Reverse Time Migration for Extended Obstacles in the Half Space: Elastic Waves

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Abstract. We consider a reverse time migration method for reconstructing extended obstacles in the half space with finite aperture data using elastic waves at a fixed frequency. We prove the resolution of the reconstruction method in terms of the aperture and the depth of the obstacle embedded in the half space. The resolution analysis studied by virtue of point spread function implies that the imaginary part of the cross-correlation imaging function always peaks on the boundary of the obstacle. Numerical experiments are included to illustrate the powerful imaging quality and to confirm our resolution results.

1. Introduction

In this paper we study a reverse time migration (RTM) algorithm to find the support of an unknown obstacle in the half space from the measurement of scattered waves on the boundary of the half space which is far away from the obstacle. The physical properties of the obstacle such as penetrable or non-penetrable, and for non-penetrable obstacles, the type of boundary conditions on the boundary of the obstacle, are not required in the algorithm.

Let the non-penetrable obstacle occupy a bounded Lipschitz domain $D \subset \mathbb{R}^2_+$ with ν the unit outer normal to its boundary Γ_D. We assume the incident wave is emitted by a point source located at x_s, explosive along the polarization direction $q \in \mathbb{R}^2$, on the surface $\Gamma_0 = \{(x_1, x_2)^T : x_1 \in \mathbb{R}, x_2 = 0\}$ which is far away from the obstacle. The measured data u_q corresponding to the polarization direction q is the solution of the following elastic scattering problem in the isotropic homogeneous medium half space with *Lamé* constant λ and μ and constant density $\rho \equiv 1$:

$$\nabla \cdot \sigma(u_q) + \rho\omega^2 u_q = -\delta_{x_s}(x) q \quad \text{in } \mathbb{R}^2_+ \backslash \bar{D} \tag{1.1}$$

$$u_q = 0 \;\text{ on } \Gamma_D \;\text{ and }\; \sigma(u_q)\cdot e_2 = 0 \;\text{ on } \Gamma_0 \tag{1.2}$$

together with the constitutive relation (Hookes law)

$$\sigma(u) = 2\mu\varepsilon(u) + \lambda \mathrm{div} u \mathbb{I}$$
$$\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$$

where ω is the circular frequency, $u(x) \in \mathbb{C}^2$ denotes the displacement fields and $\sigma(u)$ is the stress tensor. We also need to define the surface traction $T^n_x(\cdot)$ on the normal direction n,

$$T^n_x u(x) := \sigma \cdot n = 2\mu \frac{\partial u}{\partial n} + \lambda n \mathrm{div} u + \mu n \times \mathrm{curl} u$$

For simplicity, let's introduce *Lamé* operator Δ_e as

$$\Delta_e u = (\lambda + 2\mu)\nabla\nabla \cdot u - \mu \nabla \times \nabla \times u = \nabla \cdot \sigma(u)$$

The equation (1.1) is understood as the limit when $x_s \in \mathbb{R}^2_+ \backslash \bar{D}$ tends to Γ_0 whose precise meaning will be given below after we introduce the Neumann Green Tensor and the definition of the radiation condition.

The reverse time migration (RTM) method, which consists of back-propagating the complex conjugated data into the background medium and computing the crosscorrelation between the incident wave field and the backpropagated field to output the final imaging profile, is nowadays widely used in exploration geophysics [5, 6, 7, 9, 16]. In [10, 11, 12], the RTM method for reconstructing extended targets using acoustic, electromagnetic and elastic waves at a fixed frequency in the free space is proposed and studied. The resolution analysis in [10, 11, 12] is achieved without using the small inclusion or geometrical optics assumption previously made in the literature (e.g. [3, 7]). In [13], a new RTM algorithm is developed for finding extended targets in a planar waveguide which is motivated by the generalized Helmholtz-Kirchhoff identity for scattering problems in waveguides.

For the isotropic elastic media, one can process the elastic data either by separating P-wave and S-wave using Helmholtz decomposition and migrating each mode using methods based on acoustic wave theory chung2012implementation,denli2008elastic [15, 18], or by migrating the whole elastic data set based on full elastic wave equation in the geophysical exploration community. In this paper, we adopt the cross-correlation between all the component of the source and receiver displacement wavefield, which is a mixture of P-wave and S-wave. Furthermore this kind condition can be easily extended to inhomogeneous elastic medium and even anisotropic elastic wave imaging. The purpose of this paper is to provide a new mathematical understanding of the RTM method by extending RTMhalf_aco [14] where RTM method for extended targets in the half space using acoustic wave is considered. Compared to the scalar acoustic wave imaging, the vector elastic wave imaging is more complex due to a mixture of P-wave and S-wave mode. However, the virtue of the latter method is no longer need to separate the scalar and vector potentials prior to the imaging condition.

The layout of the paper is as follows. To Be Decided...

2. Green Tensor in the half space

In this section we will study the elastic Green Tensor in the half-space with Neumann boundary nedelec2011 [20]:

$$\Delta_e \mathbb{N}(x;y) + \omega^2 \mathbb{N}(x,y) = -\delta_y(x)\mathbb{I} \quad \text{in} \quad \mathbb{R}^2_+, \tag{2.1}$$ eq_n1

$$\sigma_x(\mathbb{N}(x,y))e_2 = 0 \qquad \text{on} \quad x_2 = 0 \tag{2.2}$$ eq_n2

and with Dirichlet Boundary arens1999 [4]

$$\Delta_e \mathbb{D}(x,y) + \omega^2 \mathbb{D}(x,y) = -\delta_y(x)\mathbb{I} \quad \text{in} \quad \mathbb{R}^2_+, \tag{2.3}$$ eq_d1

$$\mathbb{D}(x,y) = 0 \qquad \text{on} \quad x_2 = 0 \tag{2.4}$$ eq_d2

where $\delta_y(x)$ is the Dirac source at $y \in R^2_+$ and $N(x,y)$, $\mathbb{D}(x,y)$ are $\mathbb{C}^{2\times 2}$ matrixes. We will first use Fourier transform to derive the formula of Green Tensor in frequency domain. Let

$$\hat{\mathbb{N}}(\xi, x_2; y_2) = \int_{-\infty}^{+\infty} \mathbb{N}(x_1, x_2; y) e^{-\mathbf{i}(x_1-y_1)\xi} dx_1 \tag{2.5}$$

Throughout the paper, we will assume that for $z \in \mathbb{C}$, $z^{1/2}$ is the analytic branch of $\sqrt{z}$ such that $\mathrm{Im}\,(z^{1/2}) \geq 0$. This corresponds to the rigt half real axis as the branch cut in the complex plane. For $z = z_1 + \mathbf{i}z_2, z_1, z_2 \in \mathbb{R}$, we have

$$z^{1/2} = sgn(z_2)\sqrt{\frac{|z| + z_1}{2}} + \mathbf{i}\sqrt{\frac{|z| - z_1}{2}} \tag{2.6}$$ convention_1

For z on the right half real axis, we take $z^{1/2}$ as the limit of $(z + \mathbf{i}\varepsilon)^{1/2}$ as $\varepsilon \to 0^+$.

Let $\mathbb{G}(x,y)$ be the fundamental solution of the elastic equation kupradze1963progress [23] and recall that

$$\hat{\mathbb{G}}(\xi, x_2; y_2) = \frac{\mathbf{i}}{2\omega^2}\left[\begin{pmatrix} \mu_s & -\xi\frac{x_2-y_2}{|x_2-y_2|} \\ -\xi\frac{x_2-y_2}{|x_2-y_2|} & \frac{\xi^2}{\mu_s} \end{pmatrix} e^{\mathbf{i}\mu_s|x_2-y_2|} + \begin{pmatrix} \frac{\xi^2}{\mu_p} & \xi\frac{x_2-y_2}{|x_2-y_2|} \\ \xi\frac{x_2-y_2}{|x_2-y_2|} & \mu_p \end{pmatrix} e^{\mathbf{i}\mu_p|x_2-y_2|}\right]$$

where

$$\mu_\alpha = (k_\alpha^2 - \xi^2)^{1/2} \qquad \text{for} \quad \alpha = s, p \tag{2.7}$$

By the standard arguement in ODEs, the Green Tensor in half-space can be deduced as

$$\hat{\mathbb{N}}(\xi, x_2; y_2) = \hat{\mathbb{G}}(\xi, x_2; y_2) - \hat{\mathbb{G}}(\xi, x_2; -y_2) + \hat{\mathbb{N}}_c(\xi, x_2; y_2) \tag{2.8}$$

$$\hat{\mathbb{N}}_c(\xi, x_2; y_2) = \; = \frac{\mathbf{i}}{\omega^2 \delta(\xi)} \Big\{ A(\xi) e^{\mathbf{i}\mu_s(x_2+y_2)} + B(\xi) e^{\mathbf{i}\mu_p(x_2+y_2)} + C(\xi) e^{\mathbf{i}\mu_s x_2 + \mu_p y_2} + D(\xi) e^{\mathbf{i}\mu_p x_2 + \mu_s y_2} \Big\} \tag{2.9}$$

where

$$A(\xi) = \begin{pmatrix} \mu_s \beta^2 & -4\xi^3 \mu_s \mu_p \\ -\xi\beta^2 & 4\xi_4 \mu_p \end{pmatrix} \qquad B(\xi) = \begin{pmatrix} 4\xi^4 \mu_s & \xi\beta^2 \\ 4\xi^3 \mu_s \mu_p & \mu_p \beta^2 \end{pmatrix}$$

$$C(\xi) = \begin{pmatrix} 2\xi^2 \mu_s \beta & -2\xi \mu_s \mu_p \beta \\ -2\xi^3 \beta & 2\xi^2 \mu_p \beta \end{pmatrix} \quad D(\xi) = \begin{pmatrix} 2\xi^2 \mu_s \beta & 2\xi^3 \beta \\ 2\xi \mu_s \mu_p \beta & 2\xi^2 \mu_p \beta \end{pmatrix}$$

and $\beta(\xi) = k_s^2 - 2\xi^2$, $\delta(\xi) = \beta^2 + 4\xi^2 \mu_s \mu_p$.

The desired Green function should be obtained by taking the inverse Fourier transform of $\hat{N}(\xi, x_2; y_2)$. Unfortunately, one cannot simply take the inverse Fourier transform in the above formula because $\delta(\xi)$ have zero points in the real axis by lemma 2.1 [1][22].

root_De1 **Lemma 2.1** *Let* Lamé *constant $\lambda, \mu \in \mathbb{R}^+$, then the Rayleigh equation $\delta(\xi) = 0$ has only two roots denoted by $\pm k_R$ in complex plane. Morever, $k_R > k_s > k_p$, $k_R \in \mathbb{R}$ and k_R is called Rayleigh wave number.*

Proof. For the sake of completeness, we include a proof here. It is well known that

$$\delta(\xi) = (k_s^2 - 2\xi^2)^2 + 4\xi^2 (k_s^2 - \xi^2)^{1/2} (k_p^2 - \xi^2)^{1/2} \tag{2.10}$$

However, $\delta(\xi)$ is rendered single-valued by selecting branch cuts along $k_p < \mathrm{Re}\,(\xi) < k_s, \mathrm{Im}\,(\xi) = 0$ which is consistent with the convention (q1 **??**). A simple computation show that $\delta(\pm k_s) > 0$ and $\delta(\pm\infty + 0\mathbf{i}) < 0$. By the continuity of $\delta(\xi)$, we can obtain that it has at least two real zero points which denoted by $\pm k_R$.

Now it turn to proof that $\delta(\xi)$ has only two roots in the complex plane by the principle of arguement which follows as a theorem of the theory of complex variables Ahlfors1979Complex [2]. Now consider the contour C consisting of Γ, and C_l and C_r where $C_r = [k_p + \mathbf{i}0^+, k_s + \mathbf{i}0^+] \cup [k_p + \mathbf{i}0^-, k_s + \mathbf{i}0^-]$ that surround $[k_p, k_s]$, $C_l = [-k_s + \mathbf{i}0^+, -k_p + \mathbf{i}0^+] \cup [-k_s + \mathbf{i}0^-, -k_p + \mathbf{i}0^-]$ that surround $[-k_s, -k_p]$ and Γ denotes a circle with enough large radius. Since the function $\delta(\xi)$ clears does not have poles in the complex ξ-plane and we find that within the contour $C = \Gamma \cup C_r \cup C_l$ the number of zeros is given by

$$Z = \frac{1}{2\pi\mathbf{i}} \int_C \frac{d\delta}{d\xi} \frac{d\xi}{\delta(\xi)} \tag{2.11}$$ zero

Since $\delta(\xi) = \delta(-\xi)$ the images of C_r and C_l are the same, and one of them, say C_r, needs to be considered. We have $\delta(k_p) = (k_s^2 - 2k_p^2)^2$ and along C_r: $\delta^{\pm}(\xi) = (k_s^2 - \xi^2)^2 \mp \mathbf{i}4\xi^2\sqrt{k_s^2 - \xi^2}\sqrt{\xi^2 - k_p^2}$, and $\delta(k_s) = k_s^4$ where the plus sign applies above the cut, and the minus sign applies below the cut for $\delta(\xi)$. Let $f_1(\xi) = (k_s^2 - \xi^2)^2$ and $f_2(\xi) = 4\xi^2\sqrt{k_s^2 - \xi^2}\sqrt{\xi^2 - k_p^2}$. Then we have

$$\int_{C_r} \frac{d\delta}{d\xi}\frac{d\xi}{\delta(\xi)} = \int_{k_p}^{k_s} \frac{\delta'_+(\xi)}{\delta_+(\xi)} - \frac{\delta'_-(\xi)}{\delta_-(\xi)} d\xi \tag{2.12}$$

$$= 2\mathbf{i}\int_{k_p}^{k_s} \operatorname{Im} \frac{(f_1'(\xi) - \mathbf{i}f_2'(\xi))f_1(\xi) + \mathbf{i}f_2(\xi))}{(f_1(\xi) - \mathbf{i}f_2(\xi))(f_1(\xi) + \mathbf{i}f_2(\xi))} d\xi \tag{2.13}$$

$$= 2\mathbf{i}\int_{k_p}^{k_s} \frac{f_1'(\xi)f_2(\xi) - f_1(\xi)f_2'(\xi)}{f_1^2(\xi) + f_2^2(\xi)} d\xi \tag{2.14}$$

$$= -2\mathbf{i}\arctan \frac{f_2(\xi)}{f_1(\xi)}\bigg|_{k_p}^{k_s} = 0 \tag{2.15}$$

For $|\xi|$ large, we find $\delta(\xi) = A\xi^2 + O(1)$, thus it is easy to see that

$$\int_{\Gamma} \frac{d\delta}{d\xi}\frac{d\xi}{\delta(\xi)} = 4\pi$$

Then we obtain $Z = 2$. This completes the proof. □

In order to overcome the ambiguity above, loss is assumed in the medium so that $k_{\alpha,\varepsilon} := k_\alpha(1+\mathbf{i}\varepsilon)$. When $\varepsilon > 0$, the branch point of $\mu_{\alpha,\varepsilon}$ are $\pm k_{\alpha,\varepsilon}$ and the branch cut are denoted by the equation $\xi_1\xi_2 = k_\alpha\varepsilon, -k_\alpha \le \xi \le k_\alpha$. In this case, the poles singularities are now located off the real axis and the Fouerier inverse transform becomes meaningful. In order to express lemma root_De2 2.2 concisely, we define

$$\Omega := \{\xi \in \mathbb{C} \mid k_p\varepsilon < \xi_1\xi_2 < k_s\varepsilon\ ,\quad \xi_2 > \xi_1\varepsilon\} \tag{2.16}$$

root_De2 **Lemma 2.2** *If the elastic medium has loss that $k_{\alpha,\varepsilon} := k_\alpha(1+\mathbf{i}\varepsilon), 0 < \varepsilon < 1$ for $\alpha = p, s$, we assert that $\delta_\varepsilon(\xi) = 0$ has only two roots in domain $\Omega^c \subset \mathbb{C}$ and exactly they are $\pm k_{R,\varepsilon}$.*

Denote $B_\sigma^{+(-)}(a)$ the upper(lower) half part of the circle centered at a of the radius σ and let path $L_\sigma = (-\infty, -k_R-\sigma) \cup B_\sigma^-(-k_R) \cup (-k_R+\sigma, k_R-\sigma) \cup B_\sigma^+(k_R) \cup (k_R+\sigma, \infty)$. Using Cauchy integral theorem and lemma root_De2 2.2, we carry out:

$$\mathbb{N}(x,y) = \lim_{\varepsilon\to 0^+} \mathbb{N}_\varepsilon(x,y) = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{\mathbb{N}}_\varepsilon(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi} d\xi \tag{2.17}$$

$$= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{L_\sigma} \hat{\mathbb{N}}_\varepsilon(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi} d\xi = \frac{1}{2\pi}\int_{L_\sigma} \hat{\mathbb{N}}(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi} d\xi \tag{2.18}$$

$$= \lim_{\sigma\to 0^+} \frac{1}{2\pi}\int_{L_\sigma/B_\sigma^-(-k_R)\cup B_\sigma^+(k_R)} + \int_{B_\sigma^-(-k_R)\cup B_\sigma^+(k_R)} \hat{\mathbb{N}}_\varepsilon(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi} d\xi \tag{2.19}$$

$$= \frac{1}{2\pi} PV\int_{\mathbb{R}} \hat{\mathbb{N}}(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi} d\xi - \frac{\mathbf{i}}{2}\frac{\mathbb{N}_\delta(-k_R)}{\delta'(-k_R)}e^{-\mathbf{i}(x_1-y_1)k_R} + \frac{\mathbf{i}}{2}\frac{\mathbb{N}_\delta(k_R)}{\delta'(k_R)}e^{\mathbf{i}(x_1-y_1)k_R} \tag{2.20}$$ Ngreen

where $\mathbb{N}_\delta(\xi) = \hat{\mathbb{N}}(\xi, x_2; y_2)\delta(\xi)$.

