Reverse Time Migration for Extended Obstacles in the Half Space: Elastic Waves

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Abstract. We consider a reverse time migration method for reconstructing extended obstacles in the half space with finite aperture data using elastic waves at a fixed frequency. We prove the resolution of the reconstruction method in terms of the aperture and the depth of the obstacle embedded in the half space. The resolution analysis implies that the imaginary part of the cross-correlation imaging function always peaks on the illuminated boundary of the obstacle. Numerical experiments are included to illustrate the powerful imaging quality and to confirm our resolution results.

1. Introduction

In this paper we study a reverse time migration (RTM) algorithm to find the support of an unknown obstacle in the half space from the measurement of scattered waves on the boundary of the half space which is far away from the obstacle. The physical properties of the obstacle such as penetrable or non-penetrable, and for non-penetrable obstacles, the type of boundary conditions on the boundary of the obstacle, are not required in the algorithm.

Let the non-penetrable obstacle occupy a bounded Lipschitz domain $D \subset \mathbb{R}^2_+$ with ν the unit outer normal to its boundary Γ_D. We assume the incident wave is emitted by a point source located at x_s, explosive along the polarization direction $q \in \mathbb{R}^2$, on the surface $\Gamma_0 = \{(x_1, x_2)^T : x_1 \in \mathbb{R}, x_2 = 0\}$ which is far away from the obstacle. The measured data u_q corresponding to the polarization direction q is the solution of the following elastic scattering problem in the isotropic homogeneous medium half space with *Lamé* constant λ and μ and constant density $\rho \equiv 1$:

$$\nabla \cdot \sigma(u_q) + \rho\omega^2 u_q = -\delta_{x_s}(x)q \quad \text{in } \mathbb{R}^2_+ \backslash \bar{D} \tag{1.1}$$

$$u_q = 0 \;\text{ on } \Gamma_D \;\text{ and }\; \sigma(u_q) \cdot e_2 = 0 \;\text{ on } \Gamma_0 \tag{1.2}$$

together with the constitutive relation (Hookes law)

$$\sigma(u) = 2\mu\varepsilon(u) + \lambda \mathrm{div} u \mathbb{I}$$

$$\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$$

where ω is the circular frequency, $u(x) \in \mathbb{C}^2$ denotes the displacement fields and $\sigma(u)$ is the stress tensor. We also need to define the surface traction $T^n_x(\cdot)$ on the normal direction n,

$$T^n_x u(x) := \sigma \cdot n = 2\mu\frac{\partial u}{\partial n} + \lambda n \mathrm{div} u + \mu n \times \mathrm{curl} u$$

For simplicity, let's introduce *Lamé* operator Δ_e as

$$\Delta_e u = (\lambda + 2\mu)\nabla\nabla \cdot u - \mu\nabla \times \nabla \times u = \nabla \cdot \sigma(u)$$

The equation (1.1) is understood as the limit when $x_s \in \mathbb{R}^2_+ \backslash \bar{D}$ tends to Γ_0 whose precise meaning will be given below after we introduce the Neumann Green Tensor and the definition of the radiation condition.

The reverse time migration (RTM) method, which consists of back-propagating the complex conjugated data into the background medium and computing the crosscorrelation between the incident wave field and the backpropagated field to output the final imaging profile, is nowadays widely used in exploration geophysics [5, 6, 7, 9, 15]. In [10, 11, 12], the RTM method for reconstructing extended targets using acoustic, electromagnetic and elastic waves at a fixed frequency in the free space is proposed and studied. The resolution analysis in [10, 11, 12] is achieved without using the small inclusion or geometrical optics assumption previously made in the literature (e.g. [3, 7]). In [13], a new RTM algorithm is developed for finding extended targets in a planar waveguide which is motivated by the generalized Helmholtz-Kirchhoff identity for scattering problems in waveguides.

The layout of the paper is as follows. In section 2 we study the two Green Tensor for the scattering problem in the half space satisfying the homogeneous Neumann condition and Dirichlet condition on Γ_0. We recall the derivation of the Green Tensor by the method of Fourier transform and derive an alternative form of the Green Tensor which is crucial for the analysis in the rest. In section 3 we introduce the RTM algorithm. In section 4 we study the point spread function. In section 5 we study the resolution analysis of the RTM method. In section 6 we report extensive numerical experiments to show the competitive performance of the RTM algorithm.

2. Green Tensor in the half space

In this section we will study the elastic Green Tensor in the half-space with Neumann boundary [17]:

$$\Delta_e N(x;y) + \omega^2 N(x,y) = -\delta_y(x)\mathbb{I} \quad \text{in} \quad \mathbb{R}^2_+, \tag{2.1}$$

$$\sigma_x(N(x,y))e_2 = 0 \qquad \text{on} \quad x_2 = 0 \tag{2.2}$$

and with Dirichlet Boundary [4]

$$\Delta_e D(x,y) + \omega^2 D(x,y) = -\delta_y(x)\mathbb{I} \quad \text{in} \quad \mathbb{R}^2_+, \tag{2.3}$$

$$D(x,y) = 0 \qquad \text{on} \quad x_2 = 0 \tag{2.4}$$

where $\delta_y(x)$ is the Dirac source at $y \in R^2_+$ and $N(x,y)$, $D(x,y)$ are $\mathbb{C}^{2\times 2}$ matrixes. We will first use Fourier transform to derive the formula of Green Tensor in frequency domain. Let

$$\hat{N}(\xi, x_2; y_2) = \int_{-\infty}^{+\infty} N(x_1, x_2; y)e^{-\mathbf{i}(x_1-y_1)\xi}dx_1 \tag{2.5}$$

Throughout the paper, we will assume that for $z \in \mathbb{C}$, $z^{1/2}$ is the analytic branch of $\sqrt{z}$ such that $\mathrm{Im}\,(z^{1/2}) \geq 0$. This corresponds to the rigt half real axis as the branch cut in the complex plane. For $z = z_1 + \mathrm{i}z_2, z_1, z_2 \in \mathbb{R}$, we have

$$z^{1/2} = sgn(z_2)\sqrt{\frac{|z| + z_1}{2}} + \mathrm{i}\sqrt{\frac{|z| - z_1}{2}} \tag{2.6}$$

For z on the right half real axis, we take $z^{1/2}$ as the limit of $(z + \mathbf{i}\varepsilon)^{1/2}$ as $\varepsilon \to 0^+$. We recall that

$$\hat{\Phi}(\xi, x_2; y_2) = \frac{\mathbf{i}}{2\omega^2}\left[\begin{pmatrix} \mu_s & -\xi\frac{x_2-y_2}{|x_2-y_2|} \\ -\xi\frac{x_2-y_2}{|x_2-y_2|} & \frac{\xi^2}{\mu_s} \end{pmatrix} e^{\mathbf{i}\mu_s|x_2-y_2|} + \begin{pmatrix} \frac{\xi^2}{\mu_p} & \xi\frac{x_2-y_2}{|x_2-y_2|} \\ \xi\frac{x_2-y_2}{|x_2-y_2|} & \mu_p \end{pmatrix} e^{\mathbf{i}\mu_p|x_2-y_2|}\right]$$

where

$$\mu_\alpha = (k_\alpha^2 - \xi^2)^{1/2} \qquad \text{for} \quad \alpha = s, p \tag{2.7}$$

By the standard arguement in ODEs, the Green Tensor in half-space can be deduced as

$$\hat{N}(\xi, x_2; y_2) = \hat{\Phi}(\xi, x_2; y_2) - \hat{\Phi}(\xi, x_2; -y_2) + \hat{N}_c(\xi, x_2; y_2) \tag{2.8}$$

$$\hat{N}_c(\xi, x_2; y_2) = = \frac{\mathrm{i}}{\omega^2\delta(\xi)}\Big\{A(\xi)e^{\mathrm{i}\mu_s(x_2+y_2)} + B(\xi)e^{\mathrm{i}\mu_p(x_2+y_2)} \tag{2.9}$$
$$+ C(\xi)e^{\mathrm{i}\mu_s x_2+\mu_p y_2} + D(\xi)e^{\mathrm{i}\mu_p x_2+\mu_s y_2}\Big\}$$

where

$$A(\xi) = \begin{pmatrix} \mu_s\beta^2 & -4\xi^3\mu_s\mu_p \\ -\xi\beta^2 & 4\xi_4\mu_p \end{pmatrix} \qquad B(\xi) = \begin{pmatrix} 4\xi^4\mu_s & \xi\beta^2 \\ 4\xi^3\mu_s\mu_p & \mu_p\beta^2 \end{pmatrix}$$
$$C(\xi) = \begin{pmatrix} 2\xi^2\mu_s\beta & -2\xi\mu_s\mu_p\beta \\ -2\xi^3\beta & 2\xi^2\mu_p\beta \end{pmatrix} \quad D(\xi) = \begin{pmatrix} 2\xi^2\mu_s\beta & 2\xi^3\beta \\ 2\xi\mu_s\mu_p\beta & 2\xi^2\mu_p\beta \end{pmatrix}$$

and $\beta(\xi) = k_s^2 - 2\xi^2$, $\delta(\xi) = \beta^2 + 4\xi^2\mu_s\mu_p$.

The desired Green function should be obtained by taking the inverse Fourier transform of $\hat{N}(\xi, x_2; y_2)$. Unfortunately, one cannot simply take the inverse Fourier transform in the above formula because $\delta(\xi)$ have zero points in the real axis by lemma 2.1 [1][19].

Lemma 2.1 *Let* Lamé *constant* $\lambda, \mu \in \mathbb{R}^+$, *then the Rayleigh equation* $\delta(\xi) = 0$ *has only two roots denoted by* $\pm k_R$ *in complex plane. Morever,* $k_R > k_s > k_p$, $k_R \in \mathbb{R}$ *and* k_R *is called Rayleigh wave number.*

In order to overcome the ambiguity above, loss is assumed in the medium so that $k_{\alpha,\varepsilon} := k_\alpha(1+\mathbf{i}\varepsilon)$. When $\varepsilon > 0$, the branch point of $\mu_{\alpha,\varepsilon}$ are $\pm k_{\alpha,\varepsilon}$ and the branch cut are denoted by the equation $\xi_1\xi_2 = k_\alpha\varepsilon, -k_\alpha \le \xi \le k_\alpha$. In this case, the poles singularities are now located off the real axis and the Fouerier inverse transform becomes meaningful. In order to express lemma 2.2 concisely, we define

$$\Omega := \{\xi \in \mathbb{C} \mid k_p\varepsilon < \xi_1\xi_2 < k_s\varepsilon\ ,\quad \xi_2 > \xi_1\varepsilon\} \tag{2.10}$$

Lemma 2.2 *If the elastic medium has loss that* $k_{\alpha,\varepsilon} := k_\alpha(1+\mathbf{i}\varepsilon), 0 < \varepsilon < 1$ *for* $\alpha = p, s$, *we assert that* $\delta_\varepsilon(\xi) = 0$ *has only two roots in domain* $\Omega^c \subset \mathbb{C}$ *and exactly they are* $\pm k_{R,\varepsilon}$.

