Scattering Coefficient and Kirchhoff Approximation

1. Reflection of a plane wave by the x_1 axis

We consider the scattering of an incident plane p-wave $\hat{u}_p$ (or s-wave $\hat{u}_s$) with the incident direction $\hat{d}_0 = (\sin t_0, -\cos t_0)^T, t_0 \in (-\pi/2, \pi/2)$, by the plane $\Gamma := \{x \in \mathbb{R}^2 : x_2 = 0\}$. The incidence angle between $\hat{d}_0$ and the positive real axis is $\theta_0 = -\pi/2 + t_0, t_0 \in (-\pi/2, \pi/2)$. Denote by $\hat{\nu} = (0,1)^T$.

1.1. The case of incident p-wave

We denote the incident p-wave [1, p172] and its stress as

$$\begin{aligned}
&\hat{u}_p = A_0(\sin t_0, -\cos t_0)^T e^{\mathbf{i}k_p(x_1 \sin t_0 - x_2 \cos t_0)},\\
&\sigma(\hat{u}_p)\hat{\nu} = \mathbf{i}k_p A_0(-\mu \sin(2t_0), \lambda + 2\mu\cos^2 t_0)^T e^{\mathbf{i}k_p(x_1 \sin t_0 - x_2 \cos t_0)}.
\end{aligned}$$

The reflected p-wave is represented as

$$\begin{aligned}
&\hat{u}_{p,p} = A_1(\sin t_1, \cos t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 + x_2 \cos t_1)},\\
&\sigma(\hat{u}_{p,p})\hat{\nu} = \mathbf{i}k_p A_1(\mu \sin(2t_1), \lambda + 2\mu\cos^2 t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 + x_2 \cos t_1)}.
\end{aligned}$$

The reflected s-wave is denoted as

$$\begin{aligned}
&\hat{u}_{p,s} = A_2(\cos t_2, -\sin t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 + x_2 \cos t_2)},\\
&\sigma(\hat{u}_{p,s})\hat{\nu} = \mathbf{i}k_s A_2(\mu \cos(2t_2), -\mu\sin(2t_2))^T e^{\mathbf{i}k_s(x_1 \sin t_2 + x_2 \cos t_2)}.
\end{aligned}$$

Under the clamped condition, the total field vanishes on Γ and thus

$$\hat{u}_p(x_1,0) + \hat{u}_{p,p}(x_1,0) + \hat{u}_{p,s}(x_1,0) = 0, \quad \forall x_1 \in \mathbb{R}.$$

A simple computation shows that

$$\begin{aligned}
&t_1 = t_0, \quad \frac{\sin t_2}{\sin t_0} = \frac{k_p}{k_s} := \kappa,\\
&A_0 = \cos(t_0 - t_2), \quad A_1 = \cos(t_0 + t_2), \quad A_2 = -\sin 2t_0.
\end{aligned}$$

In summary, the total field is

$$\hat{u}_p^{\text{total}} = A_0 \hat{d}_0 e^{\mathbf{i}k_p x\cdot\hat{d}_0} + A_1 \hat{d}_1 e^{\mathbf{i}k_p x\cdot\hat{d}_1} + A_2 \hat{d}_2^{\perp} e^{\mathbf{i}k_s x\cdot\hat{d}_2}, \tag{1.1}$$

where for any $\tau = (\tau_1, \tau_2)^T \in \mathbb{R}^2$, $\tau^\perp = (\tau_2, -\tau_1)^T$, and

$$\hat{d}_1 = \hat{d}_0 - 2(\hat{d}_0\cdot\hat{\nu})\hat{\nu}, \hat{d}_2 = \kappa\hat{d}_0 - \left[\kappa(\hat{d}_0\cdot\hat{\nu}) - \sqrt{1-\kappa^2(\hat{d}_0\cdot\hat{\nu}^\perp)^2}\right]\hat{\nu}, \tag{1.2}$$

$$A_0 = \hat{d}_1\cdot\hat{d}_2, A_1 = -\hat{d}_0\cdot\hat{d}_2, A_2 = 2(\hat{d}_0\cdot\hat{\nu})(\hat{d}_0\cdot\hat{\nu}^\perp). \tag{1.3}$$

1.2. The case of incident s-wave

We denote the incident s-wave as

$$\begin{aligned}
&\hat{u}_s = A_0(-\cos t_0, -\sin t_0)^T e^{\mathbf{i}k_s(x_1 \sin t_0 - x_2 \cos t_0)},\\
&\sigma(\hat{u}_s)\hat{\nu} = \mathbf{i}k_s A_0(-\mu \cos(2t_0), \mu\sin(2t_0))^T e^{\mathbf{i}k_p(x_1 \sin t_0 - x_2 \cos t_0)}.
\end{aligned}$$

The reflected p-wave is represented as

$$\hat{u}_{s,p} = A_1(\sin t_1, \cos t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 + x_2 \cos t_1)},$$
$$\sigma(\hat{u}_{s,p})\hat{\nu} = \mathbf{i}k_p A_1(\mu \sin(2t_1), \lambda + 2\mu \cos^2 t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 + x_2 \cos t_1)}.$$

The reflected s-wave is denoted as

$$\hat{u}_{s,s} = A_2(\cos t_2, -\sin t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 + x_2 \cos t_2)},$$
$$\sigma(\hat{u}_{s,s}) = \mathbf{i}k_s A_2(\mu \cos(2t_2), -\mu \sin(2t_2))^T e^{\mathbf{i}k_s(x_1 \sin t_2 + x_2 \cos t_2)}.$$

