1. Estimate of Dirichlet Green Tensor

We need the following slight generalization of Van der Corput lemma for the oscillatory integral [3, P.152].

Lemma 1.1 *Let* $-\infty < a < b < \infty$, *and* u *is a* C^k *function* u *in* (a, b).
1. If $|u'(t)| \geq 1$ *for* $t \in (a, b)$ *and* u' *is monotone in* (a,b), *then for any* $\phi(t)$ *in* (a, b)
with integrable derivatives

$$\left| \int_a^b e^{\mathbf{i}\lambda u(t)} \phi(t) dt \right| \leq 3\lambda^{-1} \left[|\phi(b)| + \int_a^b |\phi'(t)| dt \right].$$

2. For all $k \geq 2$, *if* $|u^{(k)}(t)| \geq 1$ *for* $t \in (a, b)$, *then for any* $\phi(t)$ *in* (a, b) *with integrable*
derivatives

$$\left| \int_a^b e^{\mathbf{i}\lambda u(t)} \phi(t) dt \right| \leq 12k\lambda^{-1/k} \left[|\phi(b)| + \int_a^b |\phi'(t)| dt \right].$$

Proof. The assertion can be proved by extending the Van der Corptut lemma in [3].
Here we omit the details. □
We recall following lemma, see e.g. [6]:

Lemma 1.2 *Let* $F(\rho, a) = \int_0^a t^{\alpha-1} f(t) e^{-\mathbf{i}\rho t} dt$ *where* $0 < a \leq +\infty$, $0 < \alpha < 1$, $\rho > 0$
and $t^{\alpha-1} f \in L^1(0, a)$, *then we have*

$$|F(\rho, a)| \leq C(\frac{1}{\rho^\alpha} f(0) + \frac{1}{\rho}(a^{\alpha-1} f(a) + |t^{\alpha-1} f|_{L^1(0,a)}) \tag{1.1}$$

Proof. Put

$$g_0(t) = t^{\alpha-1} e^{-\mathbf{i}\rho t} \tag{1.2}$$

and define

$$g_1(t) = -\int_t^{t-\mathbf{i}\infty} x^{\alpha-1} e^{-\mathbf{i}\rho x} dx \tag{1.3}$$

where the path of integration is the vertical line $x = t - \mathbf{i}y, y \geq 0$. It is easy to show
that $g_1(t)' = g_0(t)$. Substituting $x = t - \mathbf{i}y$ into $g_1(t)$, we have

$$g_1(t) = \mathbf{i} \int_0^\infty (t - \mathbf{i}y)^{\alpha-1} e^{-\mathbf{i}\rho t} e^{-\rho y} dy \tag{1.4}$$

Upon integration by parts, we have

$$F(\rho, a) = \int_0^a f(t) dg_1(t)$$

$$= e^{-\mathbf{i}\frac{\alpha\pi}{2}} f(0)\Gamma(\alpha)\frac{1}{\rho^\alpha} + f(a)g_1(a) - \int_0^a f'(t)g_1(t) dt$$

$$= e^{-\mathbf{i}\frac{\alpha\pi}{2}} f(0)\Gamma(\alpha)\frac{1}{\rho^\alpha} - \mathbf{i} \int_0^\infty e^{-\rho y} dy \int_0^a f'(t)(t - \mathbf{i}y)^{\alpha-1} e^{-\mathbf{i}\rho t} dt$$

Let

$$h(y) = \int_0^a f'(t)(t - \mathbf{i}y)^{\alpha-1} e^{-\mathbf{i}\rho t} dt$$

and observe that

$$|h(y)| \leq \int_0^a |f'(t)|(t^2 + y^2)^{\frac{\alpha-1}{2}} dt$$

□

Lemma 1.3 *Let* $F(\rho, a) = \int_0^a t^{-1/2} f(t) e^{-\mathbf{i}\rho t} dt$ *where* $0 < a \leq +\infty$ *and* $\rho > 0$*, then we have*

$$|F(\rho, a) - e^{-\mathbf{i}\frac{\pi}{4}} f(0)\Gamma(1/2)\frac{1}{\rho^{1/2}}| \tag{1.5}$$

$$\leq C\left(\int_0^\infty e^{-\rho y} dy \int_0^a |f'(t)|(t^2 + y^2)^{-\frac{1}{4}} dt + \frac{1}{\rho} a^{-1/2} f(a)\right) \tag{1.6}$$

Proof. Put

$$g_0(t) = t^{-1/2} e^{-\mathbf{i}\rho t} \tag{1.7}$$

and define

$$g_1(t) = -\int_t^{t-\mathbf{i}\infty} x^{-1/2} e^{-\mathbf{i}\rho x} dx \tag{1.8}$$

where the path of integration is the vertical line $x = t - \mathbf{i}y, y \geq 0$. It is easy to show that $g_1(t) = g_0(t)$. Substituting $x = t - \mathbf{i}y$ into $g_1(t)$, we have

$$g_1(t) = \mathbf{i} \int_0^\infty (t - \mathbf{i}y)^{-1/2} e^{-\mathbf{i}\rho t} e^{-\rho y} dy \tag{1.9}$$

Upon integration by parts, we have

$$F(\rho, a) = \int_0^a f(t) dg_1(t)$$

$$= e^{-\mathbf{i}\frac{\pi}{4}} f(0)\Gamma(1/2)\frac{1}{\rho^{1/2}} + f(a)g_1(a) - \int_0^a f'(t)g_1(t) dt$$

$$= e^{-\mathbf{i}\frac{\pi}{4}} f(0)\Gamma(1/2)\frac{1}{\rho^{1/2}} + \mathbf{i}f(a) \int_0^\infty (a - \mathbf{i}y)^{-1/2} e^{-\mathbf{i}\rho t} e^{-\rho y} dy$$

$$- \mathbf{i} \int_0^\infty e^{-\rho y} dy \int_0^a f'(t)(t - \mathbf{i}y)^{-1/2} e^{-\mathbf{i}\rho t} dt$$

Let

$$h(y) = \int_0^a f'(t)(t - \mathbf{i}y)^{-1/2} e^{-\mathbf{i}\rho t} dt$$

and observe that

$$|h(y)| \leq \int_0^a |f'(t)|(t^2 + y^2)^{-\frac{1}{4}} dt$$

It is easy to see that

$$|g_1(a)| \leq a^{-1/2} \int_0^\infty e^{-\rho y} dy \leq C\frac{1}{\rho}$$

□

Lemma 1.4 *Assume that $0 < \kappa := \sin\phi_\kappa < 1, 0 < \phi_\kappa < \pi/2, 0 \le \phi \le \pi/2$ and $-\pi/2 < t_1 < t_2 < \pi/2$ satisfy that $\kappa^2 = \sin^2(\phi + t_1) = \sin^2(\phi + t_2)$. Let $f(\theta)$:*

$$f(t, \phi) := F(\sin(t + \phi), \cos(t + \phi), (\kappa^2 - \sin^2(t + \phi))^{1/2}) \tag{1.10}$$

be a function in $C^\infty(([-\pi/2, \pi/2]\backslash\{t_1, t_2\}) \times [0, \pi/2])$. Moreover, there exits $\epsilon > 0$ such that $f(\theta)$ can be represented as

$$f(t, \phi) = g_1(t, \phi) + g_2(t, \phi)(\kappa^2 - \sin^2(t + \phi))^{1/2})^{N/2} \tag{1.11}$$

where $g_1, g_2 \in C^\infty((\bigcup_{i=1,2}(t_i - \epsilon, t_i + \epsilon)) \times [0, \pi/2]))$ and $N = \pm 1$. Then for any $\rho \ge 1$, we have

$$\left|I(\rho, \phi) := \int_{-\pi/2}^{\pi/2} f(\theta)e^{i\rho\cos\theta}d\theta - \frac{N+1}{2}\left(\frac{2\pi}{\rho}\right)^{1/2}f(0)e^{i\rho - i\pi/4}\right|$$

$$\le C\frac{1}{\rho^{(2+N)/4}} \tag{1.12}$$

Proof. The proof will be split into two parts about whether ϕ equal to ϕ_κ.

If $\phi \ne \phi_\kappa$, there exists $0 < \delta < \pi/4$ such that

$$|(\kappa^2 - \sin^2(t + \phi))^{1/2}| > \frac{1}{2}|(\kappa^2 - \sin^2\phi)^{1/2}| \tag{1.13}$$

for any $t \in (-\delta, \delta)$. Let $\chi_\delta \in C_0^\infty(-\pi/2, \pi/2)$ be the cut-off function with that $0 \le \chi_\delta \le 1$, $\chi_\delta = 1$ in $(-\delta/2, \delta/2)$ and $\chi_\delta = 0$ in $(-\pi/2, \pi/2)\backslash(-\delta, \delta)$. Then we can divide I into two parts such that

$$I = \int_{-\delta}^{\delta} f(t)\chi_\delta(t)e^{i\rho\cos t}dt + \int_{-\pi/2}^{\pi/2} f(t)(1 - \chi_\delta(t))e^{i\rho\cos t}dt$$

$$=: I_1 + I_2$$

Subtitating $t(s) = 2\arcsin s/2$ for t in I_1, we can obtain

$$I_1 = \int_{\mathbb{R}} f(t(s))\chi_\delta(t(s))\frac{1}{\sqrt{1 - s^2/4}}e^{i\rho}e^{-i\rho s^2/2}ds \tag{1.14}$$

$$= \int_{\mathbb{R}} h_\delta(s)e^{i\rho}e^{-i\rho s^2/2}ds \tag{1.15}$$

It is easy to see that $h_\delta(s) \in C_0^4(\mathbb{R})$. By the lemma of the stationary phase for quadratic term in [2], we have

$$I_1 = e^{i\rho}\int_{\mathbb{R}} h_\delta(s)e^{-i\frac{\rho}{2}s^2}ds = e^{i\rho}\int_{\mathbb{R}} \widehat{h_\delta}(y)\alpha(-y)dy \tag{1.16}$$

where

$$\alpha(y) = \left(\frac{1}{2\pi\rho}\right)^{1/2}e^{-i\pi/4}e^{\frac{i}{2\rho}y^2} \tag{1.17}$$

$$= \left(\frac{1}{2\pi\rho}\right)^{1/2}e^{-i\pi/4}(1 + O(\frac{y^2}{\rho})) \tag{1.18}$$

Consequently

$$I_1 = \left(\frac{1}{2\pi\rho}\right)^{1/2}e^{i\rho - i\pi/4}\int_{\mathbb{R}} \widehat{h_\delta}(y)(1 + \frac{1}{\rho}O(y^2))dy \tag{1.19}$$

Moreover, $\int_{\mathbb{R}} \widehat{h}_\delta(y)dy = 2\pi h_\delta(0)$ and $|\int_{\mathbb{R}} \widehat{h}_\delta(y)y^2 dy| < C$ since $\widehat{h}_\delta(y) = O(1/y^4)$. Now, it turns to estimate I_2.

When $N = 1$, using integration by parts, we have

$$|I_2| = \left| \int_{(-\frac{\pi}{2},\frac{\pi}{2})\backslash(-\frac{\delta}{2},\frac{\delta}{2})} f(t)(1 - \chi_\delta(t))/\sin t \; de^{\mathrm{i}\rho\cos t} \right| \tag{1.20}$$

$$\tag{1.21}$$

$$\leq C\frac{1}{\rho} + \left| \int_{(-\frac{\pi}{2},\frac{\pi}{2})\backslash(-\frac{\delta}{2},\frac{\delta}{2})} (f(t)(1 - \chi_\delta(t))/\sin t)' \; e^{\mathrm{i}\rho\cos t} dt \right| \tag{1.22}$$

$$\leq C\frac{1}{\rho} \tag{1.23}$$

From above analysis, we obtain

$$\left| I(\rho, \phi) - \left(\frac{2\pi}{\rho}\right)^{1/2} f(0)e^{\mathrm{i}\rho - \mathrm{i}\pi/4} \right| \leq C(\phi)\frac{1}{\rho} \tag{1.24}$$

When $N = -1$, we can not use integration by parts again since $f'(\theta)$ is not integrable. However, for any $0 < \lambda_1 < 1$ and $1 < \lambda_2 < 1/\kappa$, there exists $0 < \sigma < \epsilon$, such that $\chi := ((t_1 - \sigma, t_1 + \sigma) \cup (t_2 - \sigma, t_2 + \sigma)) \cap (-\delta, \delta) = 0$, dependent on λ_1, λ_2 and

$$\lambda_1 \kappa < |\sin(t + \phi)| < \lambda_2 \kappa. \tag{1.25}$$

for any $t \in \chi$.

