1. Estimate of Dirichlet Green Tensor

We need the following slight generalization of Van der Corput lemma for the oscillatory integral [2, P.152].

Lemma 1.1 *Let $-\infty < a < b < \infty$, and u is a C^k function u in (a,b).*
1. If $|u'(t)| \geq 1$ for $t \in (a,b)$ and u' is monotone in (a,b), then for any $\phi(t)$ in (a,b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 3\lambda^{-1}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

2. For all $k \geq 2$, if $|u^{(k)}(t)| \geq 1$ for $t \in (a,b)$, then for any $\phi(t)$ in (a,b) with integrable derivatives

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 12k\lambda^{-1/k}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

Proof. The assertion can be proved by extending the Van der Corptut lemma in [2]. Here we omit the details. □

We recall following lemma, see e.g. [3]:

Lemma 1.2 *Let $F(\lambda,a) = \int_0^a t^{\alpha-1}f(t)e^{\mathbf{i}\lambda t}dt$ where $0 < a \leq +\infty$, $0 < \alpha < 1$ and $t^{\alpha-1}f \in L^1(0,a)$, then we have*

$$|F(\lambda,a)| \leq C(\frac{1}{\lambda^\alpha}f(0) + \frac{1}{\lambda}(a^{\alpha-1}f(a) + |t^{\alpha-1}f|_{L^1(0,a)}) \tag{1.1}$$

Lemma 1.3 *Let $f(\xi,\mu_s,\mu_p) = g(\xi,\mu_s,\mu_p)/\gamma(\xi,\mu_s,\mu_p)$ where g(x,y,z) is a homogeneous quadratic polynomial with respect to x,y,z. Let $a,b > 0$ and $\rho = \sqrt{a^2+b^2}$. Assume $\kappa = k_p/k_s$ and $k_s\rho > 1$, then we have*

$$\begin{aligned}&\Big|\int_{\mathbb{R}} f(\xi,\mu_s,\mu_p)e^{\mathbf{i}(\mu_s b+\xi a)}d\xi - f_{\xi=\frac{k_s a}{\rho}}\frac{k_s b}{\rho}(\frac{2\pi}{k_s\rho})^{1/2}e^{\mathbf{i}(k_s\rho-\frac{\pi}{4})}\Big| \\ &\leq C(\frac{k_s b}{\rho(k_s\rho)^{3/4}} + \frac{k_s a}{\rho(k_s\rho)^{5/4}})\end{aligned} \tag{1.2}$$

where C is only dependent on κ.

Proof. Let $I(a,b) = \int_{\mathbb{R}} f(\xi,\mu_s,\mu_p)e^{\mathbf{i}(\mu_s b+\xi a)}d\xi$. To simplify the integral, the standard substitution $\xi = k_s \sin t$ is made, taking the ξ-plane to a strip $-\pi/2 < \mathrm{Re}\, t < \pi/2$ in the t-plane, and the real axis in the ξ-plane onto the path L from $-\pi/2 + \mathbf{i}\infty \to -\pi/2 \to \pi/2 \to \pi/2 - \mathbf{i}\infty$ in the t-plane. Then $I(a,b) := I(\rho,\phi)$ becomes(Let a$=\rho\sin\phi$ and b$=\rho\cos\phi$, $0 < \phi < \pi/2$)

$$k_s\int_L f(\sin t, \cos t, (\kappa^2 - \sin^2 t)^{1/2})\cos t\; e^{\mathbf{i}k_s\rho(\cos(t-\phi))}dt \tag{1.3}$$

Taking the shift transformation of t and using cauchy integral theorem, we can obtain the representation of I(a,b):

$$k_s\int_L f(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2})\cos(t+\phi)e^{\mathbf{i}k_s\rho(\cos t)}dt$$

$$= k_s \cos\phi \int_L f(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2}) \cos t \; e^{\mathbf{i}k_s\rho(\cos t)} dt$$
$$- k_s \sin\phi \int_L f(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2}) \sin t \; e^{\mathbf{i}k_s\rho(\cos t)} dt$$
$$:= k_s(\cos\phi \; I_1 + \sin\phi \; I_2)$$

For I_2, using integration by parts on path L first, we have

$$I_2 = \frac{1}{\mathbf{i}k_s\rho} \int_L f(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2}) d \; e^{\mathbf{i}(k_s\rho\cos t)} \tag{1.4}$$
$$= -\frac{1}{\mathbf{i}k_s\rho} \int_L \frac{\partial f(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2})}{\partial t} \; e^{\mathbf{i}(k_s\rho\cos t)} dt \tag{1.5}$$
$$:= \frac{1}{k_s\rho} I_3 \tag{1.6}$$

Therefore,

$$I(\rho, \phi) = k_s(\cos\phi I_1 + \frac{\sin\phi}{k_s\rho} I_3) \tag{1.7}$$

First, we define $0 < \phi_1 < \phi_\kappa < \phi_2 < \pi/2$ such that $\sin(\phi_\kappa) = \kappa, \sin(\phi_1) < \kappa/2, \sin(\phi_2) > (1+\kappa)/2$ and $\cos(\phi_1) > (1+\kappa)/2, \cos(\phi_2) < \kappa/2$. Now, we claim that

1. If $\phi \in (0, \phi_1) \cup (\phi_2, \pi/2)$, we have

$$\left| I_1 - f(\sin\phi, \cos\phi, (\kappa^2 - \sin^2\phi)^{1/2}) (\frac{2\pi}{k_s\rho})^{1/2} e^{\mathbf{i}(k_s\rho - \frac{\pi}{4})} \right| \le C \frac{1}{k_s\rho} \tag{1.8}$$
$$|I_3| \le C \frac{1}{(k_s\rho)^{1/2}} \tag{1.9}$$

where C is independant of ϕ.