The result is

$$t_2 = t_0, \quad \frac{\sin t_1}{\sin t_0} = \frac{k_s}{k_p} = \kappa_1,$$
$$A_0 = \cos(t_0 - t_1), \quad A_1 = \sin 2t_0, \quad A_2 = \cos(t_0 + t_1).$$

In summary, the total field is

$$\hat{u}_s^{\text{total}} = A_0 \hat{d}_0^{\perp} e^{\mathbf{i}k_s x\cdot\hat{d}_0} + A_1 \hat{d}_1 e^{\mathbf{i}k_p x\cdot\hat{d}_1} + A_2 \hat{d}_2^{\perp} e^{\mathbf{i}k_s x\cdot\hat{d}_2}, \tag{1.4}$$

where

$$\hat{d}_1 = \kappa_1 \hat{d}_0 - \left[\kappa_1(\hat{d}_0 \cdot \hat{\nu}) - \sqrt{1 - \kappa_1^2(\hat{d}_0 \cdot \hat{\nu}^{\perp})^2}\right]\hat{\nu}, \hat{d}_2 = \hat{d}_0 - 2(\hat{d}_0 \cdot \hat{\nu})\hat{\nu}, \tag{1.5}$$
$$A_0 = \hat{d}_1 \cdot \hat{d}_2, A_1 = -2(\hat{d}_0 \cdot \hat{\nu})(\hat{d}_0 \cdot \hat{\nu}^{\perp}), A_2 = -\hat{d}_0 \cdot \hat{d}_1. \tag{1.6}$$

2. Reflection of a plane wave in the general case

We consider the scattering of an incident plane p-wave u_p or s-wave u_s with the incident direction $d = (\sin\theta, -\cos\theta)^T, \theta \in (-\pi/2, \pi/2)$, by the plane $\Gamma := \{x \in \mathbb{R}^2 : x \cdot \nu = 0\}$ through the origin with the normal vector $\nu = (\sin\phi, \cos\phi)^T, \phi \in (0, 2\pi)$. The angle between ν and the positive real axis is $\pi/2 - \phi$. The total fields are

$$u_p^{\text{total}} = A_0 d_0 e^{\mathbf{i}k_p x\cdot d_0} + A_1 d_1 e^{\mathbf{i}k_p x\cdot d_1} + A_2 d_2^{\perp} e^{\mathbf{i}k_s x\cdot d_2}, \tag{2.1}$$
$$u_s^{\text{total}} = A_0 d_0^{\perp} e^{\mathbf{i}k_s x\cdot d_0} + A_1 d_1 e^{\mathbf{i}k_p x\cdot d_1} + A_2 d_2^{\perp} e^{\mathbf{i}k_s x\cdot d_2}, \tag{2.2}$$

where for $i = 0, 1, 2$, d_i is the unit vector and A_i is the corresponding amplitude. We impose $u_p^{\text{total}} = 0, u_s^{\text{total}} = 0$ on Γ. Let $\hat{x} = Sx$, where $S \in \mathbb{R}^{2\times 2}$ is the rotation matrix with rotation angle ϕ,

$$S = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

We have $\hat{\nu} = S\nu$.

Theorem 2.1 *Let $u(x) \in \mathbb{C}^2$ and*

$$\Delta_e^x := \begin{pmatrix} (\lambda + 2\mu)\frac{\partial^2}{\partial x_1^2} + (\lambda + \mu)\frac{\partial^2}{\partial x_1 \partial x_2} + \mu\frac{\partial^2}{\partial x_2^2} \\ \mu\frac{\partial^2}{\partial x_1^2} + (\lambda + \mu)\frac{\partial^2}{\partial x_1 \partial x_2} + (\lambda + 2\mu)\frac{\partial^2}{\partial x_1 2^2} \end{pmatrix}.$$

Assume that $u(x)$ satisfies $\Delta_e^x u(x) + \omega^2 u(x) = 0$, then we have $\Delta_e^{\hat{x}} \hat{u}(\hat{x}) + \omega^2 \hat{u}(\hat{x}) = 0$ where $\hat{u}(\hat{x}) := Su(S^T\hat{x})$ or $u(x) = S^T\hat{u}(S\hat{x})$.

Proof. Since

$$\begin{aligned}
\frac{\partial^2}{\partial\hat{x}_1^2} &= \cos^2\phi\frac{\partial^2}{\partial x_1^2} - 2\cos\phi\sin\phi\frac{\partial^2}{\partial x_1\partial x_2} + \sin^2\phi\frac{\partial^2}{\partial x_2^2}\\
\frac{\partial^2}{\partial\hat{x}_2^2} &= \sin^2\phi\frac{\partial^2}{\partial x_1^2} + 2\cos\phi\sin\phi\frac{\partial^2}{\partial x_1\partial x_2} + \cos^2\phi\frac{\partial^2}{\partial x_2^2}\\
\frac{\partial^2}{\partial\hat{x}_1\partial\hat{x}_2} &= \cos\phi\sin\phi\frac{\partial^2}{\partial x_1^2} + (\cos^2\phi - \sin^2\phi)\frac{\partial^2}{\partial x_1\partial x_2} - \cos\phi\sin\phi\frac{\partial^2}{\partial x_2^2}
\end{aligned}$$

This completes proof after substituting above equation into $\Delta_e^{\hat{x}}\hat{u}(\hat{x})$. □

