1. Estimate of Dirichlet Green Tensor

We need the following slight generalization of Van der Corput lemma for the oscillatory integral [2, P.152].

Lemma 1.1 *Let $-\infty < a < b < \infty$, and u is a C^k function u in (a,b).*
1. If $|u'(t)| \geq 1$ for $t \in (a,b)$ and u' is monotone in (a,b), then for any $\phi(t)$ in (a,b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 3\lambda^{-1}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

2. For all $k \geq 2$, if $|u^{(k)}(t)| \geq 1$ for $t \in (a,b)$, then for any $\phi(t)$ in (a,b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 12k\lambda^{-1/k}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

Proof. The assertion can be proved by extending the Van der Corptut lemma in [2]. Here we omit the details. □

We recall following lemma, see e.g. [5]:

Lemma 1.2 *Let $F(\rho,a) = \int_0^a t^{\alpha-1}f(t)e^{-\mathbf{i}\rho t}dt$ where $0 < a \leq +\infty$, $0 < \alpha < 1$, $\rho > 0$ and $t^{\alpha-1}f \in L^1(0,a)$, then we have*

$$|F(\rho,a)| \leq C(\frac{1}{\rho^\alpha}f(0) + \frac{1}{\rho}(a^{\alpha-1}f(a) + |t^{\alpha-1}f|_{L^1(0,a)}) \tag{1.1}$$

Proof. Put

$$g_0(t) = t^{\alpha-1}e^{-\mathbf{i}\rho t} \tag{1.2}$$

and define

$$g_1(t) = -\int_t^{t-\mathbf{i}\infty} x^{\alpha-1}e^{-\mathbf{i}\rho x}dx \tag{1.3}$$

where the path of integration is the vertical line $x = t - \mathbf{i}y, y \geq 0$. It is easy to show that $g_1(t)' = g_0(t)$. Substituting $x = t - \mathbf{i}y$ into $g_1(t)$, we have

$$g_1(t) = \mathbf{i}\int_0^\infty (t-\mathbf{i}y)^{\alpha-1}e^{-\mathbf{i}\rho t}e^{-\rho y}dy \tag{1.4}$$

Upon integration by parts, we have

$$\begin{aligned} F(\rho,a) &= \int_0^a f(t)dg_1(t) \\ &= e^{-\mathbf{i}\frac{\alpha\pi}{2}}f(0)\Gamma(\alpha)\frac{1}{\rho^\alpha} + f(a)g_1(a) - \int_0^a f'(t)g_1(t)dt \\ &= e^{-\mathbf{i}\frac{\alpha\pi}{2}}f(0)\Gamma(\alpha)\frac{1}{\rho^\alpha} - \mathbf{i}\int_0^\infty e^{-\rho y}dy\int_0^a f'(t)(t-\mathbf{i}y)^{\alpha-1}e^{-\mathbf{i}\rho t}dt \end{aligned}$$

Let

$$h(y) = \int_0^a f'(t)(t-\mathbf{i}y)^{\alpha-1}e^{-\mathbf{i}\rho t}dt$$

and observe that

$$|h(y)| \le \int_0^a |f'(t)|(t^2+y^2)^{\frac{\alpha-1}{2}} dt$$

□

Lemma 1.3 *Let* $F(\rho, a) = \int_0^a t^{-1/2} f(t) e^{-\mathbf{i}\rho t} dt$ *where* $0 < a \le +\infty$ *and* $\rho > 0$, *then we have*

$$|F(\rho, a) - e^{-\mathbf{i}\frac{\pi}{4}} f(0)\Gamma(1/2)\frac{1}{\rho^{1/2}}| \tag{1.5}$$

$$\le C(\int_0^\infty e^{-\rho y} dy \int_0^a |f'(t)|(t^2+y^2)^{-\frac{1}{4}} dt + \frac{1}{\rho} a^{-1/2} f(a)) \tag{1.6}$$

Proof. Put

$$g_0(t) = t^{-1/2} e^{-\mathbf{i}\rho t} \tag{1.7}$$

and define

$$g_1(t) = -\int_t^{t-\mathbf{i}\infty} x^{-1/2} e^{-\mathbf{i}\rho x} dx \tag{1.8}$$

where the path of integration is the vertical line $x = t - \mathbf{i}y, y \ge 0$. It is easy to show that $g_1(t) = g_0(t)$. Substituting $x = t - \mathbf{i}y$ into $g_1(t)$, we have

$$g_1(t) = \mathbf{i}\int_0^\infty (t-\mathbf{i}y)^{-1/2} e^{-\mathbf{i}\rho t} e^{-\rho y} dy \tag{1.9}$$

Upon integration by parts, we have

$$\begin{aligned} F(\rho, a) &= \int_0^a f(t) dg_1(t) \\ &= e^{-\mathbf{i}\frac{\pi}{4}} f(0)\Gamma(1/2)\frac{1}{\rho^{1/2}} + f(a)g_1(a) - \int_0^a f'(t) g_1(t) dt \\ &= e^{-\mathbf{i}\frac{\pi}{4}} f(0)\Gamma(1/2)\frac{1}{\rho^{1/2}} + \mathbf{i} f(a)\int_0^\infty (a-\mathbf{i}y)^{-1/2} e^{-\mathbf{i}\rho t} e^{-\rho y} dy \\ &\quad - \mathbf{i}\int_0^\infty e^{-\rho y} dy \int_0^a f'(t)(t-\mathbf{i}y)^{-1/2} e^{-\mathbf{i}\rho t} dt \end{aligned}$$

Let

$$h(y) = \int_0^a f'(t)(t-\mathbf{i}y)^{-1/2} e^{-\mathbf{i}\rho t} dt$$

and observe that

$$|h(y)| \le \int_0^a |f'(t)|(t^2+y^2)^{-\frac{1}{4}} dt$$

It is easy to see that

$$|g_1(a)| \le a^{-1/2}\int_0^\infty e^{-\rho y} dy \le C\frac{1}{\rho}$$

□

Lemma 1.4 *Assume that* $0 < \kappa := \sin\phi_\kappa < 1, 0 < \phi_\kappa < \pi/2$, $0 \le \phi \le \pi/2$ *and* $-\pi/2 < t_1 < t_2 < \pi/2$ *satisfy that* $\kappa^2 = \sin^2(\phi + t_1) = \sin^2(\phi + t_2)$. *Let* $f(\theta)$:

$$f(t,\phi) := F(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2}) \tag{1.10}$$

be a function in $C^\infty(([-\pi/2,\pi/2]\backslash\{t_1,t_2\}) \times [0,\pi/2])$. *Moreover, there exits* $\epsilon > 0$ *such that* $f(\theta)$ *can be represented as*

$$f(t,\phi) = g_1(t,\phi) + g_2(t,\phi)(\kappa^2 - \sin^2(t+\phi))^{1/2})^{N/2} \tag{1.11}$$

where $g_1, g_2 \in C^\infty((\bigcup_{i=1,2}(t_i - \epsilon, t_i + \epsilon)) \times [0,\pi/2]))$ *and* $N = \pm 1$. *Then for any* $\rho \ge 1$, *we have*

$$\begin{aligned} &\Big| I(\rho,\phi) := \int_{-\pi/2}^{\pi/2} f(\theta) e^{\mathbf{i}\rho\cos\theta} d\theta - \frac{N+1}{2}\Big(\frac{2\pi}{\rho}\Big)^{1/2} f(0) e^{\mathbf{i}\rho - \mathbf{i}\pi/4}\Big| \\ &\le C\frac{1}{\rho^{(2+N)/4}} \end{aligned} \tag{1.12}$$

Proof. The proof will be split into two parts about whether ϕ equal to ϕ_κ.

If $\phi \ne \phi_\kappa$, there exists $0 < \delta < \pi/4$ such that

$$|(\kappa^2 - \sin^2(t+\phi))^{1/2}| > \frac{1}{2}|(\kappa^2 - \sin^2\phi)^{1/2}| \tag{1.13}$$

for any $t \in (-\delta,\delta)$. Let $\chi_\delta \in C_0^\infty(-\pi/2,\pi/2)$ be the cut-off function with that $0 \le \chi_\delta \le 1$, $\chi_\delta = 1$ in $(-\delta/2,\delta/2)$ and $\chi_\delta = 0$ in $(-\pi/2,\pi/2)\backslash(-\delta,\delta)$. Then we can divide I into two parts such that

$$\begin{aligned} I &= \int_{-\delta}^{\delta} f(t)\chi_\delta(t) e^{\mathbf{i}\rho\cos t} dt + \int_{-\pi/2}^{\pi/2} f(t)(1-\chi_\delta(t)) e^{\mathbf{i}\rho\cos t} dt \\ &=: I_1 + I_2 \end{aligned}$$

Subtitating $t(s) = 2\arcsin s/2$ for t in I_1, we can obtain

$$I_1 = \int_{\mathbb{R}} f(t(s))\chi_\delta(t(s)) \frac{1}{\sqrt{1-s^2/4}} e^{\mathbf{i}\rho} e^{-\mathbf{i}\rho s^2/2} ds \tag{1.14}$$

$$= \int_{\mathbb{R}} h_\delta(s) e^{\mathbf{i}\rho} e^{-\mathbf{i}\rho s^2/2} ds \tag{1.15}$$

It is easy to see that $h_\delta(s) \in C_0^4(\mathbb{R})$. By the lemma of the stationary phase for quadratic term in [1], we have

$$I_1 = e^{\mathbf{i}\rho}\int_{\mathbb{R}} h_\delta(s) e^{-\mathbf{i}\frac{\rho}{2}s^2} ds = e^{\mathbf{i}\rho}\int_{\mathbb{R}} \widehat{h_\delta}(y)\alpha(-y) dy \tag{1.16}$$

where

$$\alpha(y) = (\frac{1}{2\pi\rho})^{1/2} e^{-\mathbf{i}\pi/4} e^{\frac{\mathbf{i}}{2\rho}y^2} \tag{1.17}$$

$$= (\frac{1}{2\pi\rho})^{1/2} e^{-\mathbf{i}\pi/4}(1 + O(\frac{y^2}{\rho})) \tag{1.18}$$

Consequently

$$I_1 = (\frac{1}{2\pi\rho})^{1/2} e^{\mathbf{i}\rho - \mathbf{i}\pi/4}\int_{\mathbb{R}} \widehat{h_\delta}(y)(1 + \frac{1}{\rho}O(y^2)) dy \tag{1.19}$$

Moreover, $\int_{\mathbb{R}} \widehat{h_\delta}(y)dy = 2\pi h_\delta(0)$ and $|\int_{\mathbb{R}} \widehat{h_\delta}(y)y^2 dy| < C$ since $\widehat{h_\delta}(y) = O(1/y^4)$. Now, it turns to estimate I_2.

