## 1. Estimate of Dirichlet Green Tensor

We need the following slight generalization of Van der Corput lemma for the oscillatory integral [4, P.152].

**Lemma 1.1** *Let $-\infty < a < b < \infty$, and $u$ is a $C^k$ function $u$ in $(a, b)$.*
*1. If $|u'(t)| \geq 1$ for $t \in (a, b)$ and $u'$ is monotone in (a,b), then for any $\phi(t)$ in $(a, b)$ with integrable derivatives*

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 3\lambda^{-1}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

*2. For all $k \geq 2$, if $|u^{(k)}(t)| \geq 1$ for $t \in (a, b)$ , then for any $\phi(t)$ in $(a, b)$ with integrable derivatives*

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 12k\lambda^{-1/k}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

**Proof.** The assertion can be proved by extending the Van der Corptut lemma in [4]. Here we omit the details. □

We recall following lemma, see e.g. [6]:

**Lemma 1.2** *Let $F(\rho, a) = \int_0^a t^{\alpha-1} f(t)e^{-\mathbf{i}\rho t}dt$ where $0 < a \leq +\infty$, $0 < \alpha < 1$, $\rho > 0$ and $t^{\alpha-1}f \in L^1(0, a)$, then we have*

$$|F(\rho, a)| \leq C(\frac{1}{\rho^\alpha}f(0) + \frac{1}{\rho}(a^{\alpha-1}f(a) + |t^{\alpha-1}f|_{L^1(0,a)}) \tag{1.1}$$

**Proof.** Put

$$g_0(t) = t^{\alpha-1}e^{-\mathbf{i}\rho t} \tag{1.2}$$

and define

$$g_1(t) = -\int_t^{t-\mathbf{i}\infty} x^{\alpha-1}e^{-\mathbf{i}\rho x}dx \tag{1.3}$$

where the path of integration is the vertical line $x = t - \mathbf{i}y, y \geq 0$. It is easy to show that $g_1(t)' = g_0(t)$. Substituting $x = t - \mathbf{i}y$ into $g_1(t)$, we have

$$g_1(t) = \mathbf{i}\int_0^\infty (t - \mathbf{i}y)^{\alpha-1}e^{-\mathbf{i}\rho t}e^{-\rho y}dy \tag{1.4}$$

Upon integration by parts, we have

$$\begin{aligned}
F(\rho, a) = & \int_0^a f(t)dg_1(t) \\
&= e^{-\mathbf{i}\frac{\alpha\pi}{2}}f(0)\Gamma(\alpha)\frac{1}{\rho^\alpha} + f(a)g_1(a) - \int_0^a f'(t)g_1(t)dt \\
&= e^{-\mathbf{i}\frac{\alpha\pi}{2}}f(0)\Gamma(\alpha)\frac{1}{\rho^\alpha} - \mathbf{i}\int_0^\infty e^{-\rho y}dy\int_0^a f'(t)(t - \mathbf{i}y)^{\alpha-1}e^{-\mathbf{i}\rho t}dt
\end{aligned}$$

Let

$$h(y) = \int_0^a f'(t)(t - \mathbf{i}y)^{\alpha-1}e^{-\mathbf{i}\rho t}dt$$

and observe that

$$|h(y)| \leq \int_0^a |f'(t)|(t^2+y^2)^{\frac{\alpha-1}{2}}dt$$

□

**Lemma 1.3** *Let* $F(\rho,a)=\int_0^a t^{-1/2}f(t)e^{-\mathbf{i}\rho t}dt$ *where* $0<a\leq+\infty$ *and* $\rho>0$, *then we have*

$$|F(\rho,a)-e^{-\mathbf{i}\frac{\pi}{4}}f(0)\Gamma(1/2)\frac{1}{\rho^{1/2}}| \tag{1.5}$$

$$\leq C(\int_0^\infty e^{-\rho y}dy\int_0^a |f'(t)|(t^2+y^2)^{-\frac{1}{4}}dt+\frac{1}{\rho}a^{-1/2}f(a)) \tag{1.6}$$

**Proof.** Put

$$g_0(t)=t^{-1/2}e^{-\mathbf{i}\rho t} \tag{1.7}$$

and define

$$g_1(t)=-\int_t^{t-\mathbf{i}\infty} x^{-1/2}e^{-\mathbf{i}\rho x}dx \tag{1.8}$$

where the path of integration is the vertical line $x=t-\mathbf{i}y, y\geq 0$. It is easy to show that $g_1(t)=g_0(t)$. Substituting $x=t-\mathbf{i}y$ into $g_1(t)$, we have

$$g_1(t)=\mathbf{i}\int_0^\infty (t-\mathbf{i}y)^{-1/2}e^{-\mathbf{i}\rho t}e^{-\rho y}dy \tag{1.9}$$

