

# Absence of Positive Eigenvalues for the Linearized Elasticity System in the Half Space

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**Abstract.** In this paper, we prove that the linearized elasticity system in the half-space with traction free boundary has no eigenvalues.

## 1. Introduction

section1

In this paper, we consider the linearized and isotropic elasticity system defined on an unbounded domain  $\Omega = \mathbb{R}_+^2 \setminus \bar{D}$  with traction free surface  $\Gamma_0 = \{(x_1, x_2)^T : x_1 \in \mathbb{R}, x_2 = 0\}$ , where  $D \subsetneq \mathbb{R}_+^2$  is a bounded Lipschitz domain with the unit outer normal  $\nu$  to its boundary  $\Gamma_D$ . We study the eigenvalues of the following elastic scattering problem in the isotropic homogeneous medium half space with *Lamé* constant  $\lambda$  and  $\mu$  and constant density  $\rho \equiv 1$ :

$$\nabla \cdot \sigma(\mathbf{u}) + \rho\omega^2 \mathbf{u} = f \quad \text{in } \mathbb{R}_+^2 \setminus \bar{D} \quad (1.1) \quad \text{elastic\_eq}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \sigma(\mathbf{u}) \cdot e_2 = 0 \quad \text{on } \Gamma_0 \quad (1.2) \quad \text{elastic\_bd}$$

together with the constitutive relation (Hookes law)

$$\begin{aligned} \sigma(\mathbf{u}) &= 2\mu\varepsilon(\mathbf{u}) + \lambda\text{div}\mathbf{u}\mathbb{I} \\ \varepsilon(\mathbf{u}) &= \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) \end{aligned}$$

where  $\omega$  is the circular frequency,  $\mathbf{u}(x_1, x_2) = (u_1(x), u_2(x))^T \in \mathbb{C}^2$  denotes the displacement fields and  $\sigma(u)$  is the stress tensor. We also need to define the surface traction  $T_x^n(\cdot)$  on the normal direction  $\mathbf{n}$ ,

$$T_x^n \mathbf{u}(x) := \sigma \cdot \mathbf{n} = 2\mu \frac{\partial \mathbf{u}}{\partial n} + \lambda n \text{div} \mathbf{u} + \mu \mathbf{n} \times \text{curl} \mathbf{u}$$

For simplicity, let's introduce *Lamé* operator  $\Delta_e$  as

$$\Delta_e \mathbf{u} = (\lambda + 2\mu) \nabla \nabla \cdot \mathbf{u} - \mu \nabla \times \nabla \times \mathbf{u} = \nabla \cdot \sigma(\mathbf{u})$$

We remark that the results in this paper can be extended to other boundary conditions such as Neumann or mixed boundary conditions on  $\Gamma_D$ , or even to penetrable obstacle.

In order to complete the definition of the problem, we introduce the domain of the operator  $\Delta_e$

$$\mathcal{D}(\Delta_e, S) = \{v \in H^1(S) : \Delta_e v \in L^2(S), \sigma(v)e_2 = 0 \quad \text{on } \Gamma_0\}$$

where  $S$  is an unbounded domain in  $\mathbb{R}_+^2$ . For the elasticity system, the study of eigenvalue is little. The layout of the paper is as follows. In section 2

## 2. Absence of Positive Eigenvalues

In this section. Throughout the paper, we will assume that for  $z \in \mathbb{C}$ ,  $z^{1/2}$  is the analytic branch of  $\sqrt{z}$  such that  $\text{Im}(z^{1/2}) \geq 0$ . This corresponds to the right half real axis as the branch cut in the complex plane. For  $z = z_1 + \mathbf{i}z_2$ ,  $z_1, z_2 \in \mathbb{R}$ , we have

$$z^{1/2} = \text{sgn}(z_2) \sqrt{\frac{|z| + z_1}{2}} + \mathbf{i} \sqrt{\frac{|z| - z_1}{2}} \quad (2.1) \quad \text{convention\_1}$$

For  $z$  on the right half real axis, we take  $z^{1/2}$  as the limit of  $(z + \mathbf{i}\varepsilon)^{1/2}$  as  $\varepsilon \rightarrow 0^+$ .