We only analysis the integral on $\chi_1 = (t_1 - \sigma, t_1 + \sigma) \cap [-\pi/2, \pi/2]$ here, which denoted by I_{χ_1}, the proof of I_{χ_2} is similar. It is easy to see that $\sin(t + \phi)$ is monotonic in χ_1. Without loss of generality, we assume that $\sin(t_1 - \sigma + \phi) < \kappa < \sin(t_1 + \sigma + \phi)$. Let $\sin(t + \phi) = \kappa \sin\theta$ and the implicit mapping from θ to t is denoted by $t(\theta)$ while the inverse mapping by $\theta(t)$, taking the interval χ_1 onto $L_\theta : \theta_1 \to \pi/2 \to \pi/2 - \mathrm{i}\theta_2$ where $\sin(t_1 - \sigma + \phi) = \kappa \sin\theta_1, \sin(t_1 + \sigma + \phi) = \kappa \sin(\pi/2 - \mathrm{i}\theta_2)$. By substituting $t(\theta)$ into I_{χ_1}, we have

$$I_{\chi_1} = \int_{t_1-\sigma}^{t_1+\sigma} \frac{f(t)(\kappa^2 - \sin^2(t + \phi))^{1/2}}{(\kappa^2 - \sin^2(t + \phi))^{1/2}} e^{\mathrm{i}\rho\cos t} \tag{1.26}$$

$$= \int_{L_\theta} \frac{\kappa f(t(\theta))\cos\theta}{(1 - \kappa^2\sin^2\theta)^{1/2}} e^{\mathrm{i}\rho(\cos(t(\theta)))} d\theta \tag{1.27}$$

$$= \int_{L_\theta} \frac{\kappa g_1(t(\theta))\cos\theta + g_2(t(\theta))}{(1 - \kappa^2\sin^2\theta)^{1/2}} e^{\mathrm{i}\rho(\cos(t(\theta)))} d\theta \tag{1.28}$$

$$:= \int_{L_\theta} \frac{h(\theta)}{(1 - \kappa^2\sin^2\theta)^{1/2}} e^{\mathrm{i}\rho(\cos(t(\theta)))} d\theta \tag{1.29}$$

Observe that $h(\theta)$ and $\partial h/\partial\theta$ are integrable on the path L_θ by (1.11). A simple computation show that

$$\frac{dt(\theta)}{d\theta} = \frac{\kappa\cos\theta}{\cos(t + \phi)} \qquad \frac{d^2t(\theta)}{dt^2} = \frac{\kappa^2\cos^2\theta\sin(t + \phi) - \kappa\sin\theta\cos^2(t + \phi)}{\cos^3(t + \phi)}$$

Then we can obtain

$$\frac{d\cos t}{d\theta} = \frac{-\kappa\sin t\cos\theta}{\cos(t + \phi)}$$

$$\frac{d^2 \cos t}{d\theta^2} = \frac{d^2 \cos t}{dt^2}(\frac{dt}{d\theta})^2 + \frac{d \cos t}{dt}\frac{d^2 t}{d\theta^2}$$

$$= \frac{-\kappa^2 \cos^2 \theta \cos t}{\cos^2(t+\phi)} + \frac{\kappa \sin \theta \cos^2(t+\phi) \sin t - \kappa^2 \cos^2 \theta \sin(t+\phi) \sin t}{\cos^3(t+\phi)}$$

$$= \frac{-\kappa^2 \cos^2 \theta \cos \phi + \kappa \sin \theta \cos^2(t+\phi) \sin t}{\cos^3(t+\phi)}$$

$$= \frac{(\sin^2(t+\phi) - \kappa^2) \cos \phi + \cos^2(t+\phi) \sin(t+\phi) \sin t}{\cos^3(t+\phi)}$$

Since $|\sin t| > |\sin \delta|$ and $1 - \lambda_2^2 \kappa^2 < \cos^2(t+\phi) < 1 - \lambda_1^2 \kappa^2$ for $t \in \chi_1$, it follows that $\theta = \pi/2$ is the only stationary point of $\cos(t(\theta))$ and

$$\left|\frac{d^2 \cos t}{d\theta^2}(\pi/2)\right| = \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}|\sin t| > \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}\sin \delta \tag{1.30}$$

Therefore, we can choose appropriate λ_1, λ_2 such that

$$|\frac{d^2 \cos t}{d\theta^2}| > \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}\sin \delta \tag{1.31}$$

for any $\theta \in \theta(\chi_1)$. According to lemma (6.1), we obtian $|I_{\chi_1}| \leq C\frac{1}{\rho^{1/2}}$, and also $|I_{\chi_2}| \leq C\frac{1}{\rho^{1/2}}$. Using integration by parts, we get

$$\left|\int_{[-\pi/2,\pi/2]\backslash((-\delta,\delta)\cup\chi)} f(t)(1-\chi_\delta(t))e^{\mathbf{i}\rho \cos t}dt\right| \leq C\frac{1}{\rho}$$

Consequently, for $N = -1$ and $\phi \neq \phi_\kappa$, we get $|I(\rho,\phi)| \leq \frac{1}{\rho^{1/2}}$.

We now turn to the case of $\phi = \phi_\kappa$. By (1.11), we can define χ_ϵ similarly and also decompose I into I_1 and I_2. Using the same agurement above, we can easily carry out that: for $N = 1$, we have $|I_2| \leq C\frac{1}{\rho}$; for $N = -1$, we have $|I_2| \leq C\frac{1}{\rho^{1/2}}$. Finally, it remains to analysis I_1. By (1.11), we have

$$I_1 = \int_{-\epsilon}^{\epsilon} g_1\chi_\epsilon + g_2\chi_\epsilon(\sin^2 \phi_\kappa - \sin^2(t+\phi_\kappa))^{N/2}e^{\mathbf{i}\rho \cos t}dt$$

$$= \int_{-\epsilon}^{\epsilon} g_1\chi_\epsilon + g_2\chi_\epsilon(-2(\sin \phi_\kappa + \sin(t+\phi_\kappa))\cos \frac{2\phi_\kappa + t}{2}\sin t/2)^{N/2}e^{\mathbf{i}\rho \cos t}dt$$

$$= \int_{\mathbb{R}} g_1\chi_\epsilon + g_2\chi_\epsilon((\sin \phi_\kappa + \sin(t+\phi_\kappa))\cos \frac{2\phi_\kappa + t}{2})^{N/2}(-2\sin t/2)^{N/2}e^{\mathbf{i}\rho \cos t}dt$$

Also, substituting $t(s) = 2\arcsin s/2$ for t in I_1, it follows that

$$I_1 = \int_{\mathbb{R}} h_1(s)e^{-\mathbf{i}\rho\frac{s^2}{2}} + h_2(s)(-s)^{N/2}e^{-\mathbf{i}\rho\frac{s^2}{2}} \tag{1.32}$$

$$= I_{11} + I_{12} \tag{1.33}$$

where

$$h_1(s) = g_1(t(s))\chi_\epsilon(t(s))\sqrt{1-s^2/4}\ e^{\mathbf{i}\rho}$$

$$h_2(s) = g_2\chi_\epsilon((\sin \phi_\kappa + \sin(t+\phi_\kappa))\cos \frac{2\phi_\kappa + t}{2})_{t=t(s)}^{N/2}\sqrt{1-s^2/4}\ e^{\mathbf{i}\rho}$$

and $h_1(s), h_2(s) \in C_c^\infty(\mathbb{R})$. Using stationary phase lemma similarly, if $N = 1$,

$$I_{11} = \left(\frac{2\pi}{\rho}\right)^{1/2} g_1(0) e^{i\rho - i\pi/4} + O(\frac{1}{\rho}) \tag{1.34}$$

$$= \left(\frac{2\pi}{\rho}\right)^{1/2} f(0) e^{i\rho - i\pi/4} + O(\frac{1}{\rho}) \tag{1.35}$$

if $N = -1$, we get $|I_{11}| \leq C \frac{1}{\rho^{1/2}}$. For I_{12}, we have

$$I_{12} = \int_0^\infty (ih_2(s) + h_2(-s)) s^{N/2} e^{-i\rho s^2/2} ds \tag{1.36}$$

$$= \frac{1}{2} \int_0^\infty (ih_2(\sqrt{s}) + h_2(-\sqrt{s})) s^{N/4 - 1/2} e^{-i\rho s/2} ds \tag{1.37}$$

By lemma (1.2), we get $|I_{12}| \leq C \frac{1}{\rho^{(N+2)/4}}$. □

2. Some draft about Green Tensor Analysis

Let substitute $\xi = k \sin \theta$ into integral and shift the variable, we have

$$I(y) = \int_\mathbb{R} f(\xi) e^{i\xi y_1 + \mu(\xi) y_2} d\xi = \int_\mathbb{R} f(\xi) e^{i\xi(y_1 - z_1) + \mu(\xi)(y_2 - z_2)} e^{i\xi z_1 + \mu(\xi) z_2} d\xi \tag{2.1}$$

$$= k \int_L f(k \sin \theta) \cos \theta e^{ik|y-z| \cos(\theta - \eta)} e^{i|z| \cos(\theta - \phi)} d\theta \tag{2.2}$$

$$= k \int_{L_\phi} f(k \sin(\theta + \phi)) \cos(\theta + \phi) e^{ik|y-z| \cos(\theta + \phi - \eta)} e^{i|z| \cos \theta} d\theta \tag{2.3}$$

$$= k \int_L f(k \sin(\theta + \phi)) \cos(\theta + \phi) e^{ik|y-z| \cos(\theta + \phi - \eta)} e^{i|z| \cos \theta} d\theta \tag{2.4}$$

where $y_1, y_2 > 0, \sin \phi = \frac{z_1}{|z|}, \cos \phi = \frac{z_2}{|z|}, 0 < \phi < \pi/2$ and $\sin \eta = \frac{y_1 - z_1}{|y-z|}, \cos \eta = \frac{y_2 - z_2}{|y-z|}, 0 < \eta < \pi$. It is easy to see that $\phi + \eta < \pi$. Roughly, using stationary phase lemma, we obtain:

$$I(y) = f(k \sin \phi) k \cos \phi e^{ik|y-z| \cos(\phi - \eta)} \left(\frac{2\pi}{|z|}\right)^{1/2} e^{i|z| - i\frac{\pi}{4}} (1 + O(\frac{1}{|z|})) \tag{2.5}$$

$$\cos(a + ib) = \frac{e^b + e^{-b}}{2} \cos a + i \frac{e^{-b} - e^b}{2} \sin a \tag{2.6}$$

$$\sin(a + ib) = \frac{e^b + e^{-b}}{2} \sin a + i \frac{e^b - e^{-b}}{2} \cos a \tag{2.7}$$

When $\theta \in (-a - \pi/2, -a - \pi/2 + i\infty)$, let $\theta = -a - \pi/2it$, where $t > 0, 0 \leq a \leq \phi$, then

$$-\text{Im}(|z| \cos \theta + |y - z| \cos(\theta + \phi - \eta))$$

$$= |z| \sin(a + \pi/2) + |y - z| \sin(a + \pi/2 - \phi + \eta) \tag{2.8}$$

$$= |z| \cos a + |y - z| \cos(a - \phi + \eta) \tag{2.9}$$

$$= |z| \cos a + \cos a |y - z| (\cos \phi \cos \eta + \sin \phi \sin \eta) \tag{2.10}$$

$$+ \sin a |y - z| (\sin \phi \cos \eta - \cos \phi \sin \eta) \tag{2.11}$$

$$= |z| \cos a + \cos a((y_2 - z_2) \cos \phi + (y_1 - z_1) \sin \phi) \tag{2.12}$$

$$+ \sin a((y_2 - z_2) \sin \phi - (y_1 - z_1) \cos \phi) \tag{2.13}$$

$$= y_1 \sin(\phi - a) + y_2 \cos(\phi - a) > 0 \tag{2.14}$$

Now, Using Cauchy Integral Theorem, we have

$$I(y) = k \int_L f(k \sin(\theta + \phi)) \cos(\theta + \phi) e^{ik|y-z| \cos(\theta+\phi-\eta)} e^{i|z| \cos \theta} d\theta \tag{2.15}$$

Let $L_1 = (-\pi/2, -\pi/2 + i\infty)$ and $\theta = -\pi/2 + it, t > 0$, then

$$I_1(y) = k \int_{L_1} f(k \sin(\theta + \phi)) \cos(\theta + \phi) e^{ik|y-z| \cos(\theta+\phi-\eta)} e^{i|z| \cos \theta} d\theta \tag{2.16}$$

$$= \tag{2.17}$$

$$I(y) = f(k \sin \phi) k \cos \phi e^{ik|y-z| \cos(\phi-\eta)} (\frac{2\pi}{|z|})^{1/2} e^{i|z| - i\frac{\pi}{4}} \tag{2.18}$$

$$+ \frac{kz_2}{|z|} O(\left(\frac{1}{k|z|}\right)^{3/4} + \frac{1}{k|y|}) + \frac{kz_1}{|z|} O(\left(\frac{1}{k|z|}\right)^{5/4} + \left(\frac{1}{k|y|}\right)^2) \tag{2.19}$$

