1. Estimate of Dirichlet Green Tensor

We need the following slight generalization of Van der Corput lemma for the oscillatory integral [2, P.152].

Lemma 1.1 *Let $-\infty < a < b < \infty$, and u is a C^k function u in (a, b).*
1. If $|u'(t)| \geq 1$ for $t \in (a, b)$ and u' is monotone in (a,b), then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 3\lambda^{-1}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

2. For all $k \geq 2$, if $|u^{(k)}(t)| \geq 1$ for $t \in (a, b)$, then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 12k\lambda^{-1/k}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

Proof. The assertion can be proved by extending the Van der Corptut lemma in [2]. Here we omit the details. □

We recall following lemma, see e.g. [3]:

Lemma 1.2 *Let $F(\lambda, a) = \int_0^a t^{\alpha-1}f(t)e^{-\mathbf{i}\rho t}dt$ where $0 < a \leq +\infty$, $0 < \alpha < 1$, $\rho > 0$ and $t^{\alpha-1}f \in L^1(0, a)$, then we have*

$$|F(\rho, a)| \leq C(\frac{1}{\rho^\alpha}f(0) + \frac{1}{\rho}(a^{\alpha-1}f(a) + |t^{\alpha-1}f|_{L^1(0,a)}) \tag{1.1}$$

Proof. Put

$$g_0(t) = t^{\alpha-1}e^{-\mathbf{i}\rho t} \tag{1.2}$$

and define

$$g_1(t) = -\int_t^{t-\mathbf{i}\infty} x^{\alpha-1}e^{-\mathbf{i}\rho x}dx \tag{1.3}$$

where the path of integration is the vertical line $x = t - \mathbf{i}y, y \geq 0$. It is easy to show that $g_1(t)' = g_0(t)$. Substituting $x = t - \mathbf{i}y$ into $g_1(t)$, we have

$$g_1(t) = \mathbf{i}\int_0^\infty (t - \mathbf{i}y)^{\alpha-1}e^{-\mathbf{i}\rho t}e^{-\rho y}dy \tag{1.4}$$

Upon integration by parts, we have

$$\begin{aligned} F(\rho, a) &= \int_0^a f(t)dg_1(t) \\ &= e^{-\mathbf{i}\frac{\alpha\pi}{2}}f(0)\Gamma(\alpha)\frac{1}{\rho^\alpha} + f(a)g_1(a) - \int_0^a f'(t)g_1(t)dt \\ &= e^{-\mathbf{i}\frac{\alpha\pi}{2}}f(0)\Gamma(\alpha)\frac{1}{\rho^\alpha} - \mathbf{i}\int_0^\infty e^{-\rho y}dy\int_0^a f'(t)(t - \mathbf{i}y)^{\alpha-1}e^{-\mathbf{i}\rho t}dt \end{aligned}$$

Let

$$h(y) = \int_0^a f'(t)(t - \mathbf{i}y)^{\alpha-1}e^{-\mathbf{i}\rho t}dt$$

and observe that

$$|h(y)| \leq \int_0^a |f'(t)|(t^2+y^2)^{\frac{\alpha-1}{2}} dt$$

□

Lemma 1.3 *Let $F(\lambda, a) = \int_0^a t^{-1/2} f(t) e^{-\mathbf{i}\rho t} dt$ where $0 < a \leq +\infty$ and $\rho > 0$, then we have*

$$|F(\rho, a) - e^{-\mathbf{i}\frac{\pi}{4}} f(0)\Gamma(1/2)\frac{1}{\rho^{1/2}}| \tag{1.5}$$

$$\leq C\int_0^\infty e^{-\rho y} dy \int_0^a |f'(t)|(t^2+y^2)^{-\frac{1}{4}} dt + \frac{1}{\rho} a^{-1/2} f(a) \tag{1.6}$$

Proof. Put

$$g_0(t) = t^{-1/4} e^{-\mathbf{i}\rho t} \tag{1.7}$$

and define

$$g_1(t) = -\int_t^{t-\mathbf{i}\infty} x^{-1/2} e^{-\mathbf{i}\rho x} dx \tag{1.8}$$

where the path of integration is the vertical line $x = t - \mathbf{i}y, y \geq 0$. It is easy to show that $g_1(t)' = g_0(t)$. Substituting $x = t - \mathbf{i}y$ into $g_1(t)$, we have

$$g_1(t) = \mathbf{i}\int_0^\infty (t-\mathbf{i}y)^{-1/2} e^{-\mathbf{i}\rho t} e^{-\rho y} dy \tag{1.9}$$

Upon integration by parts, we have

$$\begin{aligned} F(\rho, a) = &\int_0^a f(t) dg_1(t) \\ &= e^{-\mathbf{i}\frac{\pi}{4}} f(0)\Gamma(1/2)\frac{1}{\rho^{1/2}} + f(a)g_1(a) - \int_0^a f'(t)g_1(t)dt \\ &= e^{-\mathbf{i}\frac{\pi}{4}} f(0)\Gamma(1/2)\frac{1}{\rho^{1/2}} + \mathbf{i}f(a)\int_0^\infty (a-\mathbf{i}y)^{-1/2} e^{-\mathbf{i}\rho t} e^{-\rho y} dy \\ &\quad - \mathbf{i}\int_0^\infty e^{-\rho y} dy \int_0^a f'(t)(t-\mathbf{i}y)^{-1/2} e^{-\mathbf{i}\rho t} dt \end{aligned}$$

Let

$$h(y) = \int_0^a f'(t)(t-\mathbf{i}y)^{-1/2} e^{-\mathbf{i}\rho t} dt$$

and observe that

$$|h(y)| \leq \int_0^a |f'(t)|(t^2+y^2)^{-\frac{1}{4}} dt$$

□

Lemma 1.4 *Assume that $0 < \kappa := \sin\phi_\kappa < 1, 0 < \phi_\kappa < \pi/2$, $0 \leq \phi \leq \pi/2$ and $-\pi/2 < t_1 < t_2 < \pi/2$ satisfy that $\kappa^2 = \sin^2(\phi + t_1) = \sin^2(\phi + t_2)$. Let $f(\theta)$:*

$$f(t, \phi) := F(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2}) \tag{1.10}$$

be a function in $C^\infty(([-\pi/2,\pi/2]\backslash\{t_1,t_2\})\times[0,\pi/2])$. *Moreover, there exits* $\epsilon>0$ *such that* $f(\theta)$ *can be represented as*

$$f(t,\phi)=g_1(t,\phi)+g_2(t,\phi)(\kappa^2-\sin^2(t+\phi))^{1/2})^{N/2} \tag{1.11}$$

where $g_1,g_2\in C^\infty((\bigcup_{i=1,2}(t_i-\epsilon,t_i+\epsilon))\times[0,\pi/2]))$ *and* $N=\pm1$. *Then for any* $\rho\geq1$, *we have*

$$\begin{aligned}&\Big|I(\rho,\phi):=\int_{-\pi/2}^{\pi/2}f(\theta)e^{\mathbf{i}\rho\cos\theta}d\theta-\frac{N+1}{2}\Big(\frac{2\pi}{\rho}\Big)^{1/2}f(0)e^{\mathbf{i}\rho-\mathbf{i}\pi/4}\Big|\\&\leq C\frac{1}{\rho^{(2+N)/4}}\end{aligned} \tag{1.12}$$

Proof. The proof will be split into two parts about whether ϕ equal to ϕ_κ.

