



## 1. Estimate of Dirichlet Green Tensor

We need the following slight generalization of Van der Corput lemma for the oscillatory integral [2, P.152].

**Lemma 1.1** *Let  $-\infty < a < b < \infty$ , and  $u$  is a  $C^k$  function  $u$  in  $(a, b)$ .*

1. *If  $|u'(t)| \geq 1$  for  $t \in (a, b)$  and  $u'$  is monotone in  $(a, b)$ , then for any  $\phi(t)$  in  $(a, b)$  with integrable derivatives*

$$\left| \int_a^b e^{i\lambda u(t)} \phi(t) dt \right| \leq 3\lambda^{-1} \left[ |\phi(b)| + \int_a^b |\phi'(t)| dt \right].$$

2. *For all  $k \geq 2$ , if  $|u^{(k)}(t)| \geq 1$  for  $t \in (a, b)$ , then for any  $\phi(t)$  in  $(a, b)$  with integrable derivatives*

$$\left| \int_a^b e^{i\lambda u(t)} \phi(t) dt \right| \leq 12k\lambda^{-1/k} \left[ |\phi(b)| + \int_a^b |\phi'(t)| dt \right].$$

**Proof.** The assertion can be proved by extending the Van der Corput lemma in [2]. Here we omit the details.  $\square$

We recall following lemma, see e.g. [3]:

**Lemma 1.2** *Let  $F(\lambda, a) = \int_0^a t^{\alpha-1} f(t) e^{-i\rho t} dt$  where  $0 < a \leq +\infty$ ,  $0 < \alpha < 1$ ,  $\rho > 0$  and  $t^{\alpha-1} f \in L^1(0, a)$ , then we have*

$$|F(\rho, a)| \leq C \left( \frac{1}{\rho^\alpha} f(0) + \frac{1}{\rho} (a^{\alpha-1} f(a) + |t^{\alpha-1} f|_{L^1(0, a)}) \right) \quad (1.1)$$

**Proof.** Put

$$g_0(t) = t^{\alpha-1} e^{-i\rho t} \quad (1.2)$$

and define

$$g_1(t) = - \int_t^{t-i\infty} x^{\alpha-1} e^{-i\rho x} dx \quad (1.3)$$

where the path of integration is the vertical line  $x = t - iy, y \geq 0$ . It is easy to show that  $g_1(t)' = g_0(t)$ . Substituting  $x = t - iy$  into  $g_1(t)$ , we have

$$g_1(t) = i \int_0^\infty (t - iy)^{\alpha-1} e^{-i\rho t} e^{-\rho y} dy \quad (1.4)$$

Upon integration by parts, we have

$$\begin{aligned} F(\rho, a) &= \int_0^a f(t) dg_1(t) \\ &= e^{-i\frac{\alpha\pi}{2}} f(0) \Gamma(\alpha) \frac{1}{\rho^\alpha} + f(a) g_1(a) - \int_0^a f'(t) g_1(t) dt \\ &= e^{-i\frac{\alpha\pi}{2}} f(0) \Gamma(\alpha) \frac{1}{\rho^\alpha} - i \int_0^\infty e^{-\rho y} dy \int_0^a f'(t) (t - iy)^{\alpha-1} e^{-i\rho t} dt \end{aligned}$$

Let

$$h(y) = \int_0^a f'(t) (t - iy)^{\alpha-1} e^{-i\rho t} dt$$

and observe that

$$|h(y)| \leq \int_0^a |f'(t)|(t^2 + y^2)^{\frac{\alpha-1}{2}} dt$$

□

**Lemma 1.3** *Let  $F(\lambda, a) = \int_0^a t^{-1/2} f(t) e^{-i\rho t} dt$  where  $0 < a \leq +\infty$  and  $\rho > 0$ , then we have*

$$|F(\rho, a) - e^{-i\frac{\pi}{4}} f(0) \Gamma(1/2) \frac{1}{\rho^{1/2}}| \quad (1.5)$$

$$\leq C \left( \int_0^\infty e^{-\rho y} dy \int_0^a |f'(t)|(t^2 + y^2)^{-\frac{1}{4}} dt + \frac{1}{\rho} a^{-1/2} f(a) \right) \quad (1.6)$$

**Proof.** Put

$$g_0(t) = t^{-1/2} e^{-i\rho t} \quad (1.7)$$

and define

$$g_1(t) = - \int_t^{t-i\infty} x^{-1/2} e^{-i\rho x} dx \quad (1.8)$$

where the path of integration is the vertical line  $x = t - iy, y \geq 0$ . It is easy to show that  $g_1(t) = g_0(t)$ . Substituting  $x = t - iy$  into  $g_1(t)$ , we have

$$g_1(t) = i \int_0^\infty (t - iy)^{-1/2} e^{-i\rho t} e^{-\rho y} dy \quad (1.9)$$

Upon integration by parts, we have

$$\begin{aligned} F(\rho, a) &= \int_0^a f(t) dg_1(t) \\ &= e^{-i\frac{\pi}{4}} f(0) \Gamma(1/2) \frac{1}{\rho^{1/2}} + f(a) g_1(a) - \int_0^a f'(t) g_1(t) dt \\ &= e^{-i\frac{\pi}{4}} f(0) \Gamma(1/2) \frac{1}{\rho^{1/2}} + i f(a) \int_0^\infty (a - iy)^{-1/2} e^{-i\rho t} e^{-\rho y} dy \\ &\quad - i \int_0^\infty e^{-\rho y} dy \int_0^a f'(t) (t - iy)^{-1/2} e^{-i\rho t} dt \end{aligned}$$

Let

$$h(y) = \int_0^a f'(t) (t - iy)^{-1/2} e^{-i\rho t} dt$$

and observe that

$$|h(y)| \leq \int_0^a |f'(t)|(t^2 + y^2)^{-\frac{1}{4}} dt$$

It is easy to see that

$$|g_1(a)| \leq a^{-1/2} \int_0^\infty e^{-\rho y} dy \leq C \frac{1}{\rho}$$

□

**Lemma 1.4** Assume that  $0 < \kappa := \sin \phi_\kappa < 1$ ,  $0 < \phi_\kappa < \pi/2$ ,  $0 \leq \phi \leq \pi/2$  and  $-\pi/2 < t_1 < t_2 < \pi/2$  satisfy that  $\kappa^2 = \sin^2(\phi + t_1) = \sin^2(\phi + t_2)$ . Let  $f(\theta)$ :

$$f(t, \phi) := F(\sin(t + \phi), \cos(t + \phi), (\kappa^2 - \sin^2(t + \phi))^{1/2}) \quad (1.10)$$

be a function in  $C^\infty(([-\pi/2, \pi/2] \setminus \{t_1, t_2\}) \times [0, \pi/2])$ . Moreover, there exists  $\epsilon > 0$  such that  $f(\theta)$  can be represented as

$$f(t, \phi) = g_1(t, \phi) + g_2(t, \phi)(\kappa^2 - \sin^2(t + \phi))^{1/2})^{N/2} \quad (1.11)$$

where  $g_1, g_2 \in C^\infty((\bigcup_{i=1,2} (t_i - \epsilon, t_i + \epsilon)) \times [0, \pi/2])$  and  $N = \pm 1$ . Then for any  $\rho \geq 1$ , we have

$$\begin{aligned} |I(\rho, \phi) &:= \int_{-\pi/2}^{\pi/2} f(\theta) e^{i\rho \cos \theta} d\theta - \frac{N+1}{2} \left(\frac{2\pi}{\rho}\right)^{1/2} f(0) e^{i\rho - i\pi/4}| \\ &\leq C \frac{1}{\rho^{(2+N)/4}} \end{aligned} \quad (1.12)$$