It is easy to see

$$\int_{-d}^{d} \frac{k}{(k|x-z|)^\alpha} \frac{1}{(k|x-y|)^\beta} dx_1 \leq C(\frac{1}{(kz_2)^{\alpha+\beta-1}} + \frac{1}{(ky_2)^{\alpha+\beta-1}}) \tag{2.20}$$

where $z, y \in \mathbb{R}_+^2$, $x \in \Gamma_0$ and $\alpha + \beta > 0$.

$$e^{i\mu y_2 + i\xi(x_1-y_1)} = e^{i\mu y_2 - iy_2/\tan\phi} = e^{iy_2(\mu-\xi/\tan\phi)} \tag{2.21}$$

Another method

$$\int_{-\pi/2}^{\pi/2} f(k \sin(\theta + \psi)) k \cos(\theta + \psi) e^{ik|x-y| \cos \theta} \tag{2.22}$$

$$= \int_{-\pi/2}^{\pi/2} f(k \sin(\theta + \psi)) k \cos(\theta + \psi) e^{ik|x-y| \cos(\theta+\psi-\psi)} \tag{2.23}$$

$$= \int_{-\pi/2}^{\pi/2} f(k \sin(\theta + \psi)) k \cos(\theta + \psi) e^{iky_2 \cos(\theta+\psi)+ik|x_1-y_1| \sin(\theta+\psi)} \tag{2.24}$$

$$= \int_{-\pi/2}^{\pi/2} f(k \sin(\theta + \psi)) k \cos(\theta + \psi) \tag{2.25}$$

$$e^{ik(y_2-z_2) \cos(\theta+\psi)+ik(|x_1-y_1|-|x_1-z_1|) \sin(\theta+\psi)+ik|z| \cos(\theta+\psi-\phi)} \tag{2.26}$$

3. Finite Aperture Point Spread Function

If $x \in \Gamma_0$ and $z, y \in \mathbb{R}_+^2$, by lemma (??) we have

$$G(x, y) = \frac{ik_s}{\mu\sqrt{2\pi}} \frac{1}{\delta(\xi)} \begin{pmatrix} \mu_s\beta & \xi\beta \\ 2\xi\mu_s\mu_p & 2\xi^2\mu_p \end{pmatrix}_{\xi=k_s \frac{x_1-y_1}{|x-y|}} \frac{y_2}{|x-y|} \frac{1}{(k_s|x-y|)^{1/2}} e^{ik_s|x-y|-i\frac{\pi}{4}}$$

$$+ \frac{ik_p}{\mu\sqrt{2\pi}} \frac{1}{\delta(\xi)} \begin{pmatrix} 2\xi^2\mu_s & -2\xi\mu_s\mu_p \\ -\xi\beta & \mu_p\beta \end{pmatrix}_{\xi=k_p \frac{x_1-y_1}{|x-y|}} \frac{y_2}{|x-y|} \frac{1}{(k_p|x-y|)^{1/2}} e^{ik_p|x-y|-i\frac{\pi}{4}} \tag{3.1}$$

$$+O(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}})$$

$$:=\mathcal{G}_s(x,y)+\mathcal{G}_p(x,y)+O(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}})$$

$$T_D(x,z)=\frac{k_s}{\sqrt{2\pi}}\frac{1}{\gamma(\xi)}\begin{pmatrix}\mu_s\mu_p & \xi\mu_p \\ \xi\mu_s & \xi^2\end{pmatrix}_{\xi=k_s\frac{x_1-z_1}{|x-z|}}\frac{z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{1/2}}e^{\mathbf{i}k_s|x-z|-\mathbf{i}\frac{\pi}{4}}$$

$$+\frac{k_p}{\sqrt{2\pi}}\frac{1}{\gamma(\xi)}\begin{pmatrix}\xi^2 & -\xi\mu_p \\ -\xi\mu_s & \mu_p\mu_s\end{pmatrix}_{\xi=k_p\frac{x_1-z_1}{|x-z|}}\frac{z_2}{|x-z|}\frac{1}{(k_p|x-z|)^{1/2}}e^{\mathbf{i}k_p|x-z|-\mathbf{i}\frac{\pi}{4}}\quad(3.2)$$

$$+O(\frac{k_sz_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{k_s|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}})$$

$$:=\mathcal{T}_s(x,z)+\mathcal{T}_p(x,z)+O(\frac{k_sz_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{k_s|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}})$$

Now we consider the finite aperture point spread function $J_d(z,y)$:

$$\int_{-d}^{d}(T_D(x_1,0;z_1,z_2))^T\overline{G(x_1,0;y_1,y_2)}dx_1\quad(3.3)$$

Recall following standard asymptotic expansion:

$$|x-y|=|x-z|+\widehat{x-z}\cdot(z-y)+O(\frac{|y-z|^2}{|x-z|})\quad(3.4)$$

$$|y|^{-\alpha}=|z|^{-\alpha}(1+\frac{|y|-|z|}{|z|})^{-\alpha}=|z|^{-\alpha}(1+O(\frac{|y-z|}{|z|}))\quad(3.5)$$

$$e^{\mathbf{i}t}=1+O(t)\quad(3.6)$$

$$|a^{1/2}-b^{1/2}|\le C\sqrt{|a-b|}\quad(3.7)$$

where $x,y,z\in\mathbb{R}^2$, $t,a,b\in\mathbb{R}$ and $\alpha>0$.

Lemma 3.1 *For any* $z,y\in\mathbb{R}_+^2$, $J_d(z,y)=F(z,y)+O((1+\frac{|y-z|}{z_2})(\frac{1}{k_sz_2})^{1/4}+\frac{(k_s|y-z|)^2}{k_sz_2}+(\frac{|y-z|}{z_2})^{1/2})$, *where*

$$F(z,y)=-\frac{\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d}f_s(\theta)\begin{pmatrix}\sin^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \cos^2\theta\end{pmatrix}e^{\mathbf{i}k_s(z_1-y_1)\cos\theta+\mathbf{i}k_s(z_2-y_2)\sin\theta}d\theta\ (3.8)$$

$$-\frac{\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d}f_p(\theta)\begin{pmatrix}\cos^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin^2\theta\end{pmatrix}e^{\mathbf{i}k_p(z_1-y_1)\cos\theta+\mathbf{i}k_p(z_2-y_2)\sin\theta}d\theta\ (3.9)$$

and

$$f_s(\theta)=\frac{\sin\theta((\kappa^2-\cos^2\theta)^{1/2}(1-2\cos^2\theta)+2\overline{(\kappa^2-\cos^2\theta)^{1/2}}\cos^2\theta)}{(\cos^2\theta+\sin\theta(\kappa^2-\cos^2\theta)^{1/2})((1-2\cos^2\theta)^2+4\cos^2\theta\sin\theta(\kappa^2-\cos\theta)^{1/2})}$$

$$f_p(\theta)=\frac{\sin\theta(1/\kappa^2-\cos^2\theta)^{1/2}}{(\cos^2\theta+\sin\theta(1/\kappa^2-\cos^2\theta)^{1/2})((1/\kappa^2-2\cos^2\theta)^2+4\cos^2\theta\sin\theta(1/\kappa^2-\cos\theta)^{1/2})}$$

where $0<\theta_1^d<\pi/2<\theta_2^d<\pi$ *and* $z_2=(d+z_1)\tan\theta_1^d=(z_1-d)\tan\theta_2^d$.

Proof.

$$\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}} + \frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}}$$
$$= \left(\frac{z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}} + \frac{|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}}\right)(1+O(\frac{|y-z|}{|x-z|}))$$

$$|\mu_{\mathbf{i}}(k_j\frac{x_1-y_1}{|x-y|}) - \mu_{\mathbf{i}}(k_j\frac{x_1-z_1}{|x-z|})|$$
$$\leq Ck_j\sqrt{\left|\frac{x_1-y_1}{|x-y|} - \frac{x_1-z_1}{|x-z|}\right|} \leq Ck_j\left(\frac{|y-z|}{|x-z|}\right)^{1/2}$$

where $i, j = s, p$. By above, we can obtain

$$\mathcal{G}_s(x,y) = \mathcal{G}_s(x,z)e^{\mathbf{i}k_s\widehat{x-z}\cdot(z-y)} + O(\frac{(k_s|y-z|)^2}{(k_s|x-z|)^{3/2}}) + O(\frac{(k_s|y-z|)^{1/2}}{k_s|x-z|}) \tag{3.10}$$

$$\mathcal{G}_p(x,y) = \mathcal{G}_p(x,z)e^{\mathbf{i}k_p\widehat{x-z}\cdot(z-y)} + O(\frac{(k_p|y-z|)^2}{(k_p|x-z|)^{3/2}}) + O(\frac{(k_p|y-z|)^{1/2}}{k_p|x-z|}) \tag{3.11}$$

For $l > 1$, a simple computation show that

$$\int_{-d}^{d}\frac{k_s}{(k_s|x-z|)^l}dx_1 = \frac{1}{(k_sz_2)^{l-1}}\int_{\frac{-d-z_1}{z_2}}^{\frac{d-z_1}{z_2}}\frac{1}{(1+t^2)^{l/2}}dt \leq C\frac{1}{(k_sz_2)^{l-1}} \tag{3.12}$$

Let

$$\mathcal{G}_\alpha(x,y) = \frac{\mathbf{i}}{\sqrt{2\pi}\mu}g_\alpha(\frac{x_1-y_1}{|x-y|},\kappa)\frac{1}{(k_\alpha|x-y|)^{1/2}}e^{\mathbf{i}k_\alpha|x-y|-\mathbf{i}\frac{\pi}{4}} \tag{3.13}$$

$$\mathcal{T}_\alpha(x,y) = \frac{k_\alpha}{\sqrt{2\pi}}t_\alpha(\frac{x_1-z_1}{|x-z|},\kappa)\frac{1}{(k_s|x-z|)^{1/2}}e^{\mathbf{i}k_\alpha|x-z|-\mathbf{i}\frac{\pi}{4}} \tag{3.14}$$

where $\alpha = s, p$. Now, by substituting (3.10-3.11) into $J_d(z,y)$ and using inequality (3.12), we have

$$J_d(z,y) = \frac{-\mathbf{i}}{2\pi\mu}\int_{-d}^{d}t_s(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_s(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_s\widehat{x-z}\cdot(y-z)}}{|x-z|}$$

$$+t_p(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_p(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_p\widehat{x-z}\cdot(y-z)}}{|x-z|}dx_1 \tag{3.15}$$

$$-\frac{\mathbf{i}}{2\pi\mu}\int_{-d}^{d}t_p(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_s(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_s\widehat{x-z}\cdot(y-z)}}{|x-z|} \tag{3.16}$$

$$+t_s(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_p(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_p\widehat{x-z}\cdot(y-z)}}{|x-z|}dx_1 \tag{3.17}$$

$$+O((1+\frac{|y-z|}{z_2})(\frac{1}{k_sz_2})^{1/4} + \frac{(k_s|y-z|)^2}{k_sz_2} + (\frac{|y-z|}{z_2})^{1/2}) \tag{3.18}$$

$$:= F(z,y) + R(z,y) \tag{3.19}$$

$$+O((1+\frac{|y-z|}{z_2})(\frac{1}{k_sz_2})^{1/4} + \frac{(k_s|y-z|)^2}{k_sz_2} + (\frac{|y-z|}{z_2})^{1/2}) \tag{3.20}$$

We denote $\widehat{x-z} = x - z/|x-z| = (\cos(\phi+\pi), \sin(\phi+\pi))$, then taking the substitution $x_1 = z_1 - z_2 \cot\phi$, we obtain

$$F(z,y) = \frac{-\mathbf{i}}{2\pi\mu} \int_{\theta_1^d}^{\theta_2^d} A_s(\phi,\kappa) e^{\mathbf{i}k_s(z_1-y_1)\cos\phi + \mathbf{i}k_s(z_2-y_2)\sin\phi} \tag{3.21}$$

$$+ \frac{-\mathbf{i}}{2\pi\mu} \int_{\theta_1^d}^{\theta_2^d} A_p(\phi,\kappa) e^{\mathbf{i}k_p(z_1-y_1)\cos\phi + \mathbf{i}k_p(z_2-y_2)\sin\phi} \tag{3.22}$$

$$R(z,y) = \frac{-\mathbf{i}}{2\pi\mu} \int_{\theta_1^d}^{\theta_2^d} B_s(\phi,\kappa) e^{\mathbf{i}k_s(z_1-y_1)\cos\phi + \mathbf{i}k_s(z_2-y_2)\sin\phi + (k_p-k_s)|x-z|} \tag{3.23}$$

$$+ \frac{-\mathbf{i}}{2\pi\mu} \int_{\theta_1^d}^{\theta_2^d} B_p(\phi,\kappa) e^{\mathbf{i}k_p(z_1-y_1)\cos\phi + \mathbf{i}k_p(z_2-y_2)\sin\phi + (k_s-k_p)|x-z|} \tag{3.24}$$

