{2\pi} \int_{-k_p}^{k_p} \frac{\mathbf{i}k_s^2\mu_p}{\mu\gamma(\xi)\delta(\xi)} \begin{pmatrix} \mu_s^2 & \xi\mu_s \\ \xi\mu_s & \xi^2 \end{pmatrix} e^{\mathbf{i}\mu_s(z_2-y_2)+\mathbf{i}\xi(y_1-z_1)} d\xi \\
& -\frac{1}{2\pi} \int_{(-k_s,k_s)\backslash[-k_p,k_p]} \frac{\mathbf{i}(k_s^2-4\xi^2)\mu_p}{\mu\gamma(\xi)\overline{\delta(\xi)}} \begin{pmatrix} \mu_s^2 & \xi\mu_s \\ \xi\mu_s & \xi^2 \end{pmatrix} e^{\mathbf{i}\mu_s(z_2-y_2)+\mathbf{i}\xi(y_1-z_1)} d\xi \\
:= \ & \text{III}_1 + \text{III}_2 + \text{III}_3. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.9) \quad \boxed{\texttt{d8}}
\end{aligned}
$$

It is easy to show that $\operatorname{Im} \mathbb{F}_{12}(z,y) = \operatorname{Im} \mathbb{F}_{21}(z,y) = 0$.

Now we show the first inequality in ($\overset{\texttt{d6}}{3.7}$) for the case of $i = j = 1$. The other case is similar.

Notice that for $\xi \in (-k_p, k_p)$, $\delta(\xi) \leq k_s^4$ and $\mu_p \leq \mu_s$. Then, if $z = y$,

$$-\operatorname{Im}(\text{III}_1 + \text{III}_2) = \frac{1}{2\pi\mu} \int_{-k_p}^{k_p} \frac{k_s^2\mu_s}{\delta(\xi)} d\xi \geq \frac{1}{2\pi\mu} \int_{-k_p}^{k_p} \frac{\mu_p}{k_s^2} d\xi = \frac{1}{4(\lambda + 2\mu)}.$$

If $\xi \in (-k_s, k_s)\backslash[-k_p, k_p]$, $\mu_p = \mathbf{i}\sqrt{\xi^2 - k_p^2}$, we have

$$-\text{III}_3 = \frac{1}{2\pi\mu} \int_{(-k_s,k_s)\backslash(-k_p,k_p)} \frac{\mu_s^2\sqrt{\xi^2-k_p^2}(k_s^2-4\xi^2)}{(\xi^2+\mathbf{i}\mu_s\sqrt{\xi^2-k_p^2})(\beta^2-\mathbf{i}4\xi^2\mu_s\sqrt{\xi^2-k_p^2})} d\xi.$$

A simple computation shows that $\operatorname{Im}[(\xi^2 + \mathbf{i}\mu_s\sqrt{\xi^2-k_p^2})(\beta^2 - \mathbf{i}4\xi^2\mu_s\sqrt{\xi^2-k_p^2})] = k_s^2\mu_s\sqrt{\xi^2-k_p^2}(k_s^2-4\xi^2)$. It is then clear that $-\operatorname{Im}(\text{III}_3) \geq 0$. This shows $-\operatorname{Im} \mathbb{F}_{11}(z,y) \geq 1/[4(\lambda+2\mu)]$ when $z = y$.

For $z \neq y$, we denote $y - z = |y-z|(\cos\phi, \sin\phi)^T$ for some $0 \leq \phi \leq \pi$. Then it is easy to see that

$$\text{III}_1 = \frac{1}{\mu}\int_0^\pi A(\theta, \kappa)e^{\mathbf{i}k_s|z-y|\cos(\theta-\phi)}d\theta,$$

for some function $A(\theta, \kappa)$. By using ($\overset{\texttt{c1}}{2.13}$)-($\overset{\texttt{c2}}{??}$), which are the consequences of the van Corput Lemma $\overset{\texttt{van}}{2.4}$, we can show easily

$$|\text{III}_1| \leq \frac{C}{\mu}[(k_s|z-y|)^{-1/2}) + (k_s|z-y|^{-1})].$$

The estimate for $\text{III}_2 + \text{III}_3$ can be proved similarly. This completes the proof. $\qquad\square$

4. The reverse time migration algorithm

We start by introducing some notation. Let $\|u\|_{H^1(D)} = (\|\nabla\phi\|_{L^2(D)}^2 + d_D^{-2}\|\phi\|_{L^2(D)}^2)^{1/2}$ be the weighted $H^1(D)$ norm and $\|v\|_{H^{1/2}(\Gamma)} = (d_D^{-1}\|v\|_{L^2(\Gamma)}^2 + |v|_{\frac{1}{2},\Gamma}^2)^{1/2}$ be the weighted $H^{1/2}(\Gamma)$ norm, where d_D is the diameter of D and

$$|v|_{\frac{1}{2},\Gamma} = \left(\int_\Gamma\int_\Gamma \frac{|v(x) - v(y)|^2}{|x - y|^2} ds(x)ds(y)\right)^{1/2}.$$

By the scaling argument and trace theorem we know that there exists a constant $C > 0$ independent of d_D such that for any $\phi \in C^1(\bar{D})^2$ [12, corollary 3.1],

$$\|\phi\|_{H^{1/2}(\Gamma_D)} + \|\sigma(\phi)\nu\|_{H^{-1/2}(\Gamma_D)} \le C \max_{x\in\bar{D}}(|\phi(x)| + d_D|\nabla\phi(x)|). \qquad (4.1) \quad \boxed{\text{q0}}$$

In this paper, for any Sobolev space X, we still denote X and the vector valued space X^2 or tensor valued space $X^{2\times2}$. The norms of $X, X^2, X^{2\times2}$ are all denoted as $\|\cdot\|_X$.

$\boxed{\text{lem:4.1}}$ **Lemma 4.1** *Let $k_s h \ge 1, d \gg h$, there exists a constant C depending only on κ but independent of k_s, h, d, d_D such that for any $z \in \Omega, j = 1, 2$,*

$$\|\mathbb{F}(z,\cdot)e_j\|_{H^{1/2}(\Gamma_D)} + \|\sigma(\mathbb{F}(z,\cdot)e_j)\nu\|_{H^{-1/2}(\Gamma_D)} \le \frac{C}{\mu}(1 + k_s d_D),$$

$$\|\mathbb{R}_d(z,\cdot)e_j\|_{H^{1/2}(\Gamma_D)} + \|\sigma(\mathbb{R}_d(z,\cdot)e_j)\nu\|_{H^{-1/2}(\Gamma_D)} \le \frac{C}{\mu}(1 + k_s d_D)\left[\left(\frac{h}{d}\right)^2 + (k_s h)^{-1/4}\right],$$

where $\mathbb{R}_d(z,\cdot) = \mathbb{J}_d(z,\cdot) - \mathbb{F}(z,\cdot)$.