**Proof.** The proof will be split into two parts about whether  $\phi$  equal to  $\phi_\kappa$ .

If  $\phi \neq \phi_\kappa$ , there exists  $0 < \delta < \pi/4$  such that

$$|(\kappa^2 - \sin^2(t + \phi))^{1/2}| > \frac{1}{2} |(\kappa^2 - \sin^2 \phi)^{1/2}| \quad (1.13)$$

for any  $t \in (-\delta, \delta)$ . Let  $\chi_\delta \in C_0^\infty(-\pi/2, \pi/2)$  be the cut-off function with that  $0 \leq \chi_\delta \leq 1$ ,  $\chi_\delta = 1$  in  $(-\delta/2, \delta/2)$  and  $\chi_\delta = 0$  in  $(-\pi/2, \pi/2) \setminus (-\delta, \delta)$ . Then we can divide  $I$  into two parts such that

$$\begin{aligned} I &= \int_{-\delta}^{\delta} f(t) \chi_\delta(t) e^{i\rho \cos t} dt + \int_{-\pi/2}^{\pi/2} f(t) (1 - \chi_\delta(t)) e^{i\rho \cos t} dt \\ &=: I_1 + I_2 \end{aligned}$$

Substituting  $t(s) = 2 \arcsin s/2$  for  $t$  in  $I_1$ , we can obtain

$$I_1 = \int_{\mathbb{R}} f(t(s)) \chi_\delta(t(s)) \frac{1}{\sqrt{1 - s^2/4}} e^{i\rho} e^{-i\rho s^2/2} ds \quad (1.14)$$

$$= \int_{\mathbb{R}} h_\delta(s) e^{i\rho} e^{-i\rho s^2/2} ds \quad (1.15)$$

It is easy to see that  $h_\delta(s) \in C_0^4(\mathbb{R})$ . By the lemma of the stationary phase for quadratic term in [1], we have

$$I_1 = e^{i\rho} \int_{\mathbb{R}} h_\delta(s) e^{-i\frac{\rho}{2}s^2} ds = e^{i\rho} \int_{\mathbb{R}} \widehat{h}_\delta(y) \alpha(-y) dy \quad (1.16)$$

where

$$\alpha(y) = \left(\frac{1}{2\pi\rho}\right)^{1/2} e^{-i\pi/4} e^{\frac{i}{2\rho}y^2} \quad (1.17)$$

$$= \left(\frac{1}{2\pi\rho}\right)^{1/2} e^{-i\pi/4} (1 + O(\frac{y^2}{\rho})) \quad (1.18)$$

Consequently

$$I_1 = \left(\frac{1}{2\pi\rho}\right)^{1/2} e^{i\rho - i\pi/4} \int_{\mathbb{R}} \widehat{h}_\delta(y) (1 + \frac{1}{\rho} O(y^2)) dy \quad (1.19)$$

Moreover,  $\int_{\mathbb{R}} \widehat{h}_\delta(y) dy = 2\pi h_\delta(0)$  and  $|\int_{\mathbb{R}} \widehat{h}_\delta(y) y^2 dy| < C$  since  $\widehat{h}_\delta(y) = O(1/y^4)$ . Now, it turns to estimate  $I_2$ .

When  $N = 1$ , using integration by parts, we have

$$|I_2| = \left| \int_{(-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (-\frac{\delta}{2}, \frac{\delta}{2})} f(t)(1 - \chi_\delta(t)) / \sin t \, de^{i\rho \cos t} \right| \quad (1.20)$$

$$(1.21)$$

$$\leq C \frac{1}{\rho} + \left| \int_{(-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (-\frac{\delta}{2}, \frac{\delta}{2})} (f(t)(1 - \chi_\delta(t)) / \sin t)' e^{i\rho \cos t} dt \right| \quad (1.22)$$

$$\leq C \frac{1}{\rho} \quad (1.23)$$

From above analysis, we obtain

$$\left| I(\rho, \phi) - \left( \frac{2\pi}{\rho} \right)^{1/2} f(0) e^{i\rho - i\pi/4} \right| \leq C(\phi) \frac{1}{\rho} \quad (1.24)$$

When  $N = -1$ , we can not use integration by parts again since  $f'(\theta)$  is not integrable. However, for any  $0 < \lambda_1 < 1$  and  $1 < \lambda_2 < 1/\kappa$ , there exists  $0 < \sigma < \epsilon$ , such that  $\chi := ((t_1 - \sigma, t_1 + \sigma) \cup (t_2 - \sigma, t_2 + \sigma)) \cap (-\delta, \delta) = \emptyset$ , dependent on  $\lambda_1, \lambda_2$  and

$$\lambda_1 \kappa < |\sin(t + \phi)| < \lambda_2 \kappa. \quad (1.25)$$

for any  $t \in \chi$ .

We only analysis the integral on  $\chi_1 = (t_1 - \sigma, t_1 + \sigma) \cap [-\pi/2, \pi/2]$  here, which denoted by  $I_{\chi_1}$ , the proof of  $I_{\chi_2}$  is similar. It is easy to see that  $\sin(t + \phi)$  is monotonic in  $\chi_1$ . Without loss of generality, we assume that  $\sin(t_1 - \sigma + \phi) < \kappa < \sin(t_1 + \sigma + \phi)$ . Let  $\sin(t + \phi) = \kappa \sin \theta$  and the implicit mapping from  $\theta$  to  $t$  is denoted by  $t(\theta)$  while the inverse mapping by  $\theta(t)$ , taking the interval  $\chi_1$  onto  $L_\theta : \theta_1 \rightarrow \pi/2 \rightarrow \pi/2 - i\theta_2$  where  $\sin(t_1 - \sigma + \phi) = \kappa \sin \theta_1, \sin(t_1 + \sigma + \phi) = \kappa \sin(\pi/2 - i\theta_2)$ . By substituting  $t(\theta)$  into  $I_{\chi_1}$ , we have

$$I_{\chi_1} = \int_{t_1 - \sigma}^{t_1 + \sigma} \frac{f(t)(\kappa^2 - \sin^2(t + \phi))^{1/2}}{(\kappa^2 - \sin^2(t + \phi))^{1/2}} e^{i\rho \cos t} \quad (1.26)$$

$$= \int_{L_\theta} \frac{\kappa f(t(\theta)) \cos \theta}{(1 - \kappa^2 \sin^2 \theta)^{1/2}} e^{i\rho(\cos(t(\theta)))} d\theta \quad (1.27)$$

$$= \int_{L_\theta} \frac{\kappa g_1(t(\theta)) \cos \theta + g_2(t(\theta))}{(1 - \kappa^2 \sin^2 \theta)^{1/2}} e^{i\rho(\cos(t(\theta)))} d\theta \quad (1.28)$$

$$:= \int_{L_\theta} \frac{h(\theta)}{(1 - \kappa^2 \sin^2 \theta)^{1/2}} e^{i\rho(\cos(t(\theta)))} d\theta \quad (1.29)$$

Observe that  $h(\theta)$  and  $\partial h / \partial \theta$  are integrable on the path  $L_\theta$  by (1.11). A simple computation show that

$$\frac{dt(\theta)}{d\theta} = \frac{\kappa \cos \theta}{\cos(t + \phi)} \quad \frac{d^2 t(\theta)}{d\theta^2} = \frac{\kappa^2 \cos^2 \theta \sin(t + \phi) - \kappa \sin \theta \cos^2(t + \phi)}{\cos^3(t + \phi)}$$

