Scattering Coefficient and Kirchhoff Approximation

1. Reflection of Plane Wave

1.1. P-wave

We denote incident P-wave [1, p172] as

$$u^0 = A_0(\sin t_0, \cos t_0)^T e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)} \tag{1.1}$$

and its stress as

$$\sigma(u^0) = \mathbf{i}k_p A_0(2\mu \sin t_0 \cos t_0, \lambda + 2\mu \cos^2 t_0)^T e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)}$$

The reflected P-wave is represented as

$$u^1 = A_1(\sin t_1, -\cos t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 - x_2 \cos t_1)}$$
$$\sigma(u^1) = \mathbf{i}k_p A_1(-2\mu \sin t_1 \cos t_1, \lambda + 2\mu \cos^2 t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 + x_2 \cos t_1)}$$

and reflected S-wave as

$$u^2 = A_2(\cos t_2, \sin t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$
$$\sigma(u^2) = \mathbf{i}k_s A_2(\mu(\sin^2 t_2 - \cos^2 t_2), -2\mu \sin t_2 \cos t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$

We consider the clamped condition, then the total field on the $x_2 = 0$ vanish:

$$u^0(x_1, 0) + u^1(x_1, 0) + u^2(x_1, 0) = 0$$

for any $x_1 \in \mathbb{R}$. A simple computation show that

$$t_1 = t_0 \qquad \text{and} \qquad \frac{\sin t_2}{\sin t_0} = \frac{k_p}{k_s} := \kappa$$
$$A_0 = \cos(t_0 - t_2) \qquad A_1 = \cos(t_0 + t_2) \qquad A_2 = -\sin 2t_0$$

1.2. S-wave

Similarly, we denote incident S-wave as

$$u^0 = A_0(-\cos t_0, \sin t_0)^T e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)} \tag{1.2}$$
$$\sigma(u^0) = \mathbf{i}k_s(\mu(\sin^2 t_0 - \cos^2 t_0), 2\mu \sin t_0 \cos t_0) e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)} \tag{1.3}$$

The reflected P-wave is represented as

$$u^1 = A_1(\sin t_1, -\cos t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 - x_2 \cos t_1)}$$
$$\sigma(u^1) = \mathbf{i}k_p A_1(-2\mu \sin t_1 \cos t_1, \lambda + 2\mu \cos^2 t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 + x_2 \cos t_1)}$$

and reflected S-wave as

$$u^2 = A_2(\cos t_2, \sin t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$
$$\sigma(u^2) = \mathbf{i}k_s A_2(\mu(\sin^2 t_2 - cos^2 t_2), -2\mu \sin t_2 \cos t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$

The result is

$$t_2 = t_0 \qquad \text{and} \qquad \frac{\sin t_1}{\sin t_0} = \frac{k_s}{k_p} = \frac{1}{\kappa}$$
$$A_0 = \cos(t_0 - t_1) \qquad A_1 = \sin 2t_0 \qquad A_2 = \cos(t_0 + t_1)$$

2. Scattering Coefficient of Elastic Wave

The solution for the scattering of a plane P-wave u_p(or S-wave u_s) with incident direction d_0 at a plane $\Gamma := x \in \mathbb{R}^2 : x \cdot \nu = 0$ through the origin with normal vector ν is described by

$$u = u_p + u_{p,p} + u_{p,s} = A_0 d_0 e^{\mathbf{i}kpx\cdot d} + A_1 d_1 e^{\mathbf{i}kpx\cdot d_1} + A_2 d_2^\perp d_0^{\mathbf{i}ksx\cdot d_2} \tag{2.1}$$

$$u = u_s + u_{s,p} + u_{s,s} = A_0 d_0^\perp e^{\mathbf{i}ksx\cdot d} + A_1 d_1 e^{\mathbf{i}kpx\cdot d_1} + A_2 d_2^\perp d^{\mathbf{i}ksx\cdot d_2} \tag{2.2}$$