Proof. The first estimate follows easily from (4.1) and the definition of $\mathbb{F}(z,\cdot)$ in (3.4). The second estimate follows from (4.1), Lemma 3.1 and Theorem 3.1. This completes the proof. $\qquad\square$

Now we briefly recall the classical argument of limiting absorption principle (see e.g. [25, 29, 18]) to define the scattering solution for the half-space elastic exterior problem:

$$\Delta_e u + \omega^2 u = 0 \qquad \text{in } \mathbb{R}_+^2\setminus\bar{D}, \qquad\qquad (4.2) \quad \boxed{\text{pp1}}$$

$$u = g \quad \text{on } \Gamma_D, \qquad \sigma(u)e_2 = 0 \quad \text{on } \Gamma_0, \qquad (4.3) \quad \boxed{\text{pp2}}$$

where $g \in H^{1/2}(\Gamma_D)$. Let $\varepsilon > 0$ and u_ε be the solution of the problem

$$\Delta_e u_\varepsilon + \omega^2(1 + \mathbf{i}\varepsilon)u_\varepsilon = 0 \qquad \text{in } \mathbb{R}_+^2\setminus\bar{D}, \qquad (4.4) \quad \boxed{\text{pp3}}$$

$$u_\varepsilon = g \quad \text{on } \Gamma_D, \qquad \sigma(u_\varepsilon)e_2 = 0 \quad \text{on } \Gamma_0. \qquad (4.5) \quad \boxed{\text{pp4}}$$

By the Lax-Milgram lemma, the problem (4.4)-(4.5) has a unique solution $u_\varepsilon \in H^1(\mathbb{R}_+^2\setminus\bar{D})$. Let $\mathcal{D}(\Delta_e) = \{v \in H^1(\mathbb{R}_+^2\setminus\bar{D}) : \Delta_e v \in L^2(\mathbb{R}_+^2\setminus\bar{D}), v = 0 \text{ on } \Gamma_D, \sigma(v)e_2 = 0 \text{ on } \Gamma_0\}$ as the domain of the operator $-\Delta_e$, it is shown in [18] that if ω^2 is not the eigenvalue for $-\Delta_e$ in the domain $\mathcal{D}(\Delta_e)$, u_ε converges to some function u satisfying (4.2)-(4.3) in $H^{1,-s}(\mathbb{R}_+^2\setminus\bar{D}), s > 1/2$, where the weighted Sobolev space $H^{1,s}(\mathbb{R}_+^2\setminus\bar{D}), s \in \mathbb{R}$, is defined as the set of functions in $L^{2,s}(\mathbb{R}_+^2\setminus\bar{D}) = \{v \in L_{\text{loc}}^2(\mathbb{R}_+^2\setminus\bar{D}) : (1 + |x|^2)^{s/2}v \in L^2(\mathbb{R}_+^2\setminus\bar{D})\}$ whose first derivative is also in $L^{2,s}(\mathbb{R}_+^2\setminus\bar{D})$. The norm $\|v\|_{H^{1,s}(D)} = $

$(\|v\|^2_{L^{2,s}(\mathbb{R}^2_+\backslash\bar{D})} + \|\nabla v\|^2_{L^{2,s}(\mathbb{R}^2_+\backslash\bar{D})})^{1/2}$, where $\|v\|_{L^{2,s}(\mathcal{D})} = (\int_{\mathcal{D}}(1+|x|^2)^s|v|^2dx)^{1/2}$. The absence of the positive eigenvalue for the operator $-\Delta_e$ is proved in [28] in the domain $\mathcal{D}'(\Delta_e) = \{v \in H^1(\mathbb{R}^2_+\backslash\bar{D}), \Delta_e v \in L^2(\mathbb{R}^2_+\backslash\bar{D}), \sigma(v)e_2 = 0 \text{ on } \Gamma_D \cup \Gamma_0\}$. One can easily extend the argument in [28] to show the absence of the positive eigenvalue for $-\Delta_e$ also in the domain $\mathcal{D}(\Delta_e)$ and thus obtain the following theorem for the forward scattering problem.

Theorem 4.1 *Let* $g \in H^{1/2}(\Gamma_D)$. *The half-space elastic scattering problem* (4.2)-(4.3) *admits a unique solution* $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2_+\backslash\bar{D})$. *Moreover, for any bounded open set* $\mathcal{O} \subset \mathbb{R}^2_+\backslash\bar{D}$ *there exists a constant* $C > 0$ *such that* $\|u\|_{H^1(\mathcal{O})} \leq C\|g\|_{H^{1/2}(\Gamma_D)}$.

For the sake of convenience, we introduce the following notation: for any $u, v \in H^1(\mathbb{R}^2\backslash\bar{D})$ such that $\Delta_e u, \Delta_e v \in L^2(\mathbb{R}^2\backslash\bar{D})$,

$$\mathcal{G}(u,v) = \int_{\Gamma_D} [u(x) \cdot \sigma(v(x))\nu - \sigma(u(x))\nu \cdot v(x)]ds(x). \tag{4.6}$$

Using this notation, the integral representation formula for the solution of the half-space elastic scattering problem reads:

$$u(y) \cdot q = \mathcal{G}(u(\cdot), \mathbb{N}(\cdot, y)q), \quad \forall q \in \mathbb{R}^2. \tag{4.7}$$

Now we introduce the RTM algorithm for the half-space inverse elastic scattering problem. Assume that there are $N_s \geq 1$ sources and $N_r \geq 1$ receivers uniformly distributed on Γ_0^d, where $\Gamma_0^d = \{(x_1, x_2)^T \in \Gamma_0 : x_1 \in (-d, d)\}, d > 0$ is the aperture.

For any $q \in \mathbb{R}^2$, let u_q^i be the incident field which satisfies

$$\Delta_e u_q^i(x, x_s) + \omega^2 u_q^i(x, x_s) = 0 \text{ in } \mathbb{R}^2_+, \quad u_q^i(x, x_s) = q\delta_{x_s}(x) \text{ on } \Gamma_0.$$

By the integral representation formula, $u_q^i(x, x_s) = \mathbb{T}_D(x_s, x)^T q$. The following algorithm extends the algorithm in [12] for acoustic waves.

Algorithm 4.1 (REVERSE TIME MIGRATION ALGORITHM)
Given the data $u_q^s(x_r, x_s)$ *which is the measurement of the scattered field at* x_r *when the source is emitted at* x_s *along the polarized direction* $q = e_1, e_2, s = 1, \ldots, N_s$, $r = 1, \ldots, N_r$.

$1°$ *Back-propagation: Compute* $v_q(x, x_s)$ *as the scattering solution of the following half-space elastic problem:*

$$\Delta_e v_q(x, x_s) + \omega^2 v_q(x, x_s) = 0 \text{ in } \mathbb{R}^2_+,$$

$$v_q(x, x_s) = \frac{|\Gamma_0^d|}{N_r} \sum_{r=1}^{N_r} \overline{u_q^s(x_r, x_s)}\delta_{x_r}(x) \text{ on } \Gamma_0.$$

$2°$ *Cross-correlation: For each* $z \in \Omega$, *compute the imaging function*

$$I_d(z) = \mathrm{Im} \sum_{q=e_1,e_2} \left\{ \frac{|\Gamma_0^d|}{N_s} \sum_{s=1}^{N_s} u_q^i(z, x_s) \cdot v_q(z, x_s) \right\}. \tag{4.8}$$

By the integral representation formula, we know that

$$v_q(x, x_s) \cdot e_j = \frac{|\Gamma_0^d|}{N_r} \sum_{r=1}^{N_r} \mathbb{T}_D(x_r, x) e_j \cdot \overline{u_q(x_r, x_s)},$$

which yields

$$I_d(z) = \text{Im} \sum_{q=e_1,e_2} \left\{ \frac{|\Gamma_0^d|^2}{N_s N_r} \sum_{s=1}^{N_s} \sum_{r=1}^{N_r} [\mathbb{T}_D(x_s, z)^T q] \cdot [\mathbb{T}_D(x_r, z)^T \overline{u_q^s(x_r, x_s)}] \right\}. \quad (4.9) \quad \boxed{\text{cor}}$$

This is the formula will be used in our numerical examples in section 6. By letting $N_s, N_r \to \infty$, we know that (4.8) $\boxed{\text{cor1}}$ can be viewed as an approximation of the following continuous integral:

$$\hat{I}_d(z) = \text{Im} \sum_{q=e_1,e_2} \int_{\Gamma_0^d} \int_{\Gamma_0^d} [\mathbb{T}_D(x_s, z)^T q] \cdot [\mathbb{T}_D(x_r, z)^T \overline{u_q^s(x_r, x_s)}] \, ds(x_r) ds(x_s). \quad (4.10) \quad \boxed{\text{cor2}}$$

The following theorem which extends [12, Theorem 4.1] $\boxed{\text{RTMhalf\_aco}}$ for acoustic waves will be proved in the Appendix of this paper. It shows that the difference between the half-space scattering solution and the full space scattering solution is small when the scatterer is far away from the boundary Γ_0.

