Absense of Positive Eigenvalues for the Linearized Elasticity System in the Half Space

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Abstract. In this paper, we prove the linearized elasticity system in the half-space with traction free boundry has no eigenvalues. We consider the constant Lame confficients and the desity outside the obstacle with different physical properties such as penetrable or non-penetrable, and for non-penetrable obstacles, the type of boundary conditions on the boundary of the obstacle.

1. Introduction

section1

In this paper, we consider the linearized and isotropic elasticity system defined on an unbounded domain $\Omega = \mathbb{R}^2_+ \backslash \bar{D}$ with traction free surface, where $D \subsetneq \mathbb{R}^2_+$ is a bounded Lipschitz domain with the unit outer normal ν to its boundary Γ_D. We study the eigenvalues of the following elastic scattering problem in the isotropic homogeneous medium half space with *Lamé* constant λ and μ and constant density $\rho \equiv 1$:

$$\nabla \cdot \sigma(\mathbf{u}) + \rho\omega^2 \mathbf{u} = f \quad \text{in } \mathbb{R}^2_+ \backslash \bar{D} \tag{1.1}$$

elastic_eq

$$\mathbf{u} = 0 \ \text{ on } \Gamma_D \ \text{ and } \ \sigma(\mathbf{u}) \cdot e_2 = 0 \ \text{ on } \Gamma_0 \tag{1.2}$$

together with the constitutive relation (Hookes law)

$$\sigma(\mathbf{u}) = 2\mu\varepsilon(\mathbf{u}) + \lambda \mathrm{div}\mathbf{u}\mathbb{I}$$
$$\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$$

where ω is the circular frequency, $\mathbf{u}(x_1, x_2) = (u_1(x), u_2(x))^T \in \mathbb{C}^2$ denotes the displacement fields and $\sigma(u)$ is the stress tensor. We also need to define the surface traction $T^n_x(\cdot)$ on the normal direction n,

$$T^n_x \mathbf{u}(x) := \sigma \cdot n = 2\mu \frac{\partial \mathbf{u}}{\partial n} + \lambda n \mathrm{div}\mathbf{u} + \mu n \times \mathrm{curl}\mathbf{u}$$

For simplicity, let's introduce *Lamé* operator Δ_e as

$$\Delta_e \mathbf{u} = (\lambda + 2\mu)\nabla\nabla \cdot \mathbf{u} - \mu \nabla \times \nabla \times u = \nabla \cdot \sigma(\mathbf{u})$$

For the elasticity system, the study of eigenvalue is little. To our knowledges. The layout of the paper is as follows. In section 2

2. Absence of Positive Eigenvalues

Throughout the paper, we will assume that for $z \in \mathbb{C}$, $z^{1/2}$ is the analytic branch of $\sqrt{z}$ such that $\mathrm{Im}\,(z^{1/2}) \geq 0$. This corresponds to the rigt half real axis as the branch cut in the complex plane. For $z = z_1 + \mathbf{i}z_2, z_1, z_2 \in \mathbb{R}$, we have

$$z^{1/2} = sgn(z_2)\sqrt{\frac{|z| + z_1}{2}} + \mathbf{i}\sqrt{\frac{|z| - z_1}{2}} \tag{2.1}$$

convention_1

For z on the right half real axis, we take $z^{1/2}$ as the limit of $(z + \mathbf{i}\varepsilon)^{1/2}$ as $\varepsilon \to 0^+$.