**2.1 Theorem 2.1** *Let  $\omega \in \mathbb{R}$  and  $\mathbf{u}$  satisfy the equations (1.1)-(1.2) in  $\mathcal{D}(\Delta_e, \Omega)$ , then we assert  $\mathbf{u} = 0$ .*

**lem2.1** **Lemma 2.1** *The Rayleigh equation  $\delta(\xi) = 0$  has only two zeros  $\pm k_R$ ,  $k_R > k_s$ , in the complex plane.*

**lem2.2** **Lemma 2.2** *Suppose that  $f \in L^2(\mathbb{R}_+^2)$  with compact support in  $B \subsetneq \mathbb{R} \times (h, +\infty)$ ,  $h > 0$ . Let  $\omega \in \mathbb{R}$  and  $\mathbf{u} \in \mathcal{D}(\Delta_e, \mathbb{R}_+^2)$  such that:*

$$\Delta_e \mathbf{u} + \omega^2 \mathbf{u} = f \quad (2.2) \quad \text{eq2}$$

then we assert  $\mathbf{u} = 0$  in  $(\mathbb{R} \times (h, +\infty))$ .

**Proof.** Let  $\mathcal{F}_{x_1}(\cdot) : L^2(\mathbb{R}_+^2) \rightarrow L^2(\mathbb{R}_+^2)$  be the Fourier tranfor given by  $\hat{g} := \mathcal{F}_{x_1}(g) := \int_{\mathbb{R}} g(x_1, x_2) e^{ix_1 \xi} dx_1$ . By taking the Fourier transform of (2.2) and (1.2), we obtain ODEs for  $x_2$  in  $\mathbb{R}_+$

$$\mu \frac{d^2 \hat{u}_1}{dx_2^2} + \mathbf{i}(\lambda + \mu) \xi \frac{d \hat{u}_2}{dx_2} + (\omega^2 - (\lambda + 2\mu) \xi^2) \hat{u}_1 = \hat{f}_1 \quad (2.3) \quad \text{pp3}$$

$$(\lambda + 2\mu) \frac{d^2 \hat{u}_2}{dx_2^2} + \mathbf{i}(\lambda + \mu) \xi \frac{d \hat{u}_1}{dx_2} + (\omega^2 - \mu \xi^2) \hat{u}_2 = \hat{f}_2 \quad (2.4) \quad \text{pp4}$$

and the boundary coditions on  $x_2 = 0$  are

$$\mu \frac{d \hat{u}_1}{dx_2} + \mathbf{i} \mu \xi \hat{u}_2 = 0 \quad (2.5) \quad \text{pp5}$$

$$(\lambda + 2\mu) \frac{d \hat{u}_2}{dx_2} + \mathbf{i} \lambda \xi \hat{u}_1 = 0 \quad (2.6) \quad \text{pp6}$$

In order to work with real coefficient, we use the change of variables:

$$\begin{aligned} v_1 &= \mathbf{i} \hat{u}_1, & v_2 &= \hat{u}_2, & \mathbf{v} &= (v_1, v_2)^T \\ q_1 &= \mathbf{i} \hat{f}_1, & q_2 &= \hat{f}_2, & \mathbf{q} &= (q_1, q_2)^T \end{aligned}$$

Then we have the following equations:

$$[\mathbb{A}_1 \frac{d^2}{dx_2^2} + (\mathbb{A}_2 - (\mathbb{A}_2)^T) \xi \frac{d}{dx_2} - \mathbb{A}_3 \xi^2 + \omega^2] \mathbf{v} = \mathbf{q} \quad \text{in } \mathbb{R}_+ \quad (2.7) \quad \text{eq3}$$

$$(\mathbb{A}_1 \frac{d}{dx_2} + \mathbb{A}_2 \xi) \mathbf{v} = 0 \quad \text{on } x_2 = 0 \quad (2.8)$$

where

$$\mathbb{A}_1 = \begin{pmatrix} \mu & 0 \\ 0 & \lambda + 2\mu \end{pmatrix}, \quad \mathbb{A}_2 = \begin{pmatrix} 0 & -\mu \\ \lambda & 0 \end{pmatrix}, \quad \mathbb{A}_3 = \begin{pmatrix} \lambda + 2\mu & 0 \\ 0 & \mu \end{pmatrix}$$