where $d_i = (d_i^1, d_i^2)^T$ are unit vectors, $d_i^\perp = (d_i^2, -d_i^1)^T$ and A_i are corresponding amplitude. For fixed boundary, we have $u = 0$ for $x \in \Gamma$. After a standard computation, we get for P-wave:

$$d_1 = d_0 - 2\alpha\nu \tag{2.3}$$

$$d_2 = \kappa d_0 - \beta\nu \tag{2.4}$$

$$A_0 = \kappa(d,\nu)^2 - \kappa(d,\nu^\perp)^2 - \beta(d,\nu) \tag{2.5}$$

$$A_1 = \kappa - \beta(d,\nu) \tag{2.6}$$

$$A_2 = -2(d,\nu)(d,\nu^\perp) \tag{2.7}$$

where $\alpha = (d,\nu)$, $\beta = \kappa\alpha - \sqrt{\kappa^2\alpha^2 - \kappa^2 + 1}$ and $\kappa = k_p/k_s$. For S-wave:

$$d_1 = \kappa_1 d_0 - \gamma\nu \tag{2.8}$$

$$d_2 = d_0 - 2\alpha\nu \tag{2.9}$$

$$A_0 = \kappa_1(d,\nu)^2 - \kappa_1(d,\nu^\perp)^2 - \gamma(d,\nu) \tag{2.10}$$

$$A_1 = 2(d,\nu)(d,\nu^\perp) \tag{2.11}$$

$$A_2 = \kappa_1 - \gamma(d,\nu) \tag{2.12}$$

where $\gamma = \kappa_1\alpha - \sqrt{\kappa_1^2\alpha^2 - \kappa_1^2 + 1}$ and $\kappa_1 = 1/\kappa$. Thus the traction of $u(x)$ on the plane Γ can be obtained. For P-wave

$$\begin{aligned}\sigma(u)\cdot\nu &= [\mathbf{i}k_p A_0(\lambda\nu + 2\mu(d_0,\nu)d_0) + \mathbf{i}k_p A_1(\lambda\nu + 2\mu(d_1,\nu)d_1)\\ &+\mathbf{i}k_s A_2\mu((d_2,\nu)d_2^\perp + (d_2^\perp,\nu)d_2)]e^{\mathbf{i}k_p x\cdot d} := \mathbf{i}k_p\hat{R}_p(x,d,\nu)e^{\mathbf{i}k_p x\cdot d}\end{aligned}$$

For S-wave

$$\begin{aligned}\sigma(u)\cdot\nu &= [\mathbf{i}k_s A_0\mu((d_0,\nu)d_0^\perp + (d_0^\perp,\nu)d_0) + \mathbf{i}k_p A_1(\lambda\nu + 2\mu(d_1,\nu)d_1)\\ &+\mathbf{i}k_s A_2\mu((d_2,\nu)d_2^\perp + (d_2^\perp,\nu)d_2)]e^{\mathbf{i}k_s x\cdot d} := \mathbf{i}k_s\hat{R}_s(x,d,\nu)e^{\mathbf{i}k_s x\cdot d}\end{aligned}$$

Definition 2.1 *For any unit vector $d \in \mathbb{R}^2$, let $u_p^i = de^{\mathbf{i}k_p x\cdot d}$ or $u_s^i = d^\perp e^{\mathbf{i}k_s x\cdot d}$ be the incident wave and $u_\alpha^s = u_\alpha^s(x;d)$ be the radiation solution of the Navier equation:*

$$u_\alpha^s + \omega^2 u_\alpha^s = 0 \quad in \quad \mathbb{R}^2\backslash\bar{D} \tag{2.13}$$

$$u_\alpha^s = -u_\alpha^i \quad on \quad \partial D \tag{2.14}$$

The scattering coecient R(x;d) for $x \in \partial D$ is defined by the relation

$$\sigma(u_\alpha^s + u_\alpha^i)\cdot\nu = \mathbf{i}k_\alpha R_\alpha(x;d)e^{\mathbf{i}k_\alpha x\cdot d} \quad on \quad \partial D$$

where $\alpha = p, s$.

