Reverse Time Migration for Extended Obstacles in the Half Space: Elastic Waves

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Abstract. We consider a reverse time migration method for reconstructing extended obstacles in the half space with finite aperture data using elastic waves at a fixed frequency. We prove the resolution of the reconstruction method in terms of the aperture and the depth of the obstacle embedded in the half space. The resolution analysis studied by virtue of point spread function implies that the imaginary part of the cross-correlation imaging function always peaks on the boundary of the obstacle. Numerical experiments are included to illustrate the powerful imaging quality and to confirm our resolution results.

1. Introduction

In this paper we study a reverse time migration (RTM) algorithm to find the support of an unknown obstacle in the half space from the measurement of scattered waves on the boundary of the half space which is far away from the obstacle. The physical properties of the obstacle such as penetrable or non-penetrable, and for non-penetrable obstacles, the type of boundary conditions on the boundary of the obstacle, are not required in the algorithm.

Let the non-penetrable obstacle occupy a bounded Lipschitz domain $D \subset \mathbb{R}^2_+$ with ν the unit outer normal to its boundary Γ_D. We assume the incident wave is emitted by a point source located at x_s, explosive along the polarization direction $q \in \mathbb{R}^2$, on the surface $\Gamma_0 = \{(x_1, x_2)^T : x_1 \in \mathbb{R}, x_2 = 0\}$ which is far away from the obstacle. The measured data u_q corresponding to the polarization direction q is the solution of the following elastic scattering problem in the isotropic homogeneous medium half space with *Lamé* constant λ and μ and constant density $\rho \equiv 1$:

$$\nabla \cdot \sigma(u_q) + \rho\omega^2 u_q = -\delta_{x_s}(x) q \quad \text{in } \mathbb{R}^2_+ \backslash \bar{D} \tag{1.1}$$

$$u_q = 0 \ \text{ on } \Gamma_D \ \text{ and } \ \sigma(u_q) \cdot e_2 = 0 \ \text{ on } \Gamma_0 \tag{1.2}$$

together with the constitutive relation (Hookes law)

$$\sigma(u) = 2\mu\varepsilon(u) + \lambda \mathrm{div} u \mathbb{I}$$
$$\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$$

where ω is the circular frequency, $u(x) \in \mathbb{C}^2$ denotes the displacement fields and $\sigma(u)$ is the stress tensor. We also need to define the surface traction $T^n_x(\cdot)$ on the normal direction n,

$$T^n_x u(x) := \sigma \cdot n = 2\mu \frac{\partial u}{\partial n} + \lambda n \mathrm{div} u + \mu n \times \mathrm{curl} u$$

For simplicity, let's introduce *Lamé* operator Δ_e as

$$\Delta_e u = (\lambda + 2\mu)\nabla\nabla \cdot u - \mu\nabla \times \nabla \times u = \nabla \cdot \sigma(u)$$

The equation (1.1) is understood as the limit when $x_s \in \mathbb{R}^2_+ \backslash \bar{D}$ tends to Γ_0 whose precise meaning will be given below after we introduce the Neumann Green Tensor and the definition of the radiation condition.

The reverse time migration (RTM) method, which consists of back-propagating the complex conjugated data into the background medium and computing the crosscorrelation between the incident wave field and the backpropagated field to output the final imaging profile, is nowadays widely used in exploration geophysics [5, 6, 7, 9, 16]. In [10, 11, 12], the RTM method for reconstructing extended targets using acoustic, electromagnetic and elastic waves at a fixed frequency in the free space is proposed and studied. The resolution analysis in [10, 11, 12] is achieved without using the small inclusion or geometrical optics assumption previously made in the literature (e.g. [3, 7]). In [13], a new RTM algorithm is developed for finding extended targets in a planar waveguide which is motivated by the generalized Helmholtz-Kirchhoff identity for scattering problems in waveguides.

For the isotropic elastic media, one can process the elastic data either by separating P-wave and S-wave using Helmholtz decomposition and migrating each mode using methods based on acoustic wave theory [15, 18], or by migrating the whole elastic data set based on full elastic wave equation in the geophysical exploration community. In this paper, we adopt the cross-correlation between all the component of the source and receiver displacement wavefield, which is a mixture of P-wave and S-wave. Furthermore this kind condition can be easily extended to inhomogeneous elastic medium and even anisotropic elastic wave imaging. The purpose of this paper is to provide a new mathematical understanding of the RTM method by extending [14] where RTM method for extended targets in the half space using acoustic wave is considered. Compared to the scalar acoustic wave imaging, the vector elastic wave imaging is more complex due to a mixture of P-wave and S-wave mode. However, the virtue of the latter method is no longer need to separate the scalar and vector potentials prior to the imaging condition.

The layout of the paper is as follows. In section 2 we study the two Green Tensor for the scattering problem in the half space satisfying the homogeneous Neumann condition and Dirichlet condition on Γ_0. We recall the derivation of the Green Tensor by the method of Fourier transform and derive an alternative form of the Green Tensor which is crucial for the analysis in the rest. In section 3 we study the direct scattering problem. In section 4 we introduce the RTM algorithm. In section 5 we study the point spread function. In section 6 we study the resolution analysis of the RTM method. In section 6 we report extensive numerical experiments to show the competitive performance of the RTM algorithm.

2. Green Tensor in the half space

In this section we will study the elastic Green Tensor in the half-space with Neumann boundary [20]:

$$\Delta_e N(x;y) + \omega^2 N(x,y) = -\delta_y(x)\mathbb{I} \quad \text{in} \quad \mathbb{R}^2_+, \tag{2.1}$$

$$\sigma_x(N(x,y))e_2 = 0 \qquad \text{on} \quad x_2 = 0 \tag{2.2}$$

and with Dirichlet Boundary [4]

$$\Delta_e D(x,y) + \omega^2 D(x,y) = -\delta_y(x)\mathbb{I} \quad \text{in} \quad \mathbb{R}^2_+, \tag{2.3}$$

$$D(x,y) = 0 \qquad \text{on} \quad x_2 = 0 \tag{2.4}$$

where $\delta_y(x)$ is the Dirac source at $y \in R^2_+$ and $N(x,y)$, $D(x,y)$ are $\mathbb{C}^{2\times 2}$ matrixes. We will first use Fourier transform to derive the formula of Green Tensor in frequency domain. Let

$$\hat{N}(\xi, x_2; y_2) = \int_{-\infty}^{+\infty} N(x_1, x_2; y)e^{-\mathbf{i}(x_1-y_1)\xi}dx_1 \tag{2.5}$$

Throughout the paper, we will assume that for $z \in \mathbb{C}$, $z^{1/2}$ is the analytic branch of $\sqrt{z}$ such that $\mathrm{Im}\,(z^{1/2}) \geq 0$. This corresponds to the rigt half real axis as the branch cut in the complex plane. For $z = z_1 + \mathbf{i}z_2, z_1, z_2 \in \mathbb{R}$, we have

$$z^{1/2} = sgn(z_2)\sqrt{\frac{|z|+z_1}{2}} + \mathbf{i}\sqrt{\frac{|z|-z_1}{2}} \tag{2.6}$$ convention_1

For z on the right half real axis, we take $z^{1/2}$ as the limit of $(z+\mathbf{i}\varepsilon)^{1/2}$ as $\varepsilon \to 0^+$.

Let $\Phi(x,y)$ be the fundamental solution of the elastic equation kupradze1963progress [23] and recall that

$$\hat{\Phi}(\xi, x_2; y_2) = \frac{\mathbf{i}}{2\omega^2}\left[\begin{pmatrix} \mu_s & -\xi\frac{x_2-y_2}{|x_2-y_2|} \\ -\xi\frac{x_2-y_2}{|x_2-y_2|} & \frac{\xi^2}{\mu_s} \end{pmatrix} e^{\mathbf{i}\mu_s|x_2-y_2|} + \begin{pmatrix} \frac{\xi^2}{\mu_p} & \xi\frac{x_2-y_2}{|x_2-y_2|} \\ \xi\frac{x_2-y_2}{|x_2-y_2|} & \mu_p \end{pmatrix} e^{\mathbf{i}\mu_p|x_2-y_2|}\right]$$

where

$$\mu_\alpha = (k_\alpha^2 - \xi^2)^{1/2} \qquad \text{for} \quad \alpha = s, p \tag{2.7}$$

By the standard arguement in ODEs, the Green Tensor in half-space can be deduced as

$$\hat{N}(\xi, x_2; y_2) = \hat{\Phi}(\xi, x_2; y_2) - \hat{\Phi}(\xi, x_2; -y_2) + \hat{N}_c(\xi, x_2; y_2) \tag{2.8}$$

$$\begin{aligned} \hat{N}_c(\xi, x_2; y_2) = &= \frac{\mathbf{i}}{\omega^2\delta(\xi)}\Big\{A(\xi)e^{\mathbf{i}\mu_s(x_2+y_2)} + B(\xi)e^{\mathbf{i}\mu_p(x_2+y_2)} \\ &\quad + C(\xi)e^{\mathbf{i}\mu_s x_2 + \mu_p y_2} + D(\xi)e^{\mathbf{i}\mu_p x_2 + \mu_s y_2}\Big\} \end{aligned} \tag{2.9}$$

where

$$A(\xi) = \begin{pmatrix} \mu_s\beta^2 & -4\xi^3\mu_s\mu_p \\ -\xi\beta^2 & 4\xi_4\mu_p \end{pmatrix} \qquad B(\xi) = \begin{pmatrix} 4\xi^4\mu_s & \xi\beta^2 \\ 4\xi^3\mu_s\mu_p & \mu_p\beta^2 \end{pmatrix}$$

$$C(\xi) = \begin{pmatrix} 2\xi^2\mu_s\beta & -2\xi\mu_s\mu_p\beta \\ -2\xi^3\beta & 2\xi^2\mu_p\beta \end{pmatrix} \quad D(\xi) = \begin{pmatrix} 2\xi^2\mu_s\beta & 2\xi^3\beta \\ 2\xi\mu_s\mu_p\beta & 2\xi^2\mu_p\beta \end{pmatrix}$$

and $\beta(\xi) = k_s^2 - 2\xi^2$, $\delta(\xi) = \beta^2 + 4\xi^2\mu_s\mu_p$.

The desired Green function should be obtained by taking the inverse Fourier transform of $\hat{N}(\xi, x_2; y_2)$. Unfortunately, one cannot simply take the inverse Fourier transform in the above formula because $\delta(\xi)$ have zero points in the real axis by lemma root_De1 achenbach2001linear 2.1 [1][22].

root_De1 **Lemma 2.1** *Let* Lamé *constant* $\lambda, \mu \in \mathbb{R}^+$, *then the Rayleigh equation* $\delta(\xi) = 0$ *has only two roots denoted by* $\pm k_R$ *in complex plane. Morever,* $k_R > k_s > k_p$, $k_R \in \mathbb{R}$ *and* k_R *is called Rayleigh wave number.*

Proof. For the sake of completeness, we include a proof here. It is well known that

$$\delta(\xi) = (k_s^2 - 2\xi^2)^2 + 4\xi^2(k_s^2 - \xi^2)^{1/2}(k_p^2 - \xi^2)^{1/2} \tag{2.10}$$

However, $\delta(\xi)$ is rendered single-valued by selecting branch cuts along $k_p < \mathrm{Re}\,(\xi) < k_s, \mathrm{Im}\,(\xi) = 0$ which is consistent with the convention (g1 ??). A simple computation show that $\delta(\pm k_s) > 0$ and $\delta(\pm\infty + 0\mathbf{i}) < 0$. By the continuity of $\delta(\xi)$, we can obtain that it has at least two real zero points which denoted by $\pm k_R$.

Now it turn to proof that $\delta(\xi)$ has only two roots in the complex plane by the principle of arguement which follows as a theorem of the theory of complex variables Ahlfors1979Complex [2]. Now consider the contour C consisting of Γ, and C_l and C_r where $C_r = [k_p + \mathbf{i}0^+, k_s + \mathbf{i}0^+] \cup [k_p + \mathbf{i}0^-, k_s + \mathbf{i}0^-]$ that surround $[k_p, k_s]$, $C_l = [-k_s + \mathbf{i}0^+, -k_p + \mathbf{i}0^+] \cup [-k_s + \mathbf{i}0^-, -k_p + \mathbf{i}0^-]$ that surround $[-k_s, -k_p]$ and Γ denotes a circle with enough large radius. Since the function $\delta(\xi)$ clears does not have poles in the complex ξ-plane and we find that within the contour $C = \Gamma \cup C_r \cup C_l$ the number of zeros is given by

$$Z = \frac{1}{2\pi\mathbf{i}} \int_C \frac{d\delta}{d\xi} \frac{d\xi}{\delta(\xi)} \tag{2.11}$$ zero

Since $\delta(\xi) = \delta(-\xi)$ the images of C_r and C_l are the same, and one of them, say C_r, needs to be considered. We have $\delta(k_p) = (k_s^2 - 2k_p^2)^2$ and along C_r: $\delta^{\pm}(\xi) = (k_s^2 - \xi^2)^2 \mp \mathbf{i}4\xi^2\sqrt{k_s^2 - \xi^2}\sqrt{\xi^2 - k_p^2}$, and $\delta(k_s) = k_s^4$ where the plus sign applies above the cut, and the minus sign applies below the cut for $\delta(\xi)$. Let $f_1(\xi) = (k_s^2 - \xi^2)^2$ and $f_2(\xi) = 4\xi^2\sqrt{k_s^2 - \xi^2}\sqrt{\xi^2 - k_p^2}$. Then we have

$$\int_{C_r} \frac{d\delta}{d\xi} \frac{d\xi}{\delta(\xi)} \tag{2.12}$$

$$= \int_{k_p}^{k_s} \frac{\delta'_+(\xi)}{\delta_+(\xi)} - \frac{\delta'_-(\xi)}{\delta_-(\xi)} d\xi \tag{2.13}$$

$$= 2\mathbf{i} \int_{k_p}^{k_s} \operatorname{Im}\left(\frac{\delta'_+(\xi)}{\delta_+(\xi)}\right) d\xi \tag{2.14}$$

$$= 2\mathbf{i} \int_{k_p}^{k_s} \operatorname{Im} \frac{(f_1'(\xi) - \mathbf{i}f_2'(\xi))f_1(\xi) + \mathbf{i}f_2(\xi))}{(f_1(\xi) - \mathbf{i}f_2(\xi))(f_1(\xi) + \mathbf{i}f_2(\xi))} d\xi \tag{2.15}$$

$$= 2\mathbf{i} \int_{k_p}^{k_s} \frac{f_1'(\xi)f_2(\xi) - f_1(\xi)f_2'(\xi)}{f_1^2(\xi) + f_2^2(\xi)} d\xi \tag{2.16}$$

$$= 2\mathbf{i} \int_{k_p}^{k_s} \frac{f_1^2(\xi)}{f_1^2(\xi) + f_2^2(\xi)} \frac{f_1'(\xi)f_2(\xi) - f_1(\xi)f_2'(\xi)}{f_1^2(\xi)} d\xi \tag{2.17}$$

$$= -2\mathbf{i} \int_{k_p}^{k_s} \frac{f_1^2(\xi)}{f_1^2(\xi) + f_2^2(\xi)} d\frac{f_2(\xi)}{f_1(\xi)} \tag{2.18}$$

$$= -2\mathbf{i} \arctan \frac{f_2(\xi)}{f_1(\xi)}\Big|_{k_p}^{k_s} = 0 \tag{2.19}$$

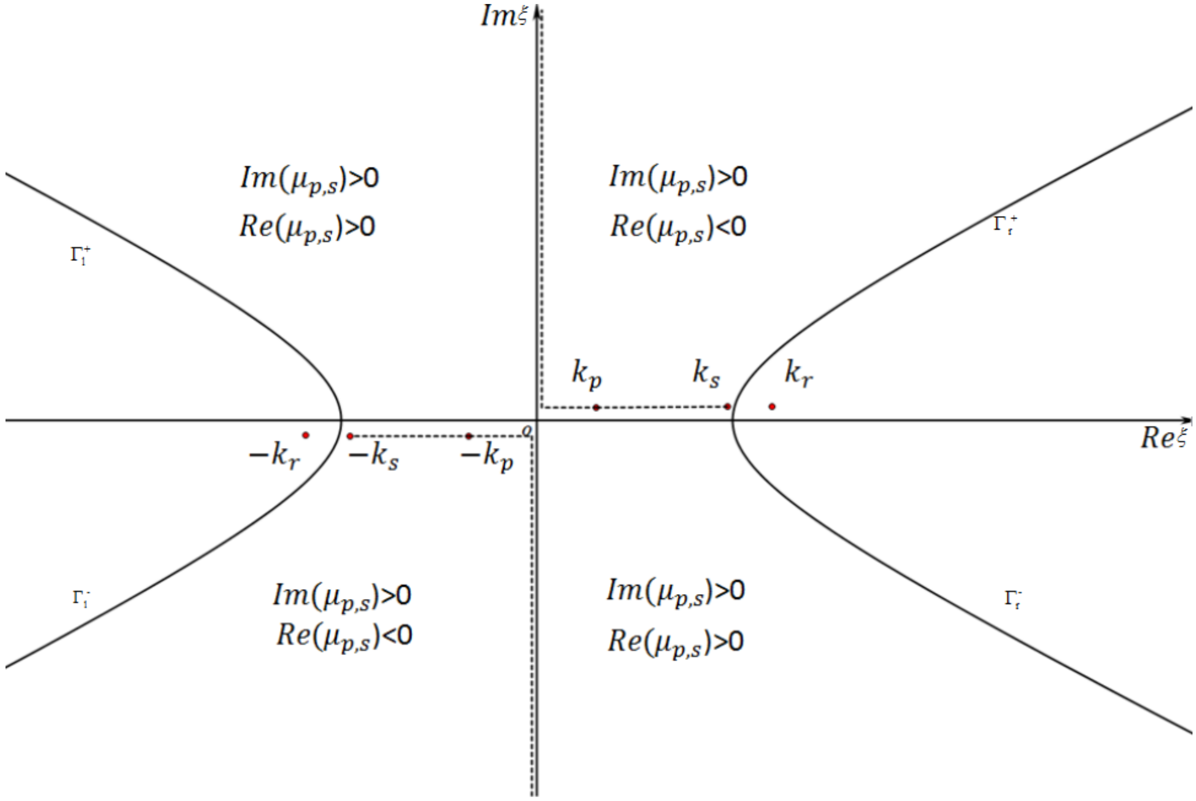
For $|\xi|$ large, we find $\delta(\xi) = A\xi^2 + O(1)$, thus it is easy to see that

$$\int_\Gamma \frac{d\delta}{d\xi} \frac{d\xi}{\delta(\xi)} = 4\pi$$

Then we obtain $Z = 2$. This completes the proof. $\square$

In order to overcome the ambiguity above, loss is assumed in the medium so that $k_{\alpha,\varepsilon} := k_\alpha(1+\mathbf{i}\varepsilon)$. When $\varepsilon > 0$, the branch point of $\mu_{\alpha,\varepsilon}$ are $\pm k_{\alpha,\varepsilon}$ and the branch cut are denoted by the equation $\xi_1\xi_2 = k_\alpha\varepsilon, -k_\alpha \le \xi \le k_\alpha$. In this case, the poles singularities are now located off the real axis and the Fouerier inverse transform becomes meaningful. In order to express lemma root_De2 2.2 concisely, we define

$$\Omega := \{\xi \in \mathbb{C} \mid k_p\varepsilon < \xi_1\xi_2 < k_s\varepsilon\ ,\quad \xi_2 > \xi_1\varepsilon\} \tag{2.20}$$



ure_newpath **Figure 1.** New Integration Path in the ξ-plane

root_De2 **Lemma 2.2** *If the elastic medium has loss that* $k_{\alpha,\varepsilon} := k_\alpha(1+\mathbf{i}\varepsilon), 0 < \varepsilon < 1$ *for* $\alpha = p, s$, *we assert that* $\delta_\varepsilon(\xi) = 0$ *has only two roots in domain* $\Omega^c \subset \mathbb{C}$ *and exactly they are* $\pm k_{R,\varepsilon}$.