When $N = 1$, using integration by parts, we have

$$|I_2| = \left| \int_{(-\frac{\pi}{2},\frac{\pi}{2})\setminus(-\frac{\delta}{2},\frac{\delta}{2})} f(t)(1-\chi_\delta(t))/\sin t \;\; de^{\mathbf{i}\rho\cos t} \right| \tag{1.20}$$

$$\tag{1.21}$$

$$\leq C\frac{1}{\rho} + \left| \int_{(-\frac{\pi}{2},\frac{\pi}{2})\setminus(-\frac{\delta}{2},\frac{\delta}{2})} (f(t)(1-\chi_\delta(t))/\sin t)' \; e^{\mathbf{i}\rho\cos t} dt \right| \tag{1.22}$$

$$\leq C\frac{1}{\rho} \tag{1.23}$$

From above analysis, we obtain

$$\left| I(\rho,\phi) - \left(\frac{2\pi}{\rho}\right)^{1/2} f(0)e^{\mathbf{i}\rho - \mathbf{i}\pi/4} \right| \leq C(\phi)\frac{1}{\rho} \tag{1.24}$$

When $N = -1$, we can not use integration by parts again since $f'(\theta)$ is not integrable. However, for any $0 < \lambda_1 < 1$ and $1 < \lambda_2 < 1/\kappa$, there exists $0 < \sigma < \epsilon$, such that $\chi := ((t_1-\sigma, t_1+\sigma) \cup (t_2-\sigma, t_2+\sigma)) \cap (-\delta,\delta) = 0$, dependent on λ_1, λ_2 and

$$\lambda_1\kappa < |\sin(t+\phi)| < \lambda_2\kappa. \tag{1.25}$$

for any $t \in \chi$.

We only analysis the integral on $\chi_1 = (t_1-\sigma, t_1+\sigma) \cap [-\pi/2, \pi/2]$ here, which denoted by I_{χ_1}, the proof of I_{χ_2} is similar. It is easy to see that $\sin(t+\phi)$ is monotonic in χ_1. Without loss of generality, we assume that $\sin(t_1-\sigma+\phi) < \kappa < \sin(t_1+\sigma+\phi)$. Let $\sin(t+\phi) = \kappa\sin\theta$ and the implicit mapping from θ to t is denoted by $t(\theta)$ while the inverse mapping by $\theta(t)$, taking the interval χ_1 onto $L_\theta : \theta_1 \to \pi/2 \to \pi/2 - \mathbf{i}\theta_2$ where $\sin(t_1-\sigma+\phi) = \kappa\sin\theta_1, \sin(t_1+\sigma+\phi) = \kappa\sin(\pi/2 - \mathbf{i}\theta_2)$. By substituting $t(\theta)$ into I_{χ_1}, we have

$$I_{\chi_1} = \int_{t_1-\sigma}^{t_1+\sigma} \frac{f(t)(\kappa^2 - \sin^2(t+\phi))^{1/2}}{(\kappa^2 - \sin^2(t+\phi))^{1/2}} e^{\mathbf{i}\rho\cos t} \tag{1.26}$$

$$= \int_{L_\theta} \frac{\kappa f(t(\theta))\cos\theta}{(1-\kappa^2\sin^2\theta)^{1/2}} e^{\mathbf{i}\rho(\cos(t(\theta)))} d\theta \tag{1.27}$$

$$= \int_{L_\theta} \frac{\kappa g_1(t(\theta))\cos\theta + g_2(t(\theta))}{(1-\kappa^2\sin^2\theta)^{1/2}} e^{\mathbf{i}\rho(\cos(t(\theta)))} d\theta \tag{1.28}$$

$$:= \int_{L_\theta} \frac{h(\theta)}{(1-\kappa^2\sin^2\theta)^{1/2}} e^{\mathbf{i}\rho(\cos(t(\theta)))} d\theta \tag{1.29}$$

Observe that $h(\theta)$ and $\partial h/\partial\theta$ are integrable on the path L_θ by (1.11). A simple computation show that

$$\frac{dt(\theta)}{d\theta} = \frac{\kappa\cos\theta}{\cos(t+\phi)} \qquad \frac{d^2t(\theta)}{dt^2} = \frac{\kappa^2\cos^2\theta\sin(t+\phi) - \kappa\sin\theta\cos^2(t+\phi)}{\cos^3(t+\phi)}$$

Then we can obtain

$$\frac{d\cos t}{d\theta} = \frac{-\kappa\sin t\cos\theta}{\cos(t+\phi)}$$

$$\begin{aligned}\frac{d^2\cos t}{d\theta^2} &= \frac{d^2\cos t}{dt^2}(\frac{dt}{d\theta})^2 + \frac{d\cos t}{dt}\frac{d^2 t}{d\theta^2} \\ &= \frac{-\kappa^2\cos^2\theta\cos t}{\cos^2(t+\phi)} + \frac{\kappa\sin\theta\cos^2(t+\phi)\sin t - \kappa^2\cos^2\theta\sin(t+\phi)\sin t}{\cos^3(t+\phi)} \\ &= \frac{-\kappa^2\cos^2\theta\cos\phi + \kappa\sin\theta\cos^2(t+\phi)\sin t}{\cos^3(t+\phi)} \\ &= \frac{(\sin^2(t+\phi)-\kappa^2)\cos\phi + \cos^2(t+\phi)\sin(t+\phi)\sin t}{\cos^3(t+\phi)}\end{aligned}$$

Since $|\sin t| > |\sin\delta|$ and $1-\lambda_2^2\kappa^2 < \cos^2(t+\phi) < 1-\lambda_1^2\kappa^2$ for $t \in \chi_1$, it follows that $\theta = \pi/2$ is the only stationary point of $\cos(t(\theta))$ and

$$\left|\frac{d^2\cos t}{d\theta^2}(\pi/2)\right| = \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}|\sin t| > \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}\sin\delta \tag{1.30}$$

Therefore, we can choose appropriate λ_1, λ_2 such that

$$|\frac{d^2\cos t}{d\theta^2}| > \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}\sin\delta \tag{1.31}$$

for any $\theta \in \theta(\chi_1)$. According to lemma (6.1), we obtian $|I_{\chi_1}| \le C\frac{1}{\rho^{1/2}}$, and also $|I_{\chi_2}| \le C\frac{1}{\rho^{1/2}}$. Using integration by parts, we get

$$\left|\int_{[-\pi/2,\pi/2]\backslash((-\delta,\delta)\cup\chi)} f(t)(1-\chi_\delta(t))e^{\mathbf{i}\rho\cos t}dt\right| \le C\frac{1}{\rho}$$

Consequently, for $N=-1$ and $\phi \ne \phi_\kappa$, we get $|I(\rho,\phi)| \le \frac{1}{\rho^{1/2}}$.

We now turn to the case of $\phi = \phi_\kappa$. By (1.11), we can define χ_ϵ similarly and also decompose I into I_1 and I_2. Using the same agurement above, we can easily carry out that: for $N=1$, we have $|I_2| \le C\frac{1}{\rho}$; for $N=-1$, we have $|I_2| \le C\frac{1}{\rho^{1/2}}$. Finally, it remains to analysis I_1. By (1.11), we have

$$\begin{aligned}I_1 &= \int_{-\epsilon}^{\epsilon} g_1\chi_\epsilon + g_2\chi_\epsilon(\sin^2\phi_\kappa - \sin^2(t+\phi_\kappa))^{N/2}e^{\mathbf{i}\rho\cos t}dt \\ &= \int_{-\epsilon}^{\epsilon} g_1\chi_\epsilon + g_2\chi_\epsilon(-2(\sin\phi_\kappa+\sin(t+\phi_\kappa))\cos\frac{2\phi_\kappa+t}{2}\sin t/2)^{N/2}e^{\mathbf{i}\rho\cos t}dt \\ &= \int_{\mathbb{R}} g_1\chi_\epsilon + g_2\chi_\epsilon((\sin\phi_\kappa+\sin(t+\phi_\kappa))\cos\frac{2\phi_\kappa+t}{2})^{N/2}(-2\sin t/2)^{N/2}e^{\mathbf{i}\rho\cos t}dt\end{aligned}$$

Also, subtituting $t(s) = 2\arcsin s/2$ for t in I_1, it follows that

$$\begin{aligned}I_1 &= \int_{\mathbb{R}} h_1(s)e^{-\mathbf{i}\rho\frac{s^2}{2}} + h_2(s)(-s)^{N/2}e^{-\mathbf{i}\rho\frac{s^2}{2}} &(1.32)\\ &= I_{11} + I_{12} &(1.33)\end{aligned}$$

where

$$\begin{aligned}h_1(s) &= g_1(t(s))\chi_\epsilon(t(s))\sqrt{1-s^2/4}\; e^{\mathbf{i}\rho} \\ h_2(s) &= g_2\chi_\epsilon((\sin\phi_\kappa+\sin(t+\phi_\kappa))\cos\frac{2\phi_\kappa+t}{2})^{N/2}_{t=t(s)}\sqrt{1-s^2/4}\; e^{\mathbf{i}\rho}\end{aligned}$$

and $h_1(s), h_2(s) \in C_c^\infty(\mathbb{R})$. Using stationary phase lemma similarly, if $N = 1$,

$$I_{11} = \Big(\frac{2\pi}{\rho}\Big)^{1/2} g_1(0) e^{\mathbf{i}\rho - \mathbf{i}\pi/4} + O(\frac{1}{\rho}) \tag{1.34}$$
$$= \Big(\frac{2\pi}{\rho}\Big)^{1/2} f(0) e^{\mathbf{i}\rho - \mathbf{i}\pi/4} + O(\frac{1}{\rho}) \tag{1.35}$$

if $N = -1$, we get $|I_{11}| \le C\frac{1}{\rho^{1/2}}$. For I_{12}, we have

$$I_{12} = \int_0^\infty (\mathbf{i}h_2(s) + h_2(-s)) s^{N/2}\ e^{-\mathbf{i}\rho s^2/2} ds \tag{1.36}$$
$$= \frac{1}{2}\int_0^\infty (\mathbf{i}h_2(\sqrt{s}) + h_2(-\sqrt{s})) s^{N/4-1/2} e^{-\mathbf{i}\rho s/2} ds \tag{1.37}$$

