1. New proof for Stationary Phase Method

Preminary:

$$\begin{aligned}
&\cos\theta = 1 - 2\sin^2\frac{\theta}{2} := 1 - t^2\\
&t = e^{-\mathbf{i}\frac{\pi}{4}}s\\
&\sin\theta(s) := S(s) = e^{-\mathbf{i}\frac{\pi}{4}}s(2+\mathbf{i}s^2)^{-1/2}\cos\phi + (1+\mathbf{i}s^2)\sin\phi\\
&\cos\theta(s) := C(s) = (1+\mathbf{i}s^2)\cos\phi - e^{-\mathbf{i}\frac{\pi}{4}}s(2+\mathbf{i}s^2)^{-1/2}\sin\phi\\
&\cos\frac{\theta(s)}{2} = (2-(t(s))^2)^{1/2} = (2+\mathbf{i}s^2)^{1/2}
\end{aligned}$$

Let $f(\xi) := h(\xi, \mu(\xi), \mu_\kappa(\xi))$ be a analytic function with respect to ξ in $\mathbb{C}\backslash\{\mathbf{i}\mathbb{R}\cup(-1,1)\}$. For any $a, b > 0$, we denote

$$I(f; a, b) = \int_{\mathbb{R}} f(\xi)e^{\mathbf{i}a\xi+\mathbf{i}b\mu(\xi)}d\xi$$

where $\mu(\xi) = (1-\xi^2)^{1/2}$, $\mu_\kappa(\xi) = (\kappa-\xi^2)^{1/2}$.

Lemma 1.1 *Let $a, b > 0$, $\rho = \sqrt{a^2+b^2}$, and $f(\xi) := h(\xi, \mu(\xi), \mu_\kappa(\xi))$ be a analytic function in $\mathbb{C}\backslash\{\mathbf{i}\mathbb{R}\cup(-1,1)\}$. Then*

$$I(f; a, b) = \sqrt{\frac{2}{\rho}}e^{\mathbf{i}\rho-\mathbf{i}\pi/4}\int_{\mathbb{R}} F(\frac{t}{\sqrt{\rho}})C(\frac{t}{\sqrt{\rho}})e^{-t^2}dt + O(\rho^{-3/2})\|F(\frac{t}{\sqrt{\rho}})C(\frac{t}{\sqrt{\rho}})t^2e^{-t^2}\|_{L^1(\mathbb{R})}$$

where $F(s) = h(S(s), C(s), \mu_\kappa(S(s))$ and $\sin\phi = a/\rho, \cos\phi = b/\rho$.

Proof. To simplify the integral, the standard substitution $\xi = k_s\sin\theta$ is made, taking the ξ-plane to a strip $-\pi/2 < \operatorname{Re}\theta < \pi/2$ in the θ-plane, and the real axis in the ξ-plane onto the path L from $-\pi/2 + \mathbf{i}\infty \to -\pi/2 \to \pi/2 \to \pi/2 - \mathbf{i}\infty$ in the $\theta - plane$. The integral $I(f; a, b)$ then becomes(Let a=$\rho\sin\phi$ and b=$\rho\cos\phi$, $0 < \phi < \pi/2$)

$$I(f; a, b) = \int_L h(\sin\theta, \cos\theta, \mu_\kappa(\sin\theta))\cos\theta\ e^{\mathbf{i}\rho(\cos(\theta-\phi))}d\theta \tag{1.1}$$

Taking the shift transformation of θ and using cauchy integral theorem, we can obtain the more useful representation of $I(f; a, b)$:

$$I(f; a, b) = \int_L f(\sin(\theta+\phi))\cos(\theta+\phi)\ e^{\mathbf{i}\rho\cos\theta}d\theta \tag{1.2}$$

Notice that $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$, by substituting $\theta(t) = 2\arctan\frac{\sqrt{2}t}{2}$, we get:

$$I(f; a, b) = e^{\mathbf{i}\rho}\int_{L_1\cup(-1,1)\cup L_2} f(\sin(\theta(t)+\phi))\cos(\theta(t)+\phi)\frac{2}{(2-t^2)^{1/2}}\ e^{-\mathbf{i}\rho t^2}dt$$

where

$$\begin{aligned}
L1 &= \{t|(\operatorname{Re}t)^2 - (\operatorname{Im}t)^2 = 1, \operatorname{Re}t < 0, \operatorname{Im}t > 0\}\\
L2 &= \{t|(\operatorname{Re}t)^2 - (\operatorname{Im}t)^2 = 1, \operatorname{Re}t > 0, \operatorname{Im}t < 0\}
\end{aligned}$$

and the geometry is depicted in Figure 1. A simple computation show that the substitution $\theta(t) = 2\arctan\frac{\sqrt{2}t}{2}$ transform the domain $\Omega_\theta = \{\theta||\operatorname{Re}\theta| < \pi, \operatorname{Re}\theta\cdot\operatorname{Im}\theta <$