Specially, $N(x,y)$ has a simple form when $x_2=0$ following a simple calculation:

$$\hat{\mathbb{N}}(\xi,0;y_2) = \frac{\mathbf{i}}{\mu\delta(\xi)}\left[\begin{pmatrix} 2\xi^2\mu_s & -2\xi\mu_s\mu_p \\ -\xi\beta & \mu_p\beta \end{pmatrix} e^{\mathbf{i}\mu_p y_2} + \begin{pmatrix} \mu_s\beta & \xi\beta \\ 2\xi\mu_s\mu_p & 2\xi^2\mu_p \end{pmatrix} e^{\mathbf{i}\mu_s y_2}\right] \quad (2.21)$$

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$$:= \frac{1}{\delta(\xi)}(\mathcal{N}_p(\xi)e^{\mathbf{i}\mu_p y_2} + \mathcal{N}_s(\xi)e^{\mathbf{i}\mu_s y_2}) \quad (2.22)$$

and let $N_f(x_1;y_1,y_2)$ denote the first part of N and $N_r(x_1;y_1,y_2)$ denote the second part of $\mathbb{N}$ in (2.20).

It remains to study Dirichlet Green Tensor $\mathbb{D}(x,y)$. We still use Fourier transform to derive the formula of Green Tensor in frequency domain. Then we can obtain $\mathbb{D}(x,y)$ similar to $\mathbb{N}(x,y)$. It follows an alternative representation for $\mathbb{D}(x,y)$

$$\hat{\mathbb{D}}(\xi,x_2;y_2) = \hat{\mathbb{G}}(\xi,x_2;y_2) - \hat{\mathbb{G}}(\xi,x_2;-y_2) + \hat{M}(\xi,x_2;y_2) \quad (2.23)$$

$$\hat{M}(\xi,x_2;y_2) = \frac{\mathbf{i}}{\omega^2\gamma(\xi)}\Big\{A(\xi)e^{\mathbf{i}\mu_s(x_2+y_2)} + B(\xi)e^{\mathbf{i}\mu_p(x_2+y_2)} - A(\xi)e^{\mathbf{i}\mu_s x_2+\mu_p y_2} - B(\xi)e^{\mathbf{i}\mu_p x_2+\mu_s y_2}\Big\} \quad (2.24)$$

where

$$A(\xi) = \begin{pmatrix} \xi^2\mu_s & -\xi\mu_s\mu_p \\ -\xi^3 & \xi^2\mu_p \end{pmatrix} \qquad B(\xi) = \begin{pmatrix} \xi^2\mu_s & \xi^3 \\ \xi\mu_s\mu_p & \xi^2\mu_p \end{pmatrix}$$

and $\gamma(\xi) = \xi^2 + \mu_s\mu_p$.

root_Ga **Lemma 2.3** *Let* Lamé *constant* $\lambda,\mu\in\mathbb{C}$ *and* $\mathrm{Im}\,(k_s)\geq 0, \mathrm{Im}\,(k_p)\geq 0$, *then equation* $\gamma(\xi)=0$ *has no root in complex plane.*

Proof. Let $F(\xi) = \gamma(\xi) * (\xi^2 - \mu_s\mu_p)$ and it is easy to see that the root of $\gamma(\xi)=0$ is also of $F(\xi)=0$. A simple computation show that $F(\xi) = (k_s^2+k_p^2)\xi^2 - k_p^2k_s^2$. However, only when $\xi^2 = k_p^2k_s^2/(k_s^2+k_p^2)$, $F(\xi)=0$ but $\gamma(\xi) = 2k_p^2k_s^2/(k_s^2+k_p^2)$. This completes the proof. □

Thus, we get the representation of Green Tensor by inverse Fourier transform

$$\mathbb{D}(x,y) = \mathbb{G}(x,y) - \mathbb{G}(x,y') + \frac{1}{2\pi}\int_{\mathbb{R}} \hat{M}(\xi,x_2;y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \quad (2.25)$$

Let $\mathbb{T}_D(x,y)$ denote the traction of $\mathbb{D}(x,y)$ in direction e_2 with respect to x such that $\mathbb{T}_D(x,y)e_i = T_x^{e_2}(\mathbb{D}(x,y))e_i = T_x^{e_2}(\mathbb{D}(x,y)e_i)$. Then we can get the representation of $T_D(x,y)$ by a trivial calculation.

$$\mathbb{T}_D(x,y) = \mathbb{T}(x,y) - \mathbb{T}(x,y') + \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\mathbb{T}}_M(\xi,x_2;y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \quad (2.26)$$

and

$$\hat{\mathbb{T}}_M(\xi,x_2;y_2) = \frac{\mu}{\omega^2\gamma(\xi)}\Big\{E(\xi)e^{\mathbf{i}\mu_s(x_2+y_2)} + F(\xi)e^{\mathbf{i}\mu_p(x_2+y_2)} - E(\xi)e^{\mathbf{i}\mu_s x_2+\mu_p y_2} - F(\xi)e^{\mathbf{i}\mu_p x_2+\mu_s y_2}\Big\} \quad (2.27)$$

where

$$E(\xi) = \begin{pmatrix} -\xi^2\beta & \xi\mu_p\beta \\ 2\xi^3\mu_s & -2\xi^2\mu_s\mu_p \end{pmatrix} \qquad F(\xi) = \begin{pmatrix} -2\xi^2\mu_s\mu_p & -2\xi^3\mu_p \\ -\xi\mu_s\beta & -\xi^2\beta \end{pmatrix}$$

Specially, $T_D(x, y)$ has a simple form when $x_2 = 0$:

$$\mathbb{T}_D(x_1, 0; y_1, y_2) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\mathbb{T}}_D(\xi, 0; y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \tag{2.28}$$

where

$$\begin{aligned} \hat{\mathbb{T}}_D(\xi, 0; y_2) = & \frac{1}{\gamma(\xi)}\left[\begin{pmatrix} \mu_s\mu_p & \xi\mu_p \\ \xi\mu_s & \xi^2 \end{pmatrix}e^{\mathbf{i}\mu_s y_2} + \begin{pmatrix} \xi^2 & -\xi\mu_p \\ -\xi\mu_s & \mu_p\mu_s \end{pmatrix}e^{\mathbf{i}\mu_p y_2}\right] && (2.29) \\ := & \mathcal{T}_p(\xi)e^{\mathbf{i}\mu_p y_2} + \mathcal{T}_s(\xi)e^{\mathbf{i}\mu_s y_2} && (2.30) \end{aligned}$$

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We should give asymptotic anslysis for $N(x_1, 0, y)$ and $T_D(x_1, 0; y)$ which will be in demand in the subsequent discussions. We need the following slight generalization of Van der Corput lemma for the oscillatory integral grafakos [21, P.152].

van **Lemma 2.4** *Let $-\infty < a < b < \infty$, and u is a C^k function u in (a, b).*
1. If $|u'(t)| \ge 1$ for $t \in (a, b)$ and u' is monotone in (a,b), then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \le 3\lambda^{-1}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

2. For all $k \ge 2$, if $|u^{(k)}(t)| \ge 1$ for $t \in (a, b)$, then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \le 12k\lambda^{-1/k}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

Proof. The assertion can be proved by extending the Van der Corptut lemma in grafakos [21]. Here we omit the details. □

gular_term **Lemma 2.5** *Assume that $0 < \kappa < 1$ and $f(t) \in C([-\frac{\pi}{2}, \frac{\pi}{2}]) \cap W^{1,1}([-\frac{\pi}{2}, \frac{\pi}{2}])$. Then for any $\rho > 1$ and $\phi \ge \phi^* > \arcsin\kappa := \phi_\kappa$, we have*

$$\Big|\int_{-\pi/2}^{\pi/2} f(t)(\kappa^2 - \sin^2(t+\phi))^{-1/2}e^{\mathbf{i}\rho\cos t}dt\Big| \le C\frac{1}{\rho^{1/2}}(|f(0)| + \int_{-\pi/2}^{\pi/2}|f'(t)|dt) \tag{2.31}$$

es_singular

where C only depend on ϕ^* and κ.

Proof. Solving the following equation:

$$\kappa^2 - \sin^2(t+\phi) = 0$$

we have, $t_1 = \phi_\kappa - \phi$ and $t_2 = -\phi_\kappa - \phi$ or $t_2 = \pi - \phi_\kappa - \phi$ depending on whether $\phi + \phi_\kappa \le \pi/2$ or $\phi + \phi_\kappa > \pi/2$. Without loss of generality, we assume the later case and thus $t_2 = \pi - \phi_\kappa - \phi$. Let $d = \min\{\frac{\phi^* - \phi_\kappa}{2}, \frac{\phi_\kappa}{2}\}$, then we can divide $[-\pi/2, \pi/2]$ into four intervals:

$$I_1 = [-d, d], I_2 = [t_1 - d, t_2 + d], I_3 = [t_2 - d, t_3 + d], I_4 = [-\frac{\pi}{2}, \frac{\pi}{2}]/I_1 \cup I_2 \cup I_3$$

We remark that if $t_2+d>\pi/2$, then let $I_3=[t_2-d,\pi/2]\cup[-\pi/2,t_2+d-\pi]$. A simple computation show that

$$\sin(t+\phi)>\sin(\phi^*-d)>\kappa \qquad t\in I_1 \tag{2.32}$$ es_I1

$$\sin(\phi_\kappa-d)\le|\sin(t+\phi)|\le\sin(\phi_\kappa+d), \qquad t\in I_2\cup I_3 \tag{2.33}$$ es_I2

$$|\sin(t+\phi)|\le\sin(\phi_\kappa-d) \text{ or } |\sin(t+\phi)|\ge\sin(\phi_\kappa+d), \qquad t\in I_4 \tag{2.34}$$ es_I_4

By lemma 2.4 and inequelity (2.32), we can obtain:

$$\begin{aligned}
&\Big|\int_{I_1} f(t)(\kappa^2-\sin^2(t+\phi))^{-1/2}e^{\mathbf{i}\rho\cos t}dt\Big| \\
&\le \frac{C}{\cos d}\rho^{-1/2}\Big(|f(d)||\kappa^2-\sin^2(d+\phi)|^{-1/2}+\int_{I_1}|f'(t)||\kappa^2-\sin^2(t+\phi)|^{-1/2}dt \\
&\int_{I_1}|f(t)||\kappa^2-\sin^2(t+\phi)|^{-3/2}dt\Big) \\
&\le C(\phi^*,\kappa)\rho^{-1/2}(|f(0)|+\int_{I_1}|f'(t)|dt)
\end{aligned}$$

and similarly we have

$$\begin{aligned}
&\Big|\int_{I_4} f(t)(\kappa^2-\sin^2(t+\phi))^{-1/2}e^{\mathbf{i}\rho\cos t}dt\Big| \\
&\le C(\phi^*,\kappa)\rho^{-1}(|f(0)|+\int_{I_1}|f'(t)|dt)
\end{aligned}$$

For interval I_2, we only analysis the integral on $[t_1-d,t_1]$. The integral on the other side can be estimated in the same method. Let $0<\delta<d$, then by lemma 2.4 we can get

$$\begin{aligned}
&\Big|\int_{t_1-d}^{t_1-\delta} f(t)(\kappa^2-\sin^2(t+\phi))^{-1/2}e^{\mathbf{i}\rho\cos t}dt\Big| \\
&\le \frac{C}{|\sin(t_1-d)|}\rho^{-1}\Big(|f(t_1-\delta)||\kappa^2-\sin^2(\phi_\kappa-\delta)|^{-1/2}+\int_{-d}^{-\delta}|f'(t_1+t)||\kappa^2-\sin^2(\phi_\kappa+t)|^{-1/2}dt \\
&+\int_{-d}^{-\delta}|f(t_1+t)||\kappa^2-\sin^2(\phi_\kappa+t)|^{-3/2}dt\Big) \\
&\le C\frac{\cos^{-3/2}(\phi_\kappa+d)}{\sin d}\rho^{-1}\Big(|f(t_1-\delta)|((\kappa+\sin(\phi_\kappa-d))\delta)^{-1/2}+\int_{-d}^{-\delta}|f'(t_1+t)|((\kappa+\sin(\phi_\kappa-d))\delta)^{-1/2}dt \\
&+\int_{-d}^{-\delta}|f(t_1+t)|(-(\kappa+\sin(\phi_\kappa-d))t)^{-3/2}dt\Big) \\
&\le C(\phi^*,\kappa)(1+\delta^{-1/2})\rho^{-1}(|f(0)|+\int_{-\pi/2}^{\pi/2}|f'(t)|dt)
\end{aligned}$$

Now, we turn to estimate the residual part of $[t_1-d,t_1]$. By inequality 2.33, we have

$$\begin{aligned}
&\Big|\int_{t_1-\delta}^{t_1} f(t)(\kappa^2-\sin^2(t+\phi))^{-1/2}e^{\mathbf{i}\rho\cos t}dt\Big| \\
&\le \|f\|_{L^\infty}(\kappa+\sin(\phi_\kappa-d))\int_{-\delta}^{0}(\kappa-\sin(\phi_\kappa+t))dt
\end{aligned}$$

$$
\begin{aligned}
&\leq \|f\|_{L^\infty}(\kappa+\sin(\phi_\kappa-d))\int_{-\delta}^{0}(\kappa-\sin(\phi_\kappa+t))^{-1/2}\frac{\cos t}{\cos d}\\
&\leq C(d,\kappa)\|f\|_{L^\infty}(\kappa-\sin(\phi_\kappa-\delta))^{1/2}\leq C(\phi^*,\kappa)\|f\|_{L^\infty}\delta^{1/2}
\end{aligned}
$$

Cosequently, let δ be equal to $d*\rho^{-1}$, we can obtain

$$
\begin{aligned}
&\Big|\int_{I_2} f(t)(\kappa^2-\sin^2(t+\phi))^{-1/2}e^{\mathbf{i}\rho\cos t}dt\Big|\\
&\leq C(\phi^*,\kappa)\rho^{-1/2}(|f(0)|+\int_{I_1}|f'(t)|dt)
\end{aligned}
$$

Similarly, we can etimate the integral on I_3. This completes the proof. □

Therefore, the estimate of $T_D(x_1,0;y_1,y_2)$ and $N(x_1,0;y_1,y_2)$ are now direct consequences of lemma singular_term 2.5.

es_dgreen **Lemma 2.6** *For every $x\in\Gamma_0$, $y\in\mathbb{R}^2_+$ such that $|x_1-y_1|/|x-y|>(1+\kappa)/2$, $y_2/|x-y|<\kappa/2$ and $k_s|x-y|>1$, we have*

$$
|T_D(x,y)|\leq C\bigg(\frac{k_sy_2}{|x-y|}\frac{1}{(k_s|x-y|)^{1/2}}+\frac{k_s|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{3/2}}\bigg) \tag{2.35}
$$

where C is only dependent on κ.