By lemma (1.2), we get $|I_{12}| \le C\frac{1}{\rho^{(N+2)/4}}$. □

2. Some draft about Green Tensor Analysis

Let substitute $\xi = k\sin\theta$ into integral and shift the variable, we have

$$I(y) = \int_{\mathbb{R}} f(\xi) e^{\mathbf{i}\xi y_1 + \mu(\xi) y_2} d\xi = \int_{\mathbb{R}} f(\xi) e^{\mathbf{i}\xi(y_1 - z_1) + \mu(\xi)(y_2 - z_2)} e^{\mathbf{i}\xi z_1 + \mu(\xi) z_2} d\xi \tag{2.1}$$
$$= k\int_L f(k\sin\theta)\cos\theta e^{\mathbf{i}k|y-z|\cos(\theta-\eta)} e^{\mathbf{i}|z|\cos(\theta-\phi)} d\theta \tag{2.2}$$
$$= k\int_{L_\phi} f(k\sin(\theta+\phi))\cos(\theta+\phi) e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)} e^{\mathbf{i}|z|\cos\theta} d\theta \tag{2.3}$$
$$= k\int_L f(k\sin(\theta+\phi))\cos(\theta+\phi) e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)} e^{\mathbf{i}|z|\cos\theta} d\theta \tag{2.4}$$

where $y_1, y_2 > 0, \sin\phi = \frac{z_1}{|z|}, \cos\phi = \frac{z_2}{|z|}, 0 < \phi < \pi/2$ and $\sin\eta = \frac{y_1 - z_1}{|y-z|}, \cos\eta = \frac{y_2 - z_2}{|y-z|}, 0 < \eta < \pi$. It is easy to see that $\phi + \eta < \pi$. Roughly, using stationary phase lemma, we obtain:

$$I(y) = f(k\sin\phi) k\cos\phi e^{\mathbf{i}k|y-z|\cos(\phi-\eta)} (\frac{2\pi}{|z|})^{1/2} e^{\mathbf{i}|z| - \mathbf{i}\frac{\pi}{4}} (1 + O(\frac{1}{|z|})) \tag{2.5}$$

$$\cos(a + \mathbf{i}b) = \frac{e^b + e^{-b}}{2}\cos a + \mathbf{i}\frac{e^{-b} - e^b}{2}\sin a \tag{2.6}$$
$$\sin(a + \mathbf{i}b) = \frac{e^b + e^{-b}}{2}\sin a + \mathbf{i}\frac{e^b - e^{-b}}{2}\cos a \tag{2.7}$$

When $\theta \in (-a - \pi/2, -a - \pi/2 + \mathbf{i}\infty)$, let $\theta = -a - \pi/2\mathbf{i}t$, where $t > 0, 0 \le a \le \phi$, then

$$-\mathrm{Im}\,(|z|\cos\theta + |y-z|\cos(\theta + \phi - \eta))$$
$$= |z|\sin(a + \pi/2) + |y-z|\sin(a + \pi/2 - \phi + \eta) \tag{2.8}$$
$$= |z|\cos a + |y-z|\cos(a - \phi + \eta) \tag{2.9}$$
$$= |z|\cos a + \cos a|y-z|(\cos\phi\cos\eta + \sin\phi\sin\eta) \tag{2.10}$$
$$+ \sin a|y-z|(\sin\phi\cos\eta - \cos\phi\sin\eta) \tag{2.11}$$

$$= |z|\cos a + \cos a((y_2 - z_2)\cos\phi + (y_1 - z_1)\sin\phi) \tag{2.12}$$
$$+ \sin a((y_2 - z_2)\sin\phi - (y_1 - z_1)\cos\phi) \tag{2.13}$$
$$= y_1 \sin(\phi - a) + y_2 \cos(\phi - a) > 0 \tag{2.14}$$

Now, Using Cauchy Integral Theorem, we have

$$I(y) = k\int_L f(k\sin(\theta+\phi))\cos(\theta+\phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \tag{2.15}$$

Let $L_1 = (-\pi/2, -\pi/2 + \mathbf{i}\infty)$ and $\theta = -\pi/2 + \mathbf{i}t, t > 0$, then

$$I_1(y) = k\int_{L_1} f(k\sin(\theta+\phi))\cos(\theta+\phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \tag{2.16}$$
$$= \tag{2.17}$$

$$I(y) = f(k\sin\phi)k\cos\phi e^{\mathbf{i}k|y-z|\cos(\phi-\eta)}(\frac{2\pi}{|z|})^{1/2}e^{\mathbf{i}|z|-\mathbf{i}\frac{\pi}{4}} \tag{2.18}$$
$$+\frac{kz_2}{|z|}O(\left(\frac{1}{k|z|}\right)^{3/4} + \frac{1}{k|y|}) + \frac{kz_1}{|z|}O(\left(\frac{1}{k|z|}\right)^{5/4} + \left(\frac{1}{k|y|}\right)^2) \tag{2.19}$$

It is easy to see

$$\int_{-d}^{d}\frac{k}{(k|x-z|)^\alpha}\frac{1}{(k|x-y|)^\beta}dx_1 \le C(\frac{1}{(kz_2)^{\alpha+\beta-1}} + \frac{1}{(ky_2)^{\alpha+\beta-1}}) \tag{2.20}$$

where $z, y \in \mathbb{R}^2_+$, $x \in \Gamma_0$ and $\alpha + \beta > 0$.

$$e^{\mathbf{i}\mu y_2 + \mathbf{i}\xi(x_1-y_1)} = e^{\mathbf{i}\mu y_2 - \mathbf{i}y_2/\tan\phi} = e^{\mathbf{i}y_2(\mu-\xi/\tan\phi)} \tag{2.21}$$

Another method

$$\int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi)e^{\mathbf{i}k|x-y|\cos\theta} \tag{2.22}$$
$$= \int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi)e^{\mathbf{i}k|x-y|\cos(\theta+\psi-\psi)} \tag{2.23}$$
$$= \int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi)e^{\mathbf{i}ky_2\cos(\theta+\psi)+\mathbf{i}k|x_1-y_1|\sin(\theta+\psi)} \tag{2.24}$$
$$= \int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi) \tag{2.25}$$
$$e^{\mathbf{i}k(y_2-z_2)\cos(\theta+\psi)+\mathbf{i}k(|x_1-y_1|-|x_1-z_1|)\sin(\theta+\psi)+\mathbf{i}k|z|\cos(\theta+\psi-\phi)} \tag{2.26}$$

3. Finite Aperture Point Spread Function

If $x \in \Gamma_0$ and $z, y \in \mathbb{R}^2_+$, by lemma (**??**) we have

$$G(x,y) = \frac{\mathbf{i}k_s}{\mu\sqrt{2\pi}}\frac{1}{\delta(\xi)}\left(\begin{array}{cc} \mu_s\beta & \xi\beta \\ 2\xi\mu_s\mu_p & 2\xi^2\mu_p \end{array}\right)_{\xi=k_s\frac{x_1-y_1}{|x-y|}}\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{1/2}}e^{\mathbf{i}k_s|x-y|-\mathbf{i}\frac{\pi}{4}}$$
$$+\frac{\mathbf{i}k_p}{\mu\sqrt{2\pi}}\frac{1}{\delta(\xi)}\left(\begin{array}{cc} 2\xi^2\mu_s & -2\xi\mu_s\mu_p \\ -\xi\beta & \mu_p\beta \end{array}\right)_{\xi=k_p\frac{x_1-y_1}{|x-y|}}\frac{y_2}{|x-y|}\frac{1}{(k_p|x-y|)^{1/2}}e^{\mathbf{i}k_p|x-y|-\mathbf{i}\frac{\pi}{4}} \tag{3.1}$$

$$+O(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}})$$
$$:=\mathcal{G}_s(x,y)+\mathcal{G}_p(x,y)+O(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}})$$

$$\begin{aligned}T_D(x,z) &= \frac{k_s}{\sqrt{2\pi}}\frac{1}{\gamma(\xi)}\begin{pmatrix}\mu_s\mu_p & \xi\mu_p\\ \xi\mu_s & \xi^2\end{pmatrix}_{\xi=k_s\frac{x_1-z_1}{|x-z|}}\frac{z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{1/2}}e^{\mathbf{i}k_s|x-z|-\mathbf{i}\frac{\pi}{4}}\\ &+\frac{k_p}{\sqrt{2\pi}}\frac{1}{\gamma(\xi)}\begin{pmatrix}\xi^2 & -\xi\mu_p\\ -\xi\mu_s & \mu_p\mu_s\end{pmatrix}_{\xi=k_p\frac{x_1-z_1}{|x-z|}}\frac{z_2}{|x-z|}\frac{1}{(k_p|x-z|)^{1/2}}e^{\mathbf{i}k_p|x-z|-\mathbf{i}\frac{\pi}{4}} \qquad (3.2)\\ &+O(\frac{k_sz_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{k_s|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}})\\ &:=\mathcal{T}_s(x,z)+\mathcal{T}_p(x,z)+O(\frac{k_sz_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{k_s|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}})\end{aligned}$$

Now we consider the finite aperture point spread function $J_d(z,y)$:

$$\int_{-d}^{d}(T_D(x_1,0;z_1,z_2))^T\overline{G(x_1,0;y_1,y_2)}dx_1 \qquad (3.3)$$

Recall following standard asymptotic expansion:

$$|x-y|=|x-z|+\widehat{x-z}\cdot(z-y)+O(\frac{|y-z|^2}{|x-z|}) \qquad (3.4)$$
$$|y|^{-\alpha}=|z|^{-\alpha}(1+\frac{|y|-|z|}{|z|})^{-\alpha}=|z|^{-\alpha}(1+O(\frac{|y-z|}{|z|})) \qquad (3.5)$$
$$e^{\mathbf{i}t}=1+O(t) \qquad (3.6)$$
$$|a^{1/2}-b^{1/2}|\le C\sqrt{|a-b|} \qquad (3.7)$$

where $x,y,z\in\mathbb{R}^2$, $t,a,b\in\mathbb{R}$ and $\alpha>0$.