Then we can obtain

$$\frac{d \cos t}{d\theta} = \frac{-\kappa \sin t \cos \theta}{\cos(t + \phi)}$$

$$\begin{aligned}
\frac{d^2 \cos t}{d\theta^2} &= \frac{d^2 \cos t}{dt^2} \left( \frac{dt}{d\theta} \right)^2 + \frac{d \cos t}{dt} \frac{d^2 t}{d\theta^2} \\
&= \frac{-\kappa^2 \cos^2 \theta \cos t}{\cos^2(t + \phi)} + \frac{\kappa \sin \theta \cos^2(t + \phi) \sin t - \kappa^2 \cos^2 \theta \sin(t + \phi) \sin t}{\cos^3(t + \phi)} \\
&= \frac{-\kappa^2 \cos^2 \theta \cos \phi + \kappa \sin \theta \cos^2(t + \phi) \sin t}{\cos^3(t + \phi)} \\
&= \frac{(\sin^2(t + \phi) - \kappa^2) \cos \phi + \cos^2(t + \phi) \sin(t + \phi) \sin t}{\cos^3(t + \phi)}
\end{aligned}$$

Since  $|\sin t| > |\sin \delta|$  and  $1 - \lambda_2^2 \kappa^2 < \cos^2(t + \phi) < 1 - \lambda_1^2 \kappa^2$  for  $t \in \chi_1$ , it follows that  $\theta = \pi/2$  is the only stationary point of  $\cos(t(\theta))$  and

$$\left| \frac{d^2 \cos t}{d\theta^2}(\pi/2) \right| = \frac{(1 - \kappa^2)\kappa}{(1 - \kappa^2)^{3/2}} |\sin t| > \frac{(1 - \kappa^2)\kappa}{(1 - \kappa^2)^{3/2}} \sin \delta \quad (1.30)$$

Therefore, we can choose appropriate  $\lambda_1, \lambda_2$  such that

$$\left| \frac{d^2 \cos t}{d\theta^2} \right| > \frac{(1 - \kappa^2)\kappa}{(1 - \kappa^2)^{3/2}} \sin \delta \quad (1.31)$$

for any  $\theta \in \theta(\chi_1)$ . According to lemma (1.1), we obtain  $|I_{\chi_1}| \leq C \frac{1}{\rho^{1/2}}$ , and also  $|I_{\chi_2}| \leq C \frac{1}{\rho^{1/2}}$ . Using integration by parts, we get

$$\left| \int_{[-\pi/2, \pi/2] \setminus ((-\delta, \delta) \cup \chi)} f(t)(1 - \chi_\delta(t)) e^{i\rho \cos t} dt \right| \leq C \frac{1}{\rho}$$

Consequently, for  $N = -1$  and  $\phi \neq \phi_\kappa$ , we get  $|I(\rho, \phi)| \leq \frac{1}{\rho^{1/2}}$ .

We now turn to the case of  $\phi = \phi_\kappa$ . By (1.11), we can define  $\chi_\epsilon$  similarly and also decompose  $I$  into  $I_1$  and  $I_2$ . Using the same argument above, we can easily carry out that: for  $N = 1$ , we have  $|I_2| \leq C \frac{1}{\rho}$ ; for  $N = -1$ , we have  $|I_2| \leq C \frac{1}{\rho^{1/2}}$ . Finally, it remains to analysis  $I_1$ . By (1.11), we have

$$\begin{aligned}
I_1 &= \int_{-\epsilon}^{\epsilon} g_1 \chi_\epsilon + g_2 \chi_\epsilon (\sin^2 \phi_\kappa - \sin^2(t + \phi_\kappa))^{N/2} e^{i\rho \cos t} dt \\
&= \int_{-\epsilon}^{\epsilon} g_1 \chi_\epsilon + g_2 \chi_\epsilon (-2(\sin \phi_\kappa + \sin(t + \phi_\kappa)) \cos \frac{2\phi_\kappa + t}{2} \sin t/2)^{N/2} e^{i\rho \cos t} dt \\
&= \int_{\mathbb{R}} g_1 \chi_\epsilon + g_2 \chi_\epsilon ((\sin \phi_\kappa + \sin(t + \phi_\kappa)) \cos \frac{2\phi_\kappa + t}{2})^{N/2} (-2 \sin t/2)^{N/2} e^{i\rho \cos t} dt
\end{aligned}$$

Also, substituting  $t(s) = 2 \arcsin s/2$  for  $t$  in  $I_1$ , it follows that

$$I_1 = \int_{\mathbb{R}} h_1(s) e^{-i\rho \frac{s^2}{2}} + h_2(s) (-s)^{N/2} e^{-i\rho \frac{s^2}{2}} \quad (1.32)$$

$$= I_{11} + I_{12} \quad (1.33)$$

where

$$\begin{aligned}
h_1(s) &= g_1(t(s)) \chi_\epsilon(t(s)) \sqrt{1 - s^2/4} e^{i\rho} \\
h_2(s) &= g_2 \chi_\epsilon((\sin \phi_\kappa + \sin(t + \phi_\kappa)) \cos \frac{2\phi_\kappa + t}{2})_{t=t(s)}^{N/2} \sqrt{1 - s^2/4} e^{i\rho}
\end{aligned}$$

and  $h_1(s), h_2(s) \in C_c^\infty(\mathbb{R})$ . Using stationary phase lemma similarly, if  $N = 1$ ,

$$I_{11} = \left(\frac{2\pi}{\rho}\right)^{1/2} g_1(0) e^{i\rho - i\pi/4} + O\left(\frac{1}{\rho}\right) \quad (1.34)$$

$$= \left(\frac{2\pi}{\rho}\right)^{1/2} f(0) e^{i\rho - i\pi/4} + O\left(\frac{1}{\rho}\right) \quad (1.35)$$

if  $N = -1$ , we get  $|I_{11}| \leq C \frac{1}{\rho^{1/2}}$ . For  $I_{12}$ , we have

$$I_{12} = \int_0^\infty (ih_2(s) + h_2(-s)) s^{N/2} e^{-i\rho s^2/2} ds \quad (1.36)$$

$$= \frac{1}{2} \int_0^\infty (ih_2(\sqrt{s}) + h_2(-\sqrt{s})) s^{N/4-1/2} e^{-i\rho s/2} ds \quad (1.37)$$

By lemma (1.2), we get  $|I_{12}| \leq C \frac{1}{\rho^{(N+2)/4}}$ .  $\square$

## 2. Some draft about Green Tensor Analysis

Let substitute  $\xi = k \sin \theta$  into integral and shift the variable, we have

$$I(y) = \int_{\mathbb{R}} f(\xi) e^{i\xi y_1 + \mu(\xi) y_2} d\xi = \int_{\mathbb{R}} f(\xi) e^{i\xi(y_1 - z_1) + \mu(\xi)(y_2 - z_2)} e^{i\xi z_1 + \mu(\xi) z_2} d\xi \quad (2.1)$$

$$= k \int_L f(k \sin \theta) \cos \theta e^{ik|y-z| \cos(\theta-\eta)} e^{i|z| \cos(\theta-\phi)} d\theta \quad (2.2)$$

$$= k \int_{L_\phi} f(k \sin(\theta + \phi)) \cos(\theta + \phi) e^{ik|y-z| \cos(\theta+\phi-\eta)} e^{i|z| \cos \theta} d\theta \quad (2.3)$$

$$= k \int_L f(k \sin(\theta + \phi)) \cos(\theta + \phi) e^{ik|y-z| \cos(\theta+\phi-\eta)} e^{i|z| \cos \theta} d\theta \quad (2.4)$$

where  $y_1, y_2 > 0$ ,  $\sin \phi = \frac{z_1}{|z|}$ ,  $\cos \phi = \frac{z_2}{|z|}$ ,  $0 < \phi < \pi/2$  and  $\sin \eta = \frac{y_1 - z_1}{|y - z|}$ ,  $\cos \eta = \frac{y_2 - z_2}{|y - z|}$ ,  $0 < \eta < \pi$ . It is easy to see that  $\phi + \eta < \pi$ . Roughly, using stationary phase lemma, we obtain:

$$I(y) = f(k \sin \phi) k \cos \phi e^{ik|y-z| \cos(\phi-\eta)} \left(\frac{2\pi}{|z|}\right)^{1/2} e^{i|z| - i\frac{\pi}{4}} (1 + O(\frac{1}{|z|})) \quad (2.5)$$

$$\cos(a + ib) = \frac{e^b + e^{-b}}{2} \cos a + i \frac{e^{-b} - e^b}{2} \sin a \quad (2.6)$$

$$\sin(a + ib) = \frac{e^b + e^{-b}}{2} \sin a + i \frac{e^b - e^{-b}}{2} \cos a \quad (2.7)$$