If $\phi\neq\phi_\kappa$, there exists $0<\delta<\pi/4$ such that

$$|(\kappa^2-\sin^2(t+\phi))^{1/2}|>\frac{1}{2}|(\kappa^2-\sin^2\phi)^{1/2}| \tag{1.13}$$

for any $t\in(-\delta,\delta)$. Let $\chi_\delta\in C_0^\infty(-\pi/2,\pi/2)$ be the cut-off function with that $0\leq\chi_\delta\leq1$, $\chi_\delta=1$ in $(-\delta/2,\delta/2)$ and $\chi_\delta=0$ in $(-\pi/2,\pi/2)\backslash(-\delta,\delta)$. Then we can divide I into two parts such that

$$\begin{aligned}I&=\int_{-\delta}^{\delta}f(t)\chi_\delta(t)e^{\mathbf{i}\rho\cos t}dt+\int_{-\pi/2}^{\pi/2}f(t)(1-\chi_\delta(t))e^{\mathbf{i}\rho\cos t}dt\\&=:I_1+I_2\end{aligned}$$

Subtitating $t(s)=2\arcsin s/2$ for t in I_1, we can obtain

$$I_1=\int_{\mathbb{R}}f(t(s))\chi_\delta(t(s))\frac{1}{\sqrt{1-s^2/4}}e^{\mathbf{i}\rho}e^{-\mathbf{i}\rho s^2/2}ds \tag{1.14}$$

$$=\int_{\mathbb{R}}h_\delta(s)e^{\mathbf{i}\rho}e^{-\mathbf{i}\rho s^2/2}ds \tag{1.15}$$

It is easy to see that $h_\delta(s)\in C_0^4(\mathbb{R})$. By the lemma of the stationary phase for quadratic term in [1], we have

$$I_1=e^{\mathbf{i}\rho}\int_{\mathbb{R}}h_\delta(s)e^{-\mathbf{i}\frac{\rho}{2}s^2}ds=e^{\mathbf{i}\rho}\int_{\mathbb{R}}\widehat{h_\delta}(y)\alpha(-y)dy \tag{1.16}$$

where

$$\alpha(y)=(\frac{1}{2\pi\rho})^{1/2}e^{-\mathbf{i}\pi/4}e^{\frac{\mathbf{i}}{2\rho}y^2} \tag{1.17}$$

$$=(\frac{1}{2\pi\rho})^{1/2}e^{-\mathbf{i}\pi/4}(1+O(\frac{y^2}{\rho})) \tag{1.18}$$

Consequently

$$I_1=(\frac{1}{2\pi\rho})^{1/2}e^{\mathbf{i}\rho-\mathbf{i}\pi/4}\int_{\mathbb{R}}\widehat{h_\delta}(y)(1+\frac{1}{\rho}O(y^2))dy \tag{1.19}$$

Moreover, $\int_{\mathbb{R}}\widehat{h_\delta}(y)dy=2\pi h_\delta(0)$ and $|\int_{\mathbb{R}}\widehat{h_\delta}(y)y^2dy|<C$ since $\widehat{h_\delta}(y)=O(1/y^4)$. Now, it turns to estimate I_2.

When $N = 1$, using integration by parts, we have

$$|I_2| = \left| \int_{(-\frac{\pi}{2},\frac{\pi}{2})\setminus(-\frac{\delta}{2},\frac{\delta}{2})} f(t)(1-\chi_\delta(t))/\sin t \;\; de^{\mathbf{i}\rho\cos t} \right| \tag{1.20}$$

$$\tag{1.21}$$

$$\leq C\frac{1}{\rho} + \left| \int_{(-\frac{\pi}{2},\frac{\pi}{2})\setminus(-\frac{\delta}{2},\frac{\delta}{2})} (f(t)(1-\chi_\delta(t))/\sin t)' \; e^{\mathbf{i}\rho\cos t} dt \right| \tag{1.22}$$

$$\leq C\frac{1}{\rho} \tag{1.23}$$

From above analysis, we obtain

$$\left| I(\rho,\phi) - \left(\frac{2\pi}{\rho}\right)^{1/2} f(0) e^{\mathbf{i}\rho - \mathbf{i}\pi/4} \right| \leq C(\phi)\frac{1}{\rho} \tag{1.24}$$

When $N = -1$, we can not use integration by parts again since $f'(\theta)$ is not integrable. However, for any $0 < \lambda_1 < 1$ and $1 < \lambda_2 < 1/\kappa$, there exists $0 < \sigma < \epsilon$, such that $\chi := ((t_1 - \sigma, t_1 + \sigma) \cup (t_2 - \sigma, t_2 + \sigma)) \cap (-\delta, \delta) = 0$, dependent on λ_1, λ_2 and

$$\lambda_1 \kappa < |\sin(t+\phi)| < \lambda_2 \kappa. \tag{1.25}$$

for any $t \in \chi$.

