1. Estimate of Dirichlet Green Tensor

We need the following slight generalization of Van der Corput lemma for the oscillatory integral [3, P.152].

Lemma 1.1 *Let $-\infty < a < b < \infty$, and u is a C^k function u in (a, b).*
1. If $|u'(t)| \geq 1$ for $t \in (a, b)$ and u' is monotone in (a,b), then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 3\lambda^{-1}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

2. For all $k \geq 2$, if $|u^{(k)}(t)| \geq 1$ for $t \in (a, b)$, then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 12k\lambda^{-1/k}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

Proof. The assertion can be proved by extending the Van der Corptut lemma in [3]. Here we omit the details. □

We recall following lemma, see e.g. [6]:

Lemma 1.2 *Let $F(\rho, a) = \int_0^a t^{\alpha-1}f(t)e^{-\mathbf{i}\rho t}dt$ where $0 < a \leq +\infty$, $0 < \alpha < 1$, $\rho > 0$ and $t^{\alpha-1}f \in L^1(0, a)$, then we have*

$$|F(\rho, a)| \leq C(\frac{1}{\rho^\alpha}f(0) + \frac{1}{\rho}(a^{\alpha-1}f(a) + |t^{\alpha-1}f|_{L^1(0,a)}) \tag{1.1}$$

Proof. Put

$$g_0(t) = t^{\alpha-1}e^{-\mathbf{i}\rho t} \tag{1.2}$$

and define

$$g_1(t) = -\int_t^{t-\mathbf{i}\infty} x^{\alpha-1}e^{-\mathbf{i}\rho x}dx \tag{1.3}$$

where the path of integration is the vertical line $x = t - \mathbf{i}y, y \geq 0$. It is easy to show that $g_1(t)' = g_0(t)$. Substituting $x = t - \mathbf{i}y$ into $g_1(t)$, we have

$$g_1(t) = \mathbf{i}\int_0^\infty (t - \mathbf{i}y)^{\alpha-1}e^{-\mathbf{i}\rho t}e^{-\rho y}dy \tag{1.4}$$

Upon integration by parts, we have

$$\begin{aligned}
F(\rho, a) &= \int_0^a f(t)dg_1(t) \\
&= e^{-\mathbf{i}\frac{\alpha\pi}{2}}f(0)\Gamma(\alpha)\frac{1}{\rho^\alpha} + f(a)g_1(a) - \int_0^a f'(t)g_1(t)dt \\
&= e^{-\mathbf{i}\frac{\alpha\pi}{2}}f(0)\Gamma(\alpha)\frac{1}{\rho^\alpha} - \mathbf{i}\int_0^\infty e^{-\rho y}dy\int_0^a f'(t)(t - \mathbf{i}y)^{\alpha-1}e^{-\mathbf{i}\rho t}dt
\end{aligned}$$

Let

$$h(y) = \int_0^a f'(t)(t - \mathbf{i}y)^{\alpha-1}e^{-\mathbf{i}\rho t}dt$$

and observe that

$$|h(y)| \le \int_0^a |f'(t)|(t^2+y^2)^{\frac{\alpha-1}{2}}dt$$

□

Lemma 1.3 *Let* $F(\rho,a)=\int_0^a t^{-1/2}f(t)e^{-\mathbf{i}\rho t}dt$ *where* $0<a\le+\infty$ *and* $\rho>0$, *then we have*

$$|F(\rho,a)-e^{-\mathbf{i}\frac{\pi}{4}}f(0)\Gamma(1/2)\frac{1}{\rho^{1/2}}| \tag{1.5}$$

$$\le C(\int_0^\infty e^{-\rho y}dy\int_0^a |f'(t)|(t^2+y^2)^{-\frac{1}{4}}dt+\frac{1}{\rho}a^{-1/2}f(a)) \tag{1.6}$$

Proof. Put

$$g_0(t)=t^{-1/2}e^{-\mathbf{i}\rho t} \tag{1.7}$$

and define

$$g_1(t)=-\int_t^{t-\mathbf{i}\infty} x^{-1/2}e^{-\mathbf{i}\rho x}dx \tag{1.8}$$

where the path of integration is the vertical line $x=t-\mathbf{i}y, y\ge 0$. It is easy to show that $g_1(t)=g_0(t)$. Substituting $x=t-\mathbf{i}y$ into $g_1(t)$, we have

$$g_1(t)=\mathbf{i}\int_0^\infty (t-\mathbf{i}y)^{-1/2}e^{-\mathbf{i}\rho t}e^{-\rho y}dy \tag{1.9}$$

Upon integration by parts, we have

$$\begin{aligned} F(\rho,a) &= \int_0^a f(t)dg_1(t) \\ &= e^{-\mathbf{i}\frac{\pi}{4}}f(0)\Gamma(1/2)\frac{1}{\rho^{1/2}}+f(a)g_1(a)-\int_0^a f'(t)g_1(t)dt \\ &= e^{-\mathbf{i}\frac{\pi}{4}}f(0)\Gamma(1/2)\frac{1}{\rho^{1/2}}+\mathbf{i}f(a)\int_0^\infty (a-\mathbf{i}y)^{-1/2}e^{-\mathbf{i}\rho t}e^{-\rho y}dy \\ &\quad -\mathbf{i}\int_0^\infty e^{-\rho y}dy\int_0^a f'(t)(t-\mathbf{i}y)^{-1/2}e^{-\mathbf{i}\rho t}dt \end{aligned}$$

Let

$$h(y)=\int_0^a f'(t)(t-\mathbf{i}y)^{-1/2}e^{-\mathbf{i}\rho t}dt$$

and observe that

$$|h(y)|\le\int_0^a |f'(t)|(t^2+y^2)^{-\frac{1}{4}}dt$$

It is easy to see that

$$|g_1(a)|\le a^{-1/2}\int_0^\infty e^{-\rho y}dy\le C\frac{1}{\rho}$$

□

Lemma 1.4 *Assume that* $0 < \kappa := \sin\phi_\kappa < 1, 0 < \phi_\kappa < \pi/2,\ 0 \le \phi \le \pi/2$ *and* $-\pi/2 < t_1 < t_2 < \pi/2$ *satisfy that* $\kappa^2 = \sin^2(\phi + t_1) = \sin^2(\phi + t_2)$. *Let* $f(\theta)$*:*

$$f(t,\phi) := F(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2}) \tag{1.10}$$

be a function in $C^\infty(([-\pi/2, \pi/2]\backslash\{t_1, t_2\}) \times [0, \pi/2])$. *Moreover, there exits* $\epsilon > 0$ *such that* $f(\theta)$ *can be represented as*

$$f(t,\phi) = g_1(t,\phi) + g_2(t,\phi)(\kappa^2 - \sin^2(t+\phi))^{1/2})^{N/2} \tag{1.11}$$

where $g_1, g_2 \in C^\infty((\bigcup_{i=1,2}(t_i - \epsilon, t_i + \epsilon)) \times [0, \pi/2]))$ *and* $N = \pm 1$. *Then for any* $\rho \ge 1$, *we have*

$$\begin{aligned} &\Big|I(\rho,\phi) := \int_{-\pi/2}^{\pi/2} f(\theta) e^{\mathbf{i}\rho\cos\theta} d\theta - \frac{N+1}{2}\Big(\frac{2\pi}{\rho}\Big)^{1/2} f(0) e^{\mathbf{i}\rho - \mathbf{i}\pi/4}\Big| \\ &\le C\frac{1}{\rho^{(2+N)/4}} \end{aligned} \tag{1.12}$$

Proof. The proof will be split into two parts about whether ϕ equal to ϕ_κ.

If $\phi \ne \phi_\kappa$, there exists $0 < \delta < \pi/4$ such that

$$|(\kappa^2 - \sin^2(t+\phi))^{1/2}| > \frac{1}{2}|(\kappa^2 - \sin^2\phi)^{1/2}| \tag{1.13}$$

for any $t \in (-\delta, \delta)$. Let $\chi_\delta \in C_0^\infty(-\pi/2, \pi/2)$ be the cut-off function with that $0 \le \chi_\delta \le 1$, $\chi_\delta = 1$ in $(-\delta/2, \delta/2)$ and $\chi_\delta = 0$ in $(-\pi/2, \pi/2)\backslash(-\delta, \delta)$. Then we can divide I into two parts such that

$$\begin{aligned} I &= \int_{-\delta}^{\delta} f(t)\chi_\delta(t) e^{\mathbf{i}\rho\cos t} dt + \int_{-\pi/2}^{\pi/2} f(t)(1-\chi_\delta(t)) e^{\mathbf{i}\rho\cos t} dt \\ &=: I_1 + I_2 \end{aligned}$$

Subtitating $t(s) = 2\arcsin s/2$ for t in I_1, we can obtain

$$I_1 = \int_{\mathbb{R}} f(t(s))\chi_\delta(t(s)) \frac{1}{\sqrt{1 - s^2/4}} e^{\mathbf{i}\rho} e^{-\mathbf{i}\rho s^2/2} ds \tag{1.14}$$

$$= \int_{\mathbb{R}} h_\delta(s) e^{\mathbf{i}\rho} e^{-\mathbf{i}\rho s^2/2} ds \tag{1.15}$$

It is easy to see that $h_\delta(s) \in C_0^4(\mathbb{R})$. By the lemma of the stationary phase for quadratic term in [2], we have

$$I_1 = e^{\mathbf{i}\rho} \int_{\mathbb{R}} h_\delta(s) e^{-\mathbf{i}\frac{\rho}{2}s^2} ds = e^{\mathbf{i}\rho} \int_{\mathbb{R}} \widehat{h_\delta}(y)\alpha(-y) dy \tag{1.16}$$

where

$$\alpha(y) = (\frac{1}{2\pi\rho})^{1/2} e^{-\mathbf{i}\pi/4} e^{\frac{\mathbf{i}}{2\rho}y^2} \tag{1.17}$$

$$= (\frac{1}{2\pi\rho})^{1/2} e^{-\mathbf{i}\pi/4}(1 + O(\frac{y^2}{\rho})) \tag{1.18}$$

Consequently

$$I_1 = (\frac{1}{2\pi\rho})^{1/2} e^{\mathbf{i}\rho - \mathbf{i}\pi/4} \int_{\mathbb{R}} \widehat{h_\delta}(y)(1 + \frac{1}{\rho}O(y^2)) dy \tag{1.19}$$

Moreover, $\int_{\mathbb{R}} \widehat{h_\delta}(y)dy = 2\pi h_\delta(0)$ and $|\int_{\mathbb{R}} \widehat{h_\delta}(y)y^2dy| < C$ since $\widehat{h_\delta}(y) = O(1/y^4)$. Now, it turns to estimate I_2.

When $N = 1$, using integration by parts, we have

$$|I_2| = \left| \int_{(-\frac{\pi}{2},\frac{\pi}{2})\setminus(-\frac{\delta}{2},\frac{\delta}{2})} f(t)(1-\chi_\delta(t))/\sin t \;\; de^{\mathbf{i}\rho\cos t} \right| \tag{1.20}$$

$$\tag{1.21}$$

$$\leq C\frac{1}{\rho} + \left| \int_{(-\frac{\pi}{2},\frac{\pi}{2})\setminus(-\frac{\delta}{2},\frac{\delta}{2})} (f(t)(1-\chi_\delta(t))/\sin t)' \; e^{\mathbf{i}\rho\cos t}dt \right| \tag{1.22}$$

$$\leq C\frac{1}{\rho} \tag{1.23}$$

From above analysis, we obtain

$$\left| I(\rho,\phi) - \left(\frac{2\pi}{\rho}\right)^{1/2} f(0)e^{\mathbf{i}\rho - \mathbf{i}\pi/4} \right| \leq C(\phi)\frac{1}{\rho} \tag{1.24}$$

When $N = -1$, we can not use integration by parts again since $f'(\theta)$ is not integrable. However, for any $0 < \lambda_1 < 1$ and $1 < \lambda_2 < 1/\kappa$, there exists $0 < \sigma < \epsilon$, such that $\chi := ((t_1-\sigma, t_1+\sigma) \cup (t_2-\sigma, t_2+\sigma)) \cap (-\delta,\delta) = 0$, dependent on λ_1, λ_2 and

$$\lambda_1\kappa < |\sin(t+\phi)| < \lambda_2\kappa. \tag{1.25}$$

for any $t \in \chi$.

We only analysis the integral on $\chi_1 = (t_1-\sigma, t_1+\sigma) \cap [-\pi/2, \pi/2]$ here, which denoted by I_{χ_1}, the proof of I_{χ_2} is similar. It is easy to see that $\sin(t+\phi)$ is monotonic in χ_1. Without loss of generality, we assume that $\sin(t_1-\sigma+\phi) < \kappa < \sin(t_1+\sigma+\phi)$. Let $\sin(t+\phi) = \kappa\sin\theta$ and the implicit mapping from θ to t is denoted by $t(\theta)$ while the inverse mapping by $\theta(t)$, taking the interval χ_1 onto $L_\theta : \theta_1 \to \pi/2 \to \pi/2 - \mathbf{i}\theta_2$ where $\sin(t_1-\sigma+\phi) = \kappa\sin\theta_1, \sin(t_1+\sigma+\phi) = \kappa\sin(\pi/2 - \mathbf{i}\theta_2)$. By substituting $t(\theta)$ into I_{χ_1}, we have

$$I_{\chi_1} = \int_{t_1-\sigma}^{t_1+\sigma} \frac{f(t)(\kappa^2 - \sin^2(t+\phi))^{1/2}}{(\kappa^2 - \sin^2(t+\phi))^{1/2}} e^{\mathbf{i}\rho\cos t} \tag{1.26}$$

$$= \int_{L_\theta} \frac{\kappa f(t(\theta))\cos\theta}{(1-\kappa^2\sin^2\theta)^{1/2}} e^{\mathbf{i}\rho(\cos(t(\theta)))}d\theta \tag{1.27}$$

$$= \int_{L_\theta} \frac{\kappa g_1(t(\theta))\cos\theta + g_2(t(\theta))}{(1-\kappa^2\sin^2\theta)^{1/2}} e^{\mathbf{i}\rho(\cos(t(\theta)))}d\theta \tag{1.28}$$

$$:= \int_{L_\theta} \frac{h(\theta)}{(1-\kappa^2\sin^2\theta)^{1/2}} e^{\mathbf{i}\rho(\cos(t(\theta)))}d\theta \tag{1.29}$$

Observe that $h(\theta)$ and $\partial h/\partial\theta$ are integrable on the path L_θ by (1.11). A simple computation show that

$$\frac{dt(\theta)}{d\theta} = \frac{\kappa\cos\theta}{\cos(t+\phi)} \qquad \frac{d^2t(\theta)}{dt^2} = \frac{\kappa^2\cos^2\theta\sin(t+\phi) - \kappa\sin\theta\cos^2(t+\phi)}{\cos^3(t+\phi)}$$

Then we can obtain

$$\frac{d\cos t}{d\theta} = \frac{-\kappa\sin t\cos\theta}{\cos(t+\phi)}$$

$$\begin{aligned}\frac{d^2\cos t}{d\theta^2} &= \frac{d^2\cos t}{dt^2}(\frac{dt}{d\theta})^2+\frac{d\cos t}{dt}\frac{d^2 t}{d\theta^2}\\ &= \frac{-\kappa^2\cos^2\theta\cos t}{\cos^2(t+\phi)}+\frac{\kappa\sin\theta\cos^2(t+\phi)\sin t-\kappa^2\cos^2\theta\sin(t+\phi)\sin t}{\cos^3(t+\phi)}\\ &= \frac{-\kappa^2\cos^2\theta\cos\phi+\kappa\sin\theta\cos^2(t+\phi)\sin t}{\cos^3(t+\phi)}\\ &= \frac{(\sin^2(t+\phi)-\kappa^2)\cos\phi+\cos^2(t+\phi)\sin(t+\phi)\sin t}{\cos^3(t+\phi)}\end{aligned}$$

Since $|\sin t|>|\sin\delta|$ and $1-\lambda_2^2\kappa^2<\cos^2(t+\phi)<1-\lambda_1^2\kappa^2$ for $t\in\chi_1$, it follows that $\theta=\pi/2$ is the only stationary point of $\cos(t(\theta))$ and

$$\left|\frac{d^2\cos t}{d\theta^2}(\pi/2)\right|=\frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}|\sin t|>\frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}\sin\delta \tag{1.30}$$

Therefore, we can choose appropriate λ_1,λ_2 such that

$$|\frac{d^2\cos t}{d\theta^2}|>\frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}\sin\delta \tag{1.31}$$

for any $\theta\in\theta(\chi_1)$. According to lemma (6.1), we obtian $|I_{\chi_1}|\le C\frac{1}{\rho^{1/2}}$, and also $|I_{\chi_2}|\le C\frac{1}{\rho^{1/2}}$. Using integration by parts, we get

$$\left|\int_{[-\pi/2,\pi/2]\setminus((-\delta,\delta)\cup\chi)} f(t)(1-\chi_\delta(t))e^{\mathbf{i}\rho\cos t}dt\right|\le C\frac{1}{\rho}$$

Consequently, for $N=-1$ and $\phi\neq\phi_\kappa$, we get $|I(\rho,\phi)|\le\frac{1}{\rho^{1/2}}$.

