

# Reverse Time Migration for Extended Obstacles in the Half Space: Elastic Waves

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**Abstract.** We consider a reverse time migration method for reconstructing extended obstacles in the half space with finite aperture data using elastic waves at a fixed frequency. We prove the resolution of the reconstruction method in terms of the aperture and the depth of the obstacle embedded in the half space. The resolution analysis studied by virtue of point spread function implies that the imaginary part of the cross-correlation imaging function always peaks on the boundary of the obstacle. Numerical experiments are included to illustrate the powerful imaging quality and to confirm our resolution results.

## 1. Introduction

section1

In this paper we study a reverse time migration (RTM) algorithm to find the support of an unknown obstacle in the half space from the measurement of scattered waves on the boundary of the half space which is far away from the obstacle. The physical properties of the obstacle such as penetrable or non-penetrable, and for non-penetrable obstacles, the type of boundary conditions on the boundary of the obstacle, are not required in the algorithm.

Let the non-penetrable obstacle occupy a bounded Lipschitz domain  $D \subset \mathbb{R}_+^2$  with  $\nu$  the unit outer normal to its boundary  $\Gamma_D$ . We assume the incident wave is emitted by a point source located at  $x_s$ , explosive along the polarization direction  $q \in \mathbb{R}^2$ , on the surface  $\Gamma_0 = \{(x_1, x_2)^T : x_1 \in \mathbb{R}, x_2 = 0\}$  which is far away from the obstacle. The measured data  $u_q$  corresponding to the polarization direction  $q$  is the solution of the following elastic scattering problem in the isotropic homogeneous medium half space with Lamé constant  $\lambda$  and  $\mu$  and constant density  $\rho \equiv 1$ :

$$\nabla \cdot \sigma(u_q) + \rho \omega^2 u_q = -\delta_{x_s}(x)q \quad \text{in } \mathbb{R}_+^2 \setminus \bar{D} \quad (1.1)$$

elastic\_eq

$$u_q = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \sigma(u_q) \cdot e_2 = 0 \quad \text{on } \Gamma_0 \quad (1.2)$$

together with the constitutive relation (Hookes law)

$$\begin{aligned} \sigma(u) &= 2\mu \varepsilon(u) + \lambda \operatorname{div} u \mathbb{I} \\ \varepsilon(u) &= \frac{1}{2}(\nabla u + (\nabla u)^T) \end{aligned}$$

where  $\omega$  is the circular frequency,  $u(x) \in \mathbb{C}^2$  denotes the displacement fields and  $\sigma(u)$  is the stress tensor. We also need to define the surface traction  $T_x^n(\cdot)$  on the normal direction  $n$ ,

$$T_x^n u(x) := \sigma \cdot n = 2\mu \frac{\partial u}{\partial n} + \lambda n \operatorname{div} u + \mu n \times \operatorname{curl} u$$

For simplicity, let's introduce Lamé operator  $\Delta_e$  as

$$\Delta_e u = (\lambda + 2\mu) \nabla \nabla \cdot u - \mu \nabla \times \nabla \times u = \nabla \cdot \sigma(u)$$

The equation (1.1) is understood as the limit when  $x_s \in \mathbb{R}_+^2 \setminus \bar{D}$  tends to  $\Gamma_0$  whose precise meaning will be given below after we introduce the Neumann Green Tensor and the definition of the radiation condition.

The reverse time migration (RTM) method, which consists of back-propagating the complex conjugated data into the background medium and computing the crosscorrelation between the incident wave field and the backpropagated field to output the final imaging profile, is nowadays widely used in exploration geophysics [5, 6, 7, 8, 15]. In [9, 10, 11], the RTM method for reconstructing extended targets using acoustic, electromagnetic and elastic waves at a fixed frequency in the free space is proposed and studied. The resolution analysis in [9, 10, 11] is achieved without using the small inclusion or geometrical optics assumption previously made in the literature (e.g. [3, 6]). In [12], a new RTM algorithm is developed for finding extended targets in a

planar waveguide which is motivated by the generalized Helmholtz-Kirchhoff identity for scattering problems in waveguides.

For the isotropic elastic media, one can process the elastic data either by separating P-wave and S-wave using Helmholtz decomposition and migrating each mode using methods based on acoustic wave theory [chung2012implementation,denli2008elastic](#) [14, 17], or by migrating the whole elastic data set based on full elastic wave equation in the geophysical exploration community. In this paper, we adopt the cross-correlation between all the component of the source and receiver displacement wavefield, which is a mixture of P-wave and S-wave. Furthermore this kind condition can be easily extended to inhomogeneous elastic medium and even anisotropic elastic wave imaging. The purpose of this paper is to provide a new mathematical understanding of the RTM method by extending [RTMhalf\\_aco](#) [13] where RTM method for extended targets in the half space using acoustic wave is considered. Compared to the scalar acoustic wave imaging, the vector elastic wave imaging is more complex due to a mixture of P-wave and S-wave mode. However, the virtue of the latter method is no longer need to separate the scalar and vector potentials prior to the imaging condition.

The layout of the paper is as follows. To Be Decided...

## 2. Green Tensor in the half space

In this section we will study the elastic Green Tensor in the half-space with Neumann boundary [nedelec2011](#) [19]:

$$\Delta_e \mathbb{N}(x; y) + \omega^2 \mathbb{N}(x, y) = -\delta_y(x) \mathbb{I} \quad \text{in } \mathbb{R}_+^2, \quad (2.1) \quad \text{eq\_n1}$$

$$\sigma_x(\mathbb{N}(x, y))e_2 = 0 \quad \text{on } x_2 = 0 \quad (2.2) \quad \text{eq\_n2}$$

and with Dirichlet Boundary [arens1999](#) [4]

$$\Delta_e \mathbb{D}(x, y) + \omega^2 \mathbb{D}(x, y) = -\delta_y(x) \mathbb{I} \quad \text{in } \mathbb{R}_+^2, \quad (2.3) \quad \text{eq\_d1}$$

$$\mathbb{D}(x, y) = 0 \quad \text{on } x_2 = 0 \quad (2.4) \quad \text{eq\_d2}$$

where  $\delta_y(x)$  is the Dirac source at  $y \in \mathbb{R}_+^2$  and  $\mathbb{N}(x, y)$ ,  $\mathbb{D}(x, y)$  are  $\mathbb{C}^{2 \times 2}$  matrixes. We will first use Fourier transform to derive the formula of Green Tensor in frequency domain. Let

$$\hat{\mathbb{N}}(\xi, x_2; y_2) = \int_{-\infty}^{+\infty} \mathbb{N}(x_1, x_2; y) e^{-i(x_1 - y_1)\xi} dx_1 \quad (2.5)$$

Throughout the paper, we will assume that for  $z \in \mathbb{C}$ ,  $z^{1/2}$  is the analytic branch of  $\sqrt{z}$  such that  $\text{Im}(z^{1/2}) \geq 0$ . This corresponds to the right half real axis as the branch cut in the complex plane. For  $z = z_1 + iz_2$ ,  $z_1, z_2 \in \mathbb{R}$ , we have

$$z^{1/2} = \text{sgn}(z_2) \sqrt{\frac{|z| + z_1}{2}} + i \sqrt{\frac{|z| - z_1}{2}} \quad (2.6) \quad \text{convention\_1}$$

For  $z$  on the right half real axis, we take  $z^{1/2}$  as the limit of  $(z + i\varepsilon)^{1/2}$  as  $\varepsilon \rightarrow 0^+$ . [kupradze1963progress](#) Let  $\mathbb{G}(x, y)$  be the fundamental solution of the elastic equation [22] and recall that

$$\hat{\mathbb{G}}(\xi, x_2; y_2) = \frac{i}{2\omega^2} \left[ \begin{pmatrix} \mu_s & -\xi \frac{x_2 - y_2}{|x_2 - y_2|} \\ -\xi \frac{x_2 - y_2}{|x_2 - y_2|} & \xi^2 \frac{1}{\mu_s} \end{pmatrix} e^{i\mu_s |x_2 - y_2|} + \begin{pmatrix} \xi^2 \frac{1}{\mu_p} & \xi \frac{x_2 - y_2}{|x_2 - y_2|} \\ \xi \frac{x_2 - y_2}{|x_2 - y_2|} & \mu_p \end{pmatrix} e^{i\mu_p |x_2 - y_2|} \right]$$

where

$$\mu_\alpha = (k_\alpha^2 - \xi^2)^{1/2} \quad \text{for } \alpha = s, p \quad (2.7)$$

By the standard argument in ODEs, the Green Tensor in half-space can be deduced as

$$\hat{N}(\xi, x_2; y_2) = \hat{G}(\xi, x_2; y_2) - \hat{G}(\xi, x_2; -y_2) + \hat{N}_c(\xi, x_2; y_2) \quad (2.8)$$

$$\begin{aligned} \hat{N}_c(\xi, x_2; y_2) = & \frac{\mathbf{i}}{\omega^2 \delta(\xi)} \left\{ A(\xi) e^{\mathbf{i}\mu_s(x_2+y_2)} + B(\xi) e^{\mathbf{i}\mu_p(x_2+y_2)} \right. \\ & \left. + C(\xi) e^{\mathbf{i}\mu_s x_2 + \mathbf{i}\mu_p y_2} + D(\xi) e^{\mathbf{i}\mu_p x_2 + \mathbf{i}\mu_s y_2} \right\} \end{aligned} \quad (2.9)$$

where

$$\begin{aligned} A(\xi) &= \begin{pmatrix} \mu_s \beta^2 & -4\xi^3 \mu_s \mu_p \\ -\xi \beta^2 & 4\xi^4 \mu_p \end{pmatrix} & B(\xi) &= \begin{pmatrix} 4\xi^4 \mu_s & \xi \beta^2 \\ 4\xi^3 \mu_s \mu_p & \mu_p \beta^2 \end{pmatrix} \\ C(\xi) &= \begin{pmatrix} 2\xi^2 \mu_s \beta & -2\xi \mu_s \mu_p \beta \\ -2\xi^3 \beta & 2\xi^2 \mu_p \beta \end{pmatrix} & D(\xi) &= \begin{pmatrix} 2\xi^2 \mu_s \beta & 2\xi^3 \beta \\ 2\xi \mu_s \mu_p \beta & 2\xi^2 \mu_p \beta \end{pmatrix} \end{aligned}$$

and  $\beta(\xi) = k_s^2 - 2\xi^2$ ,  $\delta(\xi) = \beta^2 + 4\xi^2 \mu_s \mu_p$ .

The desired Green function should be obtained by taking the inverse Fourier transform of  $\hat{N}(\xi, x_2; y_2)$ . Unfortunately, one cannot simply take the inverse Fourier transform in the above formula because  $\delta(\xi)$  have zero points in the real axis by lemma [root\\_De1](#) [1][21].

**Lemma 2.1** *Let Lamé constant  $\lambda, \mu \in \mathbb{R}^+$ , then the Rayleigh equation  $\delta(\xi) = 0$  has only two roots denoted by  $\pm k_R$  in complex plane. Moreover,  $k_R > k_s > k_p$ ,  $k_R \in \mathbb{R}$  and  $k_R$  is called Rayleigh wave number.*

**Proof.** For the sake of completeness, we include a proof here. It is well known that

$$\delta(\xi) = (k_s^2 - 2\xi^2)^2 + 4\xi^2 (k_s^2 - \xi^2)^{1/2} (k_p^2 - \xi^2)^{1/2} \quad (2.10)$$

However,  $\delta(\xi)$  is rendered single-valued by selecting branch cuts along  $k_p < \text{Re}(\xi) < k_s, \text{Im}(\xi) = 0$  which is consistent with the convention [convention\\_1](#) (2.6). A simple computation show that  $\delta(\pm k_s) > 0$  and  $\delta(\pm\infty + \mathbf{0i}) < 0$ . By the continuity of  $\delta(\xi)$ , we can obtain that it has at least two real zero points which denoted by  $\pm k_R$ .

Now it turn to proof that  $\delta(\xi)$  has only two roots in the complex plane by the principle of argument which follows as a theorem of the theory of complex variables [Ahlfors1979Complex](#) [2]. Now consider the contour  $C$  consisting of  $\Gamma$ , and  $C_l$  and  $C_r$  where  $C_r = [k_p + \mathbf{i}0^+, k_s + \mathbf{i}0^+] \cup [k_p + \mathbf{i}0^-, k_s + \mathbf{i}0^-]$  that surround  $[k_p, k_s]$ ,  $C_l = [-k_s + \mathbf{i}0^+, -k_p + \mathbf{i}0^+] \cup [-k_s + \mathbf{i}0^-, -k_p + \mathbf{i}0^-]$  that surround  $[-k_s, -k_p]$  and  $\Gamma$  denotes a circle with enough large radius. Since the function  $\delta(\xi)$  clearly does not have poles in the complex  $\xi$ -plane and we find that within the contour  $C = \Gamma \cup C_r \cup C_l$  the number of zeros is given by

$$Z = \frac{1}{2\pi\mathbf{i}} \int_C \frac{d\delta}{d\xi} \frac{d\xi}{\delta(\xi)} \quad (2.11) \quad \text{zero}$$

Since  $\delta(\xi) = \delta(-\xi)$  the images of  $C_r$  and  $C_l$  are the same, and one of them, say  $C_r$ , needs to be considered. We have  $\delta(k_p) = (k_s^2 - 2k_p^2)^2$  and along  $C_r$ :  $\delta^\pm(\xi) = (k_s^2 - \xi^2)^2 \mp i4\xi^2 \sqrt{k_s^2 - \xi^2} \sqrt{\xi^2 - k_p^2}$ , and  $\delta(k_s) = k_s^4$  where the plus sign applies above the cut, and the minus sign applies below the cut for  $\delta(\xi)$ . Let  $f_1(\xi) = (k_s^2 - \xi^2)^2$  and  $f_2(\xi) = 4\xi^2 \sqrt{k_s^2 - \xi^2} \sqrt{\xi^2 - k_p^2}$ . Then we have

$$\int_{C_r} \frac{d\delta}{d\xi} \frac{d\xi}{\delta(\xi)} = \int_{k_p}^{k_s} \frac{\delta'_+(\xi)}{\delta_+(\xi)} - \frac{\delta'_-(\xi)}{\delta_-(\xi)} d\xi \quad (2.12)$$

$$= 2i \int_{k_p}^{k_s} \text{Im} \frac{(f'_1(\xi) - i f'_2(\xi)) f_1(\xi) + i f_2(\xi)}{(f_1(\xi) - i f_2(\xi))(f_1(\xi) + i f_2(\xi))} d\xi \quad (2.13)$$

$$= 2i \int_{k_p}^{k_s} \frac{f'_1(\xi) f_2(\xi) - f_1(\xi) f'_2(\xi)}{f_1^2(\xi) + f_2^2(\xi)} d\xi \quad (2.14)$$

$$= -2i \arctan \frac{f_2(\xi)}{f_1(\xi)} \Big|_{k_p}^{k_s} = 0 \quad (2.15)$$

For  $|\xi|$  large, we find  $\delta(\xi) = A\xi^2 + O(1)$ , thus it is easy to see that

$$\int_{\Gamma} \frac{d\delta}{d\xi} \frac{d\xi}{\delta(\xi)} = 4\pi$$

Then we obtain  $Z = 2$ . This completes the proof.  $\square$

In order to overcome the ambiguity above, loss is assumed in the medium so that  $k_{\alpha,\varepsilon} := k_\alpha(1+i\varepsilon)$ . When  $\varepsilon > 0$ , the branch point of  $\mu_{\alpha,\varepsilon}$  are  $\pm k_{\alpha,\varepsilon}$  and the branch cut are denoted by the equation  $\xi_1 \xi_2 = k_{\alpha,\varepsilon}$ ,  $-k_{\alpha,\varepsilon} \leq \xi \leq k_{\alpha,\varepsilon}$ . In this case, the poles singularities are now located off the real axis and the Fourier inverse transform becomes meaningful. In order to express lemma [root\\_De2](#) 2.2 concisely, we define

$$\Omega := \{\xi \in \mathbb{C} \mid k_p \varepsilon < \xi_1 \xi_2 < k_s \varepsilon, \quad \xi_2 > \xi_1 \varepsilon\} \quad (2.16)$$

**Lemma 2.2** *If the elastic medium has loss that  $k_{\alpha,\varepsilon} := k_\alpha(1+i\varepsilon)$ ,  $0 < \varepsilon < 1$  for  $\alpha = p, s$ , we assert that  $\delta_\varepsilon(\xi) = 0$  has only two roots in domain  $\Omega^c \subset \mathbb{C}$  and exactly they are  $\pm k_{R,\varepsilon}$ .*

Denote  $B_\sigma^{+(-)}(a)$  the upper(lower) half part of the circle centered at  $a$  of the radius  $\sigma$  and let path  $L_\sigma = (-\infty, -k_R - \sigma) \cup B_\sigma^-( -k_R) \cup (-k_R + \sigma, k_R - \sigma) \cup B_\sigma^+(k_R) \cup (k_R + \sigma, \infty)$ . Using Cauchy integral theorem and lemma [root\\_De2](#) 2.2, we carry out:

$$\mathbb{N}(x, y) = \lim_{\varepsilon \rightarrow 0^+} \mathbb{N}_\varepsilon(x, y) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\mathbb{N}}_\varepsilon(\xi, x_2; y_2) e^{i(x_1 - y_1)\xi} d\xi \quad (2.17)$$

$$= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi} \int_{L_\sigma} \hat{\mathbb{N}}_\varepsilon(\xi, x_2; y_2) e^{i(x_1 - y_1)\xi} d\xi = \frac{1}{2\pi} \int_{L_\sigma} \hat{\mathbb{N}}(\xi, x_2; y_2) e^{i(x_1 - y_1)\xi} d\xi \quad (2.18)$$

$$= \lim_{\sigma \rightarrow 0^+} \frac{1}{2\pi} \int_{L_\sigma / B_\sigma^-( -k_R) \cup B_\sigma^+(k_R)} + \int_{B_\sigma^-( -k_R) \cup B_\sigma^+(k_R)} \hat{\mathbb{N}}(\xi, x_2; y_2) e^{i(x_1 - y_1)\xi} d\xi \quad (2.19)$$

$$= \frac{1}{2\pi} \text{P.V} \int_{\mathbb{R}} \hat{\mathbb{N}}(\xi, x_2; y_2) e^{i(x_1 - y_1)\xi} d\xi - \frac{i}{2} \frac{\mathbb{N}_\delta(-k_R)}{\delta'(-k_R)} e^{-i(x_1 - y_1)k_R} + \frac{i}{2} \frac{\mathbb{N}_\delta(k_R)}{\delta'(k_R)} e^{i(x_1 - y_1)k_R} \quad (2.20)$$

**Ngreen**

where  $\mathbb{N}_\delta(\xi) = \hat{\mathbb{N}}(\xi, x_2; y_2) \delta(\xi)$ .