We only analysis the integral on $\chi_1 = (t_1 - \sigma, t_1 + \sigma) \cap [-\pi/2, \pi/2]$ here, which denoted by I_{χ_1}, the proof of I_{χ_2} is similar. It is easy to see that $\sin(t+\phi)$ is monotonic in χ_1. Without loss of generality, we assume that $\sin(t_1 - \sigma + \phi) < \kappa < \sin(t_1 + \sigma + \phi)$. Let $\sin(t+\phi) = \kappa \sin\theta$ and the implicit mapping from θ to t is denoted by $t(\theta)$ while the inverse mapping by $\theta(t)$, taking the interval χ_1 onto $L_\theta : \theta_1 \to \pi/2 \to \pi/2 - \mathbf{i}\theta_2$ where $\sin(t_1 - \sigma + \phi) = \kappa \sin\theta_1, \sin(t_1 + \sigma + \phi) = \kappa \sin(\pi/2 - \mathbf{i}\theta_2)$. By substituting $t(\theta)$ into I_{χ_1}, we have

$$I_{\chi_1} = \int_{t_1-\sigma}^{t_1+\sigma} \frac{f(t)(\kappa^2 - \sin^2(t+\phi))^{1/2}}{(\kappa^2 - \sin^2(t+\phi))^{1/2}} e^{\mathbf{i}\rho\cos t} \tag{1.26}$$

$$= \int_{L_\theta} \frac{\kappa f(t(\theta))\cos\theta}{(1-\kappa^2\sin^2\theta)^{1/2}} e^{\mathbf{i}\rho(\cos(t(\theta)))} d\theta \tag{1.27}$$

$$= \int_{L_\theta} \frac{\kappa g_1(t(\theta))\cos\theta + g_2(t(\theta))}{(1-\kappa^2\sin^2\theta)^{1/2}} e^{\mathbf{i}\rho(\cos(t(\theta)))} d\theta \tag{1.28}$$

$$:= \int_{L_\theta} \frac{h(\theta)}{(1-\kappa^2\sin^2\theta)^{1/2}} e^{\mathbf{i}\rho(\cos(t(\theta)))} d\theta \tag{1.29}$$

Observe that $h(\theta)$ and $\partial h/\partial\theta$ are integrable on the path L_θ by (1.11). A simple computation show that

$$\frac{dt(\theta)}{d\theta} = \frac{\kappa\cos\theta}{\cos(t+\phi)} \qquad \frac{d^2t(\theta)}{dt^2} = \frac{\kappa^2\cos^2\theta\sin(t+\phi) - \kappa\sin\theta\cos^2(t+\phi)}{\cos^3(t+\phi)}$$

Then we can obtain

$$\frac{d\cos t}{d\theta} = \frac{-\kappa\sin t\cos\theta}{\cos(t+\phi)}$$

$$\frac{d^2\cos t}{d\theta^2} = \frac{d^2\cos t}{dt^2}(\frac{dt}{d\theta})^2 + \frac{d\cos t}{dt}\frac{d^2t}{d\theta^2}$$

$$= \frac{-\kappa^2 \cos^2\theta \cos t}{\cos^2(t+\phi)} + \frac{\kappa \sin\theta \cos^2(t+\phi)\sin t - \kappa^2\cos^2\theta \sin(t+\phi)\sin t}{\cos^3(t+\phi)}$$
$$= \frac{-\kappa^2 \cos^2\theta \cos\phi + \kappa\sin\theta\cos^2(t+\phi)\sin t}{\cos^3(t+\phi)}$$
$$= \frac{(\sin^2(t+\phi)-\kappa^2)\cos\phi + \cos^2(t+\phi)\sin(t+\phi)\sin t}{\cos^3(t+\phi)}$$

Since $|\sin t| > |\sin\delta|$ and $1-\lambda_2^2\kappa^2 < \cos^2(t+\phi) < 1-\lambda_1^2\kappa^2$ for $t \in \chi_1$, it follows that $\theta = \pi/2$ is the only stationary point of $\cos(t(\theta))$ and

$$\left|\frac{d^2\cos t}{d\theta^2}(\pi/2)\right| = \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}|\sin t| > \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}\sin\delta \tag{1.30}$$

Therefore, we can choose appropriate λ_1, λ_2 such that

$$|\frac{d^2\cos t}{d\theta^2}| > \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}\sin\delta \tag{1.31}$$

for any $\theta \in \theta(\chi_1)$. According to lemma (1.1), we obtian $|I_{\chi_1}| \leq C\frac{1}{\rho^{1/2}}$, and also $|I_{\chi_2}| \leq C\frac{1}{\rho^{1/2}}$. Using integration by parts, we get

$$\left|\int_{[-\pi/2,\pi/2]\backslash((-\delta,\delta)\cup\chi)} f(t)(1-\chi_\delta(t))e^{\mathbf{i}\rho\cos t}dt\right| \leq C\frac{1}{\rho}$$

Consequently, for $N=-1$ and $\phi \neq \phi_\kappa$, we get $|I(\rho,\phi)| \leq \frac{1}{\rho^{1/2}}$.

We now turn to the case of $\phi = \phi_\kappa$. By (1.11), we can define χ_ϵ similarly and also decompose I into I_1 and I_2. Using the same agurement above, we can easily carry out that: for $N=1$, we have $|I_2| \leq C\frac{1}{\rho}$; for $N=-1$, we have $|I_2| \leq C\frac{1}{\rho^{1/2}}$. Finally, it remains to analysis I_1. By (1.11), we have

$$\begin{aligned} I_1 &= \int_{-\epsilon}^{\epsilon} g_1\chi_\epsilon + g_2\chi_\epsilon(\sin^2\phi_\kappa - \sin^2(t+\phi_\kappa))^{N/2}e^{\mathbf{i}\rho\cos t}dt \\ &= \int_{-\epsilon}^{\epsilon} g_1\chi_\epsilon + g_2\chi_\epsilon(-2(\sin\phi_\kappa + \sin(t+\phi_\kappa))\cos\frac{2\phi_\kappa+t}{2}\sin t/2)^{N/2}e^{\mathbf{i}\rho\cos t}dt \\ &= \int_{\mathbb{R}} g_1\chi_\epsilon + g_2\chi_\epsilon((\sin\phi_\kappa + \sin(t+\phi_\kappa))\cos\frac{2\phi_\kappa+t}{2})^{N/2}(-2\sin t/2)^{N/2}e^{\mathbf{i}\rho\cos t}dt \end{aligned}$$

Also, subtituting $t(s) = 2\arcsin s/2$ for t in I_1, it follows that

$$I_1 = \int_{\mathbb{R}} h_1(s)e^{-\mathbf{i}\rho\frac{s^2}{2}} + h_2(s)(-s)^{N/2}e^{-\mathbf{i}\rho\frac{s^2}{2}} \tag{1.32}$$
$$= I_{11} + I_{12} \tag{1.33}$$

where

$$h_1(s) = g_1(t(s))\chi_\epsilon(t(s))\sqrt{1-s^2/4}\ e^{\mathbf{i}\rho}$$
$$h_2(s) = g_2\chi_\epsilon((\sin\phi_\kappa + \sin(t+\phi_\kappa))\cos\frac{2\phi_\kappa+t}{2})^{N/2}_{t=t(s)}\sqrt{1-s^2/4}\ e^{\mathbf{i}\rho}$$

and $h_1(s), h_2(s) \in C_c^\infty(\mathbb{R})$. Using stationary phase lemma similarly, if $N=1$,

$$I_{11} = \left(\frac{2\pi}{\rho}\right)^{1/2} g_1(0)e^{\mathbf{i}\rho - \mathbf{i}\pi/4} + O(\frac{1}{\rho}) \tag{1.34}$$
$$= \left(\frac{2\pi}{\rho}\right)^{1/2} f(0)e^{\mathbf{i}\rho - \mathbf{i}\pi/4} + O(\frac{1}{\rho}) \tag{1.35}$$

if $N=-1$, we get $|I_{11}|\le C\frac{1}{\rho^{1/2}}$. For I_{12}, we have

$$I_{12} = \int_0^\infty (\mathbf{i}h_2(s)+h_2(-s))s^{N/2}\ e^{-\mathbf{i}\rho s^2/2}ds \tag{1.36}$$

$$= \frac{1}{2}\int_0^\infty (\mathbf{i}h_2(\sqrt{s})+h_2(-\sqrt{s}))s^{N/4-1/2}e^{-\mathbf{i}\rho s/2}ds \tag{1.37}$$

