1. Estimate of Dirichlet Green Tensor

We need the following slight generalization of Van der Corput lemma for the oscillatory integral [2, P.152].

Lemma 1.1 *Let $-\infty < a < b < \infty$, and u is a C^k function u in (a, b).*
1. If $|u'(t)| \geq 1$ for $t \in (a, b)$ and u' is monotone in (a,b), then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 3\lambda^{-1}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

2. For all $k \geq 2$, if $|u^{(k)}(t)| \geq 1$ for $t \in (a, b)$, then for any $\phi(t)$ in (a, b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 12k\lambda^{-1/k}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

Proof. The assertion can be proved by extending the Van der Corptut lemma in [2]. Here we omit the details. □

We recall following lemma, see e.g. [3]:

Lemma 1.2 *Let $F(\lambda, a) = \int_0^a t^{\alpha-1}f(t)e^{-\mathbf{i}\rho t}dt$ where $0 < a \leq +\infty$, $0 < \alpha < 1$, $\rho > 0$ and $t^{\alpha-1}f \in L^1(0, a)$, then we have*

$$|F(\rho, a)| \leq C(\frac{1}{\rho^\alpha}f(0) + \frac{1}{\rho}(a^{\alpha-1}f(a) + |t^{\alpha-1}f|_{L^1(0,a)}) \tag{1.1}$$

Proof. Put

$$g_0(t) = t^{\alpha-1}e^{-\mathbf{i}\rho t} \tag{1.2}$$

and define

$$g_1(t) = -\int_t^{t-\mathbf{i}\infty} x^{\alpha-1}e^{-\mathbf{i}\rho x}dx \tag{1.3}$$

where the path of integration is the vertical line $x = t - \mathbf{i}y, y \geq 0$. It is easy to show that $g_1(t)' = g_0(t)$. Substituting $x = t - \mathbf{i}y$ into $g_1(t)$, we have

$$g_1(t) = \mathbf{i}\int_0^\infty (t - \mathbf{i}y)^{\alpha-1}e^{-\mathbf{i}\rho t}e^{-\rho y}dy \tag{1.4}$$

Upon integration by parts, we have

$$\begin{aligned} F(\rho, a) = & \int_0^a f(t)dg_1(t) \\ & = e^{-\mathbf{i}\frac{\alpha\pi}{2}}f(0)\Gamma(\alpha)\frac{1}{\rho^\alpha} + f(a)g_1(a) - \int_0^a f'(t)g_1(t)dt \\ & = e^{-\mathbf{i}\frac{\alpha\pi}{2}}f(0)\Gamma(\alpha)\frac{1}{\rho^\alpha} - \mathbf{i}\int_0^\infty e^{-\rho y}dy\int_0^a f'(t)(t - \mathbf{i}y)^{\alpha-1}e^{-\mathbf{i}\rho t}dt \end{aligned}$$

Let

$$h(y) = \int_0^a f'(t)(t - \mathbf{i}y)^{\alpha-1}e^{-\mathbf{i}\rho t}dt$$

and observe that

$$|h(y)| \leq \int_0^a |f'(t)|(t^2+y^2)^{\frac{\alpha-1}{2}}dt$$

□

Lemma 1.3 *Let* $F(\lambda, a) = \int_0^a t^{-1/2} f(t) e^{-\mathbf{i}\rho t} dt$ *where* $0 < a \leq +\infty$ *and* $\rho > 0$*, then we have*

$$|F(\rho, a) - e^{-\mathbf{i}\frac{\pi}{4}} f(0)\Gamma(1/2)\frac{1}{\rho^{1/2}}| \tag{1.5}$$

$$\leq C(\int_0^\infty e^{-\rho y}dy \int_0^a |f'(t)|(t^2+y^2)^{-\frac{1}{4}}dt + \frac{1}{\rho}a^{-1/2}f(a)) \tag{1.6}$$

Proof. Put

$$g_0(t) = t^{-1/2}e^{-\mathbf{i}\rho t} \tag{1.7}$$

and define

$$g_1(t) = -\int_t^{t-\mathbf{i}\infty} x^{-1/2}e^{-\mathbf{i}\rho x}dx \tag{1.8}$$

where the path of integration is the vertical line $x = t - \mathbf{i}y, y \geq 0$. It is easy to show that $g_1(t) = g_0(t)$. Substituting $x = t - \mathbf{i}y$ into $g_1(t)$, we have

$$g_1(t) = \mathbf{i}\int_0^\infty (t-\mathbf{i}y)^{-1/2}e^{-\mathbf{i}\rho t}e^{-\rho y}dy \tag{1.9}$$

Upon integration by parts, we have

$$\begin{aligned} F(\rho, a) = & \int_0^a f(t)dg_1(t) \\ & = e^{-\mathbf{i}\frac{\pi}{4}} f(0)\Gamma(1/2)\frac{1}{\rho^{1/2}} + f(a)g_1(a) - \int_0^a f'(t)g_1(t)dt \\ & = e^{-\mathbf{i}\frac{\pi}{4}} f(0)\Gamma(1/2)\frac{1}{\rho^{1/2}} + \mathbf{i}f(a)\int_0^\infty (a-\mathbf{i}y)^{-1/2}e^{-\mathbf{i}\rho t}e^{-\rho y}dy \\ & \quad - \mathbf{i}\int_0^\infty e^{-\rho y}dy\int_0^a f'(t)(t-\mathbf{i}y)^{-1/2}e^{-\mathbf{i}\rho t}dt \end{aligned}$$

Let

$$h(y) = \int_0^a f'(t)(t-\mathbf{i}y)^{-1/2}e^{-\mathbf{i}\rho t}dt$$

and observe that

$$|h(y)| \leq \int_0^a |f'(t)|(t^2+y^2)^{-\frac{1}{4}}dt$$

It is easy to see that

$$|g_1(a)| \leq a^{-1/2}\int_0^\infty e^{-\rho y}dy \leq C\frac{1}{\rho}$$

□

Lemma 1.4 *Assume that* $0 < \kappa := \sin\phi_\kappa < 1, 0 < \phi_\kappa < \pi/2$, $0 \le \phi \le \pi/2$ *and* $-\pi/2 < t_1 < t_2 < \pi/2$ *satisfy that* $\kappa^2 = \sin^2(\phi + t_1) = \sin^2(\phi + t_2)$. *Let* $f(\theta)$:

$$f(t,\phi) := F(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2}) \tag{1.10}$$

be a function in $C^\infty(([-\pi/2,\pi/2]\backslash\{t_1,t_2\}) \times [0,\pi/2])$. *Moreover, there exits* $\epsilon > 0$ *such that* $f(\theta)$ *can be represented as*

$$f(t,\phi) = g_1(t,\phi) + g_2(t,\phi)(\kappa^2 - \sin^2(t+\phi))^{1/2})^{N/2} \tag{1.11}$$

where $g_1, g_2 \in C^\infty((\bigcup_{i=1,2}(t_i - \epsilon, t_i + \epsilon)) \times [0,\pi/2]))$ *and* $N = \pm 1$. *Then for any* $\rho \ge 1$, *we have*

$$\begin{aligned}
&\Big| I(\rho,\phi) := \int_{-\pi/2}^{\pi/2} f(\theta) e^{\mathbf{i}\rho\cos\theta} d\theta - \frac{N+1}{2}\Big(\frac{2\pi}{\rho}\Big)^{1/2} f(0) e^{\mathbf{i}\rho - \mathbf{i}\pi/4}\Big| \\
&\le C\frac{1}{\rho^{(2+N)/4}}
\end{aligned} \tag{1.12}$$

Proof. The proof will be split into two parts about whether ϕ equal to ϕ_κ.