Specially,  $N(x, y)$  has a simple form when  $x_2 = 0$  following an easy calculation:

$$\hat{\mathbb{N}}(\xi, 0; y_2) = \frac{\mathbf{i}}{\mu\delta(\xi)} \left[ \begin{pmatrix} 2\xi^2\mu_s & -2\xi\mu_s\mu_p \\ -\xi\beta & \mu_p\beta \end{pmatrix} e^{\mathbf{i}\mu_p y_2} + \begin{pmatrix} \mu_s\beta & \xi\beta \\ 2\xi\mu_s\mu_p & 2\xi^2\mu_p \end{pmatrix} e^{\mathbf{i}\mu_s y_2} \right] \quad (2.21) \quad \boxed{\text{ngreen}}$$

$$:= \frac{1}{\delta(\xi)} (\mathcal{N}_p(\xi) e^{\mathbf{i}\mu_p y_2} + \mathcal{N}_s(\xi) e^{\mathbf{i}\mu_s y_2}) \quad (2.22)$$

and let  $\mathbb{N}_f(x_1; y_1, y_2)$  denote the first part of  $\mathbb{N}$  and  $\mathbb{N}_r(x_1; y_1, y_2)$  denote the second part of  $\mathbb{N}$  in (2.20).  $\boxed{\text{ngreen}}$

It remains to study Dirichlet Green Tensor  $\mathbb{D}(x, y)$ . We still use Fourier transform to derive the formula of Green Tensor in frequency domain. Then we can obtain  $\mathbb{D}(x, y)$  similar to  $\mathbb{N}(x, y)$ . It follows an alternative representation for  $\mathbb{D}(x, y)$

$$\hat{\mathbb{D}}(\xi, x_2; y_2) = \hat{\mathbb{G}}(\xi, x_2; y_2) - \hat{\mathbb{G}}(\xi, x_2; -y_2) + \hat{M}(\xi, x_2; y_2) \quad (2.23)$$

$$\begin{aligned} \hat{M}(\xi, x_2; y_2) = \frac{\mathbf{i}}{\omega^2 \gamma(\xi)} & \left\{ A(\xi) e^{\mathbf{i}\mu_s(x_2+y_2)} + B(\xi) e^{\mathbf{i}\mu_p(x_2+y_2)} \right. \\ & \left. - A(\xi) e^{\mathbf{i}\mu_s x_2 + \mu_p y_2} - B(\xi) e^{\mathbf{i}\mu_p x_2 + \mu_s y_2} \right\} \end{aligned} \quad (2.24)$$

where

$$A(\xi) = \begin{pmatrix} \xi^2\mu_s & -\xi\mu_s\mu_p \\ -\xi^3 & \xi^2\mu_p \end{pmatrix} \quad B(\xi) = \begin{pmatrix} \xi^2\mu_s & \xi^3 \\ \xi\mu_s\mu_p & \xi^2\mu_p \end{pmatrix}$$

and  $\gamma(\xi) = \xi^2 + \mu_s\mu_p$ .

$\boxed{\text{root\_Ga}}$

**Lemma 2.3** *Let Lamé constant  $\lambda, \mu \in \mathbb{C}$  and  $\text{Im}(k_s) \geq 0, \text{Im}(k_p) \geq 0$ , then equation  $\gamma(\xi) = 0$  has no root in complex plane.*

**Proof.** Let  $F(\xi) = \gamma(\xi) * (\xi^2 - \mu_s\mu_p)$  and it is easy to see that the root of  $\gamma(\xi) = 0$  is also of  $F(\xi) = 0$ . A simple computation show that  $F(\xi) = (k_s^2 + k_p^2)\xi^2 - k_p^2 k_s^2$ . However, only when  $\xi^2 = k_p^2 k_s^2 / (k_s^2 + k_p^2)$ ,  $F(\xi) = 0$  but  $\gamma(\xi) = 2k_p^2 k_s^2 / (k_s^2 + k_p^2)$ . This completes the proof.  $\square$

Thus, we get the representation of Green Tensor by inverse Fourier transform

$$\mathbb{D}(x, y) = \mathbb{G}(x, y) - \mathbb{G}(x, y') + \frac{1}{2\pi} \int_{\mathbb{R}} \hat{M}(\xi, x_2; y_2) e^{\mathbf{i}(x_1 - y_1)\xi} d\xi \quad (2.25)$$

Let  $\mathbb{T}_D(x, y)$  denote the traction of  $\mathbb{D}(x, y)$  in direction  $e_2$  with respect to  $x$  such that  $\mathbb{T}_D(x, y)e_i = \sigma_x(\mathbb{D}(x, y)e_i)e_2$ . Then we can get the representation of  $T_D(x, y)$  by a trivial calculation.

$$\mathbb{T}_D(x, y) = \mathbb{T}(x, y) - \mathbb{T}(x, y') + \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\mathbb{T}}_M(\xi, x_2; y_2) e^{\mathbf{i}(x_1 - y_1)\xi} d\xi \quad (2.26)$$

and

$$\begin{aligned} \hat{\mathbb{T}}_M(\xi, x_2; y_2) = \frac{\mu}{\omega^2 \gamma(\xi)} & \left\{ E(\xi) e^{\mathbf{i}\mu_s(x_2+y_2)} + F(\xi) e^{\mathbf{i}\mu_p(x_2+y_2)} \right. \\ & \left. - E(\xi) e^{\mathbf{i}\mu_s x_2 + \mu_p y_2} - F(\xi) e^{\mathbf{i}\mu_p x_2 + \mu_s y_2} \right\} \end{aligned} \quad (2.27)$$

where

$$E(\xi) = \begin{pmatrix} -\xi^2\beta & \xi\mu_p\beta \\ 2\xi^3\mu_s & -2\xi^2\mu_s\mu_p \end{pmatrix} \quad F(\xi) = \begin{pmatrix} -2\xi^2\mu_s\mu_p & -2\xi^3\mu_p \\ -\xi\mu_s\beta & -\xi^2\beta \end{pmatrix}$$

Specially,  $T_D(x, y)$  has a simple form when  $x_2 = 0$ :

$$\mathbb{T}_D(x_1, 0; y_1, y_2) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\mathbb{T}}_D(\xi, 0; y_2) e^{i(x_1 - y_1)\xi} d\xi \quad (2.28)$$

where

$$\hat{\mathbb{T}}_D(\xi, 0; y_2) = \frac{1}{\gamma(\xi)} \left[ \begin{pmatrix} \xi^2 & -\xi\mu_p \\ -\xi\mu_s & \mu_p\mu_s \end{pmatrix} e^{i\mu_p y_2} + \begin{pmatrix} \mu_s\mu_p & \xi\mu_p \\ \xi\mu_s & \xi^2 \end{pmatrix} e^{i\mu_s y_2} \right] \quad (2.29) \quad \text{tgreen}$$

$$:= \mathcal{T}_p(\xi) e^{i\mu_p y_2} + \mathcal{T}_s(\xi) e^{i\mu_s y_2} \quad (2.30)$$

We should give asymptotic anlysis for  $\mathbb{N}(x_1, 0, y)$  and  $\mathbb{T}_D(x_1, 0; y)$  which will be in demand in the subsequent discussions. First we need the following slight generalization of Van der Corput lemma for the oscillatory integral [\[20, P.152\]](#) [grafakos](#).

**van** **Lemma 2.4** *Let  $-\infty < a < b < \infty$ , and  $u$  is a  $C^k$  function  $u$  in  $(a, b)$ .*

1. *If  $|u'(t)| \geq 1$  for  $t \in (a, b)$  and  $u'$  is monotone in  $(a, b)$ , then for any  $\phi(t)$  in  $(a, b)$  with integrable derivatives*

$$\left| \int_a^b e^{i\lambda u(t)} \phi(t) dt \right| \leq 3\lambda^{-1} \left[ |\phi(b)| + \int_a^b |\phi'(t)| dt \right].$$

2. *For all  $k \geq 2$ , if  $|u^{(k)}(t)| \geq 1$  for  $t \in (a, b)$ , then for any  $\phi(t)$  in  $(a, b)$  with integrable derivatives*

$$\left| \int_a^b e^{i\lambda u(t)} \phi(t) dt \right| \leq 12k\lambda^{-1/k} \left[ |\phi(b)| + \int_a^b |\phi'(t)| dt \right].$$

**Proof.** The assertion can be proved by extending the Van der Corput lemma in [\[20\]](#) [grafakos](#). Here we omit the details.  $\square$

**singular\_term**

**Lemma 2.5** *Assume that  $0 < \kappa < 1$  and  $f(t) \in C([- \frac{\pi}{2}, \frac{\pi}{2}]) \cap W^{1,1}([- \frac{\pi}{2}, \frac{\pi}{2}])$ . Then for any  $\rho > 1$  and  $\phi \geq \phi^* > \arcsin \kappa := \phi_\kappa$ , we have*

$$\left| \int_{-\pi/2}^{\pi/2} f(t) (\kappa^2 - \sin^2(t + \phi))^{-1/2} e^{i\rho \cos t} dt \right| \leq C \frac{1}{\rho^{1/2}} (|f(0)| + \int_{-\pi/2}^{\pi/2} |f'(t)| dt) \quad (2.31) \quad \text{es_singular}$$

where  $C$  only depend on  $\phi^*$  and  $\kappa$ .

**Proof.** Solving the following equation:

$$\kappa^2 - \sin^2(t + \phi) = 0$$

we have,  $t_1 = \phi_\kappa - \phi$  and  $t_2 = -\phi_\kappa - \phi$  or  $t_2 = \pi - \phi_\kappa - \phi$  depending on whether  $\phi + \phi_\kappa \leq \pi/2$  or  $\phi + \phi_\kappa > \pi/2$ . Without loss of generality, we assume the later case and thus  $t_2 = \pi - \phi_\kappa - \phi$ . Let  $d = \min\{\frac{\phi^* - \phi_\kappa}{2}, \frac{\phi_\kappa}{2}\}$ , then we can divide  $[-\pi/2, \pi/2]$  into four intervals:

$$I_1 = [-d, d], I_2 = [t_1 - d, t_2 + d], I_3 = [t_2 - d, t_3 + d], I_4 = [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus I_1 \cup I_2 \cup I_3$$

We remark that if  $t_2 + d > \pi/2$ , then let  $I_3 = [t_2 - d, \pi/2] \cup [-\pi/2, t_2 + d - \pi]$ . A simple computation show that

$$\sin(t + \phi) > \sin(\phi^* - d) > \kappa \quad t \in I_1 \quad (2.32) \quad \boxed{\text{es\_I1}}$$

$$\sin(\phi_\kappa - d) \leq |\sin(t + \phi)| \leq \sin(\phi_\kappa + d), \quad t \in I_2 \cup I_3 \quad (2.33) \quad \boxed{\text{es\_I2}}$$

$$|\sin(t + \phi)| \leq \sin(\phi_\kappa - d) \text{ or } |\sin(t + \phi)| \geq \sin(\phi_\kappa + d), \quad t \in I_4 \quad (2.34) \quad \boxed{\text{es\_I\_4}}$$

By lemma [2.4](#) and inequality [\(2.32\)](#), we can obtain:

$$\begin{aligned} & \left| \int_{I_1} f(t) (\kappa^2 - \sin^2(t + \phi))^{-1/2} e^{i\rho \cos t} dt \right| \\ & \leq \frac{C}{\cos d} \rho^{-1/2} \left( |f(d)| |\kappa^2 - \sin^2(d + \phi)|^{-1/2} + \int_{I_1} |f'(t)| |\kappa^2 - \sin^2(t + \phi)|^{-1/2} dt \right. \\ & \quad \left. + \frac{1}{2} \int_{I_1} |f(t)| |\kappa^2 - \sin^2(t + \phi)|^{-3/2} dt \right) \\ & \leq C(\phi^*, \kappa) \rho^{-1/2} (|f(0)| + \int_{I_1} |f'(t)| dt) \end{aligned}$$

and similarly we have

$$\begin{aligned} & \left| \int_{I_4} f(t) (\kappa^2 - \sin^2(t + \phi))^{-1/2} e^{i\rho \cos t} dt \right| \\ & \leq C(\phi^*, \kappa) \rho^{-1} (|f(0)| + \int_{I_1} |f'(t)| dt) \end{aligned}$$

For interval  $I_2$ , we only analysis the integral on  $[t_1 - d, t_1]$ . The integral on the other side can be estimated in the same method. Let  $0 < \delta < d$ , then by lemma [2.4](#) we can get

$$\begin{aligned} & \left| \int_{t_1-d}^{t_1-\delta} f(t) (\kappa^2 - \sin^2(t + \phi))^{-1/2} e^{i\rho \cos t} dt \right| \\ & \leq \frac{C}{|\sin(t_1 - d)|} \rho^{-1} \left( |f(t_1 - \delta)| |\kappa^2 - \sin^2(\phi_\kappa - \delta)|^{-1/2} + \int_{-d}^{-\delta} |f'(t_1 + t)| |\kappa^2 - \sin^2(\phi_\kappa + t)|^{-1/2} dt \right. \\ & \quad \left. + \int_{-d}^{-\delta} |f(t_1 + t)| |\kappa^2 - \sin^2(\phi_\kappa + t)|^{-3/2} dt \right) \\ & \leq C \frac{\cos^{-3/2}(\phi_\kappa + d)}{\sin d} \rho^{-1} \left( |f(t_1 - \delta)| ((\kappa + \sin(\phi_\kappa - d))\delta)^{-1/2} + \int_{-d}^{-\delta} |f'(t_1 + t)| ((\kappa + \sin(\phi_\kappa - d))\delta)^{-1/2} dt \right. \\ & \quad \left. + \frac{1}{2} \int_{-d}^{-\delta} |f(t_1 + t)| (-(\kappa + \sin(\phi_\kappa - d))t)^{-3/2} dt \right) \\ & \leq C(\phi^*, \kappa) (1 + \delta^{-1/2}) \rho^{-1} (|f(0)| + \int_{-\pi/2}^{\pi/2} |f'(t)| dt) \end{aligned}$$

Now, we turn to estimate the residual part of  $[t_1 - d, t_1]$ . By inequality [2.33](#), we have

$$\begin{aligned} & \left| \int_{t_1-\delta}^{t_1} f(t) (\kappa^2 - \sin^2(t + \phi))^{-1/2} e^{i\rho \cos t} dt \right| \\ & \leq \|f\|_{L^\infty} (\kappa + \sin(\phi_\kappa - d)) \int_{-\delta}^0 (\kappa - \sin(\phi_\kappa + t)) dt \end{aligned}$$



$$\begin{aligned}
&\leq \|f\|_{L^\infty}(\kappa + \sin(\phi_\kappa - d)) \int_{-\delta}^0 (\kappa - \sin(\phi_\kappa + t))^{-1/2} \frac{\cos t}{\cos d} \\
&\leq C(d, \kappa) \|f\|_{L^\infty}(\kappa - \sin(\phi_\kappa - \delta))^{1/2} \leq C(\phi^*, \kappa) \|f\|_{L^\infty} \delta^{1/2}
\end{aligned}$$

Cosequently, let  $\delta$  be equal to  $d * \rho^{-1}$ , we can obtain

$$\begin{aligned}
&\left| \int_{I_2} f(t) (\kappa^2 - \sin^2(t + \phi))^{-1/2} e^{i\rho \cos t} dt \right| \\
&\leq C(\phi^*, \kappa) \rho^{-1/2} (|f(0)| + \int_{I_1} |f'(t)| dt)
\end{aligned}$$

Similarly, we can etimate the integral on  $I_3$ . This completes the proof.  $\square$

The estimate of  $T_D(x_1, 0; y_1, y_2)$  and  $N(x_1, 0; y_1, y_2)$  now are direct consequences of lemma 2.5.

es\_dgreen

**Lemma 2.6** For every  $x \in \Gamma_0$ ,  $y \in \mathbb{R}_+^2$  such that  $|x_1 - y_1|/|x - y| > (1 + \kappa)/2$ ,  $y_2/|x - y| < \kappa/2$  and  $k_s|x - y| > 1$ , we have

$$|T_D(x, y)| \leq C \left( \frac{k_s y_2}{|x - y|} \frac{1}{(k_s|x - y|)^{1/2}} + \frac{k_s|x_1 - y_1|}{|x - y|} \frac{1}{(k_s|x - y|)^{3/2}} \right) \quad (2.35)$$

where  $C$  is only dependent on  $\kappa$ .