We now turn to the case of $\phi=\phi_\kappa$. By (1.11), we can define χ_ϵ similarly and also decompose I into I_1 and I_2. Using the same agurement above, we can easily carry out that: for $N=1$, we have $|I_2|\le C\frac{1}{\rho}$; for $N=-1$, we have $|I_2|\le C\frac{1}{\rho^{1/2}}$. Finally, it remains to analysis I_1. By (1.11), we have

$$\begin{aligned}I_1 &= \int_{-\epsilon}^{\epsilon} g_1\chi_\epsilon+g_2\chi_\epsilon(\sin^2\phi_\kappa-\sin^2(t+\phi_\kappa))^{N/2}e^{\mathbf{i}\rho\cos t}dt\\ &= \int_{-\epsilon}^{\epsilon} g_1\chi_\epsilon+g_2\chi_\epsilon(-2(\sin\phi_\kappa+\sin(t+\phi_\kappa))\cos\frac{2\phi_\kappa+t}{2}\sin t/2)^{N/2}e^{\mathbf{i}\rho\cos t}dt\\ &= \int_{\mathbb{R}} g_1\chi_\epsilon+g_2\chi_\epsilon((\sin\phi_\kappa+\sin(t+\phi_\kappa))\cos\frac{2\phi_\kappa+t}{2})^{N/2}(-2\sin t/2)^{N/2}e^{\mathbf{i}\rho\cos t}dt\end{aligned}$$

Also, subtituting $t(s)=2\arcsin s/2$ for t in I_1, it follows that

$$\begin{aligned}I_1 &= \int_{\mathbb{R}} h_1(s)e^{-\mathbf{i}\rho\frac{s^2}{2}}+h_2(s)(-s)^{N/2}e^{-\mathbf{i}\rho\frac{s^2}{2}} &(1.32)\\ &= I_{11}+I_{12} &(1.33)\end{aligned}$$

where

$$\begin{aligned}h_1(s) &= g_1(t(s))\chi_\epsilon(t(s))\sqrt{1-s^2/4}\ e^{\mathbf{i}\rho}\\ h_2(s) &= g_2\chi_\epsilon((\sin\phi_\kappa+\sin(t+\phi_\kappa))\cos\frac{2\phi_\kappa+t}{2})^{N/2}_{t=t(s)}\sqrt{1-s^2/4}\ e^{\mathbf{i}\rho}\end{aligned}$$

and $h_1(s), h_2(s) \in C_c^\infty(\mathbb{R})$. Using stationary phase lemma similarly, if $N = 1$,

$$I_{11} = \Big(\frac{2\pi}{\rho}\Big)^{1/2} g_1(0) e^{\mathbf{i}\rho - \mathbf{i}\pi/4} + O(\frac{1}{\rho}) \tag{1.34}$$

$$= \Big(\frac{2\pi}{\rho}\Big)^{1/2} f(0) e^{\mathbf{i}\rho - \mathbf{i}\pi/4} + O(\frac{1}{\rho}) \tag{1.35}$$

if $N = -1$, we get $|I_{11}| \leq C\frac{1}{\rho^{1/2}}$. For I_{12}, we have

$$I_{12} = \int_0^\infty (\mathbf{i}h_2(s) + h_2(-s)) s^{N/2}\ e^{-\mathbf{i}\rho s^2/2} ds \tag{1.36}$$

$$= \frac{1}{2}\int_0^\infty (\mathbf{i}h_2(\sqrt{s}) + h_2(-\sqrt{s})) s^{N/4-1/2} e^{-\mathbf{i}\rho s/2} ds \tag{1.37}$$

By lemma (1.2), we get $|I_{12}| \leq C\frac{1}{\rho^{(N+2)/4}}$. □

2. Some draft about Green Tensor Analysis

Let substitute $\xi = k\sin\theta$ into integral and shift the variable, we have

$$I(y) = \int_{\mathbb{R}} f(\xi) e^{\mathbf{i}\xi y_1 + \mu(\xi) y_2} d\xi = \int_{\mathbb{R}} f(\xi) e^{\mathbf{i}\xi(y_1 - z_1) + \mu(\xi)(y_2 - z_2)} e^{\mathbf{i}\xi z_1 + \mu(\xi) z_2} d\xi \tag{2.1}$$

$$= k\int_L f(k\sin\theta)\cos\theta e^{\mathbf{i}k|y-z|\cos(\theta-\eta)} e^{\mathbf{i}|z|\cos(\theta-\phi)} d\theta \tag{2.2}$$

$$= k\int_{L_\phi} f(k\sin(\theta+\phi))\cos(\theta+\phi) e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)} e^{\mathbf{i}|z|\cos\theta} d\theta \tag{2.3}$$

$$= k\int_L f(k\sin(\theta+\phi))\cos(\theta+\phi) e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)} e^{\mathbf{i}|z|\cos\theta} d\theta \tag{2.4}$$

where $y_1, y_2 > 0, \sin\phi = \frac{z_1}{|z|}, \cos\phi = \frac{z_2}{|z|}, 0 < \phi < \pi/2$ and $\sin\eta = \frac{y_1 - z_1}{|y-z|}, \cos\eta = \frac{y_2 - z_2}{|y-z|}, 0 < \eta < \pi$. It is easy to see that $\phi + \eta < \pi$. Roughly, using stationary phase lemma, we obtain:

$$I(y) = f(k\sin\phi) k\cos\phi e^{\mathbf{i}k|y-z|\cos(\phi-\eta)} (\frac{2\pi}{|z|})^{1/2} e^{\mathbf{i}|z| - \mathbf{i}\frac{\pi}{4}} (1 + O(\frac{1}{|z|})) \tag{2.5}$$

$$\cos(a + \mathbf{i}b) = \frac{e^b + e^{-b}}{2}\cos a + \mathbf{i}\frac{e^{-b} - e^b}{2}\sin a \tag{2.6}$$

$$\sin(a + \mathbf{i}b) = \frac{e^b + e^{-b}}{2}\sin a + \mathbf{i}\frac{e^b - e^{-b}}{2}\cos a \tag{2.7}$$

When $\theta \in (-a - \pi/2, -a - \pi/2 + \mathbf{i}\infty)$, let $\theta = -a - \pi/2\mathbf{i}t$, where $t > 0, 0 \leq a \leq \phi$, then

$$-\mathrm{Im}\left(|z|\cos\theta + |y-z|\cos(\theta + \phi - \eta)\right)$$

$$= |z|\sin(a + \pi/2) + |y-z|\sin(a + \pi/2 - \phi + \eta) \tag{2.8}$$

$$= |z|\cos a + |y-z|\cos(a - \phi + \eta) \tag{2.9}$$

$$= |z|\cos a + \cos a|y-z|(\cos\phi\cos\eta + \sin\phi\sin\eta) \tag{2.10}$$

$$+ \sin a|y-z|(\sin\phi\cos\eta - \cos\phi\sin\eta) \tag{2.11}$$

$$= |z|\cos a + \cos a((y_2 - z_2)\cos\phi + (y_1 - z_1)\sin\phi) \tag{2.12}$$
$$+\sin a((y_2 - z_2)\sin\phi - (y_1 - z_1)\cos\phi) \tag{2.13}$$
$$= y_1\sin(\phi - a) + y_2\cos(\phi - a) > 0 \tag{2.14}$$

Now, Using Cauchy Integral Theorem, we have

$$I(y) = k\int_L f(k\sin(\theta+\phi))\cos(\theta+\phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \tag{2.15}$$

Let $L_1 = (-\pi/2, -\pi/2 + \mathbf{i}\infty)$ and $\theta = -\pi/2 + \mathbf{i}t, t > 0$, then

$$I_1(y) = k\int_{L_1} f(k\sin(\theta+\phi))\cos(\theta+\phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \tag{2.16}$$
$$= \tag{2.17}$$

$$I(y) = f(k\sin\phi)k\cos\phi e^{\mathbf{i}k|y-z|\cos(\phi-\eta)}(\frac{2\pi}{|z|})^{1/2}e^{\mathbf{i}|z|-\mathbf{i}\frac{\pi}{4}} \tag{2.18}$$
$$+\frac{kz_2}{|z|}O(\left(\frac{1}{k|z|}\right)^{3/4} + \frac{1}{k|y|}) + \frac{kz_1}{|z|}O(\left(\frac{1}{k|z|}\right)^{5/4} + \left(\frac{1}{k|y|}\right)^2) \tag{2.19}$$

It is easy to see

$$\int_{-d}^{d}\frac{k}{(k|x-z|)^\alpha}\frac{1}{(k|x-y|)^\beta}dx_1 \le C(\frac{1}{(kz_2)^{\alpha+\beta-1}} + \frac{1}{(ky_2)^{\alpha+\beta-1}}) \tag{2.20}$$

where $z, y \in \mathbb{R}^2_+$, $x \in \Gamma_0$ and $\alpha + \beta > 0$.

$$e^{\mathbf{i}\mu y_2 + \mathbf{i}\xi(x_1-y_1)} = e^{\mathbf{i}\mu y_2 - \mathbf{i}y_2/\tan\phi} = e^{\mathbf{i}y_2(\mu - \xi/\tan\phi)} \tag{2.21}$$

Another method

$$\int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi)e^{\mathbf{i}k|x-y|\cos\theta} \tag{2.22}$$
$$= \int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi)e^{\mathbf{i}k|x-y|\cos(\theta+\psi-\psi)} \tag{2.23}$$
$$= \int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi)e^{\mathbf{i}ky_2\cos(\theta+\psi)+\mathbf{i}k|x_1-y_1|\sin(\theta+\psi)} \tag{2.24}$$
$$= \int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi) \tag{2.25}$$
$$e^{\mathbf{i}k(y_2-z_2)\cos(\theta+\psi)+\mathbf{i}k(|x_1-y_1|-|x_1-z_1|)\sin(\theta+\psi)+\mathbf{i}k|z|\cos(\theta+\psi-\phi)} \tag{2.26}$$

3. Finite Aperture Point Spread Function

If $x \in \Gamma_0$ and $z, y \in \mathbb{R}^2_+$, by lemma (**??**) we have

$$G(x,y) = \frac{\mathbf{i}k_s}{\mu\sqrt{2\pi}}\frac{1}{\delta(\xi)}\begin{pmatrix} \mu_s\beta & \xi\beta \\ 2\xi\mu_s\mu_p & 2\xi^2\mu_p \end{pmatrix}_{\xi=k_s\frac{x_1-y_1}{|x-y|}}\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{1/2}}e^{\mathbf{i}k_s|x-y|-\mathbf{i}\frac{\pi}{4}}$$
$$+\frac{\mathbf{i}k_p}{\mu\sqrt{2\pi}}\frac{1}{\delta(\xi)}\begin{pmatrix} 2\xi^2\mu_s & -2\xi\mu_s\mu_p \\ -\xi\beta & \mu_p\beta \end{pmatrix}_{\xi=k_p\frac{x_1-y_1}{|x-y|}}\frac{y_2}{|x-y|}\frac{1}{(k_p|x-y|)^{1/2}}e^{\mathbf{i}k_p|x-y|-\mathbf{i}\frac{\pi}{4}} \tag{3.1}$$

$$+O(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}})$$
$$:=\mathcal{G}_s(x,y)+\mathcal{G}_p(x,y)+O(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}})$$

$$\begin{aligned} T_D(x,z) &= \frac{k_s}{\sqrt{2\pi}}\frac{1}{\gamma(\xi)}\begin{pmatrix}\mu_s\mu_p & \xi\mu_p\\ \xi\mu_s & \xi^2\end{pmatrix}_{\xi=k_s\frac{x_1-z_1}{|x-z|}}\frac{z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{1/2}}e^{\mathbf{i}k_s|x-z|-\mathbf{i}\frac{\pi}{4}}\\ &+\frac{k_p}{\sqrt{2\pi}}\frac{1}{\gamma(\xi)}\begin{pmatrix}\xi^2 & -\xi\mu_p\\ -\xi\mu_s & \mu_p\mu_s\end{pmatrix}_{\xi=k_p\frac{x_1-z_1}{|x-z|}}\frac{z_2}{|x-z|}\frac{1}{(k_p|x-z|)^{1/2}}e^{\mathbf{i}k_p|x-z|-\mathbf{i}\frac{\pi}{4}} \qquad (3.2)\\ &+O(\frac{k_s z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{k_s|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}})\\ &:=\mathcal{T}_s(x,z)+\mathcal{T}_p(x,z)+O(\frac{k_s z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{k_s|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}})\end{aligned}$$

Now we consider the finite aperture point spread function $J_d(z,y)$:

$$\int_{-d}^{d}(T_D(x_1,0;z_1,z_2))^T\overline{G(x_1,0;y_1,y_2)}dx_1 \qquad (3.3)$$

Recall following standard asymptotic expansion:

$$|x-y|=|x-z|+\widehat{x-z}\cdot(z-y)+O(\frac{|y-z|^2}{|x-z|}) \qquad (3.4)$$
$$|y|^{-\alpha}=|z|^{-\alpha}(1+\frac{|y|-|z|}{|z|})^{-\alpha}=|z|^{-\alpha}(1+O(\frac{|y-z|}{|z|})) \qquad (3.5)$$
$$e^{\mathbf{i}t}=1+O(t) \qquad (3.6)$$
$$|a^{1/2}-b^{1/2}|\le C\sqrt{|a-b|} \qquad (3.7)$$

where $x,y,z\in\mathbb{R}^2$, $t,a,b\in\mathbb{R}$ and $\alpha>0$.

Lemma 3.1 *For any* $z,y\in\mathbb{R}^2_+$, $J_d(z,y)=F(z,y)+O((1+\frac{|y-z|}{z_2})(\frac{1}{k_s z_2})^{1/4}+\frac{(k_s|y-z|)^2}{k_s z_2}+(\frac{|y-z|}{z_2})^{1/2})$, *where*

$$F(z,y)=-\frac{\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d}f_s(\theta)\begin{pmatrix}\sin^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \cos^2\theta\end{pmatrix}e^{\mathbf{i}k_s(z_1-y_1)\cos\theta+\mathbf{i}k_s(z_2-y_2)\sin\theta}d\theta \qquad (3.8)$$
$$-\frac{\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d}f_p(\theta)\begin{pmatrix}\cos^2\theta & -\sin\theta\cos\theta\\ -\sin\theta\cos\theta & \sin^2\theta\end{pmatrix}e^{\mathbf{i}k_p(z_1-y_1)\cos\theta+\mathbf{i}k_p(z_2-y_2)\sin\theta}d\theta \qquad (3.9)$$

and

$$f_s(\theta)=\frac{\sin\theta((\kappa^2-\cos^2\theta)^{1/2}(1-2\cos^2\theta)+2\overline{(\kappa^2-\cos^2\theta)^{1/2}}\cos^2\theta)}{(\cos^2\theta+\sin\theta(\kappa^2-\cos^2\theta)^{1/2})\overline{((1-2\cos^2\theta)^2+4\cos^2\theta\sin\theta(\kappa^2-\cos\theta)^{1/2})}}$$
$$f_p(\theta)=\frac{\sin\theta(1/\kappa^2-\cos^2\theta)^{1/2}}{(\cos^2\theta+\sin\theta(1/\kappa^2-\cos^2\theta)^{1/2})((1/\kappa^2-2\cos^2\theta)^2+4\cos^2\theta\sin\theta(1/\kappa^2-\cos\theta)^{1/2})}$$

where $0<\theta_1^d<\pi/2<\theta_2^d<\pi$ *and* $z_2=(d+z_1)\tan\theta_1^d=(z_1-d)\tan\theta_2^d$.

