Absense of Positive Eigenvalues for the Linearized Elasticity System in the Half Space

Zhiming Chen, Shiqi Zhou

LSEC, Institute of Computational Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

Abstract. In this paper, we prove that the linearized elasticity system in the half-space with traction free boundry has no eigenvalues. ....

1. Introduction

section1

In this paper, we consider the linearized and isotropic elasticity system defined on an unbounded domain $\Omega = \mathbb{R}^2_+ \backslash \bar{D}$ with traction free surface $\Gamma_0 = \{(x_1, x_2)^T : x_1 \in \mathbb{R}, x_2 = 0\}$, where $D \subsetneq \mathbb{R}^2_+$ is a bounded Lipschitz domain with the unit outer normal ν to its boundary Γ_D. We study the eigenvalues of the following elastic scattering problem in the isotropic homogeneous medium half space with *Lamé* constant λ and μ and constant density $\rho \equiv 1$:

$$\nabla \cdot \sigma(\mathbf{u}) + \rho\omega^2 \mathbf{u} = f \qquad \text{in } \mathbb{R}^2_+ \backslash \bar{D} \tag{1.1}$$

elastic_eq

$$\mathbf{u} = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \sigma(\mathbf{u}) \cdot e_2 = 0 \qquad \text{on } \Gamma_0 \tag{1.2}$$

elastic_bd

together with the constitutive relation (Hookes law)

$$\sigma(\mathbf{u}) = 2\mu\varepsilon(\mathbf{u}) + \lambda \mathrm{div}\mathbf{u}\mathbb{I}$$

$$\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$$

where ω is the circular frequency, $\mathbf{u}(x_1, x_2) = (u_1(x), u_2(x))^T \in \mathbb{C}^2$ denotes the displacement fields and $\sigma(u)$ is the stress tensor. We also need to define the surface traction $T^n_x(\cdot)$ on the normal direction n,

$$T^n_x \mathbf{u}(x) := \sigma \cdot n = 2\mu \frac{\partial \mathbf{u}}{\partial n} + \lambda n \mathrm{div}\mathbf{u} + \mu n \times \mathrm{curl}\mathbf{u}$$

For simplicity, let's introduce *Lamé* operator Δ_e as

$$\Delta_e \mathbf{u} = (\lambda + 2\mu)\nabla\nabla \cdot \mathbf{u} - \mu\nabla \times \nabla \times u = \nabla \cdot \sigma(\mathbf{u})$$

We remark that the results in this paper can be extended to other boundary conditions such as Neumann or mixed boundary conditions on Γ_D, or even to penetrable obstacle.

In order to complete the definition of the problem, we introduce the domain of the operater Δ_e

$$\mathcal{D}(\Delta_e, S) = \{v \in H^1(S) : \Delta_e v \in L^2(S), \sigma(v)e_2 = 0 \quad \text{on } \Gamma_0\}$$

where S is an unbounded domain in $\mathbb{R}^2_+$. For the elasticity system, the study of eigenvalue is little sini2004 [2]. The layout of the paper is as follows. In section 2

2. Absence of Positive Eigenvalues

In this section. Throughout the paper, we will assume that for $z \in \mathbb{C}$, $z^{1/2}$ is the analytic branch of $\sqrt{z}$ such that $\mathrm{Im}\,(z^{1/2}) \geq 0$. This corresponds to the rigt half real axis as the branch cut in the complex plane. For $z = z_1 + \mathbf{i}z_2, z_1, z_2 \in \mathbb{R}$, we have

$$z^{1/2} = sgn(z_2)\sqrt{\frac{|z| + z_1}{2}} + \mathbf{i}\sqrt{\frac{|z| - z_1}{2}} \tag{2.1}$$

convention_1

For z on the right half real axis, we take $z^{1/2}$ as the limit of $(z + \mathbf{i}\varepsilon)^{1/2}$ as $\varepsilon \to 0^+$.