**Proof.** Put

$$I(|x_1 - y_1|, y_2) = \int_{\mathbb{R}} \mathcal{T}_s(\xi) e^{i(\mu_s y_2 + \xi|x_1 - y_1|)} d\xi \quad (2.36)$$

$$J(|x_1 - y_1|, y_2) = \int_{\mathbb{R}} \mathcal{T}_p(\xi) e^{i(\mu_p y_2 + \xi|x_1 - y_1|)} d\xi \quad (2.37)$$

To simplify the integral  $I$ , the standard substitution  $\xi = k_s \sin t$  is made, taking the  $\xi$ -plane to a strip  $-\pi < \text{Re } t < \pi$  in the  $t$ -plane, and the real axis in the  $\xi$ -plane onto the path  $L$  from  $-\pi/2 + i\infty \rightarrow -\pi/2 \rightarrow \pi/2 \rightarrow \pi/2 - i\infty$  in the  $t$ -plane. The integral  $I(|x_1 - y_1|, y_2)$  then becomes (Let  $|x_1 - y_1| = \rho \sin \phi$  and  $y_2 = \rho \cos \phi$ ,  $0 < \phi < \pi/4$ ):

$$k_s \int_L F(\sin t, \cos t, (\kappa^2 - \sin^2 t)^{1/2}) \cos t e^{ik_s \rho (\cos(t - \phi))} dt \quad (2.38)$$

where  $F(\sin t, \cos t, (\kappa^2 - \sin^2 t)^{1/2}) = \mathcal{T}_s(k_s \sin t)$ . Taking the shift transformation of  $t$  and using cauchy integral theorem, we can get the representation of  $I$ :

$$\begin{aligned}
&k_s \int_L F(\sin(t + \phi), \cos(t + \phi), (\kappa^2 - \sin^2(t + \phi))^{1/2}) \cos(t + \phi) e^{ik_s \rho (\cos t)} dt \\
&= k_s \cos \phi \int_L F(\sin(t + \phi), \cos(t + \phi), (\kappa^2 - \sin^2(t + \phi))^{1/2}) \cos t e^{ik_s \rho (\cos t)} dt \\
&- k_s \sin \phi \int_L F(\sin(t + \phi), \cos(t + \phi), (\kappa^2 - \sin^2(t + \phi))^{1/2}) \sin t e^{ik_s \rho (\cos t)} dt \\
&:= k_s (\cos \phi I_1 + \sin \phi I_2)
\end{aligned}$$

For  $I_1$ , we split the integral path  $L$  into  $L_1 = (-\pi/2, \pi/2)$  and  $L_2 = (-\pi/2 + i\infty, -\pi/2) \cup (\pi/2, \pi/2 - i\infty)$ , then we have corresponding representation:  $I_1 = I_{11} + I_{12}$ . Since  $F(t) \cos t \in W^{1,1}([-\frac{\pi}{2}, \frac{\pi}{2}])$ , we can obtain  $|I_{11}| \leq 1/(k_s \rho)^{1/2}$  by lemma 2.4. Using

integration by parts, it follows that  $|I_{12}| \leq 1/(k_s \rho)$ . For  $I_2$ , using integration by parts on path  $L$  first, we have

$$I_2 = \frac{1}{\mathbf{i}k_s \rho} \int_L F(\sin(t + \phi), \cos(t + \phi), (\kappa^2 - \sin^2(t + \phi))^{1/2}) d e^{\mathbf{i}(k_s \rho \cos t)} \quad (2.39)$$

$$= -\frac{1}{\mathbf{i}k_s \rho} \int_{L_1 \cup L_2} \frac{\partial F(\sin(t + \phi), \cos(t + \phi), (\kappa^2 - \sin^2(t + \phi))^{1/2})}{\partial t} e^{\mathbf{i}(k_s \rho \cos t)} dt \quad (2.40)$$

$$= -\frac{1}{\mathbf{i}k_s \rho} (I_{21} + I_{22}) \quad (2.41)$$

It is easy to check that  $\partial F / \partial t (\kappa^2 - \sin^2(t + \phi))^{1/2} \in C([- \frac{\pi}{2}, \frac{\pi}{2}]) \cap W^{1,1}([- \frac{\pi}{2}, \frac{\pi}{2}])$ . Therefore, we have  $|I_{21}| \leq 1/(k_s \rho)^{1/2}$  by lemma 2.5. <sup>singular term</sup> Similarly, we can obtain  $|I_{22}| < C \rho^{-1}$ . The estimate of  $J(|x_1 - y_1|, y_2)$  can be proved by the same method as employed above. Here we omit the details. This completes the proof.  $\square$

es\_ngreen

**Lemma 2.7** For every  $x \in \Gamma_0$ ,  $y \in \mathbb{R}_+^2$  such that  $|x_1 - y_1|/|x - y| > (1 + \kappa)/2$ ,  $y_2/|x - y| < \kappa/2$  and  $k_s|x - y| > 1$ , we have

$$|\mathbb{N}(x, y)| \leq \frac{C}{\mu} \left( \frac{y_2}{|x - y|} \frac{1}{(k_s|x - y|)^{1/2}} + \frac{|x_1 - y_1|}{|x - y|} \frac{1}{(k_s|x - y|)^{3/2}} + e^{-\sqrt{k_R^2 - k_s^2} y_2} \right) \quad (2.42)$$

where  $C$  is only dependent on  $\kappa$ .

**Proof.** For  $\mathbb{N}(x, y)$ , it suffice to estimate the first part  $\mathbb{N}_f$ . The proof is similar to lemma 2.6. <sup>es\_dgreen</sup> Here we only point out the different parts. Denote integral path  $P_\sigma^1 = (-\infty, -k_R - \sigma) \cup (-k_R + \sigma, k_R - \sigma) \cup (k_R + \sigma, \infty)$  and  $P_\sigma = P_\sigma^1 \cup B_\sigma^+(-k_R) \cup B_\sigma^+(k_R)$ . By the definition of Cauchy principal value, we have

$$\begin{aligned} \mathbb{N}_f(x, y) &= \frac{1}{2\pi} \text{P.V} \int_{\mathbb{R}} \hat{\mathbb{N}}(\xi, x_2; y_2) e^{\mathbf{i}(x_1 - y_1)\xi} d\xi = \lim_{\sigma \rightarrow 0^+} \frac{1}{2\pi} \int_{P_\sigma^1} \hat{\mathbb{N}}(\xi, x_2; y_2) e^{\mathbf{i}(x_1 - y_1)\xi} d\xi \\ &= \frac{1}{2\pi} \int_{P_\sigma} \hat{\mathbb{N}}(\xi, x_2; y_2) e^{\mathbf{i}(x_1 - y_1)\xi} d\xi + \frac{\mathbf{i}}{2} \frac{\mathbb{N}_\delta(-k_R)}{\delta'(-k_R)} e^{-\mathbf{i}(x_1 - y_1)k_R} + \frac{\mathbf{i}}{2} \frac{\mathbb{N}_\delta(k_R)}{\delta'(k_R)} e^{\mathbf{i}(x_1 - y_1)k_R} \end{aligned} \quad (2.43)$$

Notice that, in the present case, the substitution  $\xi = k_s \sin t$ , taking  $P_\sigma$  in the  $\xi$ -plane onto  $\Gamma_\sigma$  in the  $t$ -plane. The geometry now is depicted in Figure 1. <sup>figure\_trans</sup>

Similarly, if  $\sigma$  small enough, we can taking the shift transformation of  $t$  and use cauchy integral theorem. Therefore, the proof of this lemma can be completed by the same method as employed in the lemma 2.6. <sup>es\_dgreen</sup> Here we omit the details.  $\square$

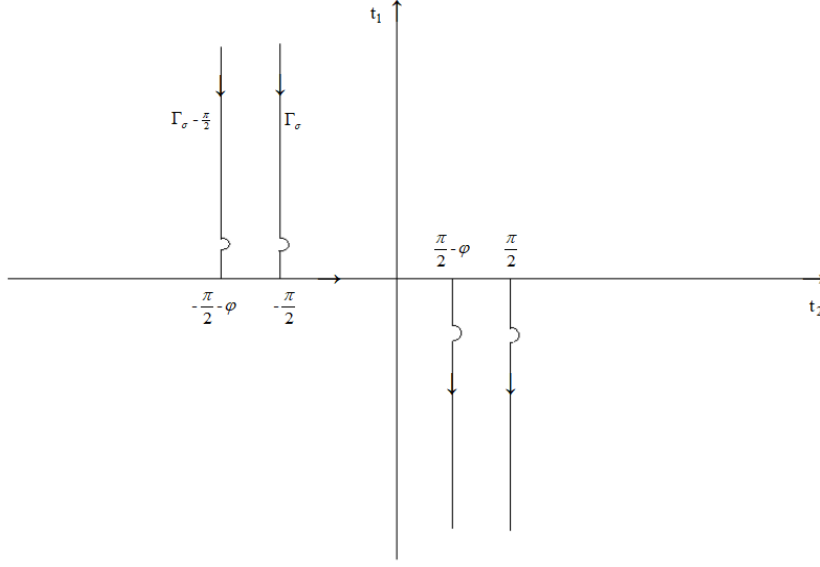
In the following we will use the weighted norm  $\|u\|_{H^1(D)} = (\|\nabla \phi\|_{L^2(D)}^2 + d_D^{-2} \|\phi\|_{L^2(D)}^2)^{1/2}$  be the weighted  $H^1(D)$  norm and  $\|v\|_{H^{1/2}(\Gamma)} = (d_D^{-1} \|v\|_{L^2(\Gamma)}^2 + |v|_{\frac{1}{2}, \Gamma}^2)^{1/2}$  be the weighted  $H^{1/2}(\Gamma)$  norm, where  $d_D$  is the diameter of  $D$  and

$$|v|_{\frac{1}{2}, \Gamma} = \left( \int_\Gamma \int_\Gamma \frac{|v(x) - v(y)|^2}{|x - y|^2} ds(x) ds(y) \right)^{1/2}.$$

By scaling argument and trace theorem we know that there exists a constant  $C > 0$  independent of  $d_D$  such that for any  $\phi \in C^1(\bar{D})^2$  <sup>RTMhalf\_aco</sup> [13, corollary 3.1],

$$\|\phi\|_{H^{1/2}(\Gamma_D)} + \|\sigma(\phi) \cdot \nu\|_{H^{-1/2}(\Gamma_D)} \leq C \max_{x \in D} (|\phi(x)| + d_D |\nabla \phi(x)|) \quad (2.44)$$

q0



**Figure 1.** Transform from  $P_\sigma$  in the  $\xi$ -plane to  $\Gamma_\sigma$  in the  $t$ -plane

We finish this section by introducing some results on forward scattering problem which will be discussed briefly in the Appendix of this paper.

**Theorem 2.1** *Let  $g \in H^{1/2}(\Gamma_D)$ , then the scattering problem of elastic equation in the half space*

$$\Delta_e u + \omega^2 u = 0 \quad \text{in } \mathbb{R}_+^2 \setminus \bar{D}, \quad (2.45)$$

$$u = g \quad \text{on } \Gamma_D, \quad (2.46)$$

$$\sigma(u)e_2 = 0 \quad \text{on } \Gamma_0, \quad (2.47)$$

*$u$  satisfies the generalized radiation condition <sup>Guzina2006</sup> [24] such that*

$$\lim_{r \rightarrow \infty} \int_{S_r^+} (\sigma(N(x, y)e_i \hat{r}) \cdot u(x) - (N(x, y)e_i) \cdot (\sigma(u)\hat{r})) ds(x) = 0 \quad (2.48)$$

where  $S_r^+ := \{x \in \mathbb{R}_+^2 \mid \|x\| = r^2\}$ ,  $\hat{r} = x/r$  and  $y \in \mathbb{R}_+^2$ . Then the problem <sup>elas\_1</sup> (2.45)-(2.48) admits a unique solution  $u \in H_{\text{loc}}^1(\mathbb{R}_+^2 \setminus \bar{D})$ . Moreover, for any bounded open set  $\mathcal{O} \subset \mathbb{R}_+^2 \setminus \bar{D}$  there exists a constant  $C > 0$  such that

$$\|u\|_{H^1(\mathcal{O})} \leq C \|g\|_{H^{-1/2}(\Gamma_D)} \quad (2.49)$$

We also give the definition of the Dirichlet-to-Neumann mapping for Elastic scattering problems. For any  $g \in H^{1/2}(\Gamma_D)$ , define  $T_h(g) = \sigma(u)\nu \in H^{-1/2}(\Gamma_D)$  where  $u$  is the corresponding solution of above equation and also  $T_f(g) = \sigma(v)\nu$  where  $v$  is the solution of full space scattering problem and their operator norm are denoted by  $\|T_h\|$  or  $\|T_f\|$  in the remainder of this paper. For the sake of convenience, we introduce the following notation:

$$\mathcal{G}(W, U) = \int_{\Gamma_D} [W(x) \cdot \sigma(U(x))\nu - \sigma(W(x))\nu \cdot U(x)] ds(x) \quad (2.50)$$

Using this notation, the integral representation formula of the solution  $U(x)$  to the scattering problem <sup>elas\_1</sup> (2.45)-(2.47) reads:  $U(x) \cdot q = \mathcal{G}(U(\cdot), \mathbb{N}(\cdot, x)q)$ .

### 3. Reverse time migration method

In this section we introduce RTM method for inverse elastic scattering problems in the half space. Assume that there are  $N_s$  sources and  $N_r$  receivers uniformly distributed on  $\Gamma_0^d$ , where  $\Gamma_0^d = \{(x_1, x_2)^T \in \Gamma_0 : x_1 \in [-d, d]\}$ ,  $d > 0$  is aperture. We denote by  $\Omega$  the sampling domain in which the obstacle is sought. Let  $h = \text{dist}(\Omega, \Gamma_0)$  be the distance of  $\Omega$  to  $\Gamma_0$ . We assume the obstacle  $D \subset \Omega$  and there exist constants  $0 < c_1 < 1, c_2 > 0, c_3 > 0$  such that

$$|x_1| \leq c_1 d, \quad |x_1 - y_1| \leq c_2 h, \quad |x_2| \leq c_3 h \quad \forall x, y \in \Omega \quad (3.1)$$

convention\_2

Before proposing RTM imaging function, we first introduce the point spread function which measures the resolution for finding point source. In [13], the point spread function has been defined in the case of acoustic wave. We now define a similar elastic point spread function  $\mathbb{J}(z, y)$ , a  $\mathbb{C}^{2 \times 2}$  matrix, which back-propagate the conjugated data  $\overline{N(x, y)}$  as the Dirichlet boundary condition. Thus, for any  $z, y \in \mathbb{R}_+^2$

$$\begin{aligned} \mathbb{J}_{ij}(z, y) &:= e_i \cdot \mathbb{J}(z, y) e_j = \int_{\Gamma_0} \sigma_x(\mathbb{D}(x, z) e_i) e_j \cdot \overline{N(x, y)} e_j ds(x) \\ &= \int_{\mathbb{R}} \sigma_x(\mathbb{D}(x_1, 0; z_1, z_2) e_i) e_j \cdot \overline{N(x_1, 0; y_1, y_2)} e_j dx_1 \end{aligned} \quad (3.2)$$

fullpsf

The estimate in lemma 2.6-2.7 show that the integral above exists. Moreover, we define

$$\begin{aligned} \mathbb{F}(z, y) &= \frac{1}{2\pi} \int_{-k_p}^{k_p} \frac{\mathcal{T}_p(\xi)^T \overline{\mathcal{N}_p(\xi)}}{\overline{\delta(\xi)}} e^{i\mu_p(z_2 - y_2) + i(y_1 - z_1)\xi} d\xi \\ &\quad + \frac{1}{2\pi} \int_{-k_s}^{k_s} \frac{\mathcal{T}_s(\xi)^T \overline{\mathcal{N}_s(\xi)}}{\overline{\delta(\xi)}} e^{i\mu_s(z_2 - y_2) + i(y_1 - z_1)\xi} d\xi \end{aligned}$$

The following theorem show that  $\mathbb{F}(z, y)$  is the main contribution to the point spread function when  $k_s h \gg 1$ . Put  $n_* = \min\{N | \kappa^{2N-1} < 1/c_3, N \in \mathbb{Z}_+\}$ . Then we present some primary results for point spread function, whose proof will be stated in section 4.

thm\_psf

**Theorem 3.1** *Let  $k_s h > 1$ . For any  $z, y \in \Omega$ , let  $\mathbb{R}(z, y) = \mathbb{J}(z, y) - \mathbb{F}(z, y)$  and it satisfy that*

$$|\mathbb{R}_{ij}(z, y)| + k_s^{-1} |\nabla_y \mathbb{R}_{ij}(z, y)| \leq \frac{C}{\mu} \left( \frac{1}{(k_s h)^{\frac{1}{2n_*}}} + k_s h e^{-k_s h \sqrt{\kappa_R^2 - 1}} \right) := \frac{C}{\mu} \epsilon_1(k_s h) \quad (3.3)$$

uniformly for  $z, y \in \Omega$ . Here  $\kappa_R := k_R/k_s$  and the constant  $C$  may dependent on  $k_s d_D$  and  $\kappa := k_p/k_s$ , but is independent of  $k_s, k_p, h, d_D$ .