Proof. Put

$$
I(|x_1-y_1|,y_2)=\int_{\mathbb{R}}\mathcal{T}_s(\xi)e^{\mathbf{i}(\mu_sy_2+\xi|x_1-y_1|)}d\xi \tag{2.36}
$$

$$
J(|x_1-y_1|,y_2)=\int_{\mathbb{R}}\mathcal{T}_p(\xi)e^{\mathbf{i}(\mu_py_2+\xi|x_1-y_1|)}d\xi \tag{2.37}
$$

To simplify the integral I, the standard substitution $\xi=k_s\sin t$ is made, taking the ξ-plane to a strip $-\pi<\mathrm{Re}\,t<\pi$ in the t-plane, and the real axis in the ξ-plane onto the path L from $-\pi/2+\mathbf{i}\infty\to-\pi/2\to\pi/2\to\pi/2-\mathbf{i}\infty$ in the t-plane. The integral $I(|x_1-y_1|,y_2)$ then becomes (Let $|x_1-y_1|=\rho\sin\phi$ and $y_2=\rho\cos\phi$, $0<\phi<\pi/4$):

$$
k_s\int_L F(\sin t,\cos t,(\kappa^2-\sin^2t)^{1/2})\cos t\ e^{\mathbf{i}k_s\rho(\cos(t-\phi))}dt \tag{2.38}
$$

where $F(\sin t,\cos t,(\kappa^2-\sin^2t)^{1/2})=\mathcal{T}_s(k_s\sin t)$. Taking the shift transformation of t and using cauchy integral theorem, we can get the representation of I:

$$
\begin{aligned}
&k_s\int_L F(\sin(t+\phi),\cos(t+\phi),(\kappa^2-\sin^2(t+\phi))^{1/2})\cos(t+\phi)e^{\mathbf{i}k_s\rho(\cos t)}dt\\
&=k_s\cos\phi\int_L F(\sin(t+\phi),\cos(t+\phi),(\kappa^2-\sin^2(t+\phi))^{1/2})\cos t\ e^{\mathbf{i}k_s\rho(\cos t)}dt\\
&-k_s\sin\phi\int_L F(\sin(t+\phi),\cos(t+\phi),(\kappa^2-\sin^2(t+\phi))^{1/2})\sin t\ e^{\mathbf{i}k_s\rho(\cos t)}dt\\
&:=k_s(\cos\phi\ I_1+\sin\phi\ I_2)
\end{aligned}
$$

For I_1, we split the integral path L into $L_1=(-\pi/2,\pi/2)$ and $L_2=(-\pi/2+\mathbf{i}\infty,-\pi/2)\cup(\pi/2,\pi/2-\mathbf{i}\infty)$, then we have corresponding representation: $I_1=I_{11}+I_{12}$. Since $F(t)\cos t\in W^{1,1}([-\frac{\pi}{2},\frac{\pi}{2}])$, we can obtain $|I_{11}|\leq 1/(k_s\rho)^{1/2}$ by lemma van 2.4. Using integration by parts, it follows that $|I_{12}| \le 1/(k_s\rho)$. For I_2, using integration by parts on path L first, we have

$$I_2 = \frac{1}{\mathbf{i}k_s\rho}\int_L F(\sin(t+\phi), \cos(t+\phi), (\kappa^2-\sin^2(t+\phi))^{1/2})d\, e^{\mathbf{i}(k_s\rho\cos t)} \tag{2.39}$$

$$= -\frac{1}{\mathbf{i}k_s\rho}\int_{L_1\cup L_2}\frac{\partial F(\sin(t+\phi), \cos(t+\phi), (\kappa^2-\sin^2(t+\phi))^{1/2})}{\partial t}\, e^{\mathbf{i}(k_s\rho\cos t)}dt \tag{2.40}$$

$$= -\frac{1}{\mathbf{i}k_s\rho}(I_{21}+I_{22}) \tag{2.41}$$

It is easy to check that $\partial F/\partial t(\kappa^2-\sin^2(t+\phi))^{1/2} \in C([-\frac{\pi}{2},\frac{\pi}{2}])\cap W^{1,1}([-\frac{\pi}{2},\frac{\pi}{2}])$. Therefore, we have $|I_{21}| \le 1/(k_s\rho)^{1/2}$ by lemma 2.5. Similarly, we can obtain $|I_{22}| < C\rho^{-1}$. The estimate of $J(|x_1-y_1|, y_2)$ can be proved by the same method as employed above. Here we omit the details. This completes the proof. □

es_ngreen **Lemma 2.7** *For every $x \in \Gamma_0$, $y \in \mathbb{R}^2_+$ such that $|x_1-y_1|/|x-y| > (1+\kappa)/2$, $y_2/|x-y| < \kappa/2$ and $k_s|x-y| > 1$, we have*

$$|\mathbb{N}(x,y)| \le \frac{C}{\mu}\left(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{1/2}} + \frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{3/2}} + e^{-\sqrt{k_R^2-k_s^2}y_2}\right) \tag{2.42}$$

where C is only dependent on κ.

Proof. For $\mathbb{N}(x,y)$, it suffice to estimate the first part $\mathbb{N}_f$. The proof is similar to lemma 2.6. Here we only point out the different parts. Denote integral path $P^1_\sigma = (-\infty, -k_R-\sigma)\cup(-k_R+\sigma, k_R-\sigma)\cup(k_R+\sigma,\infty)$ and $P_\sigma = P^1_\sigma \cup B^+_\sigma(-k_R)\cup B^+_\sigma(k_R)$. By the defination of Cauchy principal value, we have

$$\mathbb{N}_f(x,y) = \frac{1}{2\pi}PV\int_{\mathbb{R}}\hat{\mathbb{N}}(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi = \lim_{\sigma\to0^+}\frac{1}{2\pi}\int_{P^1_\sigma}\hat{\mathbb{N}}(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi$$

$$= \frac{1}{2\pi}\int_{P_\sigma}\hat{\mathbb{N}}(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi + \frac{\mathbf{i}}{2}\frac{\mathbb{N}_\delta(-k_R)}{\delta'(-k_R)}e^{-\mathbf{i}(x_1-y_1)k_R} + \frac{\mathbf{i}}{2}\frac{\mathbb{N}_\delta(k_R)}{\delta'(k_R)}e^{\mathbf{i}(x_1-y_1)k_R} \tag{2.43}$$

Notice that, in the present case, the substitution $\xi = k_s\sin t$, taking P_σ in the ξ-plane onto Γ_σ in the t-plane. The geometry now is depicted in Figure 1.

Similarly, if σ small enough, we can taking the shift transformation of t and use cauchy integral theorem. Therefore, the proof of this lemma can be completed by the same method as employed in the lemma 2.6. Here we omit the details. □

In the following we will use the weighted norm $\|u\|_{H^1(D)} = (\|\nabla\phi\|^2_{L^2(D)} + d_D^{-2}\|\phi\|^2_{L^2(D)})^{1/2}$ be the weighted $H^1(D)$ norm and $\|v\|_{H^{1/2}(\Gamma)} = (d_D^{-1}\|v\|^2_{L^2(\Gamma)} + |v|^2_{\frac{1}{2},\Gamma})^{1/2}$ be the weighted $H^{1/2}(\Gamma)$ norm, where d_D is the diameter of D and

$$|v|_{\frac{1}{2},\Gamma} = \left(\int_\Gamma\int_\Gamma\frac{|v(x)-v(y)|^2}{|x-y|^2}ds(x)ds(y)\right)^{1/2}.$$

By scaling argument and trace theorem we know that there exists a constant $C > 0$ independent of d_D such that for any $\phi \in C^1(\bar{D})$ [14, corollary 3.1],

$$\|\phi\|_{H^{1/2}(\Gamma_D)} + \|\sigma(\phi)\cdot\nu\|_{H^{-1/2}(\Gamma_D)} \le C\max_{x\in D}(|\phi(x)| + d_D|\nabla\phi(x)|) \tag{2.44}$$ q0

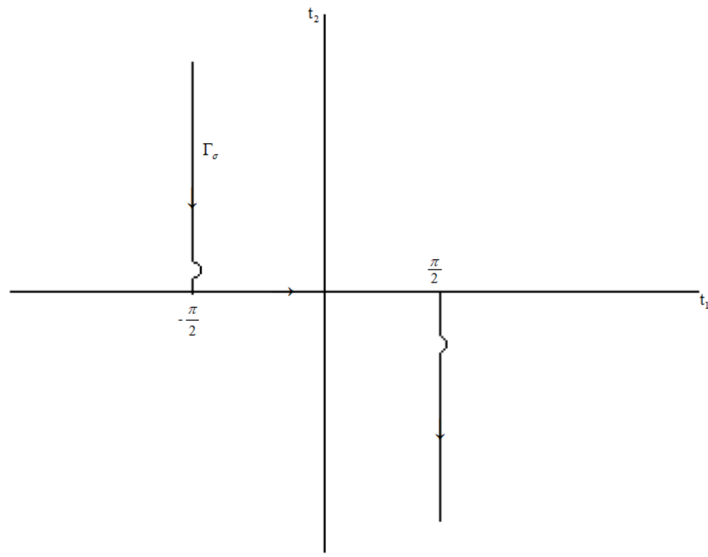

Figure 1. Transform from P_σ in the ξ-plane to Γ_σ in the t-plane

We finish this section by introducing some results on foward scattering problem which will be discussed briefly in the Appendix of this paper.

Theorem 2.1 *Let $g \in H^{1/2}(\Gamma_D)$, then the scattering problem of elastic equation in the half space*

$$\Delta_e u + \omega^2 u = 0 \qquad \text{in } \mathbb{R}^2_+ \backslash \bar{D}, \tag{2.45}$$

$$u = g \quad \text{on } \Gamma_D, \tag{2.46}$$

$$\sigma(u)e_2 = 0 \quad \text{on} \Gamma_0, \tag{2.47}$$

u satisfies the generalized radiation codition [25] such that

$$\lim_{r \to \infty} \int_{S_r^+} (\sigma(N(x,y)e_i)\hat{r}) \cdot u(x) - (N(x,y)e_i) \cdot (\sigma(u)\hat{r}) ds(x) = 0 \tag{2.48}$$

where $S_r^+ := \{x \in \mathbb{R}^2_+ \mid \|x\| = r^2\}$, $\hat{r} = x/r$ and $y \in \mathbb{R}^2_+$. Then the problem (2.45)-(2.48) admits a unique solution $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2_+ \backslash \bar{D})$. Moreover, for any bounded open set $\mathcal{O} \subset \mathbb{R}^2_+ \backslash \bar{D}$ there exists a constant $C > 0$ such that

$$\|u\|_{H^1(\mathcal{O})} \leq C \|g\|_{H^{-1/2}(\Gamma_D)} n \tag{2.49}$$

Moreover, we give the definition of the Dirichlet-to-Neumann mapping for Elastic scattering problems. For any $g \in H^{1/2}(\Gamma_D)$, define $T_e(g) = \sigma(u)\nu \in H^{-1/2}(\Gamma_D)$ where u is the corresponding solution of above equation. we will denote $\|T_e\|$ by its operator norm in the remainder of this paper.

For the sake of convenience, we introduce the following notation:

$$\mathcal{G}(W, U) = \int_{\Gamma_D} [W(x) \cdot \sigma(U(x))\nu - \sigma(W(x))\nu \cdot U(x)] ds(x)$$

Using this notation, the integral representation formula of the solution U(x) to the scattering problem (2.45)-(2.47) reads: $u(x) \cdot q = \mathcal{G}(u, \mathbb{N}(\cdot, x)q)$.

Lemma 2.8 *Let $W \in C^1(\bar{D})^2$ and $U \in H^1_{loc}(R^2\backslash\bar{D})$ be the solution of the scattering problem (2.45)-(2.48) with the boundary condition $g = u$ on Γ_D for some function $u \in C^1(\bar{D})^2$. Then we have*

$$\mathcal{G}(W,U) \leq C(1+\|T_e\|)\max_{x\in\bar{D}}[(|W(x)|+|\nabla W(x)|)(|u(x)|+|\nabla u(x)|)]$$

Proof. □

3. Reverse time migration method

In this section we introduce RTM method for inverse elastic scattering problems in the half space. Assume that there N_s sources and N_r receivers uniformly distributed on Γ_0^d, where $\Gamma_0^d = \{(x_1,x_2)^T \in \Gamma_0 : x_1 \in [-d,d]\}, d > 0$ is aperture. We denote by Ω the sampling domain in which the obstacle is sought. Let $h = dist(\Omega,\Gamma_0)$ be the distance of Ω to Γ_0. We assume the obstacle $D \subset \Omega$ and there exist constants $0 < c_1 < 1, c_2 > 0, c_3 > 0$ such that

$$|x_1| \leq c_1 d, \quad |x_1 - y_1| \leq c_2 h, \quad |x_2| \leq c_3 h \quad \forall x,y \in \Omega \tag{3.1}$$ convention_2

Before proposing RTM imaging function, we first introduce the point spred function which measures the resolution for finding point source[3]. In [14], the point spread function has been defined in the case of acoustic wave. We now define elastic point spread function $\mathbb{J}(z,y)$, a $\mathbb{C}^{2\times 2}$ matrix, which back-propagate the conjugated data $\overline{N(x,y)}$ as the Dirichlet boundary condition. Thus, for any $z,y \in \mathbb{R}^2_+$

$$e_i \cdot \mathbb{J}e_j := \mathbb{J}_{ij}(z,y) = \int_{\Gamma_0} \sigma_x(\mathbb{D}(x,y)e_i)e_2 \cdot \overline{\mathbb{N}(x,y)}e_j ds(x) \tag{3.2}$$ fullpsf

$$= \int_{\mathbb{R}} \sigma_x(\mathbb{D}(x_1,0;z_1,z_2)e_i)e_2 \cdot \overline{\mathbb{N}(x_1,0;z_1,z_2)}e_j dx_1 \tag{3.3}$$

The estimate in lemma 2.6-2.7 show that the integral above exists. Moreove, we define

$$\mathbb{F}(z,y) = \frac{1}{2\pi}\int_{-k_p}^{k_p} \frac{\mathcal{T}_p(\xi)^T\overline{\mathcal{N}_p}(\xi)}{\overline{\delta(\xi)}} e^{\mathbf{i}\mu_p(z_2-y_2)+\mathbf{i}(y_1-z_1)\xi} d\xi \tag{3.4}$$

$$+ \frac{1}{2\pi}\int_{-k_s}^{k_s} \frac{\mathcal{T}_s(\xi)^T\overline{\mathcal{N}_s}(\xi)}{\overline{\delta(\xi)}} e^{\mathbf{i}\mu_s(z_2-y_2)+\mathbf{i}(y_1-z_1)\xi} d\xi \tag{3.5}$$

The following theorem show that $\mathbb{F}(z,y)$ is the main contribution to the point spread funciton when $k_s h \gg 1$. Put $n_* = \min\{N|\kappa^{2N-1} < 1/c_3, N \in \mathbb{Z}_+\}$. Then we present some primary results for point spread function, whose proof will be stated in seciton 4.

thm_psf **Theorem 3.1** *Let $k_s h > 1$. For any $z,y \in \Omega$, let $\mathbb{R}(z,y) = \mathbb{J}(z,y) - \mathbb{F}(z,y)$ and it satisfy that*

$$|\mathbb{R}_{ij}(z,y)| + k_s^{-1}|\nabla_y \mathbb{R}_{ij}(z,y)| \leq \frac{C}{\mu}\left(\frac{1}{(k_s h)^{\frac{1}{2n^*}}} + e^{-k_s h\sqrt{\kappa_R^2-1}}\right) := \frac{C}{\mu}\epsilon_1(k_s h) \tag{3.6}$$

uniformly for $z,y \in \Omega$. Here $\kappa_R := k_R/k_s$ and the constant C may dependent on $k_s d_D$ and $\kappa := k_p/k_s$, but is independent of k_s, k_p, h, d_D.

Based on the above argument, we know that $R(z,y)$ becomes small when z,y move away from Γ_0. Our goal is to show $F(z,y)$ has the similar decay to the elastic fundamental solution $\operatorname{Im}\Phi(z,y)$ as $|z-y| \to \infty$.

festimate1 **Lemma 3.1** *For any $z, y \in \mathbb{R}^2_+$, when $z = y$*

$$
\begin{aligned}
&|\operatorname{Im}\mathbb{F}_{ii}(z,y)| \geq \frac{1}{4(\lambda+2\mu)} \ , \ i = 1,2 \\
&\operatorname{Im}\mathbb{F}_{12}(z,y) = \operatorname{Im}\mathbb{F}_{21}(z,y) = 0
\end{aligned}
$$

and for $z \neq y$

$$
|\mathbb{F}_{ij}(z,y)| \leq \frac{C}{\mu}[(k_s|z-y|)^{-1/2}) + (k_s|z-y|^{-1})]
$$

where constant C is only dependent on $\kappa := k_p/k_s$.

By (2.44), we obtain the following consequence of theorem 3.1 and Lemma 3.1 whcih will be used in the resolution analysis.

cor_psf **Corollary 3.1** *There exists a constant C independent of k_s, h such that*

$$
\begin{aligned}
&\|F(z,\cdot)\|_{H^{1/2}(\Gamma_D)} + \|\sigma(F(z,\cdot))\cdot\nu\|_{H^{-1/2}(\Gamma_D)} \leq \frac{C}{\mu}(1+k_s d_D) \\
&\|R(z,\cdot)\|_{H^{1/2}(\Gamma_D)} + \|\sigma(R(z,\cdot))\cdot\nu\|_{H^{-1/2}(\Gamma_D)} \leq \frac{C}{\mu}(1+k_s d_D)\epsilon_1(k_s h)
\end{aligned}
$$

uniformly for $z \in \Omega$, where d_D is the diameter of the obstacle D.