2. If $\phi \in [\phi_1, \phi_2]$ and $\phi \ne \phi_\kappa$, we have

$$\left| I_1 - f(\sin\phi, \cos\phi, (\kappa^2 - \sin^2\phi)^{1/2}) (\frac{2\pi}{k_s\rho})^{1/2} e^{\mathbf{i}(k_s\rho - \frac{\pi}{4})} \right| \le C(\phi) \frac{1}{k_s\rho} \tag{1.10}$$
$$|I_3| \le C(\phi) \frac{1}{(k_s\rho)^{1/2}} \tag{1.11}$$

3. If $\phi = \phi_\kappa$, we have

$$\left| I_1 - f(\sin\phi, \cos\phi, 0) (\frac{2\pi}{k_s\rho})^{1/2} e^{\mathbf{i}(k_s\rho - \frac{\pi}{4})} \right| \le C \frac{1}{(k_s\rho)^{3/4}} \tag{1.12}$$
$$|I_3| \le C(\phi) \frac{1}{(k_s\rho)^{1/4}} \tag{1.13}$$

Since $I(\rho, \phi)$ is a continuous function, we can obtain estimate (1.2) soon by the claim and equality (1.7). We now proceed with the proof of the claim above.

1. For the claim 1, wo only give the proof when $\phi \in (\phi_2, \pi/2)$ since the similar proof can be adjusted to the other case. By the convention of ϕ_2, for any $\phi \in (\phi_2, \pi/2)$, there exists $0 < \delta < \pi/4$ only dependent on κ such that

$$|\sin(t+\phi)| > (1+2\kappa)/3, |\cos(t+\phi)| < 2\kappa/3 \tag{1.14}$$

for any $t \in (-\delta, \delta)$ while

$$|\cos(t+\phi)| > (1+2\kappa)/3, |\sin(t+\phi)| < 2\kappa/3 \tag{1.15}$$

for any $t \in (-\pi/2, -\pi/2+\delta) \cup (\pi/2-\delta, \pi/2)$. Let $\chi_\delta \in C_0^\infty(-\pi/2, \pi/2)$ be the cut-off function with that $0 \le \chi_\delta \le 1$, $\chi_\delta = 1$ in $(-\delta/2, \delta/2)$ and $\chi_\delta = 0$ in $L\backslash(-\delta, \delta)$. Then we can divide I_1 into two parts such that

$$\begin{aligned} I_1 &= \int_L f(t)\cos t e^{\mathbf{i}k_s\rho\cos t}dt \\ &= \int_{\mathbb{R}} f(t)\cos t\chi_\delta(t) e^{\mathbf{i}k_s\rho\cos t}dt + \int_{L'} f(t)\cos t(1-\chi_\delta(t)) e^{\mathbf{i}k_s\rho\cos t}dt \\ &=: I_{11} + I_{12} \end{aligned} \tag{1.16}$$

where $L' = L\backslash(-\delta/2, \delta/2)$ and $f(t) := f(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2})$. Let $g_\delta(t) = f(t)\cos t\chi_\delta(t)$ and subtitating $t(s) = 2\arcsin s/2$ for t in I_{11}, we have

$$I_{11} = \int_{\mathbb{R}} g_\delta(t(s)) \frac{1}{\sqrt{1-s^2/4}} e^{\mathbf{i}k_s\rho} e^{-\mathbf{i}k_s\rho s^2/2} ds \tag{1.17}$$

Let $h_\delta(s) = g_\delta(t(s))\frac{1}{\sqrt{1-s^2/4}}$. It is easy to see that $h_\delta(s) \in C_0^\infty(-2\sin\delta/2, 2\sin\delta/2)$. By the lemma of the stationary phase for quadratic term in [1], we have

$$I_{11} = e^{\mathbf{i}k_s\rho} \int_{\mathbb{R}} h_\delta(s) e^{-\mathbf{i}\frac{k_s\rho}{2}s^2} ds = e^{\mathbf{i}k_s\rho} \int_{\mathbb{R}} \widehat{h_\delta}(y)\alpha(-y)dy \tag{1.18}$$

where

$$\alpha(y) = (\frac{1}{2\pi k_s\rho})^{1/2} e^{-\mathbf{i}\pi/4} e^{\frac{\mathbf{i}}{2k_s\rho}y^2} \tag{1.19}$$

$$= (\frac{1}{2\pi k_s\rho})^{1/2} e^{-\mathbf{i}\pi/4} (1 + O(\frac{y^2}{k_s\rho})) \tag{1.20}$$

Consequently

$$I_{11} = (\frac{1}{2\pi k_s\rho})^{1/2} e^{\mathbf{i}k_s\rho - \mathbf{i}\pi/4} \int_{\mathbb{R}} \widehat{h_\delta}(y)(1 + \frac{1}{k_s\rho}O(y^2))dy \tag{1.21}$$