By lemma (1.2), we get $|I_{12}|\le C\frac{1}{\rho^{(N+2)/4}}$. □

2. Some draft about Green Tensor Analysis

Let substitute $\xi = k\sin\theta$ into integral and shift the variable, we have

$$I(y) = \int_{\mathbb{R}} f(\xi)e^{\mathbf{i}\xi y_1+\mu(\xi)y_2}d\xi = \int_{\mathbb{R}} f(\xi)e^{\mathbf{i}\xi(y_1-z_1)+\mu(\xi)(y_2-z_2)}e^{\mathbf{i}\xi z_1+\mu(\xi)z_2}d\xi \tag{2.1}$$

$$= k\int_L f(k\sin\theta)\cos\theta e^{\mathbf{i}k|y-z|\cos(\theta-\eta)}e^{\mathbf{i}|z|\cos(\theta-\phi)}d\theta \tag{2.2}$$

$$= k\int_{L_\phi} f(k\sin(\theta+\phi))\cos(\theta+\phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \tag{2.3}$$

$$= k\int_{L} f(k\sin(\theta+\phi))\cos(\theta+\phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \tag{2.4}$$

where $y_1, y_2>0, \sin\phi = \frac{z_1}{|z|}, \cos\phi=\frac{z_2}{|z|}, 0<\phi<\pi/2$ and $\sin\eta = \frac{y_1-z_1}{|y-z|}, \cos\eta = \frac{y_2-z_2}{|y-z|}, 0<\eta<\pi$. It is easy to see that $\phi+\eta<\pi$. Roughly, using stationary phase lemma, we obtain:

$$I(y) = f(k\sin\phi)k\cos\phi e^{\mathbf{i}k|y-z|\cos(\phi-\eta)}(\frac{2\pi}{|z|})^{1/2}e^{\mathbf{i}|z|-\mathbf{i}\frac{\pi}{4}}(1+O(\frac{1}{|z|})) \tag{2.5}$$

$$\cos(a+\mathbf{i}b) = \frac{e^b+e^{-b}}{2}\cos a+\mathbf{i}\frac{e^{-b}-e^b}{2}\sin a \tag{2.6}$$

$$\sin(a+\mathbf{i}b) = \frac{e^b+e^{-b}}{2}\sin a+\mathbf{i}\frac{e^{b}-e^{-b}}{2}\cos a \tag{2.7}$$

When $\theta\in(-a-\pi/2, -a-\pi/2+\mathbf{i}\infty)$, let $\theta = -a-\pi/2\mathbf{i}t$, where $t>0, 0\le a\le\phi$, then

$$-\mathrm{Im}\,(|z|\cos\theta+|y-z|\cos(\theta+\phi-\eta))$$

$$= |z|\sin(a+\pi/2)+|y-z|\sin(a+\pi/2-\phi+\eta) \tag{2.8}$$

$$= |z|\cos a+|y-z|\cos(a-\phi+\eta) \tag{2.9}$$

$$= |z|\cos a+\cos a|y-z|(\cos\phi\cos\eta+\sin\phi\sin\eta) \tag{2.10}$$

$$+\sin a|y-z|(\sin\phi\cos\eta-\cos\phi\sin\eta) \tag{2.11}$$

$$= |z|\cos a+\cos a((y_2-z_2)\cos\phi+(y_1-z_1)\sin\phi) \tag{2.12}$$

$$+\sin a((y_2-z_2)\sin\phi-(y_1-z_1)\cos\phi) \tag{2.13}$$

$$= y_1\sin(\phi-a)+y_2\cos(\phi-a)>0 \tag{2.14}$$

Now, Using Cauchy Integral Theorem, we have

$$I(y) = k\int_L f(k\sin(\theta+\phi))\cos(\theta+\phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \tag{2.15}$$

Let $L_1 = (-\pi/2, -\pi/2 + \mathbf{i}\infty)$ and $\theta = -\pi/2 + \mathbf{i}t, t > 0$, then

$$I_1(y) = k\int_{L_1} f(k\sin(\theta+\phi))\cos(\theta+\phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \tag{2.16}$$

$$= \tag{2.17}$$

$$I(y) = f(k\sin\phi)k\cos\phi e^{\mathbf{i}k|y-z|\cos(\phi-\eta)}(\frac{2\pi}{|z|})^{1/2}e^{\mathbf{i}|z|-\mathbf{i}\frac{\pi}{4}} \tag{2.18}$$

$$+\frac{kz_2}{|z|}O(\left(\frac{1}{k|z|}\right)^{3/4}+\frac{1}{k|y|})+\frac{kz_1}{|z|}O(\left(\frac{1}{k|z|}\right)^{5/4}+\left(\frac{1}{k|y|}\right)^{2}) \tag{2.19}$$

It is easy to see

$$\int_{-d}^{d}\frac{k}{(k|x-z|)^{\alpha}}\frac{1}{(k|x-y|)^{\beta}}dx_1 \le C(\frac{1}{(kz_2)^{\alpha+\beta-1}}+\frac{1}{(ky_2)^{\alpha+\beta-1}}) \tag{2.20}$$

where $z, y \in \mathbb{R}^2_+$, $x \in \Gamma_0$ and $\alpha+\beta > 0$.

$$e^{\mathbf{i}\mu y_2+\mathbf{i}\xi(x_1-y_1)} = e^{\mathbf{i}\mu y_2-\mathbf{i}y_2/\tan\phi} = e^{\mathbf{i}y_2(\mu-\xi/\tan\phi)} \tag{2.21}$$