$\boxed{\text{thm:4.2}}$ **Theorem 4.2** *Let* $g \in H^{1/2}(\Gamma_D)$ *and* u_1, u_2 *be the scattering solution of following problems:*

$$\Delta_e u_1 + \omega^2 u_1 = 0 \ \text{ in } \mathbb{R}_+^2 \backslash \bar{D}, \quad u_1 = g \ \text{ on } \Gamma_D, \quad \sigma(u_1) e_2 = 0 \ \text{ on } \Gamma_0, \quad (4.11) \quad \boxed{\text{e1}}$$

and

$$\Delta_e u_2 + \omega^2 u_2 = 0 \ \text{ in } \mathbb{R}^2 \backslash \bar{D}, \quad u_2 = g \ \text{ on } \Gamma_D. \quad (4.12) \quad \boxed{\text{e2}}$$

Then there exits a constant C *depending only on* κ *but independant of* k_s, h, d_D *such that*

$$\|\sigma(u_1 - u_2)\nu\|_{H^{-1/2}(\Gamma_D)} \le \frac{C}{\mu} (1 + \|T_1\|)(1 + \|T_2\|)(1 + k_s d_D)^2 (k_s h)^{-1/2} \|g\|_{H^{1/2}(\Gamma_D)}.$$

Here $T_1, T_2 : H^{1/2}(\Gamma_D) \to H^{-1/2}(\Gamma_D)$ *are the Dirichlet to Neumann mapping associated to the elastic scattering problem* (4.11) $\boxed{\text{e1}}$ *and* (4.12) $\boxed{\text{e2}}$, *respectively.* $\|T_1\|, \|T_2\|$ *denote their operator norms.*

We remark that the well-posedness of the full space elastic scattering problem (4.12) $\boxed{\text{e2}}$ under the so-called Sommerfeld-Kupradze radiation condition is well known (cf. e.g. [23] $\boxed{\text{ku63}}$). It is equivalent to the solution defined by the limiting absorption principle [25, 13] $\boxed{\text{leis, cxz2016}}$.

The following theorem, which relates the imaging function $\hat{I}_d(z)$ to the point spread function in section 3, shows the resolution of our RTM imaging algorithm for the half-space inverse elastic scattering problems.

$\boxed{\text{thm:4.3}}$ **Theorem 4.3** *For any* $z \in \Omega$, *let* $\mathbb{U}(z, x) \in \mathbb{C}^{2 \times 2}$ *such that* $\mathbb{U}(z, x) e_j$, $j = 1, 2$, *is the scattering solution of the problem:*

$$\Delta_e[\mathbb{U}(z, x) e_j] + \omega^2[\mathbb{U}(z, x) e_j] = 0 \ \text{ in } \mathbb{R}^2 \backslash \bar{D}, \quad \mathbb{U}(z, x) e_j = -\overline{\mathbb{F}(z, x)} e_j \ \text{ on } \Gamma_D.$$

Then, we have

$$\hat{I}_d(z) = \text{Im} \sum_{j=1}^{2} \int_{\Gamma_D} [\sigma(\mathbb{U}(z,x)e_j + \overline{\mathbb{F}(z,x)}e_j)\nu] \cdot [\overline{\mathbb{F}(z,x)}e_j]ds(x) + R_d(z),$$

where $|R_d(z)| \leq C\mu^{-2}(1 + \|T_1\|)(1 + \|T_2\|)(1 + k_s d_D)^3 \left[\left(\frac{h}{d}\right)^2 + (k_s h)^{-1/4}\right]$ *for some constant* C *depending only on* κ *but independent of* k_s, k_p, h, d, d_D.

Proof. From (4.10) we know that

$$\hat{I}_d(z) = \text{Im} \sum_{q=e_1,e_2} \int_{\Gamma_0^d} [\mathbb{T}_D(x_s, z)^T q] \cdot \hat{\mathbb{V}}_q(z, x_s)ds(x_s), \qquad (4.13) \quad \boxed{\text{g5}}$$

where for $j = 1, 2$,

$$\hat{\mathbb{V}}_q(z, x_s) \cdot e_j = \int_{\Gamma_0^d} \mathbb{T}_D(x_r, z)e_j \cdot \overline{u_q^s(x_r, x_s)}ds(x_r).$$

By (4.7) we know that $u_q^s(x_r, x_s) \cdot e_i = \mathcal{G}(u_q^s(\cdot, x_s), \mathbb{N}(\cdot, x_r)e_i), i = 1, 2$, and thus

$$\hat{\mathbb{V}}(z, x_s) \cdot e_j = \mathcal{G}\left(\overline{u_q^s(\cdot, x_s)}, \left[\int_{\Gamma_0^d} \sum_{i=1}^{2} [\mathbb{T}_D(x_r, z)]_{ij} \overline{\mathbb{N}(\cdot, x_r)}e_i ds(x_r)\right]\right).$$

By using the reciprocity relation $\mathbb{N}(x, x_r) = \mathbb{N}(x_r, x)^T$ and the definition of $\mathbb{J}_d(\cdot, \cdot)$ in (3.1), we obtain

$$\int_{\Gamma_0^d} \sum_{i=1}^{2} [\mathbb{T}_D(x_r, z)]_{ij} \overline{\mathbb{N}(\cdot, x_r)}e_i ds(x_r) = \mathbb{J}_d(z, x)^T e_j. \qquad (4.14) \quad \boxed{\text{g6}}$$

This implies $\hat{\mathbb{V}}_q(z, x_s) = \mathcal{G}(\overline{u_q^s(\cdot, x_s)}, \mathbb{J}_d(z, \cdot)^T e_j)$. Substitute it into (4.13) we have

$$\hat{I}_d(z) = \text{Im} \sum_{j=1}^{2} \mathcal{G}(\mathbb{W}(z, \cdot)e_j, \mathbb{J}_d(z, \cdot)^T e_j), \qquad (4.15) \quad \boxed{\text{g3}}$$

where $\mathbb{W}(z, x) \in \mathbb{C}^{2 \times 2}$ is the tensor defined by

$$\mathbb{W}(z, x)e_j = \int_{\Gamma_0^d} \sum_{k=1}^{2} [\mathbb{T}_D(x_s, z)]_{kj} \overline{u_{e_k}^s(x, x_s)}ds(x_s), \quad j = 1, 2.$$

Notice that $\overline{\mathbb{W}(z, x)}e_j$ can be viewed as the weighted superposition of $u_{e_k}^s(x, x_s)$ and thus it satisfies

$$\Delta_e[\overline{\mathbb{W}(z, x)}e_j] + \omega^2[\overline{\mathbb{W}(z, x)}e_j] = 0 \ \text{ in } \mathbb{R}_+^2 \backslash \bar{D}, \qquad \sigma(\overline{\mathbb{W}(z, x)}e_j)e_2 = 0 \ \text{ on } \Gamma_0. \quad (4.16) \quad \boxed{\text{g7}}$$

On the boundary of the obstacle Γ_D, by using (4.14) we have

$$\overline{\mathbb{W}(z, x)}e_j = -\int_{\Gamma_0^d} \sum_{k=1}^{2} [\mathbb{T}_D(x_s, z)]_{kj} \overline{\mathbb{N}(x, x_s)}e_k ds(x_s) = -\overline{\mathbb{J}_d(z, x)}^T e_j. \qquad (4.17) \quad \boxed{\text{g8}}$$