It is easy to see that $|R(z,y)| \le C\frac{|z-y|}{z_2}$. $\qquad\square$

Let
$$g(x_1) = \frac{1}{((x_1-z_1)^2 + z_2^2)^{3/4}((x_1-y_1)^2 + y_2^2)^{1/4}}$$
$$\phi(x_1) = ((x_1-z_1)^2 + z_2^2)^{1/2} - ((x_1-y_1)^2 + y_2^2)^{1/2}$$

Then, we have
$$g'(x_1) = -g(x_1)\left[\frac{3(x_1-z_1)}{2((x_1-z_1)^2 + z_2^2)} + \frac{(x_1-y_1)}{2((x_1-y_1)^2 + y_2^2)}\right]$$

$$\phi'(x_1) = \frac{x_1 - z_1}{((x_1-z_1)^2 + z_2^2)^{1/2}} - \frac{x_1 - y_1}{((x_1-y_1)^2 + y_2^2)^{1/2}}$$

$$= \frac{\frac{(x_1-z_1)^2}{(x_1-z_1)^2+z_2^2} - \frac{(x_1-y_1)^2}{(x_1-y_1)^2+y_2^2}}{\frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}} + \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}}}$$

$$= \frac{(x_1-z_1)^2 y_2^2 - (x_1-y_1)^2 z_2^2}{\left(\frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}} + \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}}\right)((x_1-z_1)^2 + z_2^2)((x_1-y_1)^2 + y_2^2)}$$

$$\phi''(x_1) = \frac{z_2^2}{((x_1-z_1)^2 + z_2^2)^{3/2}} - \frac{y_2^2}{((x_1-y_1)^2 + y_2^2)^{3/2}}$$

Using integration by parts, we can obtain
$$\int_{-d}^{d} g(x_1) e^{\mathbf{i}\phi(x_1)} dx_1$$

$$= \left(\frac{g(d)}{\phi'(d)} e^{\mathbf{i}\phi(d)} - \frac{g(-d)}{\phi'(-d)} e^{\mathbf{i}\phi(-d)}\right) - \int_{-d}^{d} \frac{g'(x_1)}{\phi'(x_1)} - \frac{g(x_1)\phi''(x_1)}{(\phi'(x_1))^2} dx_1$$

Assume that
$$|y_1| \le c_0 d \quad |z_1| \le c_0 d \quad h \le y_2, z_2 \le c_1 h \quad d \le c_2 h$$
where $0 < c_0 < 1$. Let define $0 < \theta_y, \theta_z < \pi$ such that
$$\cos\theta_y = \frac{x_1 - y_1}{((x_1-y_1)^2 + y_2^2)^{1/2}}$$
$$\cos\theta_z = \frac{x_1 - z_1}{((x_1-z_1)^2 + z_2^2)^{1/2}}$$

By mean value theorem and the law of sines, we get

$$
\begin{aligned}
|\phi'(x_1)| = |\cos\theta_z - \cos\theta_y| &= |\sin\theta'||\theta_z - \theta_y| \\
&\geq \frac{h}{(1+c_0)d}|\sin(\theta_z - \theta_y)| \\
&= \frac{h}{(1+c_0)d}\frac{|z-y|}{|x-y|}\sin\theta_{|x-y|} \\
&= \frac{h}{(1+c_0)d}\frac{|z-y|}{|x-z|}\sin\theta_{|x-z|} \\
&\geq \frac{h^2}{(1+c_0)^2d^2}\frac{|z-y|}{|x-y|} \\
\text{or}\qquad &\geq \frac{h^2}{(1+c_0)^2d^2}\frac{|z-y|}{|x-z|}
\end{aligned}
$$

Then we have

$$
\begin{aligned}
|\frac{g(x_1)}{\phi'(x_1)}| &\leq \frac{(1+c_0)^2d^2}{h^2}\frac{1}{|z-y||x-y|^{1/2}|x-z|^{1/2}} \\
&\leq C\frac{d^2}{h^3}\frac{1}{|z-y|}
\end{aligned}
$$

Moreover, by mean value theorem agian, we have

$$
\begin{aligned}
|\phi''(x_1)| = |\frac{\sin^2\theta_z}{|x-z|} - \frac{\sin^2\theta_y}{|x-y|}| &\\
&= |\frac{2\sin\theta'\cos\theta'}{|x-y'|}(\theta_z - \theta_y) - \frac{\sin^2\theta'}{|x-y'|^2}(|x-z|-|x-y|)| \\
&\leq \pi\frac{|\sin(\theta_z - \theta_y)|}{h} + \frac{|z-y|}{h^2} \\
&\leq \pi\frac{|\sin\theta_{|x-z|}||z-y|}{h|x-z|} + \frac{|z-y|}{h^2} \\
&\leq C\frac{|z-y|}{h^2}
\end{aligned}
$$

Now, it is easy to see that

$$
\begin{aligned}
&|\int_{-d}^{d} \frac{g'(x_1)}{\phi'(x_1)} - \frac{g(x_1)\phi''(x_1)}{(\phi'(x_1))^2}dx_1| \\
&\leq C\frac{d^3}{h^4}\frac{1}{|z-y|} + C\frac{d^3}{h^3}\frac{1}{|z-y|}\frac{d^2}{h^3}
\end{aligned}
$$

Based on the above analysis, we can obtain

$$
|\int_{-d}^{d} z_2 g(x_1)e^{i\phi(x_1)}| \leq C((\frac{d}{h})^2 + (\frac{d}{h})^3 + (\frac{d}{h})^5)\frac{1}{|z-y|}
$$

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$$
\sin\phi_\kappa - \sin(t+\phi) = -2\cos(\frac{\phi_\kappa + \phi + t}{2})\sin(\frac{t+\phi - \phi_\kappa}{2})
$$

$$
\sin(\frac{t+\phi - \phi_\kappa}{2}) = \sin\frac{t}{2}\cos(\frac{\phi - \phi_\kappa}{2}) + \cos\frac{t}{2}\sin(\frac{\phi - \phi_\kappa}{2})
$$

Some think, substituting $t = 2\arcsin s/2$ into following integral

$$\int_0^\infty \chi(t)(\sin\phi_\kappa - \sin(t+\phi))^{1/2}e^{-i\rho\cos t}$$

$$= \int_0^\infty \chi(t(s))(-s\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-s^2}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2}e^{-i\rho s^2/2}$$

$$= \int_0^\infty \chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2}t^{-1/2}e^{-i\rho t/2}$$

Let

$$f(t) = t^{-1/2}e^{-i\rho t/2}$$

$$g(t) = -\int_t^{t-i\infty} x^{-1/2}e^{-i\rho x/2}dx$$

$$= i\int_0^\infty (t-ix)^{-1/2}e^{-i\rho t - \rho x}dx$$

It is easy to see that $g'(t) = f(t)$. Then we have

$$= \int_0^\infty \chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2}t^{-1/2}e^{-i\rho t/2}$$

$$= \chi(0)(-2\sin(\frac{\phi-\phi_\kappa}{2}))^{1/2}g(0)$$

$$- \int_0^\infty (\chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2})'g(t)dt$$

We get

$$g(x) = \int_0^\infty \chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{-1/2}t^{-1/2}(t-ix)^{-1/2}e^{-i\rho t}dt$$

$$R(\rho) = \int_0^\infty g(x)e^{-\rho x}dx$$

Because $\chi(t)$ has compact support $(-\delta, \delta)$, we obtain

$$gg(x) = \int_0^\delta (\sqrt{t}\cos(\theta) - \sqrt{4-t}\sin\theta)^{-1/2}t^{-1/2}(t^2+x^2)^{-1/4}dt$$

where $\theta = \frac{\phi-\phi_\kappa}{2}$. For $x > 0$, Put L(x):

$$\int_0^a \frac{1}{t^{3/4}}\frac{1}{(t^2+x^2)^{1/4}}dt$$

$$= 4\int_0^a \frac{1}{(t^2+x^2)^{1/4}}dt^{1/4}$$

$$= 4\int_0^{a^{1/4}} \frac{1}{(t^8+x^2)^{1/4}}dt$$

$$= 4x^{-1/4}\int_0^{(\frac{a}{x})^{1/4}} \frac{1}{(t^8+1)^{1/4}}dt$$

$$= 4x^{-1/4}\int_0^{(\frac{a}{x})^{1/4}} \frac{1}{(t^8+1)^{1/4}}dt$$

$$\leq 4x^{-1/4}\int_0^\infty \frac{1}{(t^8+1)^{1/4}}dt$$

Back to analysis $gg(x)$, we have

$$gg(x) \leq \int_0^\delta \left| \frac{\sqrt{t} + 2|\sin\theta|}{t - 4\sin^2\theta} \right|^{1/2} t^{-1/2}(t^2 + x^2)^{-1/4}dt$$

$$= \int_0^\delta \left| \frac{1}{\sqrt{t} - 2|\sin\theta|} \right|^{1/2} t^{-1/2}(t^2 + x^2)^{-1/4}dt$$

$$= 2\int_0^{\sqrt{\delta}} \left| \frac{1}{t - 2|\sin\theta|} \right|^{1/2} (t^4 + x^2)^{-1/4}dt$$

$$= 2\int_{-2|\sin\theta|}^{\sqrt{\delta}-2|\sin\theta|} |t|^{-1/2}((t + 2|\sin\theta|)^4 + x^2)^{-1/4}dt$$

$$\leq 4\int_0^{\delta^{1/4}} (t^8 + x^2)^{-1/4}dt + 4\int_0^{\sqrt{2|\sin\theta|}} ((t^2 - 2|\sin\theta|)^4 + x^2)^{-1/4}dt$$

$$\leq Cx^{-1/4}(1 + \int_0^{\sqrt{2|\sin\theta|}} ((t^2 - 2|\sin\theta|)^4/x + x)^{-1/4}dt)$$

$$\leq Cx^{-1/4}(1 + \int_0^{\sqrt{2|\sin\theta|}} (t^2 - 2|\sin\theta|)^{-1/2}dt)$$

$$= Cx^{-1/4}(1 + \int_0^1 (1 - t^2)^{-1/2}dt) \leq Cx^{-1/4}$$

Immediately, we can obtain

$$|g(x)| \leq Cx^{-1/4}$$

It follows that

$$R(\rho) \leq \int_0^\infty x^{-1/4}e^{-\rho x} \leq C\rho^{-3/4}$$

5. stationary of phase lemma

Lemma 5.1 *Assume that $0 < \kappa := \sin\phi_\kappa < 1, 0 < \phi_\kappa < \pi/2, 0 \leq \phi \leq \pi/2$. Let*

$$f(t, \phi) := F(\sin(t + \phi), \cos(t + \phi), (\kappa^2 - \sin^2(t + \phi))^{1/2}) \tag{5.1}$$

be a complexed function in $C([-\pi/2, \pi/2] \times [0, \pi/2])$. Moreover, its partial derivative with respect to t can be represented as

$$\frac{\partial f(t, \phi)}{\partial t} = g(t, \phi)(\kappa^2 - \sin^2(t + \phi))^{-1/2} \tag{5.2}$$

where $g(t, \phi)$ is uniformly bounded. Then for any $\rho \geq 1$, we have

$$\left| I(\rho, \phi) := \int_{-\pi/2}^{\pi/2} f(t)e^{i\rho\cos t}dt - \left(\frac{2\pi}{\rho}\right)^{1/2} f(0)e^{i\rho - i\pi/4} \right|$$

$$\leq C\frac{1}{\rho^{3/4}} \tag{5.3}$$

Proof. Solving the following equation:

$$\kappa^2 - \sin^2(t + \phi) = 0$$

we have, if $0 < \phi < \pi/2 - \phi_\kappa$,

$$t_1(\phi) = \phi_\kappa - \phi \qquad t_2(\phi) = -\phi_\kappa - \phi$$

and if $\pi/2 - \phi_\kappa \leq \phi < \pi/2$,

$$t_1(\phi) = \phi_\kappa - \phi \qquad t_2(\phi) = \pi - \phi_\kappa - \phi$$

Since $|t_2(\phi)| < \phi_\kappa$ or $|t_2(\phi)| < \pi/2 - \phi_\kappa$, we now define $\delta := \min(\frac{\phi_\kappa}{2}, \frac{\pi/2 - \phi_\kappa}{2})$ and it is easy to see that

$$\kappa + \sin(t + \phi) \neq 0 \tag{5.4}$$

$$\cos(\frac{t + \phi + \phi_\kappa}{2}) \neq 0 \tag{5.5}$$

for any $(t, \phi) \in [-\delta, \delta] \times [0, \pi/2]$. Let $\chi_\delta \in C_0^\infty(-\pi/2, \pi/2)$ be the cut-off function with that $0 \leq \chi_\delta \leq 1$, $\chi_\delta = 1$ in $(-\delta/2, \delta/2)$ and $\chi_\delta = 0$ in $(-\pi/2, \pi/2)\backslash(-\delta, \delta)$. Then we can divide I into two parts such that

$$I = \int_{-\delta}^{\delta} f(t)\chi_\delta(t)e^{i\rho\cos t}dt + \int_{-\pi/2}^{\pi/2} f(t)(1 - \chi_\delta(t))e^{i\rho\cos t}dt$$

$$=: I_1 + I_2$$

Subtitating $t(s) = 2\arcsin s/2$ for t in I_1, we can obtain

$$I_1 = \int_{-2\sin\frac{\delta}{2}}^{2\sin\frac{\delta}{2}} f(t(s))\chi_\delta(t(s))\frac{1}{\sqrt{1 - s^2/4}}e^{i\rho}e^{-i\rho s^2/2}ds \tag{5.6}$$

$$= \int_0^{2\sin\frac{\delta}{2}} (f(t(s))\chi_\delta(t(s)) + f(-t(s))\chi_\delta(-t(s)))\frac{1}{\sqrt{1 - s^2/4}}e^{i\rho}e^{-i\rho s^2/2}ds \tag{5.7}$$

$$:= I_{11} + I_{12} \tag{5.8}$$

Taking substitution $s = \sqrt{x}$, we get

$$I_{11} = \frac{1}{2}\int_0^{(2\sin\frac{\delta}{2})^2} f(t(\sqrt{x}))\chi_\delta(t(\sqrt{x}))\frac{1}{\sqrt{1 - x/4}}x^{-1/2}e^{i\rho}e^{-i\rho x/2}dx$$