Another method

$$\int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi)e^{\mathbf{i}k|x-y|\cos\theta} \tag{2.22}$$

$$= \int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi)e^{\mathbf{i}k|x-y|\cos(\theta+\psi-\psi)} \tag{2.23}$$

$$= \int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi)e^{\mathbf{i}ky_2\cos(\theta+\psi)+\mathbf{i}k|x_1-y_1|\sin(\theta+\psi)} \tag{2.24}$$

$$= \int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi) \tag{2.25}$$

$$e^{\mathbf{i}k(y_2-z_2)\cos(\theta+\psi)+\mathbf{i}k(|x_1-y_1|-|x_1-z_1|)\sin(\theta+\psi)+\mathbf{i}k|z|\cos(\theta+\psi-\phi)} \tag{2.26}$$

3. Finite Aperture Point Spread Function

If $x \in \Gamma_0$ and $z, y \in \mathbb{R}^2_+$, by lemma (**??**) we have

$$\begin{aligned} G(x,y) &= \frac{\mathbf{i}k_s}{\mu\sqrt{2\pi}}\frac{1}{\delta(\xi)}\left(\begin{array}{cc} \mu_s\beta & \xi\beta \\ 2\xi\mu_s\mu_p & 2\xi^2\mu_p \end{array}\right)_{\xi=k_s\frac{x_1-y_1}{|x-y|}}\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{1/2}}e^{\mathbf{i}k_s|x-y|-\mathbf{i}\frac{\pi}{4}} \\ &+\frac{\mathbf{i}k_p}{\mu\sqrt{2\pi}}\frac{1}{\delta(\xi)}\left(\begin{array}{cc} 2\xi^2\mu_s & -2\xi\mu_s\mu_p \\ -\xi\beta & \mu_p\beta \end{array}\right)_{\xi=k_p\frac{x_1-y_1}{|x-y|}}\frac{y_2}{|x-y|}\frac{1}{(k_p|x-y|)^{1/2}}e^{\mathbf{i}k_p|x-y|-\mathbf{i}\frac{\pi}{4}} \\ &+O(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}}) \\ &:= \mathcal{G}_s(x,y)+\mathcal{G}_p(x,y)+O(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}}) \end{aligned} \tag{3.1}$$

$$T_D(x,z) = \frac{k_s}{\sqrt{2\pi}}\frac{1}{\gamma(\xi)}\left(\begin{array}{cc} \mu_s\mu_p & \xi\mu_p \\ \xi\mu_s & \xi^2 \end{array}\right)_{\xi=k_s\frac{x_1-z_1}{|x-z|}}\frac{z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{1/2}}e^{\mathbf{i}k_s|x-z|-\mathbf{i}\frac{\pi}{4}}$$

$$+\frac{k_p}{\sqrt{2\pi}}\frac{1}{\gamma(\xi)}\left(\begin{array}{cc}\xi^2 & -\xi\mu_p \\ -\xi\mu_s & \mu_p\mu_s\end{array}\right)_{\xi=k_p\frac{x_1-z_1}{|x-z|}}\frac{z_2}{|x-z|}\frac{1}{(k_p|x-z|)^{1/2}}e^{\mathbf{i}k_p|x-z|-\mathbf{i}\frac{\pi}{4}} \tag{3.2}$$

$$+O(\frac{k_s z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{k_s|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}})$$

$$:=\mathcal{T}_s(x,z)+\mathcal{T}_p(x,z)+O(\frac{k_s z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{k_s|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}})$$

Now we consider the finite aperture point spread function $J_d(z,y)$:

$$\int_{-d}^{d}(T_D(x_1,0;z_1,z_2))^T\overline{G(x_1,0;y_1,y_2)}dx_1 \tag{3.3}$$

Recall following standard asymptotic expansion:

$$|x-y|=|x-z|+\widehat{x-z}\cdot(z-y)+O(\frac{|y-z|^2}{|x-z|}) \tag{3.4}$$

$$|y|^{-\alpha}=|z|^{-\alpha}(1+\frac{|y|-|z|}{|z|})^{-\alpha}=|z|^{-\alpha}(1+O(\frac{|y-z|}{|z|})) \tag{3.5}$$

$$e^{\mathbf{i}t}=1+O(t) \tag{3.6}$$

$$|a^{1/2}-b^{1/2}|\le C\sqrt{|a-b|} \tag{3.7}$$

where $x,y,z\in\mathbb{R}^2$, $t,a,b\in\mathbb{R}$ and $\alpha>0$.

Lemma 3.1 *For any $z,y\in\mathbb{R}^2_+$, $J_d(z,y)=F(z,y)+O((1+\frac{|y-z|}{z_2})(\frac{1}{k_s z_2})^{1/4}+\frac{(k_s|y-z|)^2}{k_s z_2}+(\frac{|y-z|}{z_2})^{1/2})$, where*

$$F(z,y)=-\frac{\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d}f_s(\theta)\left(\begin{array}{cc}\sin^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \cos^2\theta\end{array}\right)e^{\mathbf{i}k_s(z_1-y_1)\cos\theta+\mathbf{i}k_s(z_2-y_2)\sin\theta}d\theta \tag{3.8}$$

$$-\frac{\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d}f_p(\theta)\left(\begin{array}{cc}\cos^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin^2\theta\end{array}\right)e^{\mathbf{i}k_p(z_1-y_1)\cos\theta+\mathbf{i}k_p(z_2-y_2)\sin\theta}d\theta \tag{3.9}$$

and

$$f_s(\theta)=\frac{\sin\theta((\kappa^2-\cos^2\theta)^{1/2}(1-2\cos^2\theta)+2\overline{(\kappa^2-\cos^2\theta)^{1/2}}\cos^2\theta)}{(\cos^2\theta+\sin\theta(\kappa^2-\cos^2\theta)^{1/2})\overline{((1-2\cos^2\theta)^2+4\cos^2\theta\sin\theta(\kappa^2-\cos\theta)^{1/2})}}$$

$$f_p(\theta)=\frac{\sin\theta(1/\kappa^2-\cos^2\theta)^{1/2}}{(\cos^2\theta+\sin\theta(1/\kappa^2-\cos^2\theta)^{1/2})((1/\kappa^2-2\cos^2\theta)^2+4\cos^2\theta\sin\theta(1/\kappa^2-\cos\theta)^{1/2})}$$

where $0<\theta_1^d<\pi/2<\theta_2^d<\pi$ and $z_2=(d+z_1)\tan\theta_1^d=(z_1-d)\tan\theta_2^d$.