Upon integration by parts, we have

$$\begin{aligned}F(\rho,a)=&\int_0^a f(t)dg_1(t)\\ &=e^{-\mathbf{i}\frac{\pi}{4}}f(0)\Gamma(1/2)\frac{1}{\rho^{1/2}}+f(a)g_1(a)-\int_0^a f'(t)g_1(t)dt\\ &=e^{-\mathbf{i}\frac{\pi}{4}}f(0)\Gamma(1/2)\frac{1}{\rho^{1/2}}+\mathbf{i}f(a)\int_0^\infty (a-\mathbf{i}y)^{-1/2}e^{-\mathbf{i}\rho t}e^{-\rho y}dy\\ &\quad-\mathbf{i}\int_0^\infty e^{-\rho y}dy\int_0^a f'(t)(t-\mathbf{i}y)^{-1/2}e^{-\mathbf{i}\rho t}dt\end{aligned}$$

Let

$$h(y)=\int_0^a f'(t)(t-\mathbf{i}y)^{-1/2}e^{-\mathbf{i}\rho t}dt$$

and observe that

$$|h(y)|\leq\int_0^a |f'(t)|(t^2+y^2)^{-\frac{1}{4}}dt$$

It is easy to see that

$$|g_1(a)|\leq a^{-1/2}\int_0^\infty e^{-\rho y}dy\leq C\frac{1}{\rho}$$

□

**Lemma 1.4** *Assume that* $0 < \kappa := \sin\phi_\kappa < 1, 0 < \phi_\kappa < \pi/2$, $0 \leq \phi \leq \pi/2$ *and* $-\pi/2 < t_1 < t_2 < \pi/2$ *satisfy that* $\kappa^2 = \sin^2(\phi + t_1) = \sin^2(\phi + t_2)$. *Let* $f(\theta)$*:*

$$f(t, \phi) := F(\sin(t + \phi), \cos(t + \phi), (\kappa^2 - \sin^2(t + \phi))^{1/2}) \tag{1.10}$$

*be a function in* $C^\infty(([-\pi/2, \pi/2]\backslash\{t_1, t_2\}) \times [0, \pi/2])$. *Moreover, there exits* $\epsilon > 0$ *such that* $f(\theta)$ *can be represented as*

$$f(t, \phi) = g_1(t, \phi) + g_2(t, \phi)(\kappa^2 - \sin^2(t + \phi))^{1/2})^{N/2} \tag{1.11}$$

*where* $g_1, g_2 \in C^\infty((\bigcup_{i=1,2} (t_i - \epsilon, t_i + \epsilon)) \times [0, \pi/2]))$ *and* $N = \pm 1$. *Then for any* $\rho \geq 1$, *we have*

$$\Big|I(\rho, \phi) := \int_{-\pi/2}^{\pi/2} f(\theta) e^{\mathbf{i}\rho\cos\theta} d\theta - \frac{N+1}{2}\Big(\frac{2\pi}{\rho}\Big)^{1/2} f(0) e^{\mathbf{i}\rho - \mathbf{i}\pi/4}\Big|$$

$$\leq C\frac{1}{\rho^{(2+N)/4}} \tag{1.12}$$

**Proof.** The proof will be split into two parts about whether $\phi$ equal to $\phi_\kappa$.

If $\phi \neq \phi_\kappa$, there exists $0 < \delta < \pi/4$ such that

$$|(\kappa^2 - \sin^2(t + \phi))^{1/2}| > \frac{1}{2}|(\kappa^2 - \sin^2\phi)^{1/2}| \tag{1.13}$$

for any $t \in (-\delta, \delta)$. Let $\chi_\delta \in C_0^\infty(-\pi/2, \pi/2)$ be the cut-off function with that $0 \leq \chi_\delta \leq 1$, $\chi_\delta = 1$ in $(-\delta/2, \delta/2)$ and $\chi_\delta = 0$ in $(-\pi/2, \pi/2)\backslash(-\delta, \delta)$. Then we can divide $I$ into two parts such that

$$I = \int_{-\delta}^{\delta} f(t)\chi_\delta(t) e^{\mathbf{i}\rho\cos t} dt + \int_{-\pi/2}^{\pi/2} f(t)(1 - \chi_\delta(t)) e^{\mathbf{i}\rho\cos t} dt$$

$$=: I_1 + I_2$$

Subtitating $t(s) = 2\arcsin s/2$ for t in $I_1$, we can obtain

$$I_1 = \int_{\mathbb{R}} f(t(s))\chi_\delta(t(s)) \frac{1}{\sqrt{1 - s^2/4}} e^{\mathbf{i}\rho} e^{-\mathbf{i}\rho s^2/2} ds \tag{1.14}$$

$$= \int_{\mathbb{R}} h_\delta(s) e^{\mathbf{i}\rho} e^{-\mathbf{i}\rho s^2/2} ds \tag{1.15}$$

It is easy to see that $h_\delta(s) \in C_0^4(\mathbb{R})$. By the lemma of the stationary phase for quadratic term in [3], we have

$$I_1 = e^{\mathbf{i}\rho} \int_{\mathbb{R}} h_\delta(s) e^{-\mathbf{i}\frac{\rho}{2}s^2} ds = e^{\mathbf{i}\rho} \int_{\mathbb{R}} \widehat{h_\delta}(y)\alpha(-y) dy \tag{1.16}$$

where

$$\alpha(y) = (\frac{1}{2\pi\rho})^{1/2} e^{-\mathbf{i}\pi/4} e^{\frac{\mathbf{i}}{2\rho}y^2} \tag{1.17}$$

$$= (\frac{1}{2\pi\rho})^{1/2} e^{-\mathbf{i}\pi/4}(1 + O(\frac{y^2}{\rho})) \tag{1.18}$$

