

Absence of Positive Eigenvalues for the Linearized Elasticity System in the Half Space

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Abstract. In this paper, we prove the linearized elasticity system in the half-space with traction free boundary has no eigenvalues. We consider the constant Lamé coefficients and the density outside the obstacle with different physical properties such as penetrable or non-penetrable, and for non-penetrable obstacles, the type of boundary conditions on the boundary of the obstacle.

1. Introduction

section1

In this paper, we consider the linearized and isotropic elasticity system defined on an unbounded domain $\Omega = \mathbb{R}_+^2 \setminus \bar{D}$ with traction free surface, where $D \subsetneq \mathbb{R}_+^2$ is a bounded Lipschitz domain with the unit outer normal ν to its boundary Γ_D . We study the eigenvalues of the following elastic scattering problem in the isotropic homogeneous medium half space with *Lamé* constant λ and μ and constant density $\rho \equiv 1$:

$$\nabla \cdot \sigma(\mathbf{u}) + \rho\omega^2 \mathbf{u} = f \quad \text{in } \mathbb{R}_+^2 \setminus \bar{D} \quad (1.1)$$

elastic_eq

$$\mathbf{u} = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \sigma(\mathbf{u}) \cdot e_2 = 0 \quad \text{on } \Gamma_0 \quad (1.2)$$

together with the constitutive relation (Hookes law)

$$\sigma(\mathbf{u}) = 2\mu\varepsilon(\mathbf{u}) + \lambda\text{div}\mathbf{u}\mathbb{I}$$

$$\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$$

where ω is the circular frequency, $\mathbf{u}(x_1, x_2) = (u_1(x), u_2(x))^T \in \mathbb{C}^2$ denotes the displacement fields and $\sigma(u)$ is the stress tensor. We also need to define the surface traction $T_x^n(\cdot)$ on the normal direction \mathbf{n} ,

$$T_x^n \mathbf{u}(x) := \sigma \cdot \mathbf{n} = 2\mu \frac{\partial \mathbf{u}}{\partial n} + \lambda n \text{div} \mathbf{u} + \mu n \times \text{curl} \mathbf{u}$$

For simplicity, let's introduce *Lamé* operator Δ_e as

$$\Delta_e \mathbf{u} = (\lambda + 2\mu) \nabla \nabla \cdot \mathbf{u} - \mu \nabla \times \nabla \times \mathbf{u} = \nabla \cdot \sigma(\mathbf{u})$$

The layout of the paper is as follows. In section 2

2. Absence of Positive Eigenvalues

Throughout the paper, we will assume that for $z \in \mathbb{C}$, $z^{1/2}$ is the analytic branch of \sqrt{z} such that $\text{Im}(z^{1/2}) \geq 0$. This corresponds to the right half real axis as the branch cut in the complex plane. For $z = z_1 + \mathbf{i}z_2$, $z_1, z_2 \in \mathbb{R}$, we have

$$z^{1/2} = \text{sgn}(z_2) \sqrt{\frac{|z| + z_1}{2}} + \mathbf{i} \sqrt{\frac{|z| - z_1}{2}} \quad (2.1)$$

convention_1

For z on the right half real axis, we take $z^{1/2}$ as the limit of $(z + \mathbf{i}\varepsilon)^{1/2}$ as $\varepsilon \rightarrow 0^+$.

By taking the Fourier transform of (??-??), we obtain ODEs for x_2 in R_+

$$\mu \frac{d^2 \hat{u}_1}{dx_2^2} + \mathbf{i}(\lambda + \mu) \xi \frac{d \hat{u}_2}{dx_2} + (\omega^2 - (\lambda + 2\mu) \xi^2) \hat{u}_1 = 0 \quad (2.2)$$

pp3

$$(\lambda + 2\mu) \frac{d^2 \hat{u}_2}{dx_2^2} + \mathbf{i}(\lambda + \mu) \xi \frac{d \hat{u}_1}{dx_2} + (\omega^2 - \mu \xi^2) \hat{u}_2 = 0 \quad (2.3)$$

pp4

and the boundary coditions on $x_2 = 0$ are

$$\mu \frac{d \hat{u}_1}{dx_2} + \mathbf{i} \mu \xi \hat{u}_2 = 0 \quad (2.4)$$

pp5

$$(\lambda + \mu) \frac{d \hat{u}_2}{dx_2} + \mathbf{i} \lambda \xi \hat{u}_1 = 0 \quad (2.5)$$

pp6

In order to work with real coefficient, we use the change of variables:

$$v_1 = \mathbf{i}u_1, \quad v_2 = u_2, \quad \mathbf{v} = (v_2, v_2)^T$$

$$[\mathbb{A}_1 \frac{d^2}{dx_2^2} + (\mathbb{A}_\xi^2 - (\mathbb{A}_\xi^2)^T) \frac{d}{dx_2} - \mathbb{A}^3] \mathbf{v}$$

$$\mathbb{A}^1 = \begin{pmatrix} \mu & 0 \\ 0 & \lambda + 2\mu \end{pmatrix}, \quad \mathbb{A}_\xi^2 = \begin{pmatrix} 0 & -(\lambda + \mu)\xi \\ (\lambda + \mu)\xi & 0 \end{pmatrix}, \quad \mathbb{A}^3 = \begin{pmatrix} \lambda + 2\mu & 0 \\ 0 & \mu \end{pmatrix}$$

References

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