When  $\theta \in (-a - \pi/2, -a - \pi/2 + i\infty)$ , let  $\theta = -a - \pi/2 + it$ , where  $t > 0$ ,  $0 \leq a \leq \phi$ , then

$$\begin{aligned} & -\text{Im}(|z| \cos \theta + |y - z| \cos(\theta + \phi - \eta)) \\ &= |z| \sin(a + \pi/2) + |y - z| \sin(a + \pi/2 - \phi + \eta) \end{aligned} \quad (2.8)$$

$$= |z| \cos a + |y - z| \cos(a - \phi + \eta) \quad (2.9)$$

$$= |z| \cos a + \cos a |y - z| (\cos \phi \cos \eta + \sin \phi \sin \eta) \quad (2.10)$$

$$+ \sin a |y - z| (\sin \phi \cos \eta - \cos \phi \sin \eta) \quad (2.11)$$

$$= |z| \cos a + \cos a((y_2 - z_2) \cos \phi + (y_1 - z_1) \sin \phi) \quad (2.12)$$

$$+ \sin a((y_2 - z_2) \sin \phi - (y_1 - z_1) \cos \phi) \quad (2.13)$$

$$= y_1 \sin(\phi - a) + y_2 \cos(\phi - a) > 0 \quad (2.14)$$

Now, Using Cauchy Integral Theorem, we have

$$I(y) = k \int_L f(k \sin(\theta + \phi)) \cos(\theta + \phi) e^{\mathbf{i}k|y-z| \cos(\theta+\phi-\eta)} e^{\mathbf{i}|z| \cos \theta} d\theta \quad (2.15)$$

Let  $L_1 = (-\pi/2, -\pi/2 + \mathbf{i}\infty)$  and  $\theta = -\pi/2 + \mathbf{i}t, t > 0$ , then

$$I_1(y) = k \int_{L_1} f(k \sin(\theta + \phi)) \cos(\theta + \phi) e^{\mathbf{i}k|y-z| \cos(\theta+\phi-\eta)} e^{\mathbf{i}|z| \cos \theta} d\theta \quad (2.16)$$

$$= \quad (2.17)$$

$$I(y) = f(k \sin \phi) k \cos \phi e^{\mathbf{i}k|y-z| \cos(\phi-\eta)} \left(\frac{2\pi}{|z|}\right)^{1/2} e^{\mathbf{i}|z| - \mathbf{i}\frac{\pi}{4}} \quad (2.18)$$

$$+ \frac{kz_2}{|z|} O\left(\left(\frac{1}{k|z|}\right)^{3/4} + \frac{1}{k|y|}\right) + \frac{kz_1}{|z|} O\left(\left(\frac{1}{k|z|}\right)^{5/4} + \left(\frac{1}{k|y|}\right)^2\right) \quad (2.19)$$

It is easy to see

$$\int_{-d}^d \frac{k}{(k|x-z|)^\alpha} \frac{1}{(k|x-y|)^\beta} dx_1 \leq C \left( \frac{1}{(kz_2)^{\alpha+\beta-1}} + \frac{1}{(ky_2)^{\alpha+\beta-1}} \right) \quad (2.20)$$

where  $z, y \in \mathbb{R}_+^2$ ,  $x \in \Gamma_0$  and  $\alpha + \beta > 0$ .

$$e^{\mathbf{i}\mu y_2 + \mathbf{i}\xi(x_1 - y_1)} = e^{\mathbf{i}\mu y_2 - \mathbf{i}y_2 / \tan \phi} = e^{\mathbf{i}y_2(\mu - \xi / \tan \phi)} \quad (2.21)$$

Another method

$$\int_{-\pi/2}^{\pi/2} f(k \sin(\theta + \psi)) k \cos(\theta + \psi) e^{\mathbf{i}k|x-y| \cos \theta} \quad (2.22)$$

$$= \int_{-\pi/2}^{\pi/2} f(k \sin(\theta + \psi)) k \cos(\theta + \psi) e^{\mathbf{i}k|x-y| \cos(\theta+\psi-\psi)} \quad (2.23)$$

$$= \int_{-\pi/2}^{\pi/2} f(k \sin(\theta + \psi)) k \cos(\theta + \psi) e^{\mathbf{i}ky_2 \cos(\theta+\psi) + \mathbf{i}k|x_1 - y_1| \sin(\theta+\psi)} \quad (2.24)$$

$$= \int_{-\pi/2}^{\pi/2} f(k \sin(\theta + \psi)) k \cos(\theta + \psi) \quad (2.25)$$

$$e^{\mathbf{i}k(y_2 - z_2) \cos(\theta+\psi) + \mathbf{i}k(|x_1 - y_1| - |x_1 - z_1|) \sin(\theta+\psi) + \mathbf{i}k|z| \cos(\theta+\psi-\phi)} \quad (2.26)$$

### 3. Finite Aperture Point Spread Function

If  $x \in \Gamma_0$  and  $z, y \in \mathbb{R}_+^2$ , by lemma (??) we have

$$\begin{aligned} G(x, y) &= \frac{\mathbf{i}k_s}{\mu\sqrt{2\pi}} \frac{1}{\delta(\xi)} \begin{pmatrix} \mu_s \beta & \xi \beta \\ 2\xi \mu_s \mu_p & 2\xi^2 \mu_p \end{pmatrix}_{\xi=k_s \frac{x_1 - y_1}{|x - y|}} \frac{y_2}{|x - y|} \frac{1}{(k_s |x - y|)^{1/2}} e^{\mathbf{i}k_s |x - y| - \mathbf{i}\frac{\pi}{4}} \\ &+ \frac{\mathbf{i}k_p}{\mu\sqrt{2\pi}} \frac{1}{\delta(\xi)} \begin{pmatrix} 2\xi^2 \mu_s & -2\xi \mu_s \mu_p \\ -\xi \beta & \mu_p \beta \end{pmatrix}_{\xi=k_p \frac{x_1 - y_1}{|x - y|}} \frac{y_2}{|x - y|} \frac{1}{(k_p |x - y|)^{1/2}} e^{\mathbf{i}k_p |x - y| - \mathbf{i}\frac{\pi}{4}} \end{aligned} \quad (3.1)$$