Consequently

$$I_1 = (\frac{1}{2\pi\rho})^{1/2} e^{\mathbf{i}\rho - \mathbf{i}\pi/4} \int_{\mathbb{R}} \widehat{h_\delta}(y)(1 + \frac{1}{\rho}O(y^2)) dy \tag{1.19}$$

Moreover, $\int_{\mathbb{R}} \widehat{h_\delta}(y)dy = 2\pi h_\delta(0)$ and $|\int_{\mathbb{R}} \widehat{h_\delta}(y)y^2dy| < C$ since $\widehat{h_\delta}(y) = O(1/y^4)$. Now, it turns to estimate $I_2$.

When $N = 1$, using integration by parts, we have

$$|I_2| = \left| \int_{(-\frac{\pi}{2},\frac{\pi}{2})\backslash(-\frac{\delta}{2},\frac{\delta}{2})} f(t)(1-\chi_\delta(t))/\sin t \;\; de^{\mathbf{i}\rho\cos t} \right| \tag{1.20}$$

$$\tag{1.21}$$

$$\leq C\frac{1}{\rho} + \left| \int_{(-\frac{\pi}{2},\frac{\pi}{2})\backslash(-\frac{\delta}{2},\frac{\delta}{2})} (f(t)(1-\chi_\delta(t))/\sin t)' \; e^{\mathbf{i}\rho\cos t}dt \right| \tag{1.22}$$

$$\leq C\frac{1}{\rho} \tag{1.23}$$

From above analysis, we obtain

$$\left| I(\rho,\phi) - \left(\frac{2\pi}{\rho}\right)^{1/2} f(0)e^{\mathbf{i}\rho - \mathbf{i}\pi/4} \right| \leq C(\phi)\frac{1}{\rho} \tag{1.24}$$

When $N = -1$, we can not use integration by parts again since $f'(\theta)$ is not integrable. However, for any $0 < \lambda_1 < 1$ and $1 < \lambda_2 < 1/\kappa$, there exists $0 < \sigma < \epsilon$, such that $\chi := ((t_1-\sigma, t_1+\sigma) \cup (t_2-\sigma, t_2+\sigma)) \cap (-\delta,\delta) = 0$, dependent on $\lambda_1, \lambda_2$ and

$$\lambda_1\kappa < |\sin(t+\phi)| < \lambda_2\kappa. \tag{1.25}$$

for any $t \in \chi$.

We only analysis the integral on $\chi_1 = (t_1-\sigma, t_1+\sigma) \cap [-\pi/2, \pi/2]$ here, which denoted by $I_{\chi_1}$, the proof of $I_{\chi_2}$ is similar. It is easy to see that $\sin(t+\phi)$ is monotonic in $\chi_1$. Without loss of generality, we assume that $\sin(t_1-\sigma+\phi) < \kappa < \sin(t_1+\sigma+\phi)$. Let $\sin(t+\phi) = \kappa\sin\theta$ and the implicit mapping from $\theta$ to $t$ is denoted by $t(\theta)$ while the inverse mapping by $\theta(t)$, taking the interval $\chi_1$ onto $L_\theta : \theta_1 \to \pi/2 \to \pi/2 - \mathbf{i}\theta_2$ where $\sin(t_1-\sigma+\phi) = \kappa\sin\theta_1, \sin(t_1+\sigma+\phi) = \kappa\sin(\pi/2 - \mathbf{i}\theta_2)$. By substituting $t(\theta)$ into $I_{\chi_1}$, we have

$$I_{\chi_1} = \int_{t_1-\sigma}^{t_1+\sigma} \frac{f(t)(\kappa^2 - \sin^2(t+\phi))^{1/2}}{(\kappa^2 - \sin^2(t+\phi))^{1/2}} e^{\mathbf{i}\rho\cos t} \tag{1.26}$$

$$= \int_{L_\theta} \frac{\kappa f(t(\theta))\cos\theta}{(1-\kappa^2\sin^2\theta)^{1/2}} e^{\mathbf{i}\rho(\cos(t(\theta)))}d\theta \tag{1.27}$$

$$= \int_{L_\theta} \frac{\kappa g_1(t(\theta))\cos\theta + g_2(t(\theta))}{(1-\kappa^2\sin^2\theta)^{1/2}} e^{\mathbf{i}\rho(\cos(t(\theta)))}d\theta \tag{1.28}$$

$$:= \int_{L_\theta} \frac{h(\theta)}{(1-\kappa^2\sin^2\theta)^{1/2}} e^{\mathbf{i}\rho(\cos(t(\theta)))}d\theta \tag{1.29}$$

Observe that $h(\theta)$ and $\partial h/\partial\theta$ are integrable on the path $L_\theta$ by (1.11). A simple computation show that

$$\frac{dt(\theta)}{d\theta} = \frac{\kappa\cos\theta}{\cos(t+\phi)} \qquad \frac{d^2t(\theta)}{dt^2} = \frac{\kappa^2\cos^2\theta\sin(t+\phi) - \kappa\sin\theta\cos^2(t+\phi)}{\cos^3(t+\phi)}$$