Observe that

$$\sin\phi_\kappa - \sin(t + \phi) = -2\cos(\frac{\phi_\kappa + \phi + t}{2})\sin(\frac{t + \phi - \phi_\kappa}{2})$$

$$\sin(\frac{t + \phi - \phi_\kappa}{2}) = \sin\frac{t}{2}\cos(\frac{\phi - \phi_\kappa}{2}) + \cos\frac{t}{2}\sin(\frac{\phi - \phi_\kappa}{2})$$

$$:= \sin\frac{t}{2}\cos\theta + \cos\frac{t}{2}\sin\theta$$

where $\theta = \frac{\phi - \phi_\kappa}{2}$. By lemma (1.3) and using representation (5.2), inequality (5.4-5.5), it follows that

$$\left|I_{11} - \frac{1}{2}\sqrt{\frac{2\pi}{\rho}}f(0)e^{i\rho - i\frac{\pi}{4}}\right|$$

$$\leq \int_0^\infty e^{-\rho y}dy \int_0^{(2\sin\frac{\delta}{2})^2} \left|\frac{\partial(f(t(\sqrt{x}))\chi_\delta(t(\sqrt{x}))\frac{1}{\sqrt{1-x/4}})}{\partial x}\right|(x^2 + y^2)^{-\frac{1}{4}}dx$$

$$\leq C \int_0^\infty e^{-\rho y} dy \int_0^{(2\sin\frac{\delta}{2})^2} |\sqrt{x}\cos\theta + \sqrt{4-x}\sin\theta|^{-1/2} x^{-1/2}(x^2+y^2)^{-\frac{1}{4}} dx$$

$$\leq C \int_0^\infty e^{-\rho y} dy \int_0^{(2\sin\frac{\delta}{2})^2} \frac{(\sqrt{x}|\cos\theta| + \sqrt{4-x}|\sin\theta|)^{1/2}}{|x - 4\sin^2\theta|^{1/2}} x^{-1/2}(x^2+y^2)^{-\frac{1}{4}} dx$$

$$\leq C \int_0^\infty e^{-\rho y} dy \int_0^{(2\sin\frac{\delta}{2})^2} \frac{1}{|\sqrt{x} - 2|\sin\theta||^{1/2}} x^{-1/2}(x^2+y^2)^{-\frac{1}{4}} dx$$

$$\leq C \int_0^\infty e^{-\rho y} dy \int_0^{2\sin\frac{\delta}{2}} \frac{1}{|x - 2\sin|\theta||^{1/2}} (x^4+y^2)^{-\frac{1}{4}} dx$$

$$\leq C \int_0^\infty e^{-\rho y} dy \int_{-2\sin|\theta|}^{2\sin\frac{\delta}{2}-2\sin|\theta|} \frac{1}{|x|^{1/2}} ((x + 2\sin|\theta|)^4 + y^2)^{-\frac{1}{4}} dx$$

$$\leq C \int_0^\infty e^{-\rho y} dy \int_{-2\sin|\theta|}^{2\sin\frac{\delta}{2}} \frac{1}{|x|^{1/2}} ((x + 2\sin|\theta|)^4 + y^2)^{-\frac{1}{4}} dx$$

$$\leq C \int_0^\infty e^{-\rho y} dy \left(\int_0^{\sqrt{2\sin\frac{\delta}{2}}} (x^8+y^2)^{-\frac{1}{4}} dx + \int_0^{\sqrt{2\sin|\theta|}} ((x^2 - 2\sin|\theta|)^4 + y^2)^{-\frac{1}{4}} dx \right)$$

$$\leq C \int_0^\infty e^{-\rho y} dy \left(y^{-\frac{1}{4}} \int_0^\infty (x^8+1)^{-\frac{1}{4}} dx + y^{-\frac{1}{4}} \int_0^{\sqrt{2\sin|\theta|}} (2\sin|\theta| - x^2)^{-\frac{1}{2}} dx \right)$$

$$\leq C \int_0^\infty y^{-\frac{1}{4}} e^{-\rho y} dy \left(\int_0^\infty (x^8+1)^{-\frac{1}{4}} dx + \int_0^1 (1 - x^2)^{-\frac{1}{2}} dx \right) \leq C \frac{1}{\rho^{3/4}}$$

Using the same agrument , we can also carry out

$$|I_{12} - \frac{1}{2}\sqrt{\frac{2\pi}{\rho}} f(0) e^{\mathbf{i}\rho - \mathbf{i}\frac{\pi}{4}}| \leq C \frac{1}{\rho^{3/4}} \tag{5.9}$$

It remains to estimate I_2. Note that there exits $m > 0$ such that $|\sin t| \geq m$ for any $t \in [-\pi/2, \pi/2]\backslash(-\delta/2, \delta/2)$. Upon integration by parts and representation (5.2) again, we have

$$|I_{12}| \leq C\rho^{-1}(1 + \left| \int_{[-\pi/2,\pi/2]\backslash(-\delta/2,\delta/2)} \frac{\partial(f(t)(1 - \chi_\delta(t)))}{\partial t} \frac{1}{\sin t} dt \right|)$$

$$\leq C\rho^{-1}(1 + \int_{-\pi/2}^{\pi/2} \left| \frac{\partial(f(t)(1 - \chi_\delta(t)))}{\partial t} \right| dt)$$

$$\leq C\rho^{-1}(1 + \int_{-\pi/2}^{\pi/2} |(\kappa^2 - \sin^2(t + \phi))^{-1/2}| dt)$$

$$\leq C\rho^{-1}(1 + \int_{-\pi/2}^{\pi/2} |(\kappa^2 - \sin^2 t)^{-1/2}| dt)$$

$$\leq C\rho^{-1}$$

This completes the proof. □

6. cross term of psf, 17.11.15

We need the following slight generalization of Van der Corput lemma for the oscillatory integral [3, P.152].

Lemma 6.1 *Let* $-\infty < a < b < \infty$, *and* u *is a* C^k *function* u *in* (a, b).
1. If $|u'(t)| \geq 1$ *for* $t \in (a, b)$ *and* u' *is monotone in* (a,b), *then for any* $\phi(t)$ *in* (a, b) *with integrable derivatives*

$$\left| \int_a^b e^{\mathbf{i}\lambda u(t)} \phi(t) dt \right| \leq 3\lambda^{-1} \left[|\phi(b)| + \int_a^b |\phi'(t)| dt \right].$$

2. For all $k \geq 2$, *if* $|u^{(k)}(t)| \geq 1$ *for* $t \in (a, b)$, *then for any* $\phi(t)$ *in* (a, b) *with integrable derivatives*

$$\left| \int_a^b e^{\mathbf{i}\lambda u(t)} \phi(t) dt \right| \leq 12k\lambda^{-1/k} \left[|\phi(b)| + \int_a^b |\phi'(t)| dt \right].$$

Proof. The assertion can be proved by extending the Van der Corptut lemma in [3]. Here we omit the details. □

Lemma 6.2 *For* $0 < \kappa < 1$, *let* $F(\lambda) = \int_0^\kappa f(t) e^{\mathbf{i}\lambda(\sqrt{1-t^2} - \tau\sqrt{\kappa^2 - t^2} + \alpha t)} dt$, *where* $\tau \geq c_0 > 0$ *and* $\alpha \in \mathbb{R}$, *then we have*

$$|F(\lambda)| \leq C(\kappa)\lambda^{-\frac{1}{2N_*}} \left[|f(\kappa)| + \int_0^\kappa |f'(t)| dt \right]$$

where $N_* = \min\{N | \kappa^{2N-1} < c_0, N \in \mathbb{Z}_+\}$.

Proof. Put $\phi(t) = -\sqrt{1 - t^2}$ and $\psi(t, \tau) = \tau\phi(t/\kappa) - \phi(t) + \alpha t$. For easy of notations, we denote the n-th partial derivative of $g(t)$ with respect to t by $g^{(n)}(t)$. Then, it is to see that, for $n > 1$

$$\psi^{(n)}(t, \tau) = \frac{\tau}{\kappa^{n-1}} \phi^{(n)}\left(\frac{t}{\kappa}\right) - \phi^{(n)}(t)$$

A standard computation show that

$$\phi^{(1)}(t) = \frac{t}{\sqrt{1 - t^2}}$$

$$\phi^{(2)}(t) = \frac{1}{(1 - t^2)^{3/2}}$$

$$\phi^{(3)}(t) = \frac{3t}{(1 - t^2)^{5/2}}$$

Moreover, for $n \geq 3$, we have

$$\phi^{(n)}(t) = \frac{p_n(t)}{(1 - t^2)^{n-1/2}} \tag{6.1}$$

where $p_n = \sum_0^{n-2} a_k^n t^k$ is a $(n - 2)$-th polynomial such that its coefficients satisfy the following recursion formula:

$$a_{n-1}^{n+1} = (n + 1)a_{n-2}^n, \quad a_{n-2}^{n+1} = (n + 2)a_{n-3}^n$$
$$a_k^{n+1} = (k + 1)a_{k+1}^n + (2n - k)a_{k-1}^n \quad \text{for } 1 \leq k \leq n - 3$$
$$a_0^{n+1} = a_1^n$$

Since the polynmial coefficients are all positive, it is obvious that for $n \geq 1$, $\phi^{(n)}(t)$ is a monotone increasing positive function. Using the recursion formula, it follows that

$$\phi^{(n)}(0) = \begin{cases} 0 & \text{n is odd,} \\ (n-1)!!(n-3)!! & \text{n is even.} \end{cases} \tag{6.2}$$

where $(2k-1)!!$ is double factorial and $n > 3$. We are now in the position to proof the inequality. Since $0 < \kappa < 1$, obersev that

$$\psi^{(2N_*+1)}(t,\tau) \geq \frac{\tau}{\kappa^{2N_*}}\phi^{(2N_*+1)}(t) - \phi^{(2N_*+1)}(t) > 0$$

Therefore, $\psi^{(2N_*)}(t,\tau)$ is monotone increasing in $[0,\kappa)$. By (6.2), we get

$$\psi^{(2N_*)}(t,\tau) \geq \psi^{(2N_*)}(0,\tau) \geq \psi^{(2N_*)}(0,c_0) = C(2N_*)(\frac{c_0}{\kappa^{2N_*-1}} - 1) > 0 \tag{6.3}$$

The lemma is now a direct consequence of lemma (6.1). $\qquad\square$

7. Other exponential decay term: 17.11.16 on G1

The parameterization of hyperbolic curve passing $(\pm 1, 0)$ is:

$$\xi_1 = \pm\sqrt{t^2 + 1} \qquad \xi_2 = t$$

where $t \in \mathbb{R}$. Substituting $\xi = \xi_1 + \mathbf{i}\xi_2$ into $\mu(\xi) := (1 - \xi^2)^{1/2}$ and $\mu_\kappa(\xi) := (\kappa^2 - \xi^2)^{1/2}$, we get

$$\begin{aligned}
\operatorname{Im}\mu(\xi) &= \operatorname{Im}(1 - (\xi_1^2 - \xi_2^2 + \mathbf{i}2\xi_1\xi_2))^{1/2} \\
&= \operatorname{Im}(-2t\sqrt{t^2 + 1}\mathbf{i})^{1/2} = t^{1/2}(t^2 + 1)^{1/4}
\end{aligned}$$

$$\begin{aligned}
\operatorname{Im}\mu_\kappa(\xi) &= \operatorname{Im}(\kappa^2 - (\xi_1^2 - \xi_2^2 + \mathbf{i}2\xi_1\xi_2))^{1/2} \\
&= \operatorname{Im}(\kappa^2 - 1 - 2t\sqrt{t^2 + 1}\mathbf{i})^{1/2} \\
&= \sqrt{\frac{\sqrt{(1 - \kappa^2)^2 + 4t(t^2 + 1)} + 1 - \kappa^2}{2}} \\
&\geq t^{1/2}(t^2 + 1)^{1/4}
\end{aligned}$$

where we only consider the branch, denoted by Γ^+, in the first quadrant here. For $a > 0, b > 0$, we have

$$|e^{\mathbf{i}\xi a + \mathbf{i}\mu(\xi)b + \mathbf{i}\mu_\kappa(\xi)c}| \leq e^{-ta - t^{1/2}(t^2+1)^{1/4}b - t^{1/2}(t^2+1)^{1/4}c} \leq e^{-t(b+c)}$$