Lemma 3.1 *For any $z,y\in\mathbb{R}^2_+$, $J_d(z,y)=F(z,y)+O((1+\frac{|y-z|}{z_2})(\frac{1}{k_sz_2})^{1/4}+\frac{(k_s|y-z|)^2}{k_sz_2}+(\frac{|y-z|}{z_2})^{1/2})$, where*

$$F(z,y)=-\frac{\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d}f_s(\theta)\begin{pmatrix}\sin^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \cos^2\theta\end{pmatrix}e^{\mathbf{i}k_s(z_1-y_1)\cos\theta+\mathbf{i}k_s(z_2-y_2)\sin\theta}d\theta \qquad (3.8)$$
$$-\frac{\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d}f_p(\theta)\begin{pmatrix}\cos^2\theta & -\sin\theta\cos\theta\\ -\sin\theta\cos\theta & \sin^2\theta\end{pmatrix}e^{\mathbf{i}k_p(z_1-y_1)\cos\theta+\mathbf{i}k_p(z_2-y_2)\sin\theta}d\theta \qquad (3.9)$$

and

$$f_s(\theta)=\frac{\sin\theta((\kappa^2-\cos^2\theta)^{1/2}(1-2\cos^2\theta)+2\overline{(\kappa^2-\cos^2\theta)^{1/2}}\cos^2\theta)}{(\cos^2\theta+\sin\theta(\kappa^2-\cos^2\theta)^{1/2})\overline{((1-2\cos^2\theta)^2+4\cos^2\theta\sin\theta(\kappa^2-\cos\theta)^{1/2})}}$$
$$f_p(\theta)=\frac{\sin\theta(1/\kappa^2-\cos^2\theta)^{1/2}}{(\cos^2\theta+\sin\theta(1/\kappa^2-\cos^2\theta)^{1/2})((1/\kappa^2-2\cos^2\theta)^2+4\cos^2\theta\sin\theta(1/\kappa^2-\cos\theta)^{1/2})}$$

where $0<\theta_1^d<\pi/2<\theta_2^d<\pi$ and $z_2=(d+z_1)\tan\theta_1^d=(z_1-d)\tan\theta_2^d$.

Proof.

$$\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}}$$
$$=(\frac{z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}})(1+O(\frac{|y-z|}{|x-z|}))$$

$$|\mu_{\mathbf{i}}(k_j\frac{x_1-y_1}{|x-y|})-\mu_{\mathbf{i}}(k_j\frac{x_1-z_1}{|x-z|})|$$
$$\le Ck_j\sqrt{\Big|\frac{x_1-y_1}{|x-y|}-\frac{x_1-z_1}{|x-z|}\Big|}\le Ck_j\Big(\frac{|y-z|}{|x-z|}\Big)^{1/2}$$

where $i,j=s,p$. By above, we can obtain

$$\mathcal{G}_s(x,y)=\mathcal{G}_s(x,z)e^{\mathbf{i}k_s\widehat{x-z}\cdot(z-y)}+O(\frac{(k_s|y-z|)^2}{(k_s|x-z|)^{3/2}})+O(\frac{(k_s|y-z|)^{1/2}}{k_s|x-z|}) \quad (3.10)$$

$$\mathcal{G}_p(x,y)=\mathcal{G}_p(x,z)e^{\mathbf{i}k_p\widehat{x-z}\cdot(z-y)}+O(\frac{(k_p|y-z|)^2}{(k_p|x-z|)^{3/2}})+O(\frac{(k_p|y-z|)^{1/2}}{k_p|x-z|}) \quad (3.11)$$

For $l>1$, a simple computation show that

$$\int_{-d}^{d}\frac{k_s}{(k_s|x-z|)^l}dx_1=\frac{1}{(k_sz_2)^{l-1}}\int_{\frac{-d-z_1}{z_2}}^{\frac{d-z_1}{z_2}}\frac{1}{(1+t^2)^{l/2}}dt\le C\frac{1}{(k_sz_2)^{l-1}} \quad (3.12)$$

Let

$$\mathcal{G}_\alpha(x,y)=\frac{\mathbf{i}}{\sqrt{2\pi}\mu}g_\alpha(\frac{x_1-y_1}{|x-y|},\kappa)\frac{1}{(k_\alpha|x-y|)^{1/2}}e^{\mathbf{i}k_\alpha|x-y|-\mathbf{i}\frac{\pi}{4}} \quad (3.13)$$

$$\mathcal{T}_\alpha(x,y)=\frac{k_\alpha}{\sqrt{2\pi}}t_\alpha(\frac{x_1-z_1}{|x-z|},\kappa)\frac{1}{(k_s|x-z|)^{1/2}}e^{\mathbf{i}k_\alpha|x-z|-\mathbf{i}\frac{\pi}{4}} \quad (3.14)$$

where $\alpha=s,p$. Now, by substituting (3.10-3.11) into $J_d(z,y)$ and using inequality (3.12), we have

$$J_d(z,y)=\frac{-\mathbf{i}}{2\pi\mu}\int_{-d}^{d}t_s(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_s(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_s\widehat{x-z}\cdot(y-z)}}{|x-z|}$$
$$+t_p(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_p(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_p\widehat{x-z}\cdot(y-z)}}{|x-z|}dx_1 \quad (3.15)$$
$$-\frac{\mathbf{i}}{2\pi\mu}\int_{-d}^{d}t_p(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_s(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_s\widehat{x-z}\cdot(y-z)}}{|x-z|} \quad (3.16)$$
$$+t_s(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_p(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_p\widehat{x-z}\cdot(y-z)}}{|x-z|}dx_1 \quad (3.17)$$
$$+O((1+\frac{|y-z|}{z_2})(\frac{1}{k_sz_2})^{1/4}+\frac{(k_s|y-z|)^2}{k_sz_2}+(\frac{|y-z|}{z_2})^{1/2}) \quad (3.18)$$
$$:=F(z,y)+R(z,y) \quad (3.19)$$
$$+O((1+\frac{|y-z|}{z_2})(\frac{1}{k_sz_2})^{1/4}+\frac{(k_s|y-z|)^2}{k_sz_2}+(\frac{|y-z|}{z_2})^{1/2}) \quad (3.20)$$

We denote $\widehat{x-z} = x - z/|x-z| = (cos(\phi+\pi), \sin(\phi+\pi))$, then taking the substitution $x_1 = z_1 - z_2 \cot\phi$, we obtain

$$F(z,y) = \frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d} A_s(\phi,\kappa)e^{\mathbf{i}k_s(z_1-y_1)\cos\phi+\mathbf{i}k_s(z_2-y_2)\sin\phi} \tag{3.21}$$

$$+\frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d} A_p(\phi,\kappa)e^{\mathbf{i}k_p(z_1-y_1)\cos\phi+\mathbf{i}k_p(z_2-y_2)\sin\phi} \tag{3.22}$$

$$R(z,y) = \frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d} B_s(\phi,\kappa)e^{\mathbf{i}k_s(z_1-y_1)\cos\phi+\mathbf{i}k_s(z_2-y_2)\sin\phi+(k_p-k_s)|x-z|} \tag{3.23}$$

$$+\frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d} B_p(\phi,\kappa)e^{\mathbf{i}k_p(z_1-y_1)\cos\phi+\mathbf{i}k_p(z_2-y_2)\sin\phi+(k_s-k_p)|x-z|} \tag{3.24}$$

It is easy to see that $|R(z,y)| \leq C\frac{|z-y|}{z_2}$. □

Let

$$g(x_1) = \frac{1}{((x_1-z_1)^2+z_2^2)^{3/4}((x_1-y_1)^2+y_2^2)^{1/4}}$$
$$\phi(x_1) = ((x_1-z_1)^2+z_2^2)^{1/2} - ((x_1-y_1)^2+y_2^2)^{1/2}$$

Then, we have

$$g'(x_1) = -g(x_1)[\frac{3(x_1-z_1)}{2((x_1-z_1)^2+z_2^2)} + \frac{(x_1-y_1)}{2((x_1-y_1)^2+y_2^2)}]$$
$$\phi'(x_1) = \frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}} - \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}}$$
$$= \frac{\frac{(x_1-z_1)^2}{(x_1-z_1)^2+z_2^2} - \frac{(x_1-y_1)^2}{(x_1-y_1)^2+y_2^2}}{\frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}} + \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}}}$$
$$= \frac{(x_1-z_1)^2y_2^2 - (x_1-y_1)^2z_2^2}{(\frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}} + \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}})((x_1-z_1)^2+z_2^2)((x_1-y_1)^2+y_2^2)}$$
$$\phi''(x_1) = \frac{z_2^2}{((x_1-z_1)^2+z_2^2)^{3/2}} - \frac{y_2^2}{((x_1-y_1)^2+y_2^2)^{3/2}}$$

Using integration by parts, we can obtain

$$\int_{-d}^{d} g(x_1)e^{\mathbf{i}\phi(x_1)}dx_1$$
$$= (\frac{g(d)}{\phi'(d)}e^{\mathbf{i}\phi(d)} - \frac{g(-d)}{\phi'(-d)}e^{\mathbf{i}\phi(-d)}) - \int_{-d}^{d}\frac{g'(x_1)}{\phi'(x_1)} - \frac{g(x_1)\phi''(x_1)}{(\phi'(x_1))^2}dx_1$$

Assume that

$$|y_1| \leq c_0 d \quad |z_1| \leq c_0 d \quad h \leq y_2, z_2 \leq c_1 h \quad d \leq c_2 h$$

where $0 < c_0 < 1$. Let define $0 < \theta_y, \theta_z < \pi$ such that

$$\cos\theta_y = \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}}$$
$$\cos\theta_z = \frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}}$$

By mean value theorem and the law of sines, we get

$$
\begin{aligned}
|\phi'(x_1)| &= |\cos\theta_z - \cos\theta_y| = |\sin\theta'||\theta_z - \theta_y| \\
&\geq \frac{h}{(1+c_0)d}|\sin(\theta_z - \theta_y)| \\
&= \frac{h}{(1+c_0)d}\frac{|z-y|}{|x-y|}\sin\theta_{|x-y|} \\
&= \frac{h}{(1+c_0)d}\frac{|z-y|}{|x-z|}\sin\theta_{|x-z|} \\
&\geq \frac{h^2}{(1+c_0)^2 d^2}\frac{|z-y|}{|x-y|} \\
\text{or} \qquad &\geq \frac{h^2}{(1+c_0)^2 d^2}\frac{|z-y|}{|x-z|}
\end{aligned}
$$

Then we have

$$
\begin{aligned}
|\frac{g(x_1)}{\phi'(x_1)}| &\leq \frac{(1+c_0)^2 d^2}{h^2}\frac{1}{|z-y||x-y|^{1/2}|x-z|^{1/2}} \\
&\leq C\frac{d^2}{h^3}\frac{1}{|z-y|}
\end{aligned}
$$