Then we can obtain

$$\frac{d\cos t}{d\theta} = \frac{-\kappa\sin t\cos\theta}{\cos(t+\phi)}$$

$$
\begin{aligned}
\frac{d^2\cos t}{d\theta^2} &= \frac{d^2\cos t}{dt^2}(\frac{dt}{d\theta})^2 + \frac{d\cos t}{dt}\frac{d^2 t}{d\theta^2} \\
&= \frac{-\kappa^2\cos^2\theta\cos t}{\cos^2(t+\phi)} + \frac{\kappa\sin\theta\cos^2(t+\phi)\sin t - \kappa^2\cos^2\theta\sin(t+\phi)\sin t}{\cos^3(t+\phi)} \\
&= \frac{-\kappa^2\cos^2\theta\cos\phi + \kappa\sin\theta\cos^2(t+\phi)\sin t}{\cos^3(t+\phi)} \\
&= \frac{(\sin^2(t+\phi)-\kappa^2)\cos\phi + \cos^2(t+\phi)\sin(t+\phi)\sin t}{\cos^3(t+\phi)}
\end{aligned}
$$

Since $|\sin t| > |\sin\delta|$ and $1-\lambda_2^2\kappa^2 < \cos^2(t+\phi) < 1-\lambda_1^2\kappa^2$ for $t\in\chi_1$, it follows that $\theta=\pi/2$ is the only stationary point of $\cos(t(\theta))$ and

$$
\left|\frac{d^2\cos t}{d\theta^2}(\pi/2)\right| = \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}|\sin t| > \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}\sin\delta \tag{1.30}
$$

Therefore, we can choose appropriate $\lambda_1,\lambda_2$ such that

$$
|\frac{d^2\cos t}{d\theta^2}| > \frac{(1-\kappa^2)\kappa}{(1-\kappa^2)^{3/2}}\sin\delta \tag{1.31}
$$

for any $\theta\in\theta(\chi_1)$. According to lemma (6.1), we obtian $|I_{\chi_1}| \le C\frac{1}{\rho^{1/2}}$, and also $|I_{\chi_2}| \le C\frac{1}{\rho^{1/2}}$. Using integration by parts, we get

$$
\left|\int_{[-\pi/2,\pi/2]\setminus((-\delta,\delta)\cup\chi)} f(t)(1-\chi_\delta(t))e^{\mathbf{i}\rho\cos t}dt\right| \le C\frac{1}{\rho}
$$

Consequently, for $N=-1$ and $\phi\ne\phi_\kappa$, we get $|I(\rho,\phi)| \le \frac{1}{\rho^{1/2}}$.

We now turn to the case of $\phi=\phi_\kappa$. By (1.11), we can define $\chi_\epsilon$ similarly and also decompose $I$ into $I_1$ and $I_2$. Using the same agurement above, we can easily carry out that: for $N=1$, we have $|I_2| \le C\frac{1}{\rho}$; for $N=-1$, we have $|I_2| \le C\frac{1}{\rho^{1/2}}$. Finally, it remains to analysis $I_1$. By (1.11), we have

$$
\begin{aligned}
I_1 &= \int_{-\epsilon}^{\epsilon} g_1\chi_\epsilon + g_2\chi_\epsilon(\sin^2\phi_\kappa - \sin^2(t+\phi_\kappa))^{N/2}e^{\mathbf{i}\rho\cos t}dt \\
&= \int_{-\epsilon}^{\epsilon} g_1\chi_\epsilon + g_2\chi_\epsilon(-2(\sin\phi_\kappa+\sin(t+\phi_\kappa))\cos\frac{2\phi_\kappa+t}{2}\sin t/2)^{N/2}e^{\mathbf{i}\rho\cos t}dt \\
&= \int_{\mathbb{R}} g_1\chi_\epsilon + g_2\chi_\epsilon((\sin\phi_\kappa+\sin(t+\phi_\kappa))\cos\frac{2\phi_\kappa+t}{2})^{N/2}(-2\sin t/2)^{N/2}e^{\mathbf{i}\rho\cos t}dt
\end{aligned}
$$

Also, subtituting $t(s) = 2\arcsin s/2$ for t in $I_1$, it follows that

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}} h_1(s)e^{-\mathbf{i}\rho\frac{s^2}{2}} + h_2(s)(-s)^{N/2}e^{-\mathbf{i}\rho\frac{s^2}{2}} \qquad (1.32)\\
&= I_{11} + I_{12} \qquad (1.33)
\end{aligned}
$$

where

$$
\begin{aligned}
h_1(s) &= g_1(t(s))\chi_\epsilon(t(s))\sqrt{1-s^2/4}\ e^{\mathbf{i}\rho} \\
h_2(s) &= g_2\chi_\epsilon((\sin\phi_\kappa+\sin(t+\phi_\kappa))\cos\frac{2\phi_\kappa+t}{2})_{t=t(s)}^{N/2}\sqrt{1-s^2/4}\ e^{\mathbf{i}\rho}
\end{aligned}
$$