Let $\xi = \xi_1 + \mathbf{i}\xi_2 \in \mathbb{C}$, $\xi_1, \xi_2 \in \mathbb{R}$, and the hyperbolic curve Γ defined by the equation $\xi_1^2 - \xi_2^2 = k_s^2$. Denote Γ_r^+, Γ_r^- respectively the parts of right branch of Γ in the upper-half complex plane and the lower-half complex plane. Similarily, we can define Γ_l^-, Γ_l^-. Now, we can select a new integral path in the complex plane

$$NP = \begin{cases} \Gamma_l^+ \cup \Gamma_r^+ \cup [-k_s, k_s] & \text{when} \quad x_1 - y_1 \geq 0 \\ \Gamma_l^- \cup \Gamma_r^- \cup [-k_s, k_s] & \text{when} \quad x_1 - y_1 < 0 \end{cases} \tag{2.21}$$

The new integration path is depicted in Figure 1. Using Cauchy integral theorem and lemma 2.2, we carry out:

$$N_\varepsilon(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{N}_\varepsilon(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \tag{2.22}$$

$$= \frac{1}{2\pi}\int_{NP} \hat{N}_\varepsilon(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \pm \mathbf{i}Res_{\xi=\pm k_R^\varepsilon} N_\varepsilon(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi} \tag{2.23}$$

As the perturbation ε have nothing to do with the integration path NP, we could take the limitation $\varepsilon \to 0$. Therefore, we get the representation of Neumann Green Tensor

$$N(x,y) = \Phi(x,y) - \Phi(x,y') + \frac{1}{2\pi}\int_{NP} \hat{N}_c(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \tag{2.24}$$
$$\pm \mathbf{i}Res_{\xi=\pm\kappa_r}\hat{N}_c(\xi, x_2; y_2)e^{\mathbf{i}(x_1-y_1)\xi}$$

where $\pm$ are corresponding $sgn(x_1 - y_1)$. Specially, $N(x, y)$ has a simple form when $x_2 = 0$:

$$N(x,y) = \frac{1}{2\pi}\int_{NP} \hat{N}(\xi,0;y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \pm \mathbf{i}Res_{\xi=\pm\kappa_r}\hat{N}(\xi,x_2;y_2)e^{\mathbf{i}(x_1-y_1)\xi} \tag{2.25}$$

Ngreen

where

$$\hat{N}(\xi,0;y_2) = \frac{\mathbf{i}}{\mu\delta(\xi)}\left[\begin{pmatrix} 2\xi^2\mu_s & -2\xi\mu_s\mu_p \\ -\xi\beta & \mu_p\beta \end{pmatrix} e^{\mathbf{i}\mu_p y_2} + \begin{pmatrix} \mu_s\beta & \xi\beta \\ 2\xi\mu_s\mu_p & 2\xi^2\mu_p \end{pmatrix} e^{\mathbf{i}\mu_s y_2}\right] \tag{2.26}$$

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$$:= \mathcal{N}_p(\xi)e^{\mathbf{i}\mu_p y_2} + \mathcal{N}_s(\xi)e^{\mathbf{i}\mu_s y_2} \tag{2.27}$$

and let $N_r(x_1; y_1, y_2)$ denote the first part of N and $N_s(x_1; y_1, y_2)$ denote the second part of N in (2.25).

It remains to study Dirichlet Green Tensor $D(x, y)$. We still use Fourier transform to derive the formula of Green Tensor in frequency domain. Then we can obtain $D(x, y)$ similar to $N(x, y)$. It follows an alternative representation for $D(x, y)$

$$\hat{D}(\xi,x_2;y_2) = \hat{\Phi}(\xi,x_2;y_2) - \hat{\Phi}(\xi,x_2;-y_2) + \hat{M}(\xi,x_2;y_2) \tag{2.28}$$

$$\hat{M}(\xi,x_2;y_2) = \frac{\mathbf{i}}{\omega^2\gamma(\xi)}\Big\{A(\xi)e^{\mathbf{i}\mu_s(x_2+y_2)} + B(\xi)e^{\mathbf{i}\mu_p(x_2+y_2)} \tag{2.29}$$

$$- A(\xi)e^{\mathbf{i}\mu_s x_2+\mu_p y_2} - B(\xi)e^{\mathbf{i}\mu_p x_2+\mu_s y_2}\Big\}$$

where

$$A(\xi) = \begin{pmatrix} \xi^2\mu_s & -\xi\mu_s\mu_p \\ -\xi^3 & \xi^2\mu_p \end{pmatrix} \qquad B(\xi) = \begin{pmatrix} \xi^2\mu_s & \xi^3 \\ \xi\mu_s\mu_p & \xi^2\mu_p \end{pmatrix}$$

and $\gamma(\xi) = \xi^2 + \mu_s\mu_p$.

root_Ga **Lemma 2.3** *Let* Lamé *constant* $\lambda, \mu \in \mathbb{C}$ *and* $\mathrm{Im}\,(k_s) \geq 0, \mathrm{Im}\,(k_p) \geq 0$, *then equation* $\gamma(\xi) = 0$ *has no root in complex plane.*

Proof. Let $F(\xi) = \gamma(\xi) * (\xi^2 - \mu_s\mu_p)$ and it is easy to see that the root of $\gamma(\xi) = 0$ is also of $F(\xi) = 0$. A simple computation show that $F(\xi) = (k_s^2 + k_p^2)\xi^2 - k_p^2k_s^2$. However, only when $\xi^2 = k_p^2k_s^2/(k_s^2 + k_p^2)$, $F(\xi) = 0$ but $\gamma(\xi) = 2k_p^2k_s^2/(k_s^2 + k_p^2)$. This completes the proof. □

Thus, we get the representation of Green Tensor by inverse Fourier transform

$$D(x,y) = \Phi(x,y) - \Phi(x,y') + \frac{1}{2\pi}\int_{\mathbb{R}} \hat{M}(\xi,x_2;y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \tag{2.30}$$

Let $T_D(x, y)$ denote the traction of $D(x, y)$ in direction e_2 with respect to x such that $T_D(x, y)e_i = T_x^{e_2}(D(x, y))e_i = T_x^{e_2}(D(x, y)e_i)$. Then we can get the representation of $T_D(x, y)$ by a trivial calculation.

$$T_D(x,y) = T(x,y) - T(x,y') + \frac{1}{2\pi}\int_{\mathbb{R}} \hat{T}_M(\xi,x_2;y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \tag{2.31}$$

and

$$\hat{T}_M(\xi, x_2; y_2) = \frac{\mu}{\omega^2\gamma(\xi)}\Big\{E(\xi)e^{\mathbf{i}\mu_s(x_2+y_2)} + F(\xi)e^{\mathbf{i}\mu_p(x_2+y_2)} \tag{2.32}$$
$$- E(\xi)e^{\mathbf{i}\mu_s x_2+\mu_p y_2} - F(\xi)e^{\mathbf{i}\mu_p x_2+\mu_s y_2}\Big\}$$

where

$$E(\xi) = \begin{pmatrix} -\xi^2\beta & \xi\mu_p\beta \\ 2\xi^3\mu_s & -2\xi^2\mu_s\mu_p \end{pmatrix} \qquad F(\xi) = \begin{pmatrix} -2\xi^2\mu_s\mu_p & -2\xi^3\mu_p \\ -\xi\mu_s\beta & -\xi^2\beta \end{pmatrix}$$

Specially, $T_D(x, y)$ has a simple form when $x_2 = 0$:

$$T_D(x_1, 0; y_1, y_2) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{T}_D(\xi, 0; y_2)e^{\mathbf{i}(x_1-y_1)\xi}d\xi \tag{2.33}$$

where

$$\hat{T}_D(\xi, 0; y_2) = \frac{1}{\gamma(\xi)}\left[\begin{pmatrix} \mu_s\mu_p & \xi\mu_p \\ \xi\mu_s & \xi^2 \end{pmatrix}e^{\mathbf{i}\mu_s y_2} + \begin{pmatrix} \xi^2 & -\xi\mu_p \\ -\xi\mu_s & \mu_p\mu_s \end{pmatrix}e^{\mathbf{i}\mu_p y_2}\right] \tag{2.34}$$

$$:= \mathcal{T}_p(\xi)e^{\mathbf{i}\mu_p y_2} + \mathcal{T}_s(\xi)e^{\mathbf{i}\mu_s y_2} \tag{2.35}$$

To analysis the point spread function in the section 5, we should give asymptotic anslysis for $N(x_1, 0, y)$ and $T_D(x_1, 0; y)$. We need the following slight generalization of Van der Corput lemma for the oscillatory integral grafakos [21, P.152].

van **Lemma 2.4** *Let $-\infty < a < b < \infty$, and u is a C^k function u in (a, b).*
1. If $|u'(t)| \geq 1$ for $t \in (a, b)$ and u' is monotone in (a,b), then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 3\lambda^{-1}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

2. For all $k \geq 2$, if $|u^{(k)}(t)| \geq 1$ for $t \in (a, b)$, then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 12k\lambda^{-1/k}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

Proof. The assertion can be proved by extending the Van der Corptut lemma in grafakos [21]. Here we omit the details. □

es_integral **Lemma 2.5** *Assume that $0 < \kappa := \sin\phi_\kappa < 1, 0 < \phi_\kappa < \pi/2$, $0 \leq \phi \leq \pi/2$. Let*

$$f(t, \phi) := F(\sin(t + \phi), \cos(t + \phi), (\kappa^2 - \sin^2(t + \phi))^{1/2}) \tag{2.36}$$

be a complexed function in $C([-\pi/2, \pi/2] \times [0, \pi/2])$. Moreover, its partial derivative with respect to t can be represented as

$$\frac{\partial f(t, \phi)}{\partial t} = g(t, \phi)(\kappa^2 - \sin^2(t + \phi))^{-1/2} \tag{2.37}$$ assume1

where $g(t,\phi)$ and $\partial g(t,\phi)/\partial t$ are uniformly bounded. Then for any $\rho \geq 1$ and $\phi > \phi^* > \phi_\kappa$, we have

$$\left| I(\rho,\phi) := \int_{-\pi/2}^{\pi/2} f(t,\phi) e^{\mathbf{i}\rho\cos t} dt \right| \leq C \frac{1}{\rho^{1/2}} \tag{2.38}$$ es_integral_1

$$\left| H(\rho,\phi) := \int_{-\pi/2}^{\pi/2} \frac{\partial f(t,\phi)}{\partial t} e^{\mathbf{i}\rho\cos t} dt \right| \leq C \frac{1}{\rho^{1/2}} \tag{2.39}$$ es_integral_2

where C is only dependent on κ and ϕ^*.

Proof. Since $\phi > \phi^* > \phi_\kappa$, there exists $0 < \delta < \pi/4$ such that

$$|(\kappa^2 - \sin^2(t+\phi))^{1/2}| > \frac{1}{2}|(\kappa^2 - \sin^2\phi)^{1/2}| \tag{2.40}$$

for any $t \in (-\delta, \delta)$. Then we can divide I into two parts such that

$$\begin{aligned} I &= \int_{-\delta}^{\delta} f(t) e^{\mathbf{i}\rho\cos t} dt + \int_{(-\frac{\pi}{2},\frac{\pi}{2})\backslash[-\delta,\delta]} f(t) e^{\mathbf{i}\rho\cos t} dt \\ &=: I_1 + I_2 \end{aligned}$$

Similarly, we have $H = H_1 + H_2$. Let phase function $p(t) = \cos t$. It is easy to see that $|p''(t)| \geq \cos\delta$ for $t \in (-\delta,\delta)$ and $|p'(t)| \geq \sin\delta$. By lemma $\overset{\texttt{van}}{2.4}$, we obtain

$$|I_1| \leq C \frac{1}{\rho^{1/2}} \left[|f(\delta,\phi)| + \int_{-\delta}^{\delta} \left|\frac{\partial f(t,\phi)}{\partial t}\right| dt \right] \leq C \frac{1}{\rho^{1/2}} \tag{2.41}$$

$$|I_2| \leq C \frac{1}{\rho} \left[|f(\frac{\pi}{2},\phi)| + |f(-\delta,\phi)| + \int_{(-\frac{\pi}{2},\frac{\pi}{2})\backslash[-\delta,\delta]} \left|\frac{\partial f(t,\phi)}{\partial t}\right| dt \right] \leq C \frac{1}{\rho} \tag{2.42}$$

$$|H_1| \leq C \frac{1}{\rho^{1/2}} \left[\left|\frac{\partial f(\delta,\phi)}{\partial t}\right| + \int_{-\delta}^{\delta} \left|\frac{\partial^2 f(t,\phi)}{\partial^2 t}\right| dt \right] \leq C \frac{1}{\rho^{1/2}} \tag{2.43}$$

For $H_2(\rho,\phi)$, we can not use lemma $\overset{\texttt{van}}{2.4}$ again since $\partial^2 f(t,\phi)/\partial^2 t$ is not integrable on $(-\frac{\pi}{2},\frac{\pi}{2})\backslash[-\delta,\delta]$. Solving the following equation:

$$\kappa^2 - \sin^2(t+\phi) = 0$$

we have, if $0 < \phi < \pi/2 - \phi_\kappa$,

$$t_1(\phi) = \phi_\kappa - \phi \qquad t_2(\phi) = -\phi_\kappa - \phi$$

and if $\pi/2 - \phi_\kappa \leq \phi < \pi/2$,

$$t_1(\phi) = \phi_\kappa - \phi \qquad t_2(\phi) = \pi - \phi_\kappa - \phi$$

However, for any $0 < \lambda_1 < 1$ and $1 < \lambda_2 < 1/\kappa$, there exists $\sigma > 0$, such that $\chi := ((t_1 - \sigma, t_1 + \sigma) \cup (t_2 - \sigma, t_2 + \sigma)) \subset (-\frac{\pi}{2},\frac{\pi}{2})\backslash[-\delta,\delta]$, dependent on λ_1, λ_2 and

$$\lambda_1\kappa < |\sin(t+\phi)| < \lambda_2\kappa. \tag{2.44}$$ assume2

for any $t \in \chi$.

We only analysis the integral on $\chi_1 = (t_1 - \sigma, t_1 + \sigma) \cap [-\pi/2, \pi/2]$ here, which denoted by H_{χ_1}, the proof of H_{χ_2} is similar. It is easy to see that $\sin(t+\phi)$ is monotonic in χ_1. Without loss of generality, we assume that $\sin(t_1-\sigma+\phi)<\kappa<\sin(t_1+\sigma+\phi)$. Let $\sin(t+\phi)=\kappa\sin\theta$ and the implicit mapping from θ to t is denoted by $t(\theta)$ while the inverse mapping by $\theta(t)$, taking the interval χ_1 onto $L_\theta:\theta_1\to\pi/2\to\pi/2-\mathbf{i}\theta_2$ where $\sin(t_1-\sigma+\phi)=\kappa\sin\theta_1, \sin(t_1+\sigma+\phi)=\kappa\sin(\pi/2-\mathbf{i}\theta_2)$. By substituting $t(\theta)$ into H_{χ_1}, we have

$$H_{\chi_1} = \int_{t_1-\sigma}^{t_1+\sigma}\frac{g(t,\phi)}{(\kappa^2-\sin^2(t+\phi))^{1/2}}e^{\mathbf{i}\rho\cos t} \tag{2.45}$$

$$= \int_{L_\theta}\frac{g(t(\theta),\phi)}{(1-\kappa^2\sin^2\theta)^{1/2}}e^{\mathbf{i}\rho(\cos(t(\theta)))}d\theta \tag{2.46}$$

Observe that $h(\theta)$ and $\partial h/\partial\theta$ are integrable on the path L_θ by (2.6). A simple computation show that

$$\frac{dt(\theta)}{d\theta}=\frac{\kappa\cos\theta}{\cos(t+\phi)} \qquad \frac{d^2t(\theta)}{dt^2}=\frac{\kappa^2\cos^2\theta\sin(t+\phi)-\kappa\sin\theta\cos^2(t+\phi)}{\cos^3(t+\phi)}$$

Then we can obtain

$$\begin{aligned}
\frac{d\cos t}{d\theta} &= \frac{-\kappa\sin t\cos\theta}{\cos(t+\phi)}\\
\frac{d^2\cos t}{d\theta^2} &= \frac{d^2\cos t}{dt^2}\Big(\frac{dt}{d\theta}\Big)^2+\frac{d\cos t}{dt}\frac{d^2t}{d\theta^2}\\
&= \frac{-\kappa^2\cos^2\theta\cos t}{\cos^2(t+\phi)}+\frac{\kappa\sin\theta\cos^2(t+\phi)\sin t-\kappa^2\cos^2\theta\sin(t+\phi)\sin t}{\cos^3(t+\phi)}\\
&= \frac{-\kappa^2\cos^2\theta\cos\phi+\kappa\sin\theta\cos^2(t+\phi)\sin t}{\cos^3(t+\phi)}\\
&= \frac{(\sin^2(t+\phi)-\kappa^2)\cos\phi+\cos^2(t+\phi)\sin(t+\phi)\sin t}{\cos^3(t+\phi)}
\end{aligned}$$

Since $|\sin t|>|\sin\delta|$ and $1-\lambda_2^2\kappa^2<\cos^2(t+\phi)<1-\lambda_1^2\kappa^2$ for $t\in\chi_1$, it follows that $\theta=\pi/2$ is the only stationary point of $\cos(t(\theta))$ and

$$\left|\frac{d^2\cos t}{d\theta^2}(\pi/2)\right| = \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}|\sin t| > \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}\sin\delta \tag{2.47}$$

Therefore, we can choose appropriate λ_1,λ_2 such that

$$\Big|\frac{d^2\cos t}{d\theta^2}\Big| > \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}\sin\delta \tag{2.48}$$

for any $\theta\in\theta(\chi_1)$. According to lemma (2.4), we obtian $|H_{\chi_1}|\le C\frac{1}{\rho^{1/2}}$, and also $|H_{\chi_2}|\le C\frac{1}{\rho^{1/2}}$. Using integration by parts, we get

$$\left|\int_{[-\pi/2,\pi/2]\backslash((-\delta,\delta)\cup\chi)}\frac{\partial f(t,\phi)}{\partial t}e^{\mathbf{i}\rho\cos t}dt\right|\le C\frac{1}{\rho}$$

Finally, combining above inequality, we arrive at the estimate. This completes the proof. □

Therefore, the estimate of $T_D(x_1,0;y_1,y_2)$ and $N(x_1,0;y_1,y_2)$ are now direct consequence of lemma 2.5.

es_dgreen **Lemma 2.6** *For every $x \in \Gamma_0$, $y \in \mathbb{R}^2_+$ such that $|x_1 - y_1|/|x-y| > (1+\kappa)/2$, $y_2/|x-y| < \kappa/2$ and $k_s|x-y| > 1$, we have*

$$|T_D(x,y)| \leq C\bigg(\frac{k_s y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{1/2}} + \frac{k_s|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{3/2}}\bigg) \tag{2.49}$$

where C is only dependent on κ.