2.1 **Theorem 2.1** *Let $\omega \in \mathbb{R}$ and $\mathbf{u}$ statisfy the equations* elastic_eq elastic_bd *(1.1-1.2) in $\mathcal{D}(\Delta_e, \Omega)$, then we assert $\mathbf{u} = 0$.*

lem2.1 **Lemma 2.1** *Suppose that $f \in L^2(\mathbb{R}^2_+)$ with compact support in $B \subsetneq \mathbb{R}\times(h,+\infty)$, $h > 0$. Let $\omega \in \mathbb{R}$ and $\mathbf{u} \in \mathcal{D}(\Delta_e, \mathbb{R}^2_+)$ such that:*

$$\Delta_e \mathbf{u} + \omega^2 \mathbf{u} = f \tag{2.2}$$ eq2

then we assert $\mathbf{u} = 0$ in $(\mathbb{R}\times(h,+\infty))$.

Proof. Let $\mathcal{F}_{x_1}(\cdot) : L^2(R^2_+) \to L^2(R^2_+)$ be the partial Fourier tranforn given by $\hat{g} := \mathcal{F}_{x_1}(g) := \int_{\mathbb{R}} g(x_1,x_2)e^{\mathbf{i}x_1\xi}dx_1$. By taking the Fourier transform of (2.2) and (1.2), we obtain ODEs for x_2 in R_+

$$\mu\frac{d^2\hat{u}_1}{dx_2^2} + \mathbf{i}(\lambda+\mu)\xi\frac{d\hat{u}_2}{dx_2} + (\omega^2 - (\lambda+2\mu)\xi^2)\hat{u}_1 = \hat{f}_1 \tag{2.3}$$ pp3

$$(\lambda+2\mu)\frac{d^2\hat{u}_2}{dx_2^2} + \mathbf{i}(\lambda+\mu)\xi\frac{d\hat{u}_1}{dx_2} + (\omega^2 - \mu\xi^2)\hat{u}_2 = \hat{f}_2 \tag{2.4}$$ pp4

and the boundary coditions on $x_2 = 0$ are

$$\mu\frac{d\hat{u}_1}{dx_2} + \mathbf{i}\mu\xi\hat{u}_2 = 0 \tag{2.5}$$ pp5

$$(\lambda+2\mu)\frac{d\hat{u}_2}{dx_2} + \mathbf{i}\lambda\xi\hat{u}_1 = 0 \tag{2.6}$$ pp6

In order to work with real coefficient, we use the change of variables:

$$\begin{aligned} &v_1 = \mathbf{i}\hat{u}_1, \quad v_2 = \hat{u}_2, \quad \mathbf{v} = (v_1, v_2)^T \\ &q_1 = \mathbf{i}\hat{f}_1, \quad q_2 = \hat{f}_2, \quad \mathbf{q} = (q_1, q_2)^T \end{aligned}$$

Then we have the following equations:

$$[\mathbb{A}_1\frac{d^2}{dx_2^2} + (\mathbb{A}_2 - (\mathbb{A}_2)^T)\xi\frac{d}{dx_2} - \mathbb{A}_3\xi^2 + \omega^2]\mathbf{v} = \mathbf{q} \quad \text{in} \quad \mathbb{R}_+ \tag{2.7}$$ eq3

$$(\mathbb{A}_1\frac{d}{dx_2} + \mathbb{A}_2\xi) \quad \mathbf{v} = 0 \qquad \text{on} \quad x_2 = 0 \tag{2.8}$$

where

$$\mathbb{A}_1 = \begin{pmatrix} \mu & 0 \\ 0 & \lambda+2\mu \end{pmatrix}, \qquad \mathbb{A}_2 = \begin{pmatrix} 0 & -\mu \\ \lambda & 0 \end{pmatrix}, \qquad \mathbb{A}_3 = \begin{pmatrix} \lambda+2\mu & 0 \\ 0 & \mu \end{pmatrix}$$