Based on the above argument, we know that  $\mathbb{R}(z, y)$  becomes small when  $z, y$  move away from  $\Gamma_0$ . Our goal is to show  $\mathbb{F}(z, y)$  has the similar decay to the elastic fundamental solution  $\text{Im } \Phi(z, y)$  as  $|z - y| \rightarrow \infty$ .

festimate1

**Lemma 3.1** *For any  $z, y \in \mathbb{R}_+^2$ , when  $z = y$*

$$\begin{aligned} |\text{Im } \mathbb{F}_{ii}(z, y)| &\geq \frac{1}{4(\lambda + 2\mu)}, \quad i = 1, 2 \\ \text{Im } \mathbb{F}_{12}(z, y) &= \text{Im } \mathbb{F}_{21}(z, y) = 0 \end{aligned}$$

and for  $z \neq y$

$$|\mathbb{F}_{ij}(z, y)| \leq \frac{C}{\mu} [(k_s|z - y|)^{-1/2} + (k_s|z - y|^{-1})]$$

where constant  $C$  is only dependent on  $\kappa := k_p/k_s$ .

By (2.44), we obtain the following consequence of theorem 3.1 and Lemma 3.1 which will be used in the resolution analysis.

**Corollary 3.1** *There exists a constant  $C$  independent of  $k_s$ ,  $h$  such that*

$$\begin{aligned} \|F(z, \cdot)e_j\|_{H^{1/2}(\Gamma_D)} + \|\sigma(F(z, \cdot)e_j) \cdot \nu\|_{H^{-1/2}(\Gamma_D)} &\leq \frac{C}{\mu}(1 + k_s d_D) \\ \|R(z, \cdot)e_j\|_{H^{1/2}(\Gamma_D)} + \|\sigma(R(z, \cdot)e_j) \cdot \nu\|_{H^{-1/2}(\Gamma_D)} &\leq \frac{C}{\mu}(1 + k_s d_D)\epsilon_1(k_s h) \end{aligned}$$

uniformly for  $z \in \Omega$ , where  $d_D$  is the diameter of the obstacle  $D$ .

Now we consider the finite aperture point spread function  $\mathbb{J}_d(z, y)$ :

$$\mathbb{J}_{ij}^d(z, y) = \int_{-d}^d \sigma_x(\mathbb{D}(x_1, 0; z_1, z_2)e_i)e_2 \cdot \overline{\mathbb{N}(x_1, 0; z_1, z_2)}e_j dx_1 \quad (3.4)$$

Our aim is to estimate the difference  $J(z, y) - J_d(z, y)$ . It is easy to see that

$$\frac{(x_1 - z_1)^2}{\rho^2} \geq \frac{(1 - c_1)^2}{(1 - c_1)^2 + c_3^2(h/d)^2} := m(h/d) \quad (3.5)$$

$$\frac{z_2^2}{\rho^2} \leq \frac{c_3^2}{(1 - c_1)^2(d/h) + c_3^2} := M(h/d) \quad (3.6)$$

where  $\rho = \sqrt{(x_1 - z_1)^2 + z_2^2}$  and  $z \in \Omega, x \in \Gamma_0 \setminus (-d, d)$ . In the subsequent discussions, we assume  $m(h/d) > (1 + \kappa)^2/4$ ,  $M(h/d) < \kappa^2/4$ .

**Theorem 3.2** *For  $k_s h \geq 1$  and  $z, y \in \Omega$ , we have*

$$|\mathbb{J}(z, y) - \mathbb{J}_d(z, y)| + k_s^{-1} |\nabla_y(\mathbb{J}(z, y) - \mathbb{J}_d(z, y))| \quad (3.7)$$

$$\leq \frac{C}{\mu} \left( \left( \frac{h}{d} \right)^2 + \frac{(k_s h)^{1/2}}{e^{k_s h \sqrt{\kappa_R^2 - 1}}} \left( \frac{h}{d} \right)^{1/2} \right) := \frac{C}{\mu} \epsilon_2(k_s h, h/d) \quad (3.8)$$

where the constant  $C$  is only dependent on  $\kappa$ .

**Proof.** By lemma 2.7, lemma 2.6 and  $k_s h \geq 1$ , we have

$$\begin{aligned} &\left| \int_d^\infty (T_D(x_1, 0; z_1, z_2))^T \overline{N(x_1, 0; y_1, y_2)} dx_1 \right| \\ &\leq \frac{C}{\mu} \int_d^\infty \frac{k_s z_2}{|x - z|} \frac{1}{(k_s |x - z|)^{1/2}} \left( \frac{y_2}{|x - y|} \frac{1}{(k_s |x - y|)^{1/2}} + e^{-\sqrt{k_R^2 - k_s^2} y_2} \right) dx_1 \\ &\leq \frac{C}{\mu} \int_{(1-c_1)d/h}^\infty \frac{1}{(1+t^2)^{3/2}} + \frac{(k_s h)^{1/2}}{(1+t^2)^{3/4}} e^{-\sqrt{k_R^2 - k_s^2} h} dt \\ &\leq \frac{C}{\mu} \left( \left( \frac{h}{d} \right)^2 + \frac{(k_s h)^{1/2}}{e^{\sqrt{k_R^2 - k_s^2} h}} \left( \frac{h}{d} \right)^{1/2} \right) \end{aligned}$$

Here we have used the first inequality in <sup>|convention\_2</sup>(3.1). Similarly, we can prove that the estimate for the integral in  $[-\infty, -d]$ . This shows the estimate for  $J(z, y) - J_d(z, y)$ . The estimate for  $\nabla_y(J(z, y) - J_d(z, y))$  can be proved similarly.  $\square$

By <sup>g0</sup>(2.44) we obtain the following corollary

cor\_dpsf

**Corollary 3.2** *Let  $\mathbb{R}_d(z, y) = \mathbb{J}(z, y) - \mathbb{J}_d(z, y)$ . For  $k_s h \geq 1$  and  $z, y \in \Omega$ , there exists a constant  $C$  independent of  $k_s, h$  such that*

$$\|\mathbb{R}_d(z, \cdot)e_j\|_{H^{1/2}(\Gamma_D)} + \|\sigma(\mathbb{R}_d(z, \cdot)e_j) \cdot \nu\|_{H^{-1/2}(\Gamma_D)} \leq \frac{C}{\mu} \epsilon_2(k_s h, h/d)(1 + k_s d_D)$$

uniformly for  $z \in \Omega$ , where  $d_D$  is the diameter of the obstacle  $D$ .

Motivated by the above discussion, we introduce the following imaging function:

alg\_rtm

**Algorithm 3.1** (REVERSE TIME MIGRATION ALGORITHM)

Given the data  $u_k^s(x_r, x_s)$ ,  $k = 1, 2$  which is the measurement of the scattered field at  $x_r$  when the source is emitted at  $x_s$  along the polarized direction  $e_k$ ,  $s = 1, \dots, N_s$  and  $r = 1, \dots, N_r$ .

$$I_d(z) = \text{Im} \sum_{k=1}^2 \left\{ \frac{|\Gamma_0^d|}{N_s} \sum_{s=1}^{N_s} \sum_{i=1}^2 [\sigma_{x_s}(\mathbb{D}(x_s, z)e_i)e_2 \cdot e_k] [v_k(z, x_s) \cdot e_i] \right\}. \quad z \in \Omega \quad (3.9) \quad \text{cor1}$$

where  $v_k(z, x_s)$  satisfy the following scattering elastic equation:

$$\begin{aligned} \Delta_e v_k(z, x_s) + \omega^2 v_k(z, x_s) &= 0 \quad \text{in } \mathbb{R}_+^2 \\ v_k(z, x_s) &= \frac{|\Gamma_0^d|}{N_r} \sum_{r=1}^{N_r} \overline{u_k^s(x_r, x_s)} \delta_{x_r}(z) \quad \text{on } \Gamma_0 \end{aligned}$$

By letting  $N_s, N_r \rightarrow \infty$ , we know that <sup>|cor1</sup>(3.9) can be viewed as an approximation of the following continuous integral:

$$\hat{I}_d(z) = \text{Im} \sum_{k=1}^2 \int_{\Gamma_0^d} \int_{\Gamma_0^d} \sum_{i=1}^2 [\sigma_{x_s}(\mathbb{D}(x_s, z)e_i)e_2 \cdot e_k] [\sigma_{x_r}(\mathbb{D}(x_r, z)e_i)e_2 \cdot \overline{u_k^s(x_r, x_s)}] ds(x_r) ds(x_s)$$

where  $z \in \Omega$ .

Now, We turn to study the resolution of the function  $\hat{I}_d(z)$ . To do this, we first show the difference between the half space scattering solution and the full space scattering solution is small if the scatterer is far away from the boundary  $\Gamma_0$ .

diff\_solu

**Theorem 3.3** *Let  $g \in H^{1/2}(\Gamma_D)$  and  $\mathbf{u}_1, \mathbf{u}_2$  be the scattering solution of following problems:*

$$\Delta_e \mathbf{u}_1 + \omega^2 \mathbf{u}_1 = 0 \quad \text{in } \mathbb{R}_+^2 \setminus \bar{D} \quad (3.10) \quad \text{elas_r1}$$

$$\mathbf{u}_1 = g \quad \text{on } \Gamma_D \quad (3.11) \quad \text{elas_rbd}$$

$$\sigma(\mathbf{u}_1)e_2 = 0 \quad \text{on } \Gamma_0 \quad (3.12) \quad \text{elas_rbd0}$$

and

$$\Delta_e \mathbf{u}_2 + \omega^2 \mathbf{u}_2 = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D} \quad (3.13) \quad \text{elas_r2}$$

$$\mathbf{u}_2 = g \quad \text{on } \Gamma_D \quad (3.14) \quad \text{elas_rbd2}$$

Then there exists a constant  $C$  independent of  $k_s, k_p$ , such that

$$\|\sigma_x(\mathbf{u}_1 - \mathbf{u}_2)\nu\|_{H^{-1/2}(\Gamma_D)} \leq \frac{C}{\mu}(1 + \|T_f\|)(1 + \|T_h\|)(1 + k_s d_D)^2 \epsilon_1(k_s h) \|g\|_{H^{1/2}(\Gamma_D)} \quad (3.15)$$

Theorem [3.3](#) <sup>diff\_solu</sup> also will be proved in the appendix of this paper. The following theorem is the main result of resolution analysis.

**resolution1**

**Theorem 3.4** For any  $z \in \Omega$ , let  $\Psi(y, z) \in \mathbb{C}^{2 \times 2}$  be the radiation solution of the problem:

$$\begin{aligned} \Delta_e \Psi(y, z) e_i + \omega^2 \Psi e_i &= 0 \quad \text{in } \mathbb{R}_+^2 \setminus \bar{D} \quad i = 1, 2 \\ \Psi(y, z) &= -\overline{\mathbb{F}(z, y)} \quad \text{on } \Gamma_D \end{aligned}$$

Then, we have

$$\hat{I}_d(z) = \text{Im} \int_{\Gamma_D} \sum_{i=1}^2 [\sigma_y(\overline{\mathbb{F}(z, y)} + \Psi(y, z)) e_i \cdot \overline{\mathbb{F}(z, y)} e_i] ds(y) + \mathbb{W}_{\hat{I}}(z) \quad (3.16)$$

where  $|\mathbb{W}_{\hat{I}}(z)| \leq \frac{C}{\mu}(1 + \|T_f\|)(1 + \|T_h\|)(1 + k_s d_D)^4 (\epsilon_1(k_s h) + \epsilon_2(k_s h, h/d))$  uniformly for  $z$  in  $\Omega$ .

**Proof.** Denote by  $v_k(z, x_s)$  the field back-propagating from  $\overline{\mathbf{u}_k^s(x_r, x_s)}$  on  $\Gamma_d$  where  $\mathbf{u}_k^s(x, x_s) + N(x, x_s) e_k = 0$ . From [\(3.2\)](#) <sup>fullpsf</sup> and [\(2.50\)](#) <sup>bi\_op</sup>, we get for any  $z \in \Omega$ ,

$$\begin{aligned} v_k(z, x_s) \cdot e_i &= \int_{\Gamma_0^d} \sigma_{x_r}(\mathbb{D}(x_r, z) e_i) e_2 \cdot \overline{\mathbf{u}_k^s(x_r, x_s)} ds(x_r) \\ &= \sum_{j=1}^2 \int_{\Gamma_0^d} e_j^T \sigma_{x_r}(\mathbb{D}(x_r, z) e_i) e_2 \mathcal{G}(\overline{\mathbf{u}_k^s(\cdot, x_s)}, \mathbb{N}(\cdot, x_r) e_j) ds(x_r) \\ &= \mathcal{G}(\overline{\mathbf{u}_k^s(\cdot, x_s)}, \sum_{j=1}^2 \int_{\Gamma_0^d} e_j^T \sigma_{x_r}(\mathbb{D}(x_r, z) e_i) e_2 \mathbb{N}(\cdot, x_r) e_j ds(x_r)) \\ &= \mathcal{G}(\overline{\mathbf{u}_k^s(\cdot, x_s)}, \mathbb{J}_d(z, \cdot)^T e_i) \end{aligned}$$

By the definition of the imaging function  $\hat{I}_d(z)$ , we have

$$\hat{I}_d(z) = \text{Im} \sum_{k=1}^2 \int_{\Gamma_0^d} \sum_{i=1}^2 (e_k \cdot \sigma_{x_s}(\mathbb{D}(x_s, z) e_i) e_2) (v_k(z, x_s) \cdot e_i) ds(x_s) \quad (3.17)$$

$$= \text{Im} \sum_{k=1}^2 \int_{\Gamma_0^d} \text{tr}[(v_k(z, x_s) \cdot e_i) (e_k^T \sigma_{x_s}(\mathbb{D}(x_s, z) e_j) e_2)]_{ij} ds(x_s) \quad (3.18)$$

$$= \text{Im} \text{tr}[\mathcal{G}(\sum_{k=1}^2 \int_{\Gamma_0^d} \overline{\mathbf{u}_k^s(\cdot, x_s)} e_k^T \sigma_{x_s}(\mathbb{D}(x_s, z) e_j) e_2 ds(x_s), \mathbb{J}_d(z, \cdot)^T e_i)]_{ij} \quad (3.19)$$

$$= \text{Im} \text{tr}[\mathcal{G}(\mathbb{W}(\cdot, z) e_j, \mathbb{J}_d(z, \cdot)^T e_i)]_{ij} \quad (3.20)$$

**resolu\_1**

where

$$\mathbb{W}(y, z) e_j = \sum_{k=1}^2 \int_{\Gamma_0^d} \overline{\mathbf{u}_k^s(\cdot, x_s)} e_k^T \sigma_{x_s}(\mathbb{D}(x_s, z) e_j) e_2 ds(x_s) \quad (3.21)$$

Therefore,  $\overline{\mathbb{W}(y, z)}e_j$  can be viewed as the weighted superposition of  $u_k^s(y, x_s)$ . On the boundary of the obstacle  $\Gamma_D$ , we have

$$\begin{aligned}\overline{\mathbb{W}(y, z)}e_j &= \sum_{k=1}^2 \int_{\Gamma_0^d} \mathbf{u}_k^s(y, x_s) e_k^T \sigma_{x_s}(\overline{\mathbb{D}(x_s, z)}e_j) e_2 ds(x_s) \\ &= \sum_{k=1}^2 \int_{\Gamma_0^d} -\mathbb{N}(y, x_s) e_k e_k^T \sigma_{x_s}(\overline{\mathbb{D}(x_s, z)}e_j) e_2 ds(x_s) \\ &= -\overline{\mathbb{J}_d^T(z, y)}e_j\end{aligned}$$