Now define the tensor $\mathbb{W}_d(z, x) \in \mathbb{C}^{2 \times 2}$ such that $\mathbb{W}_d(z, x)e_j, j = 1, 2$, is the scattering solution of the problem

$$\Delta_e[\mathbb{W}_d(z, x)e_j] + \omega^2[\mathbb{W}_d(z, x)e_j] = 0 \quad \text{in } \mathbb{R}_+^2 \backslash \bar{D}, \qquad (4.18) \quad \boxed{\text{g9}}$$

$$\mathbb{W}_d(z, x)e_j = -\overline{\mathbb{F}(z, x)}e_j \ \text{ on } \Gamma_D, \qquad \sigma(\mathbb{W}_d(z, x)e_j)e_2 = 0 \ \text{ on } \Gamma_0. \qquad (4.19) \quad \boxed{\text{g10}}$$

By (4.15) we deduce

$$\hat{I}_d(z) = \mathrm{Im} \sum_{j=1}^{2} \mathcal{G}(\mathbb{W}(z,\cdot)e_j, J_d(z,\cdot)^T e_j - \mathbb{F}(z,\cdot)e_j)$$

$$+ \mathrm{Im} \sum_{j=1}^{2} \mathcal{G}(\mathbb{W}(z,\cdot)e_j - \overline{\mathbb{W}_d(z,\cdot)}e_j, \mathbb{F}(z,\cdot)e_j)$$

$$+ \mathrm{Im} \sum_{j=1}^{2} \mathcal{G}(\overline{\mathbb{W}_d(z,\cdot)}e_j - \overline{\mathbb{U}(z,\cdot)}e_j, \mathbb{F}(z,\cdot)e_j)$$

$$+ \mathrm{Im} \sum_{j=1}^{2} \mathcal{G}(\overline{\mathbb{U}(z,\cdot)}e_j, \mathbb{F}(z,\cdot)e_j) := \mathrm{VI}_1 + \cdots + \mathrm{VI}_4. \qquad (4.20)$$

Recall that $\mathbb{F}(z,y)^T = \mathbb{F}(z,y)$ by Theorem 3.2. By Lemma 4.1,

$$\|\mathbb{J}_d(z,\cdot)e_j\|_{H^{1/2}(\Gamma_D)} + \|\sigma(\mathbb{J}_d(z,\cdot)e_j)\nu\|_{H^{-1/2}(\Gamma_D)} \le \frac{C}{\mu}(1 + k_s d_D).$$

This implies, by (4.16)-(4.17), that

$$|\mathrm{VI}_1| \le \frac{C}{\mu^2}(1 + \|T_1\|)(1 + k_s d_D)^2 \left[\left(\frac{h}{d}\right)^2 + (k_s h)^{-1/4}\right].$$

From (4.16)-(4.17) and (4.18)-(4.19), we obtain by using Lemma 4.1 that

$$|\mathrm{VI}_2| \le \frac{C}{\mu^2}(1 + \|T_1\|)(1 + k_s d_D)^2 \left[\left(\frac{h}{d}\right)^2 + (k_s h)^{-1/4}\right].$$

To estimate the third term, we use Theoremn 4.2 and Lemma 4.1 to obtain

$$|\mathrm{VI}_3| \le \frac{C}{\mu^2}(1 + \|T_1\|)(1 + \|T_2\|)(1 + k_s d_D)^3 (k_s h)^{-1/2}.$$

Finally, since $\mathbb{U}(z,x)e_j = -\mathbb{F}(z,x)e_j$ on Γ_D,

$$\mathrm{IV}_4 = \mathrm{Im} \sum_{j=1}^{2} \int_{\Gamma_D} (\overline{\mathbb{U}(z,x)}e_j \cdot \sigma(\mathbb{F}(z,x)e_j) - \sigma(\overline{\mathbb{U}(z,x)}e_j) \cdot \mathbb{F}(z,x)e_j)ds(x)$$

$$= -\mathrm{Im} \sum_{j=1}^{2} \int_{\Gamma_D} \sigma(\overline{\mathbb{U}(z,x)}e_j + \mathbb{F}(z,x)e_j) \cdot \mathbb{F}(z,x)e_j ds(x)$$

$$= \mathrm{Im} \sum_{j=1}^{2} \int_{\Gamma_D} \sigma(\mathbb{U}(z,x)e_j + \overline{\mathbb{F}(z,x)}e_j) \cdot \overline{\mathbb{F}(z,x)}e_j ds(x).$$

The lemma follows now from (4.20). $\qquad\qquad\square$

By (3.9) we know that for any fixed $z \in \Omega$ and some functions $A_j(\xi), B_j(\xi)$, $j = 1, 2$,

$$\mathbb{F}(z,x)e_j = \int_{-k_p}^{k_p} A_j(\xi) \begin{pmatrix} -\xi \\ \mu_p \end{pmatrix} e^{\mathbf{i}(z-x)\cdot(-\xi,\mu_p)^T} d\xi + \int_{-k_s}^{k_s} B_j(\xi) \begin{pmatrix} \mu_s \\ \xi \end{pmatrix} e^{\mathbf{i}(z-x)\cdot(-\xi,\mu_s)^T} d\xi$$

$$= \int_{0}^{\pi} \left[\tilde{A}_j(\theta)\tau(\theta)e^{\mathbf{i}k_p(z-x)\cdot\tau(\theta)} + \tilde{B}_j(\theta)\tau(\theta)^{\perp}e^{\mathbf{i}k_s(z-x)\cdot\tau(\theta)}\right] d\theta,$$

where $\tilde{A}_j(\theta) = k_p A_j(k_p \cos\theta) \sin\theta$, $\tilde{B}_j(\theta) = k_s B_j(k_s \cos\theta) \sin\theta$, $\tau(\theta) = (-\cos\theta, \sin\theta)^T$ and $\tau(\theta)^\perp = (\sin\theta, \cos\theta)^T$. Thus $\overline{\mathbb{F}(z,x)}e_j$ is the weighted superposition of planar p-waves and s-waves and thus satisfies the elastic wave equation. Therefore $\mathbb{U}(z,x)e_j$ can be viewed as the scattering solution of the elastic equation with the incident wave $\overline{\mathbb{F}(z,x)}e_j$. By Theorem 3.2 we know that $\overline{\mathbb{F}(z,x)}$ decays as $|x-z|$ becomes large. Therefore the imaging function $\hat{I}_d(z)$ becomes small when z moves away from the boundary Γ_D if $k_s h \gg 1$ and $d \gg h$.

To understand the behavior of the imaging function when z is close to the boundary of the scatterer, we introduce the concept of the scattering coefficient for incident plane waves.

`scarr_con` **Definition 4.1** *For any unit vector $\tau \in \mathbb{R}^2$, let $u_p^i = \tau e^{\mathbf{i}k_p x \cdot \tau}$, $u_s^i = \tau^\perp e^{\mathbf{i}k_s x \cdot \tau}$ be the incident planar p-wave and s-wave. Let $u_\alpha^s(x) = u_\alpha^s(x;\tau), \alpha = p, s$, be the corresponding scattering solution of the elastic wave equation:*

$$\Delta_e u_\alpha^s + \omega^2 u_\alpha^s = 0 \quad in \ \mathbb{R}^2 \backslash \bar{D}, \qquad u_\alpha^s = -u_\alpha^i \quad on \ \Gamma_D.$$

The scattering coefficient $R_\alpha(x;\tau)$, $x \in \Gamma_D$, is defined by the relation

$$\sigma(u_\alpha^s(x) + u_\alpha^i(x))\nu = \mathbf{i}k_\alpha R_\alpha(x;d) e^{\mathbf{i}k_\alpha x \cdot d} \quad on \ \Gamma_D.$$

Here for $\tau = (\tau_1, \tau_2)^T \in \mathbb{R}^2$, $\tau^\perp = (\tau_2, -\tau_1)^T$.