Let $\mathbf{w}$ be the solution of the following equations:

$$\begin{aligned} &[\mathbb{A}_1\frac{d^2}{dx_2^2} + (\mathbb{A}_2 - (\mathbb{A}_2)^T)\xi\frac{d}{dx_2} - \mathbb{A}_3\xi^2 + \omega^2]\mathbf{w} = \mathbf{q} \qquad \text{in} \quad (0,h) \\ &\mathbf{w} = 0, \quad (\mathbb{A}_1\frac{d}{dx_2} + \mathbb{A}_2\xi) \quad \mathbf{w} = 0 \qquad \text{on} \quad x_2 = h \end{aligned}$$

It is easy to transform above equations into a simpler form by variables substitution $\mathbf{W} = (\mathbf{w}, (\mathbb{A}_1\frac{d}{dx_2} + \mathbb{A}_2\xi)\mathbf{w})^T$, $\mathbf{Q} = (0, 0, \mathbf{q})^T$

$$\begin{aligned} &\frac{d}{dx_2}\mathbf{W} = \mathbb{A}\mathbf{W} + \mathbf{Q} \qquad \text{in} \quad (0,h) \\ &\mathbf{W} = 0 \quad \text{on} \quad x_2 = h \end{aligned}$$

where

$$\mathbb{A} = \begin{pmatrix} -\mathbb{A}_1^{-1}\mathbb{A}_2\xi & \mathbb{A}_1^{-1} \\ -\mathbb{A}_2^T\mathbb{A}_1^{-1}\mathbb{A}_2\xi^2 + \mathbb{A}_3\xi^2 - \omega^2 & \mathbb{A}_2^T\mathbb{A}_1^{-1}\xi \end{pmatrix}$$

By the standard arguments in ODEs, we can obtain

$$\mathbf{W}(\xi, x_2) = -\Phi(\xi, x_2)\int_h^{x_2} \Phi^{-1}(\xi, t)\mathbf{Q}(\xi, t)dt$$

where

$$\Phi(\xi, t) = \begin{pmatrix} -\mu_s(\xi)e^{\mathbf{i}\mu_s t} & -\xi e^{\mathbf{i}\mu_p(\xi)t} & -\mu_s(\xi)e^{-\mathbf{i}\mu_s(\xi)t} & \xi e^{-\mathbf{i}\mu_p t} \\ -\mathbf{i}\xi e^{\mathbf{i}\mu_s(\xi)t} & \mathbf{i}\mu_p(\xi)e^{\mathbf{i}\mu_p(\xi)t} & \mathbf{i}\xi e^{-\mathbf{i}\mu_s(\xi)t} & \mathbf{i}\mu_p(\xi)e^{-\mathbf{i}\mu_p(\xi)t} \\ -\mathbf{i}\mu\beta(\xi)e^{\mathbf{i}\mu_s t} & -2\mathbf{i}\mu\xi\mu_p(\xi)e^{\mathbf{i}\mu_p(\xi)t} & \mathbf{i}\mu\beta(\xi)e^{-\mathbf{i}\mu_s(\xi)t} & -2\mathbf{i}\mu\xi\mu_p(\xi)e^{-\mathbf{i}\mu_p t} \\ 2\mu\xi\mu_s(\xi)e^{\mathbf{i}\mu_s t} & -\mu\beta(\xi)e^{\mathbf{i}\mu_p(\xi)t} & 2\mu\xi\mu_s(\xi)e^{-\mathbf{i}\mu_s t} & \mu\beta(\xi)e^{-\mathbf{i}\mu_p(\xi)t} \end{pmatrix}$$

Here $k_p = \omega/\sqrt{\lambda + 2\mu}, k_s = \omega/\sqrt{\mu}$ are wave number of p-wave and s-wave, and $\mu_\alpha = (k_\alpha^2 - \xi^2)^{1/2}$ for $\alpha = s, p$.

We extend $\mathbf{w}(\xi, x_2)$ by zero in (h, ∞). Therefore. $\mathbf{w}(\xi, x_2)$ satisfy equation eq3 2.7 in $\mathbb{R}_+$. Since $\Phi(\xi, t)$ are analytic w.r.t ξ in $\mathbb{R}\backslash\{k_p, k_s\}$ and f(x) have compact support, we deduce that for almost every $\xi \in \mathbb{R}$, $\mathbf{w}(\xi, x_2)$ are analytic and so $(\mathbb{A}_1\frac{d}{dx_2} + \mathbb{A}_2\xi)\mathbf{w}$ are.