If $\phi \ne \phi_\kappa$, there exists $0 < \delta < \pi/4$ such that

$$|(\kappa^2 - \sin^2(t+\phi))^{1/2}| > \frac{1}{2}|(\kappa^2 - \sin^2\phi)^{1/2}| \tag{1.13}$$

for any $t \in (-\delta,\delta)$. Let $\chi_\delta \in C_0^\infty(-\pi/2,\pi/2)$ be the cut-off function with that $0 \le \chi_\delta \le 1$, $\chi_\delta = 1$ in $(-\delta/2,\delta/2)$ and $\chi_\delta = 0$ in $(-\pi/2,\pi/2)\backslash(-\delta,\delta)$. Then we can divide I into two parts such that

$$\begin{aligned}
I &= \int_{-\delta}^{\delta} f(t)\chi_\delta(t) e^{\mathbf{i}\rho\cos t} dt + \int_{-\pi/2}^{\pi/2} f(t)(1-\chi_\delta(t)) e^{\mathbf{i}\rho\cos t} dt \\
&=: I_1 + I_2
\end{aligned}$$

Subtitating $t(s) = 2\arcsin s/2$ for t in I_1, we can obtain

$$I_1 = \int_{\mathbb{R}} f(t(s))\chi_\delta(t(s)) \frac{1}{\sqrt{1-s^2/4}} e^{\mathbf{i}\rho} e^{-\mathbf{i}\rho s^2/2} ds \tag{1.14}$$

$$= \int_{\mathbb{R}} h_\delta(s) e^{\mathbf{i}\rho} e^{-\mathbf{i}\rho s^2/2} ds \tag{1.15}$$

It is easy to see that $h_\delta(s) \in C_0^4(\mathbb{R})$. By the lemma of the stationary phase for quadratic term in [1], we have

$$I_1 = e^{\mathbf{i}\rho} \int_{\mathbb{R}} h_\delta(s) e^{-\mathbf{i}\frac{\rho}{2}s^2} ds = e^{\mathbf{i}\rho}\int_{\mathbb{R}} \widehat{h_\delta}(y)\alpha(-y) dy \tag{1.16}$$

where

$$\alpha(y) = (\frac{1}{2\pi\rho})^{1/2} e^{-\mathbf{i}\pi/4} e^{\frac{\mathbf{i}}{2\rho}y^2} \tag{1.17}$$

$$= (\frac{1}{2\pi\rho})^{1/2} e^{-\mathbf{i}\pi/4}(1 + O(\frac{y^2}{\rho})) \tag{1.18}$$

Consequently

$$I_1 = (\frac{1}{2\pi\rho})^{1/2} e^{\mathbf{i}\rho - \mathbf{i}\pi/4} \int_{\mathbb{R}} \widehat{h_\delta}(y)(1 + \frac{1}{\rho}O(y^2)) dy \tag{1.19}$$

Moreover, $\int_{\mathbb{R}} \widehat{h_\delta}(y)dy = 2\pi h_\delta(0)$ and $|\int_{\mathbb{R}} \widehat{h_\delta}(y)y^2dy| < C$ since $\widehat{h_\delta}(y) = O(1/y^4)$. Now, it turns to estimate I_2.

When $N = 1$, using integration by parts, we have

$$|I_2| = \left| \int_{(-\frac{\pi}{2},\frac{\pi}{2})\backslash(-\frac{\delta}{2},\frac{\delta}{2})} f(t)(1-\chi_\delta(t))/\sin t \;\; de^{\mathbf{i}\rho\cos t} \right| \tag{1.20}$$

$$\tag{1.21}$$

$$\leq C\frac{1}{\rho} + \left| \int_{(-\frac{\pi}{2},\frac{\pi}{2})\backslash(-\frac{\delta}{2},\frac{\delta}{2})} (f(t)(1-\chi_\delta(t))/\sin t)' \; e^{\mathbf{i}\rho\cos t}dt \right| \tag{1.22}$$

$$\leq C\frac{1}{\rho} \tag{1.23}$$

From above analysis, we obtain

$$\left| I(\rho,\phi) - \left(\frac{2\pi}{\rho}\right)^{1/2} f(0)e^{\mathbf{i}\rho-\mathbf{i}\pi/4} \right| \leq C(\phi)\frac{1}{\rho} \tag{1.24}$$

When $N = -1$, we can not use integration by parts again since $f'(\theta)$ is not integrable. However, for any $0 < \lambda_1 < 1$ and $1 < \lambda_2 < 1/\kappa$, there exists $0 < \sigma < \epsilon$, such that $\chi := ((t_1-\sigma, t_1+\sigma) \cup (t_2-\sigma, t_2+\sigma)) \cap (-\delta,\delta) = 0$, dependent on λ_1, λ_2 and

$$\lambda_1\kappa < |\sin(t+\phi)| < \lambda_2\kappa. \tag{1.25}$$

for any $t \in \chi$.

We only analysis the integral on $\chi_1 = (t_1-\sigma, t_1+\sigma) \cap [-\pi/2, \pi/2]$ here, which denoted by I_{χ_1}, the proof of I_{χ_2} is similar. It is easy to see that $\sin(t+\phi)$ is monotonic in χ_1. Without loss of generality, we assume that $\sin(t_1-\sigma+\phi) < \kappa < \sin(t_1+\sigma+\phi)$. Let $\sin(t+\phi) = \kappa\sin\theta$ and the implicit mapping from θ to t is denoted by $t(\theta)$ while the inverse mapping by $\theta(t)$, taking the interval χ_1 onto $L_\theta : \theta_1 \to \pi/2 \to \pi/2 - \mathbf{i}\theta_2$ where $\sin(t_1-\sigma+\phi) = \kappa\sin\theta_1, \sin(t_1+\sigma+\phi) = \kappa\sin(\pi/2-\mathbf{i}\theta_2)$. By substituting $t(\theta)$ into I_{χ_1}, we have

$$I_{\chi_1} = \int_{t_1-\sigma}^{t_1+\sigma} \frac{f(t)(\kappa^2-\sin^2(t+\phi))^{1/2}}{(\kappa^2-\sin^2(t+\phi))^{1/2}} e^{\mathbf{i}\rho\cos t} \tag{1.26}$$

$$= \int_{L_\theta} \frac{\kappa f(t(\theta))\cos\theta}{(1-\kappa^2\sin^2\theta)^{1/2}} e^{\mathbf{i}\rho(\cos(t(\theta)))}d\theta \tag{1.27}$$

$$= \int_{L_\theta} \frac{\kappa g_1(t(\theta))\cos\theta + g_2(t(\theta))}{(1-\kappa^2\sin^2\theta)^{1/2}} e^{\mathbf{i}\rho(\cos(t(\theta)))}d\theta \tag{1.28}$$

$$:= \int_{L_\theta} \frac{h(\theta)}{(1-\kappa^2\sin^2\theta)^{1/2}} e^{\mathbf{i}\rho(\cos(t(\theta)))}d\theta \tag{1.29}$$