But $\int_{\mathbb{R}} \widehat{h_\delta}(y)dy = 2\pi h_\delta(0)$ and $|\int_{\mathbb{R}} \widehat{h_\delta}(y)y^2dy| < C$ since $|\widehat{h_\delta}(y)| < C_1$ and $|\widehat{h_\delta}(y)| < C_2/y^4$ where C, C_1, C_2 is independent of ϕ. It turns to estimate I_{12}. Using integration by parts, we obtain

$$|I_{12}| = \left| \frac{1}{k_s\rho} \int_{L'} (f(t)\cos t(1-\chi_\delta(t)/\sin t)' e^{\mathbf{i}k_s\rho\cos t}dt \right| \tag{1.22}$$

$$\le \frac{1}{k_s\rho} \Bigg(\int_{L\backslash(-\frac{\pi}{2}, \frac{\pi}{2})} |(f(t)\cos t(1-\chi_\delta(t)/\sin t)'| e^{\mathbf{i}\cos t}dt \tag{1.23}$$

$$+ \int_{(-\frac{\pi}{2}, \frac{\pi}{2})\backslash(-\frac{\delta}{2}, \frac{\delta}{2})} |(f(t)\cos t(1-\chi_\delta(t)/\sin t)'|dt \Bigg) \tag{1.24}$$

$$\le C\frac{1}{k_s\rho} \tag{1.25}$$

Then inequality (1.8) follows.

For I_3, we split the integral path L into $L_1 = (-\pi/2, \pi/2)$ and $L_2 = (-\pi/2 + \mathbf{i}\infty, -\pi/2) \cup (\pi/2, \pi/2 - \mathbf{i}\infty)$, then we have corresponding representation: $I_3 = I_{31} + I_{32}$.

Then $|I_{32}| \leq C/(k_s\rho)$ can be proved by the same method used above. Following a tedious computation, we obtain a simple form of $\partial f/\partial t$:

$$\frac{\partial f}{\partial t} = \frac{(\gamma\partial_t g - g\partial_t\gamma)(\kappa^2 - \sin^2 t)^{1/2}}{(\sin^2 t + \cos t(\kappa^2 - \sin^2 t)^{1/2})^2}\frac{1}{(\kappa^2 - \sin^2 t)^{1/2}} \tag{1.26}$$

$$:= \frac{h(\sin(t+\phi), \cos(t+\phi), (\kappa^2 - \sin^2(t+\phi))^{1/2})}{(\kappa^2 - \sin^2 t)^{1/2}} \tag{1.27}$$

where h and $\partial h/\partial t$ are integrable on path L_1. Let define $t_1, t_2 \in \chi_1 = (-\pi/2 + \delta, -\delta) \cup (\delta, \pi/2 - \delta)$ which satisfy $\kappa^2 = \sin^2(t_i + \phi)$, $i = 1, 2$. Moreover, for any $0 < \lambda_1 < 1$ and $1 < \lambda_2 < 1/\kappa$, there exists $\sigma > 0$, which satisfy that $\chi_2 = (t_1 - \sigma, t_1 + \sigma) \cup (t_2 - \sigma, t_2 + \sigma) \subset \chi_1$ and is only dependent on $\lambda_1, \lambda_2, \kappa$, such that

$$\lambda_1\kappa < |\sin(t+\phi)| < \lambda_2\kappa. \tag{1.28}$$

for any $t \in \chi_2$. We are now in a position to estimate I_{21}. Similarly, we split the path L_1 into χ_2 and $L_1\backslash\chi_2$, then we have the corresponding representation: $I_{21} = I_{\chi_2} + I_{L_1\backslash\chi_2}$.

For I_{χ_2}, we only analysis the integral on $\chi_{21} = (t_1 - \sigma, t_1 + \sigma)$ denoted by $I^1_{\chi_2}$, the procedure of the another is same. Without loss of generality, we assume that $\sin(t_1 - \sigma + \phi) < \kappa < \sin(t_1 + \sigma + \phi)$. It is easy to see that $\sin(t+\phi)$ is monotonic increasing in χ_{21}. Let $\sin(t+\phi) = \kappa\sin\theta$ and the implicit mapping from θ to t is denoted by $t(\theta)$ while the inverse mapping by $\theta(t)$, taking the interval χ_{21} onto $L_\theta : \theta_1 \to \pi/2 \to \pi/2 - \mathbf{i}\theta_2$ where $\sin(t_1 - \sigma + \phi) = \kappa\sin\theta_1, \sin(t_1 + \sigma + \phi) = \kappa\sin(\pi/2 - \mathbf{i}\theta_2)$. By substituting $t(\theta)$ into $I^1_{\chi_2}$, we have

$$I^1_{\chi_2} = \int_{L_\theta} \frac{h(\kappa\sin\theta, (1 - \kappa^2\sin^2\theta)^{1/2}, \kappa\cos\theta)}{(1 - \kappa^2\sin^2\theta)^{1/2}}\, e^{\mathbf{i}k_s\rho(\cos(t(\theta)))} d\theta \tag{1.29}$$

Because of inequality 1.28, we assert that h and $\partial h/\partial\theta$ are integrable on the path L_θ. A simple computation show that

$$\frac{dt(\theta)}{d\theta} = \frac{\kappa\cos\theta}{\cos(t+\phi)} \quad \frac{d^2t(\theta)}{dt^2} = \frac{\kappa^2\cos^2\theta\sin(t+\phi) - \kappa\sin\theta\cos^2(t+\phi)}{\cos^3(t+\phi)}$$

Then we can obtain

$$\begin{aligned}
\frac{d\cos t}{d\theta} &= \frac{-\kappa\sin t\cos\theta}{\cos(t+\phi)} \\
\frac{d^2\cos t}{d\theta^2} &= \frac{d^2\cos t}{dt^2}\left(\frac{dt}{d\theta}\right)^2 + \frac{d\cos t}{dt}\frac{d^2t}{d\theta^2} \\
&= \frac{-\kappa^2\cos^2\theta\cos t}{\cos^2(t+\phi)} + \frac{\kappa\sin\theta\cos^2(t+\phi)\sin t - \kappa^2\cos^2\theta\sin(t+\phi)\sin t}{\cos^3(t+\phi)} \\
&= \frac{-\kappa^2\cos^2\theta\cos\phi + \kappa\sin\theta\cos^2(t+\phi)\sin t}{\cos^3(t+\phi)} \\
&= \frac{(\sin^2(t+\phi) - \kappa^2)\cos\phi + \cos^2(t+\phi)\sin(t+\phi)\sin t}{\cos^3(t+\phi)}
\end{aligned}$$