Moreover, by mean value theorem agian, we have

$$
\begin{aligned}
|\phi''(x_1)| &= |\frac{\sin^2\theta_z}{|x-z|} - \frac{\sin^2\theta_y}{|x-y|}| \\
&= |\frac{2\sin\theta'\cos\theta'}{|x-y'|}(\theta_z - \theta_y) - \frac{\sin^2\theta'}{|x-y'|^2}(|x-z| - |x-y|)| \\
&\leq \pi\frac{|\sin(\theta_z-\theta_y)|}{h} + \frac{|z-y|}{h^2} \\
&\leq \pi\frac{|\sin\theta_{|x-z|}||z-y|}{h|x-z|} + \frac{|z-y|}{h^2} \\
&\leq C\frac{|z-y|}{h^2}
\end{aligned}
$$

Now, it is easy to see that

$$
\begin{aligned}
&\Big|\int_{-d}^{d}\frac{g'(x_1)}{\phi'(x_1)} - \frac{g(x_1)\phi''(x_1)}{(\phi'(x_1))^2}dx_1\Big| \\
\leq\, & C\frac{d^3}{h^4}\frac{1}{|z-y|} + C\frac{d^3}{h^3}\frac{1}{|z-y|}\frac{d^2}{h^3}
\end{aligned}
$$

Based on the above analysis, we can obtain

$$
|\int_{-d}^{d} z_2 g(x_1)e^{\mathbf{i}\phi(x_1)}| \leq C((\frac{d}{h})^2 + (\frac{d}{h})^3 + (\frac{d}{h})^5)\frac{1}{|z-y|}
$$

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$$
\begin{aligned}
&\sin\phi_\kappa - \sin(t+\phi) = -2\cos(\frac{\phi_\kappa+\phi+t}{2})\sin(\frac{t+\phi-\phi_\kappa}{2}) \\
&\sin(\frac{t+\phi-\phi_\kappa}{2}) = \sin\frac{t}{2}\cos(\frac{\phi-\phi_\kappa}{2}) + \cos\frac{t}{2}\sin(\frac{\phi-\phi_\kappa}{2})
\end{aligned}
$$

Some think, substituting $t = 2\arcsin s/2$ into following integral

$$\int_0^\infty \chi(t)(\sin\phi_\kappa - \sin(t+\phi))^{1/2} e^{-\mathbf{i}\rho\cos t}$$
$$= \int_0^\infty \chi(t(s))(-s\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-s^2}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2} e^{-\mathbf{i}\rho s^2/2}$$
$$= \int_0^\infty \chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2} t^{-1/2} e^{-\mathbf{i}\rho t/2}$$

Let

$$f(t) = t^{-1/2} e^{-\mathbf{i}\rho t/2}$$
$$g(t) = -\int_t^{t-\mathbf{i}\infty} x^{-1/2} e^{-\mathbf{i}\rho x/2} dx$$
$$= \mathbf{i}\int_0^\infty (t-\mathbf{i}x)^{-1/2} e^{-\mathbf{i}\rho t - \rho x} dx$$

It is easy to see that $g'(t) = f(t)$. Then we have

$$= \int_0^\infty \chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2} t^{-1/2} e^{-\mathbf{i}\rho t/2}$$
$$= \chi(0)(-2\sin(\frac{\phi-\phi_\kappa}{2}))^{1/2} g(0)$$
$$- \int_0^\infty (\chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2})' g(t) dt$$

We get

$$g(x) = \int_0^\infty \chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{-1/2} t^{-1/2} (t-\mathbf{i}x)^{-1/2} e^{-\mathbf{i}\rho t} dt$$
$$R(\rho) = \int_0^\infty g(x) e^{-\rho x} dx$$

Because $\chi(t)$ has compact support $(-\delta, \delta)$, we obtain

$$gg(x) = \int_0^\delta (\sqrt{t}\cos(\theta) - \sqrt{4-t}\sin\theta)^{-1/2} t^{-1/2} (t^2+x^2)^{-1/4} dt$$

where $\theta = \frac{\phi-\phi_\kappa}{2}$. For $x > 0$, Put L(x):

$$\int_0^a \frac{1}{t^{3/4}} \frac{1}{(t^2+x^2)^{1/4}} dt$$
$$= 4\int_0^a \frac{1}{(t^2+x^2)^{1/4}} dt^{1/4}$$
$$= 4\int_0^{a^{1/4}} \frac{1}{(t^8+x^2)^{1/4}} dt$$
$$= 4x^{-1/4} \int_0^{(\frac{a}{x})^{1/4}} \frac{1}{(t^8+1)^{1/4}} dt$$
$$= 4x^{-1/4} \int_0^{(\frac{a}{x})^{1/4}} \frac{1}{(t^8+1)^{1/4}} dt$$
$$\le 4x^{-1/4} \int_0^\infty \frac{1}{(t^8+1)^{1/4}} dt$$

Back to analysis $gg(x)$, we have

$$\begin{aligned}
gg(x) &\leq \int_0^\delta \Big|\frac{\sqrt{t}+2|\sin\theta|}{t-4\sin^2\theta}\Big|^{1/2} t^{-1/2}(t^2+x^2)^{-1/4}dt \\
&= \int_0^\delta \Big|\frac{1}{\sqrt{t}-2|\sin\theta|}\Big|^{1/2} t^{-1/2}(t^2+x^2)^{-1/4}dt \\
&= 2\int_0^{\sqrt{\delta}} \Big|\frac{1}{t-2|\sin\theta|}\Big|^{1/2}(t^4+x^2)^{-1/4}dt \\
&= 2\int_{-2|\sin\theta|}^{\sqrt{\delta}-2|\sin\theta|} |t|^{-1/2}((t+2|\sin\theta|)^4+x^2)^{-1/4}dt \\
&\leq 4\int_0^{\delta^{1/4}}(t^8+x^2)^{-1/4}dt + 4\int_0^{\sqrt{2|\sin\theta|}}((t^2-2|\sin\theta|)^4+x^2)^{-1/4}dt \\
&\leq Cx^{-1/4}(1+\int_0^{\sqrt{2|\sin\theta|}}((t^2-2|\sin\theta|)^4/x+x)^{-1/4}dt) \\
&\leq Cx^{-1/4}(1+\int_0^{\sqrt{2|\sin\theta|}}(t^2-2|\sin\theta|)^{-1/2}dt) \\
&= Cx^{-1/4}(1+\int_0^1(1-t^2)^{-1/2}dt) \leq Cx^{-1/4}
\end{aligned}$$

Immediately, we can obtain

$$|g(x)| \leq Cx^{-1/4}$$

It follows that

$$R(\rho) \leq \int_0^\infty x^{-1/4}e^{-\rho x} \leq C\rho^{-3/4}$$

5. stationary of phase lemma

Lemma 5.1 *Assume that* $0<\kappa:=\sin\phi_\kappa<1, 0<\phi_\kappa<\pi/2$, $0\leq\phi\leq\pi/2$. *Let*

$$f(t,\phi):=F(\sin(t+\phi),\cos(t+\phi),(\kappa^2-\sin^2(t+\phi))^{1/2}) \tag{5.1}$$

be a complexed function in $C([-\pi/2,\pi/2]\times[0,\pi/2])$. *Moreover, its partial derivative with respect to* t *can be represented as*

$$\frac{\partial f(t,\phi)}{\partial t} = g(t,\phi)(\kappa^2-\sin^2(t+\phi))^{-1/2} \tag{5.2}$$

where $g(t,\phi)$ *is uniformly bounded. Then for any* $\rho\geq 1$, *we have*

$$\begin{aligned}
&\Big|I(\rho,\phi):=\int_{-\pi/2}^{\pi/2} f(t)e^{\mathbf{i}\rho\cos t}dt - \Big(\frac{2\pi}{\rho}\Big)^{1/2} f(0)e^{\mathbf{i}\rho-\mathbf{i}\pi/4}\Big| \\
&\leq C\frac{1}{\rho^{3/4}}
\end{aligned} \tag{5.3}$$

Proof. Solving the following equation:

$$\kappa^2-\sin^2(t+\phi)=0$$

we have, if $0 < \phi < \pi/2 - \phi_\kappa$,

$$t_1(\phi) = \phi_\kappa - \phi \qquad t_2(\phi) = -\phi_\kappa - \phi$$

and if $\pi/2 - \phi_\kappa \le \phi < \pi/2$,

$$t_1(\phi) = \phi_\kappa - \phi \qquad t_2(\phi) = \pi - \phi_\kappa - \phi$$

Since $|t_2(\phi)| < \phi_\kappa$ or $|t_2(\phi)| < \pi/2 - \phi_\kappa$, we now define $\delta := \min(\frac{\phi_\kappa}{2}, \frac{\pi/2-\phi_\kappa}{2})$ and it is easy to see that

$$\kappa + \sin(t+\phi) \neq 0 \tag{5.4}$$

$$\cos(\frac{t+\phi+\phi_\kappa}{2}) \neq 0 \tag{5.5}$$

for any $(t,\phi) \in [-\delta,\delta] \times [0,\pi/2]$. Let $\chi_\delta \in C_0^\infty(-\pi/2,\pi/2)$ be the cut-off function with that $0 \le \chi_\delta \le 1$, $\chi_\delta = 1$ in $(-\delta/2,\delta/2)$ and $\chi_\delta = 0$ in $(-\pi/2,\pi/2)\backslash(-\delta,\delta)$. Then we can divide I into two parts such that

$$\begin{aligned} I &= \int_{-\delta}^{\delta} f(t)\chi_\delta(t) e^{\mathbf{i}\rho\cos t} dt + \int_{-\pi/2}^{\pi/2} f(t)(1-\chi_\delta(t)) e^{\mathbf{i}\rho\cos t} dt \\ &=: I_1 + I_2 \end{aligned}$$

Subtitating $t(s) = 2\arcsin s/2$ for t in I_1, we can obtain

$$\begin{aligned} I_1 = & \int_{-2\sin\frac{\delta}{2}}^{2\sin\frac{\delta}{2}} f(t(s))\chi_\delta(t(s)) \frac{1}{\sqrt{1-s^2/4}} e^{\mathbf{i}\rho} e^{-\mathbf{i}\rho s^2/2} ds & (5.6) \\ = & \int_0^{2\sin\frac{\delta}{2}} (f(t(s))\chi_\delta(t(s)) + f(-t(s))\chi_\delta(-t(s))) \frac{1}{\sqrt{1-s^2/4}} e^{\mathbf{i}\rho} e^{-\mathbf{i}\rho s^2/2} ds & (5.7) \\ := & I_{11} + I_{12} & (5.8) \end{aligned}$$