Moreover,  $\sigma_y(\overline{\mathbb{W}(y, z)}e_j)e_2 = 0$  on  $\Gamma_0$  since  $\sigma_y(\overline{u_k^s(y, x_s)})e_2 = 0$  on  $\Gamma_0$ . Let  $\mathbb{W}_d(y, z)$  be the scattering solution of the problem

$$\Delta_e \mathbb{W}_d(y, z)e_j + \omega^2 \mathbb{W}_d(y, z)e_j = 0 \quad \text{in } \mathbb{R}_+^2 \setminus \bar{D} \quad (3.22)$$

$$\mathbb{W}_d(y, z) = \overline{\mathbb{F}(z, y)} - \overline{\mathbb{J}_d^T(z, y)} \quad \text{on } \Gamma_D \quad (3.23)$$

$$\sigma_y(\mathbb{W}_d(y, z)e_j)e_2 = 0 \quad \text{on } \Gamma_0 \quad (3.24)$$

By theorem [2.1](#) and Corollaries [3.1-3.2](#) we have

$$\|\sigma_y(\mathbb{W}_d(y, z)e_j)\nu\|_{H^{-1/2}(\Gamma_D)} \leq \|T_h\| \|\mathbb{F}(z, \cdot)e_j - \mathbb{J}_d^T(z, \cdot)e_j\|_{H^{1/2}(\Gamma_D)} \quad (3.25) \quad \boxed{\text{W\_ineq}}$$

Let  $\mathbb{V}(y, z) := \overline{\mathbb{W}(y, z)} - \mathbb{W}_d(y, z) - \Psi(y, z)$ . Since  $\overline{\mathbb{W}(y, z)} - \mathbb{W}_d(y, z)$  satisfy the half-space scattering problem [3.10](#) with  $g(y) = -\overline{\mathbb{F}(z, y)}$ , by Theorem [3.3](#), we have

$$\|\sigma_y(\mathbb{V}(y, z)e_j)\nu\|_{H^{1/2}(\Gamma_D)} \quad (3.26)$$

$$\leq \frac{C}{\mu} (1 + \|T_f\|)(1 + \|T_h\|)(1 + k_s d_D)^2 \epsilon_1(k_s h) \|\mathbb{F}(z, \cdot)e_j\|_{H^{1/2}(\Gamma_D)} \quad (3.27)$$

Now we substitute  $\overline{\mathbb{W}(y, z)} = \mathbb{V}(y, z) + \mathbb{W}_d(y, z) + \Psi(y, z)$  into [\(3.20\)](#) to obtain

$$\hat{I}_d(z) = \text{Im } \mathbf{tr}[\mathcal{G}(\overline{\Psi(\cdot, z)}e_j, \mathbb{J}_d(z, \cdot)^T e_i)]_{ij} + R_{\hat{f}}(z) \quad (3.28) \quad \boxed{\text{I\_d}}$$

where

$$R_{\hat{f}}(z) = \text{Im } \mathbf{tr}[\mathcal{G}(\overline{\mathbb{V}(\cdot, z)}e_j, \mathbb{J}_d(z, \cdot)^T e_i)]_{ij} + \text{Im } \mathbf{tr}[\mathcal{G}(\overline{\mathbb{W}_d(\cdot, z)}e_j, \mathbb{J}_d(z, \cdot)^T e_i)]_{ij} \quad (3.29)$$

By [\(3.25\)](#) and Corollaries [3.1-3.2](#) it is easy to see that

$$|R_{\hat{f}}(z)| \leq \frac{C}{\mu} (1 + \|T_f\|)(1 + \|T_h\|)(1 + k_s d_D)^4 (\epsilon_1(k_s h) + \epsilon_2(k_s h, h/d)) \quad (3.30)$$

Finally, by [\(3.28\)](#) and  $\Psi(y, z) = -\overline{\mathbb{F}(z, y)}$  on  $\Gamma_D$

$$\begin{aligned}\hat{I}_d(z) &= \text{Im } \mathbf{tr}[\mathcal{G}(\overline{\Psi(\cdot, z)}e_j, \mathbb{F}(z, \cdot)e_i)]_{ij} + \text{Im } \mathbf{tr}[\mathcal{G}(\overline{\Psi(\cdot, z)}e_j, (\mathbb{J}_d^T(z, y) - \mathbb{F}(z, y))e_i)]_{ij} + R_{\hat{f}}(z) \\ &= \text{Im } \sum_{i=1}^2 \int_{\Gamma_D} [\sigma_y(\overline{(\mathbb{F}(z, y) + \Psi(y, z))}e_i) \cdot \overline{\mathbb{F}(z, y)}e_i] ds(y) + w_{\hat{f}}(z) + R_{\hat{f}}(z)\end{aligned}$$

By Corollaries [3.1-3.2](#) we have

$$|w_{\hat{f}}(z)| \leq \frac{C}{\mu} (1 + \|T_h\|)(1 + k_s d_D)^2 (\epsilon_1(k_s h) + \epsilon_2(k_s h, h/d)) \quad (3.31)$$

□

By [\(4.6\)-\(4.7\)](#) we know that for any fixed  $z \in \Omega$ ,  $\overline{\mathbb{F}(z, \cdot)}$  satisfies the Elastic wave



equation. Thus  $\Psi(y, z)$  can be viewed as the scattering solution of the Elastic equation with the incident wave  $\overline{\mathbb{F}(z, \cdot)}$ . By lemma 3.1 we know that  $\overline{\mathbb{F}(z, \cdot)}$  decays as  $|y - z|$  becomes large. Therefore the imaging function  $\hat{I}_d(z)$  becomes small when  $z$  moves away from the boundary  $\Gamma_D$  outside the scatterer  $D$  if  $k_s h \gg 1$  and  $d \gg h$ .

To understand the behavior of the imaging function when  $z$  is close to the boundary of the scatterer, we extend the concept of the scattering coefficient for incident plane waves [13].

**Definition 3.1** For any unit vector  $d \in \mathbb{R}^2$ , let  $u_p^i = d e^{i k_p x \cdot d}$  or  $u_s^i = d^\perp e^{i k_s x \cdot d}$  be the incident wave and  $u_\alpha^s = u_\alpha^s(x; d)$  be the corresponding radiation solution of the Navier equation:

$$u_\alpha^s + \omega^2 u_\alpha^s = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D} \quad (3.32)$$

$$u_\alpha^s = -u_\alpha^i \quad \text{on } \partial D \quad (3.33)$$

The scattering coefficient  $R(x; d)$  for  $x \in \partial D$  is defined by the relation

$$\sigma(u_\alpha^s + u_\alpha^i) \cdot \nu = i k_\alpha R_\alpha(x; d) e^{i k_\alpha x \cdot d} \quad \text{on } \partial D$$

where  $\alpha = p, s$ ,  $d = (d^1, d^2)^T$  is unit vectors,  $d^\perp = (d^2, -d^1)^T$ .

In the case of high frequency approximation, the scattering coefficient can be approximated by

$$R_\alpha(x; d) = \begin{cases} R F_\alpha(d; \nu(x)) & \text{if } x \in \partial D_d^- = \{x \in \partial D, \nu(x) \cdot d < 0\}, \\ 0 & \text{if } x \in \partial D_d^+ = \{x \in \partial D, \nu(x) \cdot d \geq 0\}. \end{cases}$$

Now we consider the physical interpretation of the imaging function  $\hat{I}_d(z)$  when  $z \in \Gamma_D$ . By (4.6) and (4.7), we have

$$\overline{F(z, y)} = i \int_0^\pi A_p(\theta) \eta_\theta e^{i k_p (y-z) \cdot \eta_\theta} + A_s(\theta) \eta_\theta^\perp e^{i k_s (y-z) \cdot \eta_\theta} d\theta, \quad \eta_\theta := (\cos \theta, \sin \theta)^T$$

where

$$A_p(\theta) = \frac{k_s^2 k_p^3 \mu_s(k_p \cos \theta) \sin \theta}{2\pi \mu \gamma(k_p \cos \theta) \delta(k_p \cos \theta)} (\cos \theta, \sin \theta)$$

and

$$A_s(\theta) = \begin{cases} \frac{k_s^5 \mu_p(k_s \cos \theta) \sin \theta}{2\pi \mu \gamma(k_s \cos \theta) \delta(k_s \cos \theta)} (\sin \theta, -\cos \theta) & \theta \in (\arccos(k_p), \arccos(-k_p)) \\ \frac{k_s^5 (1-4 \cos^2 \theta) \mu_p(k_s \cos \theta) \sin \theta}{2\pi \gamma(k_s \cos \theta) \delta(k_s \cos \theta)} (\sin \theta, -\cos \theta) & \theta \in (0, \pi) \setminus [\arccos(k_p), \arccos(-k_p)] \end{cases}$$

Let  $y(s)$  be the arc length parametrization of the boundary  $\Gamma_D$  and  $y_\pm(\eta_\theta) = y(s_\pm)$  be the points such that  $\nu(y(s_\pm)) = \pm \eta_\theta$ . By the same procedure used in [13, p11, 12], we can obtain

$$\begin{aligned} \hat{I}_d(z) &\approx \sqrt{8\pi k_p} \operatorname{Im} \operatorname{tr} \int_0^\pi ((\lambda + 2\mu) A_p(\theta) \eta_\theta e^{i k_p (y_-(\eta_\theta) - z) \cdot \eta_\theta - i \frac{\pi}{4}})^T \frac{\overline{F(z, y_-(\eta_\theta))}}{\sqrt{\vartheta(y_-(\eta_\theta))}} d\theta \\ &\quad + \sqrt{8\pi k_s} \operatorname{Im} \operatorname{tr} \int_0^\pi (\mu A_s(\theta) \eta_\theta^\perp e^{i k_s (y_-(\eta_\theta) - z) \cdot \eta_\theta - i \frac{\pi}{4}})^T \frac{\overline{F(z, y_-(\eta_\theta))}}{\sqrt{\vartheta(y_-(\eta_\theta))}} d\theta \end{aligned}$$

Now for  $z$  in the part of  $\Gamma_D$  which is back to  $\Gamma_0$ , ie  $\nu(z) \cdot \eta_\theta > 0$  for any  $\theta \in [0, \pi]$ , we know that  $z$  and  $y_-(\eta_\theta)$  are far away and thus  $\hat{I}_d(z) \approx 0$ . By above formula and lemma [3.1](#), we can explain that one cannot image the back part of the obstacle with only the data collected on  $\Gamma_0$ . This confirmed in our numerical examples.

#### 4. Analysis of point spread function

In this section we give the proof of theorem [3.1](#) and lemma [3.1](#) which depend on several lemmas that follow. Without loss of generality, we assume  $z_1 - y_1 \geq 0$  in the subsequent. Otherwise, we can take substitution  $\xi = -\xi$ .

We split the spectral terms into components associated with pressure and shearing waves.

$$\hat{\mathbb{D}} = \hat{\mathbb{D}}^p + \hat{\mathbb{D}}^s \quad \hat{\mathbb{N}} = \hat{\mathbb{N}}^p + \hat{\mathbb{N}}^s$$

and we define

$$\mathbb{J}_{ij}^{\alpha\eta}(z, y) = \int_{\Gamma_0} \sigma_x(\mathbb{D}^\alpha(x, y)e_i)e_2 \cdot \overline{\mathbb{N}^\eta(x, y)d}e_j ds(x), \quad \alpha, \eta \in \{s, p\} \quad (4.1)$$

It's easy to see

$$\mathbb{J}(z, y) = \sum_{\alpha=p,s}^{\eta=p,s} \mathbb{J}^{\alpha\eta}(z, y)$$

In order to analysis the PSF, loss is assumed in the medium that  $k_{\alpha,\varepsilon} := k_\alpha(1 + \mathbf{i}\varepsilon)$ . For  $\alpha, \beta \in \{s, p\}$ , define  $b_{\alpha\beta} = k_s$  only if  $\alpha = \beta = s$ , or otherwise  $b_{\alpha\beta} = k_p$ . Then by Parseval identity, we carry out

$$\begin{aligned} \mathbb{J}^{\alpha\beta}(z, y) &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi} \int_R \frac{\mathcal{T}_\alpha^\varepsilon(\xi)^T \overline{\mathcal{N}_\beta^\varepsilon(\xi)}}{\overline{\delta^\varepsilon(\xi)}} e^{\mathbf{i}(\mu_\alpha^\varepsilon z_2 - \overline{\mu_\beta^\varepsilon} y_2) + \mathbf{i}(y_1 - z_1)\xi} d\xi \\ &= \frac{1}{2\pi} \int_{-b_{\alpha\beta}}^{b_{\alpha\beta}} \frac{\mathcal{T}_\alpha(\xi)^T \overline{\mathcal{N}_\beta(\xi)}}{\overline{\delta(\xi)}} e^{\mathbf{i}(\mu_\alpha z_2 - \overline{\mu_\beta} y_2) + \mathbf{i}(y_1 - z_1)\xi} d\xi \\ &\quad + \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi} \int_{R \setminus [-b_{\alpha\beta}, b_{\alpha\beta}]} \frac{\mathcal{T}_\alpha^\varepsilon(\xi)^T \overline{\mathcal{N}_\beta^\varepsilon(\xi)}}{\overline{\delta^\varepsilon(\xi)}} e^{\mathbf{i}(\mu_\alpha^\varepsilon z_2 - \overline{\mu_\beta^\varepsilon} y_2) + \mathbf{i}(y_1 - z_1)\xi} d\xi \\ &:= \mathbb{F}^{\alpha\beta}(z, y) + \mathbb{R}^{\alpha\beta}(z, y) \end{aligned}$$

By the definition of Cauchy principle value, we can obtain

$$\begin{aligned} \mathbb{R}^{\alpha\beta}(z, y) &= \frac{1}{2\pi} \int_{[-k_s, k_s] \setminus [-b_{\alpha\beta}, b_{\alpha\beta}]} + \text{P.V} \int_{R \setminus [-k_s, k_s]} \frac{\mathcal{T}_\alpha(\xi)^T \overline{\mathcal{N}_\beta(\xi)}}{\overline{\delta(\xi)}} e^{\mathbf{i}(\mu_\alpha z_2 - \overline{\mu_\beta} y_2) + \mathbf{i}(y_1 - z_1)\xi} d\xi \\ &\quad - \frac{\mathbf{i}}{2} \left( \frac{\mathcal{T}_\alpha(k_R)^T \overline{\mathcal{N}_\beta(k_R)}}{\overline{\delta'(k_R)}} e^{\mathbf{i}(y_1 - z_1)k_R} - \frac{\mathcal{T}_\alpha(-k_R)^T \overline{\mathcal{N}_\beta(-k_R)}}{\overline{\delta'(-k_R)}} e^{-\mathbf{i}(y_1 - z_1)k_R} \right) e^{-\sqrt{k_R^2 - k_\alpha^2} z_2 - \sqrt{k_R^2 - k_\beta^2} y_2} \\ &:= \mathbb{R}_1^{\alpha\beta}(z, y) + \mathbb{R}_2^{\alpha\beta}(z, y) + \mathbb{R}_3^{\alpha\beta}(z, y) \end{aligned}$$

**Lemma 4.1** *Let  $d = (\kappa_R - 1)/2$ , then we have*

$$|\text{P.V} \int_1^\infty \frac{k_s^4 g(t)}{\delta(k_s t)}| \leq C \left( \int_1^\infty |g(t)| dt + \max_{t \in [\kappa_R - d, \kappa_R + d]} (|g(t)| + |g'(t)|) \right)$$

where  $\kappa_R = k_R/k_s$  and  $C$  only depend on  $\kappa$ .