Proof. Put

$$I(|x_1-y_1|, y_2) = \int_{\mathbb{R}} \mathcal{T}_s(\xi) e^{\mathbf{i}(\mu_s y_2 + \xi|x_1-y_1|)} d\xi \tag{2.50}$$

$$J(|x_1-y_1|, y_2) = \int_{\mathbb{R}} \mathcal{T}_p(\xi) e^{\mathbf{i}(\mu_p y_2 + \xi|x_1-y_1|)} d\xi \tag{2.51}$$

To simplify the integral I, the standard substitution $\xi = k_s \sin t$ is made, taking the ξ-plane to a strip $-\pi < \mathrm{Re}\, t < \pi$ in the t-plane, and the real axis in the ξ-plane onto the path L from $-\pi/2 + \mathbf{i}\infty \to -\pi/2 \to \pi/2 \to \pi/2 - \mathbf{i}\infty$ in the t-plane. The integral $I(|x_1-y_1|, y_2)$ then becomes (Let $|x_1 - y_1| = \rho \sin\phi$ and $y_2 = \rho\cos\phi$, $0 < \phi < \pi/4$):

$$k_s \int_L F(\sin t, \cos t, (\kappa^2 - \sin^2 t)^{1/2}) \cos t \; e^{\mathbf{i}k_s\rho(\cos(t-\phi))} dt \tag{2.52}$$

where $F(\sin t, \cos t, (\kappa^2 - \sin^2 t)^{1/2}) = \mathcal{T}_s(k_s \sin t)$. Taking the shift transformation of t and using cauchy integral theorem, we can get the representation of I:

$$\begin{aligned}
&k_s \int_L F(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2}) \cos(t+\phi) e^{\mathbf{i}k_s\rho(\cos t)} dt \\
&= k_s \cos\phi \int_L F(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2}) \cos t \; e^{\mathbf{i}k_s\rho(\cos t)} dt \\
&-k_s \sin\phi \int_L F(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2}) \sin t \; e^{\mathbf{i}k_s\rho(\cos t)} dt \\
&:= k_s(\cos\phi \; I_1 + \sin\phi \; I_2)
\end{aligned}$$

For I_1, we split the integral path L into $L_1 = (-\pi/2, \pi/2)$ and $L_2 = (-\pi/2 + \mathbf{i}\infty, -\pi/2) \cup (\pi/2, \pi/2 - \mathbf{i}\infty)$, then we have corresponding representation: $I_1 = I_{11} + I_{12}$. Since $F(t)\cos t$ satisfy (assume1 2.37), we can obtain $|I_{11}| \leq 1/(k_s\rho)^{1/2}$ by lemma es_integral 2.5. Using integration by parts, it follows that $|I_{12}| \leq 1/(k_s\rho)$. For I_2, using integration by parts on path L first, we have

$$I_2 = \frac{1}{\mathbf{i}k_s\rho} \int_L F(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2}) d\; e^{\mathbf{i}(k_s\rho\cos t)} \tag{2.53}$$

$$= -\frac{1}{\mathbf{i}k_s\rho} \int_{L_1 \cup L_2} \frac{\partial F(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2})}{\partial t} \; e^{\mathbf{i}(k_s\rho\cos t)} dt \tag{2.54}$$

$$= -\frac{1}{\mathbf{i}k_s\rho}(I_{21} + I_{22}) \tag{2.55}$$

Similarly, we have $|I_{21} + I_{22}| \leq 1/(k_s\rho)^{1/2}$ by lemma es_integral 2.5. The estimate of $J(|x_1 - y_1|, y_2)$ can be proved by the same method as employed above. This completes the proof. $\square$

es_ngreen **Lemma 2.7** *For every* $x \in \Gamma_0$, $y \in \mathbb{R}^2_+$ *such that* $|x_1 - y_1|/|x-y| > (1+\kappa)/2$, $y_2/|x-y| < \kappa/2$ *and* $k_s|x-y| > 1$, *we have*

$$|N(x,y)| \le \frac{C}{\mu}\left(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{1/2}} + \frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{3/2}} + e^{-\sqrt{k_R^2-k_s^2}y_2}\right) \tag{2.56}$$

where C *is only dependent on* κ.

Proof. The proof is similar to lemma es_dgreen 2.6. Here we only point out the different parts. Notice that, in this case, the substitution $\xi = k_s \sin t$, taking Γ_r^+ in the ξ-plane onto $L_r := \{t = t_1 + \mathbf{i}t_2 | \cos(2t_1)\cosh(2t_2) = -1, \frac{\pi}{2} \le t_1 < \frac{3\pi}{4}, t_2 \le 0\}$ in the t-plane and Γ_l^+ in the ξ-plane onto $L_l := \{t = t_1 + \mathbf{i}t_2 | \cos(2t_1)\cosh(2t_2) = -1, -\frac{\pi}{2} \le t_1 < -\frac{1\pi}{4}, t_2 \ge 0\}$ in the t-plane. To see this transformation, observe that since

$$\sin t = \cosh t_2 \sin t_1 + \mathbf{i}\sinh t_2 \cos t_1$$
$$\cos t = \cosh t_2 \cos t_1 - \mathbf{i}\sinh t_2 \sin t_1$$

we have

$$\begin{aligned} L_r &= \{t = t_1 + \mathbf{i}t_2 \mid \cosh^2 t_2 \sin^2 t_1 - \sinh^2 t_2 \cos^2 t_1 = 1, \frac{\pi}{2} \le t_1 < \pi, t_2 \le 0\} \\ &= \{t = t_1 + \mathbf{i}t_2 \mid \frac{e^{2t_2} + e^{-2t_2}}{2}(\sin^2 t_1 - \cos^2 t_1) = 1, \frac{\pi}{2} \le t_1 < \pi, t_2 \le 0\} \\ &= \{t = t_1 + \mathbf{i}t_2 | \cos(2t_1)\cosh(2t_2) = -1, \frac{\pi}{2} \le t_1 < \frac{3\pi}{4}, t_2 \le 0\} \end{aligned}$$

The same procedure is adopted for L_l. The geometry now is depicted in Figure figure_trans 2. Put

$$I = \int_{L_r} k_s \mathcal{N}_s(k_s \sin(t+\phi)) \cos t e^{\mathbf{i}k_s\rho\cos t} dt$$

By equation (ngreen 2.26), it is easy to see that for $t \in L_r$

$$k_s|\mathcal{N}_s(k_s\sin(t+\phi))\cos t| \le C(1 + |\sin^3(t+\phi)\cos(t+\phi)|) \le C(1+\cosh^4(t_2))$$

Direct computation show that for $t \in L_r$

$$\frac{dt_1}{dt_2} = \frac{\cos(2t_1)\sinh(2t_2)}{\sin(2t_1)\cosh(2t_2)} = -\frac{\cos(2t_1)\sqrt{\cosh^2(2t_2) - 1}}{\sin(2t_1)\cosh(2t_2)} = \frac{1}{\cosh(2t_2)}$$

Thus

$$|I| \le C\int_{-\infty}^{0} (1+\cosh^4(t_2))\sqrt{\frac{1}{\cosh^2(2t_2)} + 1}\ \ e^{\frac{\sqrt{2}}{2}k_s\rho\sinh(t_2)} dt_2$$

Let $s = -\sinh(t_2)$, then for $k_s\rho > 1$ we have

$$|I| \le C\int_0^\infty (1 + (1+s^2)^2)/\sqrt{1+s^2} e^{-\frac{\sqrt{2}}{2}k_s\rho s} ds \le C\frac{1}{k_s\rho} \tag{2.57}$$

Similarly, we can also obtain

$$|\int_{L_r} \frac{\partial \mathcal{N}_s(k_s\sin(t+\phi))}{\partial t} e^{\mathbf{i}k_s\rho\cos t} dt| \le C\frac{1}{k_s\rho}$$

Therefore, the proof of this lemma can be completed by the same method as employed in the lemma es_dgreen 2.6. Here we omit the details. $\square$

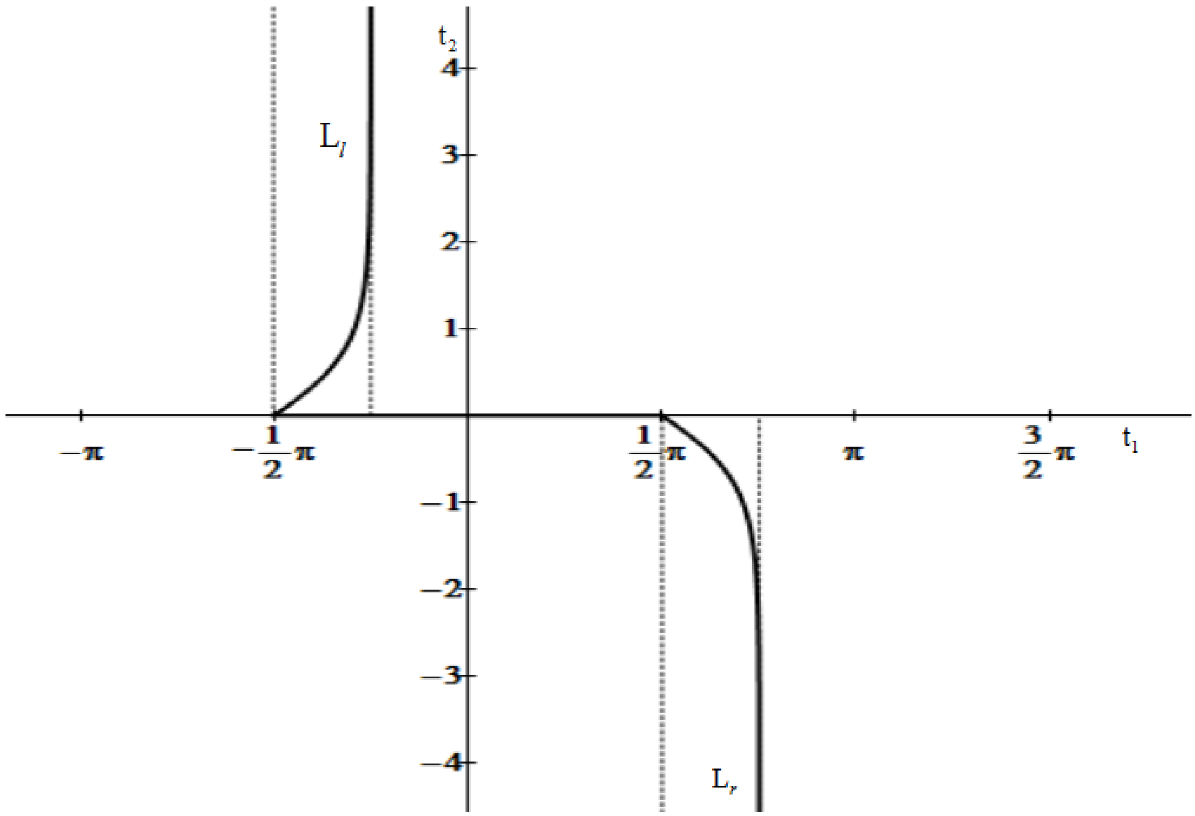


Figure 2. Transform from $\Gamma_l^+ \cup (-k_s, k_s) \cup \Gamma_r^+$ to $L_l \cup (-\pi/2, \pi/2) \cup L_r$

3. The forward scattering problem

In this section we introduce the following stability estimate of the forward elastic scattering problem in the half space which can be proved by the limiting absorption principle by extending the classical argument in [24, 27, 19].

Theorem 3.1 *Let $g \in H^{1/2}(\Gamma_D)$, then the scattering problem of elastic equation in the half space*

$$\Delta_e u + \omega^2 u = 0 \qquad \text{in } \mathbb{R}_+^2 \backslash \bar{D}, \tag{3.1}$$

$$u = g \qquad \text{on } \Gamma_D, \tag{3.2}$$

$$\sigma(u) e_2 = 0 \qquad \text{on} \Gamma_0, \tag{3.3}$$

u satisfies the generalized radiation codition [25] such that

$$\lim_{r \to \infty} \int_{S_r^+} (\sigma(N(x,y)e_i)\hat{r}) \cdot u(x) - (N(x,y)e_i) \cdot (\sigma(u)\hat{r}) ds(x) = 0 \tag{3.4}$$

where $S_r^+ := \{x \in \mathbb{R}_+^2 \mid \|x\| = r^2\}$, $\hat{r} = x/r$ and $y \in \mathbb{R}_+^2$. Then the problem (3.1)-(3.4) admits a unique solution $u \in H^1_{\mathrm{loc}}(\mathbb{R}_+^2 \backslash \bar{D})$. Moreover, for any bounded open set $\mathcal{O} \subset \mathbb{R}_+^2 \backslash \bar{D}$ there exists a constant $C > 0$ such that

$$\|u\|_{H^1(\mathcal{O})} \le C \|g\|_{H^{-1/2}(\Gamma_D)} \tag{3.5}$$

The existence of the solution can be proved by the method of limiting absorption principle. The argument is standard and we give several lemmas below, see e.g. [24]

for the consideration for Helmholtz equation. For any $z = 1 + \mathbf{i}\varepsilon, \varepsilon > 0$, $f \in H^1(\mathbb{R}^2_+)'$ with compact support in $B_R = \{x||x|^2 < R^2, x \in \mathbb{R}^2_+\} \subsetneq \mathbb{R}^2_+$ where B_R is an half disc of radius R , we consider the problem

$$\Delta_e u_z + z\omega^2 u = -f \qquad \text{in } \mathbb{R}^2_+ \tag{3.6}$$

$$\sigma(u_z)e_2 = 0 \quad \text{on} \quad \Gamma_0 \tag{3.7}$$

By Lax-Milgrim lemma we know that (3.6-3.7) has a unique solution $u_z \in H^1(\mathbb{R}^2_+)$. For any domain $D \subset \mathbb{R}^2_+$, we define the weighted space $L^{2,s}(D), s \in \mathbb{R}$, by

$$L^{2,s}(D) = \{v \in L^2_{\text{loc}}(D) : (1 + |x|^2)^{s/2} v \in L^2(D)\}$$

with the norm $\|v\|_{L^{2,s}(\mathcal{D})} = (\int_{\mathcal{D}} (1 + |x|^2)^s |v|^2 dx)^{1/2}$. The weighted Sobolev space $H^{1,s}(D), s \in \mathbb{R}$, is defined as the set of functions in $L^{2,s}(D)$ whose first derivative is also in $L^{2,s}(D)$. The norm $\|v\|_{H^{1,s}(D)} = (\|v\|^2_{L^{2,s}(D)} + \|\nabla v\|^2_{L^{2,s}(D)})^{1/2}$.

We need the following sligt generalization of Rellich Theorem:

Lemma 3.1 *Let Ω be an open Lipschitz domain, then the sobolev space $H^{1,-s}(\Omega)$ is compactly embeded in $L^{2,-s'}(\Omega)$ for every $s' > s > 0$.*

Lemma 3.2 *Let $f \in L^2(\mathbb{R}^2_+)$ with compact support in B_R. For any $z = 1+\mathbf{i}\varepsilon$, $0 < \varepsilon < 1$, we have, for any $s > 1/2$, $\|u_z\|_{H^{1,-s}(\mathbb{R}^2_+)} \leq C\|f\|_{L^2(\mathbb{R}^2_+)}$ for some constant independent of ε, u_z, and f.*

Proof. Let R_z denote the map from $L^2_c(\mathbb{R}^2_+)$ to $H^{1,-s}(\mathbb{R}^2_+)$ such that $R_z(f) = u_z$ where $L^2_c(\mathbb{R}^2_+)$ is denoted by all $f \in L^2(\mathbb{R}^2_+)$ with compact support in B_R, then it is easy to see that R_z is a linear bounded operator. It follows from theorem 3.7 in [19] that R_z is a uniformly continuous operator continues valued function on $z = 1 + \mathbf{i}\varepsilon$, $0 < \varepsilon < 1$ with value in $B(L^2_c(\mathbb{R}^2_+), H^{1,-s}(\mathbb{R}^2_+))$. Then, we can obtain that R_z is uniformly bounded in $B(L^2_c(\mathbb{R}^2_+), H^{1,-s}(\mathbb{R}^2_+))$. This complete the proof by the defintion of the operator norm. □

We next recall the following lemma which states the absence of positive eigenvalues for the linear elasticity system in half space [26].