Lemma 7.1 *For $\xi \in \Gamma_0$, let $f(\xi)$ is a complex valued function in $L^1(\Gamma^+)$ such that $|f(\xi)| \leq C(1 + \xi^k)$, $k \in \mathbb{Z}_+$. Then we have*

$$|I(a,b,c) := \int_{\Gamma^+} f(\xi)\, e^{\mathbf{i}\xi a + \mathbf{i}\mu(\xi)b + \mathbf{i}\mu_\kappa(\xi)c}\, d\xi|$$

$$\leq C(\frac{1}{b+c} + \frac{1}{(b+c)^k})$$

Proof.

$$\frac{d\xi(t)}{dt} = \frac{t}{\sqrt{t^2+1}} + \mathbf{i}$$

Substituting $\xi(t)$ into $I(a,b,c)$, we hvae

$$|I(a,b,c)| = \left| \int_0^\infty |f(\xi(t))\frac{d\xi(t)}{dt} e^{\mathbf{i}\xi(t)a+\mathbf{i}\mu(\xi(t))b+\mathbf{i}\mu_\kappa(\xi(t))c} dt \right|$$

$$\leq C \int_0^\infty (1+t^k)e^{-t(b+c)} dt$$

$$\leq C(\frac{1}{b+c} + \frac{1}{(b+c)^k})$$

$\square$

Lemma 7.2 *Let $f(\xi)$ is a bounded complex valued function in $L^1((\kappa,1))$. Then we have*

$$|I(a,b) := \int_\kappa^1 |f(\xi)e^{\mathbf{i}\xi a+\mathbf{i}\mu_\kappa(\xi)b} d\xi|$$

$$\leq C\frac{1}{b}$$

Proof. It is simple to see that

$$|I(a,b)| \leq C \int_\kappa^1 e^{-b\sqrt{\xi^2-\kappa^2}} d\xi$$

$$\leq C \int_0^{\sqrt{1-\kappa^2}} \frac{t}{\sqrt{t^2+\kappa^2}} e^{-bt} dt$$

$$\leq C\frac{1}{b}$$

$\square$

8. about principle of arguement

Put

$$\delta_\pm(t) = (\kappa - 2t^2)^2 \mp \mathbf{i}4t^2\sqrt{1-t^2}\sqrt{t^2-\kappa}$$

$$:= f_1(t) \mp \mathbf{i}f_2(t)$$

where $0 < \kappa < 1$ and we have

$$\delta'_\pm(t) = f_1'(t) \mp \mathbf{i}f_2'(t)$$

It is easy to see $f_2(1) = f_2(\kappa) = 0$ and $f_1(t) > 0$ for any $\kappa \leq t \leq 1$. Then

$$\int_\kappa^1 \frac{\delta'_+(t)}{\delta_+(t)} - \frac{\delta'_-(t)}{\delta_-(t)} dt$$

$$= 2\mathbf{i} \int_\kappa^1 \mathrm{Im}\,(\frac{\delta'_+(t)}{\delta_+(t)}) dt$$

$$= 2\mathbf{i} \int_\kappa^1 \text{Im}\, \frac{(f_1'(t) - \mathbf{i}f_2'(t))f_1(t) + \mathbf{i}f_2(t))}{(f_1(t) - \mathbf{i}f_2(t))(f_1(t) + \mathbf{i}f_2(t))}\, dt$$

$$= 2\mathbf{i} \int_\kappa^1 \frac{f_1'(t)f_2(t) - f_1(t)f_2'(t)}{f_1^2(t) + f_2^2(t)}\, dt$$

$$= 2\mathbf{i} \int_\kappa^1 \frac{f_1^2(t)}{f_1^2(t) + f_2^2(t)} \frac{f_1'(t)f_2(t) - f_1(t)f_2'(t)}{f_1^2(t)}\, dt$$

$$= -2\mathbf{i} \int_\kappa^1 \frac{f_1^2(t)}{f_1^2(t) + f_2^2(t)} d\frac{f_2(t)}{f_1(t)}$$

$$= -2\mathbf{i} \arctan \frac{f_2(t)}{f_1(t)}\Big|_\kappa^1 = 0$$

Notic that, the condition only used above are $f_2(1) = f_2(\kappa) = 0$ and $f_1(t) > 0$.

9. Fundamental solution of Elastic wave

$$G(x; y) = \frac{1}{\omega^2}(\nabla \times \nabla \cdot (g_s(x; y)\mathbb{I}) - \nabla\nabla g_p(x; y)) \tag{9.1}$$

$$= \frac{1}{\omega^2}(k_s^2 g_s(x, y) + \nabla\nabla(g_s(x; y) - g_p(x; y))) \tag{9.2}$$

where y is the Dirac source, $g_p(x; y)$ or $g_s(x; y)$ is the fundamental solution of the scalar Helmhotlz equation with wavenumbers $k_p = \omega/c_p$ or $k_s = \omega/cs$.

$$g_\alpha = \frac{\mathbf{i}}{4} H_0^{(1)}(k_\alpha|x - y|) \tag{9.3}$$

where $H_0^{(1)}(t)$ is the Hankel function of the first type and order zero. By straight calculation using $H_1^{(1)}(t) = -dH_0^{(1)}(t)/dt$ and $dH_1^{(1)}(t)/dt = H_0^{(1)}(t) - H_1^{(1)}(t)/t$, we have

$$G_{ij}(x; y) = \frac{\mathbf{i}}{4}\{(\frac{k_s^2}{\omega^2} H_0^{(1)}(k_s|x - y|) - \frac{1}{\omega^2} \frac{k_s H_1^{(1)}(k_s|x - y| - k_p H_1^{(1)}(k_p|x - y|)}{|x - y|})\delta_{ij}$$

$$+ \frac{1}{\omega^2}[(\frac{2k_s H_1^{(1)}(k_s|x - y| - 2k_p H_1^{(1)}(k_p|x - y|)}{|x - y|} - (k_s^2 H_0^{(1)}(k_s|x - y|) - k_p^2 H_0^{(1)}(k_p|x - y|))]\frac{(x_i - y_i)(x_j - y_j)}{|x - y|^2}\}$$

The definition of hankal function is $H_k^{(1)}(t) = J_k(t) + \mathbf{i}Y_k(t)$ where

$$J_k(t) = \sum_{p=0}^\infty \frac{(-1)^p}{p!(k + p)!}(t/2)^{k+2p}$$

Specially

$$J_0(t) = \sum_{p=0}^\infty \frac{(-1)^p}{p!p!}(t/2)^{2p} = 1 + \cdots$$

$$J_1(t) = \sum_{p=0}^\infty \frac{(-1)^p}{p!(1 + p)!}(t/2)^{1+2p} = \frac{t}{2} + \cdots$$

and

$$Y_k(t) = \frac{1}{\pi}\{\ln t^2 - 2\ln 2 + 2C_{euler}\}J_k(t) - \frac{1}{\pi}\sum_{p=0}^{k-1}\frac{(k-1-p)!}{p!}(2/t)^{k-2p}$$

$$- \frac{1}{\pi}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!(k+p)!}(t/2)^{k+2p}\{\psi(p+k)+\psi(p)\}$$

Speceilly

$$Y_0(t) = \frac{1}{\pi}\{\ln t^2 - 2\ln 2 + 2C_{euler}\}J_0(t) - \frac{1}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^p}{p!p!}(t/2)^{2p}\{2\psi(p)\}$$

$$Y_1(t) = \frac{1}{\pi}\{\ln t^2 - 2\ln 2 + 2C_{euler}\}J_1(t) - \frac{1}{\pi}\frac{2}{t} - \frac{t}{2\pi}$$

$$- \frac{1}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^p}{p!(1+p)!}(t/2)^{1+2p}\{\psi(p+1)+\psi(p)\}$$

Thus, we have

$$H_0^{(1)}(kr) = 1 + \mathbf{i}\frac{2}{\pi}(C_{euler}+\ln k-\ln 2) + \mathbf{i}\frac{1}{\pi}\ln r^2 + o(kr)$$

$$H_1^{(1)}(kr) = \frac{kr}{2} + \mathbf{i}\frac{1}{\pi}(C_{euler}+\ln k-\ln 2-\frac{1}{2})kr - \mathbf{i}\frac{1}{\pi}\frac{2}{kr} + \mathbf{i}\frac{1}{\pi}\ln r^2\frac{kr}{2} + o(k^2r^2)$$

$$\frac{\mathbf{i}}{4}H_0^{(1)}(kr) = \frac{\mathbf{i}}{4} - \frac{1}{2\pi}(C_{euler}+\ln k-\ln 2) - \frac{1}{4\pi}\ln r^2 + o(kr)$$

$$\frac{\mathbf{i}}{4}H_1^{(1)}(kr) = \mathbf{i}\frac{kr}{8} - \frac{1}{4\pi}(C_{euler}+\ln k-\ln 2-\frac{1}{2})kr + \frac{1}{4\pi}\frac{2}{kr} - \frac{1}{4\pi}\ln r^2\frac{kr}{2} + o(k^2r^2)$$

We also need to define the surface traction $T_x^n(\cdot)$ on the normal direction n,

$$T_x^n u(x) := \sigma \cdot n = 2\mu\frac{\partial u}{\partial n} + \lambda n\mathrm{div}u + \mu n \times \mathrm{curl}u$$

where

$$\sigma(u) = \begin{pmatrix} (\lambda+2\mu)\partial u_1/\partial x_1 + \lambda\partial u_2/\partial x_2 & \mu\partial u_1/\partial x_2 + \mu\partial u_2/\partial x_1 \\ \mu\partial u_1/\partial x_2 + \mu\partial u_2/\partial x_1 & (\lambda+2\mu)\partial u_2/\partial(x_2) + \lambda\partial u_1/\partial x_1 \end{pmatrix}$$

A simple computation show that

$$\frac{\partial^3 H_0^{(1)}(k|x-y|)}{\partial x_i^2\partial x_j} = (1+2\delta_i j)(-k^2 H_0^{(1)}(kr)\frac{r_j}{r^2} + 2kH_1^{(1)}(kr)\frac{r_j}{r^3})$$

$$+ k^3 H_1^{(1)}(kr)\frac{r_i^2 r_j}{r^3} + 4k^2 H_0^{(1)}(kr)\frac{r_i^2 r_j}{r^4} - 8kH_1^{(1)}(kr)\frac{r_i^2 r_j}{r^5}$$

where $r = |x-y|$ and $r_i = x_i - y_i$.

$$\frac{\mathbf{i}}{4}H_0^{(1)}(kr) = -\frac{1}{2\pi}(\ln\frac{kr}{2}+C_{euler})(1-(\frac{kr}{2})^2+\cdots) + \frac{1}{4\pi}(2(\frac{kr}{2})^2+\cdots) + \frac{\mathbf{i}}{4}(1-(\frac{kr}{2})^2+\cdots)$$

$$= -\frac{1}{2\pi}(\ln\frac{kr}{2})(1+O(r^2)) - \frac{1}{2\pi}C_{euler} + \frac{\mathbf{i}}{4} + O(r^2)$$

$$\frac{\mathbf{i}}{4}H_1^{(1)}(kr) = -\frac{1}{2\pi}(\ln\frac{kr}{2}+C_{euler})(\frac{kr}{2}-\frac{1}{2}(\frac{kr}{2})^3+\cdots) + \frac{1}{4\pi}(\frac{kr}{2}+O(r^3)) + \frac{\mathbf{i}}{4}(\frac{kr}{2}-\frac{1}{2}(\frac{kr}{2})^3+\cdots) + \frac{1}{2\pi}\frac{1}{kr}$$

$$= -\frac{1}{4\pi}(\ln\frac{kr}{2})(kr+O(r^3)) - \frac{kr}{4\pi}C_{euler} + \frac{kr}{8\pi} + \frac{\mathbf{i}kr}{8} + \frac{1}{2\pi}\frac{1}{kr} + O(r^3)$$

$$A(kr) := \frac{\mathbf{i}}{4}(k^2 H_0^{(1)}(kr) - 2kH_1^{(1)}(kr)/r) = \frac{k^2}{4\pi}(\ln\frac{kr}{2})(\frac{kr}{2})^2 - \frac{1}{\pi r^2} - \frac{k^2}{4\pi} + O(r^2)$$

and

$$A_{sp}(r) = A(k_s r) - A(k_p r) = \frac{k_s^2}{4\pi}(\ln\frac{kr}{2})(\frac{kr}{2})^2 - \frac{k_p^2}{4\pi}(\ln\frac{kr}{2})(\frac{kr}{2})^2 - (\frac{k_s^2}{4\pi} - \frac{k_p^2}{4\pi}) + O(r^2)$$