Observe that $h(\theta)$ and $\partial h/\partial\theta$ are integrable on the path L_θ by (1.11). A simple computation show that

$$\frac{dt(\theta)}{d\theta} = \frac{\kappa\cos\theta}{\cos(t+\phi)} \qquad \frac{d^2t(\theta)}{dt^2} = \frac{\kappa^2\cos^2\theta\sin(t+\phi) - \kappa\sin\theta\cos^2(t+\phi)}{\cos^3(t+\phi)}$$

Then we can obtain

$$\frac{d\cos t}{d\theta} = \frac{-\kappa\sin t\cos\theta}{\cos(t+\phi)}$$

$$\begin{aligned}\frac{d^2\cos t}{d\theta^2} &= \frac{d^2\cos t}{dt^2}(\frac{dt}{d\theta})^2 + \frac{d\cos t}{dt}\frac{d^2t}{d\theta^2}\\ &= \frac{-\kappa^2\cos^2\theta\cos t}{\cos^2(t+\phi)} + \frac{\kappa\sin\theta\cos^2(t+\phi)\sin t - \kappa^2\cos^2\theta\sin(t+\phi)\sin t}{\cos^3(t+\phi)}\\ &= \frac{-\kappa^2\cos^2\theta\cos\phi + \kappa\sin\theta\cos^2(t+\phi)\sin t}{\cos^3(t+\phi)}\\ &= \frac{(\sin^2(t+\phi)-\kappa^2)\cos\phi + \cos^2(t+\phi)\sin(t+\phi)\sin t}{\cos^3(t+\phi)}\end{aligned}$$

Since $|\sin t| > |\sin\delta|$ and $1-\lambda_2^2\kappa^2 < \cos^2(t+\phi) < 1-\lambda_1^2\kappa^2$ for $t\in\chi_1$, it follows that $\theta=\pi/2$ is the only stationary point of $\cos(t(\theta))$ and

$$\left|\frac{d^2\cos t}{d\theta^2}(\pi/2)\right| = \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}|\sin t| > \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}\sin\delta \tag{1.30}$$

Therefore, we can choose appropriate λ_1,λ_2 such that

$$|\frac{d^2\cos t}{d\theta^2}| > \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}\sin\delta \tag{1.31}$$

for any $\theta\in\theta(\chi_1)$. According to lemma (1.1), we obtian $|I_{\chi_1}|\le C\frac{1}{\rho^{1/2}}$, and also $|I_{\chi_2}|\le C\frac{1}{\rho^{1/2}}$. Using integration by parts, we get

$$\left|\int_{[-\pi/2,\pi/2]\setminus((-\delta,\delta)\cup\chi)} f(t)(1-\chi_\delta(t))e^{\mathbf{i}\rho\cos t}dt\right| \le C\frac{1}{\rho}$$

Consequently, for $N=-1$ and $\phi\neq\phi_\kappa$, we get $|I(\rho,\phi)|\le\frac{1}{\rho^{1/2}}$.

We now turn to the case of $\phi=\phi_\kappa$. By (1.11), we can define χ_ϵ similarly and also decompose I into I_1 and I_2. Using the same agurement above, we can easily carry out that: for $N=1$, we have $|I_2|\le C\frac{1}{\rho}$; for $N=-1$, we have $|I_2|\le C\frac{1}{\rho^{1/2}}$. Finally, it remains to analysis I_1. By (1.11), we have

$$\begin{aligned}I_1 &= \int_{-\epsilon}^{\epsilon} g_1\chi_\epsilon + g_2\chi_\epsilon(\sin^2\phi_\kappa - \sin^2(t+\phi_\kappa))^{N/2}e^{\mathbf{i}\rho\cos t}dt\\ &= \int_{-\epsilon}^{\epsilon} g_1\chi_\epsilon + g_2\chi_\epsilon(-2(\sin\phi_\kappa+\sin(t+\phi_\kappa))\cos\frac{2\phi_\kappa+t}{2}\sin t/2)^{N/2}e^{\mathbf{i}\rho\cos t}dt\\ &= \int_{\mathbb{R}} g_1\chi_\epsilon + g_2\chi_\epsilon((\sin\phi_\kappa+\sin(t+\phi_\kappa))\cos\frac{2\phi_\kappa+t}{2})^{N/2}(-2\sin t/2)^{N/2}e^{\mathbf{i}\rho\cos t}dt\end{aligned}$$

Also, subtituting $t(s)=2\arcsin s/2$ for t in I_1, it follows that

$$\begin{aligned}I_1 &= \int_{\mathbb{R}} h_1(s)e^{-\mathbf{i}\rho\frac{s^2}{2}} + h_2(s)(-s)^{N/2}e^{-\mathbf{i}\rho\frac{s^2}{2}} &(1.32)\\ &= I_{11}+I_{12} &(1.33)\end{aligned}$$

where

$$\begin{aligned}h_1(s) &= g_1(t(s))\chi_\epsilon(t(s))\sqrt{1-s^2/4}\ e^{\mathbf{i}\rho}\\ h_2(s) &= g_2\chi_\epsilon((\sin\phi_\kappa+\sin(t+\phi_\kappa))\cos\frac{2\phi_\kappa+t}{2})_{t=t(s)}^{N/2}\sqrt{1-s^2/4}\ e^{\mathbf{i}\rho}\end{aligned}$$

and $h_1(s), h_2(s) \in C_c^\infty(\mathbb{R})$. Using stationary phase lemma similarly, if $N = 1$,

$$I_{11} = \left(\frac{2\pi}{\rho}\right)^{1/2} g_1(0)e^{\mathbf{i}\rho-\mathbf{i}\pi/4} + O(\frac{1}{\rho}) \tag{1.34}$$

$$= \left(\frac{2\pi}{\rho}\right)^{1/2} f(0)e^{\mathbf{i}\rho-\mathbf{i}\pi/4} + O(\frac{1}{\rho}) \tag{1.35}$$

if $N = -1$, we get $|I_{11}| \le C\frac{1}{\rho^{1/2}}$. For I_{12}, we have

$$I_{12} = \int_0^\infty (\mathbf{i}h_2(s) + h_2(-s))s^{N/2}\ e^{-\mathbf{i}\rho s^2/2}ds \tag{1.36}$$

$$= \frac{1}{2}\int_0^\infty (\mathbf{i}h_2(\sqrt{s}) + h_2(-\sqrt{s}))s^{N/4-1/2}e^{-\mathbf{i}\rho s/2}ds \tag{1.37}$$