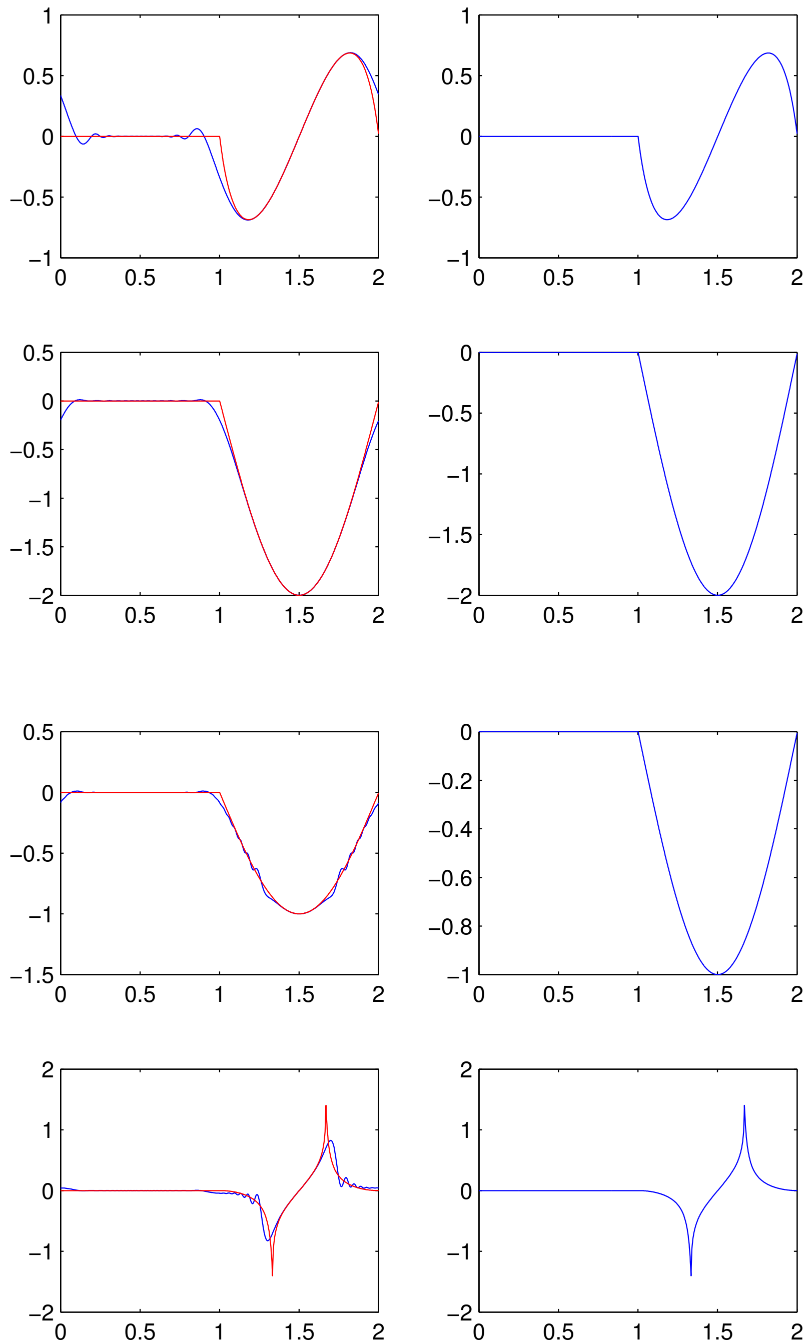

Figure 1. $\theta = 0\pi$

A convex object D locally may be cosidered at each point x as a plane with normal $\nu(x)$. Then the scattering coefficient can be approximated by

$$R_\alpha(x; d) \approx \begin{cases} \hat{R}_\alpha(x; d, \nu) & \text{if } \ x \in \partial D_d^- = \{x \in \partial D, \nu(x) \cdot d < 0\}, \\ 0 & \quad \text{if } \ x \in \partial D_d^- = \{x \in \partial D, \nu(x) \cdot d \geq 0\}. \end{cases}$$