and $h_1(s), h_2(s) \in C_c^\infty(\mathbb{R})$. Using stationary phase lemma similarly, if $N = 1$,

$$I_{11} = \left(\frac{2\pi}{\rho}\right)^{1/2} g_1(0)e^{\mathbf{i}\rho - \mathbf{i}\pi/4} + O(\frac{1}{\rho}) \tag{1.34}$$

$$= \left(\frac{2\pi}{\rho}\right)^{1/2} f(0)e^{\mathbf{i}\rho - \mathbf{i}\pi/4} + O(\frac{1}{\rho}) \tag{1.35}$$

if $N = -1$, we get $|I_{11}| \leq C\frac{1}{\rho^{1/2}}$. For $I_{12}$, we have

$$I_{12} = \int_0^\infty (\mathbf{i}h_2(s) + h_2(-s))s^{N/2}\ e^{-\mathbf{i}\rho s^2/2}ds \tag{1.36}$$

$$= \frac{1}{2}\int_0^\infty (\mathbf{i}h_2(\sqrt{s}) + h_2(-\sqrt{s}))s^{N/4-1/2}e^{-\mathbf{i}\rho s/2}ds \tag{1.37}$$

By lemma (1.2), we get $|I_{12}| \leq C\frac{1}{\rho^{(N+2)/4}}$. □

## 2. Some draft about Green Tensor Analysis

Let substitute $\xi = k\sin\theta$ into integral and shift the variable, we have

$$I(y) = \int_{\mathbb{R}} f(\xi)e^{\mathbf{i}\xi y_1 + \mu(\xi)y_2}d\xi = \int_{\mathbb{R}} f(\xi)e^{\mathbf{i}\xi(y_1-z_1)+\mu(\xi)(y_2-z_2)}e^{\mathbf{i}\xi z_1+\mu(\xi)z_2}d\xi \tag{2.1}$$

$$= k\int_L f(k\sin\theta)\cos\theta e^{\mathbf{i}k|y-z|\cos(\theta-\eta)}e^{\mathbf{i}|z|\cos(\theta-\phi)}d\theta \tag{2.2}$$

$$= k\int_{L_\phi} f(k\sin(\theta+\phi))\cos(\theta+\phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \tag{2.3}$$

$$= k\int_L f(k\sin(\theta+\phi))\cos(\theta+\phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \tag{2.4}$$

where $y_1, y_2 > 0, \sin\phi = \frac{z_1}{|z|}, \cos\phi = \frac{z_2}{|z|}, 0 < \phi < \pi/2$ and $\sin\eta = \frac{y_1-z_1}{|y-z|}, \cos\eta = \frac{y_2-z_2}{|y-z|}, 0 < \eta < \pi$. It is easy to see that $\phi + \eta < \pi$. Roughly, using stationary phase lemma, we obtain:

$$I(y) = f(k\sin\phi)k\cos\phi e^{\mathbf{i}k|y-z|\cos(\phi-\eta)}(\frac{2\pi}{|z|})^{1/2}e^{\mathbf{i}|z|-\mathbf{i}\frac{\pi}{4}}(1+O(\frac{1}{|z|})) \tag{2.5}$$

$$\cos(a+\mathbf{i}b) = \frac{e^b+e^{-b}}{2}\cos a + \mathbf{i}\frac{e^{-b}-e^b}{2}\sin a \tag{2.6}$$

$$\sin(a+\mathbf{i}b) = \frac{e^b+e^{-b}}{2}\sin a + \mathbf{i}\frac{e^b-e^{-b}}{2}\cos a \tag{2.7}$$

When $\theta \in (-a-\pi/2, -a-\pi/2+\mathbf{i}\infty)$, let $\theta = -a-\pi/2\mathbf{i}t$, where $t > 0, 0 \leq a \leq \phi$, then

$$-\mathrm{Im}\,(|z|\cos\theta + |y-z|\cos(\theta+\phi-\eta))$$

$$= |z|\sin(a+\pi/2) + |y-z|\sin(a+\pi/2-\phi+\eta) \tag{2.8}$$

$$= |z|\cos a + |y-z|\cos(a-\phi+\eta) \tag{2.9}$$

$$= |z|\cos a + \cos a|y-z|(\cos\phi\cos\eta + \sin\phi\sin\eta) \tag{2.10}$$

$$+\sin a|y-z|(\sin\phi\cos\eta - \cos\phi\sin\eta) \tag{2.11}$$

$$= |z|\cos a + \cos a((y_2 - z_2)\cos\phi + (y_1 - z_1)\sin\phi) \quad (2.12)$$
$$+ \sin a((y_2 - z_2)\sin\phi - (y_1 - z_1)\cos\phi) \quad (2.13)$$
$$= y_1 \sin(\phi - a) + y_2 \cos(\phi - a) > 0 \quad (2.14)$$

Now, Using Cauchy Integral Theorem, we have

$$I(y) = k\int_L f(k\sin(\theta+\phi))\cos(\theta+\phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \quad (2.15)$$