With this definition we deduce from Theorem 4.3 that for any $z \in \Gamma_D$,

$$\hat{I}_d(z) \approx \mathrm{Im} \sum_{j=1}^{2} \int_{\Gamma_D} \left[\int_0^\pi \overline{\tilde{A}_j(\theta)} \mathbf{i}k_p R_p(x;\tau(\theta)) e^{\mathbf{i}k_p(x-z)\cdot\tau(\theta)} d\theta \right] \cdot \overline{\mathbb{F}(z,x)} e_j ds(x)$$

$$+ \mathrm{Im} \sum_{j=1}^{2} \int_{\Gamma_D} \left[\int_0^\pi \overline{\tilde{B}_j(\theta)} \mathbf{i}k_s R_s(x;\tau(\theta)) e^{\mathbf{i}k_s(x-z)\cdot\tau(\theta)} d\theta \right] \cdot \overline{\mathbb{F}(z,x)} e_j ds(x).$$

In the case of Kirchhoff high-frequency approximation, the scattering coefficient is approximately zero in the shadow region of the obstacle:

$$R_\alpha(x;\tau) \approx 0 \ \ \text{if } x \in \Gamma_D^+(d) = \{x \in \Gamma_D, \nu(x)\cdot\tau > 0\}, \quad \alpha = p, s.$$

Let $x(s)$, $0 < s < L$, be the arc length parametrization of the boundary Γ_D and $x_\pm(\theta)$ be the points on Γ_D such that $\nu(x_\pm(\theta)) = \pm\tau(\theta)$. By using the method of the stationary phase and the above Kirchhoff approximation we can obtain as in [12] that

$$\hat{I}_d(z) \approx \mathrm{Im} \sum_{j=1}^{2} \sqrt{2\pi k_p} \int_0^\pi \overline{\tilde{A}_j(\theta)} e^{\mathbf{i}k_p(x_-(\theta)-z)\cdot\tau(\theta)+\mathbf{i}\frac{\pi}{4}} \frac{R_p(x_-(\theta);\tau(\theta)) \cdot \overline{\mathbb{F}(z,x_-(\theta))} e_j}{\sqrt{\kappa(x_-(\theta))}} d\theta$$

$$+ \mathrm{Im} \sum_{j=1}^{2} \sqrt{2\pi k_s} \int_0^\pi \overline{\tilde{B}_j(\theta)} e^{\mathbf{i}k_s(x_-(\theta)-z)\cdot\tau(\theta)+\mathbf{i}\frac{\pi}{4}} \frac{R_s(x_-(\theta);\tau(\theta)) \cdot \overline{\mathbb{F}(z,x_-(\theta))} e_j}{\sqrt{\kappa(x_-(\theta))}} d\theta.$$

Here $\kappa(x)$ is the curvature of Γ_D. Now for z in the part of Γ_D which is back to Γ_0, i.e., $\nu(z)\cdot\tau(\theta) > 0$ for any $\theta \in (0, \pi)$, we know that z and $x_-(\theta)$ are far away and thus $\hat{I}_d(z) \approx 0$. This indicates that one cannot image the back part of the obstacle with only the data collected on Γ_0. This is confirmed in our numerical examples in section 6.

5. Extensions

In this section we consider the reconstruction of non-penetrable obstacles with the impedance boundary condition and penetrable obstacles in the half space by the RTM algorithm 4.1. For non-penetrable obstacles with the impedance boundary condition on the obstacle, the measured data $u_q(x_r.x_s) = u_q^s(x_r, x_s) + \mathbb{N}(x_r, x_s)q$, $q = e_1, e_2$, where $u_q^s(x, x_s)$ is the scattering solution of the following problem:

$$\Delta_e u_q^s(x, x_s) + \omega^2 u_q^s(x, x_s) = 0 \ \ \text{in } \mathbb{R}_+^2 \backslash \bar{D},$$
$$\sigma(u_q^s(x, x_s))\nu + \mathbf{i}\eta(x)u_q^s(x, x_s) = -[\sigma(\mathbb{N}(x, x_s)q)\nu + \mathbf{i}\eta(x)\mathbb{N}(x, x_s)q] \ \ \text{on } \Gamma_D,$$
$$\sigma(u_q^s(x, x_s))e_2 = 0 \ \ \text{on } \Gamma_0.$$

By modifying the argument in the proof of Theorem 4.3, we can show the following theorem whose proof is omitted.

Theorem 5.1 *For any $z \in \Omega$, let $\mathbb{U}(z, x) \in \mathbb{C}^{2\times 2}$ such that $\mathbb{U}(z, x)e_j$, $j = 1, 2$, is the scattering solution of the problem:*

$$\Delta_e[\mathbb{U}(z, x)e_j] + \omega^2[\mathbb{U}(z, x)e_j] = 0 \ \ \text{in } \mathbb{R}^2 \backslash \bar{D},$$
$$\sigma(\mathbb{U}(z, x)e_j)\nu + \mathbf{i}\eta(x)[\mathbb{U}(z, x)e_j] = -[\sigma(\overline{\mathbb{F}(z, x)}e_j)\nu + \mathbf{i}\eta(x)\overline{\mathbb{F}(z, x)}e_j] \ \ \text{on } \Gamma_D.$$

Then the imaging function (4.10) for the half-space elastic scattering data $u_q^s(x_r, x_s)$ of the non-penetrable obstacle with the impedance boundary condition satisfies

$$\hat{I}_d(z) = - \text{Im} \sum_{j=1}^{2} \int_{\Gamma_D} [\mathbb{U}(z, x)e_j + \overline{\mathbb{F}(z, x)}e_j] \cdot [\sigma(\overline{\mathbb{F}(z, x)}e_j)\nu + \mathbf{i}\eta(x)\overline{\mathbb{F}(z, x)}e_j]ds(x)$$
$$+ R_d(z), \ \ \forall z \in \Omega,$$

where $|R_d(z)| \le C\mu^{-2}(1 + k_s d_D)^3 \left[\left(\frac{h}{d}\right)^2 + (k_s h)^{-1/4}\right]$ for some constant C depending only on κ but independent of k_s, k_p, h, d, d_D.

For the penetrable obstacle, the measured data is $u_q(x_r.x_s) = u_q^s(x_r, x_s) + \mathbb{N}(x_r, x_s)q$, $q = e_1, e_2$, where $u_q^s(x, x_s)$ is the scattering solution of the following problem:

$$\Delta_e u_q^s(x, x_s) + \omega^2 n(x)u_q^s(x, x_s) = -\omega^2(n(x) - 1)\mathbb{N}(x, x_s)q \ \ \text{in } \mathbb{R}^2,$$
$$\sigma(u_q^s(x, x_s))e_2 = 0 \ \ \text{on } \Gamma_0,$$

where $n(x) \in L^\infty(\mathbb{R}^2)$ is a positive function which is equal to 1 outside D. By modifying the argument in Theorem 4.3, the following theorem can be proved.