Proof.

$$\begin{aligned}
&\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}}\\
&=(\frac{z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}})(1+O(\frac{|y-z|}{|x-z|}))
\end{aligned}$$

$$\begin{aligned}
&|\mu_{\mathbf{i}}(k_j\frac{x_1-y_1}{|x-y|})-\mu_{\mathbf{i}}(k_j\frac{x_1-z_1}{|x-z|})|\\
&\le Ck_j\sqrt{\left|\frac{x_1-y_1}{|x-y|}-\frac{x_1-z_1}{|x-z|}\right|}\le Ck_j\Big(\frac{|y-z|}{|x-z|}\Big)^{1/2}
\end{aligned}$$

where $i, j = s, p$. By above, we can obtain

$$\mathcal{G}_s(x,y)=\mathcal{G}_s(x,z)e^{\mathbf{i}k_s\widehat{x-z}\cdot(z-y)}+O(\frac{(k_s|y-z|)^2}{(k_s|x-z|)^{3/2}})+O(\frac{(k_s|y-z|)^{1/2}}{k_s|x-z|}) \tag{3.10}$$

$$\mathcal{G}_p(x,y)=\mathcal{G}_p(x,z)e^{\mathbf{i}k_p\widehat{x-z}\cdot(z-y)}+O(\frac{(k_p|y-z|)^2}{(k_p|x-z|)^{3/2}})+O(\frac{(k_p|y-z|)^{1/2}}{k_p|x-z|}) \tag{3.11}$$

For $l>1$, a simple computation show that

$$\int_{-d}^{d}\frac{k_s}{(k_s|x-z|)^l}dx_1=\frac{1}{(k_sz_2)^{l-1}}\int_{\frac{-d-z_1}{z_2}}^{\frac{d-z_1}{z_2}}\frac{1}{(1+t^2)^{l/2}}dt\le C\frac{1}{(k_sz_2)^{l-1}} \tag{3.12}$$

Let

$$\mathcal{G}_\alpha(x,y)=\frac{\mathbf{i}}{\sqrt{2\pi}\mu}g_\alpha(\frac{x_1-y_1}{|x-y|},\kappa)\frac{1}{(k_\alpha|x-y|)^{1/2}}e^{\mathbf{i}k_\alpha|x-y|-\mathbf{i}\frac{\pi}{4}} \tag{3.13}$$

$$\mathcal{T}_\alpha(x,y)=\frac{k_\alpha}{\sqrt{2\pi}}t_\alpha(\frac{x_1-z_1}{|x-z|},\kappa)\frac{1}{(k_s|x-z|)^{1/2}}e^{\mathbf{i}k_\alpha|x-z|-\mathbf{i}\frac{\pi}{4}} \tag{3.14}$$

where $\alpha = s, p$. Now, by substituting (3.10-3.11) into $J_d(z,y)$ and using inequality (3.12), we have

$$\begin{aligned}
J_d(z,y)=&\frac{-\mathbf{i}}{2\pi\mu}\int_{-d}^{d}t_s(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_s(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_s\widehat{x-z}\cdot(y-z)}}{|x-z|}\\
&+t_p(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_p(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_p\widehat{x-z}\cdot(y-z)}}{|x-z|}dx_1 &(3.15)\\
&-\frac{\mathbf{i}}{2\pi\mu}\int_{-d}^{d}t_p(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_s(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_s\widehat{x-z}\cdot(y-z)}}{|x-z|} &(3.16)\\
&+t_s(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_p(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_p\widehat{x-z}\cdot(y-z)}}{|x-z|}dx_1 &(3.17)\\
&+O((1+\frac{|y-z|}{z_2})(\frac{1}{k_sz_2})^{1/4}+\frac{(k_s|y-z|)^2}{k_sz_2}+(\frac{|y-z|}{z_2})^{1/2}) &(3.18)\\
&:=F(z,y)+R(z,y) &(3.19)\\
&+O((1+\frac{|y-z|}{z_2})(\frac{1}{k_sz_2})^{1/4}+\frac{(k_s|y-z|)^2}{k_sz_2}+(\frac{|y-z|}{z_2})^{1/2}) &(3.20)
\end{aligned}$$

We denote $\widehat{x-z} = x - z/|x-z| = (cos(\phi+\pi), \sin(\phi+\pi))$, then taking the substitution $x_1 = z_1 - z_2 \cot\phi$, we obtain

$$F(z,y) = \frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d} A_s(\phi,\kappa)e^{\mathbf{i}k_s(z_1-y_1)\cos\phi+\mathbf{i}k_s(z_2-y_2)\sin\phi} \tag{3.21}$$

$$+\frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d} A_p(\phi,\kappa)e^{\mathbf{i}k_p(z_1-y_1)\cos\phi+\mathbf{i}k_p(z_2-y_2)\sin\phi} \tag{3.22}$$

$$R(z,y) = \frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d} B_s(\phi,\kappa)e^{\mathbf{i}k_s(z_1-y_1)\cos\phi+\mathbf{i}k_s(z_2-y_2)\sin\phi+(k_p-k_s)|x-z|} \tag{3.23}$$

$$+\frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d} B_p(\phi,\kappa)e^{\mathbf{i}k_p(z_1-y_1)\cos\phi+\mathbf{i}k_p(z_2-y_2)\sin\phi+(k_s-k_p)|x-z|} \tag{3.24}$$

It is easy to see that $|R(z,y)| \le C\frac{|z-y|}{z_2}$. □

Let

$$g(x_1) = \frac{1}{((x_1-z_1)^2+z_2^2)^{3/4}((x_1-y_1)^2+y_2^2)^{1/4}}$$

$$\phi(x_1) = ((x_1-z_1)^2+z_2^2)^{1/2} - ((x_1-y_1)^2+y_2^2)^{1/2}$$

Then, we have

$$g'(x_1) = -g(x_1)[\frac{3(x_1-z_1)}{2((x_1-z_1)^2+z_2^2)} + \frac{(x_1-y_1)}{2((x_1-y_1)^2+y_2^2)}]$$

$$\phi'(x_1) = \frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}} - \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}}$$

$$= \frac{\frac{(x_1-z_1)^2}{(x_1-z_1)^2+z_2^2} - \frac{(x_1-y_1)^2}{(x_1-y_1)^2+y_2^2}}{\frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}} + \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}}}$$

$$= \frac{(x_1-z_1)^2y_2^2 - (x_1-y_1)^2z_2^2}{(\frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}} + \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}})((x_1-z_1)^2+z_2^2)((x_1-y_1)^2+y_2^2)}$$

$$\phi''(x_1) = \frac{z_2^2}{((x_1-z_1)^2+z_2^2)^{3/2}} - \frac{y_2^2}{((x_1-y_1)^2+y_2^2)^{3/2}}$$

Using integration by parts, we can obtain

$$\int_{-d}^{d} g(x_1)e^{\mathbf{i}\phi(x_1)}dx_1$$

$$= (\frac{g(d)}{\phi'(d)}e^{\mathbf{i}\phi(d)} - \frac{g(-d)}{\phi'(-d)}e^{\mathbf{i}\phi(-d)}) - \int_{-d}^{d}\frac{g'(x_1)}{\phi'(x_1)} - \frac{g(x_1)\phi''(x_1)}{(\phi'(x_1))^2}dx_1$$

Assume that

$$|y_1| \le c_0 d \quad |z_1| \le c_0 d \quad h \le y_2, z_2 \le c_1 h \quad d \le c_2 h$$

where $0 < c_0 < 1$. Let define $0 < \theta_y, \theta_z < \pi$ such that

$$\cos\theta_y = \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}}$$

$$\cos\theta_z = \frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}}$$

By mean value theorem and the law of sines, we get

$$\begin{aligned}
|\phi'(x_1)| &= |\cos\theta_z - \cos\theta_y| = |\sin\theta'||\theta_z - \theta_y| \\
&\geq \frac{h}{(1+c_0)d}|\sin(\theta_z-\theta_y)| \\
&= \frac{h}{(1+c_0)d}\frac{|z-y|}{|x-y|}\sin\theta_{|x-y|} \\
&= \frac{h}{(1+c_0)d}\frac{|z-y|}{|x-z|}\sin\theta_{|x-z|} \\
&\geq \frac{h^2}{(1+c_0)^2d^2}\frac{|z-y|}{|x-y|} \\
\text{or} \qquad &\geq \frac{h^2}{(1+c_0)^2d^2}\frac{|z-y|}{|x-z|}
\end{aligned}$$

Then we have

$$\begin{aligned}
|\frac{g(x_1)}{\phi'(x_1)}| &\leq \frac{(1+c_0)^2d^2}{h^2}\frac{1}{|z-y||x-y|^{1/2}|x-z|^{1/2}} \\
&\leq C\frac{d^2}{h^3}\frac{1}{|z-y|}
\end{aligned}$$

Moreover, by mean value theorem agian, we have

$$\begin{aligned}
|\phi''(x_1)| &= |\frac{\sin^2\theta_z}{|x-z|} - \frac{\sin^2\theta_y}{|x-y|}| \\
&= |\frac{2\sin\theta'\cos\theta'}{|x-y'|}(\theta_z-\theta_y) - \frac{\sin^2\theta'}{|x-y'|^2}(|x-z|-|x-y|)| \\
&\leq \pi\frac{|\sin(\theta_z-\theta_y)|}{h} + \frac{|z-y|}{h^2} \\
&\leq \pi\frac{|\sin\theta_{|x-z|}||z-y|}{h|x-z|} + \frac{|z-y|}{h^2} \\
&\leq C\frac{|z-y|}{h^2}
\end{aligned}$$

Now, it is easy to see that

$$\begin{aligned}
&\Big|\int_{-d}^{d}\frac{g'(x_1)}{\phi'(x_1)} - \frac{g(x_1)\phi''(x_1)}{(\phi'(x_1))^2}dx_1\Big| \\
&\leq C\frac{d^3}{h^4}\frac{1}{|z-y|} + C\frac{d^3}{h^3}\frac{1}{|z-y|}\frac{d^2}{h^3}
\end{aligned}$$

Based on the above analysis, we can obtain

$$|\int_{-d}^{d} z_2 g(x_1)e^{\mathbf{i}\phi(x_1)}| \leq C((\frac{d}{h})^2 + (\frac{d}{h})^3 + (\frac{d}{h})^5)\frac{1}{|z-y|}$$

4. 2017.11.08

$$\sin\phi_\kappa - \sin(t+\phi) = -2\cos(\frac{\phi_\kappa+\phi+t}{2})\sin(\frac{t+\phi-\phi_\kappa}{2})$$

$$\sin(\frac{t+\phi-\phi_\kappa}{2}) = \sin\frac{t}{2}\cos(\frac{\phi-\phi_\kappa}{2}) + \cos\frac{t}{2}\sin(\frac{\phi-\phi_\kappa}{2})$$

Some think, substituting $t = 2\arcsin s/2$ into following integral

$$\int_0^\infty \chi(t)(\sin\phi_\kappa - \sin(t+\phi))^{1/2} e^{-\mathbf{i}\rho\cos t}$$
$$= \int_0^\infty \chi(t(s))(-s\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-s^2}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2} e^{-\mathbf{i}\rho s^2/2}$$
$$= \int_0^\infty \chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2} t^{-1/2} e^{-\mathbf{i}\rho t/2}$$

Let

$$f(t) = t^{-1/2} e^{-\mathbf{i}\rho t/2}$$
$$g(t) = -\int_t^{t-\mathbf{i}\infty} x^{-1/2} e^{-\mathbf{i}\rho x/2} dx$$
$$= \mathbf{i}\int_0^\infty (t-\mathbf{i}x)^{-1/2} e^{-\mathbf{i}\rho t - \rho x} dx$$

It is easy to see that $g'(t) = f(t)$. Then we have

$$= \int_0^\infty \chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2} t^{-1/2} e^{-\mathbf{i}\rho t/2}$$
$$= \chi(0)(-2\sin(\frac{\phi-\phi_\kappa}{2}))^{1/2} g(0)$$
$$- \int_0^\infty (\chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2})' g(t) dt$$

We get

$$g(x) = \int_0^\infty \chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{-1/2} t^{-1/2}(t-\mathbf{i}x)^{-1/2} e^{-\mathbf{i}\rho t} dt$$
$$R(\rho) = \int_0^\infty g(x) e^{-\rho x} dx$$

Because $\chi(t)$ has compact support $(-\delta, \delta)$, we obtain

$$gg(x) = \int_0^\delta (\sqrt{t}\cos(\theta) - \sqrt{4-t}\sin\theta)^{-1/2} t^{-1/2} (t^2+x^2)^{-1/4} dt$$

where $\theta = \frac{\phi-\phi_\kappa}{2}$. For $x > 0$, Put L(x):

$$\int_0^a \frac{1}{t^{3/4}} \frac{1}{(t^2+x^2)^{1/4}} dt$$
$$= 4\int_0^a \frac{1}{(t^2+x^2)^{1/4}} dt^{1/4}$$
$$= 4\int_0^{a^{1/4}} \frac{1}{(t^8+x^2)^{1/4}} dt$$
$$= 4x^{-1/4}\int_0^{(\frac{a}{x})^{1/4}} \frac{1}{(t^8+1)^{1/4}} dt$$
$$= 4x^{-1/4}\int_0^{(\frac{a}{x})^{1/4}} \frac{1}{(t^8+1)^{1/4}} dt$$
$$\le 4x^{-1/4}\int_0^\infty \frac{1}{(t^8+1)^{1/4}} dt$$

Back to analysis $gg(x)$, we have

$$
\begin{aligned}
gg(x) &\leq \int_0^{\delta} \Big|\frac{\sqrt{t}+2|\sin\theta|}{t-4\sin^2\theta}\Big|^{1/2} t^{-1/2}(t^2+x^2)^{-1/4}dt \\
&= \int_0^{\delta} \Big|\frac{1}{\sqrt{t}-2|\sin\theta|}\Big|^{1/2} t^{-1/2}(t^2+x^2)^{-1/4}dt \\
&= 2\int_0^{\sqrt{\delta}} \Big|\frac{1}{t-2|\sin\theta|}\Big|^{1/2} (t^4+x^2)^{-1/4}dt \\
&= 2\int_{-2|\sin\theta|}^{\sqrt{\delta}-2|\sin\theta|} |t|^{-1/2}((t+2|\sin\theta|)^4+x^2)^{-1/4}dt \\
&\leq 4\int_0^{\delta^{1/4}} (t^8+x^2)^{-1/4}dt + 4\int_0^{\sqrt{2|\sin\theta|}} ((t^2-2|\sin\theta|)^4+x^2)^{-1/4}dt \\
&\leq Cx^{-1/4}(1+\int_0^{\sqrt{2|\sin\theta|}} ((t^2-2|\sin\theta|)^4/x+x)^{-1/4}dt) \\
&\leq Cx^{-1/4}(1+\int_0^{\sqrt{2|\sin\theta|}} (t^2-2|\sin\theta|)^{-1/2}dt) \\
&= Cx^{-1/4}(1+\int_0^{1} (1-t^2)^{-1/2}dt) \leq Cx^{-1/4}
\end{aligned}
$$

Immediately, we can obtain

$$|g(x)| \leq Cx^{-1/4}$$

It follows that

$$R(\rho) \leq \int_0^{\infty} x^{-1/4}e^{-\rho x} \leq C\rho^{-3/4}$$

5. stationary of phase lemma

Lemma 5.1 *Assume that* $0 < \kappa := \sin\phi_\kappa < 1, 0 < \phi_\kappa < \pi/2$, $0 \leq \phi \leq \pi/2$. *Let*

$$f(t,\phi) := F(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2}) \tag{5.1}$$

be a complexed function in $C([-\pi/2,\pi/2]\times[0,\pi/2])$. *Moreover, its partial derivative with respect to* t *can be represented as*

$$\frac{\partial f(t,\phi)}{\partial t} = g(t,\phi)(\kappa^2-\sin^2(t+\phi))^{-1/2} \tag{5.2}$$

where $g(t,\phi)$ *is uniformly bounded. Then for any* $\rho \geq 1$, *we have*

$$
\begin{aligned}
&\Big|I(\rho,\phi) := \int_{-\pi/2}^{\pi/2} f(t)e^{\mathbf{i}\rho\cos t}dt - \Big(\frac{2\pi}{\rho}\Big)^{1/2} f(0)e^{\mathbf{i}\rho-\mathbf{i}\pi/4}\Big| \\
&\leq C\frac{1}{\rho^{3/4}}
\end{aligned}
\tag{5.3}
$$