Lemma 3.3 *Let $u \in L^2(\mathbb{R}^2_+ \backslash \bar{D})$ such that u satisfies (3.1) and (3.3), than we assert that $u = 0$ in $\mathbb{R}^2_+ \backslash \bar{D}$*

Proof. The asserting above can be proved by extending [26, theorem 3.1], here we omit the details. □

For any $0 < \varepsilon < 1$, we consider the problem

$$\Delta_e u_\varepsilon + (1 + \mathbf{i}\varepsilon)\omega^2 u_\varepsilon = 0 \qquad \text{in } \mathbb{R}^2_+ \backslash \bar{D} \tag{3.8}$$

$$u_\varepsilon = g \quad \text{on } \Gamma_D \tag{3.9}$$

$$\sigma(u_\varepsilon)e_2 = 0 \quad \text{on} \Gamma_0 \tag{3.10}$$

We know that the above problem has a unique solution $u_\varepsilon \in H^1(\mathbb{R}^2_+ \backslash \bar{D})$ by the Lax-Milgram Lemma. Thus, we have next lemma

Lemma 3.4 *Let $g \in H^{1/2}(\Gamma_D)$. For any $0 < \varepsilon < 1$, we have, for any $s > 1/2$, $\|u_\varepsilon\|_{H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})} \le C\|g\|_{H^{1/2}(\Gamma_D)}$ for some constant independent of $\varepsilon, u_\varepsilon$, and g.*

Proof. Because $h = dist(D, \Gamma_0) > 0$, we can find three concentric circles $B_{R_1}, B_{R_2}, B_{R_3}$ such that $D \subsetneq B_{R_1} \subsetneq B_{R_2} \subsetneq B_{R_3} \subsetneq \mathbb{R}^2_+$. Let $\chi \in C_0^\infty(\mathbb{R}^2_+)$ be the cut-off function such that $0 \le \chi \le 1$, $\chi = 0$ in B_{R_1}, and $\chi = 1$ outside of B_{R_2}. Let $v_\varepsilon = \chi u_\varepsilon$. Then v_ε satisfies (3.6) with $z = 1 + \mathbf{i}\varepsilon$ and $q = \sigma(u_\varepsilon)\nabla\chi + (\lambda+\mu)(\nabla^2\chi u_\varepsilon + \nabla u_\varepsilon \nabla\chi) + \mu\Delta\chi u_\varepsilon + \mu \mathrm{div} u_\varepsilon \nabla\chi$, where $\nabla^2\chi$ is the Hessian matrix of χ. Clearly q has compact support. By lemma 3.2 we can obtain

$$\|v_\varepsilon\|_{H^{1,-s}(\mathbb{R}^2_+)} \le C\|u_\varepsilon\|_{H^1(B_{R_2}\backslash\bar{D})} \tag{3.11}$$

for some constant C independent of $\varepsilon > 0$. Now let $\chi_1 \in C_0^\infty(\mathbb{R}^2_+)$ be the cut-off function with that $0 \le \chi_1 \le 1$, $\chi_1 = 1$ in B_{R_2}, and $\chi_1 = 0$ outside of B_{R_3}. For $g \in H^{1/2}(\Gamma_D)$, let $u_g \in H^1(\mathbb{R}^2_+\backslash\bar{D})$ be the lifting function such that $u_g = g$ on Γ_D and $\|u_g\|_{H^1(\mathbb{R}^2_+\backslash\bar{D})} \le C\|g\|_{H^{1/2}(\Gamma_D)}$. By testing 3.8 with $\chi_1^2(\overline{u_\varepsilon - u_g})$ and using the standard argument we have

$$\|u_\varepsilon\|_{H^1(B_{R_2}\backslash\bar{D})} \le C(\|u_\varepsilon\|_{L^2(B_{R_3}\backslash\bar{D})} + \|g\|_{H^{1/2}(\Gamma_D)}). \tag{3.12}$$

A combination of (3.11) and the above estimate yields

$$\|u_\varepsilon\|_{H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})} \le C(\|u_\varepsilon\|_{L^2(B_{R_3}\backslash\bar{D})} + \|g\|_{H^{1/2}(\Gamma_D)}). \tag{3.13}$$

Now we claim

$$\|u_\varepsilon\|_{L^2(B_{R_3}\backslash\bar{D})} \le C\|g\|_{H^{1/2}(\Gamma_D)}, \tag{3.14}$$

for any $g \in H^{1/2}(\Gamma_D)$ and $\varepsilon > 0$. If it were false, there would exist sequences $\{g_m\} \subset H^{1/2}(\Gamma_D)$ and $\{\varepsilon_m\} \subset (0,1)$, and $\{u_{\varepsilon_m}\}$ be the corresponding solution of (3.8)-(3.10) such that

$$\|u_{\varepsilon_m}\|_{L^2(B_{R_3}\backslash\bar{D})} = 1 \text{ and } \|g_m\|_{H^{-1/2}(\Gamma_D)} \le \frac{1}{m}. \tag{3.15}$$

Then $\|u_{\varepsilon_m}\|_{H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})} \le C$, and thus there is a subsequence of $\{\varepsilon_m\}$, which is still denoted by $\{\varepsilon_m\}$, such that $\varepsilon_m \to \varepsilon' \in [0,1]$, and a subsequence of $\{u_{\varepsilon_m}\}$, which is still denoted by $\{u_{\varepsilon_m}\}$, such that it converges to some $u_{\varepsilon'}$ in $H^{1,-s'}(\mathbb{R}^2_+\backslash\bar{D})$ by choosing $s' > s$. This is a consequence of Korn's inequality and lemma 3.1. So $u_{\varepsilon'} \in H^{1,-s'}(\mathbb{R}^2_+\backslash\bar{D})$ satisfies (3.8-3.10) with $g = 0$ and $\varepsilon = \varepsilon'$.

By the integral representation satistied by u_{ε_m}, we know that for $y \in \mathbb{R}^2_+\backslash\bar{B}_{R_1}$ and $i = 1, 2$

$$u_{\varepsilon'}(y)\cdot e^i = \int_{\partial B_{R_1}} (\sigma(N_{\varepsilon'}(x,y)e_i)\nu)\cdot u_{\varepsilon'}(x) - (N_{\varepsilon'}(x,y)e_i)\cdot(\sigma(u_{\varepsilon'})_{\varepsilon'}\nu)ds(x) \tag{3.16}$$

If $\varepsilon' > 0$, we deduce from (3.16) that $u_{\varepsilon'}$ decays exponentially and thus $u_{\varepsilon'} \in H^1(\mathbb{R}^2_+\backslash\bar{D})$, then $u_{\varepsilon'} = 0$ by the uniqueness of the solution in $H^1(\mathbb{R}^2_+\backslash\bar{D})$ with positive absorption. If $\varepsilon' = 0$, by the [19, theorem 5.2], we have $u_{\varepsilon'} \in L^2(\mathbb{R}^2_+\backslash\bar{D})$. Then we conclude $u_{\varepsilon'} = 0$ by the lemma 3.3 Therefore, in any case $u_{\epsilon'} = 0$, which , however contradicts to 3.15. This complete the proof. □

Now we are in the position to prove the exsitence of Theorem 3.1.

elas_exis **Lemma 3.5** *For any $s > 1/2$, $u_\varepsilon : (0,1) \to H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})$ is a uniformly continuous operator valued function. Immediately, u_ε converges to some u_0 in $H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})$ and u_0 is a solution of (3.1-3.5).*

Proof. We also give a indirect prove here. Let $\delta_0 > 0$ and $\{\mu_n\}$ and $\{\nu_n\}$ be sequences in $(0,1)$ such that

$$|\mu_n - \nu_n| \leq 1/n \qquad \text{and} \qquad \|u_{\mu_n} - u_{\nu_n}\|_{H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})} \geq \delta_0 \tag{3.17}$$

Thus there is a subsequence of $\{\mu_n\}$, which is still denoted by $\{\mu_n\}$, such that $\{\mu_n\} \to \epsilon \in [0,1]$ and also $\{\nu_n\} \to \epsilon$. Then using lemma 3.4 and the procedure proving it, we get the $u_\epsilon, v_\epsilon \in H^{1,-s'}(\mathbb{R}^2_+\backslash\bar{D})$, by choosing $s' > s$, such that

$$\|u_{\mu_n} - u_\epsilon\|_{H^{1,-s'}(\mathbb{R}^2_+\backslash\bar{D})} \to 0$$
$$\|u_{\nu_n} - v_\epsilon\|_{H^{1,-s'}(\mathbb{R}^2_+\backslash\bar{D})} \to 0$$

and $u_\epsilon = v_\epsilon$ by the same arguement in lemma 3.4 which leads to a contradiction. Thus we have proved u_ε is uniformly continuously for $\varepsilon \in (0,1)$. Then it is easy to see u_ε has a limitation in $H^{1,-s}(\mathbb{R}^2_+\backslash\bar{D})$ and the estimation of u_0 can be obtained by (3.14). This completes the proof. □

It is remain to prove the uniqueness in theorem 3.1. Actually, it can be obtained following the existence of solution with any $g \in H^{1/2}(\Gamma_D)$.

proof of Theorem 3.1 By the linearity of the problem, it is sufficient to prove that any u_0 satisfies the system (3.1-3.3) with the corresponding homogeneous boundary-value vanishes identically in $\mathbb{R}^2_+\backslash\bar{D}$. For any $y \in \mathbb{R}^2_+\backslash\bar{D}$, there exists $U^s(x,y)$ sataifies (3.1-3.3) with $g(x) = -N(x,y)$ on Γ_D following the lemma 3.5 and we define $U(x,y) = N(x,y) + U^s(x,y)$. It is easy to see that $U(x,y)$ satifies the generalized radiation condition (3.4). Thus by the integral representaion of u_0, we have

$$\lim_{r\to\infty} \int_{S_r^+} (\sigma(U(x,y)e_i)\nu) \cdot u_0(x) - (U(x,y)e_i) \cdot (\sigma(u_0)\nu) ds(x) = 0$$

Finally, combining $U(x,y) = 0, u_0(x) = 0$ on Γ_D and the Green integral theorem we find that

$$\begin{aligned} u_0(y)e_i &= \int_{\mathbb{R}^2_+\backslash\bar{D}} -(\Delta_e(N(x,y)e_i) + \omega^2 N(x,y)e_i) \cdot u_0(x)dx \\ &= \int_{\mathbb{R}^2_+\backslash\bar{D}} \Delta u_0(x) \cdot (N(x,y)e_i) - \Delta_e(N(x,y)e_i) \cdot u_0(x) \\ &= \int_{\Gamma_D} (\sigma(U(x,y)e_i)\nu) \cdot u_0(x) - (U(x,y)e_i) \cdot (\sigma(u_0)\nu) ds(x) = 0 \end{aligned}$$

Then the desired unique exsitence follows lemma 3.5. This completes the proof of theorem 3.1. □

4. Reverse time migration method

In this section we introduce RTM method for inverse elastic scattering problems in the half space. Assume that there N_s sources and N_r receivers uniformly distributed on Γ_0^d, where $\Gamma_0^d = \{(x_1, x_2)^T \in \Gamma_0 : x_1 \in [-d, d]\}, d > 0$ is aperture. We denote by Ω the sampling domain in which the obstacle is sought. Let $h = dist(\Omega, \Gamma_0)$ be the distance of Ω to Γ_0. We assume the obstacle $D \subset \Omega$ and there exist constants $0 < c_1 < 1, c_2 > 0, c_3 > 0$ such that

$$|x_1| \leq c_1 d, \quad |x_1 - y_1| \leq c_2 h, \quad |x_2| \leq c_3 h \quad \forall x, y \in \Omega \tag{4.1}$$ convention_2

Our RTM algorithm consists of two steps ela_reverse,Zhang08,Zhang2007 [8, 28, 29]. The first step is the back-propagation in which we back-propagate the complex conjugated data $\overline{u^s(x_r, x_s)}$ as the Dirichlet boundary condition into the domain. The second step is the cross-correlation in which we compute the imaginary part of the cross-correlation of the back-propagated field and the incoming wave which uses the source as the boundary codition on Γ_0.

Algorithm 4.1 (REVERSE TIME MIGRATION ALGORITHM)
Given the data $u_k^s(x_r, x_s)$, $k = 1, 2$ which is the measurement of the scattered field at x_r when the source is emitted at x_s along the polarized direction e_k, $s = 1, \ldots, N_s$ and $r = 1, \ldots, N_r$.

1° *Back-propagation: For $s = 1, \ldots, N_s$ and k=1,2, compute the back-propagation field*

$$v_k(z, x_s) = \frac{|\Gamma_0^d|}{N_r} \sum_{r=1}^{N_r} (T_{x_r}^{e_2} D(x_r, z))^T \overline{u_k^s(x_r, x_s)}, \qquad \forall z \in \Omega \tag{4.2}$$

2° *Cross-correlation: For $z \in \Omega$, compute*

$$I_d(z) = \mathrm{Im} \sum_{k=1}^{2} \left\{ \frac{|\Gamma_0^d|}{N_s} \sum_{s=1}^{N_s} [(T_{x_s}^{e_2} D(x_s, z))^T e_k] \cdot v_k(z, x_s) \right\}. \tag{4.3}$$ cor1

It is easy to that for $z \in \Omega$

$$I_d(z) = \mathrm{Im} \sum_{k=1}^{2} \left\{ \frac{|\Gamma_0^d|}{N_s} \frac{|\Gamma_0^d|}{N_r} \sum_{s=1}^{N_s} \sum_{r=1}^{N_r} [(T_{x_s}^{e_2} D(x_s, z))^T e_k] \cdot [(T_{x_r}^{e_2} D(x_r, z))^T \overline{u_k^s(x_r, x_s)}] \right\} \tag{4.4}$$ cor2

This formula is used in all our numerical experiments in section. By letting $N_s, N_r \to \infty$, we know that (cor2 4.4) can be viewed as an approximation of the following continuous integral:

$$\hat{I}_d(z) = \mathrm{Im} \sum_{k=1}^{2} \int_{\Gamma_0^d} \int_{\Gamma_0^d} [(T_{x_s}^{e_2} D(x_s, z))^T e_k] \cdot [(T_{x_r}^{e_2} D(x_r, z))^T \overline{u_k^s(x_r, x_s)}] ds(x_r) ds(x_s) \tag{4.5}$$ cor3

where $z \in \Omega$. We will study the resolution of the function $\hat{I}_d(z)$ in the section 5. To this end we will first consider the resolution of the finite aperture point source function in the next function.

5. The point spread function

We start by introducing some notation. For any bounded domain $U \subset \mathbb{R}^2$ with Lipschitz boundary Γ_U and the unit outer normal vector ν, let $\|u\|_{H^1(U)} = (\|\nabla\phi\|_{L^2(U)}^2 +$

$d_U^{-2}\|\phi\|^2_{L^2(U)})^{1/2}$ be the weighted $H^1(U)$ norm and $\|v\|_{H^{1/2}(\Gamma)} = (d_U^{-1}\|v\|^2_{L^2(\Gamma)} + |v|^2_{\frac{1}{2},\Gamma})^{1/2}$ be the weighted $H^{1/2}(\Gamma)$ norm, where d_U is the diameter of U and

$$|v|_{\frac{1}{2},\Gamma} = \left(\int_\Gamma\int_\Gamma \frac{|v(x)-v(y)|^2}{|x-y|^2}ds(x)ds(y)\right)^{1/2}.$$

By scaling argument and trace theorem we know that there exists a constant $C > 0$ independent of d_D such that for any $\phi \in C^1(\bar{U})$ [14, corollary 3.1],

$$\|\phi\|_{H^{1/2}(\Gamma_U)} + \|\sigma(\phi)\cdot\nu\|_{H^{-1/2}(\Gamma_U)} \le C\max_{x\in U}(|\phi(x)| + d_U|\nabla\phi(x)|) \tag{5.1}$$

The point spread function measures the resolution for finding point source[3]. In [14], the point spread function has been defined in the case of acoustic wave. We now define elastic point spread function $J(z,y)$, a $\mathbb{C}^{2\times 2}$ matrix, which back-propagate the conjugated data $\overline{N(x,y)}$ as the Dirichlet boundary condition. Thus, for any $z, y \in \mathbb{R}^2_+$

$$J(z,y) = \int_{\Gamma_0} (T_D(x,z))^T\overline{N(x,y)}ds(x) \tag{5.2}$$

$$= \int_{\mathbb{R}} (T_D(x_1,0;z_1,z_2))^T\overline{N(x_1,0;y_1,y_2)}dx_1 \tag{5.3}$$

The estimate in lemma 2.6-2.7 show that the integral above exists. Now, we define functions

$$\Theta(\xi;y_1,y_2) = \frac{1}{\gamma(\xi)}\left[\begin{pmatrix} \mu_s\mu_p & -\xi\mu_p \\ -\xi\mu_s & \xi^2 \end{pmatrix}e^{\mathbf{i}\mu_s y_2} + \begin{pmatrix} \xi^2 & \xi\mu_p \\ \xi\mu_s & \mu_p\mu_s \end{pmatrix}e^{\mathbf{i}\mu_p y_2}\right]e^{\mathbf{i}\xi y_1} \tag{5.4}$$

Let $\underline{\hat{N}(\xi;y)} = \hat{N}(\xi;y_2)e^{-\mathbf{i}\xi y_1}$ and $\hat{T}_D(\xi;y) = \hat{T}_D(\xi;y_2)e^{-\mathbf{i}\xi y_1}$. It is easy to see that $\Theta = \overline{\hat{T}_D(\xi;y)}$ when $\xi \in \mathbb{R}\backslash[-k_s,k_s]$.