Let  $\mathbf{w}$  be the solution of the following equations:

$$[\mathbb{A}_1 \frac{d^2}{dx_2^2} + (\mathbb{A}_2 - (\mathbb{A}_2)^T) \xi \frac{d}{dx_2} - \mathbb{A}_3 \xi^2 + \omega^2] \mathbf{w} = \mathbf{q} \quad \text{in } (0, h)$$

$$\mathbf{w} = 0, \quad (\mathbb{A}_1 \frac{d}{dx_2} + \mathbb{A}_2 \xi) \mathbf{w} = 0 \quad \text{on } x_2 = h$$

It is easy to transform above equations into a simpler form by variables substitution

$$\mathbf{W} = (\mathbf{w}, (\mathbb{A}_1 \frac{d}{dx_2} + \mathbb{A}_2 \xi) \mathbf{w})^T, \quad \mathbf{Q} = (0, 0, \mathbf{q})^T$$

$$\frac{d}{dx_2} \mathbf{W} = \mathbb{A} \mathbf{W} + \mathbf{Q} \quad \text{in } (0, h)$$

$$\mathbf{W} = 0 \quad \text{on } x_2 = h$$

where

$$\mathbb{A} = \begin{pmatrix} -\mathbb{A}_1^{-1}\mathbb{A}_2\xi & \mathbb{A}_1^{-1} \\ -\mathbb{A}_2^T\mathbb{A}_1^{-1}\mathbb{A}_2\xi^2 + \mathbb{A}_3\xi^2 - \omega^2 & \mathbb{A}_2^T\mathbb{A}_1^{-1}\xi \end{pmatrix}$$

By the standard arguments in ODEs, we can obtain

$$\mathbf{W}(\xi, x_2) = -\Phi(\xi, x_2) \int_h^{x_2} \Phi^{-1}(\xi, t) \mathbf{Q}(\xi, t) dt$$

where

$$\Phi(\xi, t) = \begin{pmatrix} -\mu_s(\xi)e^{i\mu_s t} & -\xi e^{i\mu_p(\xi)t} & -\mu_s(\xi)e^{-i\mu_s(\xi)t} & \xi e^{-i\mu_p t} \\ -i\xi e^{i\mu_s(\xi)t} & i\mu_p(\xi)e^{i\mu_p(\xi)t} & i\xi e^{-i\mu_s(\xi)t} & i\mu_p(\xi)e^{-i\mu_p(\xi)t} \\ -i\mu\beta(\xi)e^{i\mu_s t} & -2i\mu\xi\mu_p(\xi)e^{i\mu_p(\xi)t} & i\mu\beta(\xi)e^{-i\mu_s(\xi)t} & -2i\mu\xi\mu_p(\xi)e^{-i\mu_p t} \\ 2\mu\xi\mu_s(\xi)e^{i\mu_s t} & -\mu\beta(\xi)e^{i\mu_p(\xi)t} & 2\mu\xi\mu_s(\xi)e^{-i\mu_s t} & \mu\beta(\xi)e^{-i\mu_p(\xi)t} \end{pmatrix}$$

Here  $k_p = \omega/\sqrt{\lambda+2\mu}$ ,  $k_s = \omega/\sqrt{\mu}$  are wave number of p-wave and s-wave, and  $\mu_\alpha = (k_\alpha^2 - \xi^2)^{1/2}$  for  $\alpha = s, p$ .

We extend  $\mathbf{w}(\xi, x_2)$  by zero in  $(h, \infty)$ . Therefore.  $\mathbf{w}(\xi, x_2)$  satisfy equation [eq3](#) in  $\mathbb{R}_+$ . Since  $\Phi(\xi, t)$  are analytic w.r.t  $\xi$  in  $\mathbb{R} \setminus \{k_p, k_s\}$  and  $\mathbf{f}(\mathbf{x})$  have compact support, we deduce that for almost every  $\xi \in \mathbb{R}$   $\mathbf{w}(\xi, x_2)$  are analytic and so  $(\mathbb{A}_1 \frac{d}{dx_2} + \mathbb{A}_2 \xi) \mathbf{w}$  are.