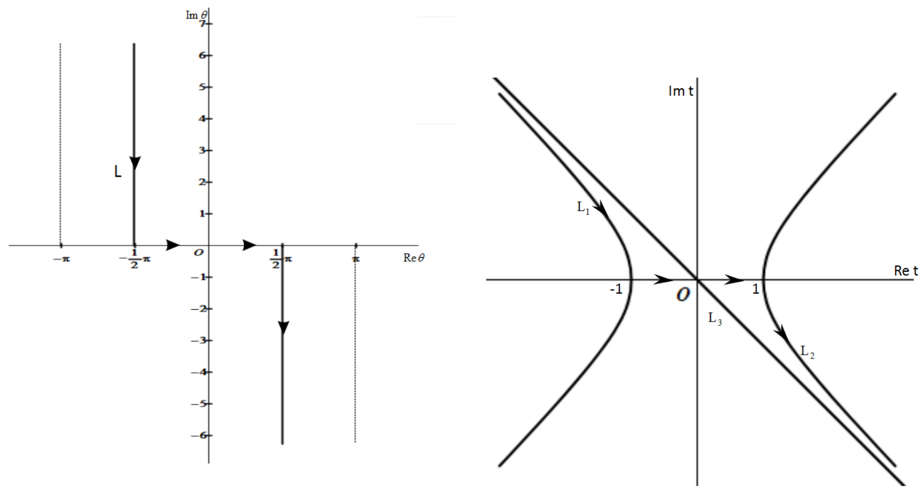

Figure 1. integral path in $\theta - plane$ and t-plane

$0\}$ in the θ-plane into $\Omega_t = \{t|\mathrm{Re}\,t \cdot \mathrm{Im}\,t < 0\}$ in t-plane. Now it is easy to see that $f(\sin(\theta(t)_\phi))$ is analytic in the domain Ω_t. Since Ω_t is surrounded by $L_1 \cup L_2 \cup (-1,1)$ and the diagonal line of the second and the fourth quadrants denote by $L3$, by using Cauchy integral theorem, we have

$$
\begin{aligned}
I(f;a,b) &= e^{\mathbf{i}\rho} \int_{L_3} f(\sin(\theta(t)+\phi)) \cos(\theta(t)+\phi) \frac{2\cos(\theta(t)+\phi)}{(2-t^2)^{1/2}}\ e^{-\mathbf{i}\rho t^2} dt \\
&= e^{\mathbf{i}\rho - \mathbf{i}\pi/4} \int_{\mathbb{R}} f(\sin(\theta(e^{-\mathbf{i}\pi/4 s})+\phi)) \frac{2\cos(\theta(e^{-\mathbf{i}\pi/4 s})+\phi)}{(2+\mathbf{i}s^2)^{1/2}}\ e^{-\rho s^2} ds \\
&= e^{\mathbf{i}\rho - \mathbf{i}\pi/4} \int_{\mathbb{R}} f(S(s)) \frac{2C(s)}{(2+\mathbf{i}s^2)^{1/2}}\ e^{-\rho s^2} ds \\
&= \sqrt{\frac{2}{\rho}} e^{\mathbf{i}\rho - \mathbf{i}\pi/4} \int_{\mathbb{R}} f(S(\frac{t}{\sqrt{\rho}})) C(\frac{t}{\sqrt{\rho}}) (1+\mathbf{i}\frac{t^2}{2\rho})^{-1/2}\ e^{-t^2} dt
\end{aligned}
$$

The lemma follows immediately from the fact that $(1+\mathbf{i}s)^{-1/2} = 1 + O(|s|), s \in \mathbb{R}$. The proof is completed. $\square$

The following lemma is a directed consequence of lemma 1.1

Lemma 1.2 *Let p(x,y,z) be a homogeneous polynomial of degree 2 and* $f(\xi) = p(\xi, \mu(\xi), \mu_\kappa(\xi))/(\xi^2 + \mu(\xi)\mu_\kappa(\xi))$. *Then for* $\rho > 1$, *we have*

$$
|I(f;a,b)| \le C(\frac{b}{\rho}\rho^{-1/2} + \frac{a}{\rho}\rho^{-5/4} + \rho^{-3/2})
$$

where C *is a constant independant of* a, b.