By lemma (1.2), we get $|I_{12}| \le C\frac{1}{\rho^{(N+2)/4}}$. □

2. Some draft about Green Tensor Analysis

Let substitute $\xi = k\sin\theta$ into integral and shift the variable, we have

$$I(y) = \int_{\mathbb{R}} f(\xi)e^{\mathbf{i}\xi y_1+\mu(\xi)y_2}d\xi = \int_{\mathbb{R}} f(\xi)e^{\mathbf{i}\xi(y_1-z_1)+\mu(\xi)(y_2-z_2)}e^{\mathbf{i}\xi z_1+\mu(\xi)z_2}d\xi \tag{2.1}$$

$$= k\int_L f(k\sin\theta)\cos\theta e^{\mathbf{i}k|y-z|\cos(\theta-\eta)}e^{\mathbf{i}|z|\cos(\theta-\phi)}d\theta \tag{2.2}$$

$$= k\int_{L_\phi} f(k\sin(\theta+\phi))\cos(\theta+\phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \tag{2.3}$$

$$= k\int_L f(k\sin(\theta+\phi))\cos(\theta+\phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \tag{2.4}$$

where $y_1, y_2 > 0, \sin\phi = \frac{z_1}{|z|}, \cos\phi = \frac{z_2}{|z|}, 0 < \phi < \pi/2$ and $\sin\eta = \frac{y_1-z_1}{|y-z|}, \cos\eta = \frac{y_2-z_2}{|y-z|}, 0 < \eta < \pi$. It is easy to see that $\phi + \eta < \pi$. Roughly, using stationary phase lemma, we obtain:

$$I(y) = f(k\sin\phi)k\cos\phi e^{\mathbf{i}k|y-z|\cos(\phi-\eta)}(\frac{2\pi}{|z|})^{1/2}e^{\mathbf{i}|z|-\mathbf{i}\frac{\pi}{4}}(1+O(\frac{1}{|z|})) \tag{2.5}$$

$$\cos(a+\mathbf{i}b) = \frac{e^b+e^{-b}}{2}\cos a + \mathbf{i}\frac{e^{-b}-e^b}{2}\sin a \tag{2.6}$$

$$\sin(a+\mathbf{i}b) = \frac{e^b+e^{-b}}{2}\sin a + \mathbf{i}\frac{e^b-e^{-b}}{2}\cos a \tag{2.7}$$

When $\theta \in (-a-\pi/2, -a-\pi/2+\mathbf{i}\infty)$, let $\theta = -a-\pi/2\mathbf{i}t$, where $t > 0, 0 \le a \le \phi$, then

$$-\mathrm{Im}\left(|z|\cos\theta + |y-z|\cos(\theta+\phi-\eta)\right)$$

$$= |z|\sin(a+\pi/2) + |y-z|\sin(a+\pi/2-\phi+\eta) \tag{2.8}$$

$$= |z|\cos a + |y-z|\cos(a-\phi+\eta) \tag{2.9}$$

$$= |z|\cos a + \cos a|y-z|(\cos\phi\cos\eta + \sin\phi\sin\eta) \tag{2.10}$$

$$+\sin a|y-z|(\sin\phi\cos\eta - \cos\phi\sin\eta) \tag{2.11}$$

$$= |z|\cos a + \cos a((y_2 - z_2)\cos\phi + (y_1 - z_1)\sin\phi) \tag{2.12}$$
$$+ \sin a((y_2 - z_2)\sin\phi - (y_1 - z_1)\cos\phi) \tag{2.13}$$
$$= y_1 \sin(\phi - a) + y_2\cos(\phi - a) > 0 \tag{2.14}$$

Now, Using Cauchy Integral Theorem, we have

$$I(y) = k\int_L f(k\sin(\theta+\phi))\cos(\theta+\phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \tag{2.15}$$

Let $L_1 = (-\pi/2, -\pi/2 + \mathbf{i}\infty)$ and $\theta = -\pi/2 + \mathbf{i}t, t > 0$, then

$$I_1(y) = k\int_{L_1} f(k\sin(\theta+\phi))\cos(\theta+\phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \tag{2.16}$$
$$= \tag{2.17}$$

$$I(y) = f(k\sin\phi)k\cos\phi e^{\mathbf{i}k|y-z|\cos(\phi-\eta)}(\frac{2\pi}{|z|})^{1/2}e^{\mathbf{i}|z|-\mathbf{i}\frac{\pi}{4}} \tag{2.18}$$

$$+\frac{kz_2}{|z|}O(\left(\frac{1}{k|z|}\right)^{3/4} + \frac{1}{k|y|}) + \frac{kz_1}{|z|}O(\left(\frac{1}{k|z|}\right)^{5/4} + \left(\frac{1}{k|y|}\right)^2) \tag{2.19}$$

It is easy to see

$$\int_{-d}^{d}\frac{k}{(k|x-z|)^\alpha}\frac{1}{(k|x-y|)^\beta}dx_1 \le C(\frac{1}{(kz_2)^{\alpha+\beta-1}} + \frac{1}{(ky_2)^{\alpha+\beta-1}}) \tag{2.20}$$

where $z, y \in \mathbb{R}_+^2$, $x \in \Gamma_0$ and $\alpha + \beta > 0$.

$$e^{\mathbf{i}\mu y_2 + \mathbf{i}\xi(x_1-y_1)} = e^{\mathbf{i}\mu y_2 - \mathbf{i}y_2/\tan\phi} = e^{\mathbf{i}y_2(\mu-\xi/\tan\phi)} \tag{2.21}$$

Another method

$$\int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi)e^{\mathbf{i}k|x-y|\cos\theta} \tag{2.22}$$
$$= \int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi)e^{\mathbf{i}k|x-y|\cos(\theta+\psi-\psi)} \tag{2.23}$$
$$= \int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi)e^{\mathbf{i}ky_2\cos(\theta+\psi)+\mathbf{i}k|x_1-y_1|\sin(\theta+\psi)} \tag{2.24}$$
$$= \int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi) \tag{2.25}$$
$$e^{\mathbf{i}k(y_2-z_2)\cos(\theta+\psi)+\mathbf{i}k(|x_1-y_1|-|x_1-z_1|)\sin(\theta+\psi)+\mathbf{i}k|z|\cos(\theta+\psi-\phi)} \tag{2.26}$$

3. Finite Aperture Point Spread Function

If $x \in \Gamma_0$ and $z, y \in \mathbb{R}_+^2$, by lemma (**??**) we have

$$G(x,y) = \frac{\mathbf{i}k_s}{\mu\sqrt{2\pi}}\frac{1}{\delta(\xi)}\begin{pmatrix} \mu_s\beta & \xi\beta \\ 2\xi\mu_s\mu_p & 2\xi^2\mu_p \end{pmatrix}_{\xi=k_s\frac{x_1-y_1}{|x-y|}}\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{1/2}}e^{\mathbf{i}k_s|x-y|-\mathbf{i}\frac{\pi}{4}}$$
$$+\frac{\mathbf{i}k_p}{\mu\sqrt{2\pi}}\frac{1}{\delta(\xi)}\begin{pmatrix} 2\xi^2\mu_s & -2\xi\mu_s\mu_p \\ -\xi\beta & \mu_p\beta \end{pmatrix}_{\xi=k_p\frac{x_1-y_1}{|x-y|}}\frac{y_2}{|x-y|}\frac{1}{(k_p|x-y|)^{1/2}}e^{\mathbf{i}k_p|x-y|-\mathbf{i}\frac{\pi}{4}} \tag{3.1}$$

$$
\begin{aligned}
&+O(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}})\\
&:=\mathcal{G}_s(x,y)+\mathcal{G}_p(x,y)+O(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}})
\end{aligned}
$$

$$
\begin{aligned}
T_D(x,z)=&\frac{k_s}{\sqrt{2\pi}}\frac{1}{\gamma(\xi)}\begin{pmatrix}\mu_s\mu_p & \xi\mu_p\\ \xi\mu_s & \xi^2\end{pmatrix}_{\xi=k_s\frac{x_1-z_1}{|x-z|}}\frac{z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{1/2}}e^{\mathbf{i}k_s|x-z|-\mathbf{i}\frac{\pi}{4}}\\
&+\frac{k_p}{\sqrt{2\pi}}\frac{1}{\gamma(\xi)}\begin{pmatrix}\xi^2 & -\xi\mu_p\\ -\xi\mu_s & \mu_p\mu_s\end{pmatrix}_{\xi=k_p\frac{x_1-z_1}{|x-z|}}\frac{z_2}{|x-z|}\frac{1}{(k_p|x-z|)^{1/2}}e^{\mathbf{i}k_p|x-z|-\mathbf{i}\frac{\pi}{4}} \qquad (3.2)\\
&+O(\frac{k_sz_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{k_s|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}})\\
&:=\mathcal{T}_s(x,z)+\mathcal{T}_p(x,z)+O(\frac{k_sz_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{k_s|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}})
\end{aligned}
$$