By this theorem, we obtain from (1.1)-(1.3) that for u_p^{total}, $d_0 = (\sin(\theta-\phi), -\cos(\theta-\phi))^T$,

$$\begin{aligned}
&d_1 = d_0 - 2(d_0\cdot\nu)\nu, d_2 = \kappa d_0 - \left[\kappa(d_0\cdot\nu) - \sqrt{1-\kappa^2(d_0\cdot\nu^\perp)^2}\right]\nu,\\
&A_0 = d_1\cdot d_2, A_1 = -d_0\cdot d_2, A_2 = 2(d_0\cdot\nu)(d_0\cdot\nu^\perp).
\end{aligned}$$

Similarly, for u_s^{total}, $d_0 = (\sin(\theta-\phi), -\cos(\theta-\phi))^T$,

$$\begin{aligned}
&d_1 = \kappa_1 d_0 - \left[\kappa_1(d_0\cdot\nu) - \sqrt{1-\kappa_1^2(d_0\cdot\nu^\perp)^2}\right]\nu, d_2 = d_0 - 2(d_0\cdot\nu)\nu,\\
&A_0 = d_1\cdot d_2, A_1 = -2(d_0\cdot\nu)(d_0\cdot\nu^\perp), A_2 = -d_0\cdot d_1.
\end{aligned}$$

The traction of $u(x)$ on the plane Γ can be obtained by simple calculation

$$\begin{aligned}
\sigma(u_p^{\text{total}})\cdot\nu = &\ [\mathbf{i}k_pA_0(\lambda\nu + 2\mu(d_0,\nu)d_0) + \mathbf{i}k_pA_1(\lambda\nu + 2\mu(d_1,\nu)d_1)\\
&+ \mathbf{i}k_sA_2\mu((d_2,\nu)d_2^\perp + (d_2^\perp,\nu)d_2)]e^{\mathbf{i}k_px\cdot d_0}\\
:= &\ \mathbf{i}k_pA_0\hat{\mathbf{R}}_p(x,d_0,\nu)e^{\mathbf{i}k_px\cdot d_0}, \qquad (2.3)\\
\sigma(u_s^{\text{total}})\cdot\nu = &\ [\mathbf{i}k_sA_0\mu((d_0,\nu)d_0^\perp + (d_0^\perp,\nu)d_0) + \mathbf{i}k_pA_1(\lambda\nu + 2\mu(d_1,\nu)d_1)\\
&+ \mathbf{i}k_sA_2\mu((d_2,\nu)d_2^\perp + (d_2^\perp,\nu)d_2)]e^{\mathbf{i}k_sx\cdot d_0}\\
:= &\ \mathbf{i}k_sA_0\hat{\mathbf{R}}_s(x,d_0,\nu)e^{\mathbf{i}k_sx\cdot d_0}. \qquad (2.4)
\end{aligned}$$

Definition 2.1 *For any unit vector $d\in\mathbb{R}^2$, let $u_p^i = de^{\mathbf{i}k_px\cdot d}$ or $u_s^i = d^\perp e^{\mathbf{i}k_sx\cdot d}$ be the incident wave and $u_\alpha^s = u_\alpha^s(x;d)$ be the radiation solution of the Navier equation:*

$$\begin{aligned}
&u_\alpha^s + \omega^2 u_\alpha^s = 0 \quad \text{in} \quad \mathbb{R}^2\backslash\bar{D}\\
&\Delta_e u_\alpha^s = -u_\alpha^i \quad \text{on} \quad \partial D
\end{aligned}$$

The scattering coefficient $\mathbf{R}_\alpha(x;d)$ for $x\in\partial D$ is defined by the relation

$$\sigma(u_\alpha^s + u_\alpha^i)\cdot\nu = \mathbf{i}k_\alpha\mathbf{R}_\alpha(x;d)e^{\mathbf{i}k_\alpha x\cdot d} \quad \text{on} \quad \partial D$$

where $\alpha = p, s$.

For a convex object D, Kirchhoff approximation approximates the scattering coefficient by considering the boundary at $x\in\partial D$ locally as a plane with normal ν to obtain

$$\mathbf{R}_\alpha(x;d) \approx \begin{cases} \hat{\mathbf{R}}_\alpha(x;d,\nu(x)) & \text{if} \quad x\in\partial D_d^- = \{x\in\partial D, \nu(x)\cdot d < 0\},\\ 0 & \text{if} \quad x\in\partial D_d^+ = \{x\in\partial D, \nu(x)\cdot d \geq 0\}.\end{cases}$$

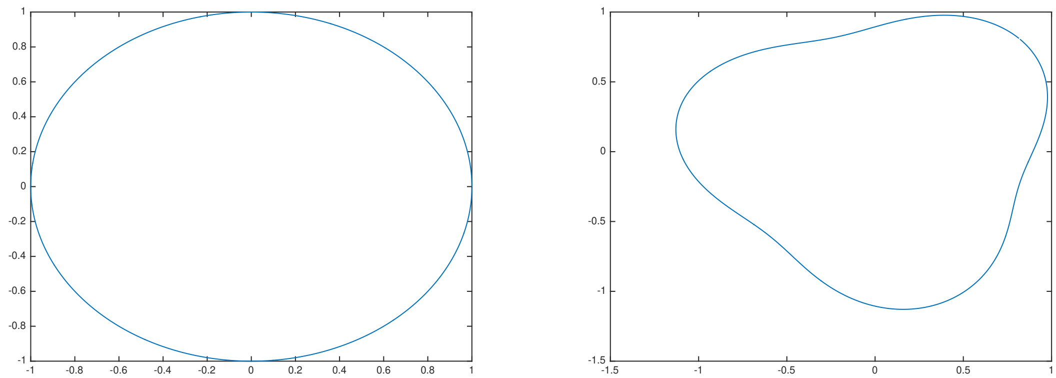

Figure 1. The shape of the obstacles.