We split the spectral terms into components associated with pressure and shearing waves.

$$\hat{T_D} = \hat{T_D^p} + \hat{T_D^s} \quad \hat{N} = \hat{N^p} + \hat{N^s} \quad \Theta = \Theta^p + \Theta^s$$

And we define

$$J^{\alpha\eta}(z,y) = \int_R (T_D^\alpha(x_1,0;z))^T\overline{N^\eta(x_1,0;y)}dx_1, \qquad \alpha = s,p \;\; \eta = s,p \tag{5.5}$$

It's esay to see

$$J(z,y) = \sum_{\alpha=p,s}^{\eta=p,s} J^{\alpha\eta}(z,y)$$

In order to analysis the PSF, loss is assumed in the medium that $k_{\alpha,\varepsilon} := k_\alpha(1+\mathbf{i}\varepsilon)$. Then by Parseval identity, we carry out

$$\begin{aligned}
J^{ss}(z,y) &= \lim_{\varepsilon\to 0^+}\int_R (T_D^s(x_1,0;z_1,z_2))^T\overline{N^{s,\varepsilon}(x_1,0;y_1,y_2)}dx_1 \\
&= \lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int_R (\hat{T}_D^s(\xi,0;z))^T\overline{\hat{N}^{s,\varepsilon}(\xi,0;y)}d\xi \\
&= \frac{1}{2\pi}\int_{-k_s}^{k_s} (\hat{T}_D^s(\xi,0;z))^T\overline{\hat{N}^{s,\varepsilon}(\xi,0;y)}d\xi
\end{aligned}$$

$$
\begin{aligned}
&+ \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{R\setminus[-k_s,k_s]} (\hat{T}_D(\xi,0;z))^T\overline{\hat{N}^{s,\varepsilon}(\xi,0;y)}d\xi \\
&:= F^{ss}(z,y) + R^{ss}(z,y)
\end{aligned}
$$

and for $(\alpha,\eta)\neq(s,s)$

$$
\begin{aligned}
J^{\alpha\eta}(z,y) &= \lim_{\varepsilon\to 0^+} \int_R (T_D^{\alpha}(x_1,0;z_1,z_2))^T\overline{N^{p,\varepsilon}(x_1,0;y_1,y_2)}dx_1 \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_R (\hat{T}_D^{\alpha}(\xi,0;z))^T\overline{\hat{N}^{\eta,\varepsilon}(\xi,0;y)}d\xi \\
&= \frac{1}{2\pi}\int_{-k_p}^{k_p} (\hat{T}_D^{s}(\xi,0;z))^T\overline{\hat{N}^{\eta,\varepsilon}(\xi,0;y)}d\xi \\
&+ \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{R\setminus[-k_p,k_p]} (\hat{T}_D^{\alpha}(\xi,0;z))^T\overline{\hat{N}^{\eta,\varepsilon}(\xi,0;y)}d\xi \\
&:= F^{\alpha\eta}(z,y) + R^{\alpha\eta}(z,y)
\end{aligned}
$$

By lemma root_De2 2.2, lemma root_Ga 2.3 and using Cauchy integral theorem, we get

$$
\begin{aligned}
\overline{R^{ss}(y,z)} &= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{R\setminus[-k_s,k_s]} \overline{(\hat{T}_D^{s}(\xi,0;z))^T}\hat{N}^{s,\varepsilon}(\xi,0;y)d\xi \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{R\setminus[-k_s,k_s]} (\Theta^{s}(\xi;z))^T\hat{N}^{s,\varepsilon}(\xi,0;y)d\xi \\
&= \frac{1}{2\pi}\int_{\Gamma_l^{\pm}\cup\Gamma_r^{\pm}} (\Theta^{s}(\xi;z))^T\hat{N}^{s}(\xi,0;y)d\xi \pm \\
&\quad \mathbf{i}\lim_{\xi\to k_R}(\xi-k_R)(\Theta^{s}(\xi;z))^T\hat{N}^{s}(\xi,0;y) \\
&:= \mathrm{I}^{ss}(z,y) + \mathrm{II}^{ss}(z,y)
\end{aligned}
$$

and for $(\alpha,\eta)\neq(s,s)$

$$
\begin{aligned}
\overline{R^{\alpha\eta}(y,z)} &= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{R\setminus[-k_s,k_s]} \overline{(\hat{T}_D^{\alpha}(\xi,0;z))^T}\hat{N}^{\eta,\varepsilon}(\xi,0;y)d\xi \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{R\setminus[-k_p,k_p]} (\Theta^{\alpha}(\xi;z))^T\hat{N}^{\eta,\varepsilon}(\xi,0;y)d\xi \\
&+ \frac{1}{2\pi}\int_{(-k_s,-k_p)\cup(k_p,k_s)} \overline{(T^{\alpha}(\xi;z))^T}\hat{N}^{\eta}(\xi,0;y)d\xi \\
&= \frac{1}{2\pi}\int_{\Gamma_l^{\pm}\cup\Gamma_r^{\pm}} (\Theta^{\alpha}(\xi;z))^T\hat{N}^{\eta}(\xi,0;y)d\xi \pm \\
&\quad \mathbf{i}\lim_{\xi\to k_R}(\xi-k_R)(\Theta^{\alpha}(\xi;z))^T\hat{N}^{\eta}(\xi,0;y) + \\
&\quad \frac{1}{2\pi}\int_{(-k_s,-k_p)\cup(k_p,k_s)} \overline{(T^{\alpha}(\xi;z))^T}\hat{N}^{\eta}(\xi,0;y)d\xi \\
&:= \mathrm{I}^{\alpha\eta}(z,y) + \mathrm{II}^{\alpha\eta}(z,y) + \mathrm{III}^{\alpha\eta}(z,y)
\end{aligned}
$$

where $\pm$ are corresponding $sgn(z_1-y_1)$. In the sequel, A^{ij} denotes the (i,j) element of a 2×2 matrix.

Our goal now is to show which is the main contribution to the point spread funciton when $k_s h \gg 1$. Put $n_* = \min\{N|\kappa^{2N-1} < 1/c_3, N \in \mathbb{Z}_+\}$. Then we claim the primary theorem in this section:

thm_psf **Theorem 5.1** *Let $k_s h > 1$. For any $z, y \in \Omega$, J(z,y)=F(z,y)+R(z,y), where*

$$F(z,y) = F_{ss}(z,y) + F_{pp}(z,y) \tag{5.6}$$

$$R(z,y) = R^{ss}(z,y) + R^{pp}(z,y) + J^{sp}(z,y) + J^{ps}(z,y) \tag{5.7}$$

Moreover,

$$|R^{ij}(z,y)| + k_s^{-1}|\nabla_y R^{ij}(z,y)| \le \frac{C}{\mu}\left(\frac{1}{(k_s h)^{\frac{1}{2n^*}}} + e^{-k_s h\sqrt{\kappa_R^2-1}}\right) := \frac{C}{\mu}\epsilon_1(k_s h) \tag{5.8}$$

uniformly for $z, y \in \Omega$. Here $\kappa_R := k_R/k_s$ and the constant C may dependent on $k_s d_D$ and $\kappa := k_p/k_s$, but is independent of k_s, k_p, h, d_D.

The proof of Theorem 5.1 depends on several lemmas that follow.

Without loss of generality. we assume $z_1 - y_1 \ge 0$ in this section. Otherwise, we can take substitution $\xi = -\xi$. Notice that the parameterization of hyperbolic curve passing $(\pm 1, 0)$ is:

$$\xi_1 = \pm\sqrt{t^2+1} \qquad \xi_2 = t$$

where $t \in \mathbb{R}$. We only consider the curve in the upper half plane, denoted by Γ^+ here. Substituting $\xi = \xi_1 + \mathbf{i}\xi_2 \in \Gamma^+$ into $\mu(\xi) := (1-\xi^2)^{1/2}$ and $\mu_\kappa(\xi) := (\kappa^2 - \xi^2)^{1/2}$, we arrive at

$$\begin{aligned}\operatorname{Im}\mu(\xi) &= \operatorname{Im}(1 - (\xi_1^2 - \xi_2^2 + \mathbf{i}2\xi_1\xi_2))^{1/2} \\ &= \operatorname{Im}(\mp 2t\sqrt{t^2+1}\mathbf{i})^{1/2} = t^{1/2}(t^2+1)^{1/4}\end{aligned} \tag{5.9}$$ mu_1

$$\begin{aligned}\operatorname{Im}\mu_\kappa(\xi) &= \operatorname{Im}(\kappa^2 - (\xi_1^2 - \xi_2^2 + \mathbf{i}2\xi_1\xi_2))^{1/2} \\ &= \operatorname{Im}(\kappa^2 - 1 \mp 2t\sqrt{t^2+1}\mathbf{i})^{1/2} \\ &= \sqrt{\frac{\sqrt{(1-\kappa^2)^2 + 4t^2(t^2+1)} + 1 - \kappa^2}{2}} \\ &\ge t^{1/2}(t^2+1)^{1/4}\end{aligned} \tag{5.10}$$ mu_2

hyper_term **Lemma 5.1** *For $\xi \in \Gamma^+$, let $f(\xi)$ be a complex valued function in $L^1(\Gamma^+)$ such that $|f(\xi)| \le C(1+\xi^k)$, $k \in \mathbb{Z}_+$. Then for $a, b, c > 0$, we have*

$$\begin{aligned}&|I(a,b,c) := \int_{\Gamma^+} f(\xi)\ e^{\mathbf{i}\xi a + \mathbf{i}\mu(\xi)b + \mathbf{i}\mu_\kappa(\xi)c}\ d\xi| \\ &\le C\left(\frac{1}{b+c} + \frac{1}{(b+c)^{k+1}}\right)\end{aligned}$$

Proof. Ovserve that

$$\frac{d\xi(t)}{dt} = \frac{t}{\sqrt{t^2+1}} + \mathbf{i}$$

By (5.9-5.10), it follows that

$$|e^{\mathbf{i}\xi a + \mathbf{i}\mu(\xi)b + \mathbf{i}\mu_\kappa(\xi)c}| \le e^{-ta - t^{1/2}(t^2+1)^{1/4}b - t^{1/2}(t^2+1)^{1/4}c} \le e^{-t(b+c)}$$

Finally, substituting $\xi(t)$ into $I(a,b,c)$, we have

$$
\begin{aligned}
|I(a,b,c)| &\le C\int_0^\infty (1+t^k)e^{-t(b+c)}dt \\
&\le C(\frac{1}{b+c}+\frac{1}{(b+c)^{k+1}})
\end{aligned}
$$

□

r_estimate1 **Lemma 5.2** *For any $z, y \in \mathbb{R}^2_+$,*

$$
|\mathrm{I}^{\alpha\beta}_{ij}(z,y)| \le \frac{C}{\mu}\sum_{j=1}^{4}(k_s(y_2+z_2))^{-j},\ \ \alpha,\beta = s,p \tag{5.11}
$$

$$
|\frac{\partial \mathrm{I}^{\alpha\beta}_{ij}(z,y)}{\partial y_k}| \le \frac{Ck_s}{\mu}\sum_{j=1}^{4}(k_s(y_2+z_2))^{-j},\ \ \alpha,\beta = s,p \tag{5.12}
$$

where C is may only dependent on κ.

Proof. Notice that

$$
\begin{aligned}
\frac{1}{\delta(\xi)} &= \frac{1}{(k_s^2-2\xi^2)+4\xi^2(k_s^2-\xi^2)^{1/2}(k_p^2-\xi^2)^{1/2}} \\
&= \frac{(k_s^2-2\xi^2)^2-4\xi^2(k_s^2-\xi^2)^{1/2}(k_p^2-\xi^2)^{1/2}}{(4k_p^2-28k_s^2)\xi^6+O(\xi^4)} = O(\frac{1}{\xi^2})
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\gamma(\xi)} &= \frac{1}{\xi^2+(k_s^2-\xi^2)^{1/2}(k_p^2-\xi^2)^{1/2}} \\
&= \frac{\xi^2-(k_s^2-\xi^2)^{1/2}(k_p^2-\xi^2)^{1/2}}{(k_s^2+k_p^2)\xi^2-k_s^2k_p^2} = O(1)
\end{aligned}
$$

as $\xi \to \infty$. Therefore, a simple computation show that the amplitude function of $\mathrm{I}^{\alpha\beta}_{ij}(z,y)$ denote by $A(\xi)$ can be written as $A(\xi) = \frac{\mu}{k_s^3}O(\xi^3)$. Now substituing $\xi = k_s t$ in the integral, the lemma now follows immediately from lemma (5.1). This completes the proof.

□

r_estimate2 **Lemma 5.3** *For any $z, y \in \mathbb{R}^2_+$,*

$$
|\mathrm{II}^{ss}_{ij}(x,y)| \le \frac{C}{\mu}e^{-\sqrt{k_R^2-k_s^2}(y_2+z_2)} \quad |\mathrm{II}^{sp}_{ij}(x,y)| \le \frac{C}{\mu}e^{-\sqrt{k_R^2-k_s^2}z_2+\sqrt{k_R^2-k_p^2}y_2} \tag{5.13}
$$

$$
|\mathrm{II}^{pp}_{ij}(x,y)| \le \frac{C}{\mu}e^{-\sqrt{k_R^2-k_p^2}(y_2+z_2)} \quad |\mathrm{II}^{ps}_{ij}(x,y)| \le \frac{C}{\mu}e^{-\sqrt{k_R^2-k_p^2}z_2+\sqrt{k_R^2-k_s^2}y_2} \tag{5.14}
$$

$$
|\frac{\partial \mathrm{II}^{ss}_{ij}(x,y)}{\partial y_k}| \le \frac{Ck_s}{\mu}e^{-\sqrt{k_R^2-k_s^2}(y_2+z_2)} \quad |\frac{\partial \mathrm{II}^{sp}_{ij}(x,y)}{\partial y_k}| \le \frac{Ck_s}{\mu}e^{-\sqrt{k_R^2-k_s^2}z_2+\sqrt{k_R^2-k_p^2}y_2} \tag{5.15}
$$

$$
|\frac{\partial \mathrm{II}^{pp}_{ij}(x,y)}{\partial y_k}| \le \frac{Ck_s}{\mu}e^{-\sqrt{k_R^2-k_p^2}(y_2+z_2)} \quad |\frac{\partial \mathrm{II}^{ps}_{ij}(x,y)}{\partial y_k}| \le \frac{Ck_s}{\mu}e^{-\sqrt{k_R^2-k_p^2}z_2+\sqrt{k_R^2-k_s^2}y_2} \tag{5.16}
$$

where C is only dependent on $\kappa := k_p/k_s$.

Proof. When $z_1 - y_1 > 0$, we have

$$\begin{aligned}\mathrm{II}_{11}^{ss} = & -\frac{1}{\mu} Res_{\xi=k_R} \frac{(k_s^2 - 4\xi^2)\mu_s^2\mu_p}{\gamma(\xi)\delta(\xi)} e^{\mathbf{i}\mu_s(z_2+y_2)+\mathbf{i}\xi(z_1-y_1)} \\ = & -\frac{(k_s^2 - 4\xi^2)\mu_s^2\mu_p}{\mu(\gamma(\xi)\delta(\xi))'} e^{\mathbf{i}\mu_s(z_2+y_2)+\mathbf{i}\xi(z_1-y_1)}|_{\xi=k_R}\end{aligned}$$

Eliminating k_s in fraction, we can obtain estimate immediately. The other terms can be proved similarly, here we omit detials. This completes the proof. □

medi_term **Lemma 5.4** *Let $f(\xi)$ be a bounded complex valued function in $L^1((\kappa,1))$. Then we have*

$$\begin{aligned}&|I(a,b) := \int_\kappa^1 |f(\xi)e^{\mathbf{i}\xi a+\mathbf{i}\mu_\kappa(\xi)b}d\xi| \\ &\leq C\frac{1}{b}\|f\|_{L^\infty(\kappa,1)}\end{aligned}$$

Proof. It is simple to see that

$$\begin{aligned}|I(a,b)| &\leq C\int_\kappa^1 e^{-b\sqrt{\xi^2-\kappa^2}}d\xi \\ &\leq C\int_0^{\sqrt{1-\kappa^2}} \frac{t}{\sqrt{t^2+\kappa^2}}e^{-bt}dt \\ &\leq C\frac{1}{b}\|f\|_{L^\infty(\kappa,1)}\end{aligned}$$

□

r_estimate3 **Lemma 5.5** *For any $z, y \in \mathbb{R}_+^2$,*

$$|\mathrm{III}_{ij}^{pp}(x,y)| \leq \frac{C}{\mu k_s(y_2+z_2)} \quad |\mathrm{III}_{ij}^{sp}(x,y)| \leq \frac{C}{\mu k_s y_2} \quad |\mathrm{III}_{ij}^{ps}(x,y)| \leq \frac{C}{\mu k_s z_2} \tag{5.17}$$

$$|\frac{\partial \mathrm{III}_{ij}^{pp}(x,y)}{\partial y_k}| \leq \frac{C}{\mu y_2+z_2} \quad |\frac{\partial \mathrm{III}_{ij}^{sp}(x,y)}{\partial y_k}| \leq \frac{C}{\mu y_2} \quad |\frac{\partial \mathrm{III}_{ij}^{ps}(x,y)}{\partial y_k}| \leq \frac{C}{\mu z_2} \tag{5.18}$$

where C is only dependent on κ.

Proof. Taking substitution $\xi = k_s t$ and using the fact that $\gamma(\xi), \delta(\xi)$ have no roots on interval $[k_p, k_s]$, then we can get supremum of amplitude function. By lemma 5.4 with $b = k_s(y+z), k_s y, k_s z$, we can get the estimate immediately. This completes the proof. □

It turn to estimate $F^{sp}(z,y)$ and $F^{ps}(z,y)$.

cross_term **Lemma 5.6** *For $0 < \kappa < 1$, let $F(\lambda) = \int_0^\kappa f(t)e^{\mathbf{i}\lambda(\sqrt{1-t^2}-\tau\sqrt{\kappa^2-t^2}+\alpha t)}dt$, where $\tau \geq c_0 > 0$ and $\alpha \in \mathbb{R}$, then we have*

$$|F(\lambda)| \leq C(\kappa)\lambda^{-\frac{1}{2N_*}}\left[|f(\kappa)| + \int_0^\kappa |f'(t)|dt\right]$$

where $N_* = \min\{N|\kappa^{2N-1} < c_0, N \in \mathbb{Z}_+\}$.

Proof. Put $\phi(t) = -\sqrt{1-t^2}$ and $\psi(t,\tau) = \tau\kappa\phi(t/\kappa) - \phi(t) + \alpha t$. For easy of notations, we denote the n-th partial derivative of $g(t)$ with respect to t by $g^{(n)}(t)$. Then, it is to see that, for $n > 1$

$$\psi^{(n)}(t,\tau) = \frac{\tau}{\kappa^{n-1}}\phi^{(n)}(\frac{t}{\kappa}) - \phi^{(n)}(t)$$

A standard computation show that

$$\phi^{(1)}(t) = \frac{t}{\sqrt{1-t^2}}$$
$$\phi^{(2)}(t) = \frac{1}{(1-t^2)^{3/2}}$$
$$\phi^{(3)}(t) = \frac{3t}{(1-t^2)^{5/2}}$$

Moreover, for $n \geq 3$, we have

$$\phi^{(n)}(t) = \frac{p_n(t)}{(1-t^2)^{n-1/2}} \tag{5.19}$$

where $p_n = \sum_0^{n-2} a_k^n t^k$ is a $(n-2)$-th polynomial such that its coefficients satisfy the following recursion formula:

$$\begin{aligned} &a_{n-1}^{n+1} = (n+1)a_{n-2}^n, \quad a_{n-2}^{n+1} = (n+2)a_{n-3}^n \\ &a_k^{n+1} = (k+1)a_{k+1}^n + (2n-k)a_{k-1}^n \quad \text{for } 1 \leq k \leq n-3 \\ &a_0^{n+1} = a_1^n \end{aligned}$$

Since the polynmial coefficients are all positive, it is obvious that for $n \geq 1$, $\phi^{(n)}(t)$ is a monotone increasing positive function. Using the recursion formula, it follows that

$$\phi^{(n)}(0) = \begin{cases} 0 & \text{n is odd,} \\ (n-1)!!(n-3)!! & \text{n is even.} \end{cases} \tag{5.20}$$

value_0

where $(2k-1)!!$ is double factorial and $n > 3$. We are now in the position to proof the inequality. Since $0 < \kappa < 1$, obersev that

$$\psi^{(2N_*+1)}(t,\tau) \geq \frac{\tau}{\kappa^{2N_*}}\phi^{(2N_*+1)}(t) - \phi^{(2N_*+1)}(t) > 0$$

Therefore, $\psi^{(2N_*)}(t,\tau)$ is monotone increasing in $[0,\kappa)$. By (5.20), we get

$$\psi^{(2N_*)}(t,\tau) \geq \psi^{(2N_*)}(0,\tau) \geq \psi^{(2N_*)}(0,c_0) = C(2N_*)(\frac{c_0}{\kappa^{2N_*-1}} - 1) > 0 \tag{5.21}$$

The lemma is now a direct consequence of lemma (2.4). □

r_estimate4

Lemma 5.7 *For any $z, y \in \Omega$,*

$$|F_{ij}^{sp}(z,y)| \leq \frac{C}{\mu}\frac{1}{(k_s h)^{\frac{1}{2n^*}}} \quad |F_{ij}^{ps}(z,y)| \leq \frac{C}{\mu}\frac{1}{(k_s h)^{\frac{1}{2n^*}}} \tag{5.22}$$

$$|\frac{\partial F_{ij}^{sp}(z,y)}{\partial y_k}| \leq \frac{Ck_s}{\mu}\frac{1}{(k_s h)^{\frac{1}{2n^*}}} \quad |\frac{\partial F_{ij}^{ps}(z,y)}{\partial y_k}| \leq \frac{Ck_s}{\mu}\frac{1}{(k_s h)^{\frac{1}{2n^*}}} \tag{5.23}$$

where C is only dependent on κ.