Let $\xi = \xi_1 + \mathbf{i}\xi_2 \in \mathbb{C}$, $\xi_1, \xi_2 \in \mathbb{R}$, and the hyperbolic curve Γ defined by the equation $\xi_1^2 - \xi_2^2 = k_s^2$. Denote Γ_r^+, Γ_r^- respectively the parts of right branch of Γ in the upper-half complex plane and the lower-half complex plane. Similarily, we can define Γ_l^-, Γ_l^-. Now, we can define a new integral path in the complex plane

$$NP = \begin{cases} \Gamma_l^+ \cup \Gamma_r^+ \cup [-k_s, k_s] & \text{when} \quad x_1 - y_1 > 0 \\ \Gamma_l^- \cup \Gamma_r^- \cup [-k_s, k_s] & \text{when} \quad x_1 - y_1 < 0 \end{cases} \tag{2.11}$$

Thus, by using Cauchy integral theorem and lemma 2.2, we have

$$N_\varepsilon(x, y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{N}_\varepsilon(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \tag{2.12}$$
$$= \frac{1}{2\pi}\int_{NP} \hat{N}_\varepsilon(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \pm \mathbf{i}Res_{\xi=\pm k_R^\varepsilon}N_\varepsilon(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi} \tag{2.13}$$

As the perturbation ε have nothing to do with the integration path NP, we could take the limitation $\varepsilon \to 0$. Thus, we get the representation of Green Tensor

$$N(x,y) = \Phi(x,y) - \Phi(x,y') + \frac{1}{2\pi}\int_{NP} \hat{N}_c(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \qquad (2.14)$$
$$\pm\, \mathbf{i}Res_{\xi=\pm\kappa_r}\hat{N}_c(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi}$$

where $\pm$ are corresponding $sgn(x_1-y_1)$. Specially, $N(x,y)$ has a simple form when $x_2 = 0$:

$$N(x,y) = \frac{1}{2\pi}\int_{NP} \hat{N}(\xi, 0; y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \pm \mathbf{i}Res_{\xi=\pm\kappa_r}\hat{N}(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi} \qquad (2.15)$$

where

$$\hat{N}(\xi, 0; y_2) = \frac{\mathrm{i}}{\mu\delta(\xi)}\left[\begin{pmatrix} 2\xi^2\mu_s & -2\xi\mu_s\mu_p \\ -\xi\beta & \mu_p\beta \end{pmatrix} e^{\mathrm{i}\mu_p y_2} + \begin{pmatrix} \mu_s\beta & \xi\beta \\ 2\xi\mu_s\mu_p & 2\xi^2\mu_p \end{pmatrix} e^{\mathrm{i}\mu_s y_2}\right] \qquad (2.16)$$

and let $N_r(x_1; y_1, y_2)$ denote the first part of N and $N_s(x_1; y_1, y_2)$ denote the second part of N in (2.15).

It remains to study Dirichlet Green Tensor $D(x,y)$. We still use Fourier transform to derive the formula of Green Tensor in frequency domain. Then we can obtain $D(x,y)$ similar to $N(x,y)$. It follows an alternative representation for $D(x,y)$

$$\hat{D}(\xi, x_2; y_2) = \hat{\Phi}(\xi, x_2; y_2) - \hat{\Phi}(\xi, x_2; -y_2) + \hat{M}(\xi, x_2; y_2) \qquad (2.17)$$

$$\hat{M}(\xi, x_2; y_2) = \frac{\mathrm{i}}{\omega^2\gamma(\xi)}\Big\{A(\xi)e^{\mathrm{i}\mu_s(x_2+y_2)} + B(\xi)e^{\mathrm{i}\mu_p(x_2+y_2)} \qquad (2.18)$$
$$- A(\xi)e^{\mathrm{i}\mu_s x_2 + \mu_p y_2} - B(\xi)e^{\mathrm{i}\mu_p x_2 + \mu_s y_2}\Big\}$$

where

$$A(\xi) = \begin{pmatrix} \xi^2\mu_s & -\xi\mu_s\mu_p \\ -\xi^3 & \xi^2\mu_p \end{pmatrix} \qquad B(\xi) = \begin{pmatrix} \xi^2\mu_s & \xi^3 \\ \xi\mu_s\mu_p & \xi^2\mu_p \end{pmatrix}$$

and $\gamma(\xi) = \xi^2 + \mu_s\mu_p$.

Lemma 2.3 *Let* Lamé *constant* $\lambda, \mu \in \mathbb{C}$ *and* $\mathrm{Im}\,(k_s) \geq 0, \mathrm{Im}\,(k_p) \geq 0$, *then equation* $\gamma(\xi) = 0$ *has no root in complex plane.*

Proof. Let $F(\xi) = \gamma(\xi) * (\xi^2 - \mu_s\mu_p)$ and it is easy to see that the root of $\gamma(\xi) = 0$ is also of $F(\xi) = 0$. A simple computation show that $F(\xi) = (k_s^2 + k_p^2)\xi^2 - k_p^2k_s^2$. However, only when $\xi^2 = k_p^2k_s^2/(K_s^2 + k_p^2)$, $F(\xi) = 0$ but $\gamma(\xi) = 2k_p^2k_s^2/(K_s^2 + k_p^2)$. This completes the proof. □

Thus, we get the representation of Green Tensor by inverse Fourier transform

$$D(x,y) = \Phi(x,y) - \Phi(x,y') + \frac{1}{2\pi}\int_{\mathbb{R}} \hat{M}(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \qquad (2.19)$$

Let $T_D(x,y)$ denote the traction of $D(x,y)$ in direction e_2 with respect to x such that $T_D(x,y)e_i = T_x^{e_2}(D(x,y))e_i = T_x^{e_2}(D(x,y)e_i)$. Then we can get the representation of $T_D(x,y)$ by a trivial calculation.

$$T_D(x,y) = T(x,y) - T(x,y') + \frac{1}{2\pi}\int_{\mathbb{R}} \hat{T}_M(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \qquad (2.20)$$

and

$$\hat{T}_M(\xi, x_2; y_2) = \frac{\mu}{\omega^2\gamma(\xi)}\Big\{E(\xi)e^{\mathrm{i}\mu_s(x_2+y_2)} + F(\xi)e^{\mathrm{i}\mu_p(x_2+y_2)} - E(\xi)e^{\mathrm{i}\mu_s x_2+\mu_p y_2} - F(\xi)e^{\mathrm{i}\mu_p x_2+\mu_s y_2}\Big\} \tag{2.21}$$

where

$$E(\xi) = \begin{pmatrix} -\xi^2\beta & \xi\mu_p\beta \\ 2\xi^3\mu_s & -2\xi^2\mu_s\mu_p \end{pmatrix} \qquad F(\xi) = \begin{pmatrix} -2\xi^2\mu_s\mu_p & -2\xi^3\mu_p \\ -\xi\mu_s\beta & -\xi^2\beta \end{pmatrix}$$

Specially, $T_D(x, y)$ has a simple form when $x_2 = 0$:

$$\hat{T}_D(\xi, 0; y_2) = \frac{1}{\gamma(\xi)}\left[\begin{pmatrix} \mu_s\mu_p & \xi\mu_p \\ \xi\mu_s & \xi^2 \end{pmatrix}e^{\mathrm{i}\mu_s y_2} + \begin{pmatrix} \xi^2 & -\xi\mu_p \\ -\xi\mu_s & \mu_p\mu_s \end{pmatrix}e^{\mathrm{i}\mu_p y_2}\right] \tag{2.22}$$

and

$$T_D(x_1, 0; y_1, y_2) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{T}_D(\xi, 0; y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \tag{2.23}$$

To analysis the point spread function in the section 5, we should give asymptotic anslysis for $N(x_1, 0, y)$ and $T_D(x_1, 0; y)$. We need the following slight generalization of Van der Corput lemma for the oscillatory integral [18, P.152].

Lemma 2.4 *Let $-\infty < a < b < \infty$, and u is a C^k function u in (a, b).*
1. If $|u'(t)| \geq 1$ for $t \in (a, b)$ and u' is monotone in (a,b), then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 3\lambda^{-1}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

2. For all $k \geq 2$, if $|u^{(k)}(t)| \geq 1$ for $t \in (a, b)$, then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 12k\lambda^{-1/k}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

Proof. The assertion can be proved by extending the Van der Corptut lemma in [18]. Here we omit the details. □

Therefore, the estimate of $T_D(x_1, 0; y_1, y_2)$ is now a direct consequence of lemma **??**.

Lemma 2.5 *For every $x \in \Gamma_0$, $y \in \mathbb{R}^2_+$ that $|x_1-y_1|/|x-y| > (1+\kappa)/2$, $y_2/|x-y| < \kappa/2$ and $k_s|x-y| > 1$, we have*

$$|T_D(x, y)| \leq C\left(\frac{k_s y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{1/2}} + \frac{k_s|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{3/2}}\right) \tag{2.24}$$

where C is only dependent on κ.

Lemma 2.6 *For every $x \in \Gamma_0$, $y \in \mathbb{R}^2_+$ that $|x_1-y_1|/|x-y| > (1+\kappa)/2$, $y_2/|x-y| < \kappa/2$ and $k_s|x-y| > 1$, we have*

$$|N(x, y)| \leq \frac{C}{\mu}\left(\frac{k_s y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{1/2}} + \frac{k_s|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{3/2}} + e^{-\sqrt{k_R^2-k_s^2}y_2}\right) \tag{2.25}$$

where C is only dependent on κ.

3. The forward scattering problem

In this section we introduce the following stability estimate of the forward elastic scattering problem in the half space which can be proved by the limiting absorption principle by extending the classical argument in [20, 24, 16]. Let the obstacle occupy a bounded Lipschitz domain $D \subset \mathbb{R}^2_+$.