Now we consider the finite aperture point spread function $\mathbb{J}_d(z,y)$:

$$
\int_{-d}^{d} (T_D(x_1, 0; z_1, z_2))^T \overline{N(x_1, 0; y_1, y_2)} dx_1 \tag{3.7}
$$

Our aim is to estimate the difference $J(z,y) - J_d(z,y)$. It is easy to see that

$$
\frac{(x_1-z_1)^2}{\rho^2} \geq \frac{(1-c_1)^2}{(1-c_1)^2 + c_3^2(h/d)^2} := m(h/d) \tag{3.8}
$$

$$
\frac{z_2^2}{\rho^2} \leq \frac{c_3^2}{(1-c_1)^2(h/d) + c_3^2} := M(h/d) \tag{3.9}
$$

where $\rho = \sqrt{(x_1-z_1)^2 + z_2^2}$ and $z \in \Omega, x \in \Gamma_0 \backslash (-d,d)$. In the sebsequent discussions, we assume $m(h/d) > (1+\kappa)^2/4$, $M(h/d) < \kappa^2/4$.

ap_psf **Theorem 3.2** *For $k_s h \geq 1$ and $z, y \in \Omega$, we have*

$$
|J(z,y) - J_d(z,y)| + k_s^{-1}|\nabla_y(J(z,y) - J_d(z,y))| \tag{3.10}
$$

$$
\leq \frac{C}{\mu}((\frac{h}{d})^2 + \frac{(k_s h)^{1/2}}{e^{k_s h\sqrt{\kappa_R^2 - 1}}}(\frac{h}{d})^{1/2}) := \frac{C}{\mu}\epsilon_2(k_s h, h/d) \tag{3.11}
$$

where the constant C is only dependent on κ.

Proof. By lemma 2.7, lemma 2.6 and $k_sh \geq 1$, we have

$$
\begin{aligned}
&\left|\int_d^{\infty}(T_D(x_1,0;z_1,z_2))^T\overline{N(x_1,0;y_1,y_2)}dx_1\right| \\
&\leq \frac{C}{\mu}\int_d^{\infty}\frac{k_sz_2}{|x-z|}\frac{1}{(k_s|x-z|)^{1/2}}\Big(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{1/2}}+e^{-\sqrt{k_R^2-k_s^2}y_2}\Big)dx_1 \\
&\leq \frac{C}{\mu}\int_{(1-c_1)d/h}^{\infty}\frac{1}{(1+t^2)^{3/2}}+\frac{(k_sh)^{1/2}}{(1+t^2)^{3/4}}e^{-\sqrt{k_R^2-k_s^2}h}dt \\
&\leq \frac{C}{\mu}((\frac{h}{d})^2+\frac{(k_sh)^{1/2}}{e^{\sqrt{k_R^2-k_s^2}h}}(\frac{h}{d})^{1/2})
\end{aligned}
$$

Here we have used the first inequeality in (3.1). Similarly, we can prove that the estimate for te integral in $[-\infty,-d]$. This shows the estimate for $J(z,y)-J_d(z,y)$. The estimate for $\nabla_y(J(z,y)-J_d(z,y))$ can be proved similarly. □

By (2.44) we obtain the following corollary

cor_dpsf **Corollary 3.2** *For $k_sh \geq 1$ and $z, y \in \Omega$, there exists a constant C independent of k_s, h such that*

$$\|J(z,\cdot)-J_d(z,\cdot)\|_{H^{1/2}(\Gamma_D)}+\|\sigma(J(z,\cdot)-J_d(z,\cdot))\cdot\nu\|_{H^{-1/2}(\Gamma_D)} \leq \frac{C}{\mu}\epsilon_2(k_sh,h/d)(1+k_sd_D)$$

uniformly for $z \in \Omega$, where d_D is the diameter of the obstacle D.

Motivated by the above discussion, we introduce the following imaging function:

alg_rtm **Algorithm 3.1** (Reverse time migration algorithm)
Given the data $u_k^s(x_r,x_s)$, $k=1,2$ which is the measurement of the scattered field at x_r when the source is emitted at x_s along the polarized direction e_k, $s=1,\ldots,N_s$ and $r=1,\ldots,N_r$.

$$I_d(z)=\mathrm{Im}\sum_{k=1}^{2}\left\{\frac{|\Gamma_0^d|}{N_s}\sum_{s=1}^{N_s}[(T_{x_s}^{e_2}D(x_s,z))^Te_k]\cdot v_k(z,x_s)\right\}. \quad z\in\Omega \qquad (3.12)$$ cor1

where $v_k(z,x_s)$ satisfy tht following scattering elastic equation:

$$
\begin{aligned}
&\Delta_e v_k(z,x_s)+\omega^2 v_k(z,x_s)=0 \qquad \text{in } \mathbb{R}_+^2 \\
&v_k(z,x_s)=\overline{u_k^s(x_r,x_s)} \qquad \text{on}\Gamma_0
\end{aligned}
$$

By letting $N_s, N_r \to \infty$, we know that (3.12) can be viewed as an approximation of the following continuous integral:

$$\hat{I}_d(z)=\mathrm{Im}\sum_{k=1}^{2}\int_{\Gamma_0^d}\int_{\Gamma_0^d}[(T_{x_s}^{e_2}D(x_s,z))^Te_k]\cdot[(T_{x_r}^{e_2}D(x_r,z))^T\overline{u_k^s(x_r,x_s)}]ds(x_r)ds(x_s) \quad (3.13)$$ cor3

where $z \in \Omega$. We will study the resolution of the function $\hat{I}_d(z)$ in the

Now, We turn to study the resolution of the function $\hat{I}_d(z)$. To do this, we first show the difference between the half space scattering solution and the full space sacttering solution is small of the sactterer is far away from the boundary Γ_0.

diff_solu **Theorem 3.3** *Let $g \in H^{1/2}(\Gamma_D)$ and $\mathbf{u}_1, \mathbf{u}_2$ be the scattering solution of following problems:*

$$\Delta_e \mathbf{u}_1 + \omega^2 \mathbf{u}_1 = 0 \qquad \text{in } \mathbb{R}^2_+\backslash\bar{D} \tag{3.14}$$ elas_r1

$$\mathbf{u}_1 = g \qquad \text{on } \Gamma_D \tag{3.15}$$ elas_rbd

$$\sigma(\mathbf{u}_1)e_2 = 0 \qquad \text{on}\Gamma_0 \tag{3.16}$$ elas_rb0

and

$$\Delta_e \mathbf{u}_2 + \omega^2 \mathbf{u}_2 = 0 \qquad \text{in } \mathbb{R}^2\backslash\bar{D} \tag{3.17}$$ elas_r2

$$\mathbf{u}_2 = g \qquad \text{on } \Gamma_D \tag{3.18}$$ elas_rbd2

Then there exits a constant C independant of k_s, k_p, such that

$$\begin{aligned} \|T^\nu_x(\mathbf{u}_1 - \mathbf{u}_2)\|_{H^{-1/2}(\Gamma_D)} &\leq \frac{C}{\mu}(1+k_s d_D)^2((k_s h)^{-1/2} + e^{-\sqrt{k_R^2-k_s^2}h})\|g\|_{H^{1/2}(\Gamma_D)} && (3.19)\\ &\leq \frac{C}{\mu}(1+k_s d_D)^2 \epsilon_1(k_s h)\|g\|_{H^{1/2}(\Gamma_D)} && (3.20)\end{aligned}$$

Theorem 3.3 (diff_solu) also will be proved in the appendix of this paper. The following theorem is the main result of resolution analysis.

resolution1 **Theorem 3.4** *For any $z \in \Omega$, let $\Psi(y,z) \in \mathbb{C}^{2\times 2}$ be the radiation solution of the problem:*

$$\begin{aligned} &\Delta_e \Psi(y,z) + \omega^2 \Psi = 0 && \text{in } \mathbb{R}^2_+\backslash\bar{D}\\ &\Psi(y,z) = -\overline{\mathbb{F}(z,y)} && \text{on } \Gamma_D \end{aligned}$$

Then, we have

$$\hat{I}_d(z) = \operatorname{Im}\mathbf{tr}\int_{\Gamma_D}(T^\nu_y(\overline{\mathbb{F}(z,y)} + \Psi(y,z))^T\overline{\mathbb{F}(z,y)}ds(y) + \mathbb{W}_{\hat{I}}(z) \tag{3.21}$$

where $|\mathbb{W}_{\hat{I}}(z)| \leq C(1+k_s d_D)^2(\epsilon_1(k_s h) + \epsilon_2(k_s h, h/d))$ uniformly for z in Ω.

Proof. By the integral representation, we have,

$$\mathbf{u}^s_k(x_r,x_s) = \int_{\Gamma_D}(T^\nu_y\mathbb{N}(y,x_r))^T\mathbf{u}^s_k(y,x_s) - \mathbb{N}(x_r,y)(T^\nu_y\mathbf{u}^s_k(y,x_s))ds(y) \tag{3.22}$$

where $\mathbf{u}^s_k(x,x_s) + N(x,x_s)e_k = 0$. From (3.2) (fullpsf) we get for any $z \in \Omega$,

$$\begin{aligned} v_k(z,x_s) &= \int_{\Gamma_0^d}(T^{e_2}_{x_r}\mathbb{D}(x_r,z))^T\overline{\mathbf{u}^s_k(x_r,x_s)}ds(x_r)\\ &= \int_{\Gamma_D}ds(y)\Big(\int_{\Gamma_0^d}(T^{e_2}_{x_r}\mathbb{D}(x_r,z))^T\overline{(T^\nu_y\mathbb{N}(y,x_r))^T}ds(x_r)\Big)\overline{\mathbf{u}^s_k(y,x_s)}\\ &\quad - \Big(\int_{\Gamma_0^d}(T^{e_2}_{x_r}\mathbb{D}(x_r,z))^T\overline{\mathbb{N}(x_r,y)}ds(x_r)\Big)\overline{(T^\nu_y\mathbf{u}^s_k(y,x_s))}\\ &= \int_{\Gamma_D}ds(y)\Big(\int_{\Gamma_0^d}(T^\nu_y\overline{\mathbb{N}(y,x_r)}T^{e_2}_{x_r}\mathbb{D}(x_r,z))^T ds(x_r)\Big)\overline{\mathbf{u}^s_k(y,x_s)} \end{aligned}$$

$$
\begin{aligned}
&- \Big(\int_{\Gamma_0^d} (T_{x_r}^{e_2}\mathbb{D}(x_r,z))^T \overline{\mathbb{N}(x_r,y)} ds(x_r) \Big) \overline{(T_y^{\nu}\mathbf{u}_k^s(y,x_s))} \\
&= \int_{\Gamma_D} ds(y) \Big(\int_{\Gamma_0^d} (T_y^{\nu}[\overline{\mathbb{N}(y,x_r)} T_{x_r}^{e_2}\mathbb{D}(x_r,z)])^T ds(x_r) \Big) \overline{u_k^s(y,x_s)} \\
&- \Big(\int_{\Gamma_0^d} (T_{x_r}^{e_2}\mathbb{D}(x_r,z))^T \overline{\mathbb{N}(x_r,y)} ds(x_r) \Big) \overline{(T_y^{\nu}\mathbf{u}_k^s(y,x_s))} \\
&= \int_{\Gamma_D} ds(y) \Big((T_y^{\nu}\mathbb{J}_d^T(z,y))^T \overline{\mathbf{u}_k^s(y,x_s)} - \mathbb{J}_d(z,y) \overline{(T_y^{\nu}\mathbf{u}_k^s(y,x_s))} \Big)
\end{aligned}
$$

where we use the fact $(\sigma_x(A(x))\nu)B = \sigma_x(A(x)B)\nu$ above. By the definition of the imaging function $\hat{I}_d(z)$, we have

$$
\begin{aligned}
\hat{I}_d(z) &= \mathrm{Im} \sum_{k=1}^{2} \int_{\Gamma_0^d} (T_{x_s}^{e_2}\mathbb{D}(x_s,z))^T e_k \cdot v_k(z,x_s) ds(x_s) && (3.23)\\
&= \int_{\Gamma_D} ds(y) \sum_{k=1}^{2} \int_{\Gamma_0^d} (T_{x_s}^{e_2}\mathbb{D}(x_s,z))^T e_k \cdot \Big((T_y^{\nu}\mathbb{J}_d^T(z,y))^T \overline{\mathbf{u}_k^s(y,x_s)} && (3.24)\\
&\quad - \mathbb{J}_d(z,y) \overline{(T_y^{\nu}\mathbf{u}_k^s(y,x_s))} \Big) && (3.25)\\
&= \mathrm{Im} \int_{\Gamma_D} ds(y) \sum_{k=1}^{2} \mathbf{tr} \Big((T_y^{\nu}\mathbb{J}_d^T(z,y))^T \int_{\Gamma_0^d} \overline{\mathbf{u}_k^s(y,x_s)} e_k^T T_{x_s}^{e_2}\mathbb{D}(x_s,z) && (3.26)\\
&\quad - \mathbb{J}_d(z,y) \int_{\Gamma_0^d} \overline{(T_y^{\nu}u_k^s(y,x_s))} e_k^T T_{x_s}^{e_2}\mathbb{D}(x_s,z) \Big) && (3.27)\\
&= \mathrm{Im} \int_{\Gamma_D} ds(y) \mathbf{tr} \Big((T_y^{\nu}\mathbb{J}_d^T(z,y))^T \sum_{k=1}^{2} \mathbb{W}_k(y,z) && (3.28)\\
&\quad - \mathbb{J}_d(z,y) (T_y^{\nu} \sum_{k=1}^{2} \mathbb{W}_k(y,z)) \Big) && (3.29)\\
&= \mathrm{Im} \int_{\Gamma_D} \mathbf{tr} \Big((T_y^{\nu}\mathbb{J}_d^T(z,y))^T \mathbb{W}(y,z) - \mathbb{J}_d(z,y)(T_y^{\nu}\mathbb{W}(y,z)) \Big) ds(y) && (3.30)
\end{aligned}
$$

resolu_1

where

$$
\begin{aligned}
\mathbb{W}(y,z) &= \sum_{k=1}^{2} \mathbb{W}_k(y,z) && (3.31)\\
\mathbb{W}_k(y,z) &= \int_{\Gamma_0^d} \overline{u_k^s(y,x_s)} e_k^T (T_{x_s}^{e_2} D(x_s,z)) ds(x_s) && (3.32)
\end{aligned}
$$

Therefore, $\overline{\mathbb{W}_k(y,z)}$ can be viewed as the weighted superposition of $u_k^s(y,x_s)$. Then $\overline{\mathbb{W}_k(y,z)}$ satisfies elastic equation

$$
\Delta_e^y \overline{\mathbb{W}_k(y,z)} + \omega^2 \overline{\mathbb{W}_k(y,z)} = 0 \qquad (3.33)
$$

On the boundary of the obstacle Γ_D, we have

$$
\overline{\mathbb{W}(y,z)} = \sum_{k=1}^{2} \int_{\Gamma_0^d} u_k^s(y,x_s) e_k^T T_{x_s}^{e_2} \overline{D(x_s,z)} ds(x_s)
$$

$$
\begin{aligned}
&= \sum_{k=1}^{2} \int_{\Gamma_0^d} -N(y, x_s) e_k e_k^T T_{x_s}^{e_2} \overline{D(x_s, z)} ds(x_s) \\
&= -\int_{\Gamma_0^d} N(y, x_s) T_{x_s}^{e_2} \overline{D(x_s, z)} ds(x_s) \\
&= -\overline{\mathbb{J}_d^T(z, y)}
\end{aligned}
$$

Moreover, $T_y^{e_2}\overline{\mathbb{W}_k(y,z)} = 0$ on Γ_0 since $T_y^{e_2} u_k^s(y, x_s) = 0$ on Γ_0. Let $\mathbb{W}_d(y, z)$ be the scattering solution of the problem

$$\Delta_e \mathbb{W}_d(y, z) + \omega^2 \mathbb{W}_d(y, z) = 0 \quad \text{in } \mathbb{R}_+^2 \backslash \bar{D} \tag{3.34}$$

$$\mathbb{W}_d(y, z) = \overline{\mathbb{F}(z, y)} - \overline{\mathbb{J}_d^T(z, y)} \quad \text{on } \Gamma_D \tag{3.35}$$

$$T_y^{e_2}(\mathbb{W}_d(y, z)) = 0 \quad \text{on } \Gamma_0 \tag{3.36}$$

By theorem 2.1 and Corollaries 3.1-3.2 we have

$$
\begin{aligned}
\|T_y^{\nu}(\mathbb{W}_d(y, z))\|_{H^{1/2}(\Gamma_D)} &\le \|\mathbb{F}(z, \cdot) - \mathbb{J}_d^T(z, \cdot)\|_{H^{1/2}(\Gamma_D)} \\
&\le C(1 + k_s d_D)(\epsilon_1(k_s h) + \epsilon_2(k_s h, h/d))
\end{aligned} \tag{3.37}
$$