Let $L_1 = (-\pi/2, -\pi/2 + \mathbf{i}\infty)$ and $\theta = -\pi/2 + \mathbf{i}t, t > 0$, then

$$I_1(y) = k\int_{L_1} f(k\sin(\theta+\phi))\cos(\theta+\phi)e^{\mathbf{i}k|y-z|\cos(\theta+\phi-\eta)}e^{\mathbf{i}|z|\cos\theta}d\theta \quad (2.16)$$
$$= \quad (2.17)$$

$$I(y) = f(k\sin\phi)k\cos\phi e^{\mathbf{i}k|y-z|\cos(\phi-\eta)}(\frac{2\pi}{|z|})^{1/2}e^{\mathbf{i}|z|-\mathbf{i}\frac{\pi}{4}} \quad (2.18)$$
$$+\frac{kz_2}{|z|}O(\left(\frac{1}{k|z|}\right)^{3/4}+\frac{1}{k|y|})+\frac{kz_1}{|z|}O(\left(\frac{1}{k|z|}\right)^{5/4}+\left(\frac{1}{k|y|}\right)^2) \quad (2.19)$$

It is easy to see

$$\int_{-d}^{d}\frac{k}{(k|x-z|)^\alpha}\frac{1}{(k|x-y|)^\beta}dx_1 \le C(\frac{1}{(kz_2)^{\alpha+\beta-1}}+\frac{1}{(ky_2)^{\alpha+\beta-1}}) \quad (2.20)$$

where $z, y \in \mathbb{R}^2_+$, $x \in \Gamma_0$ and $\alpha + \beta > 0$.

$$e^{\mathbf{i}\mu y_2+\mathbf{i}\xi(x_1-y_1)} = e^{\mathbf{i}\mu y_2-\mathbf{i}y_2/\tan\phi} = e^{\mathbf{i}y_2(\mu-\xi/\tan\phi)} \quad (2.21)$$

Another method

$$\int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi)e^{\mathbf{i}k|x-y|\cos\theta} \quad (2.22)$$
$$= \int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi)e^{\mathbf{i}k|x-y|\cos(\theta+\psi-\psi)} \quad (2.23)$$
$$= \int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi)e^{\mathbf{i}ky_2\cos(\theta+\psi)+\mathbf{i}k|x_1-y_1|\sin(\theta+\psi)} \quad (2.24)$$
$$= \int_{-\pi/2}^{\pi/2} f(k\sin(\theta+\psi))k\cos(\theta+\psi) \quad (2.25)$$
$$e^{\mathbf{i}k(y_2-z_2)\cos(\theta+\psi)+\mathbf{i}k(|x_1-y_1|-|x_1-z_1|)\sin(\theta+\psi)+\mathbf{i}k|z|\cos(\theta+\psi-\phi)} \quad (2.26)$$

## 3. Finite Aperture Point Spread Function

If $x \in \Gamma_0$ and $z, y \in \mathbb{R}^2_+$, by lemma (**??**) we have

$$G(x,y) = \frac{\mathbf{i}k_s}{\mu\sqrt{2\pi}}\frac{1}{\delta(\xi)}\begin{pmatrix} \mu_s\beta & \xi\beta \\ 2\xi\mu_s\mu_p & 2\xi^2\mu_p \end{pmatrix}_{\xi=k_s\frac{x_1-y_1}{|x-y|}}\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{1/2}}e^{\mathbf{i}k_s|x-y|-\mathbf{i}\frac{\pi}{4}}$$
$$+\frac{\mathbf{i}k_p}{\mu\sqrt{2\pi}}\frac{1}{\delta(\xi)}\begin{pmatrix} 2\xi^2\mu_s & -2\xi\mu_s\mu_p \\ -\xi\beta & \mu_p\beta \end{pmatrix}_{\xi=k_p\frac{x_1-y_1}{|x-y|}}\frac{y_2}{|x-y|}\frac{1}{(k_p|x-y|)^{1/2}}e^{\mathbf{i}k_p|x-y|-\mathbf{i}\frac{\pi}{4}} \quad (3.1)$$

$$+O(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}})$$
$$:=\mathcal{G}_s(x,y)+\mathcal{G}_p(x,y)+O(\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}})$$

$$\begin{aligned}T_D(x,z) &= \frac{k_s}{\sqrt{2\pi}}\frac{1}{\gamma(\xi)}\begin{pmatrix}\mu_s\mu_p & \xi\mu_p\\ \xi\mu_s & \xi^2\end{pmatrix}_{\xi=k_s\frac{x_1-z_1}{|x-z|}}\frac{z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{1/2}}e^{\mathbf{i}k_s|x-z|-\mathbf{i}\frac{\pi}{4}}\\ &+\frac{k_p}{\sqrt{2\pi}}\frac{1}{\gamma(\xi)}\begin{pmatrix}\xi^2 & -\xi\mu_p\\ -\xi\mu_s & \mu_p\mu_s\end{pmatrix}_{\xi=k_p\frac{x_1-z_1}{|x-z|}}\frac{z_2}{|x-z|}\frac{1}{(k_p|x-z|)^{1/2}}e^{\mathbf{i}k_p|x-z|-\mathbf{i}\frac{\pi}{4}} \qquad (3.2)\\ &+O(\frac{k_sz_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{k_s|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}})\\ &:=\mathcal{T}_s(x,z)+\mathcal{T}_p(x,z)+O(\frac{k_sz_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{k_s|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}})\end{aligned}$$