We set  $\mathbf{U} = \mathbf{v} - \mathbf{w}$  and  $\mathbf{U} = (U_1, U_2)^T$ . Then  $\mathbf{U}$  satisfy the following Cauchy problem

$$[\mathbb{A}_1 \frac{d^2}{dx_2^2} + (\mathbb{A}_2 - (\mathbb{A}_2)^T) \xi \frac{d}{dx_2} - \mathbb{A}_3 \xi^2 + \omega^2] \mathbf{U} = 0 \quad \text{in } \mathbb{R}_+ \quad (2.9) \quad \boxed{\text{eq4}}$$

$$(\mathbb{A}_1 \frac{d}{dx_2} + \mathbb{A}_2 \xi) \mathbf{U} = (\mathbb{A}_1 \frac{d}{dx_2} + \mathbb{A}_2 \xi) \mathbf{w} \quad \text{on } x_2 = 0 \quad (2.10)$$

Since the coefficients of above equations are constants, we can represent  $\mathbf{U}(\xi, x_2)$  in the following form:

$$\mathbf{U}(\xi, x_2) = c_1(\xi) \begin{pmatrix} -\mu_s \\ -i\xi \end{pmatrix} e^{i\mu_s x_2} + c_2(\xi) \begin{pmatrix} -\xi \\ i\mu_p \end{pmatrix} e^{i\mu_p x_2} + c_3(\xi) \begin{pmatrix} -\mu_s \\ i\xi \end{pmatrix} e^{-i\mu_s x_2} + c_4(\xi) \begin{pmatrix} \xi \\ i\mu_p \end{pmatrix} e^{-i\mu_p x_2}$$

If  $\xi^2 \leq k_p^2$ , then it's simple to see that  $\mathbf{U} = 0$  in  $L^2_{x_2}(\mathbb{R}_+)$ . So, for  $\xi^2 < k_p^2$ ,  $(\mathbb{A}_1 \frac{d}{dx_2} + \mathbb{A}_2 \xi) \mathbf{U} = 0$  which implis  $(\mathbb{A}_1 \frac{d}{dx_2} + \mathbb{A}_2 \xi) \mathbf{w} = 0$ . Since  $(\mathbb{A}_1 \frac{d}{dx_2} + \mathbb{A}_2 \xi) \mathbf{w}$  are analytic for almost every  $\xi \in \mathbb{R}$ , we deduce that

$$(\mathbb{A}_1 \frac{d}{dx_2} + \mathbb{A}_2 \xi) \mathbf{U} = 0 \quad \text{on } x_2 = 0 \quad (2.11) \quad \boxed{\text{bd}_1}$$

for almost every  $\xi \in \mathbb{R}$ . Therefore, we can obtain

$$\mathbf{U}(\xi, x_2) = \begin{cases} c(\xi) \begin{pmatrix} -\xi \\ i\mu_p \end{pmatrix} e^{i\mu_p x_2}, & k_p^2 < \xi^2 \leq k_s^2 \\ c_1(\xi) \begin{pmatrix} -\mu_s \\ -i\xi \end{pmatrix} e^{i\mu_s x_2} + c_2(\xi) \begin{pmatrix} -\xi \\ i\mu_p \end{pmatrix} e^{i\mu_p x_2}, & \xi^2 > k_s^2 \end{cases}$$

$$(\mathbb{A}_1 \frac{d}{dx_2} + \mathbb{A}_2 \xi) \mathbf{U} = \begin{cases} c(\xi) \begin{pmatrix} -2i\mu\xi\mu_p \\ -\mu\beta \end{pmatrix} e^{i\mu_p x_2}, & k_p^2 < \xi^2 \leq k_s^2 \\ c_1(\xi) \begin{pmatrix} -i\mu\beta \\ 2\mu\xi\mu_s \end{pmatrix} e^{i\mu_s x_2} + c_2(\xi) \begin{pmatrix} -2i\mu\xi\mu_p \\ -\mu\beta \end{pmatrix} e^{i\mu_p x_2}, & \xi^2 > k_s^2 \end{cases}$$