Proof.

$$\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}}$$
$$=(\frac{z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}})(1+O(\frac{|y-z|}{|x-z|}))$$

$$|\mu_{\mathbf{i}}(k_j\frac{x_1-y_1}{|x-y|})-\mu_{\mathbf{i}}(k_j\frac{x_1-z_1}{|x-z|})|$$
$$\le Ck_j\sqrt{\left|\frac{x_1-y_1}{|x-y|}-\frac{x_1-z_1}{|x-z|}\right|}\le Ck_j\Big(\frac{|y-z|}{|x-z|}\Big)^{1/2}$$

where $i, j = s, p$. By above, we can obtain

$$\mathcal{G}_s(x,y) = \mathcal{G}_s(x,z)e^{\mathbf{i}k_s\widehat{x-z}\cdot(z-y)} + O(\frac{(k_s|y-z|)^2}{(k_s|x-z|)^{3/2}}) + O(\frac{(k_s|y-z|)^{1/2}}{k_s|x-z|}) \tag{3.10}$$

$$\mathcal{G}_p(x,y) = \mathcal{G}_p(x,z)e^{\mathbf{i}k_p\widehat{x-z}\cdot(z-y)} + O(\frac{(k_p|y-z|)^2}{(k_p|x-z|)^{3/2}}) + O(\frac{(k_p|y-z|)^{1/2}}{k_p|x-z|}) \tag{3.11}$$

For $l > 1$, a simple computation show that

$$\int_{-d}^{d} \frac{k_s}{(k_s|x-z|)^l} dx_1 = \frac{1}{(k_s z_2)^{l-1}} \int_{\frac{-d-z_1}{z_2}}^{\frac{d-z_1}{z_2}} \frac{1}{(1+t^2)^{l/2}} dt \le C \frac{1}{(k_s z_2)^{l-1}} \tag{3.12}$$

Let

$$\mathcal{G}_\alpha(x,y) = \frac{\mathbf{i}}{\sqrt{2\pi}\mu} g_\alpha(\frac{x_1-y_1}{|x-y|}, \kappa) \frac{1}{(k_\alpha|x-y|)^{1/2}} e^{\mathbf{i}k_\alpha|x-y| - \mathbf{i}\frac{\pi}{4}} \tag{3.13}$$

$$\mathcal{T}_\alpha(x,y) = \frac{k_\alpha}{\sqrt{2\pi}} t_\alpha(\frac{x_1-z_1}{|x-z|}, \kappa) \frac{1}{(k_s|x-z|)^{1/2}} e^{\mathbf{i}k_\alpha|x-z| - \mathbf{i}\frac{\pi}{4}} \tag{3.14}$$

where $\alpha = s, p$. Now, by substituting (3.10-3.11) into $J_d(z,y)$ and using inequality (3.12), we have

$$J_d(z,y) = \frac{-\mathbf{i}}{2\pi\mu} \int_{-d}^{d} t_s(\frac{x_1-z_1}{|x-z|}, \kappa)^T \overline{g_s(\frac{x_1-z_1}{|x-z|}, \kappa)} \frac{e^{\mathbf{i}k_s\widehat{x-z}\cdot(y-z)}}{|x-z|}$$

$$+ t_p(\frac{x_1-z_1}{|x-z|}, \kappa)^T \overline{g_p(\frac{x_1-z_1}{|x-z|}, \kappa)} \frac{e^{\mathbf{i}k_p\widehat{x-z}\cdot(y-z)}}{|x-z|} dx_1 \tag{3.15}$$

$$- \frac{\mathbf{i}}{2\pi\mu} \int_{-d}^{d} t_p(\frac{x_1-z_1}{|x-z|}, \kappa)^T \overline{g_s(\frac{x_1-z_1}{|x-z|}, \kappa)} \frac{e^{\mathbf{i}k_s\widehat{x-z}\cdot(y-z)}}{|x-z|} \tag{3.16}$$

$$+ t_s(\frac{x_1-z_1}{|x-z|}, \kappa)^T \overline{g_p(\frac{x_1-z_1}{|x-z|}, \kappa)} \frac{e^{\mathbf{i}k_p\widehat{x-z}\cdot(y-z)}}{|x-z|} dx_1 \tag{3.17}$$

$$+ O((1 + \frac{|y-z|}{z_2})(\frac{1}{k_s z_2})^{1/4} + \frac{(k_s|y-z|)^2}{k_s z_2} + (\frac{|y-z|}{z_2})^{1/2}) \tag{3.18}$$

$$:= F(z,y) + R(z,y) \tag{3.19}$$

$$+ O((1 + \frac{|y-z|}{z_2})(\frac{1}{k_s z_2})^{1/4} + \frac{(k_s|y-z|)^2}{k_s z_2} + (\frac{|y-z|}{z_2})^{1/2}) \tag{3.20}$$

We denote $\widehat{x-z} = x - z/|x-z| = (cos(\phi+\pi), \sin(\phi+\pi))$, then taking the substitution $x_1 = z_1 - z_2 \cot\phi$, we obtain

$$F(z,y) = \frac{-\mathbf{i}}{2\pi\mu} \int_{\theta_1^d}^{\theta_2^d} A_s(\phi,\kappa) e^{\mathbf{i}k_s(z_1-y_1)\cos\phi + \mathbf{i}k_s(z_2-y_2)\sin\phi} \tag{3.21}$$

$$+ \frac{-\mathbf{i}}{2\pi\mu} \int_{\theta_1^d}^{\theta_2^d} A_p(\phi,\kappa) e^{\mathbf{i}k_p(z_1-y_1)\cos\phi + \mathbf{i}k_p(z_2-y_2)\sin\phi} \tag{3.22}$$

$$R(z,y) = \frac{-\mathbf{i}}{2\pi\mu} \int_{\theta_1^d}^{\theta_2^d} B_s(\phi,\kappa) e^{\mathbf{i}k_s(z_1-y_1)\cos\phi + \mathbf{i}k_s(z_2-y_2)\sin\phi + (k_p-k_s)|x-z|} \tag{3.23}$$

$$+ \frac{-\mathbf{i}}{2\pi\mu} \int_{\theta_1^d}^{\theta_2^d} B_p(\phi,\kappa) e^{\mathbf{i}k_p(z_1-y_1)\cos\phi + \mathbf{i}k_p(z_2-y_2)\sin\phi + (k_s-k_p)|x-z|} \tag{3.24}$$

It is easy to see that $|R(z,y)| \le C\frac{|z-y|}{z_2}$. $\square$

Let

$$g(x_1) = \frac{1}{((x_1-z_1)^2+z_2^2)^{3/4}((x_1-y_1)^2+y_2^2)^{1/4}}$$
$$\phi(x_1) = ((x_1-z_1)^2+z_2^2)^{1/2} - ((x_1-y_1)^2+y_2^2)^{1/2}$$