Now we consider the finite aperture point spread function $J_d(z,y)$:

$$\int_{-d}^{d}(T_D(x_1,0;z_1,z_2))^T\overline{G(x_1,0;y_1,y_2)}dx_1 \qquad (3.3)$$

Recall following standard asymptotic expansion:

$$|x-y|=|x-z|+\widehat{x-z}\cdot(z-y)+O(\frac{|y-z|^2}{|x-z|}) \qquad (3.4)$$
$$|y|^{-\alpha}=|z|^{-\alpha}(1+\frac{|y|-|z|}{|z|})^{-\alpha}=|z|^{-\alpha}(1+O(\frac{|y-z|}{|z|})) \qquad (3.5)$$
$$e^{\mathbf{i}t}=1+O(t) \qquad (3.6)$$
$$|a^{1/2}-b^{1/2}|\le C\sqrt{|a-b|} \qquad (3.7)$$

where $x,y,z\in\mathbb{R}^2$, $t,a,b\in\mathbb{R}$ and $\alpha>0$.

**Lemma 3.1** *For any $z,y\in\mathbb{R}^2_+$, $J_d(z,y)=F(z,y)+O((1+\frac{|y-z|}{z_2})(\frac{1}{k_sz_2})^{1/4}+\frac{(k_s|y-z|)^2}{k_sz_2}+(\frac{|y-z|}{z_2})^{1/2})$, where*

$$F(z,y)=-\frac{\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d}f_s(\theta)\begin{pmatrix}\sin^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \cos^2\theta\end{pmatrix}e^{\mathbf{i}k_s(z_1-y_1)\cos\theta+\mathbf{i}k_s(z_2-y_2)\sin\theta}d\theta \qquad (3.8)$$
$$-\frac{\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d}f_p(\theta)\begin{pmatrix}\cos^2\theta & -\sin\theta\cos\theta\\ -\sin\theta\cos\theta & \sin^2\theta\end{pmatrix}e^{\mathbf{i}k_p(z_1-y_1)\cos\theta+\mathbf{i}k_p(z_2-y_2)\sin\theta}d\theta \qquad (3.9)$$

*and*

$$f_s(\theta)=\frac{\sin\theta((\kappa^2-\cos^2\theta)^{1/2}(1-2\cos^2\theta)+2\overline{(\kappa^2-\cos^2\theta)^{1/2}}\cos^2\theta)}{(\cos^2\theta+\sin\theta(\kappa^2-\cos^2\theta)^{1/2})\overline{((1-2\cos^2\theta)^2+4\cos^2\theta\sin\theta(\kappa^2-\cos\theta)^{1/2})}}$$
$$f_p(\theta)=\frac{\sin\theta(1/\kappa^2-\cos^2\theta)^{1/2}}{(\cos^2\theta+\sin\theta(1/\kappa^2-\cos^2\theta)^{1/2})((1/\kappa^2-2\cos^2\theta)^2+4\cos^2\theta\sin\theta(1/\kappa^2-\cos\theta)^{1/2})}$$

*where $0<\theta_1^d<\pi/2<\theta_2^d<\pi$ and $z_2=(d+z_1)\tan\theta_1^d=(z_1-d)\tan\theta_2^d$.*

**Proof.**

$$\begin{aligned}&\frac{y_2}{|x-y|}\frac{1}{(k_s|x-y|)^{3/4}}+\frac{|x_1-y_1|}{|x-y|}\frac{1}{(k_s|x-y|)^{5/4}}\\=&\left(\frac{z_2}{|x-z|}\frac{1}{(k_s|x-z|)^{3/4}}+\frac{|x_1-z_1|}{|x-z|}\frac{1}{(k_s|x-z|)^{5/4}}\right)(1+O(\frac{|y-z|}{|x-z|}))\end{aligned}$$

$$\begin{aligned}&|\mu_{\mathbf{i}}(k_j\frac{x_1-y_1}{|x-y|})-\mu_{\mathbf{i}}(k_j\frac{x_1-z_1}{|x-z|})|\\&\le Ck_j\sqrt{\Big|\frac{x_1-y_1}{|x-y|}-\frac{x_1-z_1}{|x-z|}\Big|}\le Ck_j\Big(\frac{|y-z|}{|x-z|}\Big)^{1/2}\end{aligned}$$

where $i,j=s,p$. By above, we can obtain

$$\mathcal{G}_s(x,y)=\mathcal{G}_s(x,z)e^{\mathbf{i}k_s\widehat{x-z}\cdot(z-y)}+O(\frac{(k_s|y-z|)^2}{(k_s|x-z|)^{3/2}})+O(\frac{(k_s|y-z|)^{1/2}}{k_s|x-z|}) \tag{3.10}$$

$$\mathcal{G}_p(x,y)=\mathcal{G}_p(x,z)e^{\mathbf{i}k_p\widehat{x-z}\cdot(z-y)}+O(\frac{(k_p|y-z|)^2}{(k_p|x-z|)^{3/2}})+O(\frac{(k_p|y-z|)^{1/2}}{k_p|x-z|}) \tag{3.11}$$