Let $V(y, z) := \overline{\mathbb{W}(y, z)} - \mathbb{W}_d(y, z) - \Psi(y, z)$. Since $\overline{\mathbb{W}(y, z)} - \mathbb{W}_d(y, z)$ satisfy the half-space scattering problem 3.14 with $g(y) = -\overline{\mathbb{F}(z, y)}$, by using Theorem 3.3 and Corollary 3.1

$$\|T_y^{\nu} V(y, z)\|_{H^{1/2}(\Gamma_D)} \le C(1 + k_s d_D)^2 \epsilon_1(k_s h) \|\mathbb{F}(z, \cdot)\|_{H^{1/2}(\Gamma_D)} \tag{3.38}$$

$$\le C(1 + k_s d_D)^3 \epsilon_1(k_s h) \tag{3.39}$$

Now we substitute $\overline{\mathbb{W}(y, z)} = V(y, z) + \mathbb{W}_d(y, z) + \Psi(y, z)$ into (3.30) to obtain

$$\hat{I}_d(z) = \operatorname{Im} \mathbf{tr} \int_{\Gamma_D} (T_y^{\nu} \mathbb{J}_d^T(z, y))^T \overline{\Psi(y, z)} - \mathbb{J}_d(z, y)(T_y^{\nu} \overline{\Psi(y, z)}) ds(y) + R_{\hat{I}}(z) \tag{3.40}$$

where

$$R_{\hat{I}}(z) = -\operatorname{Im} \mathbf{tr} \int_{\Gamma_D} \mathbb{J}_d(z, y)(T_y^{\nu} V(y, z) ds(y) \tag{3.41}$$

$$+ \operatorname{Im} \mathbf{tr} \int_{\Gamma_D} (T_y^{\nu} \mathbb{J}_d^T(z, y))^T \mathbb{W}_d(y, z) - \mathbb{J}_d(z, y)(T_y^{\nu} \mathbb{W}_d(y, z)) ds(y) \tag{3.42}$$

By (3.37) and Corollaries 3.1-3.2 it is easy to see that

$$|R_{\hat{I}}(z)| \le C(1 + k_s d_D)^4 (\epsilon_1(k_s h) + \epsilon_2(k_s h, h/d)) \tag{3.43}$$

Finally, by (3.40) and $\Psi(y, z) = -\overline{\mathbb{F}(z, y)}$

$$
\begin{aligned}
\hat{I}_d(z) &= \operatorname{Im} \mathbf{tr} \int_{\Gamma_D} (T_y^{\nu}(\mathbb{F}(z, y))^T \overline{\Psi(y, z)} - \mathbb{F}(z, y)(T_y^{\nu} \overline{\Psi(y, z)}) ds(y) + w_{\hat{I}}(z) \\
&= -\operatorname{Im} \mathbf{tr} \int_{\Gamma_D} (T_y^{\nu}(\mathbb{F}(z, y))^T \mathbb{F}(z, y) + \mathbb{F}(z, y)(T_y^{\nu} \overline{\Psi(y, z)}) ds(y) + w_{\hat{I}}(z) \\
&= \operatorname{Im} \mathbf{tr} \int_{\Gamma_D} (T_y^{\nu}(\overline{\mathbb{F}(z, y)} + \Psi(y, z))^T \overline{\mathbb{F}(z, y)} ds(y) + w_{\hat{I}}(z)
\end{aligned}
$$

where

$$w_{\hat{I}}(z) = \operatorname{Im} \mathbf{tr} \int_{\Gamma_D} (T_y^{\nu}(\mathbb{J}_d^T(z, y) - \mathbb{F}(z, y))^T \overline{\Psi(y, z)} - (\mathbb{J}_d(z, y) - \mathbb{F}(z, y))(T_y^{\nu} \overline{\Psi(y, z)}) ds(y)$$

By Corollaries 3.1-3.2 we have

$$|w_{\hat{I}}(z)| \leq C(1+k_s d_D)^2(\epsilon_1(k_s h) + \epsilon_2(k_s h, h/d)) \tag{3.44}$$

□

By (4.5)-(4.6) we know that for any fixed $z \in \Omega$, $\overline{\mathbb{F}(z,\cdot)}$ satisfies the Elastic wave equation. Thus $\mathbb{G}(y,z)$ can be viewed as the scattering solution of the Elastic equation with the incident wave $\overline{\mathbb{F}(z,\cdot)}$. By lemma 3.1 we konw that $\overline{\mathbb{F}(z,\cdot)}$ decays as $|y-z|$ becomes large. Therefore the imaging function $\hat{I}_d(z)$ becomes small when z moves away from the boundary Γ_D outside the scatterer D if $k_s h \gg 1$ and $d \gg h$.

4. Some proof; Section name TBD

In this section we give the proof of theorem 3.1 and lemma 3.1 which depend on several lemmas that follow. Without loss of generality. we assume $z_1 - y_1 \geq 0$ in the sebsequent. Otherwise, we can take substitution $\xi = -\xi$.

We split the spectral terms into components associated with pressure and shearing waves.

$$\hat{\mathbb{D}} = \hat{\mathbb{D}}^p + \hat{\mathbb{D}}^s \qquad\qquad \hat{\mathbb{N}} = \hat{\mathbb{N}}^p + \hat{\mathbb{N}}^s$$

and we define

$$\mathbb{J}_{ij}^{\alpha\eta}(z,y) = \int_{\Gamma_0} \sigma_x(\mathbb{D}^\alpha(x,y)e_i)e_2 \cdot \overline{\mathbb{N}^\eta(x,y)}d)e_j ds(x), \qquad \alpha,\eta \in \{s,p\} \tag{4.1}$$

It's esay to see

$$\mathbb{J}(z,y) = \sum_{\alpha=p,s}^{\eta=p,s} \mathbb{J}^{\alpha\eta}(z,y)$$

In order to analysis the PSF, loss is assumed in the medium that $k_{\alpha,\varepsilon} := k_\alpha(1+\mathbf{i}\varepsilon)$. For $\alpha,\beta \in \{s,p\}$, define $b_{\alpha\beta} = k_s$ only if $\alpha=\beta=s$, or otherwise $b_{\alpha\beta} = k_p$.Then by Parseval identity, we carry out

$$\begin{aligned}
\mathbb{J}^{\alpha\beta}(z,y) = {} & \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_R \frac{\mathcal{T}_\alpha^\varepsilon(\xi)^T\overline{\mathcal{N}_\beta^\varepsilon}(\xi)}{\overline{\delta^\varepsilon(\xi)}} e^{\mathbf{i}(\mu_\alpha^\varepsilon z_2 - \overline{\mu_\beta^\varepsilon} y_2) + \mathbf{i}(y_1-z_1)\xi} d\xi \\
= {} & \frac{1}{2\pi}\int_{-b_{\alpha\beta}}^{b_{\alpha\beta}} \frac{\mathcal{T}_\alpha(\xi)^T\overline{\mathcal{N}_\beta}(\xi)}{\overline{\delta(\xi)}} e^{\mathbf{i}(\mu_\alpha z_2 - \overline{\mu_\beta} y_2) + \mathbf{i}(y_1-z_1)\xi} d\xi \\
+ {} & \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{R\setminus[-b_{\alpha\beta},b_{\alpha\beta}]} \frac{\mathcal{T}_\alpha^\varepsilon(\xi)^T\overline{\mathcal{N}_\beta^\varepsilon}(\xi)}{\overline{\delta^\varepsilon(\xi)}} e^{\mathbf{i}(\mu_\alpha^\varepsilon z_2 - \overline{\mu_\beta^\varepsilon} y_2) + \mathbf{i}(y_1-z_1)\xi} d\xi \\
:= {} & \mathbb{F}^{\alpha\beta}(z,y) + \mathbb{R}^{\alpha\beta}(z,y)
\end{aligned}$$

By the defination of Cauchy principle value, we can obtian

$$\begin{aligned}
\mathbb{R}^{\alpha\beta}(z,y) = {} & \frac{1}{2\pi}\int_{[-k_s,k_s]\setminus[-b_{\alpha\beta},b_{\alpha\beta}]} + P.V\int_{R\setminus[-k_s,k_s]} \frac{\mathcal{T}_\alpha(\xi)^T\overline{\mathcal{N}_\beta}(\xi)}{\overline{\delta(\xi)}} e^{\mathbf{i}(\mu_\alpha z_2 - \overline{\mu_\beta} y_2) + \mathbf{i}(y_1-z_1)\xi} d\xi \\
- {} & \frac{\mathbf{i}}{2}\Big(\frac{\mathcal{T}_\alpha(k_R)^T\overline{\mathcal{N}_\beta}(k_R)}{\overline{\delta'(k_R)}} e^{\mathbf{i}(y_1-z_1)k_R} - \frac{\mathcal{T}_\alpha(-k_R)^T\overline{\mathcal{N}_\beta}(-k_R)}{\overline{\delta'(-k_R)}} e^{-\mathbf{i}(y_1-z_1)k_R}\Big) e^{-\sqrt{k_R^2-k_\alpha^2}z_2 - \sqrt{k_R^2-k_\beta^2}y_2} \\
:= {} & \mathbb{R}_1^{\alpha\beta}(z,y) + \mathbb{R}_2^{\alpha\beta}(z,y) + \mathbb{R}_3^{\alpha\beta}(z,y)
\end{aligned}$$

pv_term **Lemma 4.1**

$$|P.V\int_1^\infty \frac{k_s^4 g(t)}{\delta(k_s t)}| \le C\Big(\int_1^\infty |g(t)|dt + \max_{t\in[1,2\kappa_R-1]}(|g(t)|+|g'(t)|)\Big)$$

where $\kappa_R = k_R/k_s$ and C only depend on κ.

Proof. Since κ_R is the first order zero point of $\delta(k_s t)$, we can choose $\sigma > 0$ small enough such that $|\delta'(k_s t)/k_s^3|$ has positive lower bound in $I_1 := (\kappa_R - \sigma, \kappa_R + \sigma)$. Let $\delta_1(t) = \frac{\delta(k_s t)}{k_s^4(t-\kappa_R)}$ and $I_2 = (1,\infty)\backslash I_1$ we have

$$\begin{aligned}
&|P.V\int_{1\infty} \frac{k_s^4 g(t)}{\delta(k_s t)}| \le |P.V\int_{I_1}\frac{k_s^4 g(t)}{\delta(k_s t)}dt| + |\int_{I_2}\frac{k_s^4 g(t)}{\delta(k_s t)}dt| \\
&\le |\int_{I_1}\frac{g(t)\delta_1(t)^{-1} - g(\kappa_R)\delta_1(\kappa_R)^{-1}}{t-\kappa_R} + P.V\int_{I_1}\frac{g(\kappa_R)\delta_1(\kappa_R)^{-1}}{t-\kappa_R}| + \max_{t\in I_2}\frac{k_s^4}{\delta(k_s t)}\int_{I_2}|g(t)|dt \\
&\le \max_{t\in I_1}|(\frac{g(t)}{\delta_1(t)})'| + C\frac{k_s^4}{\delta(k_s t)}\int_{I_2}|g(t)|dt
\end{aligned}$$

By the mean value theorem, For $t \in (1, 2\kappa_R - 1)$ we carry out

$$\begin{aligned}
&(\frac{g(t)}{\delta_1(t)})' = \frac{g'(t)\delta'(k_s t_1)k_s^5 + g(t)\delta''(k_s t_2)k_s^6}{\delta'(k_s t_1)} \quad t_1, t_2 \in [\kappa_R, t] \text{ or } [t, \kappa_R] \\
&\le C\max_{t\in I_1}(|g(t)|+|g'(t)|)
\end{aligned}$$

This completes the proof. $\square$

medi_term **Lemma 4.2** *Let $f(t)$ be a complex valued function in (κ,∞) and satisfy that $|f(t)| < C(1+t^k), k\in\mathbb{Z}_+$. Then for $\rho > 1$ we have*

$$\begin{aligned}
&|\int_\kappa^1 |f(t)e^{-\rho\sqrt{t^2-\kappa^2}}d\xi| \le C\frac{1}{\rho} \\
&|\int_1^\infty |f(t)e^{-\rho\sqrt{t^2-1}}d\xi| \le C\frac{1}{\rho}
\end{aligned}$$

Proof. It is simple to see that

$$\begin{aligned}
&|\int_\kappa^1 |f(t)e^{-\rho\sqrt{(t^2-\kappa^2)}}dt| \le C\int_\kappa^1 (1+t^k)e^{-\rho\sqrt{t^2-\kappa^2}}dt \\
&\le C\int_0^{\sqrt{(1-\kappa^2)}} \frac{t}{\sqrt{t^2+\kappa^2}}e^{-\rho t}dt \le C\frac{1}{\rho}
\end{aligned}$$

and

$$\begin{aligned}
&|\int_1^\infty |f(t)e^{-\rho\sqrt{t^2-1}}dt| \le C\int_1^\infty (1+t^k)e^{-\rho\sqrt{t^2-1}}dt \\
&\le C\int_0^\infty \frac{(1+(t^2+1)^{k/2})t}{\sqrt{t^2+1}}e^{-\rho t}dt \le C\frac{1}{\rho}
\end{aligned}$$

This completes the proof. $\square$

Lemma 4.3 *For $0<\kappa<1$, let $F(\lambda)=\int_0^\kappa f(t)e^{\mathbf{i}\lambda(\sqrt{1-t^2}-\tau\sqrt{\kappa^2-t^2}+\alpha t)}dt$, where $\tau\ge c_0>0$ and $\alpha\in\mathbb{R}$, then we have*

$$|F(\lambda)|\le C(\kappa)\lambda^{-\frac{1}{2N_*}}\left[|f(\kappa)|+\int_0^\kappa |f'(t)|dt\right]$$

where $N_*=\min\{N|\kappa^{2N-1}<c_0, N\in\mathbb{Z}_+\}$.

Proof. Put $\phi(t)=-\sqrt{1-t^2}$ and $\psi(t,\tau)=\tau\kappa\phi(t/\kappa)-\phi(t)+\alpha t$. For easy of notations, we denote the n-th partial derivative of $g(t)$ with respect to t by $g^{(n)}(t)$. Then, it is to see that, for $n>1$

$$\psi^{(n)}(t,\tau)=\frac{\tau}{\kappa^{n-1}}\phi^{(n)}(\frac{t}{\kappa})-\phi^{(n)}(t)$$

A standard computation show that

$$\phi^{(1)}(t)=\frac{t}{\sqrt{1-t^2}}\qquad \phi^{(2)}(t)=\frac{1}{(1-t^2)^{3/2}}$$

Moreover, for $n\ge 3$, we have

$$\phi^{(n)}(t)=\frac{p_n(t)}{(1-t^2)^{n-1/2}} \tag{4.2}$$

where $p_n=\sum_0^{n-2}a_k^n t^k$ is a $(n-2)$-th polynomial such that its coefficients satisfy the following recursion formula:

$$\begin{aligned}
&a_{n-1}^{n+1}=(n+1)a_{n-2}^n,\quad a_{n-2}^{n+1}=(n+2)a_{n-3}^n\\
&a_k^{n+1}=(k+1)a_{k+1}^n+(2n-k)a_{k-1}^n\quad \text{for } 1\le k\le n-3\\
&a_0^{n+1}=a_1^n
\end{aligned}$$

Since the polynmial coefficients are all positive, it is obvious that for $n\ge 1$, $\phi^{(n)}(t)$ is a monotone increasing positive function. Using the recursion formula, it follows that

$$\phi^{(n)}(0)=\begin{cases} 0 & \text{n is odd,}\\ (n-1)!!(n-3)!! & \text{n is even.}\end{cases} \tag{4.3}$$

where $(2k-1)!!$ is double factorial and $n>3$. We are now in the position to proof the inequality. Since $0<\kappa<1$, obersev that

$$\psi^{(2N_*+1)}(t,\tau)\ge\frac{\tau}{\kappa^{2N_*}}\phi^{(2N_*+1)}(t)-\phi^{(2N_*+1)}(t)>0$$

Therefore, $\psi^{(2N_*)}(t,\tau)$ is monotone increasing in $[0,\kappa)$. By (4.3), we get

$$\psi^{(2N_*)}(t,\tau)\ge\psi^{(2N_*)}(0,\tau)\ge\psi^{(2N_*)}(0,c_0)=C(2N_*)(\frac{c_0}{\kappa^{2N_*-1}}-1)>0 \tag{4.4}$$

The lemma is now a direct consequence of lemma (2.4). □

Now we are in the position to prove the main theorem of this section.

proof of Theorem 3.1. The theorem now follows from lemma 4.1, lemma 4.2 and lemma 4.3.