Then, we have

$$g'(x_1) = -g(x_1)[\frac{3(x_1-z_1)}{2((x_1-z_1)^2+z_2^2)} + \frac{(x_1-u_1)}{2((x_1-y_1)^2+y_2^2)}]$$
$$\phi'(x_1) = \frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}} - \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}}$$
$$= \frac{\frac{(x_1-z_1)^2}{(x_1-z_1)^2+z_2^2} - \frac{(x_1-y_1)^2}{(x_1-y_1)^2+y_2^2}}{\frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}} + \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}}}$$
$$= \frac{(x_1-z_1)^2y_2^2 - (x_1-y_1)^2z_2^2}{(\frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}} + \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}})((x_1-z_1)^2+z_2^2)((x_1-y_1)^2+y_2^2)}$$
$$\phi''(x_1) = \frac{z_2^2}{((x_1-z_1)^2+z_2^2)^{3/2}} - \frac{y_2^2}{((x_1-y_1)^2+y_2^2)^{3/2}}$$

Using integration by parts, we can obtain

$$\int_{-d}^{d} g(x_1)e^{\mathbf{i}k\phi(x_1)}dx_1$$
$$= \frac{1}{\mathbf{i}k}(\frac{g(d)}{\phi'(d)}e^{\mathbf{i}k\phi(d)} - \frac{g(-d)}{\phi'(-d)}e^{\mathbf{i}k\phi(-d)}) - \frac{1}{\mathbf{i}k}\int_{-d}^{d}\frac{g'(x_1)}{\phi'(x_1)} - \frac{g(x_1)\phi''(x_1)}{(\phi'(x_1))^2}dx_1$$

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$$\sin\phi_\kappa - \sin(t+\phi) = -2\cos(\frac{\phi_\kappa+\phi+t}{2})\sin(\frac{t+\phi-\phi_\kappa}{2})$$
$$\sin(\frac{t+\phi-\phi_\kappa}{2}) = \sin\frac{t}{2}\cos(\frac{\phi-\phi_\kappa}{2}) + \cos\frac{t}{2}\sin(\frac{\phi-\phi_\kappa}{2})$$

Some think, substituting $t = 2\arcsin s/2$ into following integral

$$\int_0^\infty \chi(t)(\sin\phi_\kappa - \sin(t+\phi))^{1/2}e^{-\mathbf{i}\rho\cos t}$$
$$= \int_0^\infty \chi(t(s))(-s\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-s^2}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2}e^{-\mathbf{i}\rho s^2/2}$$
$$= \int_0^\infty \chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2}t^{-1/2}e^{-\mathbf{i}\rho t/2}$$

Let

$$f(t) = t^{-1/2}e^{-\mathbf{i}\rho t/2}$$
$$g(t) = -\int_t^{t-\mathbf{i}\infty} x^{-1/2}e^{-\mathbf{i}\rho x/2}dx$$
$$= \mathbf{i}\int_0^\infty (t-\mathbf{i}x)^{-1/2}e^{-\mathbf{i}\rho t-\rho x}dx$$

It is easy to see that $g'(t) = f(t)$. Then we have

$$
\begin{aligned}
&= \int_0^{\infty} \chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2} t^{-1/2} e^{-\mathbf{i}\rho t/2} \\
&= \chi(0)(-2\sin(\frac{\phi-\phi_\kappa}{2}))^{1/2} g(0) \\
&- \int_0^{\infty} (\chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2})' g(t) dt
\end{aligned}
$$

We get

$$
g(x) = \int_0^{\infty} \chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{-1/2} t^{-1/2} (t-\mathbf{i}x)^{-1/2} e^{-\mathbf{i}\rho t} dt
$$

$$
R(\rho) = \int_0^{\infty} g(x) e^{-\rho x} dx
$$

Because $\chi(t)$ has compact support $(-\delta, \delta)$, we obtain

$$
gg(x) = \int_0^{\delta} (\sqrt{t}\cos(\theta) - \sqrt{4-t}\sin\theta)^{-1/2} t^{-1/2} (t^2+x^2)^{-1/4} dt
$$

where $\theta = \frac{\phi-\phi_\kappa}{2}$. For $x > 0$, Put L(x):

$$
\begin{aligned}
&\int_0^{a} \frac{1}{t^{3/4}} \frac{1}{(t^2+x^2)^{1/4}} dt \\
&= 4\int_0^{a} \frac{1}{(t^2+x^2)^{1/4}} dt^{1/4} \\
&= 4\int_0^{a^{1/4}} \frac{1}{(t^8+x^2)^{1/4}} dt \\
&= 4x^{-1/4} \int_0^{(\frac{a}{x})^{1/4}} \frac{1}{(t^8+1)^{1/4}} dt \\
&= 4x^{-1/4} \int_0^{(\frac{a}{x})^{1/4}} \frac{1}{(t^8+1)^{1/4}} dt \\
&\le 4x^{-1/4} \int_0^{\infty} \frac{1}{(t^8+1)^{1/4}} dt
\end{aligned}
$$

Back to analysis $gg(x)$, we have

$$
\begin{aligned}
gg(x) &\le \int_0^{\delta} \Big|\frac{\sqrt{t}+2|\sin\theta|}{t-4\sin^2\theta}\Big|^{1/2} t^{-1/2} (t^2+x^2)^{-1/4} dt \\
&= \int_0^{\delta} \Big|\frac{1}{\sqrt{t}-2|\sin\theta|}\Big|^{1/2} t^{-1/2} (t^2+x^2)^{-1/4} dt \\
&= 2\int_0^{\sqrt{\delta}} \Big|\frac{1}{t-2|\sin\theta|}\Big|^{1/2} (t^4+x^2)^{-1/4} dt \\
&= 2\int_{-2|\sin\theta|}^{\sqrt{\delta}-2|\sin\theta|} |t|^{-1/2} ((t+2|\sin\theta|)^4+x^2)^{-1/4} dt \\
&\le 4\int_0^{\delta^{1/4}} (t^8+x^2)^{-1/4} dt + 4\int_0^{\sqrt{2|\sin\theta|}} ((t^2-2|\sin\theta|)^4+x^2)^{-1/4} dt
\end{aligned}
$$

$$\leq Cx^{-1/4}(1+\int_0^{\sqrt{2|\sin\theta|}}((t^2-2|\sin\theta|)^4/x+x)^{-1/4}dt)$$
$$\leq Cx^{-1/4}(1+\int_0^{\sqrt{2|\sin\theta|}}(t^2-2|\sin\theta|)^{-1/2}dt)$$
$$= Cx^{-1/4}(1+\int_0^{1}(1-t^2)^{-1/2}dt)\leq Cx^{-1/4}$$