Theorem 5.2 *For any $z \in \Omega$, let $\mathbb{U}(z, x) \in \mathbb{C}^{2\times 2}$ such that $\mathbb{U}(z, x)e_j$, $j = 1, 2$, is the scattering solution of the problem:*

$$\Delta_e[\mathbb{U}(z, x)e_j] + \omega^2 n(x)[\mathbb{U}(z, x)e_j] = -\omega^2(n(x) - 1)\overline{\mathbb{F}(z, x)}e_j \ \ \text{in } \mathbb{R}^2.$$

Then the imaging function (4.10) for the half-space elastic scattering data $u_q^s(x_r, x_s)$ of the penetrable obstacle satisfies

$$\hat{I}_d(z) = \text{Im} \sum_{j=1}^{2} \int_D \omega^2(n(x) - 1)[(\mathbb{U}(z, x)e_j + \overline{\mathbb{F}(z, x)}e_j), \cdot \overline{\mathbb{F}(z, x)}e_j]dx + R_d(z), \quad (5.1)$$
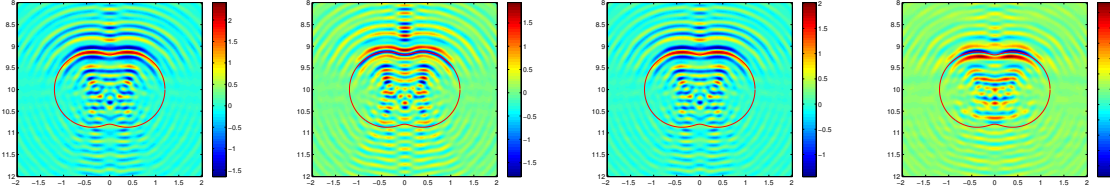
Figure 2. Example 1: From left to right: imaging results of a Dirichlet, a Neumann, an impedance with $\eta(x) = 1$, and a penetrable obstacle with diffractive index $n(x) = 0.25$.

where $|R_d(z)| \leq C\mu^{-2}(1 + k_s d_D)^3 \left[\left(\frac{h}{d}\right)^2 + (k_s h)^{-1/4}\right]$ for some constant C depending only on κ but independent of k_s, k_p, h, d, d_D.

6. Numerical examples

In this section we present several numerical examples to show the effectiveness of our RTM algorithm. To synthesize the scattering data we compute the solution $u^s(x; x_s)$ of the scattering problems by representing the ansatz solution as the single layer potential with the Neumann Green function $\mathbb{N}(x, y)$ as the kernel and discretizing the integral equation by standard Nyström methods [16]. The boundary integral equations on Γ_D are solved on a uniform mesh over the boundary with ten points per probe wavelength. The sources and receivers are both equally placed on the surface Γ_0^d, where d is the aperture. In all our numerical examples we choose $h = 10, d = 50$ and Lamé constant $\lambda = 1/2, \mu = 1/4$. The boundaries of the obstacles used in our numerical experiments are parameterized as follows,

Circle: $x_1 = \rho\cos(\theta), \quad x_2 = \rho\sin(\theta)$;

Kite: $x_1 = \cos(\theta) + 0.65\cos(2\theta) - 0.65, \quad x_2 = 1.5\sin(\theta)$;

p-leaf: $r(\theta) = 1 + 0.2\cos(p\theta)$;

peanut: $x_1 = \cos\theta + 0 : 2\cos 3\theta; x_2 = \sin\theta + 0 : 2\sin 3\theta$;

square: $x_1 = \cos 3\theta + \cos\theta; x_2 = \sin 3\theta + \sin\theta$.

where $\theta \in [0, 2\pi]$. The numerical imaging function is (4.9) in section 4.

In the following by Dirichlet, Neumann or impedance obstacle we mean the non-penetrable obstacle that satisfies Dirichlet, Neumann or impedance boundary conditions on the boundary of the obstacle.

Example 1. We consider imaging of a Dirichlet, a Neumann, an impedance, and a penetrable obstacle. The imaging domain $\Omega = (-2, 2) \times (8, 12)$ with the sampling grid 201×201. We set $N_s = N_r = 401$. The angular frequency $\omega = 2\pi$.

The imaging results are shown in Figure 2. It demonstrates clearly that our RTM algorithm can effectively image the upper boundary illuminated by the sources and receivers distributed along the boundary Γ_0 for non-penetrable obstacles. The imaging
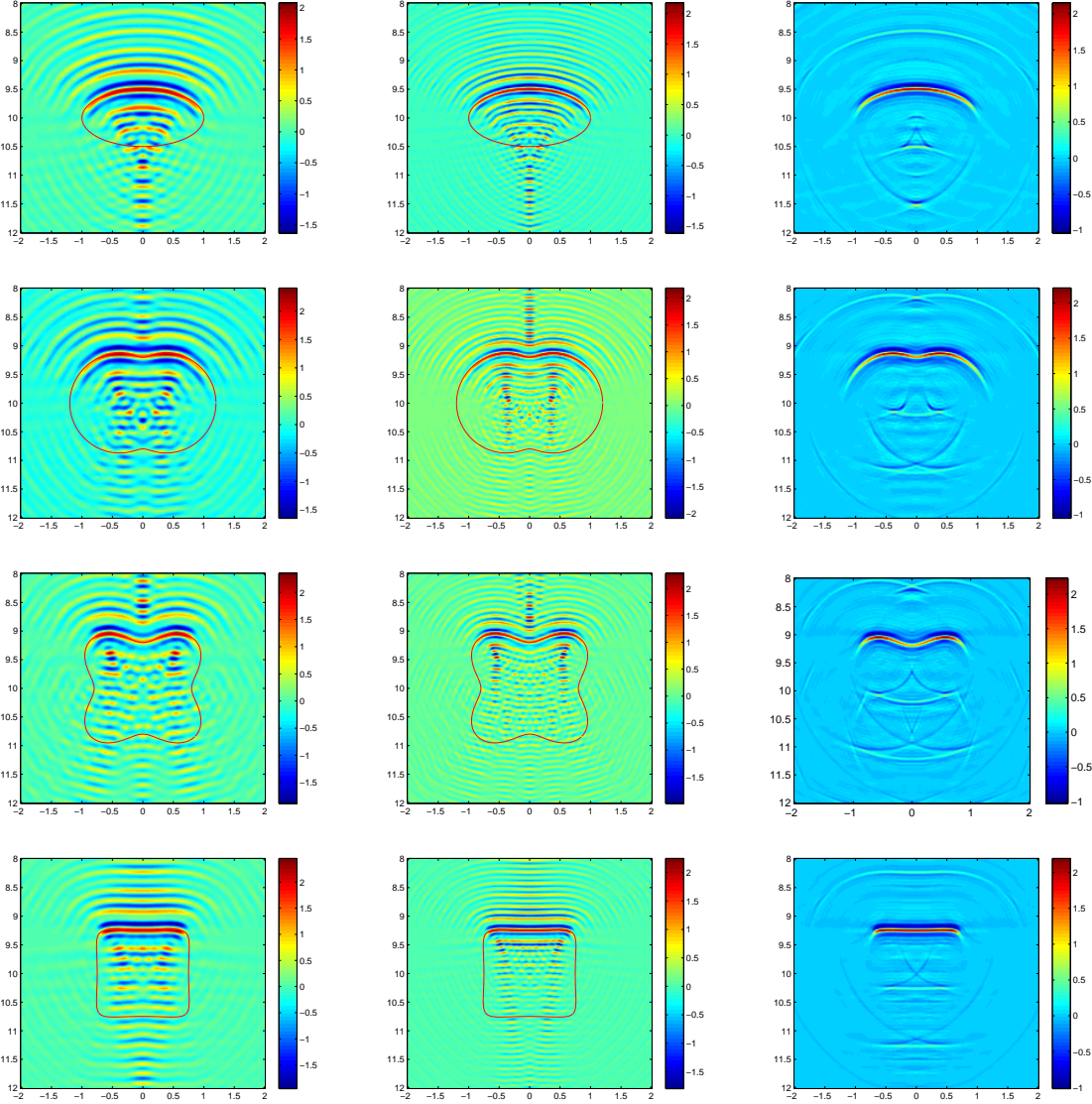
Figure 3. Example 2: Imaging results of clamped obstacles with different shapes from top to below. The left column is imaged with single frequency data where $\omega = 3\pi$, The middle column is imaged with single frequency data where $\omega = 5\pi$ and the right column is imaged with multi frequency data.

values decrease on the shadow part of the obstacles and at the points away from the boundary of the obstacle.