It is simple to see that $\theta = \pi/2$ is the only stationary point of $\cos(t(\theta))$ and we can obtain

$$\left|\frac{d^2\cos t}{d\theta^2}(\pi/2)\right| = \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}|\sin t| > \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}\sin\delta \tag{1.30}$$

Therefore, we can choose appropriate λ_1, λ_2, only dependent on κ, such that $|\frac{d^2\cos t}{d\theta^2}| > \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}\sin\delta$ for any $\theta \in \theta(\chi_{21})$. Therefore, we can decompose $\theta(\chi_{21})$ into several intervals such that in each either $|\partial\cos(t(\theta))/\partial\theta|$ or $|\partial^2\cos(t(\theta))/\partial\theta^2|$ has positive lower bound and $\partial\cos(t(\theta))/\partial\theta$ is monotonous. Since the amplitude function of integrand in $I^1_{\chi_2}$ and its derivative with respect to θ are both integrable on L_θ, the estimation $|I^1_{\chi_2}| \le C/(k_s\rho)^{1/2}$ can be obtained immediately by lemma 1.1. Then the estimate $|I_{L_1\setminus\chi_2}| \le C/(k_s\rho)^{1/2}$ also follows lemma 1.1. This completes the proof of the claim 1.

2. For the claim 2, since $\phi \neq \phi_\kappa$ we always can find some δ small enough such that $\sin^2(t+\phi) \neq \kappa^2$ for any $t \in (-\delta,\delta)$ .Thus, the proof is similar to the claim 1, here we omit the details.

3. To prove the claim 3, observe that 0 and ϕ' are the only two movable singular points of $f'(t)$ for on L where $\sin^2(\phi'+\phi_\kappa) = \kappa^2$ and $\phi' \neq 0$. However, we can not use stationary phase lemma directly because the fourth derivatives of amplitude function has singularity when $t = 0$. Let $0 < \delta < \pi/4$ such that $\sin(t+\phi_\kappa)$ is monotonic as $t \in (-\delta,\delta)$ and $2\kappa/3 \le |\sin(\pm\delta+\phi_\kappa)| \le (1+2\kappa)/3$, then we have $I_1 = I_{11} + I_{12}$, $I_3 = I_{31} + I_{32}$ similar to (1.16). Using the same argument as in the proof of the claim 1, we can easily obtain

$$|I_{12}| \le C\frac{1}{k_s\rho} \qquad |I_{32}| \le C\frac{1}{(k_s\rho)^{1/2}} \tag{1.31}$$

Obseve that $f(t)$ can be always represented as

$$f(t) = \frac{g(t)}{\sin^2(t+\phi_\kappa) + \cos(t+\phi_\kappa)(\kappa^2 - \sin^2(t+\phi_\kappa))^{1/2}} \tag{1.32}$$

$$= \frac{g(t)(\sin^2(t+\phi_\kappa) - \cos(t+\phi_\kappa)(\kappa^2 - \sin^2(t+\phi_\kappa))^{1/2})}{(1+\kappa^2)\sin^2(t+\phi_\kappa) - \kappa^2} \tag{1.33}$$

$$= f_1(t) + f_2(t)(\sin^2\phi_\kappa - \sin^2(t+\phi_\kappa))^{1/2} \tag{1.34}$$

$$= f_1(t) + \mathbf{i}f_2(t)(\sin\phi_\kappa + \sin(t+\phi_\kappa))^{1/2}\cos^{1/2}(t/2+\phi_\kappa)(2\sin\frac{t}{2})^{1/2} \tag{1.35}$$

$$= f_1(t) + g_1(t)(2\sin\frac{t}{2})^{1/2} \tag{1.36}$$

where $f_1, f_2, g_1 \in C^\infty(-\delta,\delta)$. It follows that

$$\begin{aligned} I_{11} &= \int_{\mathbb{R}} f_1(t)\cos t\chi_\delta(t)e^{\mathbf{i}k_s\rho\cos t}dt \\ &+ \int_{\mathbb{R}} g_1(t)\cos t(2\sin t/2)^{1/2}\chi_\delta(t)e^{\mathbf{i}k_s\rho\cos t}dt \\ &:= I_{111} + I_{112} \end{aligned} \tag{1.37}$$

$$\begin{aligned} I_{31} &= \int_{\mathbb{R}} f_1'(t)\chi_\delta(t)e^{\mathbf{i}k_s\rho\cos t}dt \\ &+ \int_{\mathbb{R}} g_1'(t)(2\sin t/2)^{1/2}\chi_\delta(t)e^{\mathbf{i}k_s\rho\cos t}dt \\ &+ \int_{\mathbb{R}} 1/2g_1(t)\cos t(2\sin t/2)^{-1/2}\chi_\delta(t)e^{\mathbf{i}k_s\rho\cos t} \\ &:= I_{311} + I_{312} + I_{313} \end{aligned} \tag{1.38}$$