Theorem 3.1 *Let $g \in H^{1/2}(\Gamma_D)$, then the scattering problem of elastic equation in the half space*

$$\Delta_e u + \omega^2 u = 0 \qquad \text{in } \mathbb{R}^2_+ \backslash \bar{D}, \tag{3.1}$$

$$u = g \quad \text{ on } \Gamma_D, \tag{3.2}$$

$$\sigma(u)e_2 = 0 \quad \text{ on}\Gamma_0, \tag{3.3}$$

u satisfies the generalized radiation codition[21] such that

$$\lim_{r\to\infty} \int_{S_r^+} (\sigma(N(x,y)e_i)\hat{r}) \cdot u(x) - (N(x,y)e_i) \cdot (\sigma(u)\hat{r})ds(x) = 0 \tag{3.4}$$

where $S_r^+ := \{x \in \mathbb{R}^2_+ \mid \|x\| = r^2\}$, $\hat{r} = x/r$ and $y \in \mathbb{R}^2_+$. Then the problem (3.1)-(3.4) admits a unique solution $u \in H^1_{\text{loc}}(\mathbb{R}^2_+ \backslash \bar{D})$. Moreover, for any bounded open set $\mathcal{O} \subset \mathbb{R}^2_+ \backslash \bar{D}$ there exists a constant $C > 0$ such that

$$\|u\|_{H^1(\mathcal{O})} \le C\|g\|_{H^{-1/2}(\Gamma_D)} \tag{3.5}$$

The existence of the solution can be proved by the method of limiting absorption principle. The argument is standard and we give several lemmas below, see e.g. [20] for the consideration for Helmholtz equation. For any $z = 1 + \mathbf{i}\varepsilon, \varepsilon > 0$, $f \in H^1(\mathbb{R}^2_+)'$ with compact support in $B_R = \{x||x|^2 < R^2, x \in \mathbb{R}^2_+\} \subsetneq \mathbb{R}^2_+$ where B_R is a disk of radius R , we consider the problem

$$\Delta_e u_z + z\omega^2 u = -f \qquad \text{in } \mathbb{R}^2_+ \tag{3.6}$$

$$\sigma(u_z)e_2 = 0 \quad \text{ on } \quad \Gamma_0 \tag{3.7}$$

By Lax-Milgrim lemma we know that (3.6-3.7) has a unique solution $u_z \in H^1(\mathbb{R}^2_+)$. For any domain $\mathcal{D} \subset \mathbb{R}^2_+$, we define the weighted space $L^{2,s}(\mathcal{D}), s \in \mathbb{R}$, by

$$L^{2,s}(\mathcal{D}) = \{v \in L^2_{\text{loc}}(\mathcal{D}) : (1 + |x|^2)^{s/2} v \in L^2(\mathcal{D})\}$$

with the norm $\|v\|_{L^{2,s}(\mathcal{D})} = (\int_{\mathcal{D}} (1 + |x|^2)^s |v|^2 dx)^{1/2}$. The weighted Sobolev space $H^{1,s}(\mathcal{D}), s \in \mathbb{R}$, is defined as the set of functions in $L^{2,s}(\mathcal{D})$ whose first derivative is also in $L^{2,s}(\mathcal{D})$. The norm $\|v\|_{H^{1,s}(\mathcal{D})} = (\|v\|^2_{L^{2,s}(\mathcal{D})} + \|\nabla v\|^2_{L^{2,s}(\mathcal{D})})^{1/2}$.

We need the following sligt generalization of Rellich Theorem:

Lemma 3.1 *Let Ω be an open Lipschitz domain, then the sobolev space $H^{1,-s}(\Omega)$ is compactly embeded in $L^{2,-s'}(\Omega)$ for every $s' > s > 0$.*

Lemma 3.2 *Let $f \in L^2(\mathbb{R}^2_+)$ with compact support in B_R. For any $z = 1+\mathbf{i}\varepsilon$, $0 < \varepsilon < 1$, we have, for any $s > 1/2$, $\|u_z\|_{H^{1,-s}(\mathbb{R}^2_+)} \le C\|f\|_{L^2(\mathbb{R}^2_+)}$ for some constant independent of ε, u_z, and f.*

Proof. Let R_z denote the map from $L^2_c(\mathbb{R}^2_+)$ to $H^{1,-s}(\mathbb{R}^2_+)$ such that $R_z(f) = u_z$ where $L^2_c(\mathbb{R}^2_+)$ is denoted by all $f \in L^2(\mathbb{R}^2_+)$ with compact support in B_R, then it is easy to see that R_z is a linear bounded operator. It follows from theorem 3.7 in [16] that R_z is a uniformly continuous operator continues valued function on $z = 1 + \mathbf{i}\varepsilon$, $0 < \varepsilon < 1$ with value in $B(L^2_c(\mathbb{R}^2_+), H^{1,-s}(\mathbb{R}^2_+))$. Then, we can obtain that R_z is uniformly bounded in $B(L^2_c(\mathbb{R}^2_+), H^{1,-s}(\mathbb{R}^2_+))$. This complete the proof by the defintion of the operator norm. □

We next recall the following lemma which states the absence of positive eigenvalues for the linear elasticity system in half space [22].

Lemma 3.3 *Let $u \in L^2(\mathbb{R}^2_+\backslash\bar{D})$ such that u satisfies (3.1) and (3.3), than we assert that $u = 0$ in $\mathbb{R}^2_+\backslash\bar{D}$*

Proof. The asserting above can be proved by extending [22, theorem 3.1], here we omit the details. □

For any $0 < \varepsilon < 1$, we consider the problem

$$\Delta_e u_\varepsilon + (1 + \mathbf{i}\varepsilon)\omega^2 u_\varepsilon = 0 \qquad \text{in } \mathbb{R}^2_+\backslash\bar{D} \tag{3.8}$$

$$u_\varepsilon = g \quad \text{on } \Gamma_D \tag{3.9}$$

$$\sigma(u_\varepsilon)e_2 = 0 \quad \text{on}\Gamma_0 \tag{3.10}$$

We know that the above problem has a unique solution $u_\varepsilon \in H^1(\mathbb{R}^2_+\backslash\bar{D})$ by the Lax-Milgram Lemma. Thus, we have next lemma

Lemma 3.4 *Let $g \in H^{1/2}(\Gamma_D)$. For any $0 < \varepsilon < 1$, we have, for any $s > 1/2$, $\|u_\varepsilon\|_{H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})} \le C\|g\|_{H^{1/2}(\Gamma_D)}$ for some constant independent of $\varepsilon, u_\varepsilon$, and g.*

Proof. Because $h = dist(D, \Gamma_0) > 0$, we can find three concentric circles $B_{R_1}, B_{R_2}, B_{R_3}$ such that $D \subsetneq B_{R_1} \subsetneq B_{R_2} \subsetneq B_{R_3} \subsetneq \mathbb{R}^2_+$. Let $\chi \in C_0^\infty(\mathbb{R}^2_+)$ be the cut-off function such that $0 \le \chi \le 1$, $\chi = 0$ in B_{R_1}, and $\chi = 1$ outside of B_{R_2}. Let $v_\varepsilon = \chi u_\varepsilon$. Then v_ε satisfies (3.6) with $z = 1+\mathbf{i}\varepsilon$ and $q = \sigma(u_\varepsilon)\nabla\chi + (\lambda+\mu)(\nabla^2\chi u_\varepsilon + \nabla u_\varepsilon \nabla\chi) + \mu\Delta\chi u_\varepsilon + \mu \mathrm{div} u_\varepsilon \nabla\chi$, where $\nabla^2\chi$ is the Hessian matrix of χ. Clearly q has compact support. By lemma 3.2 we can obtain

$$\|v_\varepsilon\|_{H^{1,-s}(\mathbb{R}^2_+)} \le C\|u_\varepsilon\|_{H^1(B_{R_2}\backslash\bar{D})} \tag{3.11}$$

for some constant C independent of $\varepsilon > 0$. Now let $\chi_1 \in C_0^\infty(\mathbb{R}^2_+)$ be the cut-off function with that $0 \le \chi_1 \le 1$, $\chi_1 = 1$ in B_{R_2}, and $\chi_1 = 0$ outside of B_{R_3}. For $g \in H^{1/2}(\Gamma_D)$, let $u_g \in H^1(\mathbb{R}^2_+\backslash\bar{D})$ be the lifting function such that $u_g = g$ on Γ_D and $\|u_g\|_{H^1(\mathbb{R}^2_+\backslash\bar{D})} \le C\|g\|_{H^{1/2}(\Gamma_D)}$. By testing 3.8 with $\chi_1^2(\overline{u_\varepsilon - u_g})$ and using the standard argument we have

$$\|u_\varepsilon\|_{H^1(B_{R_2}\backslash\bar{D})} \le C(\|u_\varepsilon\|_{L^2(B_{R_3}\backslash\bar{D})} + \|g\|_{H^{1/2}(\Gamma_D)}). \tag{3.12}$$

A combination of (3.11) and the above estimate yields

$$\|u_\varepsilon\|_{H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})} \le C(\|u_\varepsilon\|_{L^2(B_{R_3}\backslash\bar{D})} + \|g\|_{H^{1/2}(\Gamma_D)}). \tag{3.13}$$

Now we claim

$$\|u_\varepsilon\|_{L^2(B_{R_3}\backslash\bar{D})} \leq C\|g\|_{H^{1/2}(\Gamma_D)}, \tag{3.14}$$

for any $g \in H^{1/2}(\Gamma_D)$ and $\varepsilon > 0$. If it were false, there would exist sequences $\{g_m\} \subset H^{1/2}(\Gamma_D)$ and $\{\varepsilon_m\} \subset (0,1)$, and $\{u_{\varepsilon_m}\}$ be the corresponding solution of (3.8)-(3.10) such that

$$\|u_{\varepsilon_m}\|_{L^2(B_{R_3}\backslash\bar{D})} = 1 \text{ and } \|g_m\|_{H^{-1/2}(\Gamma_D)} \leq \frac{1}{m}. \tag{3.15}$$

Then $\|u_{\varepsilon_m}\|_{H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})} \leq C$, and thus there is a subsequence of $\{\varepsilon_m\}$, which is still denoted by $\{\varepsilon_m\}$, such that $\varepsilon_m \to \varepsilon' \in [0,1]$, and a subsequence of $\{u_{\varepsilon_m}\}$, which is still denoted by $\{u_{\varepsilon_m}\}$, such that it converges to some $u_{\varepsilon'}$ in $H^{1,-s'}(\mathbb{R}^2_+\backslash\bar{D})$ by choosing $s' > s$. This is a consequence of Korn's inequality and Rellich theorem. So $u_{\varepsilon'} \in H^{1,-s'}(\mathbb{R}^2_+\backslash\bar{D})$ satisfies (3.8-3.10) with $g = 0$ and $\varepsilon = \varepsilon'$.

By the integral representation satistied by u_{ε_m}, we know that for $y \in \mathbb{R}^2_+\backslash\bar{B}_{R_1}$ and $i = 1,2$

$$u_{\varepsilon'}(y)\cdot e^i = \int_{\partial B_{R_1}} (\sigma(N_{\varepsilon'}(x,y)e_i)\nu)\cdot u_{\varepsilon'}(x) - (N_{\varepsilon'}(x,y)e_i)\cdot(\sigma(u_{\varepsilon'})_{\varepsilon'}\nu)ds(x) \tag{3.16}$$

If $\varepsilon' > 0$, we deduce from (3.16) that $u_{\varepsilon'}$ decays exponentially and thus $u_{\varepsilon'} \in H^1(\mathbb{R}^2_+\backslash\bar{D})$, then $u_{\varepsilon'} = 0$ by the uniqueness of the solution in $H^1(\mathbb{R}^2_+\backslash\bar{D})$ with positive absorption. If $\varepsilon' = 0$, by the [16, theorem 5.2], we have $u_{\varepsilon'} \in L^2(\mathbb{R}^2_+\backslash\bar{D})$. Then we conclude $u_{\varepsilon'} = 0$ by the lemma 3.3 Therefore, in any case $u_{\epsilon'} = 0$, which , however contradicts to 3.15. This complete the proof. □

Now we are in the position to prove the exsitence of Theorem 3.1.