**Proof.** Since  $\delta(k_s t)$  has a zero at  $\kappa_R$  of order one, we can choose  $\sigma > 0$  small enough such that

$$0 < C_1 \leq \frac{\delta'(k_s t)}{k_s^3} < C_2, \quad \frac{\delta''(k_s t)}{k_s^2} < C_2 \quad (4.2) \quad \boxed{\text{de\_eq}}$$

for any  $t \in I_1 = (\kappa_R - \sigma, \kappa_R + \sigma)$ . Let  $\delta_1(t) = \frac{\delta(k_s t)}{k_s^4(t - \kappa_R)}$  and  $I_2 = (1, \infty) \setminus I_1$  we have

$$\begin{aligned} |\text{P.V} \int_{1^\infty} \frac{k_s^4 g(t)}{\delta(k_s t)}| &\leq |\text{P.V} \int_{I_1} \frac{k_s^4 g(t)}{\delta(k_s t)} dt| + |\int_{I_2} \frac{k_s^4 g(t)}{\delta(k_s t)} dt| \\ &\leq |\int_{I_1} \frac{g(t)\delta_1(t)^{-1} - g(\kappa_R)\delta_1(\kappa_R)^{-1}}{t - \kappa_R} + \text{P.V} \int_{I_1} \frac{g(\kappa_R)\delta_1(\kappa_R)^{-1}}{t - \kappa_R}| + \max_{t \in I_2} \frac{k_s^4}{\delta(k_s t)} \int_{I_2} |g(t)| dt \\ &\leq \max_{t \in I_1} |(\frac{g(t)}{\delta_1(t)})'| + C \frac{k_s^4}{\delta(k_s t)} \int_{I_2} |g(t)| dt \end{aligned}$$

By the mean value theorem and [4.2](#), for any  $t \in (1, 2\kappa_R - 1)$  we have

$$\begin{aligned} (\frac{g(t)}{\delta_1(t)})' &= \frac{g'(t)\delta'(k_s t_1)k_s^5 - g(t)\delta''(k_s t_2)k_s^6/2}{\delta'(k_s t_1)} \quad t_1, t_2 \in [\kappa_R, t] \text{ or } [t, \kappa_R] \\ &\leq C \max_{t \in I_1} (|g(t)| + |g'(t)|) \end{aligned}$$

This completes the proof.  $\square$

**Lemma 4.2** *Let  $f(t)$  be a complex valued function in  $(\kappa, \infty)$  and satisfy that  $|f(t)| < C(1 + t^k)$ ,  $k \in \mathbb{Z}_+$ . Then for  $\rho > 1$  we have*

$$\begin{aligned} |\int_{\kappa}^1 |f(t)e^{-\rho\sqrt{t^2 - \kappa^2}} d\xi| &\leq C \frac{1}{\rho} \\ |\int_1^\infty |f(t)e^{-\rho\sqrt{t^2 - 1}} d\xi| &\leq C \frac{1}{\rho} \end{aligned}$$

**Proof.** It is simple to see that

$$\begin{aligned} |\int_{\kappa}^1 |f(t)e^{-\rho\sqrt{t^2 - \kappa^2}} dt| &\leq C \int_{\kappa}^1 (1 + t^k) e^{-\rho\sqrt{t^2 - \kappa^2}} dt \\ &\leq C \int_0^{\sqrt{(1 - \kappa^2)}} \frac{t}{\sqrt{t^2 + \kappa^2}} e^{-\rho t} dt \leq C \frac{1}{\rho} \end{aligned}$$

and

$$\begin{aligned} |\int_1^\infty |f(t)e^{-\rho\sqrt{t^2 - 1}} dt| &\leq C \int_1^\infty (1 + t^k) e^{-\rho\sqrt{t^2 - 1}} dt \\ &\leq C \int_0^\infty \frac{(1 + (t^2 + 1)^{k/2})t}{\sqrt{t^2 + 1}} e^{-\rho t} dt \leq C \frac{1}{\rho} \end{aligned}$$

This completes the proof.  $\square$

**Lemma 4.3** *For  $0 < \kappa < 1$ , let  $F(\lambda) = \int_0^\kappa f(t) e^{i\lambda(\sqrt{1 - t^2} - \tau\sqrt{\kappa^2 - t^2} + \alpha t)} dt$ , where  $\tau \geq c_0 > 0$  and  $\alpha \in \mathbb{R}$ , then we have*

$$|F(\lambda)| \leq C(\kappa) \lambda^{-\frac{1}{2N_*}} \left[ |f(\kappa)| + \int_0^\kappa |f'(t)| dt \right]$$

where  $N_* = \min\{N | \kappa^{2N-1} < c_0, N \in \mathbb{Z}_+\}$ .

**Proof.** Put  $\phi(t) = -\sqrt{1-t^2}$  and  $\psi(t, \tau) = \tau\kappa\phi(t/\kappa) - \phi(t) + \alpha t$ . For easy of notations, we denote the  $n$ -th partial derivative of  $g(t)$  with respect to  $t$  by  $g^{(n)}(t)$ . Then, it is to see that, for  $n > 1$

$$\psi^{(n)}(t, \tau) = \frac{\tau}{\kappa^{n-1}} \phi^{(n)}\left(\frac{t}{\kappa}\right) - \phi^{(n)}(t)$$

A standard computation show that

$$\phi^{(1)}(t) = \frac{t}{\sqrt{1-t^2}} \quad \phi^{(2)}(t) = \frac{1}{(1-t^2)^{3/2}}$$

Moreover, for  $n \geq 3$ , we have

$$\phi^{(n)}(t) = \frac{p_n(t)}{(1-t^2)^{n-1/2}} \quad (4.3)$$

where  $p_n = \sum_0^{n-2} a_k^n t^k$  is a  $(n-2)$ -th polynomial such that its coefficients satisfy the following recursion formula:

$$\begin{aligned} a_{n-1}^{n+1} &= (n+1)a_{n-2}^n, & a_{n-2}^{n+1} &= (n+2)a_{n-3}^n \\ a_k^{n+1} &= (k+1)a_{k+1}^n + (2n-k)a_{k-1}^n & \text{for } 1 \leq k \leq n-3 \\ a_0^{n+1} &= a_1^n \end{aligned}$$

Since the polynomial coefficients are all positive, it is obvious that for  $n \geq 1$ ,  $\phi^{(n)}(t)$  is a monotone increasing positive function. Using the recursion formula, it follows that

$$\phi^{(n)}(0) = \begin{cases} 0 & n \text{ is odd,} \\ (n-1)!!(n-3)!! & n \text{ is even.} \end{cases} \quad (4.4) \quad \boxed{\text{value\_0}}$$

where  $(2k-1)!!$  is double factorial and  $n > 3$ . We are now in the position to proof the inequality. Since  $0 < \kappa < 1$ , obersev that

$$\psi^{(2N_*+1)}(t, \tau) \geq \frac{\tau}{\kappa^{2N_*}} \phi^{(2N_*+1)}(t) - \phi^{(2N_*+1)}(t) > 0$$

Therefore,  $\psi^{(2N_*)}(t, \tau)$  is monotone increasing in  $[0, \kappa)$ . By  $\boxed{\text{value\_0}}$ , we get

$$\psi^{(2N_*)}(t, \tau) \geq \psi^{(2N_*)}(0, \tau) \geq \psi^{(2N_*)}(0, c_0) = C(2N_*) \left( \frac{c_0}{\kappa^{2N_*-1}} - 1 \right) > 0 \quad (4.5)$$

The lemma is now a direct consequence of lemma  $\boxed{\text{van}}$ .  $\square$

Now we are in the position to prove the main theorem of this section.

**proof of Theorem 3.1.** By lemma  $\boxed{\text{thm\_psf}}$  and expresision  $\boxed{\text{root\_Ga}}$   $\boxed{\text{ngreen}}$   $\boxed{\text{green}}$  (2.21), (2.29), it is easy to see that

$$\begin{aligned} |e_i \cdot \mathcal{N}_\alpha(k_s t) e_j| &\leq C k_s^3 (1+t^3), \quad |e_i \cdot \mathcal{T}_\alpha(k_s t) e_j| \leq C(1+t^2) \quad t \in \mathbb{R} \\ |(\mathcal{T}_\alpha(k_s t) e_i \cdot \overline{\mathcal{N}_\beta(k_s t) e_j}) e^{i k_s (y_1 - z_1) t - z_2 \sqrt{k_\alpha^2 - k_s^2 t^2} - y_2 \sqrt{k_\beta^2 - k_s^2 t^2}}| &\leq C k_s h e^{-\sqrt{k_R^2 - k_s^2}} \quad , t \in (\kappa_R - d, \kappa_R + d) \end{aligned}$$

Substituting  $\xi = k_s t$  into  $\mathbb{R}_1^{\alpha\beta}(z, y)$ ,  $\mathbb{R}_2^{\alpha\beta}(z, y)$ ,  $\mathbb{R}_3^{\alpha\beta}(z, y)$ , the theorem now follows from lemma  $\boxed{\text{pv\_term}}$ , lemma  $\boxed{\text{medi\_term}}$  and lemma  $\boxed{\text{cross\_term}}$ .  $\square$

To prove the lemma <sup>festimate1</sup>3.1, substitute (2.29) and (2.21) into  $F_{ss}(z, y), F_{pp}(z, y)$ , then we have

$$F^{pp}(z, y) = -\frac{1}{2\pi} \int_{(-k_p, k_p)} \frac{\mathbf{i}k_s^2 \mu_s}{\mu \gamma(\xi) \delta(\xi)} \begin{pmatrix} \xi^2 & -\xi \mu_p \\ -\xi \mu_p & \mu_p^2 \end{pmatrix} e^{\mathbf{i} \mu_p (z_2 - y_2) + \mathbf{i} \xi (y_1 - z_1)} \quad (4.6) \quad \boxed{\text{F\_p}}$$

$$\begin{aligned} F^{ss}(z, y) &= -\frac{1}{2\pi} \int_{(-k_p, k_p)} \frac{\mathbf{i}k_s^2 \mu_p}{\mu \gamma(\xi) \delta(\xi)} \begin{pmatrix} \mu_s^2 & \xi \mu_s \\ \xi \mu_s & \xi^2 \end{pmatrix} e^{\mathbf{i} \mu_s (z_2 - y_2) + \mathbf{i} \xi (y_1 - z_1)} \\ &\quad - \frac{1}{2\pi} \int_{(-k_s, k_s) \setminus (-k_p, k_p)} \frac{\mathbf{i}(k_s^2 - 4\xi^2) \mu_p}{\mu \gamma(\xi) \delta(\xi)} \begin{pmatrix} \mu_s^2 & \xi \mu_s \\ \xi \mu_s & \xi^2 \end{pmatrix} e^{\mathbf{i} \mu_s (z_2 - y_2) + \mathbf{i} \xi (y_1 - z_1)} \\ &:= F^{ss1}(z, y) + F^{ss2}(z, y) \end{aligned} \quad (4.7) \quad \boxed{\text{F\_s}}$$

**proof of Lemma <sup>festimate1</sup>3.1.** We only proof the case of  $i = j = 1$ , the other ones are similar. First, we have  $\gamma(\xi) \leq k_s^2$ ,  $\delta(\xi) \leq k_s^4$  and  $\mu_p \leq \mu_s$  when  $\xi \in (-k_p, k_p)$ . Then, if  $z = y$

$$-\text{Im}(F_{11}^{pp} + F_{11}^{ss1}) \geq \frac{1}{2\pi \mu} \int_{(-k_p, k_p)} \frac{\mu_p}{k_s^2} d\xi \quad (4.8)$$

$$= \frac{k_p^2}{2\pi \mu k_s^2} \int_0^\pi \sin^2(t) dt = \frac{1}{4(\lambda + 2\mu)} \quad (4.9)$$

It's left to proof  $-\text{Im} F_{11}^{ss2} > 0$ . If  $\xi \in (-k_s, k_s) \setminus (-k_p, k_p)$ ,  $\mu_p = \mathbf{i} \sqrt{\xi^2 - k_p^2}$ . Substituting it into  $F^{ss2}$ , we have

$$F_{11}^{ss2} = \frac{1}{2\pi \mu} \int_{(-k_s, k_s) \setminus (-k_p, k_p)} \frac{\mu_s^2 \sqrt{\xi^2 - k_p^2} (k_s^2 - 4\xi^2)}{(\xi^2 + \mathbf{i} \mu_s \sqrt{\xi^2 - k_p^2}) (\beta^2 - \mathbf{i} 4\xi^2 \mu_s \sqrt{\xi^2 - k_p^2})} d\xi \quad (4.10)$$

let  $\alpha = (\xi^2 + \mathbf{i} \mu_s \sqrt{\xi^2 - k_p^2}) (\beta^2 - \mathbf{i} 4\xi^2 \mu_s \sqrt{\xi^2 - k_p^2})$ . A simple computation show that  $\text{Im} \alpha = k_s^2 \mu_s \sqrt{\xi^2 - k_p^2} (k_s^2 - 4\xi^2)$ . It is easy to see that

$$-\text{Im} F_{11}^{ss2} = \frac{k_s^2}{2\pi \mu} \int_{(-k_s, k_s) \setminus (-k_p, k_p)} \frac{\mu_s^3 (\xi^2 - k_p^2) (k_s^2 - 4\xi^2)^2}{|\alpha|^2} d\xi > 0$$

For  $z \neq y$ , we denot  $y - z = |y - z|(\cos \phi, \sin \phi)^T$  for some  $0 \leq \phi \leq 2\pi$ . Then it is easy to see that

$$F^{pp}(z, y) = \frac{1}{\mu} \int_0^\pi A(\theta, \kappa) e^{\mathbf{i} k_s |z - y| \cos(\theta - \phi)}$$

The phase function  $f(\theta) = \cos(\theta - \phi)$  satisfies  $f'(\theta) = -\sin(\theta - \phi)$ ,  $f''(\theta) = -\cos(\theta - \phi)$ . For any given  $0 \leq \phi \leq 2\pi$ , we can decompose  $[0, \pi]$  into several intervals such that in each either  $|f''(\theta)| \geq 1/2$  or  $|f'(\theta)| \geq 1/2$  and  $f'(\theta)$  is monotonous. The amplitude function  $A(\theta, \kappa)$  and their derirates are integrable on  $[0, \pi]$ . Then the estimate for  $F_{pp}(z, y)$  follows by using lemma <sup>van</sup>2.4. The estimation of  $F^{ss}(z, y)$  can be proved similarly. This completes the proof.  $\square$

## 5. Extensions

In this section we consider the reconstruction of non-penetrable obstacles with the impedance boundary condition and penetrable obstacle in the half space by the RTM

algorithm <sup>alg\_rtm</sup>3.1. For non-penetrable obstacles with the impedance boundary condition on the obstacle, the measured data  $u_k^s(x_r, x_s) = u_k(x_r, x_s) - N(x_r, x_s)e_k$ ,  $k = 1, 2$ , where  $u_k^s(x_r, x_s)$  is the radiation solution of the following problem:

$$\Delta_e u_k^s(x) + \omega^2 u_k^s(x) = 0 \quad \text{in } \mathbb{R}_+^2 \setminus \bar{D}, \quad (5.1) \quad \text{elas\_4}$$

$$\sigma(u_k^s(x))\nu + \mathbf{i}\eta(x)u_k^s(x) = -(\sigma(N(x, x_s)e_k)\nu + \mathbf{i}\eta(x)N(x, x_s)e_k) \quad \text{on } \Gamma_D, \quad (5.2) \quad \text{elas\_bd2}$$

$$\sigma(u_k^s)e_2 = 0 \quad \text{on } \Gamma_0, \quad (5.3) \quad \text{elas\_b02}$$

By modifying the argument in Theorem <sup>resolution1</sup>3.4, we can show the following result whose proof is omitted.

resolution2

**Theorem 5.1** *For any  $z \in \Omega$ , let  $\Phi(y, z) \in \mathbb{C}^{2 \times 2}$  be the radiation solution of the problem:*

$$\begin{aligned} \Delta_e \Psi(y, z)e_j + \omega^2 \Psi e_j &= 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D} \\ \sigma_y(\Phi(y, z)e_j)\nu + \mathbf{i}\eta(y)\Phi(y, z)e_j &= -(\sigma_y(\overline{F(z, y)}e_j)\nu + \mathbf{i}\eta(y)\overline{F(z, y)}e_j) \quad \text{on } \Gamma_D \end{aligned}$$

Then, we have

$$\hat{I}_d(z) = -\text{Im} \int_{\Gamma_D} \sum_{i=1}^2 [(\overline{F(z, y)} + \Psi(y, z))e_i \cdot (\sigma_y(\overline{F(z, y)}e_j)\nu + \mathbf{i}\eta(y)\overline{F(z, y)}e_i)] ds(y) + W_{\hat{f}}(z)$$

where  $|W_{\hat{f}}(z)| \leq C(1 + k_s d_D)^4 (\epsilon_1(k_s h) + \epsilon_2(k_s h, h/d))$  uniformly for  $z$  in  $\Omega$ .

For the penetrable obstacle, the measured data  $u_k^s(x_r, x_s) = u_k(x_r, x_s) - N(x_r, x_s)e_k$ ,  $k = 1, 2$ , where  $u_k^s(x_r, x_s)$  is the radiation solution of the following problem:

$$\Delta_e u_k^s(x) + \omega^2 n(x)u_k^s(x) = -\omega^2(n(x) - 1)N(x, x_s)e_k \quad \text{in } \mathbb{R}^2 \quad (5.4) \quad \text{elas\_5}$$

$$\sigma(u_k^s)e_2 = 0 \quad \text{on } \Gamma_0, \quad (5.5) \quad \text{elas\_b03}$$

where  $n(x) \in L^\infty(\mathbb{R}^2)$  is a positive function which is equal to 1 outside  $D$ . By modifying the argument in Theorem <sup>resolution1</sup>3.4, the following theorem can be proved.

resolution3

**Theorem 5.2** *For any  $z \in \Omega$ , let  $\Psi(y, z) \in \mathbb{C}^{2 \times 2}$  be the radiation solution of the problem:*

$$\Delta_e \Psi(y, z)e_j + \omega^2 n(y)\Psi e_j = -\omega^2(n(y) - 1)\overline{F(z, y)}e_j \quad \text{in } \mathbb{R}^2$$

Then, we have

$$\hat{I}_d(z) = -\text{Im} \int_D \omega^2(1 - n(y)) \sum_{i=1}^2 [(\overline{F(z, y)} + \Psi(y, z))e_i \cdot \overline{F(z, y)}e_i] dy + W_{\hat{f}}(z) \quad (5.6)$$

where  $|W_{\hat{f}}(z)| \leq C(1 + \|n(\cdot)\|_{L^\infty(D)})(1 + k_s d_D)^4 (\epsilon_1(k_s h) + \epsilon_2(k_s h, h/d))$  uniformly for  $z$  in  $\Omega$ .