$$\begin{aligned}
& +O\left(\frac{y_2}{|x-y|} \frac{1}{(k_s|x-y|)^{3/4}} + \frac{|x_1-y_1|}{|x-y|} \frac{1}{(k_s|x-y|)^{5/4}}\right) \\
& := \mathcal{G}_s(x, y) + \mathcal{G}_p(x, y) + O\left(\frac{y_2}{|x-y|} \frac{1}{(k_s|x-y|)^{3/4}} + \frac{|x_1-y_1|}{|x-y|} \frac{1}{(k_s|x-y|)^{5/4}}\right) \\
T_D(x, z) &= \frac{k_s}{\sqrt{2\pi}} \frac{1}{\gamma(\xi)} \begin{pmatrix} \mu_s \mu_p & \xi \mu_p \\ \xi \mu_s & \xi^2 \end{pmatrix}_{\xi=k_s \frac{x_1-z_1}{|x-z|}} \frac{z_2}{|x-z|} \frac{1}{(k_s|x-z|)^{1/2}} e^{i k_s |x-z| - i \frac{\pi}{4}} \\
& + \frac{k_p}{\sqrt{2\pi}} \frac{1}{\gamma(\xi)} \begin{pmatrix} \xi^2 & -\xi \mu_p \\ -\xi \mu_s & \mu_p \mu_s \end{pmatrix}_{\xi=k_p \frac{x_1-z_1}{|x-z|}} \frac{z_2}{|x-z|} \frac{1}{(k_p|x-z|)^{1/2}} e^{i k_p |x-z| - i \frac{\pi}{4}} \quad (3.2) \\
& + O\left(\frac{k_s z_2}{|x-z|} \frac{1}{(k_s|x-z|)^{3/4}} + \frac{k_s |x_1-z_1|}{|x-z|} \frac{1}{(k_s|x-z|)^{5/4}}\right) \\
& := \mathcal{T}_s(x, z) + \mathcal{T}_p(x, z) + O\left(\frac{k_s z_2}{|x-z|} \frac{1}{(k_s|x-z|)^{3/4}} + \frac{k_s |x_1-z_1|}{|x-z|} \frac{1}{(k_s|x-z|)^{5/4}}\right)
\end{aligned}$$

Now we consider the finite aperture point spread function  $J_d(z, y)$ :

$$\int_{-d}^d (T_D(x_1, 0; z_1, z_2))^T \overline{G(x_1, 0; y_1, y_2)} dx_1 \quad (3.3)$$

Recall following standard asymptotic expansion:

$$|x-y| = |x-z| + \widehat{x-z} \cdot (z-y) + O\left(\frac{|y-z|^2}{|x-z|}\right) \quad (3.4)$$

$$|y|^{-\alpha} = |z|^{-\alpha} \left(1 + \frac{|y|-|z|}{|z|}\right)^{-\alpha} = |z|^{-\alpha} \left(1 + O\left(\frac{|y-z|}{|z|}\right)\right) \quad (3.5)$$

$$e^{it} = 1 + O(t) \quad (3.6)$$

$$|a^{1/2} - b^{1/2}| \leq C \sqrt{|a-b|} \quad (3.7)$$

where  $x, y, z \in \mathbb{R}^2$ ,  $t, a, b \in \mathbb{R}$  and  $\alpha > 0$ .

**Lemma 3.1** For any  $z, y \in \mathbb{R}_+^2$ ,  $J_d(z, y) = F(z, y) + O\left((1 + \frac{|y-z|}{z_2}) \left(\frac{1}{k_s z_2}\right)^{1/4} + \frac{(k_s |y-z|)^2}{k_s z_2} + \left(\frac{|y-z|}{z_2}\right)^{1/2}\right)$ , where

$$F(z, y) = -\frac{\mathbf{i}}{2\pi\mu} \int_{\theta_1^d}^{\theta_2^d} f_s(\theta) \begin{pmatrix} \sin^2 \theta & \sin \theta \cos \theta \\ \sin \theta \cos \theta & \cos^2 \theta \end{pmatrix} e^{i k_s (z_1 - y_1) \cos \theta + i k_s (z_2 - y_2) \sin \theta} d\theta \quad (3.8)$$

$$-\frac{\mathbf{i}}{2\pi\mu} \int_{\theta_1^d}^{\theta_2^d} f_p(\theta) \begin{pmatrix} \cos^2 \theta & -\sin \theta \cos \theta \\ -\sin \theta \cos \theta & \sin^2 \theta \end{pmatrix} e^{i k_p (z_1 - y_1) \cos \theta + i k_p (z_2 - y_2) \sin \theta} d\theta \quad (3.9)$$

and

$$\begin{aligned}
f_s(\theta) &= \frac{\sin \theta ((\kappa^2 - \cos^2 \theta)^{1/2} (1 - 2 \cos^2 \theta) + 2(\kappa^2 - \cos^2 \theta)^{1/2} \cos^2 \theta)}{(\cos^2 \theta + \sin \theta (\kappa^2 - \cos^2 \theta)^{1/2}) ((1 - 2 \cos^2 \theta)^2 + 4 \cos^2 \theta \sin \theta (\kappa^2 - \cos^2 \theta)^{1/2})} \\
f_p(\theta) &= \frac{\sin \theta (1/\kappa^2 - \cos^2 \theta)^{1/2}}{(\cos^2 \theta + \sin \theta (1/\kappa^2 - \cos^2 \theta)^{1/2}) ((1/\kappa^2 - 2 \cos^2 \theta)^2 + 4 \cos^2 \theta \sin \theta (1/\kappa^2 - \cos^2 \theta)^{1/2})}
\end{aligned}$$

where  $0 < \theta_1^d < \pi/2 < \theta_2^d < \pi$  and  $z_2 = (d + z_1) \tan \theta_1^d = (z_1 - d) \tan \theta_2^d$ .

**Proof.**

$$\begin{aligned}
& \frac{y_2}{|x-y|} \frac{1}{(k_s|x-y|)^{3/4}} + \frac{|x_1-y_1|}{|x-y|} \frac{1}{(k_s|x-y|)^{5/4}} \\
&= \left( \frac{z_2}{|x-z|} \frac{1}{(k_s|x-z|)^{3/4}} + \frac{|x_1-z_1|}{|x-z|} \frac{1}{(k_s|x-z|)^{5/4}} \right) (1 + O(\frac{|y-z|}{|x-z|})) \\
& \quad |\mu_i(k_j \frac{x_1-y_1}{|x-y|}) - \mu_i(k_j \frac{x_1-z_1}{|x-z|})| \\
& \leq Ck_j \sqrt{\left| \frac{x_1-y_1}{|x-y|} - \frac{x_1-z_1}{|x-z|} \right|} \leq Ck_j \left( \frac{|y-z|}{|x-z|} \right)^{1/2}
\end{aligned}$$

where  $i, j = s, p$ . By above, we can obtain

$$\mathcal{G}_s(x, y) = \mathcal{G}_s(x, z) e^{\widehat{ik_s x - z} \cdot (z-y)} + O\left(\frac{(k_s|y-z|)^2}{(k_s|x-z|)^{3/2}}\right) + O\left(\frac{(k_s|y-z|)^{1/2}}{k_s|x-z|}\right) \quad (3.10)$$

$$\mathcal{G}_p(x, y) = \mathcal{G}_p(x, z) e^{\widehat{ik_p x - z} \cdot (z-y)} + O\left(\frac{(k_p|y-z|)^2}{(k_p|x-z|)^{3/2}}\right) + O\left(\frac{(k_p|y-z|)^{1/2}}{k_p|x-z|}\right) \quad (3.11)$$

For  $l > 1$ , a simple computation show that

$$\int_{-d}^d \frac{k_s}{(k_s|x-z|)^l} dx_1 = \frac{1}{(k_s z_2)^{l-1}} \int_{\frac{-d-z_1}{z_2}}^{\frac{d-z_1}{z_2}} \frac{1}{(1+t^2)^{l/2}} dt \leq C \frac{1}{(k_s z_2)^{l-1}} \quad (3.12)$$