Lemma 3.5 *For any $s > 1/2$, $u_\varepsilon : (0,1) \to H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})$ is a uniformly continuous operator valued function. Immediately, u_ε converges to some u_0 in $H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})$ and u_0 is a solution of (3.1-3.5).*

Proof. We also give a indirect prove here. Let $\delta_0 > 0$ and $\{\mu_n\}$ and $\{\nu_n\}$ be sequences in $(0,1)$ such that

$$|\mu_n - \nu_n| \leq 1/n \qquad \text{and} \qquad \|u_{\mu_n} - u_{\nu_n}\|_{H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})} \geq \delta_0 \tag{3.17}$$

Thus there is a subsequence of $\{\mu_n\}$, which is still denoted by $\{\mu_n\}$, such that $\{\mu_n\} \to \epsilon \in [0,1]$ and also $\{\nu_n\} \to \epsilon$. Then using lemma 3.4 and the procedure proving it, we get the $u_\epsilon, v_\epsilon \in H^{1,-s'}(\mathbb{R}^2_+\backslash\bar{D})$, by choosing $s' > s$, such that

$$\|u_{\mu_n} - u_\epsilon\|_{H^{1,-s'}(\mathbb{R}^2_+\backslash\bar{D})} \to 0$$
$$\|u_{\nu_n} - v_\epsilon\|_{H^{1,-s'}(\mathbb{R}^2_+\backslash\bar{D})} \to 0$$

and $u_\epsilon = v_\epsilon$ by the same arguement in lemma 3.4 which is contradict a contradiction. Thus we have proved u_ε is uniformly continuously for $\varepsilon \in (0,1)$. Then it is easy to see u_ε has a limitation in $H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})$ and the estimation of u_0 can be obtained by (3.14). This completes the proof. □

It is remain to prove the uniqueness in theorem 3.1. Actually, it can be obtained following the existence of solution with any $g \in H^{1/2}(\Gamma_D)$.

proof of Theorem 3.1 By the linearity of the problem, it is sufficient to prove that any u_0 satisfies the system (3.1-3.3) with the corresponding homogeneous boundary-value vanishes identically in $\mathbb{R}^2_+\backslash\bar{D}$. For any $y\in\mathbb{R}^2_+\backslash\bar{D}$, there exists $U^s(x,y)$ sataifies (3.1-3.3) with $g(x)=-N(x,y)$ on Γ_D following the lemma 3.5 and we define $U(x,y)=N(x,y)+U^s(x,y)$. It is easy to see that $U(x,y)$ satifies the generalized radiation condition (3.4). Thus by the integral representaion of u_0, we have

$$\lim_{r\to\infty}\int_{S_r^+}(\sigma(U(x,y)e_i)\nu)\cdot u_0(x)-(U(x,y)e_i)\cdot(\sigma(u_0)\nu)ds(x)=0$$

Finally, combining $U(x,y)=0, u_0(x)=0$ on Γ_D and the Green integral theorem we find that

$$\begin{aligned}
u_0(y)e_i &= \int_{\mathbb{R}^2_+\backslash\bar{D}}-(\Delta_e(N(x,y)e_i)+\omega^2N(x,y)e_i)\cdot u_0(x)dx\\
&= \int_{\mathbb{R}^2_+\backslash\bar{D}}\Delta u_0(x)\cdot(N(x,y)e_i)-\Delta_e(N(x,y)e_i)\cdot u_0(x)\\
&= \int_{\Gamma_D}(\sigma(U(x,y)e_i)\nu)\cdot u_0(x)-(U(x,y)e_i)\cdot(\sigma(u_0)\nu)ds(x)=0
\end{aligned}$$

Then the desired unique exsitence follows with lemma 3.5. This completes the proof of theorem 3.1. □

4. Reverse time migration method

In this section we introduce RTM method for inverse elastic scattering problems in the half space. Assume that there N_s sources and N_r receivers uniformly distributed on Γ_0^d, where $\Gamma_0^d=\{(x_1,x_2)^T\in\Gamma_0 : x_1\in[-d,d]\}, d>0$ is aperture. We denote by Ω the sampling domain in which the obstacle is sought. Let $h=dist(\Omega,\Gamma_0)$ be the distance of Ω to Γ_0. We assume the obstacle $D\subset\Omega$ and there exist constants $0<c_1<1, c_2>0, c_3>0$ such that

$$|x_1|\le c_1d,\quad |x_1-y_1|\le c_2h,\quad |x_2|\le c_3h\quad \forall x,y\in\Omega \tag{4.1}$$

Our RTM algorithm consists of two steps [23, 25]. The first step [8] is the back-propagation in which we back-propagate the complex conjugated data $\overline{u^s(x_r,x_s)}$ as the Dirichlet boundary condition into the domain. The second step is the cross-correlation in which we compute the imaginary part of the cross-correlation of the back-propagated field and the incoming wave which uses the source as the boundary codition on Γ_0.

Algorithm 4.1 (Reverse time migration algorithm)
Given the data $u_k^s(x_r,x_s)$, $k=1,2$ which is the measurement of the scattered field at x_r when the source is emitted at x_s in the polarized direction e_k, $s=1,\dots,N_s$ and $r=1,\dots,N_r$.

1° *Back-propagation: For $s=1,\dots,N_s$ and k=1,2, compute the back-propagation field*

$$v_k(z,x_s)=\frac{|\Gamma_0^d|}{N_r}\sum_{r=1}^{N_r}(T_{x_r}^{e_2}D(x_r,z))^T\overline{u_k^s(x_r,x_s)},\qquad \forall z\in\Omega \tag{4.2}$$

2° *Cross-correlation: For $z \in \Omega$, compute*

$$I_d(z) = \mathrm{Im} \sum_{k=1}^{2} \left\{ \frac{|\Gamma_0^d|}{N_s} \sum_{s=1}^{N_s} [(T_{x_s}^{e_2} D(x_s, z))^T e_k] \cdot v_k(z, x_s) \right\}. \tag{4.3}$$

It is easy to that for $z \in \Omega$

$$I_d(z) = \mathrm{Im} \sum_{k=1}^{2} \left\{ \frac{|\Gamma_0^d|}{N_s} \frac{|\Gamma_0^d|}{N_r} \sum_{s=1}^{N_s} \sum_{r=1}^{N_r} [(T_{x_s}^{e_2} D(x_s, z))^T e_k] \cdot [(T_{x_r}^{e_2} D(x_r, z))^T \overline{u_k^s(x_r, x_s)}] \right\} \tag{4.4}$$

This formula is used in all our numerical experiments in section. By letting $N_s, N_r \to \infty$, we know that (4.4) can be viewed as an approximation of the following continuous integral:

$$\hat{I}_d(z) = \mathrm{Im} \sum_{k=1}^{2} \int_{\Gamma_0^d} \int_{\Gamma_0^d} [(T_{x_s}^{e_2} D(x_s, z))^T e_k] \cdot [(T_{x_r}^{e_2} D(x_r, z))^T \overline{u_k^s(x_r, x_s)}] ds(x_r) ds(x_s) \tag{4.5}$$

where $z \in \Omega$. We will study the resolution of the function $\hat{I}_d(z)$ in the section 5. To this end we will first consider the resolution of the finite aperture point source function in the next function.

5. The point spread function

We start by introducing some notation. For any bounded domain $U \subset \mathbb{R}^2$ with Lipschitz boundary Γ_U and the unit outer normal vector ν, let $\|u\|_{H^1(U)} = (\|\nabla\phi\|_{L^2(U)}^2 + d_U^{-2}\|\phi\|_{L^2(U)}^2)^{1/2}$ be the weighted $H^1(U)$ norm and $\|v\|_{H^{1/2}(\Gamma)} = (d_U^{-1}\|v\|_{L^2(\Gamma)}^2 + |v|_{\frac{1}{2},\Gamma}^2)^{1/2}$ be the weighted $H^{1/2}(\Gamma)$ norm, where d_U is the diameter of U and

$$|v|_{\frac{1}{2},\Gamma} = \left(\int_\Gamma \int_\Gamma \frac{|v(x) - v(y)|^2}{|x-y|^2} ds(x) ds(y) \right)^{1/2}.$$

By scaling argument and trace theorem we know that there exists a constant $C > 0$ independent of d_D such that for any $\phi \in H^1(U)$ [14, corollary 3.1],

$$\|\phi\|_{H^{1/2}(\Gamma_U)} + \|\sigma(\phi) \cdot \nu\|_{H^{-1/2}(\Gamma_U)} \le C \max_{x \in U} (|\phi(x)| + d_U |\nabla\phi(x)|) \tag{5.1}$$

The point spread function measures the resolution for finding point source[3]. In [14], the point spread function has been defined in the case of acoustic wave. We now define elastic point spread function $J(z, y)$, a $\mathbb{C}^{2\times 2}$ matrix, which back-propagate the conjugated data $\overline{N(x, y)}$ as the Dirichlet boundary condition. Thus, for any $z, y \in \mathbb{R}_+^2$

$$J(z, y) = \int_{\Gamma_0} (T_D(x, z))^T \overline{N(x, y)} ds(x) \tag{5.2}$$

$$= \int_{\mathbb{R}} (T_D(x_1, 0; z_1, z_2))^T \overline{N(x_1, 0; y_1, y_2)} dx_1 \tag{5.3}$$

The estimate in lemma 2.5-2.6 show that the integral above exists. Now, we define functions

$$\Theta(\xi; y_1, y_2) = \frac{1}{\gamma(\xi)} \left[\begin{pmatrix} \mu_s \mu_p & -\xi \mu_p \\ -\xi \mu_s & \xi^2 \end{pmatrix} e^{\mathrm{i}\mu_s y_2} + \begin{pmatrix} \xi^2 & \xi \mu_p \\ \xi \mu_s & \mu_p \mu_s \end{pmatrix} e^{\mathrm{i}\mu_p y_2} \right] e^{\mathbf{i}\xi y_1} \tag{5.4}$$

Let $\underline{\hat{N}(\xi;y)} = \hat{N}(\xi;y_2)e^{-\mathbf{i}\xi y_1}$ and $\hat{T}_D(\xi;y) = \hat{T}_D(\xi;y_2)e^{-\mathbf{i}\xi y_1}$. It is easy to see that $\Theta = \overline{\hat{T}_D(\xi;y)}$ when $\xi \in \mathbb{R}\backslash[-k_s,k_s]$.