Proof. Solving the following equation:

$$\kappa^2 - \sin^2(t+\phi) = 0$$

we have, if $0 < \phi < \pi/2 - \phi_\kappa$,

$$t_1(\phi) = \phi_\kappa - \phi \qquad t_2(\phi) = -\phi_\kappa - \phi$$

and if $\pi/2 - \phi_\kappa \le \phi < \pi/2$,

$$t_1(\phi) = \phi_\kappa - \phi \qquad t_2(\phi) = \pi - \phi_\kappa - \phi$$

Since $|t_2(\phi)| < \phi_\kappa$ or $|t_2(\phi)| < \pi/2 - \phi_\kappa$, we now define $\delta := \min(\frac{\phi_\kappa}{2}, \frac{\pi/2-\phi_\kappa}{2})$ and it is easy to see that

$$\kappa + \sin(t+\phi) \neq 0 \tag{5.4}$$

$$\cos(\frac{t+\phi+\phi_\kappa}{2}) \neq 0 \tag{5.5}$$

for any $(t,\phi) \in [-\delta,\delta] \times [0,\pi/2]$. Let $\chi_\delta \in C_0^\infty(-\pi/2,\pi/2)$ be the cut-off function with that $0 \le \chi_\delta \le 1$, $\chi_\delta = 1$ in $(-\delta/2,\delta/2)$ and $\chi_\delta = 0$ in $(-\pi/2,\pi/2)\backslash(-\delta,\delta)$. Then we can divide I into two parts such that

$$\begin{aligned} I &= \int_{-\delta}^{\delta} f(t)\chi_\delta(t)e^{\mathbf{i}\rho\cos t}dt + \int_{-\pi/2}^{\pi/2} f(t)(1-\chi_\delta(t))e^{\mathbf{i}\rho\cos t}dt \\ &=: I_1 + I_2 \end{aligned}$$

Subtitating $t(s) = 2\arcsin s/2$ for t in I_1, we can obtain

$$I_1 = \int_{-2\sin\frac{\delta}{2}}^{2\sin\frac{\delta}{2}} f(t(s))\chi_\delta(t(s))\frac{1}{\sqrt{1-s^2/4}}e^{\mathbf{i}\rho}e^{-\mathbf{i}\rho s^2/2}ds \tag{5.6}$$

$$= \int_0^{2\sin\frac{\delta}{2}} (f(t(s))\chi_\delta(t(s)) + f(-t(s))\chi_\delta(-t(s)))\frac{1}{\sqrt{1-s^2/4}}e^{\mathbf{i}\rho}e^{-\mathbf{i}\rho s^2/2}ds \tag{5.7}$$

$$:= I_{11} + I_{12} \tag{5.8}$$

Taking substitution $s = \sqrt{x}$, we get

$$I_{11} = \frac{1}{2}\int_0^{(2\sin\frac{\delta}{2})^2} f(t(\sqrt{x}))\chi_\delta(t(\sqrt{x}))\frac{1}{\sqrt{1-x/4}}x^{-1/2}e^{\mathbf{i}\rho}e^{-\mathbf{i}\rho x/2}dx$$

Observe that

$$\begin{aligned} \sin\phi_\kappa - \sin(t+\phi) &= -2\cos(\frac{\phi_\kappa+\phi+t}{2})\sin(\frac{t+\phi-\phi_\kappa}{2}) \\ \sin(\frac{t+\phi-\phi_\kappa}{2}) &= \sin\frac{t}{2}\cos(\frac{\phi-\phi_\kappa}{2}) + \cos\frac{t}{2}\sin(\frac{\phi-\phi_\kappa}{2}) \\ &:= \sin\frac{t}{2}\cos\theta + \cos\frac{t}{2}\sin\theta \end{aligned}$$

where $\theta = \frac{\phi-\phi_\kappa}{2}$. By lemma (1.3) and using representation (5.2), inequality (5.4-5.5), it follows that

$$\begin{aligned} &|I_{11} - \frac{1}{2}\sqrt{\frac{2\pi}{\rho}}f(0)e^{\mathbf{i}\rho-\mathbf{i}\frac{\pi}{4}}| \\ \le &\int_0^\infty e^{-\rho y}dy\int_0^{(2\sin\frac{\delta}{2})^2} \Big|\frac{\partial(f(t(\sqrt{x}))\chi_\delta(t(\sqrt{x}))\frac{1}{\sqrt{1-x/4}})}{\partial x}\Big|(x^2+y^2)^{-\frac{1}{4}}dx \end{aligned}$$

$$\le C\int_0^\infty e^{-\rho y}dy\int_0^{(2\sin\frac{\delta}{2})^2} |\sqrt{x}\cos\theta+\sqrt{4-x}\sin\theta|^{-1/2}x^{-1/2}(x^2+y^2)^{-\frac{1}{4}}dx$$
$$\le C\int_0^\infty e^{-\rho y}dy\int_0^{(2\sin\frac{\delta}{2})^2} \frac{(\sqrt{x}|\cos\theta|+\sqrt{4-x}|\sin\theta|)^{1/2}}{|x-4\sin^2\theta|^{1/2}}x^{-1/2}(x^2+y^2)^{-\frac{1}{4}}dx$$
$$\le C\int_0^\infty e^{-\rho y}dy\int_0^{(2\sin\frac{\delta}{2})^2} \frac{1}{|\sqrt{x}-2|\sin\theta||^{1/2}}x^{-1/2}(x^2+y^2)^{-\frac{1}{4}}dx$$
$$\le C\int_0^\infty e^{-\rho y}dy\int_0^{2\sin\frac{\delta}{2}} \frac{1}{|x-2\sin|\theta||^{1/2}}(x^4+y^2)^{-\frac{1}{4}}dx$$
$$\le C\int_0^\infty e^{-\rho y}dy\int_{-2\sin|\theta|}^{2\sin\frac{\delta}{2}-2\sin|\theta|} \frac{1}{|x|^{1/2}}((x+2\sin|\theta|)^4+y^2)^{-\frac{1}{4}}dx$$
$$\le C\int_0^\infty e^{-\rho y}dy\int_{-2\sin|\theta|}^{2\sin\frac{\delta}{2}} \frac{1}{|x|^{1/2}}((x+2\sin|\theta|)^4+y^2)^{-\frac{1}{4}}dx$$
$$\le C\int_0^\infty e^{-\rho y}dy\bigg(\int_0^{\sqrt{2\sin\frac{\delta}{2}}}(x^8+y^2)^{-\frac{1}{4}}dx+\int_0^{\sqrt{2\sin|\theta|}}((x^2-2\sin|\theta|)^4+y^2)^{-\frac{1}{4}}dx\bigg)$$
$$\le C\int_0^\infty e^{-\rho y}dy\bigg(y^{-\frac{1}{4}}\int_0^{\infty}(x^8+1)^{-\frac{1}{4}}dx+y^{-\frac{1}{4}}\int_0^{\sqrt{2\sin|\theta|}}(2\sin|\theta|-x^2)^{-\frac{1}{2}}dx\bigg)$$
$$\le C\int_0^\infty y^{-\frac{1}{4}}e^{-\rho y}dy\bigg(\int_0^{\infty}(x^8+1)^{-\frac{1}{4}}dx+\int_0^{1}(1-x^2)^{-\frac{1}{2}}dx\bigg)\le C\frac{1}{\rho^{3/4}}$$

Using the same agrument , we can also carry out

$$|I_{12}-\frac{1}{2}\sqrt{\frac{2\pi}{\rho}}f(0)e^{\mathbf{i}\rho-\mathbf{i}\frac{\pi}{4}}|\le C\frac{1}{\rho^{3/4}} \tag{5.9}$$

It remains to estimate I_2. Note that there exits $m>0$ such that $|\sin t|\ge m$ for any $t\in[-\pi/2,\pi/2]\backslash(-\delta/2,\delta/2)$. Upon integration by parts and representation (5.2) again, we have

$$|I_{12}|\le C\rho^{-1}(1+\bigg|\int_{[-\pi/2,\pi/2]\backslash(-\delta/2,\delta/2)}\frac{\partial(f(t)(1-\chi_\delta(t)))}{\partial t}\frac{1}{\sin t}dt\bigg|)$$
$$\le C\rho^{-1}(1+\int_{-\pi/2}^{\pi/2}\bigg|\frac{\partial(f(t)(1-\chi_\delta(t)))}{\partial t}\bigg|dt)$$
$$\le C\rho^{-1}(1+\int_{-\pi/2}^{\pi/2}|(\kappa^2-\sin^2(t+\phi))^{-1/2}|dt)$$
$$\le C\rho^{-1}(1+\int_{-\pi/2}^{\pi/2}|(\kappa^2-\sin^2 t)^{-1/2}|dt)$$
$$\le C\rho^{-1}$$

This completes the proof. □

6. cross term of psf, 17.11.15

We need the following slight generalization of Van der Corput lemma for the oscillatory integral [3, P.152].

Lemma 6.1 *Let $-\infty < a < b < \infty$, and u is a C^k function u in (a, b).*
1. If $|u'(t)| \geq 1$ for $t \in (a, b)$ and u' is monotone in (a,b), then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 3\lambda^{-1}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

2. For all $k \geq 2$, if $|u^{(k)}(t)| \geq 1$ for $t \in (a, b)$, then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 12k\lambda^{-1/k}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

Proof. The assertion can be proved by extending the Van der Corptut lemma in [3]. Here we omit the details. □

Lemma 6.2 *For $0 < \kappa < 1$, let $F(\lambda) = \int_0^\kappa f(t)e^{\mathbf{i}\lambda(\sqrt{1-t^2}-\tau\sqrt{\kappa^2-t^2}+\alpha t)}dt$, where $\tau \geq c_0 > 0$ and $\alpha \in \mathbb{R}$, then we have*

$$|F(\lambda)| \leq C(\kappa)\lambda^{-\frac{1}{2N_*}}\left[|f(\kappa)| + \int_0^\kappa |f'(t)|dt\right]$$

where $N_* = \min\{N|\kappa^{2N-1} < c_0, N \in \mathbb{Z}_+\}$.

Proof. Put $\phi(t) = -\sqrt{1-t^2}$ and $\psi(t, \tau) = \tau\phi(t/\kappa) - \phi(t) + \alpha t$. For easy of notations, we denote the n-th partial derivative of $g(t)$ with respect to t by $g^{(n)}(t)$. Then, it is to see that, for $n > 1$

$$\psi^{(n)}(t, \tau) = \frac{\tau}{\kappa^{n-1}}\phi^{(n)}(\frac{t}{\kappa}) - \phi^{(n)}(t)$$

A standard computation show that

$$\phi^{(1)}(t) = \frac{t}{\sqrt{1-t^2}}$$
$$\phi^{(2)}(t) = \frac{1}{(1-t^2)^{3/2}}$$
$$\phi^{(3)}(t) = \frac{3t}{(1-t^2)^{5/2}}$$

Moreover, for $n \geq 3$, we have

$$\phi^{(n)}(t) = \frac{p_n(t)}{(1-t^2)^{n-1/2}} \tag{6.1}$$

where $p_n = \sum_0^{n-2} a_k^n t^k$ is a $(n-2)$-th polynomial such that its coefficients satisfy the following recursion formula:

$$a_{n-1}^{n+1} = (n+1)a_{n-2}^n, \quad a_{n-2}^{n+1} = (n+2)a_{n-3}^n$$
$$a_k^{n+1} = (k+1)a_{k+1}^n + (2n-k)a_{k-1}^n \quad \text{for } 1 \leq k \leq n-3$$
$$a_0^{n+1} = a_1^n$$

Since the polynmial coefficients are all positive, it is obvious that for $n \geq 1$, $\phi^{(n)}(t)$ is a monotone increasing positive function. Using the recursion formula, it follows that

$$\phi^{(n)}(0) = \begin{cases} 0 & \text{n is odd,} \\ (n-1)!!(n-3)!! & \text{n is even.} \end{cases} \tag{6.2}$$

where $(2k-1)!!$ is double factorial and $n > 3$. We are now in the position to proof the inequality. Since $0 < \kappa < 1$, obersev that

$$\psi^{(2N_*+1)}(t,\tau) \geq \frac{\tau}{\kappa^{2N_*}} \phi^{(2N_*+1)}(t) - \phi^{(2N_*+1)}(t) > 0$$

Therefore, $\psi^{(2N_*)}(t,\tau)$ is monotone increasing in $[0,\kappa)$. By (6.2), we get

$$\psi^{(2N_*)}(t,\tau) \geq \psi^{(2N_*)}(0,\tau) \geq \psi^{(2N_*)}(0,c_0) = C(2N_*)(\frac{c_0}{\kappa^{2N_*-1}} - 1) > 0 \tag{6.3}$$

The lemma is now a direct consequence of lemma (6.1). □

7. Other exponential decay term: 17.11.16 on G1

The parameterization of hyperbolic curve passing $(\pm 1, 0)$ is:

$$\xi_1 = \pm\sqrt{t^2+1} \qquad \xi_2 = t$$

where $t \in \mathbb{R}$. Substituting $\xi = \xi_1 + \mathbf{i}\xi_2$ into $\mu(\xi) := (1-\xi^2)^{1/2}$ and $\mu_\kappa(\xi) := (\kappa^2 - \xi^2)^{1/2}$, we get

$$\begin{aligned} \operatorname{Im} \mu(\xi) &= \operatorname{Im} (1 - (\xi_1^2 - \xi_2^2 + \mathbf{i}2\xi_1\xi_2))^{1/2} \\ &= \operatorname{Im} (-2t\sqrt{t^2+1}\mathbf{i})^{1/2} = t^{1/2}(t^2+1)^{1/4} \end{aligned}$$

$$\begin{aligned} \operatorname{Im} \mu_\kappa(\xi) &= \operatorname{Im} (\kappa^2 - (\xi_1^2 - \xi_2^2 + \mathbf{i}2\xi_1\xi_2))^{1/2} \\ &= \operatorname{Im} (\kappa^2 - 1 - 2t\sqrt{t^2+1}\mathbf{i})^{1/2} \\ &= \sqrt{\frac{\sqrt{(1-\kappa^2)^2 + 4t(t^2+1)} + 1 - \kappa^2}{2}} \\ &\geq t^{1/2}(t^2+1)^{1/4} \end{aligned}$$

where we only consider the branch, denoted by Γ^+, in the first quadrant here. For $a > 0, b > 0$, we have

$$|e^{\mathbf{i}\xi a + \mathbf{i}\mu(\xi)b + \mathbf{i}\mu_\kappa(\xi)c}| \leq e^{-ta - t^{1/2}(t^2+1)^{1/4}b - t^{1/2}(t^2+1)^{1/4}c} \leq e^{-t(b+c)}$$

Lemma 7.1 *For $\xi \in \Gamma_0$, let $f(\xi)$ is a complex valued function in $L^1(\Gamma^+)$ such that $|f(\xi)| \leq C(1+\xi^k)$, $k \in \mathbb{Z}_+$. Then we have*

$$\begin{aligned} &|I(a,b,c) := \int_{\Gamma^+} f(\xi)\ e^{\mathbf{i}\xi a + \mathbf{i}\mu(\xi)b + \mathbf{i}\mu_\kappa(\xi)c}\ d\xi| \\ &\leq C(\frac{1}{b+c} + \frac{1}{(b+c)^k}) \end{aligned}$$

Proof.

$$\frac{d\xi(t)}{dt} = \frac{t}{\sqrt{t^2+1}} + \mathbf{i}$$

Substituting $\xi(t)$ into $I(a,b,c)$, we hvae

$$\begin{aligned}|I(a,b,c)| &= \Big| \int_0^\infty |f(\xi(t)) \frac{d\xi(t)}{dt} e^{\mathbf{i}\xi(t)a+\mathbf{i}\mu(\xi(t))b+\mathbf{i}\mu_\kappa(\xi(t))c} dt \Big| \\ &\le C \int_0^\infty (1+t^k) e^{-t(b+c)} dt \\ &\le C(\frac{1}{b+c} + \frac{1}{(b+c)^k})\end{aligned}$$

□

Lemma 7.2 *Let $f(\xi)$ is a bounded complex valued function in $L^1((\kappa,1))$. Then we have*

$$\begin{aligned}&|I(a,b) := \int_\kappa^1 |f(\xi) e^{\mathbf{i}\xi a+\mathbf{i}\mu_\kappa(\xi)b} d\xi| \\ &\le C\frac{1}{b}\end{aligned}$$

Proof. It is simple to see that

$$\begin{aligned}|I(a,b)| &\le C \int_\kappa^1 e^{-b\sqrt{\xi^2-\kappa^2}} d\xi \\ &\le C \int_0^{\sqrt{1-\kappa^2}} \frac{t}{\sqrt{t^2+\kappa^2}} e^{-bt} dt \\ &\le C\frac{1}{b}\end{aligned}$$

□

8. about principle of arguement

Put

$$\begin{aligned}\delta_\pm(t) = &\ (\kappa-2t^2)^2 \mp \mathbf{i}4t^2\sqrt{1-t^2}\sqrt{t^2-\kappa} \\ &:= f_1(t) \mp \mathbf{i} f_2(t)\end{aligned}$$

where $0<\kappa<1$ and we have

$$\delta'_\pm(t) = f_1'(t) \mp \mathbf{i} f_2'(t)$$

It is easy to see $f_2(1) = f_2(\kappa) = 0$ and $f_1(t) > 0$ for any $\kappa \le t \le 1$. Then

$$\begin{aligned}&\int_\kappa^1 \frac{\delta'_+(t)}{\delta_+(t)} - \frac{\delta'_-(t)}{\delta_-(t)} dt \\ &= 2\mathbf{i} \int_\kappa^1 \operatorname{Im}(\frac{\delta'_+(t)}{\delta_+(t)}) dt\end{aligned}$$

$$
\begin{aligned}
&= 2\mathbf{i}\int_{\kappa}^{1} \mathrm{Im}\,\frac{(f_1'(t)-\mathbf{i}f_2'(t))f_1(t)+\mathbf{i}f_2(t))}{(f_1(t)-\mathbf{i}f_2(t))(f_1(t)+\mathbf{i}f_2(t))}dt \\
&= 2\mathbf{i}\int_{\kappa}^{1} \frac{f_1'(t)f_2(t)-f_1(t)f_2'(t)}{f_1^2(t)+f_2^2(t)}dt \\
&= 2\mathbf{i}\int_{\kappa}^{1} \frac{f_1^2(t)}{f_1^2(t)+f_2^2(t)}\frac{f_1'(t)f_2(t)-f_1(t)f_2'(t)}{f_1^2(t)}dt \\
&= \; -2\mathbf{i}\int_{\kappa}^{1} \frac{f_1^2(t)}{f_1^2(t)+f_2^2(t)}d\frac{f_2(t)}{f_1(t)} \\
&= \; -2\mathbf{i}\arctan\frac{f_2(t)}{f_1(t)}\bigg|_{\kappa}^{1} = 0
\end{aligned}
$$

Notic that, the condition only used above are $f_2(1) = f_2(\kappa) = 0$ and $f_1(t) > 0$.