Let

$$\mathcal{G}_\alpha(x, y) = \frac{\mathbf{i}}{\sqrt{2\pi\mu}} g_\alpha\left(\frac{x_1-y_1}{|x-y|}, \kappa\right) \frac{1}{(k_\alpha|x-y|)^{1/2}} e^{\mathbf{i}k_\alpha|x-y| - \mathbf{i}\frac{\pi}{4}} \quad (3.13)$$

$$\mathcal{T}_\alpha(x, y) = \frac{k_\alpha}{\sqrt{2\pi}} t_\alpha\left(\frac{x_1-z_1}{|x-z|}, \kappa\right) \frac{1}{(k_s|x-z|)^{1/2}} e^{\mathbf{i}k_\alpha|x-z| - \mathbf{i}\frac{\pi}{4}} \quad (3.14)$$

where  $\alpha = s, p$ . Now, by substituting (3.10-3.11) into  $J_d(z, y)$  and using inequality (3.12), we have

$$\begin{aligned}
J_d(z, y) &= \frac{-\mathbf{i}}{2\pi\mu} \int_{-d}^d t_s\left(\frac{x_1-z_1}{|x-z|}, \kappa\right) \overline{g_s\left(\frac{x_1-z_1}{|x-z|}, \kappa\right)} \frac{e^{\widehat{ik_s x - z} \cdot (y-z)}}{|x-z|} \\
& \quad + t_p\left(\frac{x_1-z_1}{|x-z|}, \kappa\right) \overline{g_p\left(\frac{x_1-z_1}{|x-z|}, \kappa\right)} \frac{e^{\widehat{ik_p x - z} \cdot (y-z)}}{|x-z|} dx_1
\end{aligned} \quad (3.15)$$

$$- \frac{\mathbf{i}}{2\pi\mu} \int_{-d}^d t_p\left(\frac{x_1-z_1}{|x-z|}, \kappa\right) \overline{g_s\left(\frac{x_1-z_1}{|x-z|}, \kappa\right)} \frac{e^{\widehat{ik_s x - z} \cdot (y-z)}}{|x-z|} \quad (3.16)$$

$$+ t_s\left(\frac{x_1-z_1}{|x-z|}, \kappa\right) \overline{g_p\left(\frac{x_1-z_1}{|x-z|}, \kappa\right)} \frac{e^{\widehat{ik_p x - z} \cdot (y-z)}}{|x-z|} dx_1 \quad (3.17)$$

$$+ O\left((1 + \frac{|y-z|}{z_2}) \left(\frac{1}{k_s z_2}\right)^{1/4} + \frac{(k_s|y-z|)^2}{k_s z_2} + \left(\frac{|y-z|}{z_2}\right)^{1/2}\right) \quad (3.18)$$

$$:= F(z, y) + R(z, y) \quad (3.19)$$

$$+ O\left((1 + \frac{|y-z|}{z_2}) \left(\frac{1}{k_s z_2}\right)^{1/4} + \frac{(k_s|y-z|)^2}{k_s z_2} + \left(\frac{|y-z|}{z_2}\right)^{1/2}\right) \quad (3.20)$$

We denote  $\widehat{x-z} = x-z/|x-z| = (\cos(\phi+\pi), \sin(\phi+\pi))$ , then taking the substitution  $x_1 = z_1 - z_2 \cot \phi$ , we obtain

$$F(z, y) = \frac{-\mathbf{i}}{2\pi\mu} \int_{\theta_1^d}^{\theta_2^d} A_s(\phi, \kappa) e^{\mathbf{i}k_s(z_1-y_1) \cos \phi + \mathbf{i}k_s(z_2-y_2) \sin \phi} \quad (3.21)$$

$$+ \frac{-\mathbf{i}}{2\pi\mu} \int_{\theta_1^d}^{\theta_2^d} A_p(\phi, \kappa) e^{\mathbf{i}k_p(z_1-y_1) \cos \phi + \mathbf{i}k_p(z_2-y_2) \sin \phi} \quad (3.22)$$

$$R(z, y) = \frac{-\mathbf{i}}{2\pi\mu} \int_{\theta_1^d}^{\theta_2^d} B_s(\phi, \kappa) e^{\mathbf{i}k_s(z_1-y_1) \cos \phi + \mathbf{i}k_s(z_2-y_2) \sin \phi + (k_p-k_s)|x-z|} \quad (3.23)$$

$$+ \frac{-\mathbf{i}}{2\pi\mu} \int_{\theta_1^d}^{\theta_2^d} B_p(\phi, \kappa) e^{\mathbf{i}k_p(z_1-y_1) \cos \phi + \mathbf{i}k_p(z_2-y_2) \sin \phi + (k_s-k_p)|x-z|} \quad (3.24)$$

It is easy to see that  $|R(z, y)| \leq C \frac{|z-y|}{z_2}$ .  $\square$

Let

$$g(x_1) = \frac{1}{((x_1 - z_1)^2 + z_2^2)^{3/4} ((x_1 - y_1)^2 + y_2^2)^{1/4}}$$

$$\phi(x_1) = ((x_1 - z_1)^2 + z_2^2)^{1/2} - ((x_1 - y_1)^2 + y_2^2)^{1/2}$$

Then, we have

$$g'(x_1) = -g(x_1) \left[ \frac{3(x_1 - z_1)}{2((x_1 - z_1)^2 + z_2^2)} + \frac{(x_1 - y_1)}{2((x_1 - y_1)^2 + y_2^2)} \right]$$

$$\phi'(x_1) = \frac{x_1 - z_1}{((x_1 - z_1)^2 + z_2^2)^{1/2}} - \frac{x_1 - y_1}{((x_1 - y_1)^2 + y_2^2)^{1/2}}$$

$$= \frac{\frac{(x_1 - z_1)^2}{(x_1 - z_1)^2 + z_2^2} - \frac{(x_1 - y_1)^2}{(x_1 - y_1)^2 + y_2^2}}{\frac{x_1 - z_1}{((x_1 - z_1)^2 + z_2^2)^{1/2}} + \frac{x_1 - y_1}{((x_1 - y_1)^2 + y_2^2)^{1/2}}}$$

$$= \frac{(x_1 - z_1)^2 y_2^2 - (x_1 - y_1)^2 z_2^2}{\left( \frac{x_1 - z_1}{((x_1 - z_1)^2 + z_2^2)^{1/2}} + \frac{x_1 - y_1}{((x_1 - y_1)^2 + y_2^2)^{1/2}} \right) ((x_1 - z_1)^2 + z_2^2) ((x_1 - y_1)^2 + y_2^2)}$$

$$\phi''(x_1) = \frac{z_2^2}{((x_1 - z_1)^2 + z_2^2)^{3/2}} - \frac{y_2^2}{((x_1 - y_1)^2 + y_2^2)^{3/2}}$$

Using integration by parts, we can obtain

$$\int_{-d}^d g(x_1) e^{\mathbf{i}k\phi(x_1)} dx_1$$

$$= \frac{1}{\mathbf{i}k} \left( \frac{g(d)}{\phi'(d)} e^{\mathbf{i}k\phi(d)} - \frac{g(-d)}{\phi'(-d)} e^{\mathbf{i}k\phi(-d)} \right) - \frac{1}{\mathbf{i}k} \int_{-d}^d \frac{g'(x_1)}{\phi'(x_1)} - \frac{g(x_1)\phi''(x_1)}{(\phi'(x_1))^2} dx_1$$

#### 4. 2017.11.08

$$\sin \phi_\kappa - \sin(t + \phi) = -2 \cos\left(\frac{\phi_\kappa + \phi + t}{2}\right) \sin\left(\frac{t + \phi - \phi_\kappa}{2}\right)$$

$$\sin\left(\frac{t + \phi - \phi_\kappa}{2}\right) = \sin \frac{t}{2} \cos\left(\frac{\phi - \phi_\kappa}{2}\right) + \cos \frac{t}{2} \sin\left(\frac{\phi - \phi_\kappa}{2}\right)$$

Some think, substituting  $t = 2 \arcsin s/2$  into following integral

$$\begin{aligned}
& \int_0^\infty \chi(t) (\sin \phi_\kappa - \sin(t + \phi))^{1/2} e^{-i\rho \cos t} \\
&= \int_0^\infty \chi(t(s)) (-s \cos(\frac{\phi - \phi_\kappa}{2}) - \sqrt{4 - s^2} \sin(\frac{\phi - \phi_\kappa}{2}))^{1/2} e^{-i\rho s^2/2} \\
&= \int_0^\infty \chi(t) (-\sqrt{t} \cos(\frac{\phi - \phi_\kappa}{2}) - \sqrt{4 - t} \sin(\frac{\phi - \phi_\kappa}{2}))^{1/2} t^{-1/2} e^{-i\rho t/2}
\end{aligned}$$