To prove the lemma 3.1, Substitute (2.29) and (2.21) into $F_{ss}(z,y), F_{pp}(z,y)$, then we have

$$F^{pp}(z,y) = -\frac{1}{2\pi}\int_{(-k_p,k_p)} \frac{\mathbf{i}k_s^2\mu_s}{\mu\gamma(\xi)\delta(\xi)}\begin{pmatrix} \xi^2 & -\xi\mu_p \\ -\xi\mu_p & \mu_p^2 \end{pmatrix} e^{\mathbf{i}\mu_p(z_2-y_2)+\mathbf{i}\xi(y_1-z_1)} \tag{4.5}$$

$$\begin{aligned} F^{ss}(z,y) &= -\frac{1}{2\pi}\int_{(-k_p,k_p)} \frac{\mathbf{i}k_s^2\mu_p}{\mu\gamma(\xi)\delta(\xi)}\begin{pmatrix} \mu_s^2 & \xi\mu_s \\ \xi\mu_s & \xi^2 \end{pmatrix} e^{\mathbf{i}\mu_s(z_2-y_2)+\mathbf{i}\xi(y_1-z_1)} \\ &\quad -\frac{1}{2\pi}\int_{(-k_s,k_s)\backslash(-k_p,k_p)} \frac{\mathbf{i}(k_s^2-4\xi^2)\mu_p}{\mu\gamma(\xi)\overline{\delta(\xi)}}\begin{pmatrix} \mu_s^2 & \xi\mu_s \\ \xi\mu_s & \xi^2 \end{pmatrix} e^{\mathbf{i}\mu_s(z_2-y_2)+\mathbf{i}\xi(y_1-z_1)} \\ &:= F^{ss1}(z,y) + F^{ss2}(z,y) \end{aligned} \tag{4.6}$$

proof of Lemma 3.1. We only proof the case of $i=1$, the other ones are similar. First, we have $\gamma(\xi)\le k_s^2$, $\delta(\xi)\le k_s^4$ and $\mu_p\le\mu_s$ when $\xi\in(-k_p,k_p)$. Then, if $z=y$

$$-\mathrm{Im}\,(F_{11}^{pp}+F_{11}^{ss1}) \ge \frac{1}{2\pi\mu}\int_{(-k_p,k_p)}\frac{\mu_p}{k_s^2}d\xi \tag{4.7}$$

$$= \frac{k_p^2}{2\pi\mu k_s^2}\int_0^\pi \sin^2(t)dt = \frac{1}{4(\lambda+2\mu)} \tag{4.8}$$

It's left to proof $-\mathrm{Im}\,F_{11}^{ss2}>0$. If $\xi\in(-k_s,k_s)\backslash(-k_p,k_p)$, $\mu_p=\mathbf{i}\sqrt{\xi^2-k_p^2}$. Substituting it into F^{ss2}, we have

$$F_{11}^{ss2} = \frac{1}{2\pi\mu}\int_{(-k_s,k_s)\backslash(-k_p,k_p)} \frac{\mu_s^2\sqrt{\xi^2-k_p^2}(k_s^2-4\xi^2)}{(\xi^2+\mathbf{i}\mu_s\sqrt{\xi^2-k_p^2})(\beta^2-\mathbf{i}4\xi^2\mu_s\sqrt{\xi^2-k_p^2})}d\xi \tag{4.9}$$

let $\alpha=(\xi^2+\mathbf{i}\mu_s\sqrt{\xi^2-k_p^2})(\beta^2-\mathbf{i}4\xi^2\mu_s\sqrt{\xi^2-k_p^2})$. A simple computation show that $\mathrm{Im}\,\alpha = k_s^2\mu_s\sqrt{\xi^2-k_p^2}(k_s^2-4\xi^2)$. It is easy to see that

$$-\mathrm{Im}\,F_{11}^{ss2} = \frac{k_s^2}{2\pi\mu}\int_{(-k_s,k_s)\backslash(-k_p,k_p)} \frac{\mu_s^3(\xi^2-k_p^2)(k_s^2-\xi^2)^2}{|\alpha|^2}d\xi > 0$$

For $z\neq y$, we denot $y-z=|y-z|(\cos\phi,\sin\phi)^T$ for some $0\le\phi\le 2\pi$. Then it is easy to see that

$$F^{pp}(z,y) = \frac{1}{\mu}\int_0^\pi A(\theta,\kappa)e^{\mathbf{i}k_s|z-y|\cos(\theta-\phi)}$$

The phase function $f(\theta)=\cos(\theta-\phi)$ satisfies $f'(\theta)=-\sin(\theta-\phi), f''(\theta)=-\cos(\theta-\phi)$. For any given $0\le\phi\le 2\pi$, we can decompose $[0,\pi]$ into several intervals such that in each either $|f''(\theta)|\ge 1/2$ or $|f'(\theta)|\ge 1/2$ and $f'(\theta)$ is monotonous. The amplitude function $A(\theta,\kappa)$ and their derirates are integrable on $[0,\pi]$. Then the estimate for $F_{pp}(z,y)$ follows by using lemma 2.4. The estimation of $F^{ss}(z,y)$ can be proved similarly. This completes the proof. □

To understand the behavior of the imaging fuction when z is close to the boundary of the scatterer, we extend the concept of the scattering coefficient fo incident plane waves[14].

scarr_con **Definition 4.1** *For any unit vector $d \in \mathbb{R}^2$, let $u_p^i = de^{\mathbf{i}k_p x\cdot d}$ or $u_s^i = d^{\perp}e^{\mathbf{i}k_s x\cdot d}$ be the incident wave and $u_\alpha^s = u_\alpha^s(x;d)$ be the corresponding radiation solution of the Navier equation:*

$$u_\alpha^s + \omega^2 u_\alpha^s = 0 \quad in \quad \mathbb{R}^2\backslash\bar{D} \tag{4.10}$$

$$u_\alpha^s = -u_\alpha^i \quad on \quad \partial D \tag{4.11}$$

The scattering coecient R(x;d) for $x \in \partial D$ is defined by the relation

$$\sigma(u_\alpha^s + u_\alpha^i)\cdot\nu = \mathbf{i}k_\alpha R_\alpha(x;d)e^{\mathbf{i}k_\alpha x\cdot d} \quad on \quad \partial D$$

where $\alpha = p, s$, $d = (d^1, d^2)^T$ is unit vectors, $d^\perp = (d^2, -d^1)^T$.

In the case of high frequency approximation, the scattering coecient can be approximated by

$$R_\alpha(x;d) = \begin{cases} RF_\alpha(d;\nu(x)) & \text{if } \ x \in \partial D_d^- = \{x \in \partial D, \nu(x)\cdot d < 0\}, \\ 0 & \text{if } \ x \in \partial D_d^+ = \{x \in \partial D, \nu(x)\cdot d \geq 0\}. \end{cases}$$

Now we consider the physical interpretation of the imaging function $\hat{I}_d(z)$ when $z \in \Gamma_D$. By the same procedure which used in RTMhalf_aco [14, p11,12], we can carry out that

$$\hat{I}_d(z) \approx \sqrt{8\pi k_p}\mathrm{Im}\,\mathbf{tr}\int_0^\pi ((\lambda+2\mu)A_p(\theta)\eta_\theta e^{\mathbf{i}k_p(y_-(\eta_\theta)-z)\cdot\eta_\theta - \mathbf{i}\frac{\pi}{4}})^T \frac{\overline{F(z,y_-(\eta_\theta))}}{\sqrt{\vartheta(y_-(\eta_\theta))}}d\theta$$
$$+ \sqrt{8\pi k_s}\mathrm{Im}\,\mathbf{tr}\int_0^\pi (\mu A_s(\theta)\eta_\theta^\perp e^{\mathbf{i}k_s(y_-(\eta_\theta)-z)\cdot\eta_\theta - \mathbf{i}\frac{\pi}{4}})^T \frac{\overline{F(z,y_-(\eta_\theta))}}{\sqrt{\vartheta(y_-(\eta_\theta))}}d\theta$$

where $\nu(y_-(\eta_\theta)) = -\eta_\theta$ and $\vartheta(y)$ is the curvature of Γ_D,

$$A_p(\theta) = \frac{k_s^2 k_p^3 \mu_s(k_p\cos\theta)\sin\theta}{2\pi\mu\gamma(k_p\cos\theta)\delta(k_p\cos\theta)}(\cos\theta, \sin\theta)$$

and

$$A_s(\theta) = \begin{cases} \frac{k_s^5\mu_p(k_s\cos\theta)\sin\theta}{2\pi\mu\gamma(k_s\cos\theta)\delta(k_s\cos\theta)}(\sin\theta, -\cos\theta) & \theta \in (\arccos(k_p), \arccos(-k_p)) \\ \frac{k_s^5(1-4\cos^2\theta)\mu_p(k_s\cos\theta)\sin\theta}{2\pi\gamma(k_s\cos\theta)\delta(k_s\cos\theta)}(\sin\theta, -\cos\theta) & \theta \in (0,\pi)\backslash[\arccos(k_p), \arccos(-k_p)] \end{cases}$$

By above formula and lemma festimate1 3.1, we can explain that one cannot image the back part of the obstacle with only the data collected on Γ_0. This confirmed in our numerical examples.

5. Extensions

In this section we consider the reconstruction of non-penetrable obstacles with the impedance boundary condition and penetrable obstacle in the half space by the RTM algorithm alg_rtm 3.1. For non-penetrable obstacles with the impedance boundary condition on the obstacle, the measured data $u_k^s(x_r.x_s) = u_k(x_r, x_s) - N(x_r, x_s)e_k$, $k = 1, 2$, where $u_k^s(x_r.x_s)$ is the radiation solution of the following problem:

$$\Delta_e u_k^s(x) + \omega^2 u_k^s(x) = 0 \qquad \text{in } \mathbb{R}_+^2\backslash\bar{D}, \tag{5.1}$$ elas_4

$$(T_x^\nu + \mathbf{i}\eta(x))u_k^s(x) = -(T_x^\nu + \mathbf{i}\eta(x))N(x_r, x_s)e_k \quad \text{on } \Gamma_D, \tag{5.2}$$ elas_bd2

$$\sigma(u_k^s)e_2 = 0 \quad \text{on} \quad \Gamma_0, \tag{5.3}$$ elas_b02

By modifying the argument in Theorem 5.2, we can show the following result whose proof is omitted.

Theorem 5.1 *For any $z \in \Omega$, let $\Phi(y,z) \in \mathbb{C}^{2\times 2}$ be the radiation solution of the problem:*

$$\begin{aligned} &\Delta_e \Phi(y,z) + \omega^2 \Phi = 0 \quad \text{in } \mathbb{R}^2 \backslash \bar{D} \\ &T_y^\nu \Phi(y,z) + \mathbf{i}\eta(y)\Phi(y,z) = -(T_y^\nu + \mathbf{i}\eta(y))\overline{F(z,y)} \qquad \text{on } \Gamma_D \end{aligned}$$

Then, we have

$$\hat{I}_d(z) = -\mathrm{Im}\,\mathbf{tr}\int_{\Gamma_D} (\overline{F(z,y)} + \Phi(y,z))^T((T_y^\nu + \mathbf{i}\eta(y))\overline{F(z,y)})ds(y) + W_{\hat{I}}(z) \tag{5.4}$$

where $|W_{\hat{I}}(z)| \le C(1+k_s d_D)^4(\epsilon_1(k_s h) + \epsilon_2(k_s h, h/d))$ uniformly for z in Ω, $m(h/d) > (1+\kappa)^2/4$, $M(h/d) < \kappa^2/4$.

For the penetrable obstacle , the measured data $u_k^s(x_r.x_s) = u_k(x_r,x_s) - N(x_r,x_s)e_k$, $k = 1,2$, where $u_k^s(x_r.x_s)$ is the radiation solution of the following problem:

$$\Delta_e u_k^s(x) + \omega^2 n(x) u_k^s(x) = -\omega^2(n(x)-1)N(x,x_s)e_k \qquad \text{in } \mathbb{R}^2 \tag{5.5}$$

$$\sigma(u_k^s)e_2 = 0 \quad \text{on} \quad \Gamma_0, \tag{5.6}$$

where $n(x) \in L^\infty(\mathbb{R}^2)$ is a positive function which is equel to 1 outside D. By modifying the argument in Theorem 5.2, the following theorem can be proved.

Theorem 5.2 *For any $z \in \Omega$, let $\Phi(y,z) \in \mathbb{C}^{2\times 2}$ be the radiation solution of the problem:*

$$\Delta_e \Phi(y,z) + \omega^2 n(y)\Phi = -\omega^2(n(y)-1)\overline{F(z,y)} \quad \text{in } \mathbb{R}^2$$

Then, we have

$$\hat{I}_d(z) = -\mathrm{Im}\,\mathbf{tr}\int_D \omega^2(1-n(y))(\overline{F(z,y)} + \Phi(y,z))^T\overline{F(z,y)})dy + W_{\hat{I}}(z) \tag{5.7}$$

where $|W_{\hat{I}}(z)| \le C(1+k_s d_D)^4(\epsilon_1(k_s h) + \epsilon_2(k_s h, h/d))$ uniformly for z in Ω, $m(h/d) > (1+\kappa)^2/4$, $M(h/d) < \kappa^2/4$.

6. Numerical experiments

In this section we present several numerical examples to show the effectiveness of our RTM method. To synthesize the scattering data we compute the solution $u^s(x_r;x_s)$ of the scattering problem by representing the ansatz solution as the single layer potential with the Green function $N(x;y)$ as the kernel and discretizing the integral equation by standard *Nyström* methods [17]. The boundary integral equations on Γ_D are solved on a uniform mesh over the boundary with ten points per probe wavelength. The sources and receivers are both placed on the surface Γ_d^0 with equal-distribution, where d is the aperture. In all our numerical examples we choose h = 10, d = 50 and *Lamé* constant

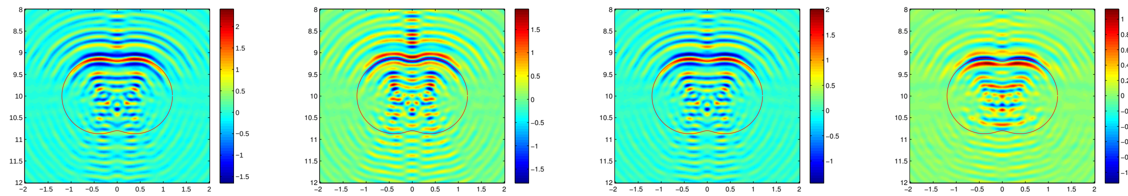

Figure 2. Example 1: From left to right: imaging results of a Dirichlet, a Neumann, a Robin bounday with impedance $\eta(x) = 1$, and a penetrable obstacle with diffractive index $n(x) = 0.25$

$\lambda = 1/2$, $\mu = 1/4$. The boundaries of the obstacles used in our numerical experiments are parameterized as follows,

$$\begin{aligned}
&\text{Circle:} && x_1 = \rho\cos(\theta), \quad x_2 = \rho\sin(\theta),\\
&\text{Kite:} && x_1 = \cos(\theta) + 0.65\cos(2\theta) - 0.65, \quad x_2 = 1.5\sin(\theta),\\
&p\text{-leaf:} && r(\theta) = 1 + 0.2\cos(p\theta),\\
&\text{peanut:} && x1 = \cos\theta + 0:2\cos 3\theta; x2 = \sin\theta + 0:2\sin 3\theta,\\
&\text{square:} && x1 = \cos 3\theta + \cos\theta; x2 = \sin 3\theta + \sin\theta.
\end{aligned}$$

where $\theta \in [0, 2\pi]$.

Example 1. We consider imaging of a Dirichlet, a Neumann, a Robin bounday, and a penetrable obstacle. The imaging domain is $(2; 2) \times (8; 12)$ with the sampling grid 201×201 and $N_s = N_r = 401$. The angular frequency is $\omega = 2\pi$.

The imaging results are shown in Figure 2. It demonstrates clearly that our RTM algorithm can effectively image the upper boundary illuminated by the sources and receivers distributed along the boundary Γ_0 for non-penetrable obstacles. The imaging values decrease on the shadow part of the obstacles and at the points away from the boundary of the obstacle.

Example 2. We consider the imaging of clamped obstacles with different shapes including circle, peanut, p-leaf and rounded square. The imaging domain is $= (2; 2) \times (8; 12)$ with the sampling grid 201×201 and $N_s = N_r = 401$. The angular frequency is $\omega = 3\pi, 4\pi$ for the sigle frequency and $\omega = \pi \times [2 : 0.5 : 8]$ for the test of multiple frequencies.