By boudary condition <sup>bd 1</sup>2.II, we have  $c(\xi) = 0$  and

$$\det \begin{pmatrix} -i\mu\beta & -2i\mu\xi\mu_p \\ 2\mu\xi\mu_s & -\mu\beta \end{pmatrix} = -i\mu(\beta^2 + 4\xi^2\mu_s\mu_p) = 0 \quad (2.12)$$

□

**proof of Theorem <sup>2.1</sup>2.1:** Since  $D \subsetneq \mathbb{R}_+^2$ , we can find two concentric circles  $B_{R_1}, B_{R_2}$  such that  $D \subsetneq B_{R_1} \subsetneq B_{R_2} \subsetneq \mathbb{R}_+^2$ . Let  $\chi \in C_0^\infty(\mathbb{R}_+^2)$  be the cut-off function such that  $0 \leq \chi \leq 1$ ,  $\chi = 0$  in  $B_{R_1}$ , and  $\chi = 1$  outside of  $B_{R_2}$ . Let  $v = \chi u$ . Then  $v$  satisfies <sup>eq2</sup>(2.2) with  $f = \sigma(u)\nabla\chi + (\lambda + \mu)(\nabla^2\chi u + \nabla u \nabla\chi) + \mu\Delta\chi u + \mu\operatorname{div}u \nabla\chi$ , where  $\nabla^2\chi$  is the Hessian matrix of  $\chi$ . Clearly  $q$  has compact support. By lemme <sup>lem2.2</sup>2.2, we have  $u = v = 0$  in  $\mathbb{R} \times (h, +\infty)$ . Finally, the unique continuation principle implies that  $u=0$  in  $\mathbb{R}_+^2$ . This completes the proof. □

## References

- |               |
|---------------|
| achenbach1980 |
| 979Complex    |
| la_reverse    |
| Yves1988      |
| grafakos      |
| 2001Linear    |
| 63progress    |
| leis          |
| Guzina2006    |
| sini2004      |
| wilcox1975    |
| Zhang08       |
| Zhang2007     |
- [1] Jan Achenbach. *Wave Propagation in Elastic Solids*. North-Holland, 1980.
  - [2] Lars V Ahlfors. *Complex Analysis: An introduction to the theory of analytic functions of one complex variable*. McGraw-Hill, 1979.
  - [3] Wen Fong Chang. Elastic reverse-time migration. *Geophysical Prospecting*, 37(3):243–256, 1987.
  - [4] Yves Dermenjian and Jean Claude Guillot. Scattering of elastic waves in a perturbed isotropic half space with a free boundary. the limiting absorption principle. *Mathematical Methods in the Applied Sciences*, 10(2):87C124, 1988.
  - [5] Loukas Grafakos. *Classical and modern Fourier analysis*. Prentice Hall, 2004.
  - [6] Johng. Harris. *Linear elastic waves*. Cambridge University Press, 2001.
  - [7] Viktor D Kupradze. *Progress in solid mechanics. 3. Dynamical problems in elasticity*. North-Holland Publishing Company, 1963.
  - [8] Rolf Leis. *Initial Boundary Value Problems in Mathematical Physics*. J. Wiley, 1986.
  - [9] Andrew I. Madyarov and Bojan B. Guzina. A radiation condition for layered elastic media. *Journal of Elasticity*, 82(1):73–98, 2006.
  - [10] Mourad Sini. Absence of positive eigenvalues for the linearized elasticity system. *Integral Equations and Operator Theory*, 49(2):255–277, 2004.
  - [11] Calvin H. Wilcox. *Scattering Theory for the d'Alembert Equation in Exterior Domains*. PhD thesis, Springer Berlin Heidelberg, 1975.
  - [12] Yu. Zhang and James. Sun. Practical issues of reverse time migration: true amplitude gathers, noise removal and harmonic-source encoding. *Aseg Extended Abstracts*, 2009(3):397–398, 2008.
  - [13] Yu Zhang, Sheng Xu, Norman Bleistein, and Guanquan Zhang. True-amplitude, angle-domain, common-image gathers from one-way wave-equation migrations. *Geophysics*, 72(1):S49–S58, 2007.