3. Numerical examples

In this section we present several numerical examples to show the effectiveness of Kirchhoff approximation. To synthesize the real scattering coefficient we compute the solution $\sigma(u^s_\alpha + u^i_\alpha)\cdot\nu$ of the scattering problems by representing the ansatz solution as the single layer potential with the Green tensor $\mathbb{G}(x,y)$ as the kernel

$$u^s(x) = \int_{\Gamma_D} -\mathbb{G}(y,x)^T \sigma(u^s(y) + u^i(y))\nu ds(y) = -u^i(x) \quad \text{ on } \ x \in \Gamma_D,$$

and discretizing the integral equation by standard Nyström methods [2]. Let $\mathbf{R}_\alpha(x;d) = (\mathbf{R}^1_\alpha(x;d), \mathbf{R}^2_\alpha(x;d))^T$, then we have

$$\mathbf{R}^j_\alpha(x;d) = \frac{\sigma(u^s(y) + u^i(y))\nu \cdot e_j}{\mathbf{i}k_\alpha e^{\mathbf{i}k_\alpha x\cdot d}}. \tag{3.1}$$

We compute $\hat{\mathbf{R}}_\alpha(x;d) = (\hat{\mathbf{R}}^1_\alpha(x;d), \hat{\mathbf{R}}^2_\alpha(x;d))^T$ by (2.3) and (2.4). In all our numerical examples we choose Lamé constant $\lambda = 1/2$, $\mu = 1/4$ and

$$\begin{aligned} u^i_p &= (\cos t, \sin t)^T e^{\mathbf{i}k_p(x_1\cos t + x_2 \sin t)} \\ u^i_s &= (\sin t, -\cos t)^T e^{\mathbf{i}k_s(x_1\cos t + x_2 \sin t)} \end{aligned}$$

where $t \in [0, 2\pi]$.. The boundaries of the obstacles used in our numerical experiments are parameterized as follows:

$$\begin{aligned} &\text{Circle:} \quad x_1 = \cos(\theta), \quad x_2 = \sin(\theta); \\ &\text{Pear:} \quad \rho = 0.5(2 + 0.3\cos(3\theta)), x_1 = \sin\frac{\pi}{4}\rho(\cos\theta - \sin\theta), x_2 = \sin\frac{\pi}{4}\rho(\cos\theta + \sin\theta), \end{aligned}$$

where $\theta \in [0, 2\pi]$ (See Figure 1).

In the following examples, we take the angular frequency $\omega = \pi, 2\pi, 4\pi, 8\pi$.

References

[1] Achenbach J 1980 *Wave Propagation in Elastic Solids* (North-Holland)

[2] Colton D and Kress R 1998 *Inverse Acoustic and Electromagnetic Scattering Problems* (Heidelberg: Springer)

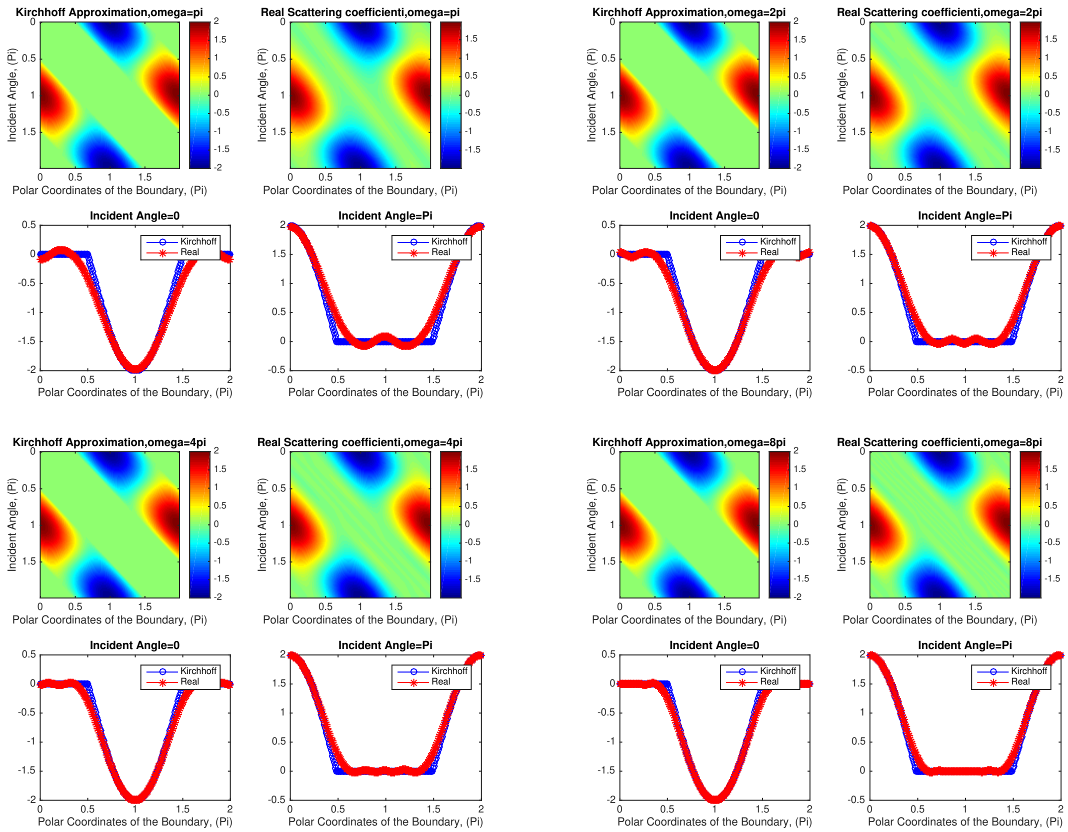

Figure 2. $\mathbf{R}_p^1$ and $\hat{\mathbf{R}}_p^1$ for the circle.