Let

$$\begin{aligned}
f(t) &= t^{-1/2} e^{-i\rho t/2} \\
g(t) &= - \int_t^{t-i\infty} x^{-1/2} e^{-i\rho x/2} dx \\
&= i \int_0^\infty (t - ix)^{-1/2} e^{-i\rho t - \rho x} dx
\end{aligned}$$

It is easy to see that  $g'(t) = f(t)$ . Then we have

$$\begin{aligned}
&= \int_0^\infty \chi(t) (-\sqrt{t} \cos(\frac{\phi - \phi_\kappa}{2}) - \sqrt{4 - t} \sin(\frac{\phi - \phi_\kappa}{2}))^{1/2} t^{-1/2} e^{-i\rho t/2} \\
&= \chi(0) (-2 \sin(\frac{\phi - \phi_\kappa}{2}))^{1/2} g(0) \\
&\quad - \int_0^\infty (\chi(t) (-\sqrt{t} \cos(\frac{\phi - \phi_\kappa}{2}) - \sqrt{4 - t} \sin(\frac{\phi - \phi_\kappa}{2}))^{1/2})' g(t) dt
\end{aligned}$$

We get

$$\begin{aligned}
g(x) &= \int_0^\infty \chi(t) (-\sqrt{t} \cos(\frac{\phi - \phi_\kappa}{2}) - \sqrt{4 - t} \sin(\frac{\phi - \phi_\kappa}{2}))^{1/2} t^{-1/2} (t - ix)^{-1/2} e^{-i\rho t} dt \\
R(\rho) &= \int_0^\infty g(x) e^{-\rho x} dx
\end{aligned}$$

Because  $\chi(t)$  has compact support  $(-\delta, \delta)$ , we obtain

$$gg(x) = \int_0^\delta (\sqrt{t} \cos(\theta) - \sqrt{4 - t} \sin \theta)^{-1/2} t^{-1/2} (t^2 + x^2)^{-1/4} dt$$

where  $\theta = \frac{\phi - \phi_\kappa}{2}$ . For  $x > 0$ , Put  $L(x)$ :

$$\begin{aligned}
& \int_0^a \frac{1}{t^{3/4}} \frac{1}{(t^2 + x^2)^{1/4}} dt \\
&= 4 \int_0^a \frac{1}{(t^2 + x^2)^{1/4}} dt^{1/4} \\
&= 4 \int_0^{a^{1/4}} \frac{1}{(t^8 + x^2)^{1/4}} dt \\
&= 4x^{-1/4} \int_0^{(\frac{a}{x})^{1/4}} \frac{1}{(t^8 + 1)^{1/4}} dt \\
&= 4x^{-1/4} \int_0^{(\frac{a}{x})^{1/4}} \frac{1}{(t^8 + 1)^{1/4}} dt \\
&\leq 4x^{-1/4} \int_0^\infty \frac{1}{(t^8 + 1)^{1/4}} dt
\end{aligned}$$

Back to analysis  $gg(x)$ , we have

$$\begin{aligned}
gg(x) &\leq \int_0^\delta \left| \frac{\sqrt{t} + 2|\sin \theta|}{t - 4\sin^2 \theta} \right|^{1/2} t^{-1/2} (t^2 + x^2)^{-1/4} dt \\
&= \int_0^\delta \left| \frac{1}{\sqrt{t} - 2|\sin \theta|} \right|^{1/2} t^{-1/2} (t^2 + x^2)^{-1/4} dt \\
&= 2 \int_0^{\sqrt{\delta}} \left| \frac{1}{t - 2|\sin \theta|} \right|^{1/2} (t^4 + x^2)^{-1/4} dt \\
&= 2 \int_{-2|\sin \theta|}^{\sqrt{\delta} - 2|\sin \theta|} |t|^{-1/2} ((t + 2|\sin \theta|)^4 + x^2)^{-1/4} dt \\
&\leq 4 \int_0^{\delta^{1/4}} (t^8 + x^2)^{-1/4} dt + 4 \int_0^{\sqrt{2|\sin \theta|}} ((t^2 - 2|\sin \theta|)^4 + x^2)^{-1/4} dt \\
&\leq Cx^{-1/4} (1 + \int_0^{\sqrt{2|\sin \theta|}} ((t^2 - 2|\sin \theta|)^4/x + x)^{-1/4} dt) \\
&\leq Cx^{-1/4} (1 + \int_0^{\sqrt{2|\sin \theta|}} (t^2 - 2|\sin \theta|)^{-1/2} dt) \\
&= Cx^{-1/4} (1 + \int_0^1 (1 - t^2)^{-1/2} dt) \leq Cx^{-1/4}
\end{aligned}$$

Immediately, we can obtain

$$|g(x)| \leq Cx^{-1/4}$$

It follows that

$$R(\rho) \leq \int_0^\infty x^{-1/4} e^{-\rho x} \leq C\rho^{-3/4}$$

## 5. stationary of phase lemma

**Lemma 5.1** Assume that  $0 < \kappa := \sin \phi_\kappa < 1$ ,  $0 < \phi_\kappa < \pi/2$ ,  $0 \leq \phi \leq \pi/2$ . Let

$$f(t, \phi) := F(\sin(t + \phi), \cos(t + \phi), (\kappa^2 - \sin^2(t + \phi))^{1/2}) \quad (5.1)$$

be a complexed function in  $C([-\pi/2, \pi/2] \times [0, \pi/2])$ . Moreover, its partial derivative with respect to  $t$  can be represented as

$$\frac{\partial f(t, \phi)}{\partial t} = g(t, \phi) (\kappa^2 - \sin^2(t + \phi))^{-1/2} \quad (5.2)$$

where  $g(t, \phi)$  is uniformly bounded. Then for any  $\rho \geq 1$ , we have

$$\begin{aligned}
&\left| I(\rho, \phi) := \int_{-\pi/2}^{\pi/2} f(t) e^{i\rho \cos t} dt - \left( \frac{2\pi}{\rho} \right)^{1/2} f(0) e^{i\rho - i\pi/4} \right| \\
&\leq C \frac{1}{\rho^{3/4}}
\end{aligned} \quad (5.3)$$

**Proof.** Solving the following equation:

$$\kappa^2 - \sin^2(t + \phi) = 0$$

we have, if  $0 < \phi < \pi/2 - \phi_\kappa$ ,

$$t_1(\phi) = \phi_\kappa - \phi \quad t_2(\phi) = -\phi_\kappa - \phi$$

and if  $\pi/2 - \phi_\kappa \leq \phi < \pi/2$ ,

$$t_1(\phi) = \phi_\kappa - \phi \quad t_2(\phi) = \pi - \phi_\kappa - \phi$$

Since  $|t_2(\phi)| < \phi_\kappa$  or  $|t_2(\phi)| < \pi/2 - \phi_\kappa$ , we now define  $\delta := \min(\frac{\phi_\kappa}{2}, \frac{\pi/2 - \phi_\kappa}{2})$  and it is easy to see that

$$\kappa + \sin(t + \phi) \neq 0 \quad (5.4)$$

$$\cos\left(\frac{t + \phi + \phi_\kappa}{2}\right) \neq 0 \quad (5.5)$$

for any  $(t, \phi) \in [-\delta, \delta] \times [0, \pi/2]$ . Let  $\chi_\delta \in C_0^\infty(-\pi/2, \pi/2)$  be the cut-off function with that  $0 \leq \chi_\delta \leq 1$ ,  $\chi_\delta = 1$  in  $(-\delta/2, \delta/2)$  and  $\chi_\delta = 0$  in  $(-\pi/2, \pi/2) \setminus (-\delta, \delta)$ . Then we can divide  $I$  into two parts such that

$$\begin{aligned} I &= \int_{-\delta}^{\delta} f(t) \chi_\delta(t) e^{i\rho \cos t} dt + \int_{-\pi/2}^{\pi/2} f(t) (1 - \chi_\delta(t)) e^{i\rho \cos t} dt \\ &=: I_1 + I_2 \end{aligned}$$