**Figure 2.** Example 1: From left to right: imaging results of a Dirichlet, a Neumann, a Robin boundary with impedance  $\eta(x) = 1$ , and a penetrable obstacle with diffractive index  $n(x) = 0.25$

figure\_1

## 6. Numerical experiments

In this section we present several numerical examples to show the effectiveness of our RTM method. To synthesize the scattering data we compute the solution  $u^s(x_r; x_s)$  of the scattering problem by representing the ansatz solution as the single layer potential with the Green function  $N(x; y)$  as the kernel and discretizing the integral equation by standard *Nyström* methods [16]. The boundary integral equations on  $\Gamma_D$  are solved on a uniform mesh over the boundary with ten points per probe wavelength. The sources and receivers are both placed on the surface  $\Gamma_d^0$  with equal-distribution, where  $d$  is the aperture. In all our numerical examples we choose  $h = 10$ ,  $d = 50$  and *Lamé* constant  $\lambda = 1/2$ ,  $\mu = 1/4$ . The boundaries of the obstacles used in our numerical experiments are parameterized as follows,

$$\begin{aligned} \text{Circle:} \quad & x_1 = \rho \cos(\theta), \quad x_2 = \rho \sin(\theta), \\ \text{Kite:} \quad & x_1 = \cos(\theta) + 0.65 \cos(2\theta) - 0.65, \quad x_2 = 1.5 \sin(\theta), \\ \text{p-leaf:} \quad & r(\theta) = 1 + 0.2 \cos(p\theta), \\ \text{peanut:} \quad & x_1 = \cos \theta + 0 : 2 \cos 3\theta; x_2 = \sin \theta + 0 : 2 \sin 3\theta, \\ \text{square:} \quad & x_1 = \cos 3\theta + \cos \theta; x_2 = \sin 3\theta + \sin \theta. \end{aligned}$$

where  $\theta \in [0, 2\pi]$ . The numerical imaging function is :

$$I_d(z) = \text{Im} \sum_{k=1}^2 \left\{ \frac{|\Gamma_0^d|}{N_s} \frac{|\Gamma_0^d|}{N_d} \sum_{s=1}^{N_s} \sum_{r=1}^{N_r} \sum_{i=1}^2 [\sigma_{x_s}(\mathbb{D}(x_s, z)e_i)e_2 \cdot e_k] [\sigma_{x_r}(\mathbb{D}(x_r, z)e_i)e_2 \cdot \overline{u_k^s(x_r, x_s)}] \right\}$$

**Example 1.** We consider imaging of a Dirichlet, a Neumann, a Robin boundary, and a penetrable obstacle. The imaging domain is  $(2; 2) \times (8; 12)$  with the sampling grid  $201 \times 201$  and  $N_s = N_r = 401$ . The angular frequency is  $\omega = 2\pi$ .

The imaging results are shown in Figure 7. It demonstrates clearly that our RTM algorithm can effectively image the upper boundary illuminated by the sources and receivers distributed along the boundary  $\Gamma_0$  for non-penetrable obstacles. The imaging values decrease on the shadow part of the obstacles and at the points away from the boundary of the obstacle.

**Example 2.** We consider the imaging of clamped obstacles with different shapes including circle, peanut, p-leaf and rounded square. The imaging domain is  $(2; 2) \times (8; 12)$  with the sampling grid  $201 \times 201$  and  $N_s = N_r = 401$ . The angular frequency is  $\omega = 3\pi, 5\pi$  for the single frequency and  $\omega = \pi \times [2 : 0.5 : 8]$  for the test of multiple frequencies.

**Figure 3.** Example 2: Imaging results of clamped obstacles with different shapes from top to below. The left row is imaged with single frequency data where  $\omega = 3\pi$ , The middle row is imaged with single frequency data where  $\omega = 5\pi$  and The left row is imaged with multi frequency data

figure\_2

**Figure 4.** Example 3: From left to right, true obstacle model with two circles. the imaging result with single frequency data where  $\omega = 3\pi$ , the imaging result with multiple frequency data.

figure\_31

**Figure 5.** Example 3: From left to right, true obstacle model with one circle and one peanut, the imaging result with single frequency data where  $\omega = 3\pi$ , the imaging result with multiple frequency data.

figure\_32

**Figure 6.** Example 4: Imaging results of a clamped obstacle with noise levels  $\mu = 0.2; 0.3; 0.4$  (from left to right). The top row is imaged with single frequency data where  $\omega = 4\pi$ , and the bottom row is imaged with multi-frequency data.

figure\_4

**Example 3** We consider the imaging of two sound soft obstacles. The first model consists of two circles along horizontal direction and the second one is a circle and a peanut along the vertical direction. The angular frequency is  $\omega = 3\pi$  for the test of the single frequency and  $\omega = \pi \times [2 : 0.5 : 8]$  for the test of multiple frequencies. Figure [figure\\_31](#) shows the imaging result of the first model. The imaging domain is  $[4, 4] \times [8, 12]$  with mesh size  $401 \times 201$  and  $N_s = N_r = 301$ . Figure [figure\\_32](#) shows the imaging result of the second model. The imaging domain is  $[4, 4] \times [8, 12]$  with mesh size  $401 \times 401$  and  $N_s = N_r = 301$ . The multi-frequency RTM imaging results in Figure [figure\\_31](#) and Figure [figure\\_32](#) are obtained by adding the imaging results from different frequencies. We observe from these two figures that imaging results can be greatly improved by stacking the multiple single frequency imaging results.

**Example 4** In this example we consider the stability of our half space RTM imaging function with respect to the complex additive Gaussian random noise. We introduce the additive Gaussian noise as follows

$$u_{\text{noise}} = u_s + \nu_{\text{noise}}$$

where  $u_s$  is the synthesized data and  $\nu_{\text{noise}}$  is the Gaussian noise with mean zero and standard deviation  $\mu$  times the maximum of the data  $|u_s|$ , i.e.  $\nu_{\text{noise}} = \frac{\mu \max |u_s|}{\sqrt{2}}(\varepsilon_1 + \mathbf{i}\varepsilon_2)$  and  $\varepsilon_i \sim \mathcal{N}(0, 1)$ .

Figure [figure\\_4](#) shows the imaging results using single frequency data added with additive Gaussian noise. The imaging quality can be improved by using multi-frequency data. as illustrated



## 7. Appendix A: proof of theorem <sup>elastic\_eq2</sup>2.1

In this section we will prove theorem <sup>elastic\_eq2</sup>2.1 by extending the classical argument in <sup>leis\_wilcox1975, Yves1988</sup>[23, 26, 18]. The existence of the solution can be proved by the method of limiting absorption principle. The argument is standard and we give several lemmas below, see e.g. <sup>leis</sup>[23] for the consideration for Helmholtz equation. For any  $z = 1 + i\varepsilon, \varepsilon > 0$ ,  $f \in H^1(\mathbb{R}_+^2)'$  with compact support in  $B_R = \{x \mid |x|^2 < R^2, x \in \mathbb{R}_+^2\} \subsetneq \mathbb{R}_+^2$  where  $B_R$  is an half disc of radius  $R$ , we consider the problem

$$\Delta_e u_z + z\omega^2 u = -f \quad \text{in } \mathbb{R}_+^2 \quad (7.1) \quad \text{elastic_eqz}$$

$$\sigma(u_z)e_2 = 0 \quad \text{on } \Gamma_0 \quad (7.2) \quad \text{elastic_b0}$$

By Lax-Milgrim lemma we know that <sup>elastic\_eqz, b0</sup>(7.1)-(7.2) has a unique solution  $u_z \in H^1(\mathbb{R}_+^2)$ . For any domain  $D \subset \mathbb{R}_+^2$ , we define the weighted space  $L^{2,s}(D)$ ,  $s \in \mathbb{R}$ , by

$$L^{2,s}(D) = \{v \in L_{\text{loc}}^2(D) : (1 + |x|^2)^{s/2} v \in L^2(D)\}$$

with the norm  $\|v\|_{L^{2,s}(D)} = (\int_D (1 + |x|^2)^s |v|^2 dx)^{1/2}$ . The weighted Sobolev space  $H^{1,s}(D)$ ,  $s \in \mathbb{R}$ , is defined as the set of functions in  $L^{2,s}(D)$  whose first derivative is also in  $L^{2,s}(D)$ . The norm  $\|v\|_{H^{1,s}(D)} = (\|v\|_{L^{2,s}(D)}^2 + \|\nabla v\|_{L^{2,s}(D)}^2)^{1/2}$ .

We need the following slight generalization of Rellich Theorem:

**Lemma 7.1** *Let  $\Omega$  be an open Lipschitz domain, then the sobolev space  $H^{1,-s}(\Omega)$  is compactly embeded in  $L^{2,-s'}(\Omega)$  for every  $s' > s > 0$ .*

**Lemma 7.2** *Let  $f \in L^2(\mathbb{R}_+^2)$  with compact support in  $B_R$ . For any  $z = 1 + i\varepsilon$ ,  $0 < \varepsilon < 1$ , we have, for any  $s > 1/2$ ,  $\|u_z\|_{H^{1,-s}(\mathbb{R}_+^2)} \leq C\|f\|_{L^2(\mathbb{R}_+^2)}$  for some constant independent of  $\varepsilon, u_z$ , and  $f$ .*

**Proof.** Let  $R_z$  denote the map from  $L_c^2(\mathbb{R}_+^2)$  to  $H^{1,-s}(\mathbb{R}_+^2)$  such that  $R_z(f) = u_z$  where  $L_c^2(\mathbb{R}_+^2)$  is denoted by all  $f \in L^2(\mathbb{R}_+^2)$  with compact support in  $B_R$ , then it is easy to see that  $R_z$  is a linear bounded operator. It follows from theorem 3.7 in <sup>Yves1988</sup>[18] that  $R_z$  is a uniformly continuous operator continues valued function on  $z = 1 + i\varepsilon$ ,  $0 < \varepsilon < 1$  with value in  $B(L_c^2(\mathbb{R}_+^2), H^{1,-s}(\mathbb{R}_+^2))$ . Then, we can obtain that  $R_z$  is uniformly bounded in  $B(L_c^2(\mathbb{R}_+^2), H^{1,-s}(\mathbb{R}_+^2))$ . This complete the proof by the defintion of the operator norm.  $\square$

We next recall the following lemma which states the absence of positive eigenvalues for the linear elasticity system in half space <sup>sini2004</sup>[25].

**Lemma 7.3** *Let  $u \in L^2(\mathbb{R}_+^2 \setminus \bar{D})$  such that  $u$  satisfies <sup>elas\_1</sup>(2.45) and <sup>elas\_b0</sup>(2.47), than we assert that  $u = 0$  in  $\mathbb{R}_+^2 \setminus \bar{D}$*

**Proof.** The asserting above can be proved by extending <sup>sini2004</sup>[25, theorem 3.1], here we omit the details.  $\square$

For any  $0 < \varepsilon < 1$ , we consider the problem

$$\Delta_\varepsilon u_\varepsilon + (1 + \mathbf{i}\varepsilon)\omega^2 u_\varepsilon = 0 \quad \text{in } \mathbb{R}_+^2 \setminus \bar{D} \quad (7.3) \quad \text{elas\_z1}$$

$$u_\varepsilon = g \quad \text{on } \Gamma_D \quad (7.4) \quad \text{elas\_zbd}$$

$$\sigma(u_\varepsilon)e_2 = 0 \quad \text{on } \Gamma_0 \quad (7.5) \quad \text{elas\_zb0}$$

We know that the above problem has a unique solution  $u_\varepsilon \in H^1(\mathbb{R}_+^2 \setminus \bar{D})$  by the Lax-Milgram Lemma. Thus, we have next lemma

**Lemma 7.4** *Let  $g \in H^{1/2}(\Gamma_D)$ . For any  $0 < \varepsilon < 1$ , we have, for any  $s > 1/2$ ,  $\|u_\varepsilon\|_{H^{1,-s}(\mathbb{R}_+^2 \setminus \bar{D})} \leq C\|g\|_{H^{1/2}(\Gamma_D)}$  for some constant independent of  $\varepsilon, u_\varepsilon$ , and  $g$ .*

**Proof.** Because  $h = \text{dist}(D, \Gamma_0) > 0$ , we can find three concentric circles  $B_{R_1}, B_{R_2}, B_{R_3}$  such that  $D \subsetneq B_{R_1} \subsetneq B_{R_2} \subsetneq B_{R_3} \subsetneq \mathbb{R}_+^2$ . Let  $\chi \in C_0^\infty(\mathbb{R}_+^2)$  be the cut-off function such that  $0 \leq \chi \leq 1$ ,  $\chi = 0$  in  $B_{R_1}$ , and  $\chi = 1$  outside of  $B_{R_2}$ . Let  $v_\varepsilon = \chi u_\varepsilon$ . Then  $v_\varepsilon$  satisfies (7.1) with  $z = 1 + \mathbf{i}\varepsilon$  and  $q = \sigma(u_\varepsilon)\nabla\chi + (\lambda + \mu)(\nabla^2\chi u_\varepsilon + \nabla u_\varepsilon \nabla\chi) + \mu\Delta\chi u_\varepsilon + \mu\text{div}u_\varepsilon \nabla\chi$ , where  $\nabla^2\chi$  is the Hessian matrix of  $\chi$ . Clearly  $q$  has compact support. By lemma 7.2 we can obtain

$$\|v_\varepsilon\|_{H^{1,-s}(\mathbb{R}_+^2)} \leq C\|u_\varepsilon\|_{H^1(B_{R_2} \setminus \bar{D})} \quad (7.6) \quad \text{elas\_ineq2}$$

for some constant  $C$  independent of  $\varepsilon > 0$ . Now let  $\chi_1 \in C_0^\infty(\mathbb{R}_+^2)$  be the cut-off function with that  $0 \leq \chi_1 \leq 1$ ,  $\chi_1 = 1$  in  $B_{R_2}$ , and  $\chi_1 = 0$  outside of  $B_{R_3}$ . For  $g \in H^{1/2}(\Gamma_D)$ , let  $u_g \in H^1(\mathbb{R}_+^2 \setminus \bar{D})$  be the lifting function such that  $u_g = g$  on  $\Gamma_D$  and  $\|u_g\|_{H^1(\mathbb{R}_+^2 \setminus \bar{D})} \leq C\|g\|_{H^{1/2}(\Gamma_D)}$ . By testing (7.3) with  $\chi_1^2(u_\varepsilon - u_g)$  and using the standard argument we have

$$\|u_\varepsilon\|_{H^1(B_{R_2} \setminus \bar{D})} \leq C(\|u_\varepsilon\|_{L^2(B_{R_3} \setminus \bar{D})} + \|g\|_{H^{1/2}(\Gamma_D)}). \quad (7.7) \quad \text{elas\_ineq3}$$

A combination of (7.6) and the above estimate yields

$$\|u_\varepsilon\|_{H^{1,-s}(\mathbb{R}_+^2 \setminus \bar{D})} \leq C(\|u_\varepsilon\|_{L^2(B_{R_3} \setminus \bar{D})} + \|g\|_{H^{1/2}(\Gamma_D)}). \quad (7.8) \quad \text{elas\_ineq4}$$

Now we claim

$$\|u_\varepsilon\|_{L^2(B_{R_3} \setminus \bar{D})} \leq C\|g\|_{H^{1/2}(\Gamma_D)}, \quad (7.9) \quad \text{elas\_ineq5}$$

for any  $g \in H^{1/2}(\Gamma_D)$  and  $\varepsilon > 0$ . If it were false, there would exist sequences  $\{g_m\} \subset H^{1/2}(\Gamma_D)$  and  $\{\varepsilon_m\} \subset (0, 1)$ , and  $\{u_{\varepsilon_m}\}$  be the corresponding solution of (7.3)-(7.5) such that

$$\|u_{\varepsilon_m}\|_{L^2(B_{R_3} \setminus \bar{D})} = 1 \text{ and } \|g_m\|_{H^{-1/2}(\Gamma_D)} \leq \frac{1}{m}. \quad (7.10) \quad \text{contradict}$$

Then  $\|u_{\varepsilon_m}\|_{H^{1,-s}(\mathbb{R}_+^2 \setminus \bar{D})} \leq C$ , and thus there is a subsequence of  $\{\varepsilon_m\}$ , which is still denoted by  $\{\varepsilon_m\}$ , such that  $\varepsilon_m \rightarrow \varepsilon' \in [0, 1]$ , and a subsequence of  $\{u_{\varepsilon_m}\}$ , which is still denoted by  $\{u_{\varepsilon_m}\}$ , such that it converges to some  $u_{\varepsilon'}$  in  $H^{1,-s'}(\mathbb{R}_+^2 \setminus \bar{D})$  by choosing  $s' > s$ . This is a consequence of Korn's inequality and lemma 7.1. So  $u_{\varepsilon'} \in H^{1,-s'}(\mathbb{R}_+^2 \setminus \bar{D})$  satisfies (7.3)-(7.5) with  $g = 0$  and  $\varepsilon = \varepsilon'$ .