We set $\mathbf{U} = \mathbf{v} - \mathbf{w}$ and $\mathbf{U} = (U_1, U_2)^T$. Then $\mathbf{U}$ satisfy the following Cauchy problem:

$$[\mathbb{A}_1\frac{d^2}{dx_2^2} + (\mathbb{A}_2 - (\mathbb{A}_2)^T)\xi\frac{d}{dx_2} - \mathbb{A}_3\xi^2 + \omega^2]\mathbf{U} = 0 \quad \text{in} \quad \mathbb{R}_+ \tag{2.9}$$ eq4

$$(\mathbb{A}_1\frac{d}{dx_2} + \mathbb{A}_2\xi)\mathbf{U} = (\mathbb{A}_1\frac{d}{dx_2} + \mathbb{A}_2\xi)\mathbf{w} \quad \text{on} \quad x_2 = 0 \tag{2.10}$$

Since the coefficients of above equations are constants, we can represent $\mathbf{U}(\xi, x_2)$ in the following form:

$$\mathbf{U}(\xi, x_2) = c_1(\xi)\begin{pmatrix} -\mu_s \\ -\mathbf{i}\xi \end{pmatrix}e^{\mathbf{i}\mu_s x_2} + c_2(\xi)\begin{pmatrix} -\xi \\ \mathbf{i}\mu_p \end{pmatrix}e^{\mathbf{i}\mu_p x_2} + c_3(\xi)\begin{pmatrix} -\mu_s \\ \mathbf{i}\xi \end{pmatrix}e^{-\mathbf{i}\mu_s x_2} + c_4(\xi)\begin{pmatrix} \xi \\ \mathbf{i}\mu_p \end{pmatrix}e^{-\mathbf{i}\mu_p x_2}$$

If $\xi^2 \le k_p^2$, then it's simple to see that $\mathbf{U} = 0$ in $L^2_{x_2}(\mathbb{R}_+)$. So, for $\xi^2 < k_p^2$, $(\mathbb{A}_1\frac{d}{dx_2} + \mathbb{A}_2\xi)\mathbf{U} = 0$ which implis $(\mathbb{A}_1\frac{d}{dx_2} + \mathbb{A}_2\xi)\mathbf{w} = 0$. Since $(\mathbb{A}_1\frac{d}{dx_2} + \mathbb{A}_2\xi)\mathbf{w}$ are analytic for almost every $\xi \in \mathbb{R}$, we deduce that

$$(\mathbb{A}_1\frac{d}{dx_2} + \mathbb{A}_2\xi)\mathbf{U} = 0 \quad \text{on} \quad x_2 = 0 \tag{2.11}$$ bd_1

for almost every $\xi \in \mathbb{R}$. Therefore, we can obtain

$$\mathbf{U}(\xi, x_2) = \begin{cases} c(\xi)\begin{pmatrix} -\xi \\ \mathbf{i}\mu_p \end{pmatrix}e^{\mathbf{i}\mu_p x_2}, & k_p^2 < \xi^2 \le k_s^2 \\ c_1(\xi)\begin{pmatrix} -\mu_s \\ -\mathbf{i}\xi \end{pmatrix}e^{\mathbf{i}\mu_s x_2} + c_2(\xi)\begin{pmatrix} -\xi \\ \mathbf{i}\mu_p \end{pmatrix}e^{\mathbf{i}\mu_p x_2}, & \xi^2 > k_s^2 \end{cases}$$

$$(\mathbb{A}_1\frac{d}{dx_2} + \mathbb{A}_2\xi)\mathbf{U} = \begin{cases} c(\xi)\begin{pmatrix} -2\mathbf{i}\mu\xi\mu_p \\ -\mu\beta \end{pmatrix}e^{\mathbf{i}\mu_p x_2}, & k_p^2 < \xi^2 \le k_s^2 \\ c_1(\xi)\begin{pmatrix} -\mathbf{i}\mu\beta \\ 2\mu\xi\mu_s \end{pmatrix}e^{\mathbf{i}\mu_s x_2} + c_2(\xi)\begin{pmatrix} -2\mathbf{i}\mu\xi\mu_p \\ -\mu\beta \end{pmatrix}e^{\mathbf{i}\mu_p x_2}, & \xi^2 > k_s^2 \end{cases}$$