9. Fundamental solution of Elastic wave

$$
\begin{aligned}
G(x;y) &= \frac{1}{\omega^2}(\nabla\times\nabla\cdot(g_s(x;y)\mathbb{I}) - \nabla\nabla g_p(x;y)) &(9.1)\\
&= \frac{1}{\omega^2}(k_s^2 g_s(x,y) + \nabla\nabla(g_s(x;y) - g_p(x;y))) &(9.2)
\end{aligned}
$$

where y is the Dirac source, $g_p(x;y)$ or $g_s(x;y)$ is the fundamental solution of the scalar Helmhotlz equation with wavenumbers $k_p = \omega/c_p$ or $k_s = \omega/cs$.

$$
g_\alpha = \frac{\mathrm{i}}{4}H_0^{(1)}(k_\alpha|x-y|) \tag{9.3}
$$

where $H_0^{(1)}(t)$ is the Hankel function of the first type and order zero. By straight calculation using $H_1^{(1)}(t) = -dH_0^{(1)}(t)/dt$ and $dH_1^{(1)}(t)/dt = H_0^{(1)}(t) - H_1^{(1)}(t)/t$, we have

$$
\begin{aligned}
G_{ij}(x;y) &= \frac{\mathbf{i}}{4}\{(\frac{k_s^2}{\omega^2}H_0^{(1)}(k_s|x-y|) - \frac{1}{\omega^2}\frac{k_sH_1^{(1)}(k_s|x-y| - k_pH_1^{(1)}(k_p|x-y|)}{|x-y|})\delta_{ij} \\
&+\frac{1}{\omega^2}[(\frac{2k_sH_1^{(1)}(k_s|x-y| - 2k_pH_1^{(1)}(k_p|x-y|)}{|x-y|} - (k_s^2H_0^{(1)}(k_s|x-y|) - k_p^2H_0^{(1)}(k_p|x-y|))]\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\}
\end{aligned}
$$

The definition of hankal function is $H_k^{(1)}(t) = J_k(t) + \mathbf{i}Y_k(t)$ where

$$
J_k(t) = \sum_{p=0}^{\infty}\frac{(-1)^p}{p!(k+p)!}(t/2)^{k+2p}
$$

Specially

$$
\begin{aligned}
J_0(t) &= \sum_{p=0}^{\infty}\frac{(-1)^p}{p!p!}(t/2)^{2p} = 1 + \cdots \\
J_1(t) &= \sum_{p=0}^{\infty}\frac{(-1)^p}{p!(1+p)!}(t/2)^{1+2p} = \frac{t}{2} + \cdots
\end{aligned}
$$

and

$$Y_k(t) = \frac{1}{\pi}\{\ln t^2 - 2\ln 2 + 2C_{euler}\}J_k(t) - \frac{1}{\pi}\sum_{p=0}^{k-1}\frac{(k-1-p)!}{p!}(2/t)^{k-2p}$$
$$-\frac{1}{\pi}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!(k+p)!}(t/2)^{k+2p}\{\psi(p+k)+\psi(p)\}$$

Speceilly

$$Y_0(t) = \frac{1}{\pi}\{\ln t^2 - 2\ln 2 + 2C_{euler}\}J_0(t) - \frac{1}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^p}{p!p!}(t/2)^{2p}\{2\psi(p)\}$$
$$Y_1(t) = \frac{1}{\pi}\{\ln t^2 - 2\ln 2 + 2C_{euler}\}J_1(t) - \frac{1}{\pi}\frac{2}{t} - \frac{t}{2\pi}$$
$$-\frac{1}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^p}{p!(1+p)!}(t/2)^{1+2p}\{\psi(p+1)+\psi(p)\}$$

Thus, we have

$$H_0^{(1)}(kr) = 1 + \mathbf{i}\frac{2}{\pi}(C_{euler} + \ln k - \ln 2) + \mathbf{i}\frac{1}{\pi}\ln r^2 + o(kr)$$
$$H_1^{(1)}(kr) = \frac{kr}{2} + \mathbf{i}\frac{1}{\pi}(C_{euler} + \ln k - \ln 2 - \frac{1}{2})kr - \mathbf{i}\frac{1}{\pi}\frac{2}{kr} + \mathbf{i}\frac{1}{\pi}\ln r^2\frac{kr}{2} + o(k^2r^2)$$
$$\frac{\mathbf{i}}{4}H_0^{(1)}(kr) = \frac{\mathbf{i}}{4} - \frac{1}{2\pi}(C_{euler} + \ln k - \ln 2) - \frac{1}{4\pi}\ln r^2 + o(kr)$$
$$\frac{\mathbf{i}}{4}H_1^{(1)}(kr) = \mathbf{i}\frac{kr}{8} - \frac{1}{4\pi}(C_{euler} + \ln k - \ln 2 - \frac{1}{2})kr + \frac{1}{4\pi}\frac{2}{kr} - \frac{1}{4\pi}\ln r^2\frac{kr}{2} + o(k^2r^2)$$

We also need to define the surface traction $T_x^n(\cdot)$ on the normal direction n,

$$T_x^n u(x) := \sigma\cdot n = 2\mu\frac{\partial u}{\partial n} + \lambda n \mathrm{div} u + \mu n \times \mathrm{curl} u$$

where

$$\sigma(u) = \begin{pmatrix} (\lambda+2\mu)\partial u_1/\partial x_1 + \lambda\partial u_2/\partial x_2 & \mu\partial u_1/\partial x_2 + \mu\partial u_2/\partial x_1 \\ \mu\partial u_1/\partial x_2 + \mu\partial u_2/\partial x_1 & (\lambda+2\mu)\partial u_2/\partial(x_2) + \lambda\partial u_1/\partial x_1 \end{pmatrix}$$

A simple computation show that

$$\frac{\partial^3 H_0^{(1)}(k|x-y|)}{\partial x_i^2\partial x_j} = (1+2\delta_i j)(-k^2H_0^{(1)}(kr)\frac{r_j}{r^2} + 2kH_1^{(1)}(kr)\frac{r_j}{r^3})$$
$$+k^3H_1^{(1)}(kr)\frac{r_i^2 r_j}{r^3} + 4k^2H_0^{(1)}(kr)\frac{r_i^2 r_j}{r^4} - 8kH_1^{(1)}(kr)\frac{r_i^2 r_j}{r^5}$$

where $r = |x-y|$ and $r_i = x_i - y_i$.

$$\frac{\mathbf{i}}{4}H_0^{(1)}(kr) = -\frac{1}{2\pi}(\ln\frac{kr}{2} + C_{euler})(1 - (\frac{kr}{2})^2 + \cdots) + \frac{1}{4\pi}(2(\frac{kr}{2})^2 + \cdots) + \frac{\mathbf{i}}{4}(1 - (\frac{kr}{2})^2 + \cdots)$$
$$= -\frac{1}{2\pi}(\ln\frac{kr}{2})(1 + O(r^2)) - \frac{1}{2\pi}C_{euler} + \frac{\mathbf{i}}{4} + O(r^2)$$
$$\frac{\mathbf{i}}{4}H_1^{(1)}(kr) = -\frac{1}{2\pi}(\ln\frac{kr}{2} + C_{euler})(\frac{kr}{2} - \frac{1}{2}(\frac{kr}{2})^3 + \cdots) + \frac{1}{4\pi}(\frac{kr}{2} + O(r^3)) + \frac{\mathbf{i}}{4}(\frac{kr}{2} - \frac{1}{2}(\frac{kr}{2})^3 + \cdots) + \frac{1}{2\pi}\frac{1}{kr}$$
$$= -\frac{1}{4\pi}(\ln\frac{kr}{2})(kr + O(r^3)) - \frac{kr}{4\pi}C_{euler} + \frac{kr}{8\pi} + \frac{\mathbf{i}kr}{8} + \frac{1}{2\pi}\frac{1}{kr} + O(r^3)$$

$$A(kr) := \frac{\mathbf{i}}{4}(k^2 H_0^{(1)}(kr) - 2kH_1^{(1)}(kr)/r) = \frac{k^2}{4\pi}(\ln\frac{kr}{2})(\frac{kr}{2})^2 - \frac{1}{\pi r^2} - \frac{k^2}{4\pi} + O(r^2)$$

and

$$A_{sp}(r) = A(k_s r) - A(k_p r) = \frac{k_s^2}{4\pi}(\ln\frac{kr}{2})(\frac{kr}{2})^2 - \frac{k_p^2}{4\pi}(\ln\frac{kr}{2})(\frac{kr}{2})^2 - (\frac{k_s^2}{4\pi} - \frac{k_p^2}{4\pi}) + O(r^2)$$

Let $g^{jkk} = \frac{\partial^3 g}{\partial x_j \partial x_k^2}$, $d = g_s - g_p$, thus

$$g^{iij} = (1+2\delta_{ij})(-A(kr)\frac{r_j}{r^2}) + \frac{\mathbf{i}k^3}{4}H_1^{(1)}(kr)\frac{r_i^2 r_j}{r^3} + 4A(kr)\frac{r_i^2 r_j}{r^4}$$

$$= (1+2\delta_{ij})(\frac{1}{\pi r^3} + \frac{k^2}{4\pi r})\frac{r_j}{r} - (\frac{4}{\pi r^3} + \frac{k^2}{2\pi r})\frac{r_i^2 r_j}{r^3} + O(r\ln r)$$

$$d^{iij} = (1+2\delta_{ij})(-A_{sp}\frac{r_j}{r^2}) + (\frac{\mathbf{i}k_s^3}{4}H_1^{(1)}(k_s r) - \frac{\mathbf{i}k_p^3}{4}H_1^{(1)}(k_p r))\frac{r_i^2 r_j}{r^3} + 4A_{sp}\frac{r_i^2 r_j}{r^4}$$

$$= (1+2\delta_{ij})(\frac{k_s^2}{4\pi r} - \frac{k_p^2}{4\pi r})\frac{r_j}{r} - (\frac{k_s^2}{2\pi r} - \frac{k_p^2}{2\pi r})\frac{r_i^2 r_j}{r^3} + O(r\ln r)$$

$$= (1+2\delta_{ij})\frac{(\lambda+\mu)\omega^2}{\mu(\lambda+2\mu)}\frac{1}{4\pi r}\frac{r_j}{r} - \frac{(\lambda+\mu)\omega^2}{\mu(\lambda+2\mu)}\frac{1}{2\pi r}\frac{r_i^2 r_j}{r^3} + O(r\ln r)$$

$$k^2 g^i = -\frac{\mathbf{i}}{4}H_1^{(1)}(kr)\frac{kr_i}{r} = -\frac{k^2}{2\pi r}\frac{r_i}{r} + O(r\ln r)$$

$$d^{iij} + d^{jjj} = 2\frac{(\lambda+\mu)\omega^2}{\mu(\lambda+2\mu)}\frac{1}{4\pi r}\frac{r_j}{r} + O(r\ln r)$$

Then we have [5, p43]

$$\sigma(Ge_1)n = \frac{1}{\omega^2}\begin{pmatrix} (\lambda+2\mu)(k_s^2 g_s^1 + d^{111}) + \lambda d^{122} & \mu(k_s^2 g_s^2 + d^{112}) + \mu d^{112} \\ \mu(k_s^2 g_s^2 + d^{112}) + \mu d^{112} & (\lambda+2\mu)d^{122} + \lambda(k_s^2 g_s^1 + d^{111}) \end{pmatrix} n$$

$$= \frac{\mu}{2\pi(\lambda+2\mu)}\left(\begin{pmatrix} \frac{2(\lambda+\mu)r_1^2}{\mu r^2} + 1 \\ \frac{2(\lambda+\mu)r_1 r_2}{\mu r^2} \end{pmatrix}(-\frac{r_1 n_1}{r^2} - \frac{r_2 n_2}{r^2}) - \begin{pmatrix} 0 \\ \frac{r_2 n_1 - r_1 n_2}{r^2} \end{pmatrix}\right) + O(r\ln r)$$

and

$$\sigma(Ge_2)n = \frac{1}{\omega^2}\begin{pmatrix} (\lambda+2\mu)d^{112} + \lambda(k_s^2 g_s^2 + d^{222}) & \mu(k_s^2 g_s^1 + d^{122}) + \mu d^{122} \\ \mu(k_s^2 g_s^1 + d^{122}) + \mu d^{122} & (\lambda+2\mu)(k_s^2 g_s^2 + d^{222}) + \lambda d^{112} \end{pmatrix} n$$

$$= \frac{\mu}{2\pi(\lambda+2\mu)}\left(\begin{pmatrix} \frac{2(\lambda+\mu)r_1 r_2}{\mu r^2} \\ \frac{2(\lambda+\mu)r_2^2}{\mu r^2} + 1 \end{pmatrix}(-\frac{r_1 n_1}{r^2} - \frac{r_2 n_2}{r^2}) - \begin{pmatrix} \frac{r_1 n_2 - r_2 n_1}{r^2} \\ 0 \end{pmatrix}\right) + O(r\ln r)$$

Now Let u be represented as single potential:

$$u = \int_{\partial D} G(x,y)\phi(y)ds(y) \tag{9.4}$$

with Neumann boundary condition

$$T_x u(x) = f(x) \quad \text{on} \quad \partial D \tag{9.5}$$

Then we obtain corresponding integral equation

$$\mathbf{P.V.}\int_{\partial D} T_x G(x,y)\phi(y)ds(y) - \frac{1}{2}\phi(x) = f(x) \tag{9.6}$$

where $x \in \partial D$. We describe the necessary parametrization of the integral equation in the two-dimensional case. We assume that the boundary curve ∂D possesses a regular analytic and 2π-periodic parametric representation of the form

$$x(t) = (x_1(t), x_2(t))$$

in counterclockwise orientation satisfying $|x'(t)| > 0$ for all t. Let [4]

$$\begin{aligned} T(x,y) &= (T_x(N_1(x,y))n, T_x(N_1(x,y))n) \\ T_0(x,y) &= -\frac{\mu}{2\pi(\lambda+2\mu)}\begin{pmatrix} 0 & \frac{r_1n_2-r_2n_1}{r^2} \\ \frac{r_2n_1-r_1n_2}{r^2} & 0 \end{pmatrix} \\ T_1(x,y) &= T(x,y) - T_0(x,y) \end{aligned}$$

Then by above analysis we have

$$\int_0^{2\pi} T_1(x(t),x(\tau))\phi(x(\tau))|x'(\tau)|dt + \mathbf{P.V.}\int_0^{2\pi} T_0(x(t),x(\tau))\phi(x(\tau))|x'(\tau)|dt - \frac{1}{2}\phi(x(t)) = f(x(t))$$

In particular, using expasion above, we can deduce the diagonal terms:

$$\lim_{\tau\to t}\frac{-r_1n_1-r_2n_2}{r^2} = \lim_{\tau\to t}\frac{(x_1(\tau)-x_1(t))x_2'(t)-(x_2(\tau)-x_2(t))x_1'(t)}{|x(t)-x(\tau)|^2|x'(t)|} = \frac{x_1''(t)x_2'(t)-x_2''(t)x_1'(t)}{2|x'(t)|^3}$$

and

$$\begin{aligned} &\lim_{\tau\to t}\frac{(r_1n_2-r_2n_1)|x'(\tau)|(\tau-t)}{r^2} \\ &= \lim_{\tau\to t}\frac{(x_1(\tau)-x_1(t))x_1'(t)+(x_2(\tau)-x_2(t))x_2'(t))|x'(\tau)|(\tau-t)}{|x(t)-x(\tau)|^2|x'(t)|} = 1 \end{aligned}$$