Subtitating $t(s) = 2\arcsin s/2$ for t in I_{112}, I_{312}, and I_{313} and by lemma (1.2), we have

$$|I_{112}| \leq C\frac{1}{(k_s\rho)^{3/4}}, |I_{312}| \leq C\frac{1}{(k_s\rho)^{3/4}}, |I_{313}| \leq C\frac{1}{(k_s\rho)^{1/4}} \tag{1.39}$$

By lemma (1.1), we have

$$|I_{311}| \leq C\frac{1}{(k_s\rho)^{1/2}} \tag{1.40}$$

Finally, the claim 3 is a direct cosequence of using stationary phase theorem for I_{111}. This completes the proof. □

Lemma 1.4 *Assume that* $0 < \kappa := \sin\phi_\kappa < 1, 0 < \phi_\kappa < \pi/2$, $0 < \phi < \pi/2$ *and* $-\pi/2 < \phi_1 < \phi_2 < \pi/2$ *satisfy that* $\kappa^2 = \sin^2(\phi + \phi_1) = \sin^2(\phi + \phi_2)$. *Let* $f(\theta)$*:*

$$f(\theta) := F(\sin(\theta + \phi), \cos(\theta + \phi), (\kappa^2 - \sin^2(\theta + \phi))^{1/2}) \tag{1.41}$$

be a function in $C^4((-\pi/2, \pi/2)\backslash\{\phi_1, \phi_2\})$. *Moreover, there exits* $\delta >$ *such that* $f(\theta)$ *can be represented as*

$$f(\theta) = g_1(\theta) + g_2(\theta)((\kappa^2 - \sin^2(\theta + \phi))^{1/2})) \tag{1.42}$$

where $g_1, g_2 \in C^4(\bigcup_{i=1,2}(\phi_i - \delta, \phi_i + \delta))$. *Then for any* $\rho > 1$, *we have*

$$\begin{aligned} &\left|I(\rho) := \int_{-\pi/2}^{\pi/2} f(\theta)\cos(\theta + \phi)e^{\mathbf{i}\rho\cos\theta}d\theta - \left(\frac{2\pi}{\rho}\right)^{1/2} f(0)\cos\phi e^{\mathbf{i}\rho - \mathbf{i}\pi/4}\right| \\ &\leq C(\cos\phi\frac{1}{\rho^{3/4}} + \sin\phi\frac{1}{\rho^{5/4}}) \end{aligned} \tag{1.43}$$

Proof. The proof will be split into two parts about whether ϕ equal to ϕ_κ.

$$f'(\theta) = g_1'(\theta) + (g_2'(\theta)(\kappa^2 - \sin^2(\theta + \phi)) - g_2(\theta))\frac{1}{(\kappa^2 - \sin^2(\theta + \phi))^{1/2}} \tag{1.44}$$

$$:= h_1(\theta) + h_2(\theta)\frac{1}{(\kappa^2 - \sin^2(\theta + \phi))^{1/2}} \tag{1.45}$$

□

2. Some draft about Green Tensor Analysis

Let substitute $\xi = k\sin\theta$ into integral and shift the variable, we have

$$I(y) = \int_{\mathbb{R}} f(\xi)e^{\mathbf{i}\xi y_1 + \mu(\xi)y_2}d\xi = \int_{\mathbb{R}} f(\xi)e^{\mathbf{i}\xi(y_1 - z_1) + \mu(\xi)(y_2 - z_2)}e^{\mathbf{i}\xi z_1 + \mu(\xi)z_2}d\xi \tag{2.1}$$

$$= k\int_L f(k\sin\theta)\cos\theta e^{\mathbf{i}k|y-z|\cos(\theta-\eta)}e^{\mathbf{i}|z|\cos(\theta-\phi)}d\theta \tag{2.2}$$

$$= k\int_{L_\phi} f(k\sin(\theta + \phi))\cos(\theta + \phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \tag{2.3}$$

$$= k\int_L f(k\sin(\theta + \phi))\cos(\theta + \phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \tag{2.4}$$

where $y_1, y_2 > 0, \sin\phi = \frac{z_1}{|z|}, \cos\phi = \frac{z_2}{|z|}, 0 < \phi < \pi/2$ and $\sin\eta = \frac{y_1-z_1}{|y-z|}, \cos\eta = \frac{y_2-z_2}{|y-z|}, 0 < \eta < \pi$. It is easy to see that $\phi + \eta < \pi$. Roughly, using stationary phase lemma, we obtain:

$$I(y) = f(k\sin\phi)k\cos\phi e^{\mathbf{i}k|y-z|\cos(\phi-\eta)}(\frac{2\pi}{|z|})^{1/2}e^{\mathbf{i}|z|-\mathbf{i}\frac{\pi}{4}}(1+O(\frac{1}{|z|})) \tag{2.5}$$

$$\cos(a+\mathbf{i}b) = \frac{e^b+e^{-b}}{2}\cos a + \mathbf{i}\frac{e^{-b}-e^b}{2}\sin a \tag{2.6}$$

$$\sin(a+\mathbf{i}b) = \frac{e^b+e^{-b}}{2}\sin a + \mathbf{i}\frac{e^b-e^{-b}}{2}\cos a \tag{2.7}$$