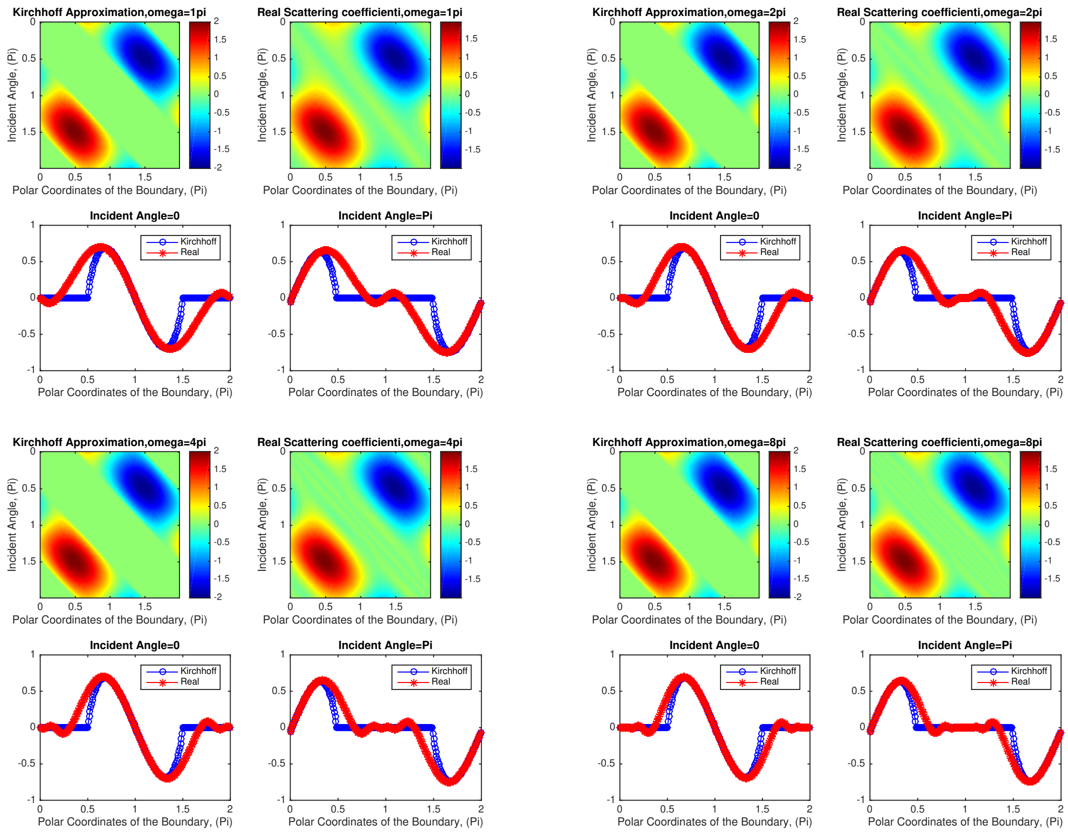

Figure 3. $\mathbf{R}_p^2$ and $\hat{\mathbf{R}}_p^2$ for the circle.