Immediately, we can obtain

$$|g(x)|\leq Cx^{-1/4}$$

It follows that

$$R(\rho)\leq\int_0^\infty x^{-1/4}e^{-\rho x}\leq C\rho^{-3/4}$$

5. stationary of phase lemma

Lemma 5.1 *Assume that* $0<\kappa:=\sin\phi_\kappa<1, 0<\phi_\kappa<\pi/2$, $0\leq\phi\leq\pi/2$. *Let*

$$f(t,\phi):=F(\sin(t+\phi),\cos(t+\phi),(\kappa^2-\sin^2(t+\phi))^{1/2}) \tag{5.1}$$

be a complexed function in $C((-\pi/2,\pi/2)\times(0,\pi/2))$. *Moreover, its partial derivative with respect to* t *can be represented as*

$$\frac{\partial f(t,\phi)}{\partial t}=g(t,\phi)(\kappa^2-\sin^2(t+\phi))^{-1/2} \tag{5.2}$$

where $g(t,\phi)$ *is uniformly bounded. Then for any* $\rho\geq1$, *we have*

$$\Big|I(\rho,\phi):=\int_{-\pi/2}^{\pi/2}f(t)e^{\mathbf{i}\rho\cos t}dt-\Big(\frac{2\pi}{\rho}\Big)^{1/2}f(0)e^{\mathbf{i}\rho-\mathbf{i}\pi/4}\Big|$$
$$\leq C\frac{1}{\rho^{3/4}} \tag{5.3}$$

Proof. Solving the following equation:

$$\kappa^2-\sin^2(t+\phi)=0$$

we have, if $0<\phi<\pi/2-\phi_\kappa$,

$$t_1(\phi)=\phi_\kappa-\phi \qquad t_2(\phi)=-\phi_\kappa-\phi$$

and if $\pi/2-\phi_\kappa\leq\phi<\pi/2$,

$$t_1(\phi)=\phi_\kappa-\phi \qquad t_2(\phi)=\pi-\phi_\kappa-\phi$$

Since $|t_2(\phi)|<\phi_\kappa$ or $|t_2(\phi)|<\pi/2-\phi_\kappa$, we now define $\delta:=\min(\frac{\phi_\kappa}{2},\frac{\pi/2-\phi_\kappa}{2})$ and it is easy to see that

$$\kappa+\sin(t+\phi)\neq0$$
$$\cos(\frac{t+\phi+\phi_\kappa}{2})\neq0$$

for any $(t,\phi)\in[-\delta,\delta]\times[0,\pi/2]$. Let $\chi_\delta\in C_0^\infty(-\pi/2,\pi/2)$ be the cut-off function with that $0\le\chi_\delta\le 1$, $\chi_\delta=1$ in $(-\delta/2,\delta/2)$ and $\chi_\delta=0$ in $(-\pi/2,\pi/2)\backslash(-\delta,\delta)$. Then we can divide I into two parts such that

$$\begin{aligned}I &= \int_{-\delta}^{\delta} f(t)\chi_\delta(t)e^{\mathbf{i}\rho\cos t}dt + \int_{-\pi/2}^{\pi/2} f(t)(1-\chi_\delta(t))e^{\mathbf{i}\rho\cos t}dt\\ &=: I_1+I_2\end{aligned}$$

Subtitating $t(s)=2\arcsin s/2$ for t in I_1, we can obtain

$$\begin{aligned}I_1 = & \int_{s(-\delta)}^{s(\delta)} f(t(s))\chi_\delta(t(s))\frac{1}{\sqrt{1-s^2/4}}e^{\mathbf{i}\rho}e^{-\mathbf{i}\rho s^2/2}ds && (5.4)\\ = & \int_0^{s(\delta)}(f(t(s))\chi_\delta(t(s))+f(-t(s))\chi_\delta(-t(s)))\frac{1}{\sqrt{1-s^2/4}}e^{\mathbf{i}\rho}e^{-\mathbf{i}\rho s^2/2}ds && (5.5)\\ := & I_{11}+I_{12} && (5.6)\end{aligned}$$

Taking substitution $s=\sqrt{x}$, we have

$$I_{11}=\frac{1}{2}\int_0^{\sqrt{2\sin\frac{\delta}{2}}} f(t(\sqrt{x}))\chi_\delta(t(\sqrt{x}))\frac{1}{\sqrt{1-x/4}}x^{-1/2}e^{\mathbf{i}\rho}e^{-\mathbf{i}\rho x/2}ds$$

Observe that there exists $0<\delta<\min(\epsilon/2,\phi_\kappa,\pi/2-\phi_\kappa)$ that $\kappa/2<\sin(t+\phi)<(1+\kappa)/2$ for any $t\in(-\delta,\delta)$ and $\phi\in(\phi_\kappa-\delta,\phi_\kappa+\delta)$.

$$\begin{aligned}&(\kappa+\sin(t+\phi))>3\kappa/2\\ &\cos(\frac{t+\phi+\phi_\kappa}{2})=\sqrt{\sin^2(\frac{t+\phi+\phi_\kappa}{2})}>\sqrt{1-((1+\kappa)/2)^2}\\ &(\kappa^2-\sin^2(t+\phi))^{1/2}\\ &=((\kappa+\sin(t+\phi))(\kappa-\sin(t+\phi))^{1/2}\\ &=(\kappa+\sin(t+\phi))^{1/2}(\cos(\frac{t+\phi+\phi_\kappa}{2}))^{1/2}(-2\sin(\frac{t+\phi-\phi_\kappa}{2}))^{1/2}\\ &\frac{\partial f(t,\phi)}{\partial t}=\frac{\partial g_1(t,\phi)}{\partial t}+\frac{\partial g_2(t,\phi)}{\partial t}(\kappa^2-\sin^2(t+\phi))^{1/2}\\ &+g_2(t,\phi)\frac{-2\cos(t+\phi)}{(\kappa^2-\sin^2(t+\phi))^{1/2}}\\ &:=\frac{h(t,\phi)}{(\kappa^2-\sin^2(t+\phi))^{1/2}}\\ &h(t,\phi)<M\end{aligned}$$

For any $\phi\in[0,\pi/2]$, we have

$$\int_{-\pi/2}^{\pi/2}\frac{\partial f(t,\phi)}{\partial t}$$

□

References

[1] Lawrence C Evans. *Partial differential equations. 2nd ed.* Marcel Dekker,, 2010.

[2] Loukas Grafakos. *Classical and modern Fourier analysis.* Prentice Hall, 2004.

[3] R. Wong, Werner Rheinboldt, and Daniel Siewiorek. *Asymptotic Approximations of Integrals.* 1989.