Proof. Let $\phi(t,\tau) = (\sqrt{1-t^2} - \tau\sqrt{\kappa^2 - t^2})$ where $\tau = y_2/z_2$. From the convention (4.1) we have $1/c_3 < \tau < c_3$. Obviously,

$$F_{ij}^{sp}(z,y) = \frac{1}{2\pi}\int_0^{k_p}\left[\mathcal{T}_s^T\mathcal{N}_p\right]_{ij}(\xi)e^{\mathbf{i}\mu_s z_2 - \mathbf{i}\mu_p y_2 - \mathbf{i}\xi(z_1-y_1)}d\xi$$
$$= \frac{k_s}{2\pi}\int_0^{\kappa}\left[\mathcal{T}_s^T\mathcal{N}_p\right]_{ij}(k_s t)e^{\mathbf{i}k_s z_2\phi(t,\tau)+\alpha t}dt$$

Now the estimate of $F_{ij}^{sp}(z,y)$ follows the lemma 5.6 with $\lambda = k_s z_2$ and $\alpha = (y_1 - z_1)/z_2$. We can obtain the estimate of $F_{ij}^{ps}(z,y)$ in the same method. This completes the proof. $\square$

Now we are in the position to prove the main theorem of this section.

proof of Theorem 5.1. The theorem now follows from lemma 5.2, lemma 5.3, lemma 5.5 and lemma 5.7.

To complete the analysis of the point spread function, Substitute (5.4) and (2.26) into $F_{ss}(z,y), F_{pp}(z,y)$:

$$F^{pp}(z,y) = -\frac{1}{2\pi}\int_{(-k_p,k_p)}\frac{\mathbf{i}k_s^2\mu_s}{\mu\gamma(\xi)\delta(\xi)}\begin{pmatrix}\xi^2 & -\xi\mu_p\\ -\xi\mu_p & \mu_p^2\end{pmatrix}e^{\mathbf{i}\mu_p(z_2-y_2)+\mathbf{i}\xi(y_1-z_1)} \tag{5.24}$$

$$F^{ss}(z,y) = -\frac{1}{2\pi}\int_{(-k_p,k_p)}\frac{\mathbf{i}k_s^2\mu_p}{\mu\gamma(\xi)\delta(\xi)}\begin{pmatrix}\mu_s^2 & \xi\mu_s\\ \xi\mu_s & \xi^2\end{pmatrix}e^{\mathbf{i}\mu_s(z_2-y_2)+\mathbf{i}\xi(y_1-z_1)} \tag{5.25}$$
$$-\frac{1}{2\pi}\int_{(-k_s,k_s)\backslash(-k_p,k_p)}\frac{\mathbf{i}(k_s^2-4\xi^2)\mu_p}{\mu\gamma(\xi)\overline{\delta(\xi)}}\begin{pmatrix}\mu_s^2 & \xi\mu_s\\ \xi\mu_s & \xi^2\end{pmatrix}e^{\mathbf{i}\mu_s(z_2-y_2)+\mathbf{i}\xi(y_1-z_1)}$$
$$:= F^{ss1}(z,y) + F^{ss2}(z,y)$$

Based on the above argument, we know that $R(z,y)$ becomes small when z,y move away from Γ_0. Our goal is to show $F(z,y)$ has the similar decay to the elastic fundamental solution $\operatorname{Im}\Phi(z,y)$ as $|z-y|\to\infty$.

Lemma 5.8 *For any $z,y\in\mathbb{R}_+^2$, when $z=y$*

$$|\operatorname{Im}F_{ii}(z,y)| \geq \frac{1}{4(\lambda+2\mu)}\ ,\ i=1,2$$
$$\operatorname{Im}F_{12}(z,y) = \operatorname{Im}F_{21}(z,y) = 0$$

and for $z\neq y$

$$|F_{ij}(z,y)| \leq \frac{C}{\mu}[(k_s|z-y|)^{-1/2}) + (k_s|z-y|^{-1})]$$

where constant C is only dependent on κ.

Proof. We only proof the case of $i=1$, the other ones are similar. First, we have $\gamma(\xi)\leq k_s^2$, $\delta(\xi)\leq k_s^4$ and $\mu_p\leq\mu_s$ when $\xi\in(-k_p,k_p)$. Then, if $z=y$

$$-\operatorname{Im}(F_{11}^{pp}+F_{11}^{ss1}) \geq \frac{1}{2\pi\mu}\int_{(-k_p,k_p)}\frac{\mu_p}{k_s^2}d\xi \tag{5.26}$$
$$= \frac{k_p^2}{2\pi\mu k_s^2}\int_0^{\pi}\sin^2(t)dt = \frac{1}{4(\lambda+2\mu)} \tag{5.27}$$

It's left to proof $-\mathrm{Im}\, F_{11}^{ss2} > 0$. If $\xi \in (-k_s, k_s)\backslash(-k_p, k_p)$, $\mu_p = \mathbf{i}\sqrt{\xi^2 - k_p^2}$. Substituting it into F^{ss2}, we have

$$F_{11}^{ss2} = \frac{1}{2\pi\mu}\int_{(-k_s,k_s)\backslash(-k_p,k_p)} \frac{\mu_s^2\sqrt{\xi^2-k_p^2}(k_s^2-4\xi^2)}{(\xi^2+\mathbf{i}\mu_s\sqrt{\xi^2-k_p^2})(\beta^2-\mathbf{i}4\xi^2\mu_s\sqrt{\xi^2-k_p^2})}d\xi \tag{5.28}$$

let $\alpha = (\xi^2+\mathbf{i}\mu_s\sqrt{\xi^2-k_p^2})(\beta^2-\mathbf{i}4\xi^2\mu_s\sqrt{\xi^2-k_p^2})$. A simple computation show that $\mathrm{Im}\,\alpha = k_s^2\mu_s\sqrt{\xi^2-k_p^2}(k_s^2-4\xi^2)$. It is easy to see that

$$-\mathrm{Im}\, F_{11}^{ss2} = \frac{k_s^2}{2\pi\mu}\int_{(-k_s,k_s)\backslash(-k_p,k_p)} \frac{\mu_s^3(\xi^2-k_p^2)(k_s^2-\xi^2)^2}{|\alpha|^2}d\xi > 0$$

For $z \neq y$, we denot $y - z = |y-z|(\cos\phi, \sin\phi)^T$ for some $0 \le \phi \le 2\pi$. Then it is easy to see that

$$F^{pp}(z,y) = \frac{1}{\mu}\int_0^\pi A(\theta,\kappa)e^{\mathbf{i}k_s|z-y|\cos(\theta-\phi)}$$

The phase function $f(\theta) = \cos(\theta-\phi)$ satisfies $f'(\theta) = -\sin(\theta-\phi)$, $f''(\theta) = -\cos(\theta-\phi)$. For any given $0 \le \phi \le 2\pi$, we can decompose $[0,\pi]$ into several intervals such that in each either $|f''(\theta)| \ge 1/2$ or $|f'(\theta)| \ge 1/2$ and $f'(\theta)$ is monotonous. The amplitude function $A(\theta,\kappa)$ and their derirates are integrable on $[0,\pi]$. Then the estimate for $F_{pp}(z,y)$ follows by using lemma van 2.4. The estimation of $F^{ss}(z,y)$ can be proved similarly. This completes the proof. □

By (g0 5.1), we obtain the following consequence of Lemma 3.1 and Lemma 3.3 whcih will be used in the next section.

cor_psf **Corollary 5.1** *There exists a constant C independent of k_s, h such that*

$$\|F(z,\cdot)\|_{H^{1/2}(\Gamma_D)} + \|\sigma(F(z,\cdot))\cdot\nu\|_{H^{1/2}(\Gamma_D)} \le \frac{C}{\mu}(1+k_s d_D)$$

$$\|R(z,\cdot)\|_{H^{1/2}(\Gamma_D)} + \|\sigma(R(z,\cdot))\cdot\nu\|_{H^{1/2}(\Gamma_D)} \le \frac{C}{\mu}(1+k_s d_D)\epsilon_1(k_s h)$$

uniformly for $z \in \Omega$, where d_D is the diameter of the obstacle D.

Now we consider the finite aperture point spread function $J_d(z,y)$:

$$\int_{-d}^{d}(T_D(x_1,0;z_1,z_2))^T\overline{N(x_1,0;y_1,y_2)}dx_1 \tag{5.29}$$

Our aim is to estimate the difference $J(z,y) - J_d(z,y)$. It is easy to see that

$$\frac{(x_1-z_1)^2}{\rho^2} = \frac{1}{1+\frac{z_2^2}{(x_1-z_1)^2}} \ge \frac{1}{1+\frac{c_3^2h^2}{(1-c_1)^2d^2}} = \frac{(1-c_1)^2}{(1-c_1)^2+c_3^2(h/d)^2} := m(h/d) \tag{5.30}$$

$$\frac{z_2^2}{\rho^2} = \frac{1}{1+\frac{(x_1-z_1)^2}{z_2^2}} \le \frac{1}{1+\frac{(1-c_1)^2d^2}{c_3^2h^2}} = \frac{c_3^2}{(1-c_1)^2(h/d)+c_3^2} := M(h/d) \tag{5.31}$$

where $\rho = \sqrt{(x_1-z_1)^2+z_2^2}$ and $z \in \Omega, x \in \Gamma_0\backslash(-d,d)$.

ap_psf **Theorem 5.2** *Assume $m(h/d) > (1+\kappa)^2/4$, $M(h/d) < \kappa^2/4$ and $k_s h \geq 1$. Then for $z, y \in \Omega$, we have*

$$|J(z,y) - J_d(z,y)| + k_s^{-1}|\nabla_y(J(z,y) - J_d(z,y))| \tag{5.32}$$

$$\leq \frac{C}{\mu}((\frac{h}{d})^2 + \frac{(k_s h)^{1/2}}{e^{k_s h\sqrt{\kappa_R^2-1}}}(\frac{h}{d})^{1/2}) := \frac{C}{\mu}\epsilon_2(k_s h, h/d) \tag{5.33}$$

where the constant C is only dependent on κ.

Proof. By lemma 2.7, lemma 2.6 and $k_s h \geq 1$, we have

$$\begin{aligned}
&\left|\int_d^\infty (T_D(x_1,0;z_1,z_2))^T\overline{N(x_1,0;y_1,y_2)}dx_1\right| \\
&\leq \frac{C}{\mu}\int_d^\infty \frac{k_s z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{1/2}}\Big(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{1/2}} + e^{-\sqrt{k_R^2-k_s^2}y_2}\Big)dx_1 \\
&\leq \frac{C}{\mu}\int_{(1-c_1)d/h}^\infty \frac{1}{(1+t^2)^{3/2}} + \frac{(k_s h)^{1/2}}{(1+t^2)^{3/4}}e^{-\sqrt{k_R^2-k_s^2}h}dt \\
&\leq \frac{C}{\mu}((\frac{h}{d})^2 + \frac{(k_s h)^{1/2}}{e^{\sqrt{k_R^2-k_s^2}h}}(\frac{h}{d})^{1/2})
\end{aligned}$$

Here we have used the first inequeality in (4.1). Similarly, we can prove that the estimate for te integral in $[-\infty, -d]$. This shows the estimate for $J(z,y) - J_d(z,y)$. The estimate for $\nabla_y(J(z,y) - J_d(z,y))$ can be proved similarly. □

By (5.1) we obtain the following corollary

cor_dpsf **Corollary 5.2** *Assume $m(h/d) > (1+\kappa)^2/4$, $M(h/d) < \kappa^2/4$ and $k_s h \geq 1$. There exists a constant C independent of k_s, h such that*

$$\|J(z,\cdot) - J_d(z,\cdot)\|_{H^{1/2}(\Gamma_D)} + \|\sigma(J(z,\cdot) - J_d(z,\cdot))\cdot\nu\|_{H^{1/2}(\Gamma_D)} \leq \frac{C}{\mu}\epsilon_2(k_s h, h/d)(1 + k_s d_D)$$

uniformly for $z \in \Omega$, where d_D is the diameter of the obstacle D.

6. The resolution analysis

In this section we study the imaging resolution of the RTM for the Dirichlet boundary obstacle in the half space. The following theorem shows that the difference between the half space scattering solution and the full space sacttering solution is small of the sactterer is far away from the boundary Γ_0.

diff_solu **Theorem 6.1** *Let $g \in H^{1/2}(\Gamma_D)$ and u_1, u_2 be the scattering solution of following problems:*

$$\Delta_e u_1 + \omega^2 u_1 = 0 \qquad \text{in } \mathbb{R}^2_+\backslash\bar{D} \tag{6.1}$$ elas_r1

$$u_1 = g \quad \text{on } \Gamma_D \tag{6.2}$$ elas_rbd

$$\sigma(u_1)e_2 = 0 \quad \text{on}\Gamma_0 \tag{6.3}$$ elas_rb0

and

$$\Delta_e u_2 + \omega^2 u_2 = 0 \qquad \text{in } \mathbb{R}^2\backslash\bar{D} \tag{6.4}$$ elas_r2

$$u_2 = g \quad \text{on } \Gamma_D \tag{6.5}$$ elas_rbd2

Then there exits a constant C such that

$$\|T_x^\nu(u_1-u_2)\|_{H^{-1/2}(\Gamma_D)} \le \frac{C}{\mu}(1+k_sd_D)^2((k_sh)^{-1/2}+e^{-\sqrt{k_R^2-k_s^2}h})\|g\|_{H^{1/2}(\Gamma_D)} \tag{6.6}$$

Before proving theorem 6.1, we need the following lemma.

es_diri_neu **Lemma 6.1** *For any $x,y\in D$, let*

$$p(x,y)=\lim_{\varepsilon\to 0^+}p^\varepsilon(x,y):=\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}\frac{f(\mu_p^\varepsilon,\mu_s^\varepsilon,\xi)}{\delta^\varepsilon(\xi)}e^{\mathbf{i}\mu_\alpha^\varepsilon x_2+\mathbf{i}\mu_\beta^\varepsilon y_2+\mathbf{i}\xi(y_1-x_1)}d\xi$$

where $f(a,b,c)$ is a homogeneous fifth order polynomial with repect to a,b,c and $\alpha,\beta\in\{p,s\}$.Then there exists a constant $C>0$ only dependent on κ such that

$$|p(x,y)|+k_s^{-1}|\nabla_xp(x,y)|+k_s^{-1}|\nabla_yp(x,y)|+k_s^{-2}|\nabla_x\nabla_yp(x,y)|\le C((k_sh)^{-1/2}+e^{-\sqrt{k_R^2-k_s^2}h})$$

uniformly for $x,y\in D$.

Proof. Without loss of generality, we assume $k_\alpha\le k_\beta$. Then we can divide $p(x,y)$ into two parts:

$$\begin{aligned}
p(x,y) &= \lim_{\varepsilon\to 0^+}\int_{I_1}+\int_{I_2}\frac{f(\mu_p^\varepsilon,\mu_s^\varepsilon,\xi)}{(k_\alpha^\varepsilon)^2\delta^\varepsilon(\xi)}e^{\mathbf{i}\mu_\alpha^\varepsilon x_2+\mathbf{i}\mu_\beta^\varepsilon y_2+\mathbf{i}\xi(y_1-x_1)}d\xi\\
&= \int_{I_1}\frac{f(\mu_p,\mu_s,\xi)}{k_\alpha^2\delta(\xi)}e^{\mathbf{i}\mu_\alpha x_2+\mathbf{i}\mu_\beta y_2+\mathbf{i}\xi(y_1-x_1)}d\xi\\
&+ \lim_{\varepsilon\to 0^+}\int_{I_2}\frac{f(\mu_p^\varepsilon,\mu_s^\varepsilon,\xi)}{(k_\alpha^\varepsilon)^2\delta^\varepsilon(\xi)}e^{\mathbf{i}\mu_\alpha^\varepsilon x_2+\mathbf{i}\mu_\beta^\varepsilon y_2+\mathbf{i}\xi(y_1-x_1)}d\xi\\
&= p_1(x,y)+p_2(x,y)
\end{aligned}$$

where $I_1=(-k_\alpha,k_\alpha)$ and $I_2=R\backslash[-k_\alpha,k_\alpha]$. Substituting $\xi=k_\alpha t$ into $p_1(x,y)$, we get

$$p_1(x,y)=\int_{-1}^1\frac{f(\mu_p(k_\alpha t),\mu_s(k_\alpha t),k_\alpha t)}{k_\alpha\delta(k_\alpha t)}e^{\mathbf{i}k_\alpha x_2(\sqrt{1-t^2}+\tau\sqrt{\varsigma^2-t^2}+\gamma t)}dt$$

where $\tau=y_2/x_2$, $\varsigma=k_\beta/k_\alpha$ and $\gamma=(y_1-x_1)/x_2$. It is easy to see that the phase function $\phi(t)=\sqrt{1-t^2}+\tau\sqrt{\varsigma^2-t^2}+\gamma t$ satifies $|\phi''(t)|\ge 1/(1-t^2)^{3/2}\ge 1$ for $t\in(-1,1)$. Then we can obtain $|p_1(x,y)|\le C1/(k_sh)^{1/2}$ by lemma 2.4.