By taking the Fourier transform of (pp1 ??-pp2 ??), we obtain ODEs for x_2 in R_+

$$\mu \frac{d^2 \hat{u}_1}{dx_2^2} + \mathbf{i}(\lambda + \mu)\xi \frac{d\hat{u}_2}{dx_2} + (\omega^2 - (\lambda + 2\mu)\xi^2)\hat{u}_1 = 0 \tag{2.2}$$

pp3

$$(\lambda + 2\mu)\frac{d^2 \hat{u}_2}{dx_2^2} + \mathbf{i}(\lambda + \mu)\xi \frac{d\hat{u}_1}{dx_2} + (\omega^2 - \mu\xi^2)\hat{u}_2 = 0 \tag{2.3}$$

pp4

and the boundary coditions on $x_2 = 0$ are

$$\mu \frac{d\hat{u}_1}{dx_2} + \mathbf{i}\mu\xi\hat{u}_2 = 0 \tag{2.4}$$

pp5

$$(\lambda + 2\mu)\frac{d\hat{u}_2}{dx_2} + \mathbf{i}\lambda\xi\hat{u}_1 = 0 \tag{2.5}$$

pp6

In order to work with real coefficient, we use the change of variables:

$$v_1 = \mathbf{i}u_1, \qquad v_2 = u_2, \qquad \mathbf{v} = (v_2, v_2)^T$$

Then we have the following equations:

$$\begin{aligned}
&[\mathbb{A}_1 \frac{d^2}{dx_2^2} + (\mathbb{A}_2 - (\mathbb{A}_2)^T)\xi \frac{d}{dx_2} - \mathbb{A}_3 \xi^2 + \omega^2]\mathbf{v} = \qquad \text{in } \mathbb{R}_+ \\
&(\mathbb{A}_1 \frac{d}{dx_2} + \mathbb{A}_2)\mathbf{v} = 0 \quad \text{on } x_2 = 0
\end{aligned}$$

where

$$\mathbb{A}_1 = \begin{pmatrix} \mu & 0 \\ 0 & \lambda + 2\mu \end{pmatrix}, \qquad \mathbb{A}_2 = \begin{pmatrix} 0 & -\mu \\ \lambda & 0 \end{pmatrix}, \qquad \mathbb{A}_3 = \begin{pmatrix} \lambda + 2\mu & 0 \\ 0 & \mu \end{pmatrix}$$

References

enbach1980 [1] Jan Achenbach. *Wave Propagation in Elastic Solids.* North-Holland, 1980.

979Complex [2] Lars V Ahlfors. *Complex Analysis: An introduction to the theory of analytic functions of one complex variable.* McGraw-Hill, 1979.

la_reverse [3] Wen Fong Chang. Elastic reverse-time migration. *Geophysical Prospecting*, 37(3):243–256, 1987.

Yves1988 [4] Yves Dermenjian and Jean Claude Guillot. Scattering of elastic waves in a perturbed isotropic half space with a free boundary. the limiting absorption principle. *Mathematical Methods in the Applied Sciences*, 10(2):87C124, 1988.

grafakos [5] Loukas Grafakos. *Classical and modern Fourier analysis.* Prentice Hall, 2004.

2001Linear [6] Johng. Harris. *Linear elastic waves.* Cambridge University Press, 2001.

63progress [7] Viktor D Kupradze. *Progress in solid mechanics. 3. Dynamical problems in elasticity.* North-Holland Publishing Company, 1963.

leis [8] Rolf Leis. *Initial Boundary Value Problems in Mathematical Physics.* J. Wiley, 1986.

Guzina2006 [9] Andrew I. Madyarov and Bojan B. Guzina. A radiation condition for layered elastic media. *Journal of Elasticity*, 82(1):73–98, 2006.

sini2004 [10] Mourad Sini. Absence of positive eigenvalues for the linearized elasticity system. *Integral Equations and Operator Theory*, 49(2):255–277, 2004.

wilcox1975 [11] Calvin H. Wilcox. *Scattering Theory for the d'Alembert Equation in Exterior Domains.* PhD thesis, Springer Berlin Heidelberg, 1975.

Zhang08 [12] Yu. Zhang and James. Sun. Practical issues of reverse time migration: true amplitude gathers, noise removal and harmonic-source encoding. *Aseg Extended Abstracts*, 2009(3):397–398, 2008.

Zhang2007 [13] Yu Zhang, Sheng Xu, Norman Bleistein, and Guanquan Zhang. True-amplitude, angle-domain, common-image gathers from one-way wave-equation migrations. *Geophysics*, 72(1):S49–S58, 2007.