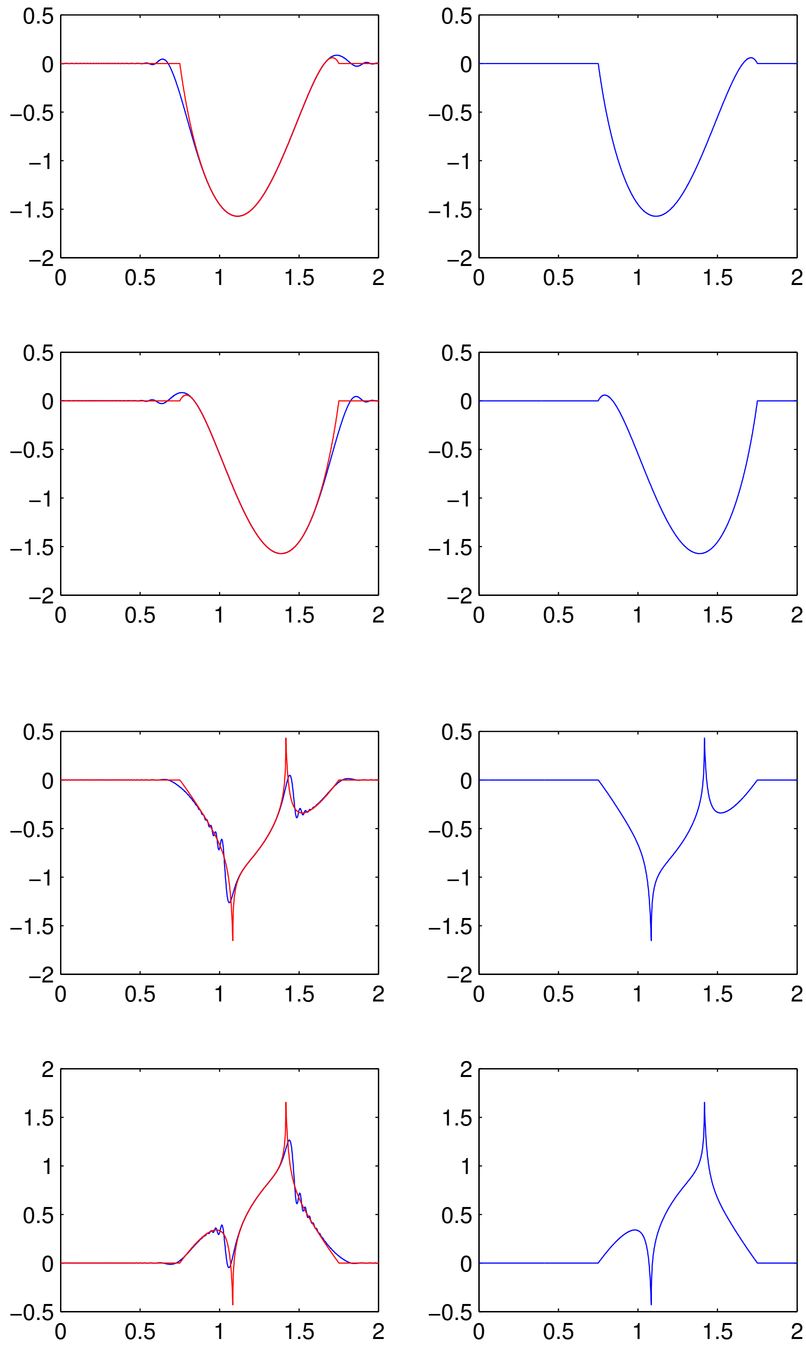

Figure 2. $\theta = pi/4$

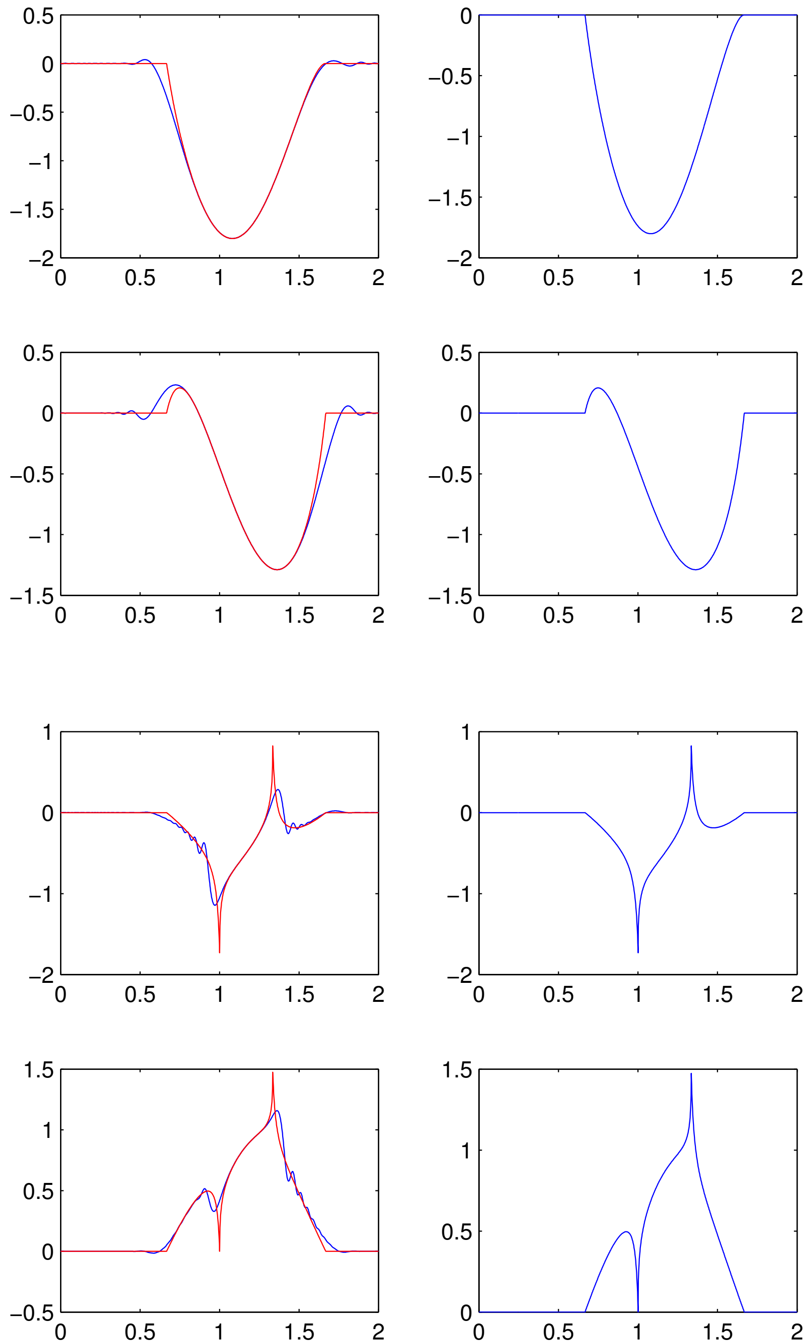

Figure 3. $\theta = \pi/3$

References

[1] Achenbach J 1980 *Wave Propagation in Elastic Solids* (North-Holland)