Taking substitution $s = \sqrt{x}$, we get

$$I_{11} = \frac{1}{2}\int_0^{(2\sin\frac{\delta}{2})^2} f(t(\sqrt{x}))\chi_\delta(t(\sqrt{x})) \frac{1}{\sqrt{1-x/4}} x^{-1/2} e^{\mathbf{i}\rho} e^{-\mathbf{i}\rho x/2} dx$$

Observe that

$$\begin{aligned} \sin\phi_\kappa - \sin(t+\phi) &= -2\cos(\frac{\phi_\kappa+\phi+t}{2})\sin(\frac{t+\phi-\phi_\kappa}{2}) \\ \sin(\frac{t+\phi-\phi_\kappa}{2}) &= \sin\frac{t}{2}\cos(\frac{\phi-\phi_\kappa}{2}) + \cos\frac{t}{2}\sin(\frac{\phi-\phi_\kappa}{2}) \\ &:= \sin\frac{t}{2}\cos\theta + \cos\frac{t}{2}\sin\theta \end{aligned}$$

where $\theta = \frac{\phi-\phi_\kappa}{2}$. By lemma (1.3) and using representation (5.2), inequality (5.4-5.5), it follows that

$$\begin{aligned} & |I_{11} - \frac{1}{2}\sqrt{\frac{2\pi}{\rho}} f(0) e^{\mathbf{i}\rho - \mathbf{i}\frac{\pi}{4}}| \\ \le & \int_0^\infty e^{-\rho y} dy \int_0^{(2\sin\frac{\delta}{2})^2} \Big| \frac{\partial (f(t(\sqrt{x}))\chi_\delta(t(\sqrt{x}))\frac{1}{\sqrt{1-x/4}})}{\partial x} \Big| (x^2+y^2)^{-\frac{1}{4}} dx \end{aligned}$$

$$\leq C\int_0^\infty e^{-\rho y}dy\int_0^{(2\sin\frac{\delta}{2})^2}|\sqrt{x}\cos\theta+\sqrt{4-x}\sin\theta|^{-1/2}x^{-1/2}(x^2+y^2)^{-\frac{1}{4}}dx$$
$$\leq C\int_0^\infty e^{-\rho y}dy\int_0^{(2\sin\frac{\delta}{2})^2}\frac{(\sqrt{x}|\cos\theta|+\sqrt{4-x}|\sin\theta|)^{1/2}}{|x-4\sin^2\theta|^{1/2}}x^{-1/2}(x^2+y^2)^{-\frac{1}{4}}dx$$
$$\leq C\int_0^\infty e^{-\rho y}dy\int_0^{(2\sin\frac{\delta}{2})^2}\frac{1}{|\sqrt{x}-2|\sin\theta||^{1/2}}x^{-1/2}(x^2+y^2)^{-\frac{1}{4}}dx$$
$$\leq C\int_0^\infty e^{-\rho y}dy\int_0^{2\sin\frac{\delta}{2}}\frac{1}{|x-2\sin|\theta||^{1/2}}(x^4+y^2)^{-\frac{1}{4}}dx$$
$$\leq C\int_0^\infty e^{-\rho y}dy\int_{-2\sin|\theta|}^{2\sin\frac{\delta}{2}-2\sin|\theta|}\frac{1}{|x|^{1/2}}((x+2\sin|\theta|)^4+y^2)^{-\frac{1}{4}}dx$$
$$\leq C\int_0^\infty e^{-\rho y}dy\int_{-2\sin|\theta|}^{2\sin\frac{\delta}{2}}\frac{1}{|x|^{1/2}}((x+2\sin|\theta|)^4+y^2)^{-\frac{1}{4}}dx$$
$$\leq C\int_0^\infty e^{-\rho y}dy\Big(\int_0^{\sqrt{2\sin\frac{\delta}{2}}}(x^8+y^2)^{-\frac{1}{4}}dx+\int_0^{\sqrt{2\sin|\theta|}}((x^2-2\sin|\theta|)^4+y^2)^{-\frac{1}{4}}dx\Big)$$
$$\leq C\int_0^\infty e^{-\rho y}dy\Big(y^{-\frac{1}{4}}\int_0^\infty(x^8+1)^{-\frac{1}{4}}dx+y^{-\frac{1}{4}}\int_0^{\sqrt{2\sin|\theta|}}(2\sin|\theta|-x^2)^{-\frac{1}{2}}dx\Big)$$
$$\leq C\int_0^\infty y^{-\frac{1}{4}}e^{-\rho y}dy\Big(\int_0^\infty(x^8+1)^{-\frac{1}{4}}dx+\int_0^1(1-x^2)^{-\frac{1}{2}}dx\Big)\leq C\frac{1}{\rho^{3/4}}$$

Using the same agrument , we can also carry out

$$|I_{12}-\frac{1}{2}\sqrt{\frac{2\pi}{\rho}}f(0)e^{\mathbf{i}\rho-\mathbf{i}\frac{\pi}{4}}|\leq C\frac{1}{\rho^{3/4}} \tag{5.9}$$

It remains to estimate I_2. Note that there exits $m>0$ such that $|\sin t|\geq m$ for any $t\in[-\pi/2,\pi/2]\backslash(-\delta/2,\delta/2)$. Upon integration by parts and representation (5.2) again, we have

$$\begin{aligned}|I_{12}|&\leq C\rho^{-1}(1+\Big|\int_{[-\pi/2,\pi/2]\backslash(-\delta/2,\delta/2)}\frac{\partial(f(t)(1-\chi_\delta(t)))}{\partial t}\frac{1}{\sin t}dt\Big|)\\&\leq C\rho^{-1}(1+\int_{-\pi/2}^{\pi/2}\Big|\frac{\partial(f(t)(1-\chi_\delta(t)))}{\partial t}\Big|dt)\\&\leq C\rho^{-1}(1+\int_{-\pi/2}^{\pi/2}|(\kappa^2-\sin^2(t+\phi))^{-1/2}|dt)\\&\leq C\rho^{-1}(1+\int_{-\pi/2}^{\pi/2}|(\kappa^2-\sin^2 t)^{-1/2}|dt)\\&\leq C\rho^{-1}\end{aligned}$$

This completes the proof. □

6. cross term of psf, 17.11.15

We need the following slight generalization of Van der Corput lemma for the oscillatory integral [2, P.152].

Lemma 6.1 *Let $-\infty < a < b < \infty$, and u is a C^k function u in (a, b).*
1. If $|u'(t)| \geq 1$ for $t \in (a, b)$ and u' is monotone in (a,b), then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 3\lambda^{-1}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

2. For all $k \geq 2$, if $|u^{(k)}(t)| \geq 1$ for $t \in (a, b)$, then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 12k\lambda^{-1/k}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

Proof. The assertion can be proved by extending the Van der Corptut lemma in [2]. Here we omit the details. □

Lemma 6.2 *For $0 < \kappa < 1$, let $F(\lambda) = \int_0^\kappa f(t)e^{\mathbf{i}\lambda(\sqrt{1-t^2}-\tau\sqrt{\kappa^2-t^2}+\alpha t)}dt$, where $\tau \geq c_0 > 0$ and $\alpha \in \mathbb{R}$, then we have*

$$|F(\lambda)| \leq C(\kappa)\lambda^{-\frac{1}{2N_*}}\left[|f(\kappa)| + \int_0^\kappa |f'(t)|dt\right]$$

where $N_* = \min\{N|\kappa^{2N-1} < c_0, N \in \mathbb{Z}_+\}$.

Proof. Put $\phi(t) = -\sqrt{1-t^2}$ and $\psi(t,\tau) = \tau\phi(t/\kappa) - \phi(t) + \alpha t$. For easy of notations, we denote the n-th partial derivative of $g(t)$ with respect to t by $g^{(n)}(t)$. Then, it is to see that, for $n > 1$

$$\psi^{(n)}(t,\tau) = \frac{\tau}{\kappa^{n-1}}\phi^{(n)}(\frac{t}{\kappa}) - \phi^{(n)}(t)$$

A standard computation show that

$$\phi^{(1)}(t) = \frac{t}{\sqrt{1-t^2}}$$
$$\phi^{(2)}(t) = \frac{1}{(1-t^2)^{3/2}}$$
$$\phi^{(3)}(t) = \frac{3t}{(1-t^2)^{5/2}}$$

Moreover, for $n \geq 3$, we have

$$\phi^{(n)}(t) = \frac{p_n(t)}{(1-t^2)^{n-1/2}} \tag{6.1}$$

where $p_n = \sum_0^{n-2} a_k^n t^k$ is a $(n-2)$-th polynomial such that its coefficients satisfy the following recursion formula:

$$a_{n-1}^{n+1} = (n+1)a_{n-2}^n, \quad a_{n-2}^{n+1} = (n+2)a_{n-3}^n$$
$$a_k^{n+1} = (k+1)a_{k+1}^n + (2n-k)a_{k-1}^n \quad \text{for } 1 \leq k \leq n-3$$
$$a_0^{n+1} = a_1^n$$

Since the polynmial coefficients are all positive, it is obvious that for $n \geq 1$, $\phi^{(n)}(t)$ is a monotone increasing positive function. Using the recursion formula, it follows that

$$\phi^{(n)}(0) = \begin{cases} 0 & \text{n is odd,} \\ (n-1)!!(n-3)!! & \text{n is even.} \end{cases} \tag{6.2}$$

where $(2k-1)!!$ is double factorial and $n > 3$. We are now in the position to proof the inequality. Since $0 < \kappa < 1$, obersev that

$$\psi^{(2N_*+1)}(t,\tau) \geq \frac{\tau}{\kappa^{2N_*}}\phi^{(2N_*+1)}(t) - \phi^{(2N_*+1)}(t) > 0$$

Therefore, $\psi^{(2N_*)}(t,\tau)$ is monotone increasing in $[0,\kappa)$. By (6.2), we get

$$\psi^{(2N_*)}(t,\tau) \geq \psi^{(2N_*)}(0,\tau) \geq \psi^{(2N_*)}(0,c_0) = C(2N_*)(\frac{c_0}{\kappa^{2N_*-1}} - 1) > 0 \tag{6.3}$$