Let $g^{jkk} = \frac{\partial^3 g}{\partial x_j \partial x_k^2}$, $d = g_s - g_p$, thus

$$g^{iij} = (1 + 2\delta_{ij})(-A(kr)\frac{r_j}{r^2}) + \frac{\mathbf{i}k^3}{4}H_1^{(1)}(kr)\frac{r_i^2 r_j}{r^3} + 4A(kr)\frac{r_i^2 r_j}{r^4}$$

$$= (1 + 2\delta_{ij})(\frac{1}{\pi r^3} + \frac{k^2}{4\pi r})\frac{r_j}{r} - (\frac{4}{\pi r^3} + \frac{k^2}{2\pi r})\frac{r_i^2 r_j}{r^3} + O(r \ln r)$$

$$d^{iij} = (1 + 2\delta_{ij})(-A_{sp}\frac{r_j}{r^2}) + (\frac{\mathbf{i}k_s^3}{4}H_1^{(1)}(k_s r) - \frac{\mathbf{i}k_p^3}{4}H_1^{(1)}(k_p r))\frac{r_i^2 r_j}{r^3} + 4A_{sp}\frac{r_i^2 r_j}{r^4}$$

$$= (1 + 2\delta_{ij})(\frac{k_s^2}{4\pi r} - \frac{k_p^2}{4\pi r})\frac{r_j}{r} - (\frac{k_s^2}{2\pi r} - \frac{k_p^2}{2\pi r})\frac{r_i^2 r_j}{r^3} + O(r \ln r)$$

$$= (1 + 2\delta_{ij})\frac{(\lambda + \mu)\omega^2}{\mu(\lambda + 2\mu)}\frac{1}{4\pi r}\frac{r_j}{r} - \frac{(\lambda + \mu)\omega^2}{\mu(\lambda + 2\mu)}\frac{1}{2\pi r}\frac{r_i^2 r_j}{r^3} + O(r \ln r)$$

$$k^2 g^i = -\frac{\mathbf{i}}{4}H_1^{(1)}(kr)\frac{kr_i}{r} = -\frac{k^2}{2\pi r}\frac{r_i}{r} + O(r \ln r)$$

$$d^{iij} + d^{jjj} = 2\frac{(\lambda + \mu)\omega^2}{\mu(\lambda + 2\mu)}\frac{1}{4\pi r}\frac{r_j}{r} + O(r \ln r)$$

Then we have [5, p43]

$$\sigma(Ge_1)n = \frac{1}{\omega^2}\begin{pmatrix} (\lambda + 2\mu)(k_s^2 g_s^1 + d^{111}) + \lambda d^{122} & \mu(k_s^2 g_s^2 + d^{112}) + \mu d^{112} \\ \mu(k_s^2 g_s^2 + d^{112}) + \mu d^{112} & (\lambda + 2\mu)d^{122} + \lambda(k_s^2 g_s^1 + d^{111}) \end{pmatrix}n$$

$$= \frac{\mu}{2\pi(\lambda + 2\mu)}\left(\begin{pmatrix} \frac{2(\lambda+\mu)r_1^2}{\mu r^2} + 1 \\ \frac{2(\lambda+\mu)r_1 r_2}{\mu r^2} \end{pmatrix}(-\frac{r_1 n_1}{r^2} - \frac{r_2 n_2}{r^2}) - \begin{pmatrix} 0 \\ \frac{r_2 n_1 - r_1 n_2}{r^2} \end{pmatrix}\right) + O(r \ln r)$$

and

$$\sigma(Ge_2)n = \frac{1}{\omega^2}\begin{pmatrix} (\lambda + 2\mu)d^{112} + \lambda(k_s^2 g_s^2 + d^{222}) & \mu(k_s^2 g_s^1 + d^{122}) + \mu d^{122} \\ \mu(k_s^2 g_s^1 + d^{122}) + \mu d^{122} & (\lambda + 2\mu)(k_s^2 g_s^2 + d^{222}) + \lambda d^{112} \end{pmatrix}n$$

$$= \frac{\mu}{2\pi(\lambda + 2\mu)}\left(\begin{pmatrix} \frac{2(\lambda+\mu)r_1 r_2}{\mu r^2} \\ \frac{2(\lambda+\mu)r_2^2}{\mu r^2} + 1 \end{pmatrix}(-\frac{r_1 n_1}{r^2} - \frac{r_2 n_2}{r^2}) - \begin{pmatrix} \frac{r_1 n_2 - r_2 n_1}{r^2} \\ 0 \end{pmatrix}\right) + O(r \ln r)$$

Now Let u be represented as single potential:

$$u = \int_{\partial D} G(x, y)\phi(y)ds(y) \tag{9.4}$$

with Neumann boundary condition

$$T_x u(x) = f(x) \quad \text{on} \quad \partial D \tag{9.5}$$

Then we obtain corresponding integral equation

$$\mathbf{P.V.}\int_{\partial D} T_x G(x, y)\phi(y)ds(y) - \frac{1}{2}\phi(x) = f(x) \tag{9.6}$$

where $x \in \partial D$. We describe the necessary parametrization of the integral equation in the two-dimensional case. We assume that the boundary curve ∂D possesses a regular analytic and 2π-periodic parametric representation of the form

$$x(t) = (x_1(t), x_2(t))$$

in counterclockwise orientation satisfying $|x'(t)| > 0$ for all t. Let [4]

$$T(x, y) = (T_x(N_1(x, y))n, T_x(N_1(x, y))n)$$

$$T_0(x, y) = -\frac{\mu}{2\pi(\lambda + 2\mu)} \begin{pmatrix} 0 & \frac{r_1 n_2 - r_2 n_1}{r^2} \\ \frac{r_2 n_1 - r_1 n_2}{r^2} & 0 \end{pmatrix}$$

$$T_1(x, y) = T(x, y) - T_0(x, y)$$

Then by above analysis we have

$$\int_0^{2\pi} T_1(x(t), x(\tau))\phi(x(\tau))|x'(\tau)|dt + \mathbf{P.V.} \int_0^{2\pi} T_0(x(t), x(\tau))\phi(x(\tau))|x'(\tau)|dt - \frac{1}{2}\phi(x(t)) = f(x(t))$$

In particular, using expasion above, we can deduce the diagonal terms:

$$\lim_{\tau \to t} \frac{-r_1 n_1 - r_2 n_2}{r^2} = \lim_{\tau \to t} \frac{(x_1(\tau) - x_1(t))x_2'(t) - (x_2(\tau) - x_2(t))x_1'(t)}{|x(t) - x(\tau)|^2|x'(t)|} = \frac{x_1''(t)x_2'(t) - x_2''(t)x_1'(t)}{2|x'(t)|^3}$$

and

$$\lim_{\tau \to t} \frac{(r_1 n_2 - r_2 n_1)|x'(\tau)|(\tau - t)}{r^2}$$

$$= \lim_{\tau \to t} \frac{(x_1(\tau) - x_1(t))x_1'(t) + (x_2(\tau) - x_2(t))x_2'(t))|x'(\tau)|(\tau - t)}{|x(t) - x(\tau)|^2|x'(t)|} = 1$$

According to above analysis, it is easy to see that T_1 has no singularity. Therefore, the numerical formulartion for T_1 requires only straightforward application of simple quadrature formula. we choose an equidistant set of knots $t_j := \pi j/n, j = 0, ..., 2n - 1$ and divide $[0, 2\pi)$ into n equivalent interval $I_i = [2j\pi/n, 2(j + 1)\pi/n)$ where $[0, 2\pi) = \bigcup_{j=1}^{j=n-1} I_i$. Each I_i is described by 3 nodes of the intrinsic variable $\xi(-1 \le \xi \le 1)$ and the quadratic shape functions are:

$$A_{-1}(\xi) = \frac{\xi(\xi - 1)}{2}$$
$$A_0(\xi) = 1 - \xi^2$$
$$A_1(\xi) = \frac{\xi(\xi + 1)}{2}$$

let

$$f(x, y) = \frac{(x_1 - y_1)n_2^x - (x_2 - y_2)n_1^x}{|x - y|^2}$$

and it is easy to see that $(n_x^1, n_x^2) = (x_2'(t), -x_1'(t))/|x'(t)|$. Now, we can repersent the integral in variable $\xi \in [-1, 1]$. For $x \in \partial D$ we have:

$$\mathbf{P.V.} \int_{\partial D} f(x, y)\phi(y)ds(y)$$

$$= \sum_{x \notin I_i} \int_{I_i} f(x, x(t))\phi(x(t))|x'(t)|dt + \mathbf{P.V.} \sum_{x \in I_i} \int_{I_i} f(x, x(t))\phi(x(t))|x'(t)|dt$$

if $x = x(2j\pi/n)$, above integral becomes to

$$\sum_{i\neq j, j-1} \frac{\pi}{n} \int_{-1}^{1} \frac{\sum_{k=1,2}(x_k^i(\xi) - x_k\frac{2j\pi}{n}))x_k'(\frac{2j\pi}{n})}{|x(\frac{2j\pi}{n}) - x^i(\xi)|^2}\phi(x^i(\xi))\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{2j\pi}{n})|}d\xi$$

$$+ \mathbf{P.V.} \sum_{i=j, j-1} \frac{\pi}{n} \int_{-1}^{1} \frac{\sum_{k=1,2}(x_k^i(\xi) - x_k\frac{2j\pi}{n}))x_k'(\frac{2j\pi}{n})}{|x(\frac{2j\pi}{n}) - x^i(\xi)|^2}\phi(x^i(\xi))\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{2j\pi}{n})|}d\xi$$

$$\approx \sum_{i\neq j, j-1}\sum_{l=-1}^{1} \phi(x^i(l))\frac{\pi}{n} \int_{-1}^{1} \frac{\sum_{k=1,2}(x_k^i(\xi) - x_k(\frac{2j\pi}{n}))x_k'(\frac{2j\pi}{n})}{|x(\frac{2j\pi}{n}) - x^i(\xi)|^2}A_l(\xi)\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{2j\pi}{n})|}d\xi$$

$$+ \phi(x(\frac{2j\pi}{n}))\mathbf{P.V.}\frac{\pi}{n}\left(\int_{-1}^{1} \frac{\sum_{k=1,2}(x_k^j(\xi) - x_k^j(-1))x_k'(\frac{2j\pi}{n})}{|x^j(-1) - x^j(\xi)|^2}A_{-1}(\xi)\frac{|x'(\frac{(2j+1+\xi)\pi}{n})|}{|x'(\frac{2j\pi}{n})|}d\xi\right.$$

$$\left.+ \int_{-1}^{1} \frac{\sum_{k=1,2}(x_k^{j-1}(\xi) - x_k^{j-1}(1))x_k'(\frac{2j\pi}{n})}{|x^{j-1}(1) - x^{j-1}(\xi)|^2}A_1(\xi)\frac{|x'(\frac{(2j-1+\xi)\pi}{n})|}{|x'(\frac{2j\pi}{n})|}d\xi\right)$$

and if $x = x((2j+1)\pi/n)$, above integral becomes to

$$\sum_{i\neq j,} \frac{\pi}{n} \int_{-1}^{1} \frac{\sum_{k=1,2}(x_k^i(\xi) - x_k\frac{2j+1\pi}{n}))x_k'(\frac{2j+1\pi}{n})}{|x(\frac{2j+1\pi}{n}) - x^i(\xi)|^2}\phi(x^i(\xi))\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{2j+1\pi}{n})|}d\xi$$

$$+ \mathbf{P.V.} \sum_{i=j, j-1} \frac{\pi}{n} \int_{-1}^{1} \frac{\sum_{k=1,2}(x_k^i(\xi) - x_k\frac{2j+1\pi}{n}))x_k'(\frac{2j+1\pi}{n})}{|x(\frac{2j+1\pi}{n}) - x^i(\xi)|^2}\phi(x^i(\xi))\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{2j+1\pi}{n})|}d\xi$$

$$\approx \sum_{i\neq j}\sum_{l=-1}^{1} \phi(x^i(l))\frac{\pi}{n} \int_{-1}^{1} \frac{\sum_{k=1,2}(x_k^i(\xi) - x_k(\frac{(2j+1)\pi}{n}))x_k'(\frac{(2j+1)\pi}{n})}{|x(\frac{(2j+1)\pi}{n}) - x^i(\xi)|^2}A_l(\xi)\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{(2j+1)\pi}{n})|}d\xi$$

$$+ \phi(x(\frac{(2j+1)\pi}{n}))\mathbf{P.V.}\frac{\pi}{n}\left(\int_{-1}^{1} \frac{\sum_{k=1,2}(x_k^j(\xi) - x_k^j(0))x_k'(\frac{(2j+1)\pi}{n})}{|x^j(0) - x^j(\xi)|^2}A_0(\xi)\frac{|x'(\frac{((2j+1)+\xi)\pi}{n})|}{|x'(\frac{(2j+1)\pi}{n})|}d\xi\right.$$

where $x^i(\xi) = x((2i+1+\xi)\pi/n)$. For simplicity, we denote a 3-D tensor M_{jil}:

$$M_{jil} = \frac{\pi}{n} \int_{-1}^{1} \frac{\sum_{k=1,2}(x_k^i(\xi) - x_k(\frac{j\pi}{n}))x_k'(\frac{j\pi}{n})}{|x(\frac{j\pi}{n}) - x^i(\xi)|^2}A_l(\xi)\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{j\pi}{n})|}d\xi$$