Example 2. We consider the imaging of Dirichlet obstacles with different shapes including a circle, a peanut, a p-leaf and a rounded square. The imaging domain $\Omega = (-2, 2) \times (8, 12)$ with the sampling grid 201×201. We set $N_s = N_r = 401$. The angular frequency $\omega = 3\pi, 4\pi$ for the single frequency and $\omega = \pi \times [2 : 0.5 : 8]$ for the test of multiple frequencies.

Example 3 We consider the imaging of two Neumann obstacles. The first model
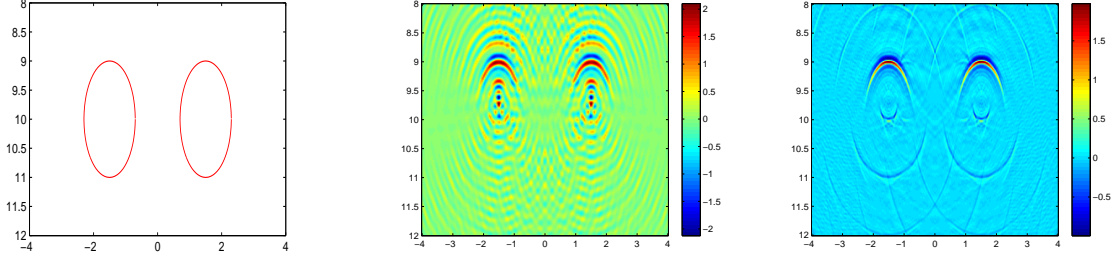
Figure 4. Example 3: From left to right, true obstacle model with two circles, the imaging result with single frequency data where $\omega = 3\pi$, the imaging result with multiple frequency data.
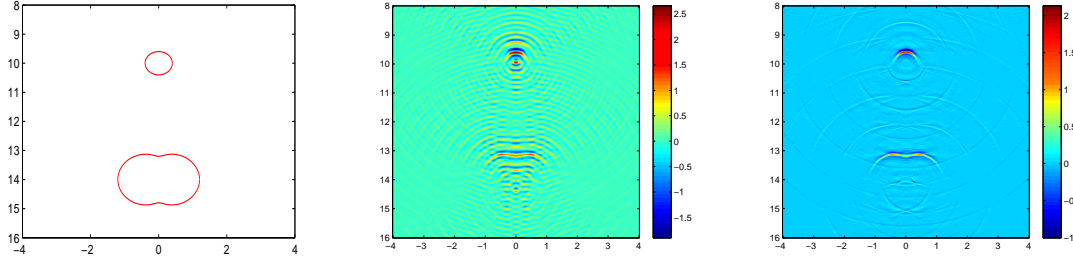

Figure 5. Example 3: From left to right, true obstacle model with one circle and one peanut, the imaging result with single frequency data where $\omega = 3\pi$, the imaging result with multiple frequency data.

consists of two circles along horizontal direction and the second one is a circle and a peanut along the vertical direction. The angular frequency $\omega = 3\pi$ for the test of the single frequency and $\omega = \pi \times [2 : 0.5 : 8]$ for the test of multiple frequencies. Figure 4 shows the imaging result of the first model. The imaging domain $\Omega = (-4, 4) \times (8, 12)$ with mesh size 401×201. We set $N_s = N_r = 301$. Figure 5 shows the imaging result of the second model. The imaging domain $\Omega = (-4, 4) \times (8, 12)$ with mesh size 401×401. We set $N_s = N_r = 301$. The multi-frequency RTM imaging results in Figure 4 and Figure 5 are obtained by adding the inmaging results from different frequencies. We observe from these two figures that imaging results can be greatly improved by stacking the multiple single frequency imaging results.

Example 4 We consider the stability of our half-space RTM imaging function with respect to the complex additive Gaussian random noise. We introduce the additive Gaussian noise as $u_{\mathrm{noise}} = u_s + \nu_{\mathrm{noise}}$, where u_s is the synthesized data and ν_{noise} is the Gaussian noise with mean zero and standard deviation σ times the maximum of the data $|u_s|$, i.e. $\nu_{\mathrm{noise}} = \frac{\sigma \max |u_s|}{\sqrt{2}}(\varepsilon_1 + \mathbf{i}\varepsilon_2)$ and $\varepsilon_i \sim \mathcal{N}(0, 1)$.

Figure 6 shows the imaging results using single frequency data added with additive Gaussian noise. The imaging quality can be greatly improved by using multi-frequency data.
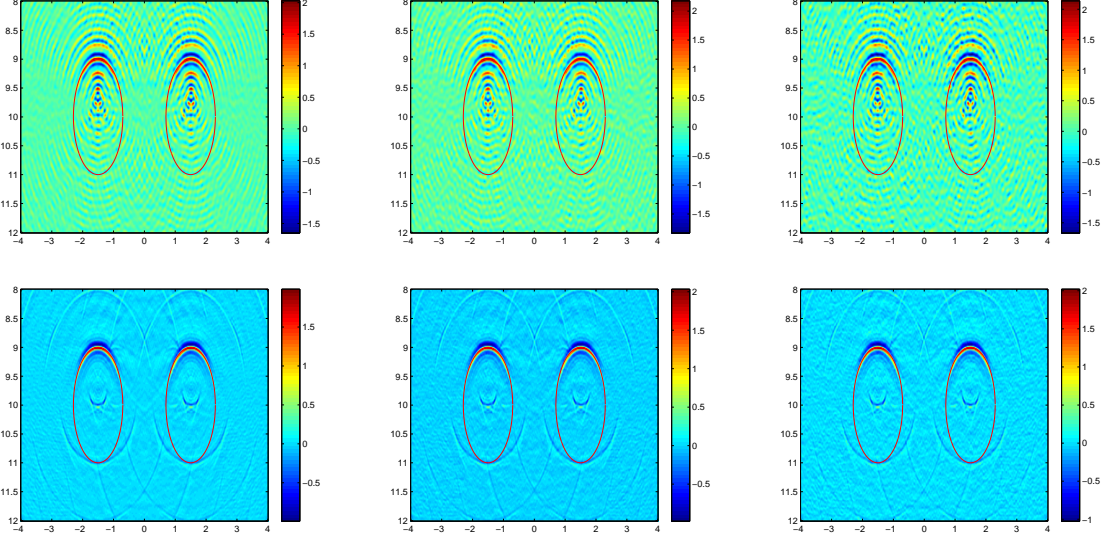
Figure 6. Example 4: Imaging results of a clamped obstacle with noise levels $\mu = 0.2; 0.3; 0.4$ (from left to right). The top row is imaged with single frequency data where $\omega = 4\pi$, and the bottom row is imaged with multi-frequency data.

7. Appendix: Proof of Theorem 4.2

In this section we prove Theorem 4.2. Let $w(x)$ be the scattering solution of the problem:

$$\Delta_e w + \omega^2 w = 0 \quad \text{in } \mathbb{R}^2_+, \quad \sigma(w)e_2 = -\sigma(u_2)e_2 \quad \text{on } \Gamma_0. \tag{7.1}$$

Then $u_1 - u_2 - w$ is the scattering solution of the problem (4.11) with the boundary condition $u_1 - u_2 - w = -w$ on Γ_D. Thus by Theorem 4.1 and (4.1), we have

$$\|\sigma(u_1 - u_2)\nu\|_{H^{-1/2}(\Gamma_D)} \leq \|T_1(u_1 - u_2 - w)\|_{H^{-1/2}(\Gamma_D)} + \|\sigma(w)\nu\|_{H^{-1/2}(\Gamma_D)}$$

$$\leq C(1 + \|T_1\|) \max_{x \in \bar{D}}(|w(x)| + d_D|\nabla w(x)|), \tag{7.2}$$

where we recall that $T_1 : H^{1/2}(\Gamma_D) \to H^{-1/2}(\Gamma_D)$ is the Dirichlet to Neumann mapping associated to the half-space elastic scattering problem (4.11) and $\|T_1\|$ denotes its operator norm.