Example 3 We consider the imaging of two sound soft obstacles. The first model consists of two circles along horizontal direction and the second one is a circle and a peanut along the vertical direction. The angular frequency is $\omega = 3\pi$ for the test of the single frequency and $\omega = \pi \times [2 : 0.5 : 8]$ for the test of multiple frequencies. Figure 4 shows the imaging result of the first model. The imaging domain is $[4, 4] \times [8, 12]$ with mesh size 401×201 and $N_s = N_r = 301$. Figure 5 shows the imaging result of the second model. The imaging domain is $[4, 4] \times [8, 12]$ with mesh size 401×401 and $N_s = N_r = 301$. The multi-frequency RTM imaging results in Figure 4 and Figure 5 are

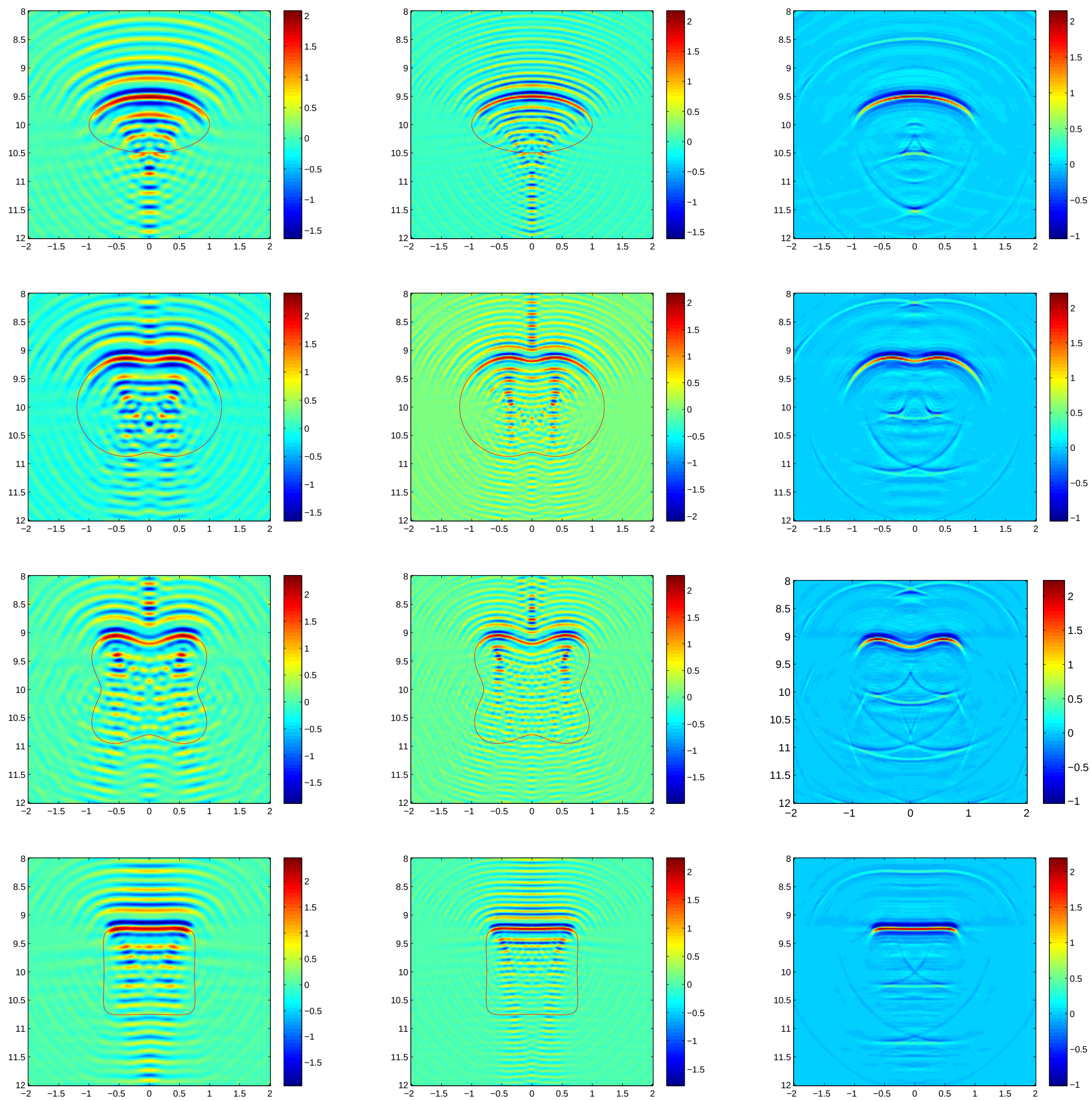

Figure 3. Example 2: Imaging results of clamped obstacles with different shapes from top to below. The left row is imaged with single frequency data where $\omega = 3\pi$, The middle row is imaged with single frequency data where $\omega = 5\pi$ and The left row is imaged with multi frequency data

figure_2

obtained by adding the inmaging results from different frequencies. We observe from these two figures that imaging results can be greatly improved by stacking the multiple single frequency imaging results.

Example 4 In this example we consider the stability of our half space RTM imaging function with respect to the complex additive Gaussian random noise. We introduce the additive Gaussian noise as follows

$$u_{\text{noise}} = u_s + \nu_{\text{noise}}$$

where u_s is the synthesized data and ν_{noise} is the Gaussian noise with mean zero and

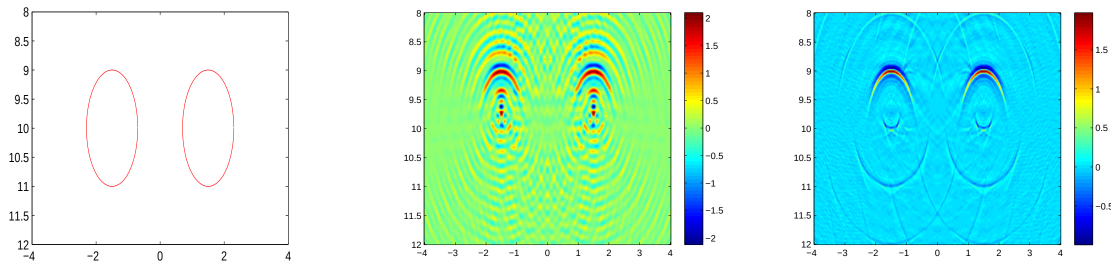

Figure 4. Example 3:From left to right, true obstacle model with two circles. the imaging result with single frequency data where $\omega = 3\pi$, the imaging result with multiple frequency data.

figure_31

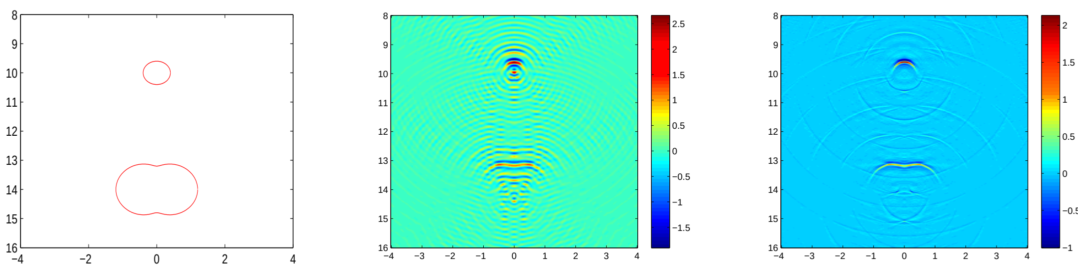

Figure 5. Example 3:From left to right, true obstacle model with one circle and one peanut, the imaging result with single frequency data where $\omega = 3\pi$, the imaging result with multiple frequency data.

figure_32

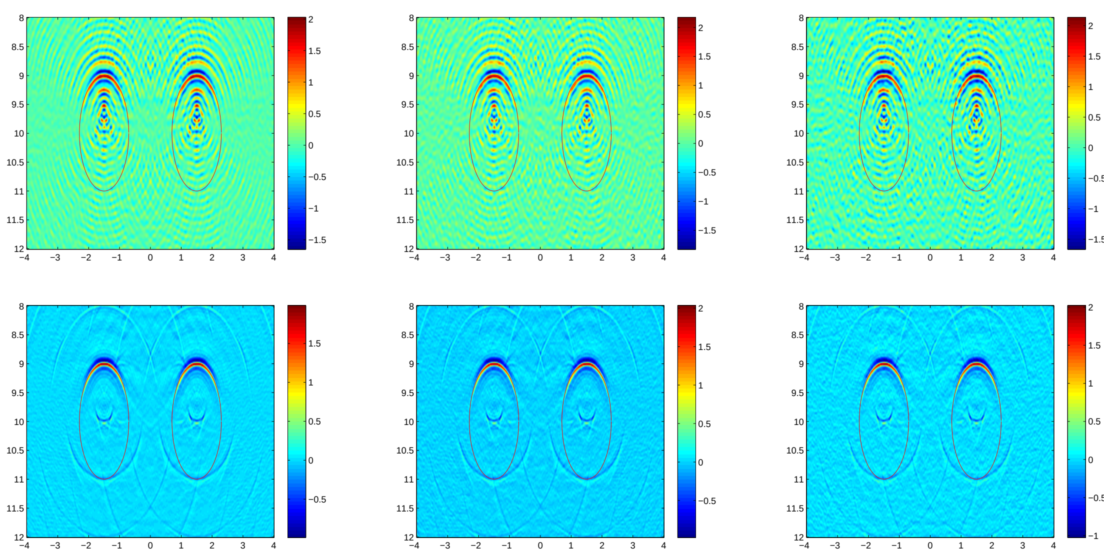

Figure 6. Example 4: Imaging results of a clamped obstacle with noise levels $\mu = 0.2; 0.3; 0.4$ (from left to right). The top row is imaged with single frequency data where $\omega = 4\pi$, and the bottom row is imaged with multi-frequency data.

figure_4

standard deviation μ times the maximum of the data $|u_s|$, i.e. $\nu_{\text{noise}} = \frac{\mu \max |u_s|}{\sqrt{2}}(\varepsilon_1 + \mathbf{i}\varepsilon_2)$ and $\varepsilon_i \sim \mathcal{N}(0,1)$.

Figure 6 shows the imaging results using single frequency data added with additive Gaussian noise. The imaging quality can be improved by using multi-frequency data. as illustrated

7. Appendix A: proof of theorem 2.1

In this section we will prove theorem 2.1 by extending the classical argument in [24, 27, 19]. The existence of the solution can be proved by the method of limiting absorption principle. The argument is standard and we give several lemmas below, see e.g. [24] for the consideration for Helmholtz equation. For any $z = 1 + \mathbf{i}\varepsilon, \varepsilon > 0$, $f \in H^1(\mathbb{R}^2_+)'$ with compact support in $B_R = \{x||x|^2 < R^2, x \in \mathbb{R}^2_+\} \subsetneq \mathbb{R}^2_+$ where B_R is an half disc of radius R , we consider the problem

$$\Delta_e u_z + z\omega^2 u = -f \qquad \text{in } \mathbb{R}^2_+ \tag{7.1}$$

$$\sigma(u_z)e_2 = 0 \quad \text{ on } \quad \Gamma_0 \tag{7.2}$$

By Lax-Milgrim lemma we know that (7.1-7.2) has a unique solution $u_z \in H^1(\mathbb{R}^2_+)$. For any domain $D \subset \mathbb{R}^2_+$, we define the weighted space $L^{2,s}(D), s \in \mathbb{R}$, by

$$L^{2,s}(D) = \{v \in L^2_{\text{loc}}(D) : (1 + |x|^2)^{s/2} v \in L^2(D)\}$$

with the norm $\|v\|_{L^{2,s}(\mathcal{D})} = (\int_{\mathcal{D}} (1 + |x|^2)^s |v|^2 dx)^{1/2}$. The weighted Sobolev space $H^{1,s}(D), s \in \mathbb{R}$, is defined as the set of functions in $L^{2,s}(D)$ whose first derivative is also in $L^{2,s}(D)$. The norm $\|v\|_{H^{1,s}(D)} = (\|v\|^2_{L^{2,s}(D)} + \|\nabla v\|^2_{L^{2,s}(D)})^{1/2}$.

We need the following sligt generalization of Rellich Theorem:

Lemma 7.1 *Let Ω be an open Lipschitz domain, then the sobolev space $H^{1,-s}(\Omega)$ is compactly embeded in $L^{2,-s'}(\Omega)$ for every $s' > s > 0$.*

Lemma 7.2 *Let $f \in L^2(\mathbb{R}^2_+)$ with compact support in B_R. For any $z = 1+\mathbf{i}\varepsilon$, $0 < \varepsilon < 1$, we have, for any $s > 1/2$, $\|u_z\|_{H^{1,-s}(\mathbb{R}^2_+)} \le C\|f\|_{L^2(\mathbb{R}^2_+)}$ for some constant independent of ε, u_z, and f.*

Proof. Let R_z denote the map from $L^2_c(\mathbb{R}^2_+)$ to $H^{1,-s}(\mathbb{R}^2_+)$ such that $R_z(f) = u_z$ where $L^2_c(\mathbb{R}^2_+)$ is denoted by all $f \in L^2(\mathbb{R}^2_+)$ with compact support in B_R, then it is easy to see that R_z is a linear bounded operator. It follows from theorem 3.7 in [19] that R_z is a uniformly continuous operator continues valued function on $z = 1 + \mathbf{i}\varepsilon$, $0 < \varepsilon < 1$ with value in $B(L^2_c(\mathbb{R}^2_+), H^{1,-s}(\mathbb{R}^2_+))$. Then, we can obtain that R_z is uniformly bounded in $B(L^2_c(\mathbb{R}^2_+), H^{1,-s}(\mathbb{R}^2_+))$. This complete the proof by the defintion of the operator norm. $\square$

We next recall the following lemma which states the absence of positive eigenvalues for the linear elasticity system in half space [26].

Lemma 7.3 *Let $u \in L^2(\mathbb{R}^2_+ \backslash \bar{D})$ such that u satisfies (2.45) and (2.47), than we assert that $u = 0$ in $\mathbb{R}^2_+ \backslash \bar{D}$*

Proof. The asserting above can be proved by extending [26, theorem 3.1], here we omit the details. $\square$

For any $0 < \varepsilon < 1$, we consider the problem

$$\Delta_e u_\varepsilon + (1 + \mathbf{i}\varepsilon)\omega^2 u_\varepsilon = 0 \qquad \text{in } \mathbb{R}^2_+ \backslash \bar{D} \tag{7.3}$$

$$u_\varepsilon = g \quad \text{ on } \Gamma_D \tag{7.4}$$

$$\sigma(u_\varepsilon)e_2 = 0 \quad \text{ on}\Gamma_0 \tag{7.5}$$

We know that the above problem has a unique solution $u_\varepsilon \in H^1(\mathbb{R}^2_+ \backslash \bar{D})$ by the Lax-Milgram Lemma. Thus, we have next lemma

Lemma 7.4 *Let $g \in H^{1/2}(\Gamma_D)$. For any $0 < \varepsilon < 1$, we have, for any $s > 1/2$, $\|u_\varepsilon\|_{H^{1,-s}(\mathbb{R}^2_+ \backslash \bar{D})} \leq C\|g\|_{H^{1/2}(\Gamma_D)}$ for some constant independent of $\varepsilon, u_\varepsilon$, and g.*

Proof. Because $h = dist(D, \Gamma_0) > 0$, we can find three concentric circles $B_{R_1}, B_{R_2}, B_{R_3}$ such that $D \subsetneq B_{R_1} \subsetneq B_{R_2} \subsetneq B_{R_3} \subsetneq \mathbb{R}^2_+$. Let $\chi \in C_0^\infty(\mathbb{R}^2_+)$ be the cut-off function such that $0 \leq \chi \leq 1$, $\chi = 0$ in B_{R_1}, and $\chi = 1$ outside of B_{R_2}. Let $v_\varepsilon = \chi u_\varepsilon$. Then v_ε satisfies (7.1) with $z = 1 + \mathbf{i}\varepsilon$ and $q = \sigma(u_\varepsilon)\nabla\chi + (\lambda+\mu)(\nabla^2\chi u_\varepsilon + \nabla u_\varepsilon \nabla\chi) + \mu\Delta\chi u_\varepsilon + \mu \mathrm{div} u_\varepsilon \nabla\chi$, where $\nabla^2\chi$ is the Hessian matrix of χ. Clearly q has compact support. By lemma 7.2 we can obtain

$$\|v_\varepsilon\|_{H^{1,-s}(\mathbb{R}^2_+)} \leq C\|u_\varepsilon\|_{H^1(B_{R_2}\backslash \bar{D})} \tag{7.6}$$

for some constant C independent of $\varepsilon > 0$. Now let $\chi_1 \in C_0^\infty(\mathbb{R}^2_+)$ be the cut-off function with that $0 \leq \chi_1 \leq 1$, $\chi_1 = 1$ in B_{R_2}, and $\chi_1 = 0$ outside of B_{R_3}. For $g \in H^{1/2}(\Gamma_D)$, let $u_g \in H^1(\mathbb{R}^2_+ \backslash \bar{D})$ be the lifting function such that $u_g = g$ on Γ_D and $\|u_g\|_{H^1(\mathbb{R}^2_+ \backslash \bar{D})} \leq C\|g\|_{H^{1/2}(\Gamma_D)}$. By testing 7.3 with $\chi_1^2(\overline{u_\varepsilon - u_g})$ and using the standard argument we have

$$\|u_\varepsilon\|_{H^1(B_{R_2}\backslash \bar{D})} \leq C(\|u_\varepsilon\|_{L^2(B_{R_3}\backslash \bar{D})} + \|g\|_{H^{1/2}(\Gamma_D)}). \tag{7.7}$$

A combination of (7.6) and the above estimate yields

$$\|u_\varepsilon\|_{H^{1,-s}(\mathbb{R}^2_+\backslash \bar{D})} \leq C(\|u_\varepsilon\|_{L^2(B_{R_3}\backslash \bar{D})} + \|g\|_{H^{1/2}(\Gamma_D)}). \tag{7.8}$$

Now we claim

$$\|u_\varepsilon\|_{L^2(B_{R_3}\backslash \bar{D})} \leq C\|g\|_{H^{1/2}(\Gamma_D)}, \tag{7.9}$$

for any $g \in H^{1/2}(\Gamma_D)$ and $\varepsilon > 0$. If it were false, there would exist sequences $\{g_m\} \subset H^{1/2}(\Gamma_D)$ and $\{\varepsilon_m\} \subset (0, 1)$, and $\{u_{\varepsilon_m}\}$ be the corresponding solution of (7.3)-(7.5) such that

$$\|u_{\varepsilon_m}\|_{L^2(B_{R_3}\backslash \bar{D})} = 1 \text{ and } \|g_m\|_{H^{-1/2}(\Gamma_D)} \leq \frac{1}{m}. \tag{7.10}$$

Then $\|u_{\varepsilon_m}\|_{H^{1,-s}(\mathbb{R}^2_+\backslash \bar{D})} \leq C$, and thus there is a subsequence of $\{\varepsilon_m\}$, which is still denoted by $\{\varepsilon_m\}$, such that $\varepsilon_m \to \varepsilon' \in [0, 1]$, and a subsequence of $\{u_{\varepsilon_m}\}$, which is still denoted by $\{u_{\varepsilon_m}\}$, such that it converges to some $u_{\varepsilon'}$ in $H^{1,-s'}(\mathbb{R}^2_+\backslash \bar{D})$ by choosing $s' > s$. This is a consequence of Korn's inequality and lemma 7.1. So $u_{\varepsilon'} \in H^{1,-s'}(\mathbb{R}^2_+\backslash \bar{D})$ satisfies (7.3-7.5) with $g = 0$ and $\varepsilon = \varepsilon'$.