For $p_2(x,y)$, by changing the integration path and using same argument as in the lemma 5.1-5.5, we can easily obtain:

$$|p_2(x,y)|\le C(\frac{1}{k_sh}+e^{-\sqrt{k_R^2-k_s^2}}h)$$

This completes the proof of the esitmate for $|p(x,y)|$. The other estimates can be proved by a similar argument. We omit the details here. □

Now we are ready to prove Theorem 6.1.

proof of Theorem 6.1 Denote by u_1^ε, u_2^ε the corresponding solution of equations 6.1,6.4 where ω is substituted by $\omega(1+\mathbf{i}\varepsilon)$ for any $0<\varepsilon<1$. Let $w^\varepsilon(x)$ be the solution of the problem:

$$\Delta_e w^\varepsilon+(1+\mathbf{i}\varepsilon)\omega^2w^\varepsilon=0 \qquad \text{in } \mathbb{R}_+^2 \tag{6.7}$$ elas_z3

$$\sigma(w^\varepsilon)e_2=-\sigma(u_2^\varepsilon)e_2 \quad \text{on}\Gamma_0 \tag{6.8}$$ elas_zb01

Then $u_1^\varepsilon - u_2^\varepsilon - w^\varepsilon$ satisfies (3.8),(3.10) with the boundary condition $u_1^\varepsilon - u_2^\varepsilon - w^\varepsilon = -w^\varepsilon$ on Γ_D. Thus by the limiting absorption principle, lemma 3.2 and trace therorem, we have

$$\begin{aligned} \|T_x^\nu(u_1^\varepsilon - u_2^\varepsilon)\|_{H^{-1/2}(\Gamma_D)} &\le C(\|w^\varepsilon\|_{H^{1/2}(\Gamma_D)} + |T_x^\nu(w^\varepsilon)\|_{H^{-1/2}(\Gamma_D)}) && (6.9) \quad \boxed{\texttt{diff1}} \\ &\le C\max_{x\in D}(|w^\varepsilon(x)| + d_D|\nabla w^\varepsilon(x|) && (6.10) \end{aligned}$$

where C is independant of ε, ω. By the integral representation formula we have for any $z \in \Gamma_0$

$$u_2^\varepsilon(z) = \int_{\Gamma_D} (T_y^\nu \Phi^\varepsilon(y,z))^T u_2^\varepsilon(y) - \Phi^\varepsilon(z,y)(T_y^\nu u_2^\varepsilon(y))ds(y) \tag{6.11}$$

which yields by using the integral representation again that for $x \in D$

$$\begin{aligned} w^\varepsilon(x) &= \int_{\Gamma_0} N^\varepsilon(x,z)(T_z^{e_2} u_2^\varepsilon(z))ds(z) && (6.12) \\ &= \int_{\Gamma_D} ds(y) \int_{\Gamma_0} N^\varepsilon(x,z)(T_z^{e_2}((T_y^\nu \Phi^\varepsilon(y,z))^T))ds(z) && (6.13) \\ &- \int_{\Gamma_D} v^\varepsilon(x,y)(T_y^\nu u_2^\varepsilon(y))ds(y) && (6.14) \\ &= \int_{\Gamma_D} ds(y) \int_{\Gamma_0} N^\varepsilon(x,z)(T_z^{e_2}(\Phi^\varepsilon(y,z)^T(T_y^\nu)^T)ds(z) && (6.15) \\ &- \int_{\Gamma_D} v^\varepsilon(x,y)(T_y^\nu u_2^\varepsilon(y))ds(y) && (6.16) \\ &= \int_{\Gamma_D} ds(y) \int_{\Gamma_0} N^\varepsilon(x,z)(T_y^\nu(T_z^{e_2}\Phi^\varepsilon(z,y))^T)^T ds(z) && (6.17) \\ &- \int_{\Gamma_D} v^\varepsilon(x,y)(T_y^\nu u_2^\varepsilon(y))ds(y) && (6.18) \\ &= \int_{\Gamma_D} (T_y^\nu(v^\varepsilon(x,y))^T)^T u_2^\varepsilon(y) - v^\varepsilon(x,y)(T_y^\nu u_2^\varepsilon(y))ds(y) && (6.19) \end{aligned}$$

where

$$v^\varepsilon(x,y) = \int_{\Gamma_0} N^\varepsilon(x,z)(T_z^{e_2}\Phi^\varepsilon(z,y))ds(z) \tag{6.20}$$

Since $\|T_x^\nu(u_2^\varepsilon)\|_{H^{-1/2}(\Gamma_D)} \le C\|g\|_{H^{1/2}(\Gamma_D)}$, we obtain

$$|w^\varepsilon(x)| \le C\|g\|_{H^{1/2}(\Gamma_D)} \max_{x\in D}(|v^\varepsilon(x,y)| + d_D|\nabla_y v^\varepsilon(x,y)|) \tag{6.21}$$

and

$$|\nabla w^\varepsilon(x)| \le C\|g\|_{H^{1/2}(\Gamma_D)} \max_{x\in D}(|\nabla_x v^\varepsilon(x,y)| + d_D|\nabla_x\nabla_y v^\varepsilon(x,y)|) \tag{6.22}$$

By (6.9) and letting $\varepsilon \to 0^+$, we have

$$\begin{aligned} &\|T_x^\nu(u_1 - u_2)\|_{H^{-1/2}(\Gamma_D)} \le C\|g\|_{H^{1/2}(\Gamma_D)} \max_{x\in D} \lim_{\varepsilon\to 0^+}(|v^\varepsilon(x,y)| && (6.23) \quad \boxed{\texttt{diff2}} \\ &+ d_D|\nabla_y v^\varepsilon(x,y)| + d_D|\nabla_x v^\varepsilon(x,y)| + d_D^2|\nabla_x\nabla_y v^\varepsilon(x,y)|) && (6.24) \end{aligned}$$

Applying the Fourier transformation to the first horizontal variable of $T_z^{e_2}\Phi^\varepsilon(z,y)$, we have

$$\mathcal{F}[T_z^{e_2}\Phi^\varepsilon](\xi,0;y)=\frac{\mu}{2\omega^2}\left[\begin{pmatrix} 2\xi^2 & -2\xi\mu_p \\ -\frac{\beta\xi}{\mu_p} & \beta \end{pmatrix} e^{\mathbf{i}\mu_p y_2}+\begin{pmatrix} \beta & \frac{\xi\beta}{\mu_s} \\ 2\xi\mu_s & 2\xi^2 \end{pmatrix} e^{\mathbf{i}\mu_s y_2}\right] e^{-\mathbf{i}\xi y_1}$$

Using Parseval identity combined with above formula and formula 2.26, we have

$$\lim_{\varepsilon\to 0^+} v^\varepsilon(x,y)=\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}\mathcal{F}[N^\varepsilon](\xi,0;x)^T\mathcal{F}[T_z^{e_2}\Phi^\varepsilon](-\xi,0;y)d\xi$$

This completes the proof by using lemma 6.1. □

The following theorem is the main result of this section.

Theorem 6.2 *For any $z\in\Omega$, let $\Psi(y,z)\in\mathbb{C}^{2\times 2}$ be the radiation solution of the problem:*

$$\begin{aligned} &\Delta_e\Psi(y,z)+\omega^2\Psi=0 && in\ \mathbb{R}^2_+\backslash\bar{D} \\ &\Psi(y,z)=-\overline{F^T(z,y)} && on\ \Gamma_D \\ &\sigma_y(\Psi(y,z))\cdot e_2=0 && on\ \Gamma_0 \end{aligned}$$

Then, we have

$$\hat{I}_d(z)=\mathrm{Im}\,\mathbf{tr}\int_{\Gamma_D}(T_y^\nu(\overline{F^T(z,y)}+\Psi(y,z))^T\overline{F^T(z,y)}ds(y)+W_{\hat{I}}(z) \tag{6.25}$$

where $|W_{\hat{I}}(z)|\le C(1+k_sd_D)^2(\epsilon_1(k_sh)+\epsilon_2(k_sh,h/d))$ uniformly for z in Ω, $m(h/d)>(1+\kappa)^2/4$, $M(h/d)<\kappa^2/4$.

Proof. By the integral representation, we have,

$$u_k^s(x_r,x_s)=\int_{\Gamma_D}(T_y^\nu N(y,x_r))^T u_k^s(y,x_s)-G(x_r,y)(T_y^\nu u_k^s(y,x_s))ds(y) \tag{6.26}$$

where $u_k^s(x,x_s)+N(x,x_s)e_k=0$. From (5.2) we get for any $z\in\Omega$,

$$\begin{aligned} v_k(z,x_s) &= \int_{\Gamma_0^d}(T_{x_r}^{e_2}D(x_r,z))^T\overline{u_k^s(x_r,x_s)}ds(x_r) \\ &= \int_{\Gamma_D}ds(y)\Big(\int_{\Gamma_0^d}(T_{x_r}^{e_2}D(x_r,z))^T\overline{(T_y^\nu N(y,x_r))^T}ds(x_r)\Big)\overline{u_k^s(y,x_s)} \\ &\quad-\Big(\int_{\Gamma_0^d}(T_{x_r}^{e_2}D(x_r,z))^T\overline{N(x_r,y)}ds(x_r)\Big)\overline{(T_y^\nu u_k^s(y,x_s))} \\ &= \int_{\Gamma_D}ds(y)\Big(\int_{\Gamma_0^d}(T_y^\nu\overline{N(y,x_r)}T_{x_r}^{e_2}D(x_r,z))^T ds(x_r)\Big)\overline{u_k^s(y,x_s)} \\ &\quad-\Big(\int_{\Gamma_0^d}(T_{x_r}^{e_2}D(x_r,z))^T\overline{N(x_r,y)}ds(x_r)\Big)\overline{(T_y^\nu u_k^s(y,x_s))} \\ &= \int_{\Gamma_D}ds(y)\Big(\int_{\Gamma_0^d}(T_y^\nu[\overline{N(y,x_r)}T_{x_r}^{e_2}D(x_r,z)])^T ds(x_r)\Big)\overline{u_k^s(y,x_s)} \\ &\quad-\Big(\int_{\Gamma_0^d}(T_{x_r}^{e_2}D(x_r,z))^T\overline{N(x_r,y)}ds(x_r)\Big)\overline{(T_y^\nu u_k^s(y,x_s))} \\ &= \int_{\Gamma_D}ds(y)\Big((T_y^\nu J_d^T(z,y))^T\overline{u_k^s(y,x_s)}-J_d(z,y)\overline{(T_y^\nu u_k^s(y,x_s))}\Big) \end{aligned}$$

where we use the fact $(\sigma_x(A(x))\nu)B = \sigma_x(A(x)B)\nu$ above. By the definition of the imaging function $\hat{I}_d(z)$, we have

$$\begin{aligned}
\hat{I}_d(z) &= \mathrm{Im}\sum_{k=1}^{2}\int_{\Gamma_0^d}(T_{x_s}^{e_2}D(x_s,z))^T e_k\cdot v_k(z,x_s)ds(x_s)\\
&= \int_{\Gamma_D}ds(y)\sum_{k=1}^{2}\int_{\Gamma_0^d}(T_{x_s}^{e_2}D(x_s,z))^T e_k\cdot\Big((T_y^{\nu}J_d^T(z,y))^T\overline{u_k^s(y,x_s)}\\
&\quad - J_d(z,y)\overline{(T_y^{\nu}u_k^s(y,x_s))}\Big)\\
&= \mathrm{Im}\int_{\Gamma_D}ds(y)\sum_{k=1}^{2}\mathbf{tr}\Big((T_y^{\nu}J_d^T(z,y))^T\int_{\Gamma_0^d}\overline{u_k^s(y,x_s)}e_k^T T_{x_s}^{e_2}D(x_s,z)\\
&\quad - J_d(z,y)\int_{\Gamma_0^d}\overline{(T_y^{\nu}u_k^s(y,x_s))}e_k^T T_{x_s}^{e_2}D(x_s,z)\Big)\\
&= \mathrm{Im}\int_{\Gamma_D}ds(y)\mathbf{tr}\Big((T_y^{\nu}J_d^T(z,y))^T\sum_{k=1}^{2}W_k(y,z)\\
&\quad - J_d(z,y)(T_y^{\nu}\sum_{k=1}^{2}W_k(y,z))\Big)\\
&= \mathrm{Im}\int_{\Gamma_D}\mathbf{tr}\Big((T_y^{\nu}J_d^T(z,y))^T W(y,z) - J_d(z,y)(T_y^{\nu}W(y,z))\Big)ds(y)
\end{aligned}$$

where

$$W(y,z) = \sum_{k=1}^{2}W_k(y,z) \tag{6.27}$$

$$W_k(y,z) = \int_{\Gamma_0^d}\overline{u_k^s(y,x_s)}e_k^T(T_{x_s}^{e_2}D(x_s,z))ds(x_s) \tag{6.28}$$

Therefore, $\overline{W_k(y,z)}$ can be viewed as the weighted superposition of $u_k^s(y,x_s)$. Then $\overline{W_k(y,z)}$ satisfies elastic equation

$$\Delta_e^y\overline{W_k(y,z)} + \omega^2\overline{W_k(y,z)} = 0 \tag{6.29}$$

On the boundary of the obstacle Γ_D, we have

$$\begin{aligned}
\overline{W(y,z)} &= \sum_{k=1}^{2}\int_{\Gamma_0^d}u_k^s(y,x_s)e_k^T T_{x_s}^{e_2}\overline{D(x_s,z)}ds(x_s)\\
&= \sum_{k=1}^{2}\int_{\Gamma_0^d}-N(y,x_s)e_k e_k^T T_{x_s}^{e_2}\overline{D(x_s,z)}ds(x_s)\\
&= -\int_{\Gamma_0^d}N(y,x_s)T_{x_s}^{e_2}\overline{D(x_s,z)}ds(x_s)\\
&= -\overline{J_d^T(z,y)}
\end{aligned}$$

Moreover, $T_y^{e_2}\overline{W_k(y,z)} = 0$ on Γ_0 since $T_y^{e_2}u_k^s(y,x_s) = 0$ on Γ_0. Let $W_d(y,z) = \overline{W(y,z)} - \Psi(y,z)$ and it follows that $W_d(y,z)$ is the scattering solution of the problem

$$\Delta_e W_d(y,z) + \omega^2 W_d(y,z) = 0 \quad \text{in } \mathbb{R}_+^2\backslash\bar{D} \tag{6.30}$$

$$W_d(y,z) = \overline{F^T(z,y)} - \overline{J_d^T(z,y)} \quad \text{on } \Gamma_D \tag{6.31}$$

$$\sigma_y(W_d(y,z)) \cdot e_2 = 0 \quad \text{on } \Gamma_0 \tag{6.32}$$

By theorem 3.1 and Corollaries 5.1-5.2 we have

$$\begin{aligned} \|\sigma(W_d(\cdot,z))\cdot \nu\|_{H^{1/2}(\Gamma_D)} &\le \|F^T(z,\cdot) - J_d^T(z,\cdot)\|_{H^{1/2}(\Gamma_D)} \\ &\le C(1+k_s d_D)(\epsilon_1(k_s h) + \epsilon_2(k_s h, h/d)) \end{aligned} \tag{6.33}$$

$$\hat{I}_d(z) = \operatorname{Im} \mathbf{tr} \int_{\Gamma_D} (T_y^\nu J_d^T(z,y))^T \overline{\Psi(y,z)} - J_d(z,y)(T_y^\nu \overline{\Psi(y,z)}) ds(y) + R_{\hat{I}}(z) \tag{6.34}$$

where

$$R_{\hat{I}}(z) = \operatorname{Im} \mathbf{tr} \int_{\Gamma_D} (T_y^\nu J_d^T(z,y))^T W_d(y,z) - J_d(z,y)(T_y^\nu W_d(y,z)) ds(y) \tag{6.35}$$

By (6.33) and Corollaries 5.1-5.2 it is easy to see that

$$|R_{\hat{I}}(z)| \le C(1+k_s d_D)^2(\epsilon_1(k_s h) + \epsilon_2(k_s h, h/d)) \tag{6.36}$$

Finally, by (6.34) and $\Psi(y,z) = -\overline{F^T(z,y)}$

$$\begin{aligned} \hat{I}_d(z) &= \operatorname{Im} \mathbf{tr} \int_{\Gamma_D} (T_y^\nu (F^T(z,y))^T \overline{\Psi(y,z)} - F(z,y)(T_y^\nu \overline{\Psi(y,z)}) ds(y) + w_{\hat{I}}(z) \\ &= -\operatorname{Im} \mathbf{tr} \int_{\Gamma_D} (T_y^\nu (F^T(z,y))^T F^T(z,y) + F(z,y)(T_y^\nu \overline{\Psi(y,z)}) ds(y) + w_{\hat{I}}(z) \\ &= \operatorname{Im} \mathbf{tr} \int_{\Gamma_D} (T_y^\nu (\overline{F^T(z,y)} + \Psi(y,z))^T \overline{F^T(z,y)} ds(y) + w_{\hat{I}}(z) \end{aligned}$$

where

$$w_{\hat{I}}(z) = \operatorname{Im} \mathbf{tr} \int_{\Gamma_D} (T_y^\nu (J_d^T(z,y) - F^T(z,y))^T \overline{\Psi(y,z)} - (J_d(z,y) - F(z,y))(T_y^\nu \overline{\Psi(y,z)}) ds(y)$$

By Corollaries 5.1-5.2 we have

$$|w_{\hat{I}}(z)| \le C(1+k_s d_D)^2(\epsilon_1(k_s h) + \epsilon_2(k_s h, h/d)) \tag{6.37}$$

□

Now, by the theorem 6.1,6.2, we have more simpler theorem:

Theorem 6.3 *For any $z \in \Omega$, let $\Phi(y,z) \in \mathbb{C}^{2\times 2}$ be the radiation solution of the problem:*

$$\begin{aligned} &\Delta_e \Phi(y,z) + \omega^2 \Phi = 0 \qquad \text{in } \mathbb{R}^2 \backslash \bar{D} \\ &\Phi(y,z) = -\overline{F^T(z,y)} \qquad \text{on } \Gamma_D \end{aligned}$$

Then, we have

$$\hat{I}_d(z) = \operatorname{Im} \mathbf{tr} \int_{\Gamma_D} (T_y^\nu (\overline{F^T(z,y)} + \Phi(y,z))^T \overline{F^T(z,y)} ds(y) + W_{\hat{I}}(z) \tag{6.38}$$

where $|W_{\hat{I}}(z)| \le C(1+k_s d_D)^4(\epsilon_1(k_s h) + \epsilon_2(k_s h, h/d))$ uniformly for z in Ω, $m(h/d) > (1+\kappa)^2/4$, $M(h/d) < \kappa^2/4$.

By (5.24)-(5.25) we know that for any fixed $z \in \Omega$, $\overline{F^T(z,\cdot)}$ satisfies the Elastic wave equation. Thus $\Phi(y,z)$ can be viewed as the scattering solution of the Elastic equation with the incident wave $\overline{F^T(z,\cdot)}$. By lemma 5.8 we konw that $\overline{F^T(z,\cdot)}$ decays as $|y-z|$ becomes large. Therefore the imaging function $\hat{I}_d(z)$ becomes small when z moves away from the boundary Γ_D outside the scatterer D if $k_s h \gg 1$ and $d \gg h$.

To understand the behavior of the imaging fuction when z is close to the boundary of the scatterer, we extend the concept of the scattering coefficient fo incident plane waves[14].