According to above analysis, it is easy to see that T_1 has no singularity. Therefore, the numerical formulartion for T_1 requires only straightforward application of simple quadrature formula. we choose an equidistant set of knots $t_j := \pi j/n, j = 0, ..., 2n-1$ and divide $[0, 2\pi)$ into n equivalent interval $I_i = [2j\pi/n, 2(j+1)\pi/n)$ where $[0, 2\pi) = \bigcup_{j=1}^{j=n-1} I_i$. Each I_i is described by 3 nodes of the intrinsic variable $\xi(-1 \le \xi \le 1)$ and the quadratic shape functions are:

$$\begin{aligned} A_{-1}(\xi) &= \frac{\xi(\xi-1)}{2} \\ A_0(\xi) &= 1-\xi^2 \\ A_1(\xi) &= \frac{\xi(\xi+1)}{2} \end{aligned}$$

let

$$f(x,y) = \frac{(x_1-y_1)n_2^x-(x_2-y_2)n_1^x}{|x-y|^2}$$

and it is easy to see that $(n_x^1, n_x^2) = (x_2'(t), -x_1'(t))/|x'(t)|$. Now, we can repersent the integral in variable $\xi \in [-1,1]$. For $x \in \partial D$ we have:

$$\begin{aligned} &\mathbf{P.V.}\int_{\partial D} f(x,y)\phi(y)ds(y) \\ &= \sum_{x\notin I_i}\int_{I_i} f(x,x(t))\phi(x(t))|x'(t)|dt + \mathbf{P.V.}\sum_{x\in I_i}\int_{I_i} f(x,x(t))\phi(x(t))|x'(t)|dt \end{aligned}$$

if $x = x(2j\pi/n)$, above integral becomes to

$$\sum_{i\neq j,j-1}\frac{\pi}{n}\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^i(\xi)-x_k\frac{2j\pi}{n}))x'_k(\frac{2j\pi}{n})}{|x(\frac{2j\pi}{n})-x^i(\xi)|^2}\phi(x^i(\xi))\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{2j\pi}{n})|}d\xi$$
$$+\,\mathbf{P}.\mathbf{V}.\sum_{i=j,j-1}\frac{\pi}{n}\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^i(\xi)-x_k\frac{2j\pi}{n}))x'_k(\frac{2j\pi}{n})}{|x(\frac{2j\pi}{n})-x^i(\xi)|^2}\phi(x^i(\xi))\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{2j\pi}{n})|}d\xi$$
$$\approx\sum_{i\neq j,j-1}\sum_{l=-1}^{1}\phi(x^i(l))\frac{\pi}{n}\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^i(\xi)-x_k(\frac{2j\pi}{n}))x'_k(\frac{2j\pi}{n})}{|x(\frac{2j\pi}{n})-x^i(\xi)|^2}A_l(\xi)\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{2j\pi}{n})|}d\xi$$
$$+\,\phi(x(\frac{2j\pi}{n}))\mathbf{P}.\mathbf{V}.\frac{\pi}{n}\Bigg(\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^j(\xi)-x_k^j(-1))x'_k(\frac{2j\pi}{n})}{|x^j(-1)-x^j(\xi)|^2}A_{-1}(\xi)\frac{|x'(\frac{(2j+1+\xi)\pi}{n})|}{|x'(\frac{2j\pi}{n})|}d\xi$$
$$+\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^{j-1}(\xi)-x_k^{j-1}(1))x'_k(\frac{2j\pi}{n})}{|x^{j-1}(1)-x^{j-1}(\xi)|^2}A_1(\xi)\frac{|x'(\frac{(2j-1+\xi)\pi}{n})|}{|x'(\frac{2j\pi}{n})|}d\xi\Bigg)$$

and if $x = x((2j+1)\pi/n)$, above integral becomes to

$$\sum_{i\neq j,}\frac{\pi}{n}\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^i(\xi)-x_k\frac{2j+1\pi}{n}))x'_k(\frac{2j+1\pi}{n})}{|x(\frac{2j+1\pi}{n})-x^i(\xi)|^2}\phi(x^i(\xi))\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{2j+1\pi}{n})|}d\xi$$
$$+\,\mathbf{P}.\mathbf{V}.\sum_{i=j,j-1}\frac{\pi}{n}\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^i(\xi)-x_k\frac{2j+1\pi}{n}))x'_k(\frac{2j+1\pi}{n})}{|x(\frac{2j+1\pi}{n})-x^i(\xi)|^2}\phi(x^i(\xi))\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{2j+1\pi}{n})|}d\xi$$
$$\approx\sum_{i\neq j}\sum_{l=-1}^{1}\phi(x^i(l))\frac{\pi}{n}\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^i(\xi)-x_k(\frac{(2j+1)\pi}{n}))x'_k(\frac{(2j+1)\pi}{n})}{|x(\frac{(2j+1)\pi}{n})-x^i(\xi)|^2}A_l(\xi)\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{(2j+1)\pi}{n})|}d\xi$$
$$+\,\phi(x(\frac{(2j+1)\pi}{n}))\mathbf{P}.\mathbf{V}.\frac{\pi}{n}\Bigg(\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^j(\xi)-x_k^j(0))x'_k(\frac{(2j+1)\pi}{n})}{|x^j(0)-x^j(\xi)|^2}A_0(\xi)\frac{|x'(\frac{((2j+1)+\xi)\pi}{n})|}{|x'(\frac{(2j+1)\pi}{n})|}d\xi$$

where $x^i(\xi) = x((2i+1+\xi)\pi/n)$. For simplicity, we denote a 3-D tensor M_{jil}:

$$M_{jil}=\frac{\pi}{n}\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^i(\xi)-x_k(\frac{j\pi}{n}))x'_k(\frac{j\pi}{n})}{|x(\frac{j\pi}{n})-x^i(\xi)|^2}A_l(\xi)\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{j\pi}{n})|}d\xi$$

Then, the general Matrix is

$$G_{ji}=\begin{cases} M_{j,i,-1}+M_{j,i-1,1} & j \text{ is even}\\ M_{j,i,0} & j \text{ is odd}\end{cases}$$

10. Traction Tenson of Neumann Green Function

$$\hat{\Phi}(\xi,x_2;y_2)=\frac{\mathbf{i}}{2\omega^2}\left[\begin{pmatrix}\mu_s & -\xi\frac{x_2-y_2}{|x_2-y_2|}\\ -\xi\frac{x_2-y_2}{|x_2-y_2|} & \frac{\xi^2}{\mu_s}\end{pmatrix}e^{\mathbf{i}\mu_s|x_2-y_2|}+\begin{pmatrix}\frac{\xi^2}{\mu_p} & \xi\frac{x_2-y_2}{|x_2-y_2|}\\ \xi\frac{x_2-y_2}{|x_2-y_2|} & \mu_p\end{pmatrix}e^{\mathbf{i}\mu_p|x_2-y_2|}\right]$$

$$\sigma(\Phi)(\xi,x_2;y_2)e_1=\frac{\mu}{2\omega^2}\left[\begin{pmatrix}-2\xi\mu_s & \frac{x_2-y_2}{|x_2-y_2|}2\xi^2\\ -\frac{x_2-y_2}{|x_2-y_2|}\beta & \frac{\xi\beta}{\mu_s}\end{pmatrix}e^{\mathbf{i}\mu_s|x_2-y_2|}+\begin{pmatrix}-\frac{\alpha\xi}{\mu_p} & -\frac{x_2-y_2}{|x_2-y_2|}\alpha\\ -\frac{x_2-y_2}{|x_2-y_2|}2\xi^2 & -2\xi\mu_p\end{pmatrix}e^{\mathbf{i}\mu_p|x_2-y_2|}\right]$$

$$\sigma(\Phi)(\xi,x_2;y_2)e_2=\frac{\mu}{2\omega^2}\left[\begin{pmatrix}-\frac{x_2-y_2}{|x_2-y_2|}\beta & \frac{\xi\beta}{\mu_s}\\ 2\xi\mu_s & -\frac{x_2-y_2}{|x_2-y_2|}2\xi^2\end{pmatrix}e^{\mathbf{i}\mu_s|x_2-y_2|}+\begin{pmatrix}-\frac{x_2-y_2}{|x_2-y_2|}2\xi^2 & -2\xi\mu_p\\ -\frac{\beta\xi}{\mu_p} & -\frac{x_2-y_2}{|x_2-y_2|}\beta\end{pmatrix}e^{\mathbf{i}\mu_p|x_2-y_2|}\right]$$

where $\alpha(\xi)=k_s^2-2\mu_p^2$, $\beta(\xi)=k_s^2-2\xi^2$.

$$\hat{N}(\xi,x_2;y_2)=\hat{\Phi}(\xi,x_2;y_2)-\hat{\Phi}(\xi,x_2;-y_2)+\hat{N}_c(\xi,x_2;y_2) \tag{10.1}$$

$$\hat{N}_c(\xi,x_2;y_2)=\frac{\mathbf{i}}{\omega^2\delta(\xi)}\Big\{A(\xi)e^{\mathbf{i}\mu_s(x_2+y_2)}+B(\xi)e^{\mathbf{i}\mu_p(x_2+y_2)} \tag{10.2}$$
$$+C(\xi)e^{\mathbf{i}\mu_s x_2+\mu_p y_2}+D(\xi)e^{\mathbf{i}\mu_p x_2+\mu_s y_2}\Big\}$$

where

$$A(\xi)=\begin{pmatrix}\mu_s\beta^2 & -4\xi^3\mu_s\mu_p\\ -\xi\beta^2 & 4\xi_4\mu_p\end{pmatrix}\qquad B(\xi)=\begin{pmatrix}4\xi^4\mu_s & \xi\beta^2\\ 4\xi^3\mu_s\mu_p & \mu_p\beta^2\end{pmatrix}$$
$$C(\xi)=\begin{pmatrix}2\xi^2\mu_s\beta & -2\xi\mu_s\mu_p\beta\\ -2\xi^3\beta & 2\xi^2\mu_p\beta\end{pmatrix}\quad D(\xi)=\begin{pmatrix}2\xi^2\mu_s\beta & 2\xi^3\beta\\ 2\xi\mu_s\mu_p\beta & 2\xi^2\mu_p\beta\end{pmatrix}$$

The traction of integral part N_c:

$$\sigma(N_c(\xi,x_2;y_2))e_1=\frac{-\mu}{\omega^2\delta(\xi)}\Big\{A(\xi)e^{\mathbf{i}\mu_s(x_2+y_2)}+B(\xi)e^{\mathbf{i}\mu_p(x_2+y_2)}$$
$$+C(\xi)e^{\mathbf{i}\mu_s x_2+\mu_p y_2}+D(\xi)e^{\mathbf{i}\mu_p x_2+\mu_s y_2}\Big\}$$

where

$$A(\xi)=\begin{pmatrix}2\xi\mu_s\beta^2 & -8\mu_s\mu_p\xi^4\\ \beta^3 & -4\xi^3\mu_p\beta\end{pmatrix}\qquad B(\xi)=\begin{pmatrix}4\xi^3\mu_s\alpha & \alpha\beta^2\\ 8\xi^4\mu_s\mu_p & 2\xi\mu_p\beta^2\end{pmatrix}$$
$$C(\xi)=\begin{pmatrix}4\xi^3\mu_s\beta & -4\xi^2\mu_s\mu_p\beta\\ 2\xi^2\beta^2 & -2\xi\mu_p\beta^2\end{pmatrix}\quad D(\xi)=\begin{pmatrix}2\xi\mu_s\alpha\beta & 2\xi^2\alpha\beta\\ 4\xi^2\mu_s\mu_p\beta & 4\xi^3\mu_p\beta\end{pmatrix}$$

In particular, for $y_2=0$, a more simpler form are deduced:

$$\sigma(N(\xi,x_2;y_2))e_1=\frac{-1}{\delta(\xi)}\left\{\begin{pmatrix}2\xi\mu_s\beta & -4\mu_s\mu_p\xi^2\\ \beta^2 & -2\xi\mu_p\beta\end{pmatrix}e^{\mathbf{i}\mu_s x_2}+\begin{pmatrix}2\xi\mu_s\alpha & \alpha\beta\\ 4\xi^2\mu_s\mu_p & 2\xi\mu_p\beta\end{pmatrix}e^{\mathbf{i}\mu_p x_2}\right\}$$

and

$$\sigma(N_c(\xi,x_2;y_2))e_2=\frac{-\mu}{\omega^2\delta(\xi)}\Big\{A(\xi)e^{\mathbf{i}\mu_s(x_2+y_2)}+B(\xi)e^{\mathbf{i}\mu_p(x_2+y_2)}$$
$$+C(\xi)e^{\mathbf{i}\mu_s x_2+\mu_p y_2}+D(\xi)e^{\mathbf{i}\mu_p x_2+\mu_s y_2}\Big\}$$

where

$$A(\xi)=\begin{pmatrix}\beta^3 & -4\xi^3\mu_p\beta\\ -2\xi\mu_s\beta^2 & 8\mu_s\mu_p\xi^4\end{pmatrix}\qquad B(\xi)=\begin{pmatrix}8\xi^4\mu_s\mu_p & 2\xi\mu_p\beta^2\\ 4\xi^3\mu_s\beta & \beta^3\end{pmatrix}$$
$$C(\xi)=\begin{pmatrix}2\xi^2\beta^2 & -2\xi\mu_p\beta^2\\ -4\xi^3\mu_s\beta & 4\xi^2\mu_s\mu_p\beta\end{pmatrix}\quad D(\xi)=\begin{pmatrix}4\xi^2\mu_s\mu_p\beta & 4\xi^3\mu_p\beta\\ 2\xi\mu_s\beta^2 & 2\xi^2\beta^2\end{pmatrix}$$

Similarly, for $y_2 = 0$, a more simpler form are deduced:

$$\sigma(N(\xi, x_2; y_2))e_2 = \frac{-1}{\delta(\xi)} \left\{ \begin{pmatrix} \beta^2 & -2\xi\mu_p\beta \\ -2\xi\mu_s\beta & 4\mu_s\mu_p\xi^2 \end{pmatrix} e^{\mathbf{i}\mu_s x_2} + \begin{pmatrix} 4\xi^2\mu_s\mu_p & 2\xi\mu_p\beta \\ 2\xi\mu_s\beta & \beta^2 \end{pmatrix} e^{\mathbf{i}\mu_p x_2} \right\}$$

where $\delta(\xi) = \beta^2 + 4\xi^2\mu_s\mu_p$

11. reflection of Plane wave

11.1. P-wave

We denote incident P-wave [1, p172] as

$$u^0 = A_0(\sin t_0, \cos t_0)^T e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)} \tag{11.1}$$

and its stress as

$$\sigma(u^0) = \mathbf{i}k_p A_0(2\mu \sin t_0 \cos t_0, \lambda + 2\mu\cos^2 t_0)^T e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)}$$

The reflected P-wave is represented as

$$u^1 = A_1(\sin t_1, -\cos t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 - x_2 \cos t_1)}$$
$$\sigma(u^1) = \mathbf{i}k_p A_1(-2\mu \sin t_1 \cos t_1, \lambda + 2\mu\cos^2 t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 + x_2 \cos t_1)}$$

and reflected S-wave as

$$u^2 = A_2(\cos t_2, \sin t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$
$$\sigma(u^2) = \mathbf{i}k_s A_2(\mu(\sin^2 t_2 - \cos^2 t_2), -2\mu \sin t_2 \cos t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$

We consider the clamped condition, then the total field on the $x_2 = 0$ vanish:

$$u^0(x_1, 0) + u^1(x_1, 0) + u^2(x_1, 0) = 0$$

for any $x_1 \in \mathbb{R}$. A simple computation show that

$$t_1 = t_0 \qquad \text{and} \qquad \frac{\sin t_2}{\sin t_0} = \frac{k_p}{k_s} := \kappa$$
$$A_0 = \cos(t_0 - t_2) \qquad A_1 = \cos(t_0 + t_2) \qquad A_2 = -\sin 2t_0$$

11.2. S-wave

Similarly, we denote incident S-wave as

$$u^0 = A_0(-\cos t_0, \sin t_0)^T e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)} \tag{11.2}$$
$$\sigma(u^0) = \mathbf{i}k_s(\mu(\sin^2 t_0 - \cos^2 t_0), 2\mu \sin t_0 \cos t_0) e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)} \tag{11.3}$$