By the integral representation satistied by u_{ε_m}, we know that for $y \in \mathbb{R}^2_+ \backslash \bar{B}_{R_1}$ and $i = 1, 2$

$$u_{\varepsilon'}(y) \cdot e^i = \int_{\partial B_{R_1}} (\sigma(N_{\varepsilon'}(x,y)e_i)\nu) \cdot u_{\varepsilon'}(x) - (N_{\varepsilon'}(x,y)e_i) \cdot (\sigma(u_{\varepsilon'})_{\varepsilon'}\nu) ds(x) \qquad (7.11)$$ green_rep

If $\varepsilon' > 0$, we deduce from (7.11) that $u_{\varepsilon'}$ decays exponentially and thus $u_{\varepsilon'} \in H^1(\mathbb{R}^2_+ \backslash \bar{D})$, then $u_{\varepsilon'} = 0$ by the uniqueness of the solution in $H^1(\mathbb{R}^2_+ \backslash \bar{D})$ with positive absorption. If $\varepsilon' = 0$, by the [19, theorem 5.2], we have $u_{\varepsilon'} \in L^2(\mathbb{R}^2_+ \backslash \bar{D})$. Then we conclude $u_{\varepsilon'} = 0$ by the lemma 7.3 Therefore, in any case $u_{\epsilon'} = 0$, which , however contradicts to 7.10. This complete the proof. □

Now we are in the position to prove the exsitence of Theorem 2.1.

elas_exis **Lemma 7.5** *For any $s > 1/2$, $u_\varepsilon : (0,1) \to H^{1,-s}(\mathbb{R}^2_+ \backslash \bar{D})$ is a uniformly continuous operator valued function. Immediately, u_ε converges to some u_0 in $H^{1,-s}(\mathbb{R}^2_+ \backslash \bar{D})$ and u_0 is a solution of (2.45-2.49).*

Proof. We also give a indirect prove here. Let $\delta_0 > 0$ and $\{\mu_n\}$ and $\{\nu_n\}$ be sequences in $(0,1)$ such that

$$|\mu_n - \nu_n| \leq 1/n \qquad \text{and} \qquad \|u_{\mu_n} - u_{\nu_n}\|_{H^{1,-s}(\mathbb{R}^2_+ \backslash \bar{D})} \geq \delta_0 \qquad (7.12)$$

Thus there is a subsequence of $\{\mu_n\}$, which is still denoted by $\{\mu_n\}$, such that $\{\mu_n\} \to \epsilon \in [0,1]$ and also $\{\nu_n\} \to \epsilon$. Then using lemma 7.4 and the procedure proving it, we get the $u_\epsilon, v_\epsilon \in H^{1,-s'}(\mathbb{R}^2_+ \backslash \bar{D})$, by choosing $s' > s$, such that

$$\|u_{\mu_n} - u_\epsilon\|_{H^{1,-s'}(\mathbb{R}^2_+ \backslash \bar{D})} \to 0$$
$$\|u_{\nu_n} - v_\epsilon\|_{H^{1,-s'}(\mathbb{R}^2_+ \backslash \bar{D})} \to 0$$

and $u_\epsilon = v_\epsilon$ by the same arguement in lemma 7.4 which leads to a contradiction. Thus we have proved u_ε is uniformly continuously for $\varepsilon \in (0,1)$. Then it is easy to see u_ε has a limitation in $H^{1,-s}(\mathbb{R}^2_+ \backslash \bar{D})$ and the estimation of u_0 can be obtained by (7.9). This completes the proof. □

It is remain to prove the uniqueness in theorem 2.1. Actually, it can be obtained following the existence of solution with any $g \in H^{1/2}(\Gamma_D)$.

proof of Theorem 2.1 By the linearity of the problem, it is sufficient to prove that any u_0 satisfies the system (2.45-2.47) with the corresponding homogeneous boundary-value vanishes identically in $\mathbb{R}^2_+ \backslash \bar{D}$. For any $y \in \mathbb{R}^2_+ \backslash \bar{D}$, there exists $U^s(x,y)$ sataifies (2.45-2.47) with $g(x) = -N(x,y)$ on Γ_D following the lemma 7.5 and we define $U(x,y) = N(x,y) + U^s(x,y)$. It is easy to see that $U(x,y)$ satifies the generalized radiation condition (2.48). Thus by the integral representaion of u_0, we have

$$\lim_{r \to \infty} \int_{S_r^+} (\sigma(U(x,y)e_i)\nu) \cdot u_0(x) - (U(x,y)e_i) \cdot (\sigma(u_0)\nu) ds(x) = 0$$

Finally, combining $U(x,y) = 0, u_0(x) = 0$ on Γ_D and the Green integral theorem we find that

$$u_0(y)e_i = \int_{\mathbb{R}^2_+ \backslash \bar{D}} -(\Delta_e(N(x,y)e_i) + \omega^2 N(x,y)e_i) \cdot u_0(x) dx$$

$$= \int_{\mathbb{R}_+^2 \setminus \bar{D}} \Delta u_0(x) \cdot (N(x,y)e_i) - \Delta_e(N(x,y)e_i) \cdot u_0(x)$$

$$= \int_{\Gamma_D} (\sigma(U(x,y)e_i)\nu) \cdot u_0(x) - (U(x,y)e_i) \cdot (\sigma(u_0)\nu) ds(x) = 0$$

Then the desired unique exsitence follows lemma 7.5. This completes the proof of theorem 2.1. □

8. Appendix B: proof of theorem 3.3

Lemma 8.1 *For any $x, y \in D$, let*

$$p(x,y) = \lim_{\varepsilon \to 0^+} p^\varepsilon(x,y) := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \frac{f(\mu_p^\varepsilon, \mu_s^\varepsilon, \xi)}{\delta^\varepsilon(\xi)} e^{\mathbf{i}\mu_\alpha^\varepsilon x_2 + \mathbf{i}\mu_\beta^\varepsilon y_2 + \mathbf{i}\xi(y_1 - x_1)} d\xi$$

where $f(a,b,c)$ is a homogeneous fifth order polynomial with repect to a,b,c and $\alpha, \beta \in \{p, s\}$.Then there exists a constant $C > 0$ only dependant on κ such that

$$|p(x,y)| + k_s^{-1}|\nabla_x p(x,y)| + k_s^{-1}|\nabla_y p(x,y)| + k_s^{-2}|\nabla_x \nabla_y p(x,y)| \le C((k_s h)^{-1/2} + e^{-\sqrt{k_R^2 - k_s^2} h})$$

uniformly for $x, y \in D$.

Proof. Without loss of generality, we assume $k_\alpha \le k_\beta$. Then we can divide $p(x,y)$ into two parts:

$$\begin{aligned}
p(x,y) =& \lim_{\varepsilon \to 0^+} \int_{I_1} + \int_{I_2} \frac{f(\mu_p^\varepsilon, \mu_s^\varepsilon, \xi)}{(k_\alpha^\varepsilon)^2 \delta^\varepsilon(\xi)} e^{\mathbf{i}\mu_\alpha^\varepsilon x_2 + \mathbf{i}\mu_\beta^\varepsilon y_2 + \mathbf{i}\xi(y_1 - x_1)} d\xi \\
=& \int_{I_1} \frac{f(\mu_p, \mu_s, \xi)}{k_\alpha^2 \delta(\xi)} e^{\mathbf{i}\mu_\alpha x_2 + \mathbf{i}\mu_\beta y_2 + \mathbf{i}\xi(y_1 - x_1)} d\xi \\
&+ \lim_{\varepsilon \to 0^+} \int_{I_2} \frac{f(\mu_p^\varepsilon, \mu_s^\varepsilon, \xi)}{(k_\alpha^\varepsilon)^2 \delta^\varepsilon(\xi)} e^{\mathbf{i}\mu_\alpha^\varepsilon x_2 + \mathbf{i}\mu_\beta^\varepsilon y_2 + \mathbf{i}\xi(y_1 - x_1)} d\xi \\
=& p_1(x,y) + p_2(x,y)
\end{aligned}$$

where $I_1 = (-k_\alpha, k_\alpha)$ and $I_2 = R \setminus [-k_\alpha, k_\alpha]$. Substituting $\xi = k_\alpha t$ into $p_1(x,y)$, we get

$$p_1(x,y) = \int_{-1}^{1} \frac{f(\mu_p(k_\alpha t), \mu_s(k_\alpha t), k_\alpha t)}{k_\alpha \delta(k_\alpha t)} e^{\mathbf{i}k_\alpha x_2(\sqrt{1-t^2} + \tau\sqrt{\varsigma^2 - t^2} + \gamma t)} dt$$

where $\tau = y_2/x_2$, $\varsigma = k_\beta/k_\alpha$ and $\gamma = (y_1 - x_1)/x_2$. It is easy to see that the phase function $\phi(t) = \sqrt{1-t^2} + \tau\sqrt{\varsigma^2 - t^2} + \gamma t$ satifies $|\phi''(t)| \ge 1/(1-t^2)^{3/2} \ge 1$ for $t \in (-1, 1)$. Then we can obtain $|p_1(x,y)| \le C1/(k_s h)^{1/2}$ by lemma 2.4.

For $p_2(x,y)$, by changing the integration path and using same argument as in the lemma ??-??, we can easily obtain:

$$|p_2(x,y)| \le C(\frac{1}{k_s h} + e^{-\sqrt{k_R^2 - k_s^2} h})$$

This completes the proof of the esitmate for $|p(x,y)|$. The other estimates can be proved by a similar argument. We omit the details here. □

Now we are ready to prove Theorem 3.3.

proof of Theorem 3.3 Denote by u_1^ε, u_2^ε the corresponding solution of equations 3.14,3.17 where ω is substituted by $\omega(1+\mathbf{i}\varepsilon)$ for any $0<\varepsilon<1$. Let $w^\varepsilon(x)$ be the solution of the problem:

$$\Delta_e w^\varepsilon + (1+\mathbf{i}\varepsilon)\omega^2 w^\varepsilon = 0 \qquad \text{in } \mathbb{R}^2_+ \tag{8.1}$$

$$\sigma(w^\varepsilon)e_2 = -\sigma(u_2^\varepsilon)e_2 \quad \text{on}\Gamma_0 \tag{8.2}$$

Then $u_1^\varepsilon - u_2^\varepsilon - w^\varepsilon$ satisfies (7.3),(7.5) with the boundary condition $u_1^\varepsilon - u_2^\varepsilon - w^\varepsilon = -w^\varepsilon$ on Γ_D. Thus by the limiting absorption principle, lemma 2.46 and trace therorem, we have

$$\|T_x^\nu(u_1^\varepsilon - u_2^\varepsilon)\|_{H^{-1/2}(\Gamma_D)} \le C(\|w^\varepsilon\|_{H^{1/2}(\Gamma_D)} + |T_x^\nu(w^\varepsilon)\|_{H^{-1/2}(\Gamma_D)}) \tag{8.3}$$

$$\le C\max_{x\in D}(|w^\varepsilon(x)| + d_D|\nabla w^\varepsilon(x|) \tag{8.4}$$

where C is independant of ε, ω. By the integral representation formula we have for any $z\in\Gamma_0$

$$u_2^\varepsilon(z) = \int_{\Gamma_D} (T_y^\nu \Phi^\varepsilon(y,z))^T u_2^\varepsilon(y) - \Phi^\varepsilon(z,y)(T_y^\nu u_2^\varepsilon(y))ds(y) \tag{8.5}$$

which yields by using the integral representation again that for $x\in D$

$$w^\varepsilon(x) = \int_{\Gamma_0} N^\varepsilon(x,z)(T_z^{e_2} u_2^\varepsilon(z))ds(z) \tag{8.6}$$

$$= \int_{\Gamma_D} ds(y) \int_{\Gamma_0} N^\varepsilon(x,z)(T_z^{e_2}((T_y^\nu \Phi^\varepsilon(y,z))^T))ds(z) \tag{8.7}$$

$$- \int_{\Gamma_D} v^\varepsilon(x,y)(T_y^\nu u_2^\varepsilon(y))ds(y) \tag{8.8}$$

$$= \int_{\Gamma_D} ds(y) \int_{\Gamma_0} N^\varepsilon(x,z)(T_y^\nu (T_z^{e_2}\Phi^\varepsilon(z,y))^T)^T ds(z) \tag{8.9}$$

$$- \int_{\Gamma_D} v^\varepsilon(x,y)(T_y^\nu u_2^\varepsilon(y))ds(y) \tag{8.10}$$

$$= \int_{\Gamma_D} (T_y^\nu(v^\varepsilon(x,y))^T)^T u_2^\varepsilon(y) - v^\varepsilon(x,y)(T_y^\nu u_2^\varepsilon(y))ds(y) \tag{8.11}$$

where

$$v^\varepsilon(x,y) = \int_{\Gamma_0} N^\varepsilon(x,z)(T_z^{e_2}\Phi^\varepsilon(z,y))ds(z) \tag{8.12}$$

Since $\|T_x^\nu(u_2^\varepsilon)\|_{H^{-1/2}(\Gamma_D)} \le C\|g\|_{H^{1/2}(\Gamma_D)}$, we obtain

$$|w^\varepsilon(x)| \le C\|g\|_{H^{1/2}(\Gamma_D)} \max_{x\in D}(|v^\varepsilon(x,y)| + d_D|\nabla_y v^\varepsilon(x,y)|) \tag{8.13}$$

and

$$|\nabla w^\varepsilon(x)| \le C\|g\|_{H^{1/2}(\Gamma_D)} \max_{x\in D}(|\nabla_x v^\varepsilon(x,y)| + d_D|\nabla_x\nabla_y v^\varepsilon(x,y)|) \tag{8.14}$$

By (8.3) and letting $\varepsilon\to 0^+$, we have

$$\|T_x^\nu(u_1-u_2)\|_{H^{-1/2}(\Gamma_D)} \le C\|g\|_{H^{1/2}(\Gamma_D)} \max_{x\in D}\lim_{\varepsilon\to 0^+}(|v^\varepsilon(x,y)| \tag{8.15}$$

$$+d_D|\nabla_y v^\varepsilon(x,y)| + d_D|\nabla_x v^\varepsilon(x,y)| + d_D^2|\nabla_x\nabla_y v^\varepsilon(x,y)|) \tag{8.16}$$

Applying the Fourier transformation to the first horizontal variable of $T_z^{e_2}\Phi^\varepsilon(z,y)$, we have

$$\mathcal{F}[T_z^{e_2}\Phi^\varepsilon](\xi,0;y) = \frac{\mu}{2\omega^2}\left[\begin{pmatrix} 2\xi^2 & -2\xi\mu_p \\ -\frac{\beta\xi}{\mu_p} & \beta \end{pmatrix} e^{\mathbf{i}\mu_p y_2} + \begin{pmatrix} \beta & \frac{\xi\beta}{\mu_s} \\ 2\xi\mu_s & 2\xi^2 \end{pmatrix} e^{\mathbf{i}\mu_s y_2}\right] e^{-\mathbf{i}\xi y_1}$$

Using Parseval identity combined with above formula and formula 2.21, we have

$$\lim_{\varepsilon\to 0^+} v^\varepsilon(x,y) = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} \mathcal{F}[N^\varepsilon](\xi,0;x)^T \mathcal{F}[T_z^{e_2}\Phi^\varepsilon](-\xi,0;y)d\xi$$

This completes the proof by using lemma 8.1. □

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