Now we consider the finite aperture point spread function $J_d(z,y)$:

$$
\int_{-d}^{d}(T_D(x_1,0;z_1,z_2))^T\overline{G(x_1,0;y_1,y_2)}dx_1 \qquad (3.3)
$$

Recall following standard asymptotic expansion:

$$
|x-y|=|x-z|+\widehat{x-z}\cdot(z-y)+O(\frac{|y-z|^2}{|x-z|}) \qquad (3.4)
$$

$$
|y|^{-\alpha}=|z|^{-\alpha}(1+\frac{|y|-|z|}{|z|})^{-\alpha}=|z|^{-\alpha}(1+O(\frac{|y-z|}{|z|})) \qquad (3.5)
$$

$$
e^{\mathbf{i}t}=1+O(t) \qquad (3.6)
$$

$$
|a^{1/2}-b^{1/2}|\le C\sqrt{|a-b|} \qquad (3.7)
$$

where $x,y,z\in\mathbb{R}^2$, $t,a,b\in\mathbb{R}$ and $\alpha>0$.

Lemma 3.1 *For any $z,y\in\mathbb{R}^2_+$, $J_d(z,y)=F(z,y)+O((1+\frac{|y-z|}{z_2})(\frac{1}{k_sz_2})^{1/4}+\frac{(k_s|y-z|)^2}{k_sz_2}+(\frac{|y-z|}{z_2})^{1/2})$, where*

$$
\begin{aligned}
F(z,y)=&-\frac{\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d}f_s(\theta)\begin{pmatrix}\sin^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \cos^2\theta\end{pmatrix}e^{\mathbf{i}k_s(z_1-y_1)\cos\theta+\mathbf{i}k_s(z_2-y_2)\sin\theta}d\theta \qquad (3.8)\\
&-\frac{\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d}f_p(\theta)\begin{pmatrix}\cos^2\theta & -\sin\theta\cos\theta\\ -\sin\theta\cos\theta & \sin^2\theta\end{pmatrix}e^{\mathbf{i}k_p(z_1-y_1)\cos\theta+\mathbf{i}k_p(z_2-y_2)\sin\theta}d\theta \qquad (3.9)
\end{aligned}
$$

and

$$
f_s(\theta)=\frac{\sin\theta((\kappa^2-\cos^2\theta)^{1/2}(1-2\cos^2\theta)+2\overline{(\kappa^2-\cos^2\theta)^{1/2}}\cos^2\theta)}{(\cos^2\theta+\sin\theta(\kappa^2-\cos^2\theta)^{1/2})\overline{((1-2\cos^2\theta)^2+4\cos^2\theta\sin\theta(\kappa^2-\cos\theta)^{1/2})}}
$$

$$
f_p(\theta)=\frac{\sin\theta(1/\kappa^2-\cos^2\theta)^{1/2}}{(\cos^2\theta+\sin\theta(1/\kappa^2-\cos^2\theta)^{1/2})((1/\kappa^2-2\cos^2\theta)^2+4\cos^2\theta\sin\theta(1/\kappa^2-\cos\theta)^{1/2})}
$$

where $0<\theta_1^d<\pi/2<\theta_2^d<\pi$ and $z_2=(d+z_1)\tan\theta_1^d=(z_1-d)\tan\theta_2^d$.

Proof.

$$\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}}$$
$$=(\frac{z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}})(1+O(\frac{|y-z|}{|x-z|}))$$

$$|\mu_{\mathbf{i}}(k_j\frac{x_1-y_1}{|x-y|})-\mu_{\mathbf{i}}(k_j\frac{x_1-z_1}{|x-z|})|$$
$$\leq Ck_j\sqrt{\Big|\frac{x_1-y_1}{|x-y|}-\frac{x_1-z_1}{|x-z|}\Big|}\leq Ck_j\Big(\frac{|y-z|}{|x-z|}\Big)^{1/2}$$

where $i,j=s,p$. By above, we can obtain

$$\mathcal{G}_s(x,y)=\mathcal{G}_s(x,z)e^{\mathbf{i}k_s\widehat{x-z}\cdot(z-y)}+O(\frac{(k_s|y-z|)^2}{(k_s|x-z|)^{3/2}})+O(\frac{(k_s|y-z|)^{1/2}}{k_s|x-z|}) \tag{3.10}$$

$$\mathcal{G}_p(x,y)=\mathcal{G}_p(x,z)e^{\mathbf{i}k_p\widehat{x-z}\cdot(z-y)}+O(\frac{(k_p|y-z|)^2}{(k_p|x-z|)^{3/2}})+O(\frac{(k_p|y-z|)^{1/2}}{k_p|x-z|}) \tag{3.11}$$

For $l>1$, a simple computation show that

$$\int_{-d}^{d}\frac{k_s}{(k_s|x-z|)^l}dx_1=\frac{1}{(k_sz_2)^{l-1}}\int_{\frac{-d-z_1}{z_2}}^{\frac{d-z_1}{z_2}}\frac{1}{(1+t^2)^{l/2}}dt\leq C\frac{1}{(k_sz_2)^{l-1}} \tag{3.12}$$

Let

$$\mathcal{G}_\alpha(x,y)=\frac{\mathbf{i}}{\sqrt{2\pi}\mu}g_\alpha(\frac{x_1-y_1}{|x-y|},\kappa)\frac{1}{(k_\alpha|x-y|)^{1/2}}e^{\mathbf{i}k_\alpha|x-y|-\mathbf{i}\frac{\pi}{4}} \tag{3.13}$$

$$\mathcal{T}_\alpha(x,y)=\frac{k_\alpha}{\sqrt{2\pi}}t_\alpha(\frac{x_1-z_1}{|x-z|},\kappa)\frac{1}{(k_s|x-z|)^{1/2}}e^{\mathbf{i}k_\alpha|x-z|-\mathbf{i}\frac{\pi}{4}} \tag{3.14}$$

where $\alpha=s,p$. Now, by substituting (3.10-3.11) into $J_d(z,y)$ and using inequality (3.12), we have

$$J_d(z,y)=\frac{-\mathbf{i}}{2\pi\mu}\int_{-d}^{d}t_s(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_s(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_s\widehat{x-z}\cdot(y-z)}}{|x-z|}$$
$$+t_p(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_p(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_p\widehat{x-z}\cdot(y-z)}}{|x-z|}dx_1 \tag{3.15}$$
$$-\frac{\mathbf{i}}{2\pi\mu}\int_{-d}^{d}t_p(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_s(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_s\widehat{x-z}\cdot(y-z)}}{|x-z|} \tag{3.16}$$
$$+t_s(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_p(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_p\widehat{x-z}\cdot(y-z)}}{|x-z|}dx_1 \tag{3.17}$$
$$+O((1+\frac{|y-z|}{z_2})(\frac{1}{k_sz_2})^{1/4}+\frac{(k_s|y-z|)^2}{k_sz_2}+(\frac{|y-z|}{z_2})^{1/2}) \tag{3.18}$$
$$:=F(z,y)+R(z,y) \tag{3.19}$$
$$+O((1+\frac{|y-z|}{z_2})(\frac{1}{k_sz_2})^{1/4}+\frac{(k_s|y-z|)^2}{k_sz_2}+(\frac{|y-z|}{z_2})^{1/2}) \tag{3.20}$$