The reflected P-wave is represented as

$$u^1 = A_1(\sin t_1, -\cos t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 - x_2 \cos t_1)}$$
$$\sigma(u^1) = \mathbf{i}k_p A_1(-2\mu \sin t_1 \cos t_1, \lambda + 2\mu\cos^2 t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 + x_2 \cos t_1)}$$

and reflected S-wave as

$$u^2 = A_2(\cos t_2, \sin t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$
$$\sigma(u^2) = \mathbf{i}k_s A_2(\mu(\sin^2 t_2 - cos^2 t_2), -2\mu \sin t_2 \cos t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$

The result is

$$t_2 = t_0 \qquad \text{and} \qquad \frac{\sin t_1}{\sin t_0} = \frac{k_s}{k_p} = \frac{1}{\kappa}$$
$$A_0 = \cos(t_0 - t_1) \qquad A_1 = \sin 2t_0 \qquad A_2 = \cos(t_0 + t_1)$$

12. scattering relation of elastic wave

The solution for the scattering of a plane P-wave u_p(or S-wave u_s) with incident direction d_0 at a plane $\Gamma := x \in \mathbb{R}^2 : x \cdot \nu = 0$ through the origin with normal vector ν is described by

$$u = u_p + u_{p,p} + u_{p,s} = A_0 d_0 e^{\mathbf{i}kpx\cdot d} + A_1 d_1 e^{\mathbf{i}kpx\cdot d_1} + A_2 d_2^\perp d_0^{\mathbf{i}ksx\cdot d_2} \tag{12.1}$$
$$u = u_s + u_{s,p} + u_{s,s} = A_0 d_0^\perp e^{\mathbf{i}ksx\cdot d} + A_1 d_1 e^{\mathbf{i}kpx\cdot d_1} + A_2 d_2^\perp d^{\mathbf{i}ksx\cdot d_2} \tag{12.2}$$

where $d_i = (d_i^1, d_i^2)^T$ are unit vectors, $d_i^\perp = (d_i^2, -d_i^1)^T$ and A_i are corresponding amplitude. For fixed boundary, we have $u = 0$ for $x \in \Gamma$. After a standard computation, we get for P-wave:

$$d_1 = d_0 - 2\alpha\nu \tag{12.3}$$
$$d_2 = \kappa d_0 - \beta\nu \tag{12.4}$$
$$A_0 = \kappa(d,\nu)^2 - \kappa(d,\nu^\perp)^2 - \beta(d,\nu) \tag{12.5}$$
$$A_1 = \kappa - \beta(d,\nu) \tag{12.6}$$
$$A_2 = -2(d,\nu)(d,\nu^\perp) \tag{12.7}$$

where $\alpha = (d,\nu)$, $\beta = \kappa\alpha - \sqrt{\kappa^2\alpha^2 - \kappa^2 + 1}$ and $\kappa = k_p/k_s$. For S-wave:

$$d_1 = \kappa_1 d_0 - \gamma\nu \tag{12.8}$$
$$d_2 = d_0 - 2\alpha\nu \tag{12.9}$$
$$A_0 = \kappa_1(d,\nu)^2 - \kappa_1(d,\nu^\perp)^2 - \gamma(d,\nu) \tag{12.10}$$
$$A_1 = 2(d,\nu)(d,\nu^\perp) \tag{12.11}$$
$$A_2 = \kappa_1 - \gamma(d,\nu) \tag{12.12}$$

where $\gamma = \kappa_1\alpha - \sqrt{\kappa_1^2\alpha^2 - \kappa_1^2 + 1}$ and $\kappa_1 = 1/\kappa$. Thus the traction of $u(x)$ on the plane Γ can be obtained. For P-wave

$$\sigma(u)\cdot\nu = [\mathbf{i}k_p A_0(\lambda\nu + 2\mu(d_0,\nu)d_0) + \mathbf{i}k_p A_1(\lambda\nu + 2\mu(d_1,\nu)d_1)$$
$$+\mathbf{i}k_s A_2\mu((d_2,\nu)d_2^\perp + (d_2^\perp,\nu)d_2)]e^{\mathbf{i}k_p x\cdot d} := \mathbf{i}k_p R f_p(x,d,\nu)e^{\mathbf{i}k_p x\cdot d}$$

For S-wave

$$\sigma(u)\cdot\nu = [\mathbf{i}k_s A_0\mu((d_0,\nu)d_0^\perp + (d_0^\perp,\nu)d_0) + \mathbf{i}k_p A_1(\lambda\nu + 2\mu(d_1,\nu)d_1)$$
$$+\mathbf{i}k_s A_2\mu((d_2,\nu)d_2^\perp + (d_2^\perp,\nu)d_2)]e^{\mathbf{i}k_s x\cdot d} := \mathbf{i}k_s R f_s(x,d,\nu)e^{\mathbf{i}k_s x\cdot d}$$

Definition 12.1 *For any unit vector $d \in \mathbb{R}^2$, let $u_p^i = de^{\mathbf{i}k_p x\cdot d}$ or $u_s^i = d^\perp e^{\mathbf{i}k_s x\cdot d}$ be the incident wave and $u_\alpha^s = u_\alpha^s(x;d)$ be the radiation solution of the Navier equation:*

$$u_\alpha^s + \omega^2 u_\alpha^s = 0 \quad in \quad \mathbb{R}^2\backslash\bar{D} \tag{12.13}$$
$$u_\alpha^s = -u_\alpha^i \quad on \quad \partial D \tag{12.14}$$

The scattering coecient R(x;d) for $x \in \partial D$ *is defined by the relation*

$$\sigma(u_\alpha^s + u_\alpha^i) \cdot \nu = \mathbf{i} k_\alpha R_\alpha(x; d) e^{\mathbf{i} k_\alpha x \cdot d} \quad on \;\; \partial D$$

where $\alpha = p, s$.

In the case of high frequency approximation, the scattering coecient can be approximated by

$$R_\alpha(x; d) = \begin{cases} Rf_\alpha(x; d, \nu) & \text{if } \; x \in \partial D_d^- = \{x \in \partial D, \nu(x) \cdot d < 0\}, \\ 0 & \text{if } \; x \in \partial D_d^- = \{x \in \partial D, \nu(x) \cdot d \geq 0\}. \end{cases}$$

13. Difference of solution of naviar equation in full-space and half-space,0105

For any $0 < \varepsilon < 1$, we consider the problem

$$\Delta_e u_1^\varepsilon + (1 + \mathbf{i}\varepsilon)\omega^2 u_1^\varepsilon = 0 \qquad \text{in } \mathbb{R}_+^2 \backslash \bar{D} \tag{13.1}$$

$$u_1^\varepsilon = g \quad \text{ on } \Gamma_D \tag{13.2}$$

$$\sigma(u_\varepsilon^1) e_2 = 0 \quad \text{ on} \Gamma_0 \tag{13.3}$$

and

$$\Delta_e u_2^\varepsilon + (1 + \mathbf{i}\varepsilon)\omega^2 u_2^\varepsilon = 0 \qquad \text{in } \mathbb{R}^2 \backslash \bar{D} \tag{13.4}$$

$$u_2^\varepsilon = g \quad \text{ on } \Gamma_D \tag{13.5}$$

Let $w^\varepsilon(x)$ be the solution of the problem:

$$\Delta_e w^\varepsilon + (1 + \mathbf{i}\varepsilon)\omega^2 w^\varepsilon = 0 \qquad \text{in } \mathbb{R}_+^2 \tag{13.6}$$

$$\sigma(w^\varepsilon) e_2 = -\sigma(u_2^\varepsilon) e_2 \quad \text{ on} \Gamma_0 \tag{13.7}$$

Then $u_1^\varepsilon - u_2^\varepsilon - w^\varepsilon$ satisfies (13.1),(13.3) with the boundary condition $u_1^\varepsilon - u_2^\varepsilon - w^\varepsilon = -w^\varepsilon$ on Γ_D. Thus by the limiting absorption principle, we have

$$\|T_x^\nu(u_1^\varepsilon - u_2^\varepsilon)\|_{H^{-1/2}(\Gamma_D)} \leq C(\|w^\varepsilon\|_{H^{1/2}(\Gamma_D)} + |T_x^\nu(w^\varepsilon)\|_{H^{-1/2}(\Gamma_D)}) \tag{13.8}$$

$$\leq C \max_{x \in D}(|w^\varepsilon(x)| + d_D |\nabla w^\varepsilon(x|) \tag{13.9}$$

where C is independant of ε, ω. By the integral representation formula we have for any $z \in \Gamma_0$

$$u_2^\varepsilon(z) = \int_{\Gamma_D} (T_y^\nu \Phi^\varepsilon(y, z))^T u_2^\varepsilon(y) - \Phi^\varepsilon(z, y)(T_y^\nu u_2^\varepsilon(y)) ds(y) \tag{13.10}$$

which yields by using the integral representation again that for $x \in D$

$$w^\varepsilon(x) = \int_{\Gamma_0} N^\varepsilon(x, z)(T_z^{e_2} u_2^\varepsilon(z)) ds(z) \tag{13.11}$$

$$= \int_{\Gamma_D} ds(y) \int_{\Gamma_0} N^\varepsilon(x, z)(T_z^{e_2}((T_y^\nu \Phi^\varepsilon(y, z))^T)) ds(z) \tag{13.12}$$

$$- \int_{\Gamma_D} v^\varepsilon(x, y)(T_y^\nu u_2^\varepsilon(y)) ds(y) \tag{13.13}$$