When $\theta \in (-a-\pi/2, -a-\pi/2+\mathbf{i}\infty)$, let $\theta = -a-\pi/2\mathbf{i}t$, where $t > 0, 0 \le a \le \phi$, then

$$\begin{aligned}
&-\mathrm{Im}\,(|z|\cos\theta + |y-z|\cos(\theta+\phi-\eta)) \\
&= |z|\sin(a+\pi/2) + |y-z|\sin(a+\pi/2-\phi+\eta) && (2.8)\\
&= |z|\cos a + |y-z|\cos(a-\phi+\eta) && (2.9)\\
&= |z|\cos a + \cos a|y-z|(\cos\phi\cos\eta + \sin\phi\sin\eta) && (2.10)\\
&+ \sin a|y-z|(\sin\phi\cos\eta - \cos\phi\sin\eta) && (2.11)\\
&= |z|\cos a + \cos a((y_2-z_2)\cos\phi + (y_1-z_1)\sin\phi) && (2.12)\\
&+ \sin a((y_2-z_2)\sin\phi - (y_1-z_1)\cos\phi) && (2.13)\\
&= y_1\sin(\phi-a) + y_2\cos(\phi-a) > 0 && (2.14)
\end{aligned}$$

Now, Using Cauchy Integral Theorem, we have

$$I(y) = k\int_L f(k\sin(\theta+\phi))\cos(\theta+\phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \tag{2.15}$$

Let $L_1 = (-\pi/2, -\pi/2+\mathbf{i}\infty)$ and $\theta = -\pi/2+\mathbf{i}t, t > 0$, then

$$\begin{aligned}
I_1(y) &= k\int_{L_1} f(k\sin(\theta+\phi))\cos(\theta+\phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta && (2.16)\\
&= && (2.17)
\end{aligned}$$

$$\begin{aligned}
&I(y) = f(k\sin\phi)k\cos\phi e^{\mathbf{i}k|y-z|\cos(\phi-\eta)}(\frac{2\pi}{|z|})^{1/2}e^{\mathbf{i}|z|-\mathbf{i}\frac{\pi}{4}} && (2.18)\\
&+\frac{kz_2}{|z|}O(\left(\frac{1}{k|z|}\right)^{3/4} + \frac{1}{k|y|}) + \frac{kz_1}{|z|}O(\left(\frac{1}{k|z|}\right)^{5/4} + \left(\frac{1}{k|y|}\right)^2) && (2.19)
\end{aligned}$$

It is easy to see

$$\int_{-d}^{d}\frac{k}{(k|x-z|)^\alpha}\frac{1}{(k|x-y|)^\beta}dx_1 \le C(\frac{1}{(kz_2)^{\alpha+\beta-1}} + \frac{1}{(ky_2)^{\alpha+\beta-1}}) \tag{2.20}$$

where $z, y \in \mathbb{R}^2_+$, $x \in \Gamma_0$ and $\alpha + \beta > 0$.

$$e^{\mathbf{i}\mu y_2 + \mathbf{i}\xi(x_1-y_1)} = e^{\mathbf{i}\mu y_2 - \mathbf{i}y_2/\tan\phi} = e^{\mathbf{i}y_2(\mu-\xi/\tan\phi)} \tag{2.21}$$

Another method

$$\int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi)e^{\mathbf{i}k|x-y|\cos\theta} \tag{2.22}$$

$$= \int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi)e^{\mathbf{i}k|x-y|\cos(\theta+\psi-\psi)} \tag{2.23}$$

$$= \int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi)e^{\mathbf{i}ky_2\cos(\theta+\psi)+\mathbf{i}k|x_1-y_1|\sin(\theta+\psi)} \tag{2.24}$$

$$= \int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi) \tag{2.25}$$

$$e^{\mathbf{i}k(y_2-z_2)\cos(\theta+\psi)+\mathbf{i}k(|x_1-y_1|-|x_1-z_1|)\sin(\theta+\psi)+\mathbf{i}k|z|\cos(\theta+\psi-\phi)} \tag{2.26}$$

3. Finite Aperture Point Spread Function

If $x \in \Gamma_0$ and $z, y \in \mathbb{R}^2_+$, by lemma (1.3) we have

$$\begin{aligned} G(x,y) &= \frac{\mathbf{i}k_s}{\mu\sqrt{2\pi}}\frac{1}{\delta(\xi)}\begin{pmatrix} \mu_s\beta & \xi\beta \\ 2\xi\mu_s\mu_p & 2\xi^2\mu_p \end{pmatrix}_{\xi=k_s\frac{x_1-y_1}{|x-y|}} \frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{1/2}}e^{\mathbf{i}k_s|x-y|-\mathbf{i}\frac{\pi}{4}} \\ &+\frac{\mathbf{i}k_p}{\mu\sqrt{2\pi}}\frac{1}{\delta(\xi)}\begin{pmatrix} 2\xi^2\mu_s & -2\xi\mu_s\mu_p \\ -\xi\beta & \mu_p\beta \end{pmatrix}_{\xi=k_p\frac{x_1-y_1}{|x-y|}} \frac{y_2}{|x-y|}\frac{1}{(k_p|x-y|)^{1/2}}e^{\mathbf{i}k_p|x-y|-\mathbf{i}\frac{\pi}{4}} \\ &+O(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}}) \\ &:= \mathcal{G}_s(x,y)+\mathcal{G}_p(x,y)+O(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}}) \end{aligned} \tag{3.1}$$

$$\begin{aligned} T_D(x,z) &= \frac{k_s}{\sqrt{2\pi}}\frac{1}{\gamma(\xi)}\begin{pmatrix} \mu_s\mu_p & \xi\mu_p \\ \xi\mu_s & \xi^2 \end{pmatrix}_{\xi=k_s\frac{x_1-z_1}{|x-z|}} \frac{z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{1/2}}e^{\mathbf{i}k_s|x-z|-\mathbf{i}\frac{\pi}{4}} \\ &+\frac{k_p}{\sqrt{2\pi}}\frac{1}{\gamma(\xi)}\begin{pmatrix} \xi^2 & -\xi\mu_p \\ -\xi\mu_s & \mu_p\mu_s \end{pmatrix}_{\xi=k_p\frac{x_1-z_1}{|x-z|}} \frac{z_2}{|x-z|}\frac{1}{(k_p|x-z|)^{1/2}}e^{\mathbf{i}k_p|x-z|-\mathbf{i}\frac{\pi}{4}} \\ &+O(\frac{k_sz_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{k_s|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}}) \\ &:= \mathcal{T}_s(x,z)+\mathcal{T}_p(x,z)+O(\frac{k_sz_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{k_s|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}}) \end{aligned} \tag{3.2}$$