We denote $\widehat{x-z} = x - z/|x-z| = (cos(\phi+\pi), \sin(\phi+\pi))$, then taking the substitution $x_1 = z_1 - z_2 \cot\phi$, we obtain

$$F(z,y) = \frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d} A_s(\phi,\kappa)e^{\mathbf{i}k_s(z_1-y_1)\cos\phi+\mathbf{i}k_s(z_2-y_2)\sin\phi} \tag{3.21}$$

$$+\frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d} A_p(\phi,\kappa)e^{\mathbf{i}k_p(z_1-y_1)\cos\phi+\mathbf{i}k_p(z_2-y_2)\sin\phi} \tag{3.22}$$

$$R(z,y) = \frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d} B_s(\phi,\kappa)e^{\mathbf{i}k_s(z_1-y_1)\cos\phi+\mathbf{i}k_s(z_2-y_2)\sin\phi+(k_p-k_s)|x-z|} \tag{3.23}$$

$$+\frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d} B_p(\phi,\kappa)e^{\mathbf{i}k_p(z_1-y_1)\cos\phi+\mathbf{i}k_p(z_2-y_2)\sin\phi+(k_s-k_p)|x-z|} \tag{3.24}$$

It is easy to see that $|R(z,y)| \le C\frac{|z-y|}{z_2}$. □

Let

$$g(x_1) = \frac{1}{((x_1-z_1)^2+z_2^2)^{3/4}((x_1-y_1)^2+y_2^2)^{1/4}}$$
$$\phi(x_1) = ((x_1-z_1)^2+z_2^2)^{1/2} - ((x_1-y_1)^2+y_2^2)^{1/2}$$

Then, we have

$$g'(x_1) = -g(x_1)[\frac{3(x_1-z_1)}{2((x_1-z_1)^2+z_2^2)} + \frac{(x_1-y_1)}{2((x_1-y_1)^2+y_2^2)}]$$
$$\phi'(x_1) = \frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}} - \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}}$$
$$= \frac{\frac{(x_1-z_1)^2}{(x_1-z_1)^2+z_2^2} - \frac{(x_1-y_1)^2}{(x_1-y_1)^2+y_2^2}}{\frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}} + \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}}}$$
$$= \frac{(x_1-z_1)^2y_2^2 - (x_1-y_1)^2z_2^2}{(\frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}} + \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}})((x_1-z_1)^2+z_2^2)((x_1-y_1)^2+y_2^2)}$$
$$\phi''(x_1) = \frac{z_2^2}{((x_1-z_1)^2+z_2^2)^{3/2}} - \frac{y_2^2}{((x_1-y_1)^2+y_2^2)^{3/2}}$$

Using integration by parts, we can obtain

$$\int_{-d}^{d} g(x_1)e^{\mathbf{i}\phi(x_1)}dx_1$$
$$= (\frac{g(d)}{\phi'(d)}e^{\mathbf{i}\phi(d)} - \frac{g(-d)}{\phi'(-d)}e^{\mathbf{i}\phi(-d)}) - \int_{-d}^{d}\frac{g'(x_1)}{\phi'(x_1)} - \frac{g(x_1)\phi''(x_1)}{(\phi'(x_1))^2}dx_1$$

Assume that

$$|y_1| \le c_0 d \quad |z_1| \le c_0 d \quad h \le y_2, z_2 \le c_1 h \quad d \le c_2 h$$

where $0 < c_0 < 1$. Let define $0 < \theta_y, \theta_z < \pi$ such that

$$\cos\theta_y = \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}}$$
$$\cos\theta_z = \frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}}$$

By mean value theorem and the law of sines, we get

$$
\begin{aligned}
|\phi'(x_1)| &= |\cos\theta_z - \cos\theta_y| = |\sin\theta'||\theta_z - \theta_y| \\
&\geq \frac{h}{(1+c_0)d}|\sin(\theta_z-\theta_y)| \\
&= \frac{h}{(1+c_0)d}\frac{|z-y|}{|x-y|}\sin\theta_{|x-y|} \\
&= \frac{h}{(1+c_0)d}\frac{|z-y|}{|x-z|}\sin\theta_{|x-z|} \\
&\geq \frac{h^2}{(1+c_0)^2d^2}\frac{|z-y|}{|x-y|} \\
\text{or} \qquad &\geq \frac{h^2}{(1+c_0)^2d^2}\frac{|z-y|}{|x-z|}
\end{aligned}
$$

Then we have

$$
\begin{aligned}
|\frac{g(x_1)}{\phi'(x_1)}| &\leq \frac{(1+c_0)^2d^2}{h^2}\frac{1}{|z-y||x-y|^{1/2}|x-z|^{1/2}} \\
&\leq C\frac{d^2}{h^3}\frac{1}{|z-y|}
\end{aligned}
$$

Moreover, by mean value theorem agian, we have

$$
\begin{aligned}
|\phi''(x_1)| &= |\frac{\sin^2\theta_z}{|x-z|} - \frac{\sin^2\theta_y}{|x-y|}| \\
&= |\frac{2\sin\theta'\cos\theta'}{|x-y'|}(\theta_z-\theta_y) - \frac{\sin^2\theta'}{|x-y'|^2}(|x-z|-|x-y|)| \\
&\leq \pi\frac{|\sin(\theta_z-\theta_y)|}{h} + \frac{|z-y|}{h^2} \\
&\leq \pi\frac{|\sin\theta_{|x-z|}||z-y|}{h|x-z|} + \frac{|z-y|}{h^2} \\
&\leq C\frac{|z-y|}{h^2}
\end{aligned}
$$

Now, it is easy to see that

$$
\begin{aligned}
&\Big|\int_{-d}^{d}\frac{g'(x_1)}{\phi'(x_1)} - \frac{g(x_1)\phi''(x_1)}{(\phi'(x_1))^2}dx_1\Big| \\
&\leq C\frac{d^3}{h^4}\frac{1}{|z-y|} + C\frac{d^3}{h^3}\frac{1}{|z-y|}\frac{d^2}{h^3}
\end{aligned}
$$

Based on the above analysis, we can obtain

$$
|\int_{-d}^{d} z_2 g(x_1)e^{\mathbf{i}\phi(x_1)}| \leq C((\frac{d}{h})^2 + (\frac{d}{h})^3 + (\frac{d}{h})^5)\frac{1}{|z-y|}
$$

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$$
\sin\phi_\kappa - \sin(t+\phi) = -2\cos(\frac{\phi_\kappa+\phi+t}{2})\sin(\frac{t+\phi-\phi_\kappa}{2})
$$

$$
\sin(\frac{t+\phi-\phi_\kappa}{2}) = \sin\frac{t}{2}\cos(\frac{\phi-\phi_\kappa}{2}) + \cos\frac{t}{2}\sin(\frac{\phi-\phi_\kappa}{2})
$$