Definition 6.1 *For any unit vector $d \in \mathbb{R}^2$, let $u_p^i = de^{\mathbf{i}k_p x\cdot d}$ or $u_s^i = d^\perp e^{\mathbf{i}k_s x\cdot d}$ be the incident wave and $u_\alpha^s = u_\alpha^s(x;d)$ be the radiation solution of the Navier equation:*

$$u_\alpha^s + \omega^2 u_\alpha^s = 0 \quad in \quad \mathbb{R}^2 \backslash \bar{D} \tag{6.39}$$

$$u_\alpha^s = -u_\alpha^i \quad on \quad \partial D \tag{6.40}$$

The scattering coecient R(x;d) for $x \in \partial D$ is defined by the relation

$$\sigma(u_\alpha^s + u_\alpha^i)\cdot\nu = \mathbf{i}k_\alpha R_\alpha(x;d)e^{\mathbf{i}k_\alpha x\cdot d} \quad on \quad \partial D$$

where $\alpha = p, s$.

In the case of high frequency approximation, the scattering coecient can be approximated by

$$R_\alpha(x;d) = \begin{cases} RF_\alpha(d;\nu(x)) & \text{if } \ x \in \partial D_d^- = \{x \in \partial D, \nu(x)\cdot d < 0\}, \\ 0 & \text{if } \ x \in \partial D_d^- = \{x \in \partial D, \nu(x)\cdot d \geq 0\}. \end{cases}$$

where

$$\begin{aligned}
&RF_p(d;\nu) = (\lambda\nu + 2\mu(d,\nu)d) + A_1/A_0(\lambda\nu + 2\mu(d_1,\nu)d_1) \\
&+\kappa A_2/A_0\mu((d_2,\nu)d_2^\perp + (d_2^\perp,\nu)d_2) \\
&d_1 = d - 2\alpha\nu \\
&d_2 = \kappa d - \beta\nu \\
&A_0 = \kappa(d,\nu)^2 - \kappa(d,\nu^\perp)^2 - \beta(d,\nu) \\
&A_1 = \kappa - \beta(d,\nu) \\
&A_2 = -2(d,\nu)(d,\nu^\perp) \\
&\alpha = (d,\nu), \beta = \kappa\alpha - \sqrt{\kappa^2\alpha^2 - \kappa^2 + 1}
\end{aligned}$$

and

$$\begin{aligned}
&RF_s(d;\nu) = \mu((d,\nu)d^\perp + (d^\perp,\nu)d) + 1/\kappa A_1/A_0(\lambda\nu + 2\mu(d_1,\nu)d_1) \\
&+A_2/A_0\mu((d_2,\nu)d_2^\perp + (d_2^\perp,\nu)d_2) \\
&d_1 = 1/\kappa d_0 - \gamma\nu \\
&d_2 = d_0 - 2\alpha\nu \\
&A_0 = 1/\kappa(d,\nu)^2 - 1/\kappa(d,\nu^\perp)^2 - \gamma(d,\nu) \\
&A_1 = 2(d,\nu)(d,\nu^\perp) \\
&A_2 = 1/\kappa - \gamma(d,\nu) \\
&\alpha = (d,\nu), \gamma = 1/\kappa\alpha - \sqrt{(1/\kappa)^2\alpha^2 - (1/\kappa)^2 + 1}
\end{aligned}$$

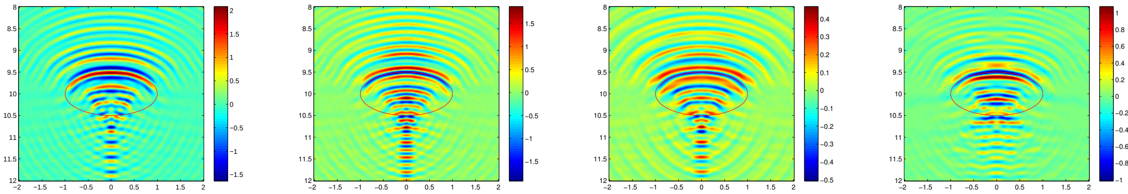


Figure 3. Example 1: From left to right: imaging results of a Neumann, a Robin bounday with impedance $\eta(x) = 1$, and a penetrable obstacle with diffractive index $n(x) = 0.25$

figure_1

7. Extensions

8. Numerical experiments

In this section we present several numerical examples to show the effectiveness of our RTM method. To synthesize the scattering data we compute the solution $u^s(x_r; x_s)$ of the scattering problem by representing the ansatz solution as the double layer potential with the Green function $N(x; y)$ as the kernel and discretizing the integral equation by standard *Nyström* methods [17]. The boundary integral equations on Γ_D are solved on a uniform mesh over the boundary with ten points per probe wavelength. The sources and receivers are both placed on the surface Γ_d^0 with equal-distribution, where d is the aperture. In all our numerical examples we choose h = 10, d = 50 and *Lamé* constant $\lambda = 1/2$, $\mu = 1/4$. The boundaries of the obstacles used in our numerical experiments are parameterized as follows,

$$
\begin{aligned}
&\text{Circle:} && x_1 = \rho\cos(\theta), \quad x_2 = \rho\sin(\theta),\\
&\text{Kite:} && x_1 = \cos(\theta) + 0.65\cos(2\theta) - 0.65, \quad x_2 = 1.5\sin(\theta),\\
&p\text{-leaf:} && r(\theta) = 1 + 0.2\cos(p\theta),\\
&\text{peanut:} && x1 = \cos\theta + 0 : 2\cos 3\theta; x2 = \sin\theta + 0 : 2\sin 3\theta,\\
&\text{square:} && x1 = \cos 3\theta + \cos\theta; x2 = \sin 3\theta + \sin\theta.
\end{aligned}
$$

where $\theta \in [0, 2\pi]$.

Example 1. We consider imaging of a Dirichlet, a Neumann, a Robin bounday, and a penetrable obstacle. The imaging domain $\omega = (2; 2) \times (8; 12)$ with the sampling grid 201×201 and $N_s = N_r = 401$. The angular frequency is $\omega = 2\pi$.

The imaging results are shown in Figure 7. It demonstrates clearly that our RTM algorithm can effectively image the upper boundary illuminated by the sources and receivers distributed along the boundary Γ_0 for non-penetrable obstacles. The imaging values decrease on the shadow part of the obstacles and at the points away from the boundary of the obstacle.

Example 2. We consider the imaging of clamped obstacles with different shapes including circle, peanut, p-leaf and rounded square. The imaging domain is $\omega =$

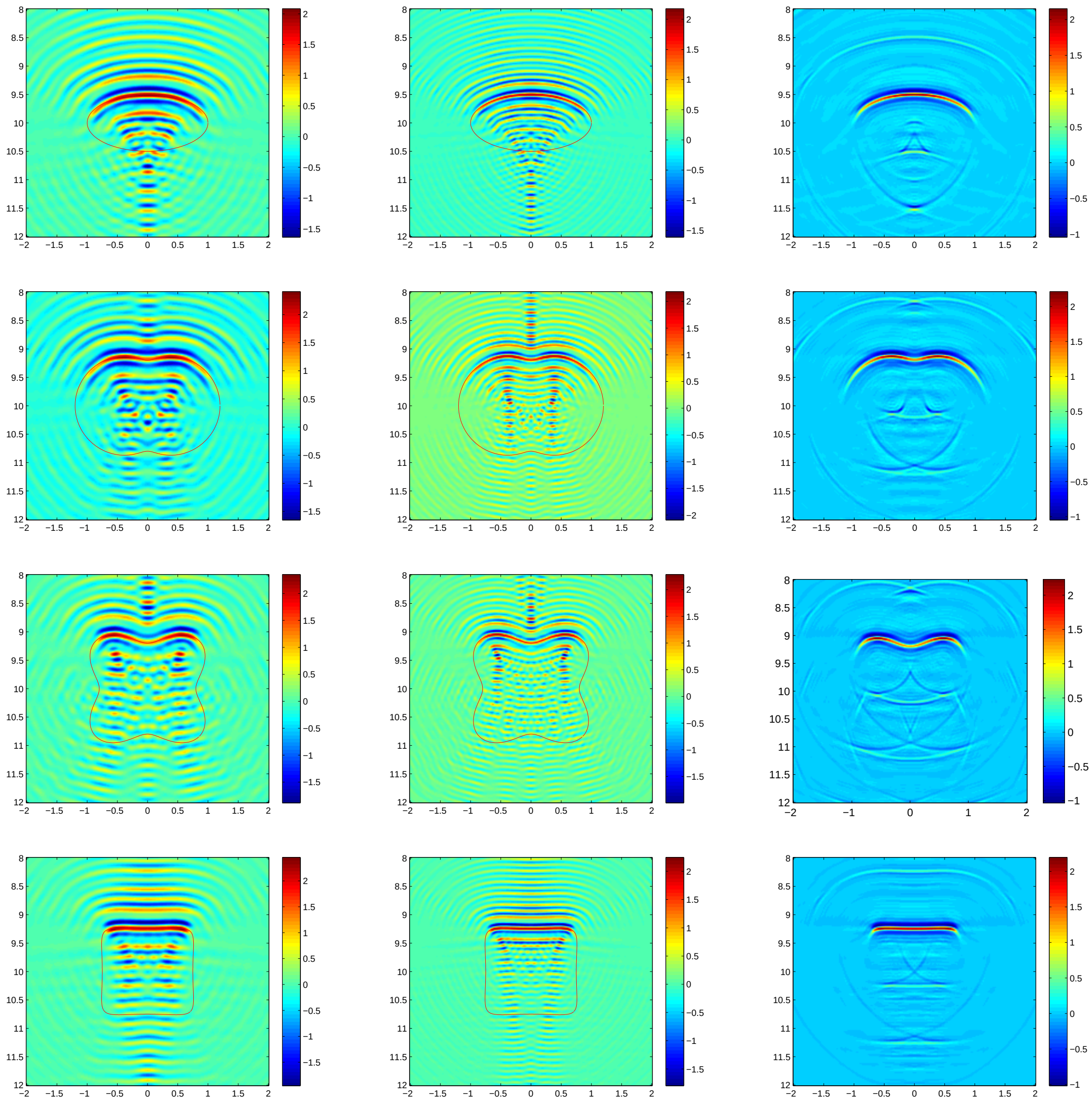


Figure 4. Example 2: Imaging results of clamped obstacles with different shapes from top to below. The left row is imaged with single frequency data where $\omega = 3\pi$, The middle row is imaged with single frequency data where $\omega = 5\pi$ and The left row is imaged with multi frequency data

figure_2

$(2; 2) \times (8; 12)$ with the sampling grid 201×201 and $N_s = N_r = 401$. The angular frequency is $\omega = 3\pi, 4\pi$ for the sigle frequency and $\omega = \pi \times [2 : 0.5 : 8]$ for the test of multiple frequencies.

Example 3 We consider the imaging of two sound soft obstacles. The first model consists of two circles along horizontal direction and the second one is a circle and a peanut along the vertical direction. The angular frequency is $\omega = 3\pi$ for the test of the single frequency and $\omega = \pi \times [2 : 0.5 : 8]$ for the test of multiple frequencies. Figure 5 figure_3 shows the imaging result of the first model. The imaging domain is $[4, 4] \times [8, 12]$ with mesh size 401×201 and $N_s = N_r = 301$. Figure shows the imaging result of the second

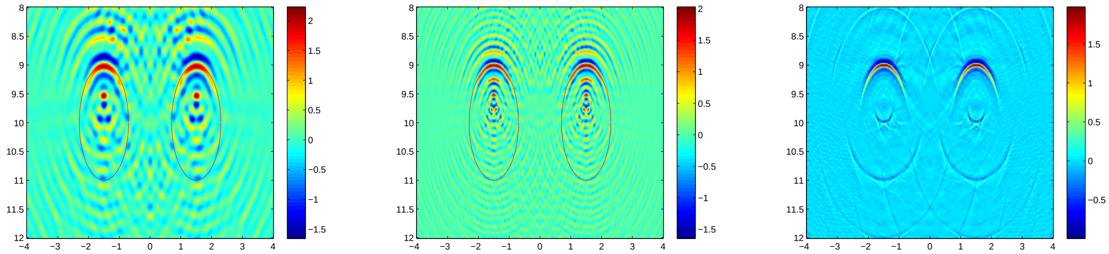


Figure 5. Example 3:From left to right, the imaging result with single frequency data where $\omega = 2\pi, 4\pi$, the imaging result with multiple frequency data.

figure_3

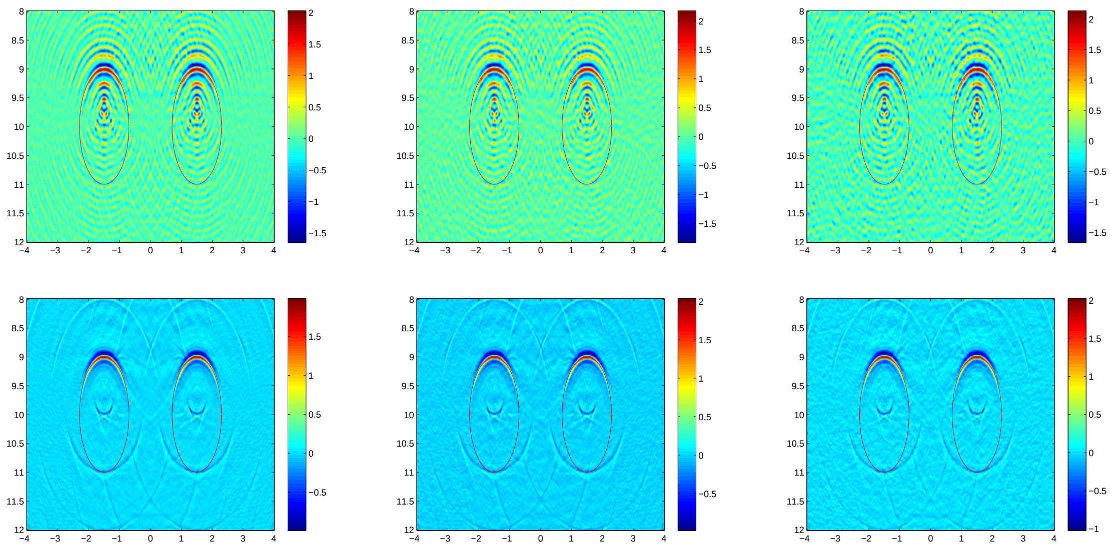


Figure 6. Example 4: Imaging results of a clamped obstacle with noise levels $\mu = 0.2; 0.3; 0.4$ (from left to right). The top row is imaged with single frequency data where $\omega = 4\pi$, and the bottom row is imaged with multi-frequency data.

figure_4

model.

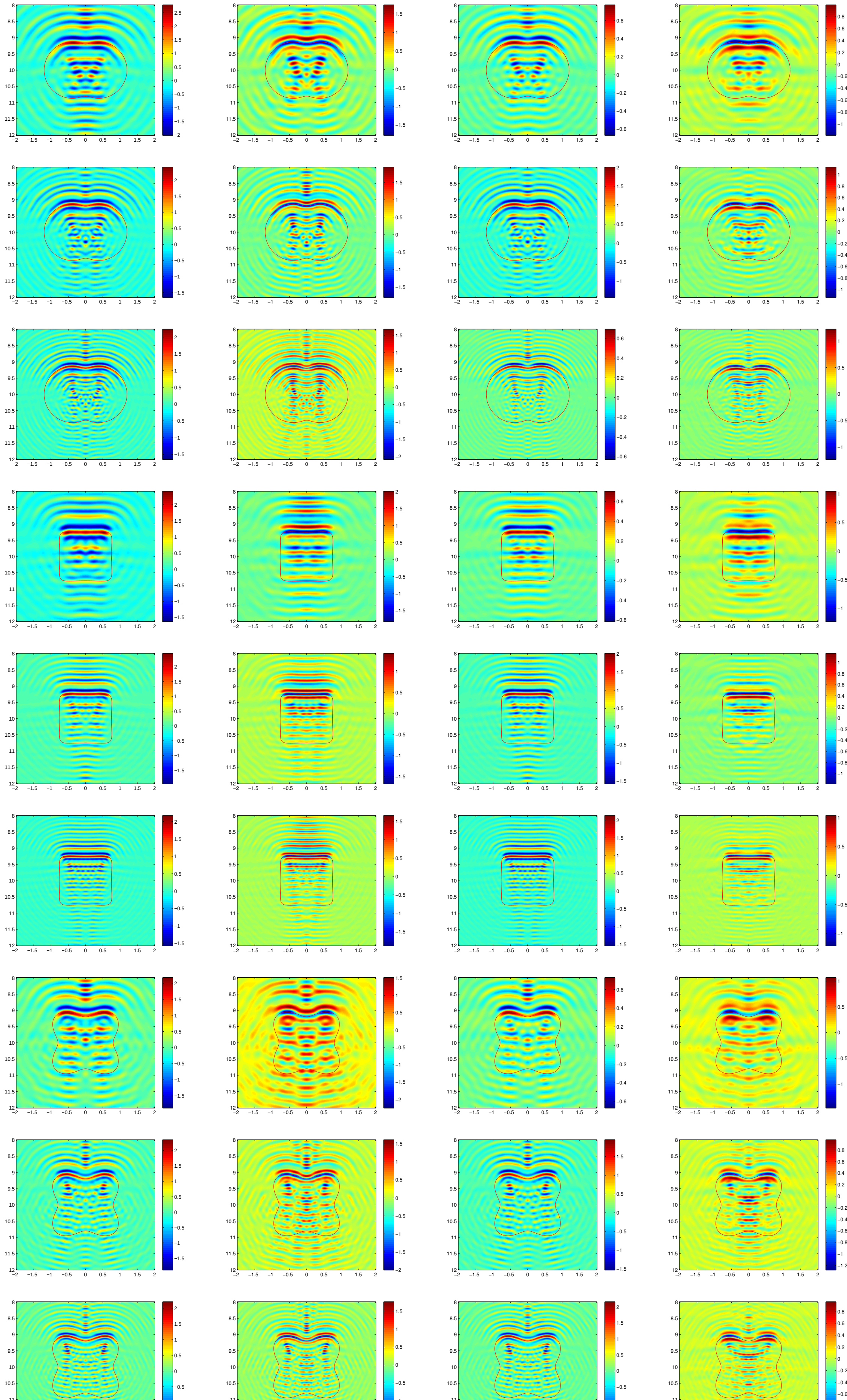
Example 4 In this example we consider the stability of our half space RTM imaging function with respect to the complex additive Gaussian random noise. We introduce the additive Gaussian noise as follows

$$u_{\text{noise}} = u_s + \nu_{\text{noise}}$$

where u_s is the synthesized data and ν_{noise} is the Gaussian noise with mean zero and standard deviation μ times the maximum of the data $|u_s|$, i.e. $\nu_{\text{noise}} = \frac{\mu \max |u_s|}{\sqrt{2}}(\varepsilon_1 + \mathbf{i}\varepsilon_2)$ and $\varepsilon_i \sim \mathcal{N}(0, 1)$.

figure_4

Figure 6 shows the imaging results using single frequency data added with additive Gaussian noise. The imaging quality can be improved by using multi-frequency data. as illustrated



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