Now we consider the finite aperture point spread function $J_d(z,y)$:

$$\int_{-d}^{d}(T_D(x_1,0;z_1,z_2))^T\overline{G(x_1,0;y_1,y_2)}dx_1 \tag{3.3}$$

Recall following standard asymptotic expansion:

$$|x-y| = |x-z| + \widehat{x-z}\cdot(z-y) + O(\frac{|y-z|^2}{|x-z|}) \tag{3.4}$$

$$|y|^{-\alpha} = |z|^{-\alpha}(1+\frac{|y|-|z|}{|z|})^{-\alpha} = |z|^{-\alpha}(1+O(\frac{|y-z|}{|z|})) \tag{3.5}$$

$$e^{\mathbf{i}t} = 1+O(t) \tag{3.6}$$

$$|a^{1/2}-b^{1/2}| \le C\sqrt{|a-b|} \tag{3.7}$$

where $x, y, z \in \mathbb{R}^2$, $t, a, b \in \mathbb{R}$ and $\alpha > 0$.

Lemma 3.1 *For any* $z, y \in \mathbb{R}^2_+$, $J_d(z,y) = F(z,y) + O((1+\frac{|y-z|}{z_2})(\frac{1}{k_s z_2})^{1/4} + \frac{(k_s|y-z|)^2}{k_s z_2} + (\frac{|y-z|}{z_2})^{1/2})$, *where*

$$F(z,y) = -\frac{\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d} f_s(\theta)\begin{pmatrix} \sin^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \cos^2\theta \end{pmatrix} e^{\mathbf{i}k_s(z_1-y_1)\cos\theta + \mathbf{i}k_s(z_2-y_2)\sin\theta} d\theta \quad (3.8)$$

$$-\frac{\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d} f_p(\theta)\begin{pmatrix} \cos^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin^2\theta \end{pmatrix} e^{\mathbf{i}k_p(z_1-y_1)\cos\theta + \mathbf{i}k_p(z_2-y_2)\sin\theta} d\theta \quad (3.9)$$

and

$$f_s(\theta) = \frac{\sin\theta((\kappa^2-\cos^2\theta)^{1/2}(1-2\cos^2\theta) + 2\overline{(\kappa^2-\cos^2\theta)^{1/2}}\cos^2\theta)}{(\cos^2\theta + \sin\theta(\kappa^2-\cos^2\theta)^{1/2})\overline{((1-2\cos^2\theta)^2 + 4\cos^2\theta\sin\theta(\kappa^2-\cos\theta)^{1/2})}}$$

$$f_p(\theta) = \frac{\sin\theta(1/\kappa^2-\cos^2\theta)^{1/2}}{(\cos^2\theta + \sin\theta(1/\kappa^2-\cos^2\theta)^{1/2})((1/\kappa^2-2\cos^2\theta)^2 + 4\cos^2\theta\sin\theta(1/\kappa^2-\cos\theta)^{1/2})}$$

where $0 < \theta_1^d < \pi/2 < \theta_2^d < \pi$ *and* $z_2 = (d+z_1)\tan\theta_1^d = (z_1-d)\tan\theta_2^d$.

Proof.

$$\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}} + \frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}}$$
$$= (\frac{z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}} + \frac{|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}})(1+O(\frac{|y-z|}{|x-z|}))$$

$$|\mu_{\mathbf{i}}(k_j\frac{x_1-y_1}{|x-y|}) - \mu_{\mathbf{i}}(k_j\frac{x_1-z_1}{|x-z|})|$$
$$\leq Ck_j\sqrt{\left|\frac{x_1-y_1}{|x-y|} - \frac{x_1-z_1}{|x-z|}\right|} \leq Ck_j\left(\frac{|y-z|}{|x-z|}\right)^{1/2}$$

where $i, j = s, p$. By above, we can obtain

$$\mathcal{G}_s(x,y) = \mathcal{G}_s(x,z)e^{\mathbf{i}k_s\widehat{x-z}\cdot(z-y)} + O(\frac{(k_s|y-z|)^2}{(k_s|x-z|)^{3/2}}) + O(\frac{(k_s|y-z|)^{1/2}}{k_s|x-z|}) \quad (3.10)$$

$$\mathcal{G}_p(x,y) = \mathcal{G}_p(x,z)e^{\mathbf{i}k_p\widehat{x-z}\cdot(z-y)} + O(\frac{(k_p|y-z|)^2}{(k_p|x-z|)^{3/2}}) + O(\frac{(k_p|y-z|)^{1/2}}{k_p|x-z|}) \quad (3.11)$$

For $l > 1$, a simple computation show that

$$\int_{-d}^{d}\frac{k_s}{(k_s|x-z|)^l}dx_1 = \frac{1}{(k_s z_2)^{l-1}}\int_{\frac{-d-z_1}{z_2}}^{\frac{d-z_1}{z_2}}\frac{1}{(1+t^2)^{l/2}}dt \leq C\frac{1}{(k_s z_2)^{l-1}} \quad (3.12)$$