Some think, substituting $t = 2\arcsin s/2$ into following integral

$$\int_0^\infty \chi(t)(\sin\phi_\kappa - \sin(t+\phi))^{1/2}e^{-\mathbf{i}\rho\cos t}$$
$$= \int_0^\infty \chi(t(s))(-s\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-s^2}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2}e^{-\mathbf{i}\rho s^2/2}$$
$$= \int_0^\infty \chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2}t^{-1/2}e^{-\mathbf{i}\rho t/2}$$

Let

$$f(t) = t^{-1/2}e^{-\mathbf{i}\rho t/2}$$
$$g(t) = -\int_t^{t-\mathbf{i}\infty} x^{-1/2}e^{-\mathbf{i}\rho x/2}dx$$
$$= \mathbf{i}\int_0^\infty (t-\mathbf{i}x)^{-1/2}e^{-\mathbf{i}\rho t-\rho x}dx$$

It is easy to see that $g'(t) = f(t)$. Then we have

$$= \int_0^\infty \chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2}t^{-1/2}e^{-\mathbf{i}\rho t/2}$$
$$= \chi(0)(-2\sin(\frac{\phi-\phi_\kappa}{2}))^{1/2}g(0)$$
$$- \int_0^\infty (\chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2})'g(t)dt$$

We get

$$g(x) = \int_0^\infty \chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{-1/2}t^{-1/2}(t-\mathbf{i}x)^{-1/2}e^{-\mathbf{i}\rho t}dt$$
$$R(\rho) = \int_0^\infty g(x)e^{-\rho x}dx$$

Because $\chi(t)$ has compact support $(-\delta,\delta)$, we obtain

$$gg(x) = \int_0^\delta (\sqrt{t}\cos(\theta) - \sqrt{4-t}\sin\theta)^{-1/2}t^{-1/2}(t^2+x^2)^{-1/4}dt$$

where $\theta = \frac{\phi-\phi_\kappa}{2}$. For $x > 0$, Put L(x):

$$\int_0^a \frac{1}{t^{3/4}}\frac{1}{(t^2+x^2)^{1/4}}dt$$
$$= 4\int_0^a \frac{1}{(t^2+x^2)^{1/4}}dt^{1/4}$$
$$= 4\int_0^{a^{1/4}} \frac{1}{(t^8+x^2)^{1/4}}dt$$
$$= 4x^{-1/4}\int_0^{(\frac{a}{x})^{1/4}} \frac{1}{(t^8+1)^{1/4}}dt$$
$$= 4x^{-1/4}\int_0^{(\frac{a}{x})^{1/4}} \frac{1}{(t^8+1)^{1/4}}dt$$
$$\le 4x^{-1/4}\int_0^\infty \frac{1}{(t^8+1)^{1/4}}dt$$

Back to analysis $gg(x)$, we have

$$\begin{aligned}
gg(x) &\le \int_0^\delta \Big|\frac{\sqrt{t}+2|\sin\theta|}{t-4\sin^2\theta}\Big|^{1/2} t^{-1/2}(t^2+x^2)^{-1/4}dt \\
&= \int_0^\delta \Big|\frac{1}{\sqrt{t}-2|\sin\theta|}\Big|^{1/2} t^{-1/2}(t^2+x^2)^{-1/4}dt \\
&= 2\int_0^{\sqrt{\delta}} \Big|\frac{1}{t-2|\sin\theta|}\Big|^{1/2}(t^4+x^2)^{-1/4}dt \\
&= 2\int_{-2|\sin\theta|}^{\sqrt{\delta}-2|\sin\theta|} |t|^{-1/2}((t+2|\sin\theta|)^4+x^2)^{-1/4}dt \\
&\le 4\int_0^{\delta^{1/4}}(t^8+x^2)^{-1/4}dt + 4\int_0^{\sqrt{2|\sin\theta|}}((t^2-2|\sin\theta|)^4+x^2)^{-1/4}dt \\
&\le Cx^{-1/4}(1+\int_0^{\sqrt{2|\sin\theta|}}((t^2-2|\sin\theta|)^4/x+x)^{-1/4}dt) \\
&\le Cx^{-1/4}(1+\int_0^{\sqrt{2|\sin\theta|}}(t^2-2|\sin\theta|)^{-1/2}dt) \\
&= Cx^{-1/4}(1+\int_0^1(1-t^2)^{-1/2}dt) \le Cx^{-1/4}
\end{aligned}$$

Immediately, we can obtain

$$|g(x)| \le Cx^{-1/4}$$

It follows that

$$R(\rho) \le \int_0^\infty x^{-1/4}e^{-\rho x} \le C\rho^{-3/4}$$

5. stationary of phase lemma

Lemma 5.1 *Assume that* $0<\kappa:=\sin\phi_\kappa<1, 0<\phi_\kappa<\pi/2$, $0\le\phi\le\pi/2$. *Let*

$$f(t,\phi) := F(\sin(t+\phi),\cos(t+\phi),(\kappa^2-\sin^2(t+\phi))^{1/2}) \tag{5.1}$$

be a complexed function in $C([-\pi/2,\pi/2]\times[0,\pi/2])$. *Moreover, its partial derivative with respect to* t *can be represented as*

$$\frac{\partial f(t,\phi)}{\partial t} = g(t,\phi)(\kappa^2-\sin^2(t+\phi))^{-1/2} \tag{5.2}$$

where $g(t,\phi)$ *is uniformly bounded. Then for any* $\rho\ge1$, *we have*

$$\begin{aligned}
&\Big|I(\rho,\phi) := \int_{-\pi/2}^{\pi/2} f(t)e^{\mathbf{i}\rho\cos t}dt - \Big(\frac{2\pi}{\rho}\Big)^{1/2} f(0)e^{\mathbf{i}\rho-\mathbf{i}\pi/4}\Big| \\
&\le C\frac{1}{\rho^{3/4}}
\end{aligned} \tag{5.3}$$

Proof. Solving the following equation:

$$\kappa^2-\sin^2(t+\phi)=0$$

we have, if $0 < \phi < \pi/2 - \phi_\kappa$,

$$t_1(\phi) = \phi_\kappa - \phi \qquad t_2(\phi) = -\phi_\kappa - \phi$$

and if $\pi/2 - \phi_\kappa \le \phi < \pi/2$,

$$t_1(\phi) = \phi_\kappa - \phi \qquad t_2(\phi) = \pi - \phi_\kappa - \phi$$

Since $|t_2(\phi)| < \phi_\kappa$ or $|t_2(\phi)| < \pi/2 - \phi_\kappa$, we now define $\delta := \min(\frac{\phi_\kappa}{2}, \frac{\pi/2-\phi_\kappa}{2})$ and it is easy to see that

$$\kappa + \sin(t+\phi) \neq 0 \tag{5.4}$$

$$\cos(\frac{t+\phi+\phi_\kappa}{2}) \neq 0 \tag{5.5}$$

for any $(t, \phi) \in [-\delta, \delta] \times [0, \pi/2]$. Let $\chi_\delta \in C_0^\infty(-\pi/2, \pi/2)$ be the cut-off function with that $0 \le \chi_\delta \le 1$, $\chi_\delta = 1$ in $(-\delta/2, \delta/2)$ and $\chi_\delta = 0$ in $(-\pi/2, \pi/2)\backslash(-\delta, \delta)$. Then we can divide I into two parts such that

$$\begin{aligned} I &= \int_{-\delta}^{\delta} f(t)\chi_\delta(t) e^{\mathbf{i}\rho\cos t} dt + \int_{-\pi/2}^{\pi/2} f(t)(1-\chi_\delta(t)) e^{\mathbf{i}\rho\cos t} dt \\ &=: I_1 + I_2 \end{aligned}$$