We split the spectral terms into components associated with pressure and shearing waves.

$$\hat{T_D} = \hat{T_D^p} + \hat{T_D^s} \quad \hat{N} = \hat{N^p} + \hat{N^s} \quad \Theta = \Theta^p + \Theta^s$$

Thus, we define

$$J^{\alpha\eta}(z,y) = \int_R (T_D^\alpha(x_1,0;z))^T \overline{N^\eta(x_1,0;y)}dx_1, \qquad \alpha = s,p \quad \eta = s,p \tag{5.5}$$

It's esay to see

$$J(z,y) = \sum_{\alpha=p,s}^{\eta=p,s} J^{\alpha\eta}(z,y)$$

In order to analysis the PSF, loss is assumed in the medium that $k_{\alpha,\varepsilon} := k_\alpha(1+\mathbf{i}\varepsilon)$. Then by Parseval identity and lemma 2.2, we carry out

$$\begin{aligned}
J^{ss}(z,y) = & \lim_{\varepsilon\to 0^+} \int_R (T_D^s(x_1,0;z_1,z_2))^T \overline{N^{s,\varepsilon}(x_1,0;y_1,y_2)}dx_1 \\
= & \lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int_R (\hat{T}_D^s(\xi,0;z))^T \overline{\hat{N}^{s,\varepsilon}(\xi,0;y)}d\xi \\
= & \frac{1}{2\pi} \int_{-k_s}^{k_s} (\hat{T}_D^s(\xi,0;z))^T \overline{\hat{N}^{s,\varepsilon}(\xi,0;y)}d\xi \\
+ & \lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int_{R\backslash[-k_s,k_s]} (\hat{T}_D(\xi,0;z))^T \overline{\hat{N}^{s,\varepsilon}(\xi,0;y)}d\xi \\
:= & F^{ss}(z,y) + R^{ss}(z,y)
\end{aligned}$$

and for $(\alpha,\eta) \neq (s,s)$

$$\begin{aligned}
J^{\alpha\eta}(z,y) = & \lim_{\varepsilon\to 0^+} \int_R (T_D^\alpha(x_1,0;z_1,z_2))^T \overline{N^{p,\varepsilon}(x_1,0;y_1,y_2)}dx_1 \\
= & \lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int_R (\hat{T}_D^\alpha(\xi,0;z))^T \overline{\hat{N}^{\eta,\varepsilon}(\xi,0;y)}d\xi \\
= & \frac{1}{2\pi} \int_{-k_p}^{k_p} (\hat{T}_D^s(\xi,0;z))^T \overline{\hat{N}^{\eta,\varepsilon}(\xi,0;y)}d\xi \\
+ & \lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int_{R\backslash[-k_p,k_p]} (\hat{T}_D^\alpha(\xi,0;z))^T \overline{\hat{N}^{\eta,\varepsilon}(\xi,0;y)}d\xi \\
:= & F^{\alpha\eta}(z,y) + R^{\alpha\eta}(z,y)
\end{aligned}$$

Using Cauchy integral theorem, we get

$$\begin{aligned}
\overline{R^{ss}(y,z)} = & \lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int_{R\backslash[-k_s,k_s]} \overline{(\hat{T}_D^s(\xi,0;z))^T} \hat{N}^{s,\varepsilon}(\xi,0;y)d\xi \\
= & \lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int_{R\backslash[-k_s,k_s]} (\Theta^s(\xi;z))^T \hat{N}^{s,\varepsilon}(\xi,0;y)d\xi
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{\Gamma_l^{\pm}\cup\Gamma_r^{\pm}} (\Theta^s(\xi;z))^T\hat{N}^s(\xi,0;y)d\xi + ResiduePart\\
&:= \mathrm{I}^{ss}(z,y)+\mathrm{II}^{ss}(z,y)
\end{aligned}
$$

and for $(\alpha,\eta)\neq(s,s)$

$$
\begin{aligned}
\overline{R^{\alpha\eta}(y,z)} &= \lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int_{R\backslash[-k_s,k_s]} \overline{(\hat{T}_D^{\alpha}(\xi,0;z))^T}\hat{N}^{\eta,\varepsilon}(\xi,0;y)d\xi\\
&= \lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int_{R\backslash[-k_p,k_p]} (\Theta^{\alpha}(\xi;z))^T\hat{N}^{\eta,\varepsilon}(\xi,0;y)d\xi\\
&+ \frac{1}{2\pi}\int_{(-k_s,-k_p)\cup(k_p,k_s)} \overline{(T^{\alpha}(\xi;z))^T}\hat{N}^{\eta}(\xi,0;y)d\xi\\
&= \frac{1}{2\pi}\int_{\Gamma_l^{\pm}\cup\Gamma_r^{\pm}} (\Theta^{\alpha}(\xi;z))^T\hat{N}^{\eta}(\xi,0;y)d\xi + ResiduePart\\
&+ \frac{1}{2\pi}\int_{(-k_s,-k_p)\cup(k_p,k_s)} \overline{(T^{\alpha}(\xi;z))^T}\hat{N}^{\eta}(\xi,0;y)d\xi\\
&:= \mathrm{I}^{\alpha\eta}(z,y)+\mathrm{II}^{\alpha\eta}(z,y)+\mathrm{III}^{\alpha\eta}(z,y)
\end{aligned}
$$

where $\pm$ are corresponding $sgn(z_1-y_1)$. In the sequel, A^{ij} denotes the (i,j) element of a 2×2 matrix.

Our goal now is to show the main contribution to the point spread funciton when $k_sh\gg 1$. Put $n_*=\min\{N|\kappa^{2N-1}<1/c_3, N\in\mathbb{Z}_+\}$. Then we claim the primary theorem in this section:

Theorem 5.1 *Let $k_sh>1$. For any $z,y\in\Omega$, J(z,y)=F(z,y)+R(z,y), where*

$$F(z,y)=F_{ss}(z,y)+F_{pp}(z,y) \tag{5.6}$$

$$R(z,y)=R^{ss}(z,y)+R^{pp}(z,y)+J^{sp}(z,y)+J^{ps}(z,y) \tag{5.7}$$

Moreover,

$$|R^{ij}(z,y)|+k_s^{-1}|\nabla_y R^{ij}(z,y)|\le\frac{C}{\mu}\left(\frac{1}{(k_sh)^{\frac{1}{2n^*}}}+e^{-\sqrt{k_R^2-k_s^2}h}\right) \tag{5.8}$$

uniformly for $z,y\in\Omega$. Here the constant C may dependent on k_sd_D and $\kappa:=k_p/k_s$, but is independent of k, h, d_D.

The proof of Theorem 5.1 depends on several lemmas that follow.

Without loss of generality. we assume $z_1-y_1\ge 0$ in this section. Otherwise, we can take substitution $\xi=-\xi$. Notice that the parameterization of hyperbolic curve passing $(\pm1,0)$ is:

$$\xi_1=\pm\sqrt{t^2+1}\qquad \xi_2=t$$

where $t\in\mathbb{R}$. We only consider the curve in the upper halp plane, denoted by Γ^+ here. Substituting $\xi=\xi_1+\mathbf{i}\xi_2\in\Gamma^+$ into $\mu(\xi):=(1-\xi^2)^{1/2}$ and $\mu_\kappa(\xi):=(\kappa^2-\xi^2)^{1/2}$, we arrive at

$$
\begin{aligned}
\operatorname{Im}\mu(\xi) &= \operatorname{Im}(1-(\xi_1^2-\xi_2^2+\mathbf{i}2\xi_1\xi_2))^{1/2} \\
&= \operatorname{Im}(-2t\sqrt{t^2+1}\mathbf{i})^{1/2}=t^{1/2}(t^2+1)^{1/4}
\end{aligned} \tag{5.9}
$$

$$\begin{aligned}\operatorname{Im}\mu_\kappa(\xi) &= \operatorname{Im}(\kappa^2 - (\xi_1^2 - \xi_2^2 + \mathbf{i}2\xi_1\xi_2))^{1/2} \\ &= \operatorname{Im}(\kappa^2 - 1 - 2t\sqrt{t^2+1}\mathbf{i})^{1/2} \\ &= \sqrt{\frac{\sqrt{(1-\kappa^2)^2 + 4t^2(t^2+1)} + 1 - \kappa^2}{2}} \\ &\geq t^{1/2}(t^2+1)^{1/4}\end{aligned} \tag{5.10}$$

Lemma 5.1 *For $\xi \in \Gamma_0$, let $f(\xi)$ is a complex valued function in $L^1(\Gamma^+)$ such that $|f(\xi)| \leq C(1+\xi^k)$, $k \in \mathbb{Z}_+$. Then for $a, b, c > 0$, we have*

$$|I(a,b,c) := \int_{\Gamma^+} f(\xi)\ e^{\mathbf{i}\xi a + \mathbf{i}\mu(\xi)b + \mathbf{i}\mu_\kappa(\xi)c}\ d\xi|$$
$$\leq C(\frac{1}{b+c} + \frac{1}{(b+c)^{k+1}})$$

Proof. Ovserve that

$$\frac{d\xi(t)}{dt} = \frac{t}{\sqrt{t^2+1}} + \mathbf{i}$$

By (5.9-5.10), it follows that

$$|e^{\mathbf{i}\xi a + \mathbf{i}\mu(\xi)b + \mathbf{i}\mu_\kappa(\xi)c}| \leq e^{-ta - t^{1/2}(t^2+1)^{1/4}b - t^{1/2}(t^2+1)^{1/4}c} \leq e^{-t(b+c)}$$

Finally, substituting $\xi(t)$ into $I(a,b,c)$, we have

$$\begin{aligned}|I(a,b,c)| &= \Big|\int_0^\infty |f(\xi(t))\frac{d\xi(t)}{dt} e^{\mathbf{i}\xi(t)a + \mathbf{i}\mu(\xi(t))b + \mathbf{i}\mu_\kappa(\xi(t))c} dt\Big| \\ &\leq C\int_0^\infty (1+t^k)e^{-t(b+c)}dt \\ &\leq C(\frac{1}{b+c} + \frac{1}{(b+c)^{k+1}})\end{aligned}$$

□

Lemma 5.2 *For any $z, y \in \mathbb{R}^2_+$,*

$$|\mathrm{I}_{ij}^{\alpha\beta}(z,y)| \leq \frac{C}{\mu}\sum_{j=1}^{4}(k_s(y_2+z_2))^{-j},\ \ \alpha,\beta = s,p \tag{5.11}$$

where C is may only dependent on κ.