By the integral representation formula, the scattering solution of the problem (7.1) satisfies

$$w(y) \cdot e_j = \int_{\Gamma_0} \sigma(u_2(x))e_2 \cdot \mathbb{N}(x,y)e_j ds(x), \quad \forall y \in \mathbb{R}^2, \; j = 1, 2. \tag{7.3}$$

On the other hand, by the integral representation formula, we have

$$u_2(x) \cdot e_j = \mathcal{G}(u_2(\cdot), \mathbb{G}(\cdot, x)e_j), \quad \forall x \in \Gamma_0, \; j = 1, 2.$$

where $\mathcal{G}(\cdot, \cdot)$ is defined in (4.6) and $\mathbb{G}(\cdot, \cdot)$ is the fundamental solution matrix of the elastic wave equation introduced in section 2. For any $x \in \Gamma_0, z \in \mathbb{R}^2$, denote by $\mathbb{T}(z, x) \in \mathbb{C}^{2 \times 2}$ be the traction tensor, $\mathbb{T}(z, x)q = \sigma(\mathbb{G}(z, x)q)e_2, \forall q \in \mathbb{R}^2$. The (i, j)-th element of $\mathbb{T}(z, x)$ is

$$[\mathbb{T}(z, x)]_{ij} = [\sigma(\mathbb{G}(z, x)e_j)e_2]e_i, \quad i, j = 1, 2.$$

Simple calculation shows that

$$\sigma(u_2(x))e_2 \cdot e_i = \mathcal{G}(u_2(\cdot), \mathbb{T}(\cdot, x)^T e_i), \quad \forall x \in \Gamma_0, i = 1, 2.$$

which yields from (7.3) that

$$w(y) \cdot e_j = \mathcal{G}(u_2(\cdot), \int_{\Gamma_0} \sum_{i=1}^{2} [\mathbb{T}(\cdot, x)^T e_i] \cdot [e_i^T \mathbb{N}(x, y) e_j] ds(x)) = \mathcal{G}(u_2(\cdot), \mathbb{V}(\cdot, y) e_j), \quad (7.4) \quad \boxed{\texttt{f4}}$$

where

$$\mathbb{V}(z, y) = \int_{\Gamma_0} \mathbb{T}(z, x)^T \mathbb{N}(x, y) ds(x), \quad \forall y \in \mathbb{R}^2, z \in \Gamma_D.$$

Notice that $\|\sigma(u_2)\nu\|_{H^{-1/2}(\Gamma_D)} \le \|T_2\| \|g\|_{H^{1/2}(\Gamma_D)}$, where $T_2 : H^{1/2}(\Gamma_D) \to H^{-1/2}(\Gamma_D)$ is the Dirichlet to Neumann mapping associated to the scattering problem (4.12) and $\|T_2\|$ denotes its operator norm. We obtain from (7.4) and (4.1) that

$$|w(y)| + d_D |\nabla w(y)| \le C(1 + \|T_2\|) \|g\|_{H^{1/2}(\Gamma_D)} \max_{z \in \Gamma_D} \sum_{i,j=0}^{1} d_D^{i+j} |\nabla_z^i \nabla_y^j \mathbb{V}(z, y)|. \quad (7.5) \quad \boxed{\texttt{f6}}$$

To estimate the term involving $\mathbb{V}(z, y)$, we use Parserval identity and Lemma 2.2 to obtain

$$\mathbb{V}(z, y) = \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}} \hat{\mathbb{T}}(z_2; \xi, 0)^T \hat{\mathbb{N}}(\xi, 0; y_2) e^{-\mathbf{i}\xi(y_1 + z_1)} d\xi$$

$$- \frac{\mathbf{i}}{2} \left[\hat{\mathbb{T}}(z_2; \xi, 0)^T \hat{\mathbb{N}}(\xi, 0; y_2) e^{-\mathbf{i}\xi(y_1 + z_1)} \right]_{-k_R}^{k_R}$$

It is easy to see that

$$\hat{\mathbb{T}}(z_2; \xi, 0) = \frac{1}{2} \begin{pmatrix} 1 & \frac{\xi}{\mu_s} \\ 2(k_p^{-2} - k_s^{-2})\xi\mu_s & 2(k_p^{-2} - k_s^{-2})\xi^2 \end{pmatrix} e^{\mathbf{i}\mu_s z_2}$$

$$+ \frac{1}{2} \begin{pmatrix} 0 & 0 \\ -[1 - 2(k_p^{-2} - k_s^{-2})\xi^2]\frac{\xi}{\mu_p^2} & 1 - 2(k_p^{-2} - k_s^{-2})\xi^2 \end{pmatrix} e^{\mathbf{i}\mu_p z_2}$$

$$:= \tilde{\mathcal{T}}_s(\xi) e^{\mathbf{i}\mu_p z_2} + \tilde{\mathcal{T}}_p(\xi) e^{\mathbf{i}\mu_s z_2}.$$

Now by using (2.8) we have

$$\mathbb{V}(z, y) = \frac{1}{2\pi} \sum_{\alpha,\beta=p,s} \text{p.v.} \int_{\mathbb{R}} \frac{\tilde{\mathbb{T}}_\alpha(\xi)^T \mathbb{N}_\beta(\xi)}{\delta(\xi)} e^{\mathbf{i}\mu_\alpha z_2 + \mathbf{i}\mu_\beta y_2 - \mathbf{i}(y_1 + z_1)} d\xi$$

$$- \frac{\mathbf{i}}{2} \sum_{\alpha,\beta=p,s} \left[\frac{\tilde{\mathbb{T}}_\alpha(\xi)^T \mathbb{N}_\beta(\xi)}{\delta'(\xi)} e^{\mathbf{i}\mu_\alpha z_2 + \mathbf{i}\mu_\beta y_2 - \mathbf{i}\xi(y_1 + z_1)} \right]_{-k_R}^{k_R} := \text{V}_1 + \text{V}_2.$$

To estimate V_1 we split the integral into two domains $(-k_s, k_s)$ and $\mathbb{R} \backslash [-k_s, k_s]$ and use the Van der Corput lemma 2.4 to estimate the integral in the first interval and the argument in Lemma 3.4 to estimate the integral in $\mathbb{R} \backslash [-k_s, k_s]$. This yields $|\text{V}_1| \le C\mu^{-1}(k_s h)^{-1/2}$. By the same argument in Lemma 3.2 we can show $|\text{V}_2| \le Ce^{-\sqrt{k_R - k_s}h}$. This shows

$$\max_{z \in \Gamma_D} |\mathbb{V}(z, y)| \le \frac{C}{\mu}(k_s h)^{-1/2}, \quad \forall y \in D.$$

A similar argument shows that

$$\max_{z\in\Gamma_D} k_s^{i+j}|\nabla_z^i \nabla_y^j \mathbb{V}(z,y)| \leq \frac{C}{\mu}(k_s h)^{-1/2}, \quad \forall y \in D, \ i,j=0,1.$$

Substitute the above two estimates into (7.5) we obtain

$$\max_{y\in\bar{D}}(|w(y)| + d_D|\nabla w(y)|) \leq \frac{C}{\mu}(1 + \|T_2\|)(1 + k_s d_D)^2 (k_s h)^{-1/2}\|g\|_{H^{1/2}(\Gamma_D)}.$$

This completes the proof of the theorem from (7.2). □

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