Subtitating $t(s) = 2\arcsin s/2$ for t in I_1, we can obtain

$$\begin{aligned} I_1 = & \int_{-2\sin\frac{\delta}{2}}^{2\sin\frac{\delta}{2}} f(t(s))\chi_\delta(t(s)) \frac{1}{\sqrt{1-s^2/4}} e^{\mathbf{i}\rho} e^{-\mathbf{i}\rho s^2/2} ds && (5.6) \\ = & \int_0^{2\sin\frac{\delta}{2}} (f(t(s))\chi_\delta(t(s)) + f(-t(s))\chi_\delta(-t(s))) \frac{1}{\sqrt{1-s^2/4}} e^{\mathbf{i}\rho} e^{-\mathbf{i}\rho s^2/2} ds && (5.7) \\ := & I_{11} + I_{12} && (5.8) \end{aligned}$$

Taking substitution $s = \sqrt{x}$, we get

$$I_{11} = \frac{1}{2}\int_0^{(2\sin\frac{\delta}{2})^2} f(t(\sqrt{x}))\chi_\delta(t(\sqrt{x})) \frac{1}{\sqrt{1-x/4}} x^{-1/2} e^{\mathbf{i}\rho} e^{-\mathbf{i}\rho x/2} dx$$

Observe that

$$\begin{aligned} \sin\phi_\kappa - \sin(t+\phi) &= -2\cos(\frac{\phi_\kappa+\phi+t}{2})\sin(\frac{t+\phi-\phi_\kappa}{2}) \\ \sin(\frac{t+\phi-\phi_\kappa}{2}) &= \sin\frac{t}{2}\cos(\frac{\phi-\phi_\kappa}{2}) + \cos\frac{t}{2}\sin(\frac{\phi-\phi_\kappa}{2}) \\ &:= \sin\frac{t}{2}\cos\theta + \cos\frac{t}{2}\sin\theta \end{aligned}$$

where $\theta = \frac{\phi-\phi_\kappa}{2}$. By lemma (1.3) and using representation (5.2), inequality (5.4-5.5), it follows that

$$\begin{aligned} & |I_{11} - \frac{1}{2}\sqrt{\frac{2\pi}{\rho}} f(0) e^{\mathbf{i}\rho - \mathbf{i}\frac{\pi}{4}}| \\ \le & \int_0^\infty e^{-\rho y} dy \int_0^{(2\sin\frac{\delta}{2})^2} \Big| \frac{\partial(f(t(\sqrt{x}))\chi_\delta(t(\sqrt{x}))\frac{1}{\sqrt{1-x/4}})}{\partial x} \Big| (x^2+y^2)^{-\frac{1}{4}} dx \end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^\infty e^{-\rho y}dy\int_0^{(2\sin\frac{\delta}{2})^2} |\sqrt{x}\cos\theta+\sqrt{4-x}\sin\theta|^{-1/2}x^{-1/2}(x^2+y^2)^{-\frac14}dx\\
&\le C\int_0^\infty e^{-\rho y}dy\int_0^{(2\sin\frac{\delta}{2})^2} \frac{\sqrt{x}|\cos\theta|+\sqrt{4-x}|\sin\theta|^{1/2}}{|x-4\sin^2\theta|^{1/2}}x^{-1/2}(x^2+y^2)^{-\frac14}dx\\
&\le C\int_0^\infty e^{-\rho y}dy\int_0^{(2\sin\frac{\delta}{2})^2} \frac{1}{|\sqrt{x}-2|\sin\theta||^{1/2}}x^{-1/2}(x^2+y^2)^{-\frac14}dx\\
&\le C\int_0^\infty e^{-\rho y}dy\int_0^{2\sin\frac{\delta}{2}} \frac{1}{|x-2\sin|\theta||^{1/2}}(x^4+y^2)^{-\frac14}dx\\
&\le C\int_0^\infty e^{-\rho y}dy\int_{-2\sin|\theta|}^{2\sin\frac{\delta}{2}-2\sin|\theta|} \frac{1}{|x|^{1/2}}((x+2\sin|\theta|)^4+y^2)^{-\frac14}dx\\
&\le C\int_0^\infty e^{-\rho y}dy\int_{-2\sin|\theta|}^{2\sin\frac{\delta}{2}} \frac{1}{|x|^{1/2}}((x+2\sin|\theta|)^4+y^2)^{-\frac14}dx\\
&\le C\int_0^\infty e^{-\rho y}dy\Big(\int_0^{\sqrt{2\sin\frac{\delta}{2}}}(x^8+y^2)^{-\frac14}dx+\int_0^{\sqrt{2\sin|\theta|}}((x^2-2\sin|\theta|)^4+y^2)^{-\frac14}dx\Big)\\
&\le C\int_0^\infty e^{-\rho y}dy\Big(y^{-\frac14}\int_0^\infty(x^8+1)^{-\frac14}dx+y^{-\frac14}\int_0^{\sqrt{2\sin|\theta|}}(2\sin|\theta|-x^2)^{-\frac12}dx\Big)\\
&\le C\int_0^\infty y^{-\frac14}e^{-\rho y}dy\Big(\int_0^\infty(x^8+1)^{-\frac14}dx+\int_0^1(1-x^2)^{-\frac12}dx\Big)\le C\frac{1}{\rho^{3/4}}
\end{aligned}
$$

Using the same agrument , we can also carry out

$$
|I_{12}-\frac12\sqrt{\frac{2\pi}{\rho}}f(0)e^{\mathbf{i}\rho-\mathbf{i}\frac{\pi}{4}}|\le C\frac{1}{\rho^{3/4}} \tag{5.9}
$$

It remains to estimate I_2. Note that there exits $m>0$ such that $|\sin t|\ge m$ for any $t\in[-\pi/2,\pi/2]\backslash(-\delta/2,\delta/2)$. Upon integration by parts and representation (5.2) again, we have

$$
\begin{aligned}
|I_{12}| &\le C\rho^{-1}(1+\Big|\int_{[-\pi/2,\pi/2]\backslash(-\delta/2,\delta/2)}\frac{\partial(f(t)(1-\chi_\delta(t)))}{\partial t}\frac{1}{\sin t}dt\Big|)\\
&\le C\rho^{-1}(1+\int_{-\pi/2}^{\pi/2}\Big|\frac{\partial(f(t)(1-\chi_\delta(t)))}{\partial t}\Big|dt)\\
&\le C\rho^{-1}(1+\int_{-\pi/2}^{\pi/2}|(\kappa^2-\sin^2(t+\phi))^{-1/2}|dt)\\
&\le C\rho^{-1}(1+\int_{-\pi/2}^{\pi/2}|(\kappa^2-\sin^2 t)^{-1/2}|dt)\\
&\le C\rho^{-1}
\end{aligned}
$$

This completes the proof. □

References

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