Proof. Notice that

$$\begin{aligned}\frac{1}{\delta(\xi)} &= \frac{1}{(k_s^2 - 2\xi^2) + 4\xi^2(k_s^2-\xi^2)^{1/2}(k_p^2-\xi^2)^{1/2}} \\ &= \frac{(k_s^2-2\xi^2)^2 - 4\xi^2(k_s^2-\xi^2)^{1/2}(k_p^2-\xi^2)^{1/2}}{(4k_p^2 - 28k_s^2)\xi^6 + O(\xi^4)} = O(\frac{1}{\xi^2})\end{aligned}$$

$$\begin{aligned}\frac{1}{\gamma(\xi)} &= \frac{1}{\xi^2 + (k_s^2-\xi^2)^{1/2}(k_p^2-\xi^2)^{1/2}} \\ &= \frac{\xi^2 - (k_s^2-\xi^2)^{1/2}(k_p^2-\xi^2)^{1/2}}{(k_s^2+k_p^2)\xi^2 - k_s^2k_p^2} = O(1)\end{aligned}$$

as $\xi \to \infty$. Therefore, a simple computation show that the amplitude function of $\mathrm{I}_{ij}^{\alpha\beta}(z,y)$ denote by $A(\xi)$ can be written as $A(\xi) = \mu/k_s^3 O(\xi^3)$. Now substituing $\xi = k_s t$ in the integral, the lemma now follows immediately from lemma (5.1). This completes the proof.

□

Lemma 5.3 *For any $z, y \in \mathbb{R}_+^2$,*

$$|\mathrm{II}_{ij}^{ss}(x,y)| \leq \frac{C}{\mu} e^{-\sqrt{k_R^2-k_s^2}(y_2+z_2)} \quad |\mathrm{II}_{ij}^{sp}(x,y)| \leq \frac{C}{\mu} e^{-\sqrt{k_R^2-k_s^2}z_2+\sqrt{k_R^2-k_p^2}y_2} \tag{5.12}$$

$$|\mathrm{II}_{ij}^{pp}(x,y)| \leq \frac{C}{\mu} e^{-\sqrt{k_R^2-k_p^2}(y_2+z_2)} \quad |\mathrm{II}_{ij}^{ps}(x,y)| \leq \frac{C}{\mu} e^{-\sqrt{k_R^2-k_p^2}z_2+\sqrt{k_R^2-k_s^2}y_2} \tag{5.13}$$

where C is only dependent on $\kappa := k_p/k_s$.

Proof. When $z_1 - y_1 > 0$, we have

$$\begin{aligned}
\mathrm{II}_{11}^{ss} &= -\frac{1}{\mu} Res_{\xi=k_R} \frac{(k_s^2-4\xi^2)\mu_s^2\mu_p}{\gamma(\xi)\delta(\xi)} e^{\mathbf{i}\mu_s(z_2+y_2)+\mathbf{i}\xi(z_1-y_1)} \\
&= -\frac{(k_s^2-4\xi^2)\mu_s^2\mu_p}{\mu(\gamma(\xi)\delta(\xi))'} e^{\mathbf{i}\mu_s(z_2+y_2)+\mathbf{i}\xi(z_1-y_1)}|_{\xi=k_R}
\end{aligned}$$

Eliminating k_s in fraction, we can obtain estimation (5.12). The other terms can be estimated similarly, here we omit detials. This completes the proof. □

Lemma 5.4 *Let $f(\xi)$ is a bounded complex valued function in $L^1((\kappa,1))$. Then we have*

$$\begin{aligned}
&|I(a,b) := \int_\kappa^1 |f(\xi)e^{\mathbf{i}\xi a+\mathbf{i}\mu_\kappa(\xi)b} d\xi| \\
&\leq C\frac{1}{b}\|f\|_{L^\infty(\kappa,1)}
\end{aligned}$$

Proof. It is simple to see that

$$\begin{aligned}
|I(a,b)| &\leq C\int_\kappa^1 e^{-b\sqrt{\xi^2-\kappa^2}} d\xi \\
&\leq C\int_0^{\sqrt{1-\kappa^2}} \frac{t}{\sqrt{t^2+\kappa^2}} e^{-bt} dt \\
&\leq C\frac{1}{b}\|f\|_{L^\infty(\kappa,1)}
\end{aligned}$$

□

Lemma 5.5 *For any $z, y \in \mathbb{R}_+^2$,*

$$|\mathrm{III}_{ij}^{pp}(x,y)| \leq \frac{C}{\mu k_s(y_2+z_2)} \quad |\mathrm{III}_{ij}^{sp}(x,y)| \leq \frac{C}{\mu k_s y_2} \quad |\mathrm{III}_{ij}^{ps}(x,y)| \leq \frac{C}{\mu k_s z_2} \tag{5.14}$$

where C is only dependent on κ.

Proof. Taking substitution $\xi = k_s t$ and using the fact that $\gamma(\xi), \delta(\xi)$ have no roots on interval $[k_p, k_s]$, then we can get supremum of amplitude function. By lemma 5.4 with $b = k_s(y+z), k_s y, k_s z$, we can get the estimate immediately. This completes the proof. □

It turn to estimate $F^{sp}(z,y)$ and $F^{ps}(z,y)$.

Lemma 5.6 *For $0 < \kappa < 1$, let $F(\lambda) = \int_0^\kappa f(t) e^{\mathbf{i}\lambda(\sqrt{1-t^2} - \tau\sqrt{\kappa^2 - t^2} + \alpha t)} dt$, where $\tau \geq c_0 > 0$ and $\alpha \in \mathbb{R}$, then we have*

$$|F(\lambda)| \leq C(\kappa)\lambda^{-\frac{1}{2N_*}}\left[|f(\kappa)| + \int_0^\kappa |f'(t)|dt\right]$$

where $N_* = \min\{N | \kappa^{2N-1} < c_0, N \in \mathbb{Z}_+\}$.

Proof. Put $\phi(t) = -\sqrt{1-t^2}$ and $\psi(t,\tau) = \tau\phi(t/\kappa) - \phi(t) + \alpha t$. For easy of notations, we denote the n-th partial derivative of $g(t)$ with respect to t by $g^{(n)}(t)$. Then, it is to see that, for $n > 1$

$$\psi^{(n)}(t,\tau) = \frac{\tau}{\kappa^{n-1}}\phi^{(n)}(\frac{t}{\kappa}) - \phi^{(n)}(t)$$

A standard computation show that

$$\begin{aligned}
\phi^{(1)}(t) &= \frac{t}{\sqrt{1-t^2}} \\
\phi^{(2)}(t) &= \frac{1}{(1-t^2)^{3/2}} \\
\phi^{(3)}(t) &= \frac{3t}{(1-t^2)^{5/2}}
\end{aligned}$$

Moreover, for $n \geq 3$, we have

$$\phi^{(n)}(t) = \frac{p_n(t)}{(1-t^2)^{n-1/2}} \tag{5.15}$$

where $p_n = \sum_0^{n-2} a_k^n t^k$ is a $(n-2)$-th polynomial such that its coefficients satisfy the following recursion formula:

$$\begin{aligned}
&a_{n-1}^{n+1} = (n+1)a_{n-2}^n, \quad a_{n-2}^{n+1} = (n+2)a_{n-3}^n \\
&a_k^{n+1} = (k+1)a_{k+1}^n + (2n-k)a_{k-1}^n \quad \text{for } 1 \leq k \leq n-3 \\
&a_0^{n+1} = a_1^n
\end{aligned}$$

Since the polynmial coefficients are all positive, it is obvious that for $n \geq 1$, $\phi^{(n)}(t)$ is a monotone increasing positive function. Using the recursion formula, it follows that

$$\phi^{(n)}(0) = \begin{cases} 0 & \text{n is odd,} \\ (n-1)!!(n-3)!! & \text{n is even.} \end{cases} \tag{5.16}$$

where $(2k-1)!!$ is double factorial and $n > 3$. We are now in the position to proof the inequality. Since $0 < \kappa < 1$, obersev that

$$\psi^{(2N_*+1)}(t,\tau) \geq \frac{\tau}{\kappa^{2N_*}}\phi^{(2N_*+1)}(t) - \phi^{(2N_*+1)}(t) > 0$$

Therefore, $\psi^{(2N_*)}(t,\tau)$ is monotone increasing in $[0,\kappa)$. By (5.16), we get

$$\psi^{(2N_*)}(t,\tau) \geq \psi^{(2N_*)}(0,\tau) \geq \psi^{(2N_*)}(0,c_0) = C(2N_*)(\frac{c_0}{\kappa^{2N_*-1}} - 1) > 0 \tag{5.17}$$

The lemma is now a direct consequence of lemma (2.4). □

Lemma 5.7 *For any $z, y \in \Omega$,*

$$|F^{sp}_{ij}(x,y)| \leq \frac{C}{\mu}\frac{1}{(k_s h)^{\frac{1}{2n^*}}} \quad |F^{ps}_{ij}(x,y)| \leq \frac{C}{\mu}\frac{1}{(k_s h)^{\frac{1}{2n^*}}} \tag{5.18}$$

where C is only dependent on κ.

Proof.

□

Now we are in the position to prove the main theorem of this section.

proof of Theorem 5.1. The theorem now follows from lemma 5.2, lemma 5.3, lemma 5.5 and lemma

To complete the analysis of the point spread function, Substitute (5.4) and (2.16) into $F_{ss}(z,y), F_{pp}(z,y)$:

$$F^{pp}(z,y) = -\frac{1}{2\pi}\int_{(-k_p,-k_p)} \frac{\mathbf{i}k_s^2\mu_s}{\mu\gamma(\xi)\delta(\xi)} \begin{pmatrix} \xi^2 & -\xi\mu_p \\ -\xi\mu_p & \mu_p^2 \end{pmatrix} e^{\mathrm{i}\mu_p(z_2-y_2)+\mathbf{i}\xi(y_1-z_1)}$$

$$\begin{aligned} F^{ss}(z,y) &= -\frac{1}{2\pi}\int_{(-k_p,-k_p)} \frac{\mathbf{i}k_s^2\mu_p}{\mu\gamma(\xi)\delta(\xi)} \begin{pmatrix} \mu_s^2 & \xi\mu_s \\ \xi\mu_s & \xi^2 \end{pmatrix} e^{\mathrm{i}\mu_p(z_2-y_2)+\mathbf{i}\xi(y_1-z_1)} \\ &\quad -\frac{1}{2\pi}\int_{(-k_s,k_s)\setminus(-k_p,k_p)} \frac{\mathbf{i}(k_s^2-4\xi^2)\mu_p}{\mu\gamma(\xi)\overline{\delta(\xi)}} \begin{pmatrix} \mu_s^2 & \xi\mu_s \\ \xi\mu_s & \xi^2 \end{pmatrix} e^{\mathrm{i}\mu_s(z_2-y_2)+\mathbf{i}\xi(y_1-z_1)} \\ &:= F^{ss1}(z,y) + F^{ss2}(z,y) \end{aligned}$$

Based on the above argument, we know that $R(z,y)$ becomes small when z,y move away from Γ_0. Our goal is to show $F(z,y)$ has the similar decay to the elastic fundamental solution $\operatorname{Im}\Phi(z,y)$ as $|z-y| \to \infty$.