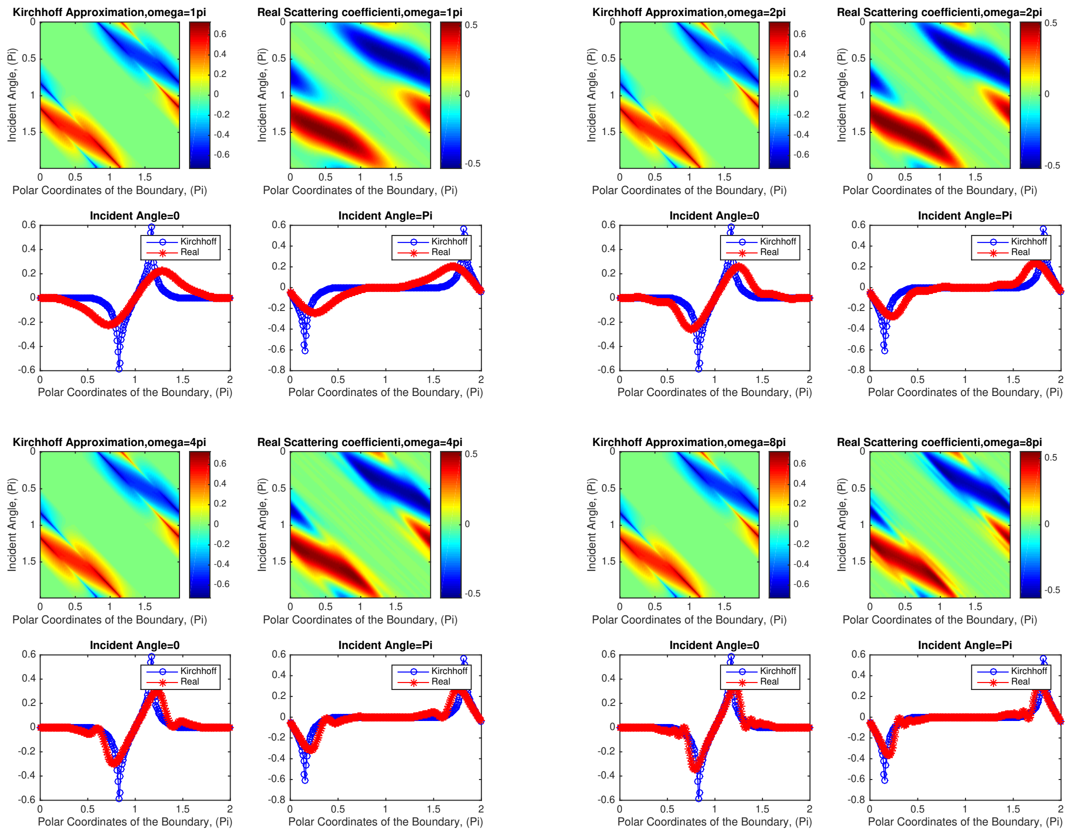

Figure 4. $\mathbf{R}_s^1$ and $\hat{\mathbf{R}}_s^1$ for the circle.

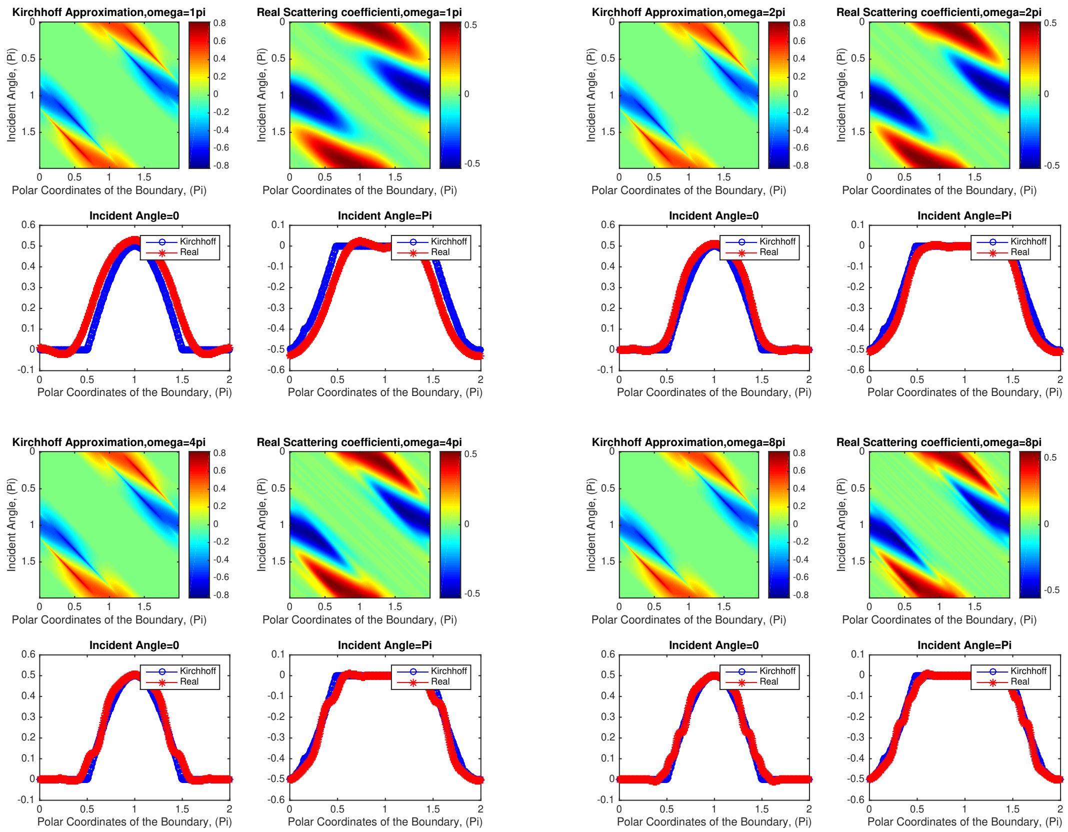

Figure 5. $\mathbf{R}_s^2$ and $\hat{\mathbf{R}}_s^2$ for the circle.