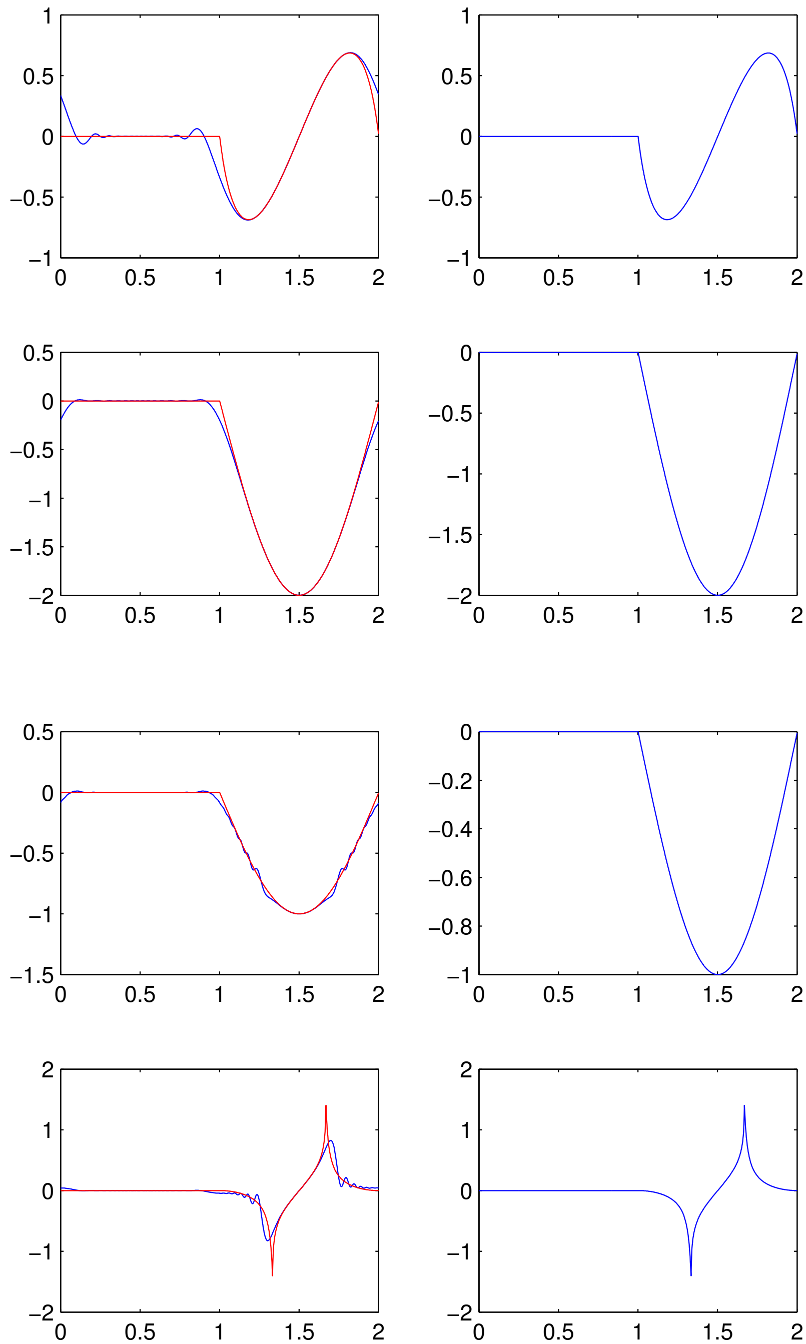

Figure 1. $\theta = 0\pi$

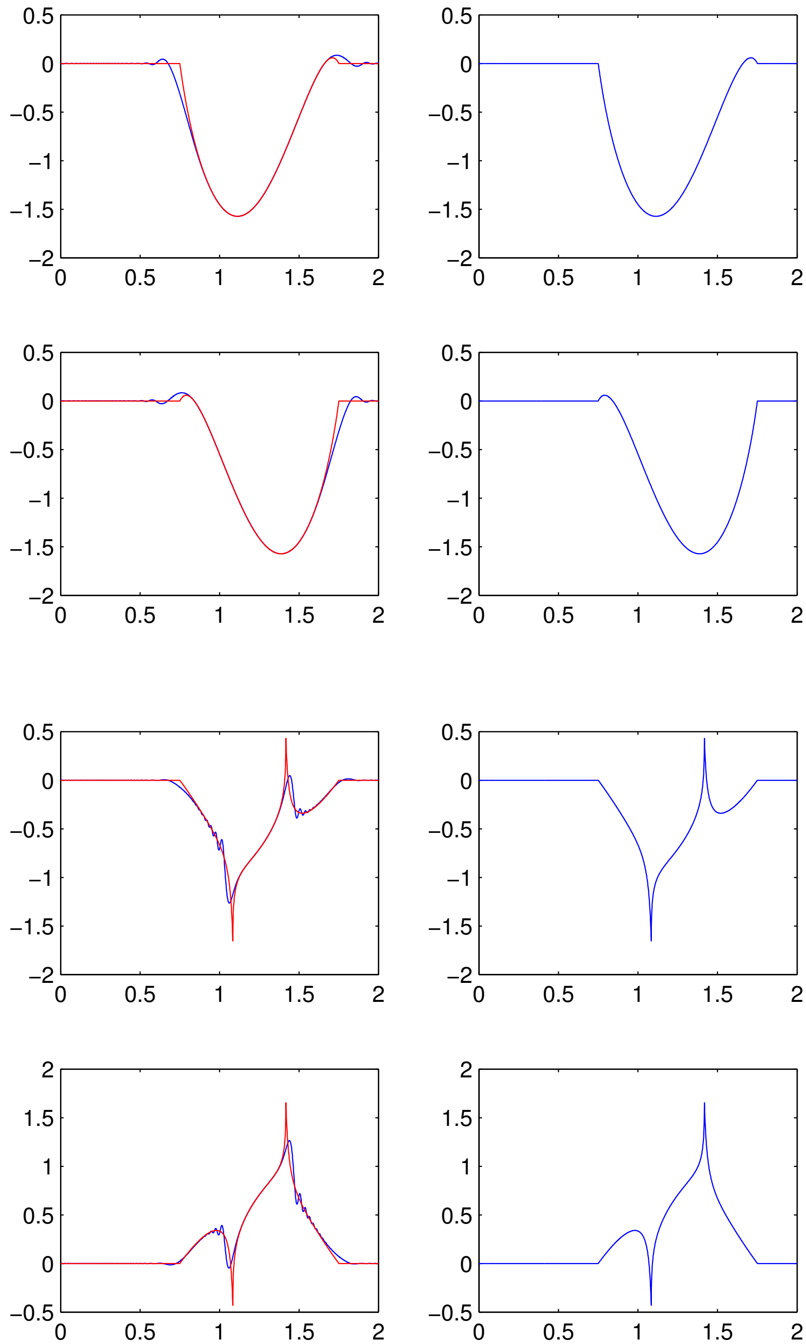

Figure 2. $\theta = pi/4$

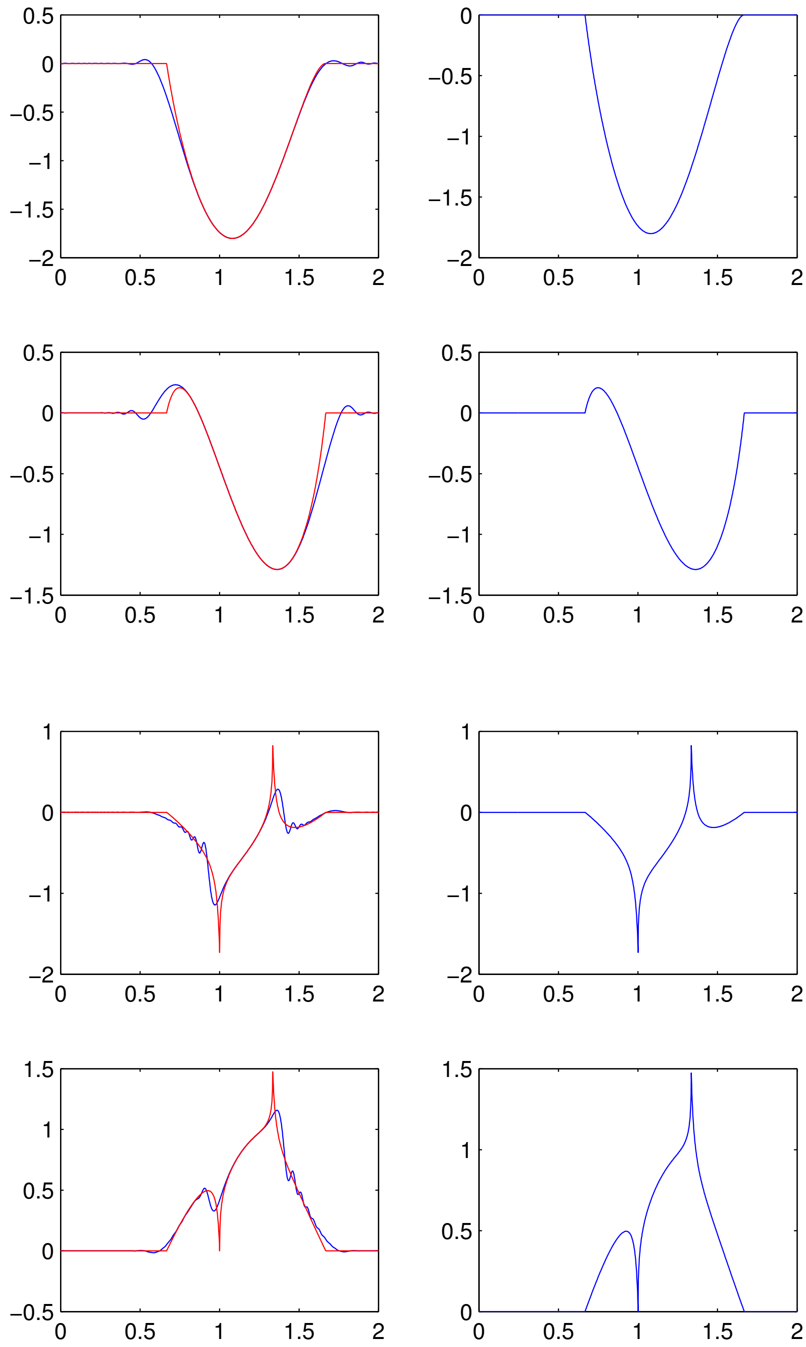

Figure 3. $\theta = \pi/3$

$$= \int_{\Gamma_D} ds(y) \int_{\Gamma_0} N^\varepsilon(x,z)(T_z^{e_2}(\Phi^\varepsilon(y,z)^T(T_y^\nu)^T)ds(z) \tag{13.14}$$

$$- \int_{\Gamma_D} v^\varepsilon(x,y)(T_y^\nu u_2^\varepsilon(y))ds(y) \tag{13.15}$$

$$= \int_{\Gamma_D} ds(y) \int_{\Gamma_0} N^\varepsilon(x,z)(T_y^\nu(T_z^{e_2}\Phi^\varepsilon(z,y))^T)^T ds(z) \tag{13.16}$$

$$- \int_{\Gamma_D} v^\varepsilon(x,y)(T_y^\nu u_2^\varepsilon(y))ds(y) \tag{13.17}$$

$$= \int_{\Gamma_D} (T_y^\nu(v^\varepsilon(x,y))^T)^T u_2^\varepsilon(y) - v^\varepsilon(x,y)(T_y^\nu u_2^\varepsilon(y))ds(y) \tag{13.18}$$

where

$$v^\varepsilon(x,y) = \int_{\Gamma_0} N^\varepsilon(x,z)(T_z^{e_2}\Phi^\varepsilon(z,y))ds(z) \tag{13.19}$$

Since $\|T_x^\nu(u_2^\varepsilon)\|_{H^{-1/2}(\Gamma_D)} \leq C\|g\|_{H^{1/2}(\Gamma_D)}$, we obtain

$$|w^\varepsilon(x)| \leq C\|g\|_{H^{1/2}(\Gamma_D)} \max_{x\in D}(|v^\varepsilon(x,y)| + d_D|\nabla_y v^\varepsilon(x,y)|) \tag{13.20}$$

and

$$|\nabla w^\varepsilon(x)| \leq C\|g\|_{H^{1/2}(\Gamma_D)} \max_{x\in D}(|\nabla_x v^\varepsilon(x,y)| + d_D|\nabla_x\nabla_y v^\varepsilon(x,y)|) \tag{13.21}$$

By (13.8) and letting $\varepsilon \to 0^+$, we have

$$\|T_x^\nu(u_1 - u_2)\|_{H^{-1/2}(\Gamma_D)} \leq C\|g\|_{H^{1/2}(\Gamma_D)} \max_{x\in D} \lim_{\varepsilon\to 0^+}(|v^\varepsilon(x,y)| \tag{13.22}$$

$$+d_D|\nabla_y v^\varepsilon(x,y)| + d_D|\nabla_x v^\varepsilon(x,y)| + d_D^2|\nabla_x\nabla_y v^\varepsilon(x,y)|) \tag{13.23}$$

where u_1 is the scattering solution in the half-space and u_2 in the full-space. Now, it turns to estimate $v^\varepsilon(x,y)$. Applying the Fourier transformation to the first horizontal variable of $N^\varepsilon(z,x)$ and$T_z^{e_2}\Phi^\varepsilon(z,y)$, we have

$$\mathcal{F}[N^\varepsilon](\xi,0;x) = \frac{\mathbf{i}}{\mu\delta(\xi)}\left[\begin{pmatrix} 2\xi^2\mu_s & -2\xi\mu_s\mu_p \\ -\xi\beta & \mu_p\beta \end{pmatrix} e^{\mathbf{i}\mu_p x_2} + \begin{pmatrix} \mu_s\beta & \xi\beta \\ 2\xi\mu_s\mu_p & 2\xi^2\mu_p \end{pmatrix} e^{\mathbf{i}\mu_s x_2}\right] e^{-\mathbf{i}\xi x_1}$$

$$\mathcal{F}[T_z^{e_2}\Phi^\varepsilon](\xi,0;y) = \frac{\mu}{2\omega^2}\left[\begin{pmatrix} 2\xi^2 & -2\xi\mu_p \\ -\frac{\beta\xi}{\mu_p} & \beta \end{pmatrix} e^{\mathbf{i}\mu_p y_2} + \begin{pmatrix} \beta & \frac{\xi\beta}{\mu_s} \\ 2\xi\mu_s & 2\xi^2 \end{pmatrix} e^{\mathbf{i}\mu_s y_2}\right] e^{-\mathbf{i}\xi y_1}$$

Using Parseval identity combined with above two formula, we have

$$\lim_{\varepsilon\to 0^+} v^\varepsilon(x,y) = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} \mathcal{F}[N^\varepsilon](\xi,0;x)^T \mathcal{F}[T_z^{e_2}\Phi^\varepsilon](-\xi,0;y)d\xi$$

Lemma 13.1 *For any $x, y \in D$, let*

$$p(x,y) = \lim_{\varepsilon\to 0^+} p^\varepsilon(x,y) := \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} \frac{f(\mu_p^\varepsilon,\mu_s^\varepsilon,\xi)}{\delta^\varepsilon(\xi)} e^{\mathbf{i}\mu_\alpha^\varepsilon x_2 + \mathbf{i}\mu_\beta^\varepsilon y_2 + \mathbf{i}\xi(y_1 - x_1)}d\xi$$

where $f(a,b,c)$ is a homogeneous fifth order polynomial with repect to a,b,c and $\alpha = s,p$, $\beta = s,p$.Then there exists a constant $C > 0$ only dependent on κ such that

$$|p(x,y)| + k_s^{-1}|\nabla_x p(x,y)| + k_s^{-1}|\nabla_y p(x,y)| + k_s^{-2}|\nabla_x\nabla_y p(x,y)| \leq C((k_s h)^{-1/2} + e^{-\sqrt{k_R^2 - k_s^2}h})$$

uniformly for $x, y \in D$.

Proof. Without loss of generality, we assume $k_\alpha \le k_\beta$. Then we can divide $p(x,y)$ into two parts:

$$
\begin{aligned}
p(x,y) &= \lim_{\varepsilon\to 0^+}\int_{I_1}+\int_{I_2}\frac{f(\mu_p^\varepsilon,\mu_s^\varepsilon,\xi)}{(k_\alpha^\varepsilon)^2\delta^\varepsilon(\xi)}e^{\mathbf{i}\mu_\alpha^\varepsilon x_2+\mathbf{i}\mu_\beta^\varepsilon y_2+\mathbf{i}\xi(y_1-x_1)}d\xi\\
&= \int_{I_1}\frac{f(\mu_p,\mu_s,\xi)}{k_\alpha^2\delta(\xi)}e^{\mathbf{i}\mu_\alpha x_2+\mathbf{i}\mu_\beta y_2+\mathbf{i}\xi(y_1-x_1)}d\xi\\
&+ \lim_{\varepsilon\to 0^+}\int_{I_2}\frac{f(\mu_p^\varepsilon,\mu_s^\varepsilon,\xi)}{(k_\alpha^\varepsilon)^2\delta^\varepsilon(\xi)}e^{\mathbf{i}\mu_\alpha^\varepsilon x_2+\mathbf{i}\mu_\beta^\varepsilon y_2+\mathbf{i}\xi(y_1-x_1)}d\xi\\
&= p_1(x,y)+p_2(x,y)
\end{aligned}
$$

where $I_1=(-k_\alpha,k_\alpha)$, $I_2=(-2k_R+k_\alpha,k_\alpha)\cup(k_\alpha,2k_R-k_\alpha)$ and $I_2=R\backslash[-k_\alpha,k_\alpha]$. Substituting $\xi=k_\alpha t$ into $p_1(x,y)$, we get

$$
p_1(x,y)=\int_{-1}^{1}\frac{f(\mu_p(k_\alpha t),\mu_s(k_\alpha t),k_\alpha t)}{k_\alpha\delta(k_\alpha t)}e^{\mathbf{i}k_\alpha x_2(\sqrt{1-t^2}+\tau\sqrt{\varsigma^2-t^2}+\gamma t)}dt
$$

where $\tau=y_2/x_2$, $\varsigma=k_\beta/k_\alpha$ and $\gamma=(y_1-x_1)/x_2$. It is easy to see that the phase function $\phi(t)=\sqrt{1-t^2}+\tau\sqrt{\varsigma^2-t^2}+\gamma t$ satifies $|\phi''(t)|\ge 1/(1-t^2)^{3/2}\ge 1$ for $t\in(-1,1)$. Then we can obtain $|p_1(x,y)|\le C1/(k_sh)^{1/2}$ by lemma 6.1.

For $p_2(x,y)$, by changing the integration path and using same argument as in the proof of estimate for psf, we can easily obtain:

$$
|p_2(x,y)|\le C(\frac{1}{k_sh}+e^{-\sqrt{k_R^2-k_s^2}}h)
$$

This completes the proof of the esitmate for $|p(x,y)|$. The other estimates can be proved by a similar argument. We omit the details □

Lemma 13.2 *For any $x,y\in D$, let*

$$
p(x,y)=\lim_{\varepsilon\to 0^+}p^\varepsilon(x,y):=\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}\frac{f(\mu_p^\varepsilon,\mu_s^\varepsilon,\xi)}{\delta^\varepsilon(\xi)}e^{\mathbf{i}\mu_\alpha^\varepsilon x_2+\mathbf{i}\mu_\beta^\varepsilon y_2+\mathbf{i}\xi(y_1-x_1)}d\xi
$$

where $f(a,b,c)$ is a homogeneous fifth order polynomial with repect to a,b,c and $\alpha=s,p$, $\beta=s,p$.Then there exists a constant $C>0$ only dependant on κ such that

$$
|p(x,y)|+k_s^{-1}|\nabla_x p(x,y)|+k_s^{-1}|\nabla_y p(x,y)|+k_s^{-2}|\nabla_x\nabla_y p(x,y)|\le C(1+k_sd_D)((k_sh)^{-1/2}+e^{-\sqrt{k_R^2-k_s^2}h})
$$

uniformly for $x,y\in D$.

Proof. Without loss of generality, we assume $k_\alpha\le k_\beta$. Then we can divide $p(x,y)$ into four parts:

$$
\begin{aligned}
p(x,y) &= \lim_{\varepsilon\to 0^+}\int_{I_1}+\int_{I_2}+\int_{I_3}\frac{f(\mu_p^\varepsilon,\mu_s^\varepsilon,\xi)}{(k_\alpha^\varepsilon)^2\delta^\varepsilon(\xi)}e^{\mathbf{i}\mu_\alpha^\varepsilon x_2+\mathbf{i}\mu_\beta^\varepsilon y_2+\mathbf{i}\xi(y_1-x_1)}d\xi\\
&= \int_{I_1}+\mathrm{PV}\int_{I_2}+\int_{I_3}\frac{f(\mu_p,\mu_s,\xi)}{k_\alpha^2\delta(\xi)}e^{\mathbf{i}\mu_\alpha x_2+\mathbf{i}\mu_\beta y_2+\mathbf{i}\xi(y_1-x_1)}d\xi\\
&+ \mathbf{i}\pi(\frac{f(\mu_p(k_R),\mu_s(k_R),k_R)}{k_\alpha^2\delta'(k_R)}e^{\mathbf{i}\mu_\alpha(k_R)x_2+\mathbf{i}\mu_\beta(k_R)y_2+\mathbf{i}k_R(y_1-x_1)}
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{f(\mu_p(k_R), \mu_s(k_R), -k_R)}{k_\alpha^2 \delta'(-k_R)} e^{\mathbf{i}\mu_\alpha(k_R)x_2 + \mathbf{i}\mu_\beta(k_R)y_2 - \mathbf{i}k_R(y_1 - x_1)}) \\
: \quad & = p_1(x,y) + p_2(x,y) + p_3(x,y) + p_4(x,y)
\end{aligned}
$$

where $I_1 = (-k_\alpha, k_\alpha)$, $I_2 = (-2k_R + k_\alpha, k_\alpha) \cup (k_\alpha, 2k_R - k_\alpha)$ and $I_2 = R \backslash [-2k_R + k_\alpha, 2k_R - k_\alpha]$. Substituting $\xi = k_\alpha t$ into $p_1(x,y)$, we get

$$
p_1(x,y) = \int_{-1}^{1} \frac{f(\mu_p(k_\alpha t), \mu_s(k_\alpha t), k_\alpha t)}{k_\alpha \delta(k_\alpha t)} e^{\mathbf{i}k_\alpha x_2(\sqrt{1-t^2} + \tau\sqrt{\varsigma^2 - t^2} + \gamma t)} dt
$$

where $\tau = y_2/x_2$, $\varsigma = k_\beta/k_\alpha$ and $\gamma = (y_1 - x_1)/x_2$. It is easy to see that the phase function $\phi(t) = \sqrt{1-t^2} + \tau\sqrt{\varsigma^2 - t^2} + \gamma t$ satifies $|\phi''(t)| \geq 1/(1-t^2)^{3/2} \geq 1$ for $t \in (-1, 1)$. Then we can obtain $|p_1(x,y)| \leq C1/(k_s h)^{1/2}$ by lemma 6.1.

Let

$$
g_\pm(\xi) = \frac{f(\mu_p, \mu_s, \xi)(\xi \pm k_R)}{\delta(\xi)} e^{\mathbf{i}\mu_\alpha x_2 + \mathbf{i}\mu_\beta y_2 + \mathbf{i}\xi(y_1 - x_1)} d\xi
$$

Then by the definition of cauchy principle value, we have

$$
p_2(x,y) = \int_{-2k_R + k_\alpha}^{-k_\alpha} \frac{g_-(\xi) - g_-(-k_R)}{x + k_R} d\xi + \int_{k_\alpha}^{2k_R - k_\alpha} \frac{g_+(\xi) - g_+(k_R)}{x - k_R} d\xi
$$

□

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