Then, the general Matrix is

$$G_{ji} = \begin{cases} M_{j,i,-1} + M_{j,i-1,1} & j \text{ is even} \\ M_{j,i,0} & j \text{ is odd} \end{cases}$$

10. Traction Tenson of Neumann Green Function

$$\hat{\Phi}(\xi, x_2; y_2) = \frac{\mathbf{i}}{2\omega^2}\left[\left(\begin{array}{cc} \mu_s & -\xi\frac{x_2-y_2}{|x_2-y_2|} \\ -\xi\frac{x_2-y_2}{|x_2-y_2|} & \frac{\xi^2}{\mu_s} \end{array}\right)e^{\mathbf{i}\mu_s|x_2-y_2|} + \left(\begin{array}{cc} \frac{\xi^2}{\mu_p} & \xi\frac{x_2-y_2}{|x_2-y_2|} \\ \xi\frac{x_2-y_2}{|x_2-y_2|} & \mu_p \end{array}\right)e^{\mathbf{i}\mu_p|x_2-y_2|}\right]$$

$$\hat{N}(\xi, x_2; y_2) = \hat{\Phi}(\xi, x_2; y_2) - \hat{\Phi}(\xi, x_2; -y_2) + \hat{N}_c(\xi, x_2; y_2) \tag{10.1}$$

$$\hat{N}_c(\xi, x_2; y_2) = \ = \frac{\mathbf{i}}{\omega^2 \delta(\xi)} \Big\{ A(\xi) e^{\mathbf{i}\mu_s(x_2+y_2)} + B(\xi) e^{\mathbf{i}\mu_p(x_2+y_2)} \tag{10.2}$$

$$+ C(\xi) e^{\mathbf{i}\mu_s x_2 + \mu_p y_2} + D(\xi) e^{\mathbf{i}\mu_p x_2 + \mu_s y_2} \Big\}$$

$$\sigma(\Phi)(\xi, x_2; y_2) e_1 = \frac{\mu}{2\omega^2} \left[\begin{pmatrix} -2\xi\mu_s & \frac{x_2-y_2}{|x_2-y_2|} 2\xi^2 \\ -\frac{x_2-y_2}{|x_2-y_2|}\beta & \frac{\xi\beta}{\mu_s} \end{pmatrix} e^{\mathbf{i}\mu_s|x_2-y_2|} + \begin{pmatrix} -\frac{\alpha\xi}{\mu_p} & -\frac{x_2-y_2}{|x_2-y_2|}\alpha \\ -\frac{x_2-y_2}{|x_2-y_2|}2\xi^2 & -2\xi\mu_p \end{pmatrix} e^{\mathbf{i}\mu_p|x_2-y_2|} \right]$$

$$\sigma(\Phi)(\xi, x_2; y_2) e_2 = \frac{\mu}{2\omega^2} \left[\begin{pmatrix} -\frac{x_2-y_2}{|x_2-y_2|}\beta & \frac{\xi\beta}{\mu_s} \\ 2\xi\mu_s & -\frac{x_2-y_2}{|x_2-y_2|}2\xi^2 \end{pmatrix} e^{\mathbf{i}\mu_s|x_2-y_2|} + \begin{pmatrix} -\frac{x_2-y_2}{|x_2-y_2|}2\xi^2 & -2\xi\mu_p \\ -\frac{\beta\xi}{\mu_p} & -\frac{x_2-y_2}{|x_2-y_2|}\beta \end{pmatrix} e^{\mathbf{i}\mu_p|x_2-y_2|} \right]$$

where $\alpha(\xi) = k_s^2 - 2\mu_p^2$, $\beta(\xi) = k_s^2 - 2\xi^2$.

$$\hat{N}(\xi, x_2; y_2) = \hat{\Phi}(\xi, x_2; y_2) - \hat{\Phi}(\xi, x_2; -y_2) + \hat{N}_c(\xi, x_2; y_2) \tag{10.3}$$

$$\hat{N}_c(\xi, x_2; y_2) = \frac{\mathbf{i}}{\omega^2 \delta(\xi)} \Big\{ A(\xi) e^{\mathbf{i}\mu_s(x_2+y_2)} + B(\xi) e^{\mathbf{i}\mu_p(x_2+y_2)} \tag{10.4}$$

$$+ C(\xi) e^{\mathbf{i}\mu_s x_2 + \mu_p y_2} + D(\xi) e^{\mathbf{i}\mu_p x_2 + \mu_s y_2} \Big\}$$

where

$$A(\xi) = \begin{pmatrix} \mu_s \beta^2 & -4\xi^3 \mu_s \mu_p \\ -\xi\beta^2 & 4\xi_4 \mu_p \end{pmatrix} \qquad B(\xi) = \begin{pmatrix} 4\xi^4 \mu_s & \xi\beta^2 \\ 4\xi^3 \mu_s \mu_p & \mu_p \beta^2 \end{pmatrix}$$

$$C(\xi) = \begin{pmatrix} 2\xi^2 \mu_s \beta & -2\xi \mu_s \mu_p \beta \\ -2\xi^3 \beta & 2\xi^2 \mu_p \beta \end{pmatrix} \qquad D(\xi) = \begin{pmatrix} 2\xi^2 \mu_s \beta & 2\xi^3 \beta \\ 2\xi \mu_s \mu_p \beta & 2\xi^2 \mu_p \beta \end{pmatrix}$$

The traction of integral part N_c:

$$\sigma(N_c(\xi, x_2; y_2)) e_1 = \frac{-\mu}{\omega^2 \delta(\xi)} \Big\{ A(\xi) e^{\mathbf{i}\mu_s(x_2+y_2)} + B(\xi) e^{\mathbf{i}\mu_p(x_2+y_2)}$$

$$+ C(\xi) e^{\mathbf{i}\mu_s x_2 + \mu_p y_2} + D(\xi) e^{\mathbf{i}\mu_p x_2 + \mu_s y_2} \Big\}$$

where

$$A(\xi) = \begin{pmatrix} 2\xi \mu_s \beta^2 & -8\mu_s \mu_p \xi^4 \\ \beta^3 & -4\xi^3 \mu_p \beta \end{pmatrix} \qquad B(\xi) = \begin{pmatrix} 4\xi^3 \mu_s \alpha & \alpha\beta^2 \\ 8\xi^4 \mu_s \mu_p & 2\xi \mu_p \beta^2 \end{pmatrix}$$

$$C(\xi) = \begin{pmatrix} 4\xi^3 \mu_s \beta & -4\xi^2 \mu_s \mu_p \beta \\ 2\xi^2 \beta^2 & -2\xi \mu_p \beta^2 \end{pmatrix} \qquad D(\xi) = \begin{pmatrix} 2\xi \mu_s \alpha\beta & 2\xi^2 \alpha\beta \\ 4\xi^2 \mu_s \mu_p \beta & 4\xi^3 \mu_p \beta \end{pmatrix}$$

and

$$\sigma(N_c(\xi, x_2; y_2)) e_2 = \frac{-\mu}{\omega^2 \delta(\xi)} \Big\{ A(\xi) e^{\mathbf{i}\mu_s(x_2+y_2)} + B(\xi) e^{\mathbf{i}\mu_p(x_2+y_2)}$$

$$+ C(\xi) e^{\mathbf{i}\mu_s x_2 + \mu_p y_2} + D(\xi) e^{\mathbf{i}\mu_p x_2 + \mu_s y_2} \Big\}$$

where

$$A(\xi) = \begin{pmatrix} \beta^3 & -4\xi^3\mu_p\beta \\ -2\xi\mu_s\beta^2 & 8\mu_s\mu_p\xi^4 \end{pmatrix} \qquad B(\xi) = \begin{pmatrix} 8\xi^4\mu_s\mu_p & 2\xi\mu_p\beta^2 \\ 4\xi^3\mu_s\beta & \beta^3 \end{pmatrix}$$

$$C(\xi) = \begin{pmatrix} 2\xi^2\beta^2 & -2\xi\mu_p\beta^2 \\ -4\xi^3\mu_s\beta & 4\xi^2\mu_s\mu_p\beta \end{pmatrix} \qquad D(\xi) = \begin{pmatrix} 4\xi^2\mu_s\mu_p\beta & 4\xi^3\mu_p\beta \\ 2\xi\mu_s\beta^2 & 2\xi^2\beta^2 \end{pmatrix}$$

where $\delta(\xi) = \beta^2 + 4\xi^2\mu_s\mu_p$

11. reflection of Plane wave

11.1. P-wave

We denote incident P-wave [1, p172] as

$$u^0 = A_0(\sin t_0, \cos t_0)^T e^{\mathbf{i}k_p(x_1\sin t_0 + x_2\cos t_0)} \tag{11.1}$$

and its stress as

$$\sigma(u^0) = \mathbf{i}k_p A_0(2\mu\sin t_0\cos t_0, \lambda + 2\mu\cos^2 t_0)^T e^{\mathbf{i}k_p(x_1\sin t_0 + x_2\cos t_0)}$$

The reflected P-wave is represented as

$$u^1 = A_1(\sin t_1, -\cos t_1)^T e^{\mathbf{i}k_p(x_1\sin t_1 - x_2\cos t_1)}$$
$$\sigma(u^1) = \mathbf{i}k_p A_1(-2\mu\sin t_1\cos t_1, \lambda + 2\mu\cos^2 t_1)^T e^{\mathbf{i}k_p(x_1\sin t_1 + x_2\cos t_1)}$$

and reflected S-wave as

$$u^2 = A_2(\cos t_2, \sin t_2)^T e^{\mathbf{i}k_s(x_1\sin t_2 - x_2\cos t_2)}$$
$$\sigma(u^2) = \mathbf{i}k_s A_2(\mu(\sin^2 t_2 - cos^2 t_2), -2\mu\sin t_2\cos t_2)^T e^{\mathbf{i}k_s(x_1\sin t_2 - x_2\cos t_2)}$$

We consider the clamped condition, then the total field on the $x_2 = 0$ vanish:

$$u^0(x_1, 0) + u^1(x_1, 0) + u^2(x_1, 0) = 0$$

for any $x_1 \in \mathbb{R}$. A simple computation show that

$$t_1 = t_0 \qquad \text{and} \qquad \frac{\sin t_2}{\sin t_0} = \frac{k_p}{k_s} := \kappa$$
$$A_0 = \cos(t_0 - t_2) \qquad A_1 = \cos(t_0 + t_2) \qquad A_2 = -\sin 2t_0$$

11.2. S-wave

Similarly, we denote incident S-wave as

$$u^0 = A_0(-\cos t_0, \sin t_0)^T e^{\mathbf{i}k_p(x_1\sin t_0 + x_2\cos t_0)} \tag{11.2}$$
$$\sigma(u^0) = \mathbf{i}k_s(\mu(\sin^2 t_0 - cos^2 t_0), 2\mu\sin t_0\cos t_0)e^{\mathbf{i}k_p(x_1\sin t_0 + x_2\cos t_0)} \tag{11.3}$$

The reflected P-wave is represented as

$$u^1 = A_1(\sin t_1, -\cos t_1)^T e^{\mathbf{i}k_p(x_1\sin t_1 - x_2\cos t_1)}$$
$$\sigma(u^1) = \mathbf{i}k_p A_1(-2\mu\sin t_1\cos t_1, \lambda + 2\mu\cos^2 t_1)^T e^{\mathbf{i}k_p(x_1\sin t_1 + x_2\cos t_1)}$$

and reflected S-wave as

$$u^2 = A_2(\cos t_2, \sin t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$

$$\sigma(u^2) = \mathbf{i}k_s A_2(\mu(\sin^2 t_2 - cos^2 t_2), -2\mu \sin t_2 \cos t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$

The result is

$$t_2 = t_0 \qquad \text{and} \qquad \frac{\sin t_1}{\sin t_0} = \frac{k_s}{k_p} = \frac{1}{\kappa}$$

$$A_0 = \cos(t_0 - t_1) \qquad A_1 = \sin 2t_0 \qquad A_2 = \cos(t_0 + t_1)$$

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