By the integral representation satisfied by  $u_{\varepsilon_m}$ , we know that for  $y \in \mathbb{R}_+^2 \setminus \bar{B}_{R_1}$  and  $i = 1, 2$

$$u_{\varepsilon'}(y) \cdot e^i = \int_{\partial B_{R_1}} (\sigma(N_{\varepsilon'}(x, y)e_i)\nu) \cdot u_{\varepsilon'}(x) - (N_{\varepsilon'}(x, y)e_i) \cdot (\sigma(u_{\varepsilon'})_{\varepsilon'}\nu) ds(x) \quad (7.11) \quad \text{green\_rep}$$

If  $\varepsilon' > 0$ , we deduce from (7.11) that  $u_{\varepsilon'}$  decays exponentially and thus  $u_{\varepsilon'} \in H^1(\mathbb{R}_+^2 \setminus \bar{D})$ , then  $u_{\varepsilon'} = 0$  by the uniqueness of the solution in  $H^1(\mathbb{R}_+^2 \setminus \bar{D})$  with positive absorption. If  $\varepsilon' = 0$ , by the [18, theorem 5.2], we have  $u_{\varepsilon'} \in L^2(\mathbb{R}_+^2 \setminus \bar{D})$ . Then we conclude  $u_{\varepsilon'} = 0$  by the lemma 7.3. Therefore, in any case  $u_{\varepsilon'} = 0$ , which, however contradicts to 7.10. This complete the proof.  $\square$

Now we are in the position to prove the existence of Theorem 2.1.  $\text{elastic\_eq2}$

**Lemma 7.5** *For any  $s > 1/2$ ,  $u_\varepsilon : (0, 1) \rightarrow H^{1,-s}(\mathbb{R}_+^2 \setminus \bar{D})$  is a uniformly continuous operator valued function. Immediately,  $u_\varepsilon$  converges to some  $u_0$  in  $H^{1,-s}(\mathbb{R}_+^2 \setminus \bar{D})$  and  $u_0$  is a solution of (2.45-2.49).  $\text{elas\_elas\_ineq}$*

**Proof.** We also give a indirect prove here. Let  $\delta_0 > 0$  and  $\{\mu_n\}$  and  $\{\nu_n\}$  be sequences in  $(0, 1)$  such that

$$|\mu_n - \nu_n| \leq 1/n \quad \text{and} \quad \|u_{\mu_n} - u_{\nu_n}\|_{H^{1,-s}(\mathbb{R}_+^2 \setminus \bar{D})} \geq \delta_0 \quad (7.12)$$

Thus there is a subsequence of  $\{\mu_n\}$ , which is still denoted by  $\{\mu_n\}$ , such that  $\{\mu_n\} \rightarrow \epsilon \in [0, 1]$  and also  $\{\nu_n\} \rightarrow \epsilon$ . Then using lemma 7.4 and the procedure proving it, we get the  $u_\epsilon, v_\epsilon \in H^{1,-s'}(\mathbb{R}_+^2 \setminus \bar{D})$ , by choosing  $s' > s$ , such that

$$\begin{aligned} \|u_{\mu_n} - u_\epsilon\|_{H^{1,-s'}(\mathbb{R}_+^2 \setminus \bar{D})} &\rightarrow 0 \\ \|u_{\nu_n} - v_\epsilon\|_{H^{1,-s'}(\mathbb{R}_+^2 \setminus \bar{D})} &\rightarrow 0 \end{aligned}$$

and  $u_\epsilon = v_\epsilon$  by the same argument in lemma 7.4 which leads to a contradiction. Thus we have proved  $u_\varepsilon$  is uniformly continuous for  $\varepsilon \in (0, 1)$ . Then it is easy to see  $u_\varepsilon$  has a limitation in  $H^{1,-s}(\mathbb{R}_+^2 \setminus \bar{D})$  and the estimation of  $u_0$  can be obtained by (7.9). This completes the proof.  $\square$

It is remain to prove the uniqueness in theorem 2.1.  $\text{elastic\_eq2}$  Actually, it can be obtained following the existence of solution with any  $g \in H^{1/2}(\Gamma_D)$ .

**proof of Theorem 2.1**  $\text{elastic\_eq2}$  By the linearity of the problem, it is sufficient to prove that any  $u_0$  satisfies the system (2.45-2.47) with the corresponding homogeneous boundary-value vanishes identically in  $\mathbb{R}_+^2 \setminus \bar{D}$ . For any  $y \in \mathbb{R}_+^2 \setminus \bar{D}$ , there exists  $U^s(x, y)$  sataifies (2.45-2.47) with  $g(x) = -N(x, y)$  on  $\Gamma_D$  following the lemma 7.5 and we define  $U(x, y) = N(x, y) + U^s(x, y)$ . It is easy to see that  $U(x, y)$  satisfies the generalized radiation condition (2.48). Thus by the integral representaion of  $u_0$ , we have

$$\lim_{r \rightarrow \infty} \int_{S_r^+} (\sigma(U(x, y)e_i)\nu) \cdot u_0(x) - (U(x, y)e_i) \cdot (\sigma(u_0)\nu) ds(x) = 0$$

Finally, combining  $U(x, y) = 0, u_0(x) = 0$  on  $\Gamma_D$  and the Green integral theorem we find that

$$\begin{aligned} u_0(y)e_i &= \int_{\mathbb{R}_+^2 \setminus \bar{D}} -(\Delta_e(N(x, y)e_i) + \omega^2 N(x, y)e_i) \cdot u_0(x) dx \\ &= \int_{\mathbb{R}_+^2 \setminus \bar{D}} \Delta u_0(x) \cdot (N(x, y)e_i) - \Delta_e(N(x, y)e_i) \cdot u_0(x) \\ &= \int_{\Gamma_D} (\sigma(U(x, y)e_i)\nu) \cdot u_0(x) - (U(x, y)e_i) \cdot (\sigma(u_0)\nu) ds(x) = 0 \end{aligned}$$

Then the desired unique existence follows lemma elas\_exis 7.5. This completes the proof of theorem elastic\_eq2 2.1.  $\square$

## 8. Appendix B: proof of theorem diff\_solu 3.3

**Lemma 8.1** For any  $x, y \in D$ , let

$$p(x, y) = \lim_{\varepsilon \rightarrow 0^+} p^\varepsilon(x, y) := \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}} \frac{f(\mu_p^\varepsilon, \mu_s^\varepsilon, \xi)}{\delta^\varepsilon(\xi)} e^{i\mu_\alpha^\varepsilon x_2 + i\mu_\beta^\varepsilon y_2 + i\xi(y_1 - x_1)} d\xi$$

where  $f(a, b, c)$  is a homogeneous fifth order polynomial with respect to  $a, b, c$  and  $\alpha, \beta \in \{p, s\}$ . Then there exists a constant  $C > 0$  only dependant on  $\kappa$  such that

$$|p(x, y)| + k_s^{-1} |\nabla_x p(x, y)| + k_s^{-1} |\nabla_y p(x, y)| + k_s^{-2} |\nabla_x \nabla_y p(x, y)| \leq C((k_s h)^{-1/2} + k_s h e^{-\sqrt{k_R^2 - k_s^2} h})$$

uniformly for  $x, y \in D$ .

**Proof.** Without loss of generality, we assume  $k_\alpha \leq k_\beta$ . Then we can divide  $p(x, y)$  into two parts:

$$\begin{aligned} p(x, y) &= \lim_{\varepsilon \rightarrow 0^+} \int_{I_1} + \int_{I_2} \frac{f(\mu_p^\varepsilon, \mu_s^\varepsilon, \xi)}{(k_\alpha^\varepsilon)^2 \delta^\varepsilon(\xi)} e^{i\mu_\alpha^\varepsilon x_2 + i\mu_\beta^\varepsilon y_2 + i\xi(y_1 - x_1)} d\xi \\ &= \int_{I_1} \frac{f(\mu_p, \mu_s, \xi)}{k_\alpha^2 \delta(\xi)} e^{i\mu_\alpha x_2 + i\mu_\beta y_2 + i\xi(y_1 - x_1)} d\xi \\ &\quad + \lim_{\varepsilon \rightarrow 0^+} \int_{I_2} \frac{f(\mu_p^\varepsilon, \mu_s^\varepsilon, \xi)}{(k_\alpha^\varepsilon)^2 \delta^\varepsilon(\xi)} e^{i\mu_\alpha^\varepsilon x_2 + i\mu_\beta^\varepsilon y_2 + i\xi(y_1 - x_1)} d\xi \\ &= p_1(x, y) + p_2(x, y) \end{aligned}$$

where  $I_1 = (-k_\alpha, k_\alpha)$  and  $I_2 = \mathbb{R} \setminus [-k_\alpha, k_\alpha]$ . Substituting  $\xi = k_\alpha t$  into  $p_1(x, y)$ , we get

$$p_1(x, y) = \int_{-1}^1 \frac{f(\mu_p(k_\alpha t), \mu_s(k_\alpha t), k_\alpha t)}{k_\alpha \delta(k_\alpha t)} e^{ik_\alpha x_2(\sqrt{1-t^2} + \tau\sqrt{\varsigma^2 - t^2} + \gamma t)} dt$$

where  $\tau = y_2/x_2$ ,  $\varsigma = k_\beta/k_\alpha$  and  $\gamma = (y_1 - x_1)/x_2$ . It is easy to see that the phase function  $\phi(t) = \sqrt{1-t^2} + \tau\sqrt{\varsigma^2 - t^2} + \gamma t$  satisfies  $|\phi''(t)| \geq 1/(1-t^2)^{3/2} \geq 1$  for  $t \in (-1, 1)$ . Then we can obtain  $|p_1(x, y)| \leq C1/(k_s h)^{1/2}$  by lemma van 2.4.

For  $p_2(x, y)$ , by the definition of Cauchy principle value and lemma pv\_term 4.1 and lemma medi\_term 4.2, we can easily obtain:

$$|p_2(x, y)| \leq C\left(\frac{1}{k_s h} + k_s h e^{-\sqrt{k_R^2 - k_s^2} h}\right)$$

This completes the proof of the estimate for  $|p(x, y)|$ . The other estimates can be proved by a similar argument. We omit the details here.  $\square$

Now we are ready to prove Theorem diff\_solu 3.3.  
**proof of Theorem diff\_solu 3.3** Denote by  $u_1^\varepsilon, u_2^\varepsilon$  the corresponding solution of equations

~~elas\_3.10,3.13~~  
~~elas\_r2~~

where  $\omega$  is substituted by  $\omega(1 + \mathbf{i}\varepsilon)$  for any  $0 < \varepsilon < 1$ . Let  $w^\varepsilon(x)$  be the solution of the problem:

$$\Delta_e w^\varepsilon + (1 + \mathbf{i}\varepsilon)\omega^2 w^\varepsilon = 0 \quad \text{in } \mathbb{R}_+^2 \quad (8.1) \quad \boxed{\text{elas\_z3}}$$

$$\sigma(w^\varepsilon)e_2 = -\sigma(u_2^\varepsilon)e_2 \quad \text{on } \Gamma_0 \quad (8.2) \quad \boxed{\text{elas\_zb01}}$$

Then  $u_1^\varepsilon - u_2^\varepsilon - w^\varepsilon$  satisfies ~~(7.3), (7.5)~~ ~~elas\_7.3, 7.5~~ with the boundary condition  $u_1^\varepsilon - u_2^\varepsilon - w^\varepsilon = -w^\varepsilon$  on  $\Gamma_D$ . Thus by the limiting absorption principle, lemma ~~2.46~~ ~~elas\_bd~~ and trace theorem, we have

$$\|\sigma_x(u_1^\varepsilon - u_2^\varepsilon)\nu\|_{H^{-1/2}(\Gamma_D)} \leq C(\|w^\varepsilon\|_{H^{1/2}(\Gamma_D)} + \|T_x^\nu(w^\varepsilon)\|_{H^{-1/2}(\Gamma_D)}) \quad (8.3) \quad \boxed{\text{diff1}}$$

$$\leq C(1 + \|T_h^\varepsilon\|) \max_{x \in D} (|w^\varepsilon(x)| + d_D |\nabla w^\varepsilon(x)|) \quad (8.4)$$

where  $C$  is independent of  $\varepsilon, \omega$ . By the integral representation formula we have for any  $z \in \Gamma_0$

$$u_2^\varepsilon(z) \cdot e_i = \mathcal{G}(u_2^\varepsilon(\cdot), \Phi(\cdot, z)e_i) \quad (8.5)$$

$$\sigma^\varepsilon(z) \cdot e_i = \mathcal{G}(u_2^\varepsilon(\cdot), [\sigma_z(\Phi^T(\cdot, z)e_1)e_2 \cdot e_i, \sigma_z(\Phi^T(\cdot, z)e_2)e_2 \cdot e_i]^T) \quad (8.6)$$

which yields by using the integral representation again that for  $x \in D$

$$w^\varepsilon(x) \cdot e_j = \int_{\Gamma_0} N^\varepsilon(z, x) e_j \cdot (\sigma_z(u_2^\varepsilon(z))e_2) ds(z) \quad (8.7)$$

$$= \mathcal{G}(u_2^\varepsilon(\cdot), v^\varepsilon(x, \cdot)e_j) \quad (8.8)$$

where

$$e_i^T v^\varepsilon(x, y) e_j = \int_{\Gamma_0} N^\varepsilon(z, x) e_j \cdot (\sigma_z(\Phi^\varepsilon(z, y)e_i)e_2) ds(z) \quad (8.9)$$

Since  $\|\sigma(u_2^\varepsilon)\nu\|_{H^{-1/2}(\Gamma_D)} \leq \|T_f^\varepsilon\| \|g\|_{H^{1/2}(\Gamma_D)}$ , we obtain

$$|w^\varepsilon(x)| \leq C(1 + \|T_f^\varepsilon\|) \|g\|_{H^{1/2}(\Gamma_D)} \max_{x \in D} (|v^\varepsilon(x, y)| + d_D |\nabla_y v^\varepsilon(x, y)|) \quad (8.10)$$

and

$$|\nabla w^\varepsilon(x)| \leq C(1 + \|T_f^\varepsilon\|) \|g\|_{H^{1/2}(\Gamma_D)} \max_{x \in D} (|\nabla_x v^\varepsilon(x, y)| + d_D |\nabla_x \nabla_y v^\varepsilon(x, y)|) \quad (8.11)$$

By ~~(8.3)~~ ~~diff1~~ and letting  $\varepsilon \rightarrow 0^+$ , we have

$$\|\sigma(u_1 - u_2)\nu\|_{H^{-1/2}(\Gamma_D)} \leq C(1 + \|T_f\|)(1 + \|T_h\|) \|g\|_{H^{1/2}(\Gamma_D)} \quad (8.12) \quad \boxed{\text{diff2}}$$

$$\max_{x \in D} \lim_{\varepsilon \rightarrow 0^+} (|v^\varepsilon(x, y)| + d_D |\nabla_y v^\varepsilon(x, y)| + d_D |\nabla_x v^\varepsilon(x, y)| + d_D^2 |\nabla_x \nabla_y v^\varepsilon(x, y)|) \quad (8.13)$$

Applying the Fourier transformation to the first horizontal variable of  $T_z^{e_2} \Phi^\varepsilon(z, y)$ , we have

$$\mathcal{F}[T_z^{e_2} \Phi^\varepsilon](\xi, 0; y) = \frac{\mu}{2\omega^2} \left[ \begin{pmatrix} 2\xi^2 & -2\xi\mu_p \\ -\frac{\beta\xi}{\mu_p} & \beta \end{pmatrix} e^{\mathbf{i}\mu_p y_2} + \begin{pmatrix} \beta & \frac{\xi\beta}{\mu_s} \\ 2\xi\mu_s & 2\xi^2 \end{pmatrix} e^{\mathbf{i}\mu_s y_2} \right] e^{-\mathbf{i}\xi y_1}$$

Using Parseval identity combined with above formula and formula ~~2.21~~ ~~ngreen~~, we have

$$\lim_{\varepsilon \rightarrow 0^+} v^\varepsilon(x, y) = \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}} \mathcal{F}[N^\varepsilon](\xi, 0; x)^T \mathcal{F}[T_z^{e_2} \Phi^\varepsilon](-\xi, 0; y) d\xi$$

This completes the proof by using lemma ~~8.1~~ ~~es\_diri\_neu~~. □

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