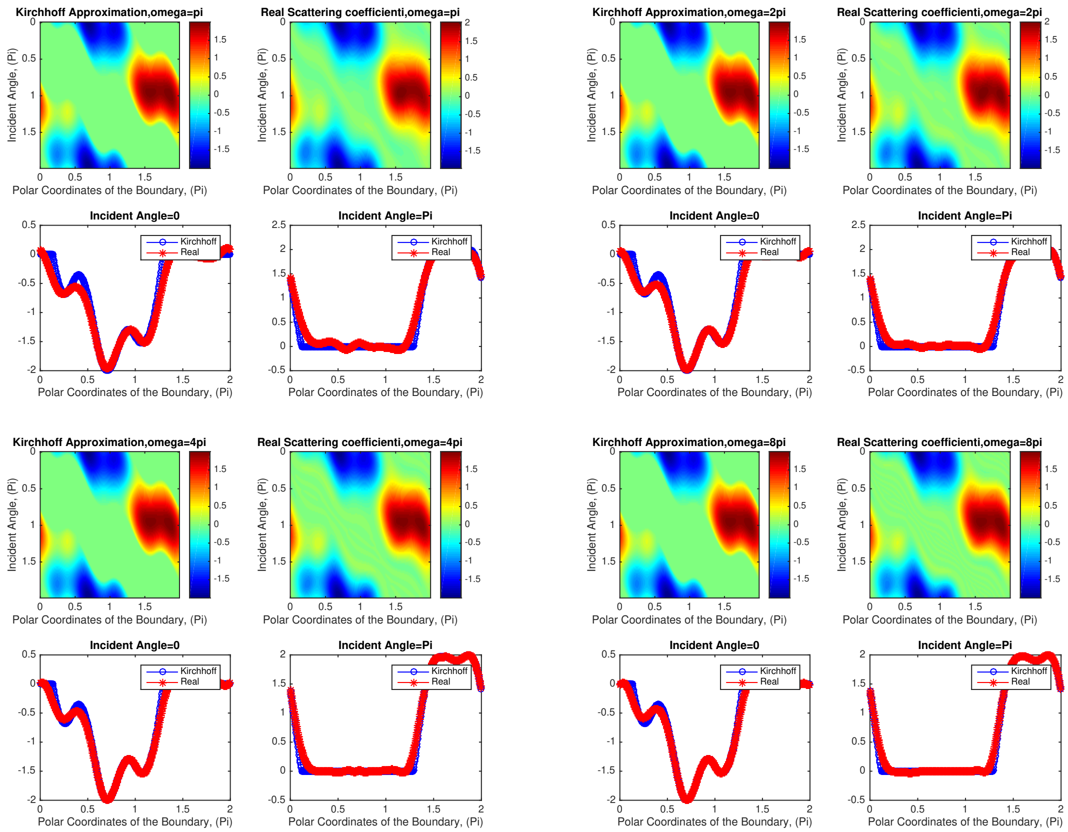

Figure 6. $\mathbf{R}_p^1$ and $\hat{\mathbf{R}}_p^1$ for the pear.

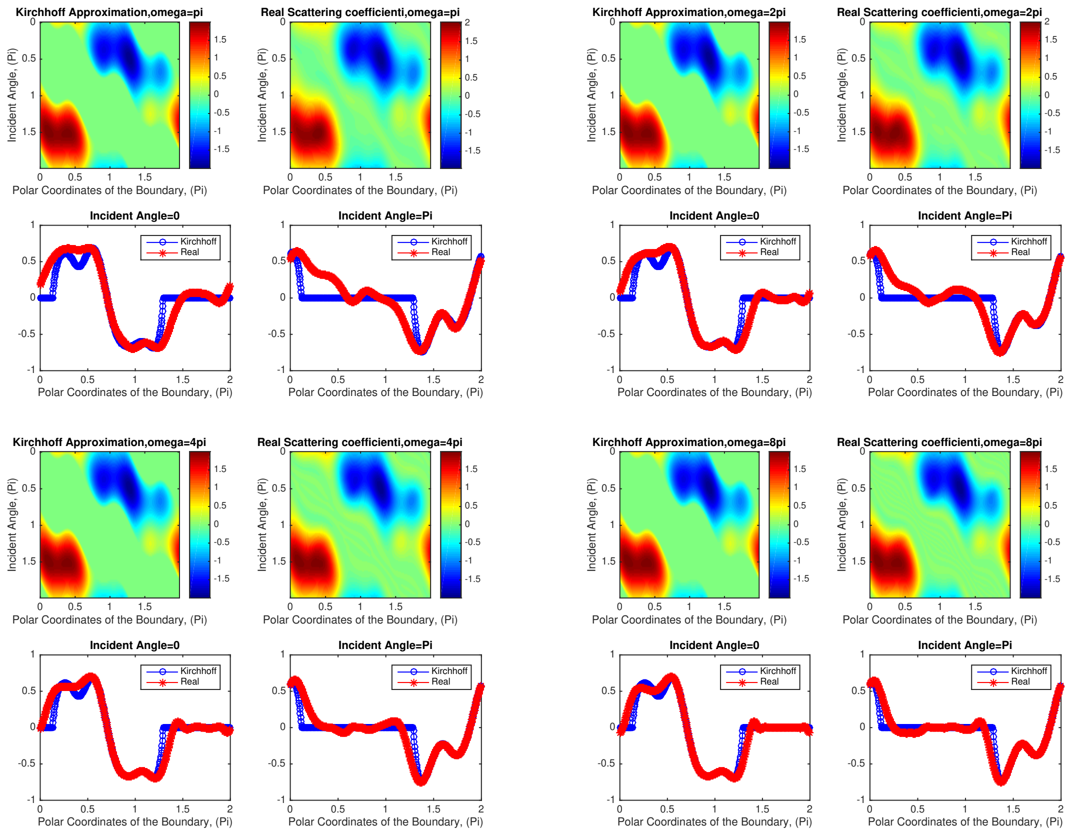

Figure 7. $\mathbf{R}_p^2$ and $\hat{\mathbf{R}}_p^2$ for the pear.