Proof. By lemme 1.1, it suffice to estimate the integral $I(\rho)$ where

$$
\begin{aligned}
I(\rho) &= \int_{\mathbb{R}} F(\frac{t}{\sqrt{\rho}}) C(\frac{t}{\sqrt{\rho}}) e^{-t^2} dt \\
&= \cos\phi \int_{\mathbb{R}} F(\frac{t}{\sqrt{\rho}})(1+\mathbf{i}\frac{t^2}{\rho}) e^{-t^2} dt - \frac{1}{\rho}\sin\phi \int_{\mathbb{R}} e^{-\mathbf{i}\pi/4} F(\frac{t}{\sqrt{\rho}})(2+\mathbf{i}\frac{t^2}{\rho})^{1/2} t e^{-t^2} dt \\
&:= \frac{b}{\rho}\phi I_1(\rho) - \frac{1}{\rho}\frac{a}{\rho} e^{-\mathbf{i}\pi/4} I_2(\rho)
\end{aligned}
$$

For $s \in \mathbb{R}$, it is easy to check that

$$\max\{|S(s)|, |C(s)|\} \leq |s(2+\mathbf{i}s^2)^{1/2}| + |1+\mathbf{i}s^2| \leq C(1+s+s^2)$$
$$|\mu_\kappa(C(s))| \leq C(1+|S(s)|) \leq C(1+s+s^2)$$

where C is independent of ϕ. Consequently, for $\rho > 1$, we obtain

$$|I_1(\rho)| \leq \int_{\mathbb{R}} |p(S(\frac{t}{\sqrt{\rho}}), (\frac{t}{\sqrt{\rho}}), \mu_\kappa(S(\frac{t}{\sqrt{\rho}})))|(1+\frac{t^2}{\rho})e^{-t^2}dt$$
$$\leq C\int_{\mathbb{R}} \sum_{k=0}^{6} \frac{t}{\sqrt{\rho}} e^{-t^2}dt \leq C$$

Before estimating $I_2(\rho)$, we need to deal with term $\mu_\kappa(S(s))$. Let $\kappa = \sin_\kappa, 0 < \theta_\kappa < \pi/2$, then we have

$$|\mu_\kappa(S(s))|^2 = |\sin^2\theta(s) - \sin^2(\theta(s)+\phi)|$$
$$= 4|\sin\frac{\theta(s)+\theta_\kappa+\phi}{2}||\cos\frac{\theta(s)+\theta_\kappa+\phi}{2}||\cos\frac{\theta(s)-\theta_\kappa+\phi}{2}||\sin\frac{\theta(s)-\theta_\kappa+\phi}{2}|$$
$$\geq C|\sin\frac{\theta(s)-\theta_\kappa+\phi}{2}|$$
$$\geq C(|s\cos\frac{\theta_\kappa-\phi}{2} + \sqrt{\sqrt{4+s^2}+2}\sin\frac{\theta_\kappa-\phi}{2}| + |s\cos\frac{\theta_\kappa-\phi}{2} - \sqrt{\sqrt{4+s^2}-2}\sin\frac{\theta_\kappa-\phi}{2}|)$$
$$\geq Cs$$

Now using integration by parts and inequality above, we get

$$|I_2(\rho)| \leq \frac{1}{\sqrt{\rho}}\int_{\mathbb{R}}\Big(|F'(\frac{t}{\sqrt{\rho}})||2+\frac{t^2}{\rho}|^{1/2} + |F(\frac{t}{\sqrt{\rho}})|\Big)e^{-t^2}dt$$
$$\leq C\frac{1}{\sqrt{\rho}}\int_{\mathbb{R}}|F'(\frac{t}{\sqrt{\rho}})|e^{-t^2}dt + C\frac{1}{\sqrt{\rho}}$$
$$\leq C\frac{1}{\sqrt{\rho}}\int_{\mathbb{R}}|\mu_\kappa(S(\frac{t}{\sqrt{\rho}}))|^{-1}e^{-t^2}dt + C\frac{1}{\sqrt{\rho}}$$
$$\leq C\frac{1}{\rho^{1/4}}$$

By the smae procedure as above, it is easy to see that

$$\|F(\frac{t}{\sqrt{\rho}})C(\frac{t}{\sqrt{\rho}})t^2e^{-t^2}\|_{L^1(\mathbb{R})} \leq C$$

This completes the proof. □

References