Lemma 5.8 *For any $z, y \in \mathbb{R}^2_+$, when $z = y$*

$$|\operatorname{Im} F_{ii}(z,y)| \geq \frac{1}{4(\lambda+2\mu)}\ , \ i = 1,2$$
$$\operatorname{Im} F_{12}(z,y) = \operatorname{Im} F_{21}(z,y) = 0$$

and for $z \neq y$

$$|F_{ij}(z,y)| \leq \frac{C}{\mu}[(k_s|z-y|)^{-1/2}) + (k_s|z-y|^{-1})]$$

where constant C is only dependent on κ.

Proof. We only proof the case of $i = 1$, the other ones are similar. First, we have $\gamma(\xi) \leq k_s^2$, $\delta(\xi) \leq k_s^4$ and $\mu_p \leq \mu_s$ when $\xi \in (-k_p,k_p)$. Then, if $z = y$

$$-\operatorname{Im}(F^{pp}_{11} + F^{ss1}_{11}) \geq \frac{1}{2\pi\mu}\int_{(-k_p,k_p)} \frac{\mu_p}{k_s^2} d\xi \tag{5.19}$$

$$= \frac{k_p^2}{2\pi\mu k_s^2}\int_0^\pi \sin^2(t)dt = \frac{1}{4(\lambda+2\mu)} \tag{5.20}$$

It's left to proof $-\mathrm{Im}\, F_{11}^{ss2} > 0$. If $\xi \in (-k_s, k_s)\backslash(-k_p, k_p)$, $\mu_p = \mathbf{i}\sqrt{\xi^2 - k_p^2}$. Substituting it into F^{ss2}, we have

$$F_{11}^{ss2} = \frac{1}{2\pi\mu}\int_{(-k_s,k_s)\backslash(-k_p,k_p)} \frac{\mu_s^2\sqrt{\xi^2-k_p^2}(k_s^2-4\xi^2)}{(\xi^2+\mathbf{i}\mu_s\sqrt{\xi^2-k_p^2})(\beta^2-\mathbf{i}4\xi^2\mu_s\sqrt{\xi^2-k_p^2})}d\xi \tag{5.21}$$

let $\alpha = (\xi^2+\mathbf{i}\mu_s\sqrt{\xi^2-k_p^2})(\beta^2-\mathbf{i}4\xi^2\mu_s\sqrt{\xi^2-k_p^2})$. A simple computation show that $\mathrm{Im}\,\alpha = k_s^2\mu_s\sqrt{\xi^2-k_p^2}(k_s^2-4\xi^2)$. It is easy to see that

$$-\mathrm{Im}\, F_{11}^{ss2} = \frac{k_s^2}{2\pi\mu}\int_{(-k_s,k_s)\backslash(-k_p,k_p)} \frac{\mu_s^3(\xi^2-k_p^2)(k_s^2-\xi^2)^2}{|\alpha|^2}d\xi > 0$$

For $z \neq y$, we denot $y - z = |y-z|(\cos\phi, \sin\phi)^T$ for some $0 \le \phi \le 2\pi$. Then it is easy to see that

$$F^{pp}(z,y) = \frac{1}{\mu}\int_0^\pi A(\theta,\kappa)e^{\mathbf{i}k_s|z-y|\cos(\theta-\phi)}$$

The phase function $f(\theta) = \cos(\theta-\phi)$ satisfies $f'(\theta) = -\sin(\theta-\phi)$, $f''(\theta) = -\cos(\theta-\phi)$. For any given $0 \le \phi \le 2\pi$, we can decompose $[0,\pi]$ into several intervals such that in each either $|f''(\theta)| \ge 1/2$ or $|f'(\theta)| \ge 1/2$ and $f'(\theta)$ is monotonous. The amplitude function $A(\theta,\kappa)$ and their derirates are integrable on $[0,\pi]$. Then the estimate for $F_{pp}(z,y)$ follows by using lemma 2.4. The estimation of $F^{ss}(z,y)$ can be proved similarly. This completes the proof. □

By (5.1), we obtain the following consequence of Lemma 3.1 and Lemma 3.3 whcih will be used in the next section.

Corollary 5.1 *There exists a constant C independent of k_s, h such that*

$$\|F(z,\cdot)\|_{H^{1/2}(\Gamma_D)} + \|\sigma(F(z,\cdot))\cdot\nu\|_{H^{1/2}(\Gamma_D)} \le \frac{C}{\mu}(1+kd_D)$$
$$\|R(z,\cdot)\|_{H^{1/2}(\Gamma_D)} + \|\sigma(R(z,\cdot))\cdot\nu\|_{H^{1/2}(\Gamma_D)} \le \frac{C}{\mu}(1+kd_D)\left(\frac{1}{(k_sh)^{\frac{1}{2n^*}}} + e^{-\sqrt{k_R^2-k_s^2}h}\right)$$

uniformly for $z \in \Omega$, where d_D is the diameter of the obstacle D.

Now we consider the finite aperture point spread function $J_d(z,y)$:

$$\int_{-d}^{d}(T_D(x_1,0;z_1,z_2))^T\overline{N(x_1,0;y_1,y_2)}dx_1 \tag{5.22}$$

Our aim is to estimate the difference $J(z,y) - J_d(z,y)$. It is easy to see that

$$\frac{(x_1-z_1)^2}{\rho^2} = \frac{1}{1+\frac{z_2^2}{(x_1-z_1)^2}} \ge \frac{1}{1+\frac{c_3^2h^2}{(1-c_1)^2d^2}} := m(h/d) \tag{5.23}$$

$$\frac{z_2^2}{\rho^2} = \frac{1}{1+\frac{(x_1-z_1)^2}{z_2^2}} \le \frac{1}{1+\frac{(1-c_1)^2d^2}{c_3^2h^2}} := M(h/d) \tag{5.24}$$

where $\rho = \sqrt{(x_1-z_1)^2+z_2^2}$ and $z \in \Omega, x \in \Gamma_0\backslash(-d,d)$.

Theorem 5.2 *Assume $m(h/d) > (1+\kappa)^2/4$, $M(h/d) < \kappa^2/4$ and $k_sh \geq 1$. Then for $z, y \in \Omega$, we have*

$$|J(z,y) - J_d(z,y)| + k_s^{-1}|\nabla_y(J(z,y) - J_d(z,y))| \leq \frac{C}{\mu}(\frac{h}{d} + \frac{(k_sh)^{1/2}}{e^{\sqrt{k_r^2-k_s^2}h}}(\frac{h}{d})^{1/2}) \tag{5.25}$$

where the constant C is only dependent on κ.

Proof. By lemma 2.6, lemma 2.5 and $k_sh \geq 1$, we have

$$\begin{aligned}
&\left|\int_d^\infty (T_D(x_1,0;z_1,z_2))^T\overline{N(x_1,0;y_1,y_2)}dx_1\right| \\
&\leq \frac{C}{\mu}\int_d^\infty \frac{k_sz_2}{|x-z|}\frac{1}{(k_s|x-z|)^{1/2}}\Big(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{1/2}} + e^{-\sqrt{k_r^2-k_s^2}y_2}\Big)dx_1 \\
&\leq \frac{C}{\mu}\int_{(1-c_1)d/h}^\infty \frac{1}{(1+t^2)^{3/2}} + \frac{(k_sh)^{1/2}}{(1+t^2)^{3/4}}e^{-\sqrt{k_r^2-k_s^2}h}dt \\
&\leq \frac{C}{\mu}((\frac{h}{d})^2 + \frac{(k_sh)^{1/2}}{e^{\sqrt{k_r^2-k_s^2}h}}(\frac{h}{d})^{1/2})
\end{aligned}$$

Here we have used the first inequeality in (4.1). Similarly, we can prove that the estimate for te integral in $[-\infty, -d]$. This shows the estimate for $J(z,y) - J_d(z,y)$. The estimate for $\nabla_y(J(z,y) - J_d(z,y))$ can be proved similarly. $\square$

By (5.1) we obtain the following corollary

Corollary 5.2 *Assume $m(h/d) > (1+\kappa)^2/4$, $M(h/d) < \kappa^2/4$ and $k_sh \geq 1$. There exists a constant C independent of k_s, h such that*

$$\begin{aligned}
&\|J(z,\cdot) - J_d(z,\cdot)\|_{H^{1/2}(\Gamma_D)} + \|\sigma(J(z,\cdot) - J_d(z,\cdot))\cdot\nu\|_{H^{1/2}(\Gamma_D)} \\
&\leq \frac{C}{\mu}((\frac{h}{d})^2 + \frac{(k_sh)^{1/2}}{e^{\sqrt{k_r^2-k_s^2}h}}(\frac{h}{d})^{1/2})(1+k_sd_D)
\end{aligned}$$

uniformly for $z \in \Omega$, where d_D is the diameter of the obstacle D.

6. The resolution analysis

In this section we study the imaging resolution of the RTM for the Dirichlet boundary obstacle in the half space.

Theorem 6.1 *For any $z \in \Omega$, let $\Phi(y,z) \in \mathbb{C}^{2\times 2}$ be the radiation solution of the problem:*

$$\begin{aligned}
&\Delta_e\Phi(y,z) + \omega^2\Phi = 0 && \text{in } \mathbb{R}^2_+\backslash\bar{D} \\
&\Phi(y,z) = -\overline{F(z,y)} && \text{on } \Gamma_D \\
&\sigma_y(\Phi(y,z))\cdot e_2 = 0 && \text{on } \Gamma_0
\end{aligned}$$

Then, we have

$$\hat{I}_d(z) = \operatorname{Im}\mathbf{tr}\{\int_{\Gamma_D}[T_y^\nu(\overline{F(z,y)} + \Phi(y,z)]\overline{F(z,y)}ds(y)\} + W_{\hat{I}}(z) \tag{6.1}$$

where $|W_{\hat{I}}(z)| \leq C(1+k_sd_D)()$ uniformly for z in Ω.