Let

$$\mathcal{G}_\alpha(x,y) = \frac{\mathbf{i}}{\sqrt{2\pi}\mu}g_\alpha(\frac{x_1-y_1}{|x-y|},\kappa)\frac{1}{(k_\alpha|x-y|)^{1/2}}e^{\mathbf{i}k_\alpha|x-y| - \mathbf{i}\frac{\pi}{4}} \quad (3.13)$$

$$\mathcal{T}_\alpha(x,y) = \frac{k_\alpha}{\sqrt{2\pi}}t_\alpha(\frac{x_1-z_1}{|x-z|},\kappa)\frac{1}{(k_s|x-z|)^{1/2}}e^{\mathbf{i}k_\alpha|x-z| - \mathbf{i}\frac{\pi}{4}} \quad (3.14)$$

where $\alpha = s, p$. Now, by substituting (3.10-3.11) into $J_d(z,y)$ and using inequality (3.12), we have

$$J_d(z,y) = \frac{-\mathbf{i}}{2\pi\mu}\int_{-d}^{d} t_s(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_s(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_s\widehat{x-z}\cdot(y-z)}}{|x-z|}$$

$$+t_p(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_p(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_p\widehat{x-z}\cdot(y-z)}}{|x-z|}dx_1 \tag{3.15}$$

$$-\frac{\mathbf{i}}{2\pi\mu}\int_{-d}^{d}t_p(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_s(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_s\widehat{x-z}\cdot(y-z)}}{|x-z|} \tag{3.16}$$

$$+t_s(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_p(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_p\widehat{x-z}\cdot(y-z)}}{|x-z|}dx_1 \tag{3.17}$$

$$+O((1+\frac{|y-z|}{z_2})(\frac{1}{k_sz_2})^{1/4}+\frac{(k_s|y-z|)^2}{k_sz_2}+(\frac{|y-z|}{z_2})^{1/2}) \tag{3.18}$$

$$:=F(z,y)+R(z,y) \tag{3.19}$$

$$+O((1+\frac{|y-z|}{z_2})(\frac{1}{k_sz_2})^{1/4}+\frac{(k_s|y-z|)^2}{k_sz_2}+(\frac{|y-z|}{z_2})^{1/2}) \tag{3.20}$$

We denote $\widehat{x-z}=x-z/|x-z|=(cos(\phi+\pi),\sin(\phi+\pi))$, then taking the substitution $x_1=z_1-z_2\cot\phi$, we obtain

$$F(z,y)=\frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d}A_s(\phi,\kappa)e^{\mathbf{i}k_s(z_1-y_1)\cos\phi+\mathbf{i}k_s(z_2-y_2)\sin\phi} \tag{3.21}$$

$$+\frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d}A_p(\phi,\kappa)e^{\mathbf{i}k_p(z_1-y_1)\cos\phi+\mathbf{i}k_p(z_2-y_2)\sin\phi} \tag{3.22}$$

$$R(z,y)=\frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d}B_s(\phi,\kappa)e^{\mathbf{i}k_s(z_1-y_1)\cos\phi+\mathbf{i}k_s(z_2-y_2)\sin\phi+(k_p-k_s)|x-z|} \tag{3.23}$$

$$+\frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d}B_p(\phi,\kappa)e^{\mathbf{i}k_p(z_1-y_1)\cos\phi+\mathbf{i}k_p(z_2-y_2)\sin\phi+(k_s-k_p)|x-z|} \tag{3.24}$$

It is easy to see that $|R(z,y)|\le C\frac{|z-y|}{z_2}$. □

Let

$$g(x_1)=\frac{1}{((x_1-z_1)^2+z_2^2)^{3/4}((x_1-y_1)^2+y_2^2)^{1/4}}$$
$$\phi(x_1)=((x_1-z_1)^2+z_2^2)^{1/2}-((x_1-y_1)^2+y_2^2)^{1/2}$$

Then, we have

$$g'(x_1)=-g(x_1)[\frac{3(x_1-z_1)}{2((x_1-z_1)^2+z_2^2)}+\frac{(x_1-u_1)}{2((x_1-y_1)^2+y_2^2)}]$$
$$\phi'(x_1)=\frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}}-\frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}}$$
$$=\frac{\frac{(x_1-z_1)^2}{(x_1-z_1)^2+z_2^2}-\frac{(x_1-y_1)^2}{(x_1-y_1)^2+y_2^2}}{\frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}}+\frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}}}$$
$$=\frac{(x_1-z_1)^2y_2^2-(x_1-y_1)^2z_2^2}{(\frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}}+\frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}})((x_1-z_1)^2+z_2^2)((x_1-y_1)^2+y_2^2)}$$
$$\phi''(x_1)=\frac{z_2^2}{((x_1-z_1)^2+z_2^2)^{3/2}}-\frac{y_2^2}{((x_1-y_1)^2+y_2^2)^{3/2}}$$

Using integration by parts, we can obtain

$$\int_{-d}^{d} g(x_1)e^{\mathbf{i}k\phi(x_1)}dx_1$$
$$= \frac{1}{\mathbf{i}k}(\frac{g(d)}{\phi'(d)}e^{\mathbf{i}k\phi(d)} - \frac{g(-d)}{\phi'(-d)}e^{\mathbf{i}k\phi(-d)}) - \frac{1}{\mathbf{i}k}\int_{-d}^{d}\frac{g'(x_1)}{\phi'(x_1)} - \frac{g(x_1)\phi''(x_1)}{(\phi'(x_1))^2}dx_1$$

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