The lemma is now a direct consequence of lemma (6.1). □

7. Other exponential decay term: 17.11.16 on G1

The parameterization of hyperbolic curve passing $(\pm 1, 0)$ is:

$$\xi_1 = \pm\sqrt{t^2+1} \qquad \xi_2 = t$$

where $t \in \mathbb{R}$. Substituting $\xi = \xi_1 + \mathbf{i}\xi_2$ into $\mu(\xi) := (1-\xi^2)^{1/2}$ and $\mu_\kappa(\xi) := (\kappa^2 - \xi^2)^{1/2}$, we get

$$\begin{aligned} \operatorname{Im}\mu(\xi) &= \operatorname{Im}(1 - (\xi_1^2 - \xi_2^2 + \mathbf{i}2\xi_1\xi_2))^{1/2} \\ &= \operatorname{Im}(-2t\sqrt{t^2+1}\mathbf{i})^{1/2} = t^{1/2}(t^2+1)^{1/4} \end{aligned}$$

$$\begin{aligned} \operatorname{Im}\mu_\kappa(\xi) &= \operatorname{Im}(\kappa^2 - (\xi_1^2 - \xi_2^2 + \mathbf{i}2\xi_1\xi_2))^{1/2} \\ &= \operatorname{Im}(\kappa^2 - 1 - 2t\sqrt{t^2+1}\mathbf{i})^{1/2} \\ &= \sqrt{\frac{\sqrt{(1-\kappa^2)^2 + 4t(t^2+1)} + 1 - \kappa^2}{2}} \\ &\geq t^{1/2}(t^2+1)^{1/4} \end{aligned}$$

where we only consider the branch, denoted by Γ^+, in the first quadrant here. For $a > 0, b > 0$, we have

$$|e^{\mathbf{i}\xi a + \mathbf{i}\mu(\xi)b + \mathbf{i}\mu_\kappa(\xi)c}| \leq e^{-ta - t^{1/2}(t^2+1)^{1/4}b - t^{1/2}(t^2+1)^{1/4}c} \leq e^{-t(b+c)}$$

Lemma 7.1 *For $\xi \in \Gamma_0$, let $f(\xi)$ is a complex valued function in $L^1(\Gamma^+)$ such that $|f(\xi)| \leq C(1+\xi^k)$, $k \in \mathbb{Z}_+$. Then we have*

$$\begin{aligned} &|I(a,b,c) := \int_{\Gamma^+} f(\xi)\ e^{\mathbf{i}\xi a + \mathbf{i}\mu(\xi)b + \mathbf{i}\mu_\kappa(\xi)c}\ d\xi| \\ &\leq C(\frac{1}{b+c} + \frac{1}{(b+c)^k}) \end{aligned}$$

Proof.

$$\frac{d\xi(t)}{dt} = \frac{t}{\sqrt{t^2+1}} + \mathbf{i}$$

Substituting $\xi(t)$ into $I(a,b,c)$, we hvae

$$\begin{aligned}|I(a,b,c)| &= \Big| \int_0^\infty |f(\xi(t))\frac{d\xi(t)}{dt} e^{\mathbf{i}\xi(t)a+\mathbf{i}\mu(\xi(t))b+\mathbf{i}\mu_\kappa(\xi(t))c}dt\Big| \\ &\le C\int_0^\infty (1+t^k)e^{-t(b+c)}dt \\ &\le C(\frac{1}{b+c} + \frac{1}{(b+c)^k})\end{aligned}$$

□

Lemma 7.2 *Let $f(\xi)$ is a bounded complex valued function in $L^1((\kappa,1))$. Then we have*

$$\begin{aligned}&|I(a,b) := \int_\kappa^1 |f(\xi)e^{\mathbf{i}\xi a+\mathbf{i}\mu_\kappa(\xi)b}d\xi| \\ &\le C\frac{1}{b}\end{aligned}$$

Proof. It is simple to see that

$$\begin{aligned}|I(a,b)| &\le C\int_\kappa^1 e^{-b\sqrt{\xi^2-\kappa^2}}d\xi \\ &\le C\int_0^{\sqrt{1-\kappa^2}} \frac{t}{\sqrt{t^2+\kappa^2}}e^{-bt}dt \\ &\le C\frac{1}{b}\end{aligned}$$

□

8. about principle of arguement

Put

$$\begin{aligned}\delta_\pm(t) = &\ (\kappa-2t^2)^2 \mp \mathbf{i}4t^2\sqrt{1-t^2}\sqrt{t^2-\kappa} \\ &:= f_1(t) \mp \mathbf{i}f_2(t)\end{aligned}$$

where $0<\kappa<1$ and we have

$$\delta'_\pm(t) = f_1'(t) \mp \mathbf{i}f_2'(t)$$

It is easy to see $f_2(1) = f_2(\kappa) = 0$ and $f_1(t) > 0$ for any $\kappa \le t \le 1$. Then

$$\begin{aligned}&\int_\kappa^1 \frac{\delta'_+(t)}{\delta_+(t)} - \frac{\delta'_-(t)}{\delta_-(t)}dt \\ &= 2\mathbf{i}\int_\kappa^1 \operatorname{Im}(\frac{\delta'_+(t)}{\delta_+(t)})dt\end{aligned}$$

$$
\begin{aligned}
&= 2\mathbf{i}\int_\kappa^1 \operatorname{Im}\frac{(f_1'(t)-\mathbf{i}f_2'(t))f_1(t)+\mathbf{i}f_2(t))}{(f_1(t)-\mathbf{i}f_2(t))(f_1(t)+\mathbf{i}f_2(t))}dt\\
&= 2\mathbf{i}\int_\kappa^1 \frac{f_1'(t)f_2(t)-f_1(t)f_2'(t)}{f_1^2(t)+f_2^2(t)}dt\\
&= 2\mathbf{i}\int_\kappa^1 \frac{f_1^2(t)}{f_1^2(t)+f_2^2(t)}\frac{f_1'(t)f_2(t)-f_1(t)f_2'(t)}{f_1^2(t)}dt\\
&= \ -2\mathbf{i}\int_\kappa^1 \frac{f_1^2(t)}{f_1^2(t)+f_2^2(t)}d\frac{f_2(t)}{f_1(t)}\\
&= \ -2\mathbf{i}\arctan\frac{f_2(t)}{f_1(t)}\bigg|_\kappa^1 = 0
\end{aligned}
$$

Notic that, the condition only used above are $f_2(1)=f_2(\kappa)=0$ and $f_1(t)>0$.

9. Fundamental solution of Elastic wave

$$
\begin{aligned}
G(x;y) &= \frac{1}{\omega^2}(\nabla\times\nabla\cdot(g_s(x;y)\mathbb{I})-\nabla\nabla g_p(x;y)) &(9.1)\\
&= \frac{1}{\omega^2}(k_s^2 g_s(x,y)+\nabla\nabla(g_s(x;y)-g_p(x;y))) &(9.2)
\end{aligned}
$$

where y is the Dirac source, $g_p(x;y)$ or $g_s(x;y)$ is the fundamental solution of the scalar Helmhotlz equation with wavenumbers $k_p=\omega/c_p$ or $k_s=\omega/cs$.

$$
g_\alpha = \frac{\mathrm{i}}{4}H_0^{(1)}(k_\alpha|x-y|) \tag{9.3}
$$

where $H_0^{(1)}(t)$ is the Hankel function of the first type and order zero. By straight calculation using $H_1^{(1)}(t)=-dH_0^{(1)}(t)/dt$ and $dH_1^{(1)}(t)/dt=H_0^{(1)}(t)-H_1^{(1)}(t)/t$, we have

$$
\begin{aligned}
G_{ij}(x;y) = \frac{\mathbf{i}}{4}\{(\frac{k_s^2}{\omega^2}H_0^{(1)}(k_s|x-y|)-\frac{1}{\omega^2}\frac{k_sH_1^{(1)}(k_s|x-y|-k_pH_1^{(1)}(k_p|x-y|)}{|x-y|})\delta_{ij}\\
+\frac{1}{\omega^2}[(\frac{2k_sH_1^{(1)}(k_s|x-y|-2k_pH_1^{(1)}(k_p|x-y|)}{|x-y|}-(k_s^2H_0^{(1)}(k_s|x-y|)-k_p^2H_0^{(1)}(k_p|x-y|))]\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\}
\end{aligned}
$$

The definition of hankal function is $H_k^{(1)}(t)=J_k(t)+\mathbf{i}Y_k(t)$ where

$$
J_k(t)=\sum_{p=0}^\infty \frac{(-1)^p}{p!(k+p)!}(t/2)^{k+2p}
$$

Specially

$$
\begin{aligned}
J_0(t) &= \sum_{p=0}^\infty \frac{(-1)^p}{p!p!}(t/2)^{2p} = 1+\cdots\\
J_1(t) &= \sum_{p=0}^\infty \frac{(-1)^p}{p!(1+p)!}(t/2)^{1+2p} = \frac{t}{2}+\cdots
\end{aligned}
$$

and

$$Y_k(t) = \frac{1}{\pi}\{\ln t^2 - 2\ln 2 + 2C_{euler}\}J_k(t) - \frac{1}{\pi}\sum_{p=0}^{k-1}\frac{(k-1-p)!}{p!}(2/t)^{k-2p}$$
$$- \frac{1}{\pi}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!(k+p)!}(t/2)^{k+2p}\{\psi(p+k)+\psi(p)\}$$

Specally

$$Y_0(t) = \frac{1}{\pi}\{\ln t^2 - 2\ln 2 + 2C_{euler}\}J_0(t) - \frac{1}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^p}{p!p!}(t/2)^{2p}\{2\psi(p)\}$$
$$Y_1(t) = \frac{1}{\pi}\{\ln t^2 - 2\ln 2 + 2C_{euler}\}J_1(t) - \frac{1}{\pi}\frac{2}{t} - \frac{t}{2\pi}$$
$$-\frac{1}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^p}{p!(1+p)!}(t/2)^{1+2p}\{\psi(p+1)+\psi(p)\}$$