Proof. By the integral representation, we have,

$$u_k^s(x_r, x_s) = \int_{\Gamma_D} (T_y^\nu N(y, x_r))^T u_k^s(y, x_s) - G(x_r, y)(T_y^\nu u_k^s(y, x_s)) ds(y) \tag{6.2}$$

where $u_k^s(x, x_s) + N(x, x_s)e_k = 0$. From (5.2) we get for any $z \in \Omega$,

$$\begin{aligned}
v_k(z, x_s) = & \int_{\Gamma_0^d} (T_{x_r}^{e_2} D(x_r, z))^T \overline{u_k^s(x_r, x_s)} ds(x_r) \\
= & \int_{\Gamma_D} ds(y) \Big(\int_{\Gamma_0^d} (T_{x_r}^{e_2} D(x_r, z))^T \overline{(T_y^\nu N(y, x_r))^T} ds(x_r) \Big) \overline{u_k^s(y, x_s)} \\
& - \Big(\int_{\Gamma_0^d} (T_{x_r}^{e_2} D(x_r, z))^T \overline{N(x_r, y)} ds(x_r) \Big) \overline{(T_y^\nu u_k^s(y, x_s))} \\
= & \int_{\Gamma_D} ds(y) \Big(\int_{\Gamma_0^d} (T_y^\nu \overline{N(y, x_r)} T_{x_r}^{e_2} D(x_r, z))^T ds(x_r) \Big) \overline{u_k^s(y, x_s)} \\
& - \Big(\int_{\Gamma_0^d} (T_{x_r}^{e_2} D(x_r, z))^T \overline{N(x_r, y)} ds(x_r) \Big) \overline{(T_y^\nu u_k^s(y, x_s))} \\
= & \int_{\Gamma_D} ds(y) \Big(\int_{\Gamma_0^d} (T_y^\nu [\overline{N(y, x_r)} T_{x_r}^{e_2} D(x_r, z)])^T ds(x_r) \Big) \overline{u_k^s(y, x_s)} \\
& - \Big(\int_{\Gamma_0^d} (T_{x_r}^{e_2} D(x_r, z))^T \overline{N(x_r, y)} ds(x_r) \Big) \overline{(T_y^\nu u_k^s(y, x_s))} \\
= & \int_{\Gamma_D} ds(y) \Big((T_y^\nu J_d^T(z, y))^T \overline{u_k^s(y, x_s)} - J_d(z, y) \overline{(T_y^\nu u_k^s(y, x_s))} \Big)
\end{aligned}$$

where we use the fact $(\sigma_x(A(x))\nu)B = \sigma_x(A(x)B)\nu$ above. By the definition of the imaging function $\hat{I}_d(z)$, we have

$$\begin{aligned}
\hat{I}_d(z) & = \mathrm{Im} \sum_{k=1}^{2} \int_{\Gamma_0^d} (T_{x_s}^{e_2} D(x_s, z))^T e_k \cdot v_k(z, x_s) ds(x_s) \\
& = \int_{\Gamma_D} ds(y) \sum_{k=1}^{2} \int_{\Gamma_0^d} (T_{x_s}^{e_2} D(x_s, z))^T e_k \cdot \Big((T_y^\nu J_d^T(z, y))^T \overline{u_k^s(y, x_s)} \\
& \quad - J_d(z, y) \overline{(T_y^\nu u_k^s(y, x_s))} \Big) \\
& = \mathrm{Im} \int_{\Gamma_D} ds(y) \sum_{k=1}^{2} \mathbf{tr} \Big((T_y^\nu J_d^T(z, y))^T \int_{\Gamma_0^d} \overline{u_k^s(y, x_s)} e_k^T T_{x_s}^{e_2} D(x_s, z) \\
& \quad - J_d(z, y) \int_{\Gamma_0^d} \overline{(T_y^\nu u_k^s(y, x_s))} e_k^T T_{x_s}^{e_2} D(x_s, z) \Big) \\
& = \mathrm{Im} \int_{\Gamma_D} ds(y) \mathbf{tr} \Big((T_y^\nu J_d^T(z, y))^T \sum_{k=1}^{2} W_k(y, z) \\
& \quad - J_d(z, y)(T_y^\nu \sum_{k=1}^{2} W_k(y, z)) \Big) \\
& = \mathrm{Im} \int_{\Gamma_D} \mathbf{tr} \Big((T_y^\nu J_d^T(z, y))^T W(y, z) - J_d(z, y)(T_y^\nu W(y, z)) \Big) ds(y)
\end{aligned}$$

where

$$W(y,z) = \sum_{k=1}^{2} W_k(y,z) \tag{6.3}$$

$$W_k(y,z) = \int_{\Gamma_0^d} \overline{u_k^s(y,x_s)} e_k^T (T_{x_s}^{e_2} D(x_s,z)) ds(x_s) \tag{6.4}$$

Therefore, $\overline{W_k(y,z)}$ can be viewed as the weighted superposition of $u_k^s(y,x_s)$. Then $\overline{W_k(y,z)}$ satisfies elastic equation

$$\Delta_e^y \overline{W_k(y,z)} + \omega^2 \overline{W_k(y,z)} = 0 \tag{6.5}$$

On the boundary of the obstacle Γ_D, we have

$$\begin{aligned} W(y,z) &= \sum_{k=1}^{2} \int_{\Gamma_0^d} u_k^s(y,x_s) e_k^T T_{x_s}^{e_2} \overline{D(x_s,z)} ds(x_s) \\ &= \sum_{k=1}^{2} \int_{\Gamma_0^d} -N(y,x_s) e_k e_k^T T_{x_s}^{e_2} \overline{D(x_s,z)} ds(x_s) \\ &= -\int_{\Gamma_0^d} N(y,x_s) T_{x_s}^{e_2} \overline{D(x_s,z)} ds(x_s) \\ &= -\overline{J_d^T(z,y)} \end{aligned}$$

Moreover, $T_y^{e_2} \overline{W_k(y,z)} = 0$ on Γ_0 since $T_y^{e_2} u_k^s(y,x_s) = 0$ on Γ_0. Let $W_d(y,z)$ be the scattering solution of the problem

$$\Delta_e W_d(y,z) + \omega^2 u_q = 0 \quad \text{in } \mathbb{R}_+^2 \backslash \bar{D} \tag{6.6}$$

$$W_d(y,z) = \overline{F(z,y)} - \overline{J_d^T(z,y)} \quad \text{on } \Gamma_D \tag{6.7}$$

$$\sigma_y(W_d(y,z)) \cdot e_2 = 0 \quad \text{on } \Gamma_0 \tag{6.8}$$

$$\hat{I}_d(z) = \operatorname{Im} \mathbf{tr} \int_{\Gamma_D} (T_y^\nu J_d^T(z,y))^T \overline{\Psi(y,z)} - J_d(z,y)(T_y^\nu \overline{\Psi(y,z)}) ds(y) + R_{\hat{I}}(z) \tag{6.9}$$

where

$$R_{\hat{I}}(z) = \operatorname{Im} \mathbf{tr} \int_{\Gamma_D} (T_y^\nu J_d^T(z,y))^T W_d(y,z) - J_d(z,y)(T_y^\nu W_d(y,z)) ds(y) \tag{6.10}$$

□

7. Extensions

8. Numerical experiments

References

[1] Jan Achenbach. *Wave Propagation in Elastic Solids.* North-Holland, 1980.

[2] Lars V Ahlfors. *Complex Analysis: An introduction to the theory of analytic functions of one complex variable.* McGraw-Hill, 1979.

[3] Habib Ammari, Josselin Garnier, Wenjia Jing, Hyeonbae Kang, Mikyoung Lim, Knut Sølna, and Han Wang. *Mathematical and statistical methods for multistatic imaging*, volume 2098. Springer, 2013.

[4] T. Arens. A new integral equation formulation for the scattering of plane elastic waves by diffraction gratings. *Journal of Integral Equations and Applications*, 11(3):232C245, 1999.

[5] Edip Baysal, Dan D Kosloff, and John WC Sherwood. Reverse time migration. *Geophysics*, 48(11):1514–1524, 1983.

[6] Augustinus Johannes Berkhout. *Seismic migration: Imaging of acoustic energy by wave field extrapolation*, volume 12. Elsevier, 2012.

[7] Norman Bleistein, Jack K Cohen, W John Jr, et al. *Mathematics of multidimensional seismic imaging, migration, and inversion*, volume 13. Springer Science & Business Media, 2013.

[8] Wen Fong Chang. Elastic reverse-time migration. *Geophysical Prospecting*, 37(3):243–256, 1987.

[9] Wen-Fong Chang and George A McMechan. Elastic reverse-time migration. *Geophysics*, 52(10):1365–1375, 1987.

[10] Junqing Chen, Zhiming Chen, and Guanghui Huang. Reverse time migration for extended obstacles: acoustic waves. *Inverse Problems*, 29(8):085005, 2013.

[11] Junqing Chen, Zhiming Chen, and Guanghui Huang. Reverse time migration for extended obstacles: electromagnetic waves. *Inverse Problems*, 29(8):085006, 2013.

[12] Zhiming CHEN and GuangHui HUANG. Reverse time migration for extended obstacles: Elastic waves. *SCIENTIA SINICA Mathematica*, 45(8):1103–1114, 2015.

[13] Zhiming Chen and GuangHui Huang. Reverse time migration for reconstructing extended obstacles in planar acoustic waveguides. *Science China Mathematics*, 58(9):1811–1834, 2015.

[14] Zhiming Chen and Guanghui Huang. Reverse time migration for reconstructing extended obstacles in the half space. *Inverse Problems*, 31(5):055007, 2015.

[15] Jon F Claerbout. Imaging the earth's interior. 1985.

[16] Yves Dermenjian and Jean Claude Guillot. Scattering of elastic waves in a perturbed isotropic half space with a free boundary. the limiting absorption principle. *Mathematical Methods in the Applied Sciences*, 10(2):87C124, 1988.

[17] Mario Durán, Ignacio Muga, and Jean-Claude Nédélec. The outgoing time-harmonic elastic wave in a half-plane with free boundary. *SIAM Journal on Applied Mathematics*, 71(2):443–464, 2011.

[18] Loukas Grafakos. *Classical and modern Fourier analysis*. Prentice Hall, 2004.

[19] Johng. Harris. *Linear elastic waves*. Cambridge University Press, 2001.

[20] Rolf Leis. *Initial Boundary Value Problems in Mathematical Physics*. J. Wiley, 1986.

[21] Andrew I. Madyarov and Bojan B. Guzina. A radiation condition for layered elastic media. *Journal of Elasticity*, 82(1):73–98, 2006.

[22] Mourad Sini. Absence of positive eigenvalues for the linearized elasticity system. *Integral Equations and Operator Theory*, 49(2):255–277, 2004.

[23] J. Sun, Y. Zhang, J. Sun, and Y. Zhang. Practical issues of reverse time migration: true amplitude gathers, noise removal and harmonic-source encoding. *Aseg Extended Abstracts*, 2009(3):397–398, 2008.

[24] Calvin H. Wilcox. *Scattering Theory for the d'Alembert Equation in Exterior Domains*. PhD thesis, Springer Berlin Heidelberg, 1975.

[25] Yu Zhang, Sheng Xu, Norman Bleistein, and Guanquan Zhang. True-amplitude, angle-domain, common-image gathers from one-way wave-equation migrations. *Geophysics*, 72(1):S49–S58, 2007.