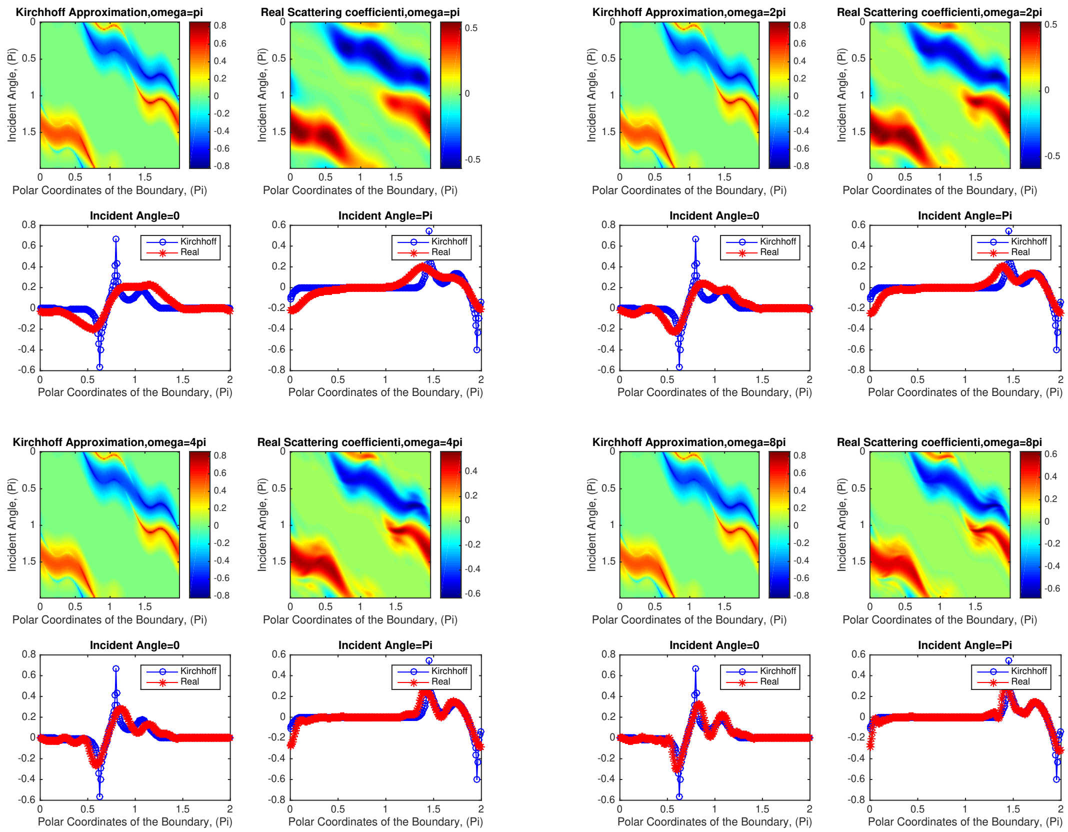

Figure 8. $\mathbf{R}_s^1$ and $\hat{\mathbf{R}}_s^1$ for the pear.

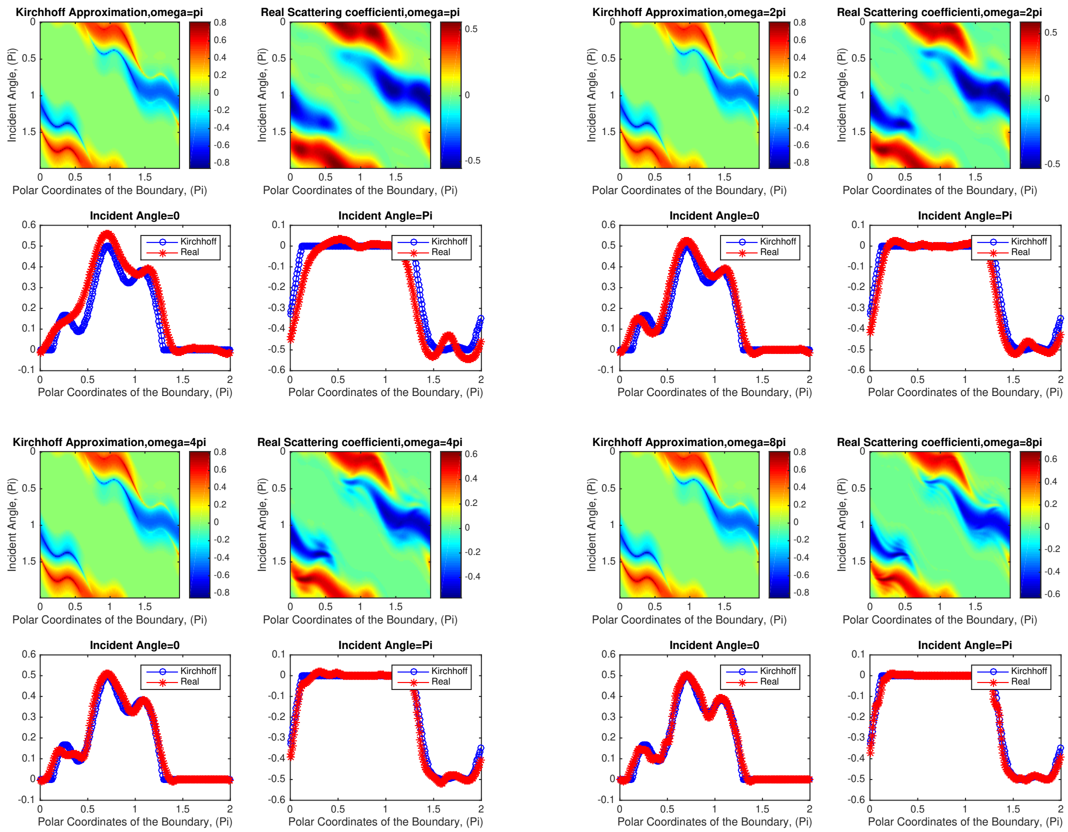

Figure 9. $\mathbf{R}_s^2$ and $\hat{\mathbf{R}}_s^2$ for the pear.