Thus, we have

$$H_0^{(1)}(kr) = 1 + \mathbf{i}\frac{2}{\pi}(C_{euler} + \ln k - \ln 2) + \mathbf{i}\frac{1}{\pi}\ln r^2 + o(kr)$$
$$H_1^{(1)}(kr) = \frac{kr}{2} + \mathbf{i}\frac{1}{\pi}(C_{euler} + \ln k - \ln 2 - \frac{1}{2})kr - \mathbf{i}\frac{1}{\pi}\frac{2}{kr} + \mathbf{i}\frac{1}{\pi}\ln r^2\frac{kr}{2} + o(k^2r^2)$$
$$\frac{\mathbf{i}}{4}H_0^{(1)}(kr) = \frac{\mathbf{i}}{4} - \frac{1}{2\pi}(C_{euler} + \ln k - \ln 2) - \frac{1}{4\pi}\ln r^2 + o(kr)$$
$$\frac{\mathbf{i}}{4}H_1^{(1)}(kr) = \mathbf{i}\frac{kr}{8} - \frac{1}{4\pi}(C_{euler} + \ln k - \ln 2 - \frac{1}{2})kr + \frac{1}{4\pi}\frac{2}{kr} - \frac{1}{4\pi}\ln r^2\frac{kr}{2} + o(k^2r^2)$$

We also need to define the surface traction $T_x^n(\cdot)$ on the normal direction n,

$$T_x^n u(x) := \sigma\cdot n = 2\mu\frac{\partial u}{\partial n} + \lambda n \mathrm{div} u + \mu n\times \mathrm{curl} u$$

where

$$\sigma(u) = \begin{pmatrix} (\lambda+2\mu)\partial u_1/\partial x_1 + \lambda\partial u_2/\partial x_2 & \mu\partial u_1/\partial x_2 + \mu\partial u_2/\partial x_1 \\ \mu\partial u_1/\partial x_2 + \mu\partial u_2/\partial x_1 & (\lambda+2\mu)\partial u_2/\partial(x_2) + \lambda\partial u_1/\partial x_1 \end{pmatrix}$$

A simple computation show that

$$\frac{\partial^3 H_0^{(1)}(k|x-y|)}{\partial x_i^2\partial x_j} = (1+2\delta_i j)(-k^2H_0^{(1)}(kr)\frac{r_j}{r^2} + 2kH_1^{(1)}(kr)\frac{r_j}{r^3})$$
$$+k^3H_1^{(1)}(kr)\frac{r_i^2r_j}{r^3} + 4k^2H_0^{(1)}(kr)\frac{r_i^2r_j}{r^4} - 8kH_1^{(1)}(kr)\frac{r_i^2r_j}{r^5}$$

where $r = |x-y|$ and $r_i = x_i - y_i$.

$$\frac{\mathbf{i}}{4}H_0^{(1)}(kr) = -\frac{1}{2\pi}(\ln\frac{kr}{2} + C_{euler})(1 - (\frac{kr}{2})^2 + \cdots) + \frac{1}{4\pi}(2(\frac{kr}{2})^2 + \cdots) + \frac{\mathbf{i}}{4}(1 - (\frac{kr}{2})^2 + \cdots)$$
$$= -\frac{1}{2\pi}(\ln\frac{kr}{2})(1 + O(r^2)) - \frac{1}{2\pi}C_{euler} + \frac{\mathbf{i}}{4} + O(r^2)$$
$$\frac{\mathbf{i}}{4}H_1^{(1)}(kr) = -\frac{1}{2\pi}(\ln\frac{kr}{2} + C_{euler})(\frac{kr}{2} - \frac{1}{2}(\frac{kr}{2})^3 + \cdots) + \frac{1}{4\pi}(\frac{kr}{2} + O(r^3)) + \frac{\mathbf{i}}{4}(\frac{kr}{2} - \frac{1}{2}(\frac{kr}{2})^3 + \cdots) + \frac{1}{2\pi}\frac{1}{kr}$$
$$= -\frac{1}{4\pi}(\ln\frac{kr}{2})(kr + O(r^3)) - \frac{kr}{4\pi}C_{euler} + \frac{kr}{8\pi} + \frac{\mathbf{i}kr}{8} + \frac{1}{2\pi}\frac{1}{kr} + O(r^3)$$

$$A(kr) := \frac{\mathbf{i}}{4}(k^2 H_0^{(1)}(kr) - 2kH_1^{(1)}(kr)/r) = \frac{k^2}{4\pi}(\ln\frac{kr}{2})(\frac{kr}{2})^2 - \frac{1}{\pi r^2} - \frac{k^2}{4\pi} + O(r^2)$$

and

$$A_{sp}(r) = A(k_s r) - A(k_p r) = \frac{k_s^2}{4\pi}(\ln\frac{kr}{2})(\frac{kr}{2})^2 - \frac{k_p^2}{4\pi}(\ln\frac{kr}{2})(\frac{kr}{2})^2 - (\frac{k_s^2}{4\pi} - \frac{k_p^2}{4\pi}) + O(r^2)$$

Let $g^{jkk} = \frac{\partial^3 g}{\partial x_j \partial x_k^2}$, $d = g_s - g_p$, thus

$$g^{iij} = (1+2\delta_{ij})(-A(kr)\frac{r_j}{r^2}) + \frac{\mathbf{i}k^3}{4}H_1^{(1)}(kr)\frac{r_i^2 r_j}{r^3} + 4A(kr)\frac{r_i^2 r_j}{r^4}$$
$$= (1+2\delta_{ij})(\frac{1}{\pi r^3} + \frac{k^2}{4\pi r})\frac{r_j}{r} - (\frac{4}{\pi r^3} + \frac{k^2}{2\pi r})\frac{r_i^2 r_j}{r^3} + O(r\ln r)$$
$$d^{iij} = (1+2\delta_{ij})(-A_{sp}\frac{r_j}{r^2}) + (\frac{\mathbf{i}k_s^3}{4}H_1^{(1)}(k_s r) - \frac{\mathbf{i}k_p^3}{4}H_1^{(1)}(k_p r))\frac{r_i^2 r_j}{r^3} + 4A_{sp}\frac{r_i^2 r_j}{r^4}$$
$$= (1+2\delta_{ij})(\frac{k_s^2}{4\pi r} - \frac{k_p^2}{4\pi r})\frac{r_j}{r} - (\frac{k_s^2}{2\pi r} - \frac{k_p^2}{2\pi r})\frac{r_i^2 r_j}{r^3} + O(r\ln r)$$
$$= (1+2\delta_{ij})\frac{(\lambda+\mu)\omega^2}{\mu(\lambda+2\mu)}\frac{1}{4\pi r}\frac{r_j}{r} - \frac{(\lambda+\mu)\omega^2}{\mu(\lambda+2\mu)}\frac{1}{2\pi r}\frac{r_i^2 r_j}{r^3} + O(r\ln r)$$
$$k^2 g^i = -\frac{\mathbf{i}}{4}H_1^{(1)}(kr)\frac{kr_i}{r} = -\frac{k^2}{2\pi r}\frac{r_i}{r} + O(r\ln r)$$
$$d^{iij} + d^{jjj} = 2\frac{(\lambda+\mu)\omega^2}{\mu(\lambda+2\mu)}\frac{1}{4\pi r}\frac{r_j}{r} + O(r\ln r)$$

Then we have [4, p43]

$$\sigma(G_1)n = \frac{1}{\omega^2}\begin{pmatrix} (\lambda+2\mu)(k_s^2 g_s^1 + d^{111}) + \lambda d^{122} & \mu(k_s^2 g_s^2 + d^{112}) + \mu d^{112} \\ \mu(k_s^2 g_s^2 + d^{112}) + \mu d^{112} & (\lambda+2\mu)d^{122} + \lambda(k_s^2 g_s^1 + d^{111}) \end{pmatrix} n$$
$$= \frac{\mu}{2\pi(\lambda+2\mu)}\left(\begin{pmatrix} \frac{2(\lambda+\mu)r_1^2}{\mu r^2} + 1 \\ \frac{2(\lambda+\mu)r_1 r_2}{\mu r^2} \end{pmatrix}(-\frac{r_1 n_1}{r^2} - \frac{r_2 n_2}{r^2}) - \begin{pmatrix} 0 \\ \frac{r_2 n_1 - r_1 n_2}{r^2} \end{pmatrix}\right)$$

and

$$\sigma(G_2)n = \frac{1}{\omega^2}\begin{pmatrix} (\lambda+2\mu)d^{112} + \lambda(k_s^2 g_s^2 + d^{222}) & \mu(k_s^2 g_s^1 + d^{122}) + \mu d^{122} \\ \mu(k_s^2 g_s^1 + d^{122}) + \mu d^{122} & (\lambda+2\mu)(k_s^2 g_s^2 + d^{222}) + \lambda d^{112} \end{pmatrix} n$$
$$= \frac{\mu}{2\pi(\lambda+2\mu)}\left(\begin{pmatrix} \frac{2(\lambda+\mu)r_1 r_2}{\mu r^2} \\ \frac{2(\lambda+\mu)r_2^2}{\mu r^2} + 1 \end{pmatrix}(-\frac{r_1 n_1}{r^2} - \frac{r_2 n_2}{r^2}) - \begin{pmatrix} \frac{r_1 n_2 - r_2 n_1}{r^2} \\ 0 \end{pmatrix}\right)$$

Now Let u be represented as single potential:

$$u = \int_{\partial D} G(x,y)\phi(y)ds(y) \tag{9.4}$$

with Neumann boundary condition

$$T_x u(x) = f(x) \quad \text{on} \quad \partial D \tag{9.5}$$

Then we obtain corresponding integral equation

$$\mathbf{P.V.}\int_{\partial D} T_x G(x,y)\phi(y)ds(y) - \frac{1}{2}\phi(x) = f(x) \tag{9.6}$$

where $x \in \partial D$. We describe the necessary parametrization of the integral equation in the two-dimensional case. We assume that the boundary curve ∂D possesses a regular analytic and 2π-periodic parametric representation of the form

$$x(t) = (x_1(t), x_2(t))$$

in counterclockwise orientation satisfying $|x'(t)| > 0$ for all t. Let [3]

$$T_0(x,y) = -\frac{\mu}{2\pi(\lambda+2\mu)} \begin{pmatrix} 0 & \frac{r_1 n_2 - r_2 n_1}{r^2} \\ \frac{r_2 n_1 - r_1 n_2}{r^2} & 0 \end{pmatrix}$$

Then by above analysis we have

$$\int_0^{2\pi} (T_x G(x(t), x(\tau) - T_0(x(t), x(\tau)))\phi(x(\tau))|x'(\tau)|dt$$
$$+\mathbf{P.V.} \int_0^{2\pi} T_0(x(t), x(\tau))\phi(x(\tau))|x'(\tau)|dt - \frac{1}{2}\phi(x(t)) = f(x(t))$$

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