By boudary condition 2.11, we have $c(\xi) = 0$ for $k_p^2 < \xi^2 \le k_s^2$ and

$$\mathbf{det}\begin{pmatrix} -\mathbf{i}\mu\beta & -2\mathbf{i}\mu\xi\mu_p \\ 2\mu\xi\mu_s & -\mu\beta \end{pmatrix} = -\mathbf{i}\mu(\beta^2 + 4\xi^2\mu_s\mu_p) = 0 \quad \text{for} \;\; \xi^2 > k_s^2 \tag{2.12}$$

Therefore, by the next lemma 2.2 we have $\mathbf{U}(\xi, x_2) = 0$ for almost every $\xi \in \mathbb{R}$ which implis $\mathbf{v}(\xi, x_2) = 0$ for almost every $\xi \in \mathbb{R}$ and $x_2 \in (h, +\infty)$. This completes the proof by taking the inverse Fourier tranformation of $\hat{\mathbf{u}}(\xi, x_2)$. □

lem2.2 **Lemma 2.2** *The Rayleigh equation $\beta(\xi)^2 + 4\xi^2\mu_s(\xi)\mu_p(\xi) = 0$ has only two zeros $\pm k_R$, $k_R > k_s$, in the complex plane.*

proof of Theorem 2.1: Since $D \subsetneq \mathbb{R}_+^2$,we can find two concentric circles B_{R_1}, B_{R_2} such that $D \subsetneq B_{R_1} \subsetneq B_{R_2} \subsetneq \mathbb{R}_+^2$. Let $\chi \in C_0^\infty(\mathbb{R}_+^2)$ be the cut-off function such that $0 \le \chi \le 1$, $\chi = 0$ in B_{R_1}, and $\chi = 1$ outside of B_{R_2}. Let $v = \chi u$. Then v satisfies (2.2) with $f = \sigma(u)\nabla\chi + (\lambda+\mu)(\nabla^2\chi u + \nabla u \nabla\chi) + \mu\Delta\chi u + \mu \mathrm{div} u \nabla\chi$, where $\nabla^2\chi$ is the Hessian matrix of χ. Clearly $q \in L^2(\mathbb{R}_+^2)$ has compact support. By lemme 2.2, we have $u = v = 0$ in $\mathbb{R} \times (h, +\infty)$. Finally, the unique continuation principle [1] implies that u=0 in $\mathbb{R}_+^2$. This completes the proof. □

2.2 **Theorem 2.2** *Let $\omega \in \mathbb{R}$ and $\mathbf{u}$ statisfy the equations (1.1 in $\mathcal{D}(\Delta_e, \Omega)$, if u satisfies the following boundary conditions*

$$\sigma(\mathbf{u})\nu + \mathbf{i}\eta(x)\mathbf{u} = 0$$

where $\eta(x)$ is a bounded function on Γ_D, then we assert $\mathbf{u} = 0$.

2.3 **Theorem 2.3** *Let $\omega \in \mathbb{R}$ and $\mathbf{u} \in \mathcal{D}(\Delta_e, \mathbb{R}_+^2)$, if u satisfies the following equations:*

$$\Delta_e \mathbf{u} + \omega^2(1 + n(x))\mathbf{u} = 0$$

where $n(x) \in L^\infty(D)$ is a positive scalar function supported in D, then we assert $\mathbf{u} = 0$.

References

2001STRONG [1] Antonino Morassi. Strong unique continuation for the lam system of elasticity. *Communications in Partial Differential Equations*, 26(9-10):1787–1810, 2001.

sini2004 [2] Mourad Sini. Absence of positive eigenvalues for the linearized elasticity system. *Integral Equations and Operator Theory*, 49(2):255–277, 2004.