Substituting  $t(s) = 2 \arcsin s/2$  for  $t$  in  $I_1$ , we can obtain

$$I_1 = \int_{-2 \sin \frac{\delta}{2}}^{2 \sin \frac{\delta}{2}} f(t(s)) \chi_\delta(t(s)) \frac{1}{\sqrt{1 - s^2/4}} e^{i\rho} e^{-i\rho s^2/2} ds \quad (5.6)$$

$$= \int_0^{2 \sin \frac{\delta}{2}} (f(t(s)) \chi_\delta(t(s)) + f(-t(s)) \chi_\delta(-t(s))) \frac{1}{\sqrt{1 - s^2/4}} e^{i\rho} e^{-i\rho s^2/2} ds \quad (5.7)$$

$$:= I_{11} + I_{12} \quad (5.8)$$

Taking substitution  $s = \sqrt{x}$ , we get

$$I_{11} = \frac{1}{2} \int_0^{(2 \sin \frac{\delta}{2})^2} f(t(\sqrt{x})) \chi_\delta(t(\sqrt{x})) \frac{1}{\sqrt{1 - x/4}} x^{-1/2} e^{i\rho} e^{-i\rho x/2} dx$$

Observe that

$$\begin{aligned} \sin \phi_\kappa - \sin(t + \phi) &= -2 \cos\left(\frac{\phi_\kappa + \phi + t}{2}\right) \sin\left(\frac{t + \phi - \phi_\kappa}{2}\right) \\ \sin\left(\frac{t + \phi - \phi_\kappa}{2}\right) &= \sin \frac{t}{2} \cos\left(\frac{\phi - \phi_\kappa}{2}\right) + \cos \frac{t}{2} \sin\left(\frac{\phi - \phi_\kappa}{2}\right) \\ &:= \sin \frac{t}{2} \cos \theta + \cos \frac{t}{2} \sin \theta \end{aligned}$$

where  $\theta = \frac{\phi - \phi_\kappa}{2}$ . By lemma (1.3) and using representation (5.2), inequality (5.4-5.5), it follows that

$$\begin{aligned} &|I_{11} - \frac{1}{2} \sqrt{\frac{2\pi}{\rho}} f(0) e^{i\rho - i\frac{\pi}{4}}| \\ &\leq \int_0^\infty e^{-\rho y} dy \int_0^{(2 \sin \frac{\delta}{2})^2} \left| \frac{\partial(f(t(\sqrt{x})) \chi_\delta(t(\sqrt{x})) \frac{1}{\sqrt{1 - x/4}})}{\partial x} \right| (x^2 + y^2)^{-\frac{1}{4}} dx \end{aligned}$$

$$\begin{aligned}
&\leq C \int_0^\infty e^{-\rho y} dy \int_0^{(2 \sin \frac{\delta}{2})^2} |\sqrt{x} \cos \theta + \sqrt{4-x} \sin \theta|^{-1/2} x^{-1/2} (x^2 + y^2)^{-\frac{1}{4}} dx \\
&\leq C \int_0^\infty e^{-\rho y} dy \int_0^{(2 \sin \frac{\delta}{2})^2} \frac{|\sqrt{x} \cos \theta| + \sqrt{4-x} |\sin \theta|^{1/2}}{|x - 4 \sin^2 \theta|^{1/2}} x^{-1/2} (x^2 + y^2)^{-\frac{1}{4}} dx \\
&\leq C \int_0^\infty e^{-\rho y} dy \int_0^{(2 \sin \frac{\delta}{2})^2} \frac{1}{|\sqrt{x} - 2 \sin \theta|^{1/2}} x^{-1/2} (x^2 + y^2)^{-\frac{1}{4}} dx \\
&\leq C \int_0^\infty e^{-\rho y} dy \int_0^{2 \sin \frac{\delta}{2}} \frac{1}{|x - 2 \sin |\theta||^{1/2}} (x^4 + y^2)^{-\frac{1}{4}} dx \\
&\leq C \int_0^\infty e^{-\rho y} dy \int_{-2 \sin |\theta|}^{2 \sin \frac{\delta}{2} - 2 \sin |\theta|} \frac{1}{|x|^{1/2}} ((x + 2 \sin |\theta|)^4 + y^2)^{-\frac{1}{4}} dx \\
&\leq C \int_0^\infty e^{-\rho y} dy \int_{-2 \sin |\theta|}^{2 \sin \frac{\delta}{2}} \frac{1}{|x|^{1/2}} ((x + 2 \sin |\theta|)^4 + y^2)^{-\frac{1}{4}} dx \\
&\leq C \int_0^\infty e^{-\rho y} dy \left( \int_0^{\sqrt{2 \sin \frac{\delta}{2}}} (x^8 + y^2)^{-\frac{1}{4}} dx + \int_0^{\sqrt{2 \sin |\theta|}} ((x^2 - 2 \sin |\theta|)^4 + y^2)^{-\frac{1}{4}} dx \right) \\
&\leq C \int_0^\infty e^{-\rho y} dy \left( y^{-\frac{1}{4}} \int_0^\infty (x^8 + 1)^{-\frac{1}{4}} dx + y^{-\frac{1}{4}} \int_0^{\sqrt{2 \sin |\theta|}} (2 \sin |\theta| - x^2)^{-\frac{1}{2}} dx \right) \\
&\leq C \int_0^\infty y^{-\frac{1}{4}} e^{-\rho y} dy \left( \int_0^\infty (x^8 + 1)^{-\frac{1}{4}} dx + \int_0^1 (1 - x^2)^{-\frac{1}{2}} dx \right) \leq C \frac{1}{\rho^{3/4}}
\end{aligned}$$

Using the same argument, we can also carry out

$$|I_{12} - \frac{1}{2} \sqrt{\frac{2\pi}{\rho}} f(0) e^{i\rho - i\frac{\pi}{4}}| \leq C \frac{1}{\rho^{3/4}} \quad (5.9)$$

It remains to estimate  $I_2$ . Note that there exists  $m > 0$  such that  $|\sin t| \geq m$  for any  $t \in [-\pi/2, \pi/2] \setminus (-\delta/2, \delta/2)$ . Upon integration by parts and representation (5.2) again, we have

$$\begin{aligned}
|I_{12}| &\leq C \rho^{-1} \left( 1 + \left| \int_{[-\pi/2, \pi/2] \setminus (-\delta/2, \delta/2)} \frac{\partial(f(t)(1 - \chi_\delta(t)))}{\partial t} \frac{1}{\sin t} dt \right| \right) \\
&\leq C \rho^{-1} \left( 1 + \int_{-\pi/2}^{\pi/2} \left| \frac{\partial(f(t)(1 - \chi_\delta(t)))}{\partial t} \right| dt \right) \\
&\leq C \rho^{-1} \left( 1 + \int_{-\pi/2}^{\pi/2} |(\kappa^2 - \sin^2(t + \phi))^{-1/2}| dt \right) \\
&\leq C \rho^{-1} \left( 1 + \int_{-\pi/2}^{\pi/2} |(\kappa^2 - \sin^2 t)^{-1/2}| dt \right) \\
&\leq C \rho^{-1}
\end{aligned}$$

This completes the proof.  $\square$

## References

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