For $l>1$, a simple computation show that

$$\int_{-d}^{d}\frac{k_s}{(k_s|x-z|)^l}dx_1=\frac{1}{(k_sz_2)^{l-1}}\int_{\frac{-d-z_1}{z_2}}^{\frac{d-z_1}{z_2}}\frac{1}{(1+t^2)^{l/2}}dt\le C\frac{1}{(k_sz_2)^{l-1}} \tag{3.12}$$

Let

$$\mathcal{G}_\alpha(x,y)=\frac{\mathbf{i}}{\sqrt{2\pi}\mu}g_\alpha(\frac{x_1-y_1}{|x-y|},\kappa)\frac{1}{(k_\alpha|x-y|)^{1/2}}e^{\mathbf{i}k_\alpha|x-y|-\mathbf{i}\frac{\pi}{4}} \tag{3.13}$$

$$\mathcal{T}_\alpha(x,y)=\frac{k_\alpha}{\sqrt{2\pi}}t_\alpha(\frac{x_1-z_1}{|x-z|},\kappa)\frac{1}{(k_s|x-z|)^{1/2}}e^{\mathbf{i}k_\alpha|x-z|-\mathbf{i}\frac{\pi}{4}} \tag{3.14}$$

where $\alpha=s,p$. Now, by substituting (3.10-3.11) into $J_d(z,y)$ and using inequality (3.12), we have

$$\begin{aligned}J_d(z,y)=&\frac{-\mathbf{i}}{2\pi\mu}\int_{-d}^{d}t_s(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_s(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_s\widehat{x-z}\cdot(y-z)}}{|x-z|}\\&+t_p(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_p(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_p\widehat{x-z}\cdot(y-z)}}{|x-z|}dx_1 &(3.15)\\&-\frac{\mathbf{i}}{2\pi\mu}\int_{-d}^{d}t_p(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_s(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_s\widehat{x-z}\cdot(y-z)}}{|x-z|} &(3.16)\\&+t_s(\frac{x_1-z_1}{|x-z|},\kappa)^T\overline{g_p(\frac{x_1-z_1}{|x-z|},\kappa)}\frac{e^{\mathbf{i}k_p\widehat{x-z}\cdot(y-z)}}{|x-z|}dx_1 &(3.17)\\&+O((1+\frac{|y-z|}{z_2})(\frac{1}{k_sz_2})^{1/4}+\frac{(k_s|y-z|)^2}{k_sz_2}+(\frac{|y-z|}{z_2})^{1/2}) &(3.18)\\&:=F(z,y)+R(z,y) &(3.19)\\&+O((1+\frac{|y-z|}{z_2})(\frac{1}{k_sz_2})^{1/4}+\frac{(k_s|y-z|)^2}{k_sz_2}+(\frac{|y-z|}{z_2})^{1/2}) &(3.20)\end{aligned}$$

We denote $\widehat{x-z} = x - z/|x-z| = (cos(\phi+\pi), \sin(\phi+\pi))$, then taking the substitution $x_1 = z_1 - z_2 \cot\phi$, we obtain

$$F(z,y) = \frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d} A_s(\phi,\kappa)e^{\mathbf{i}k_s(z_1-y_1)\cos\phi+\mathbf{i}k_s(z_2-y_2)\sin\phi} \tag{3.21}$$

$$+ \frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d} A_p(\phi,\kappa)e^{\mathbf{i}k_p(z_1-y_1)\cos\phi+\mathbf{i}k_p(z_2-y_2)\sin\phi} \tag{3.22}$$

$$R(z,y) = \frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d} B_s(\phi,\kappa)e^{\mathbf{i}k_s(z_1-y_1)\cos\phi+\mathbf{i}k_s(z_2-y_2)\sin\phi+(k_p-k_s)|x-z|} \tag{3.23}$$

$$+ \frac{-\mathbf{i}}{2\pi\mu}\int_{\theta_1^d}^{\theta_2^d} B_p(\phi,\kappa)e^{\mathbf{i}k_p(z_1-y_1)\cos\phi+\mathbf{i}k_p(z_2-y_2)\sin\phi+(k_s-k_p)|x-z|} \tag{3.24}$$

It is easy to see that $|R(z,y)| \le C\frac{|z-y|}{z_2}$. □

Let

$$g(x_1) = \frac{1}{((x_1-z_1)^2+z_2^2)^{3/4}((x_1-y_1)^2+y_2^2)^{1/4}}$$
$$\phi(x_1) = ((x_1-z_1)^2+z_2^2)^{1/2} - ((x_1-y_1)^2+y_2^2)^{1/2}$$

Then, we have

$$g'(x_1) = -g(x_1)[\frac{3(x_1-z_1)}{2((x_1-z_1)^2+z_2^2)} + \frac{(x_1-y_1)}{2((x_1-y_1)^2+y_2^2)}]$$
$$\phi'(x_1) = \frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}} - \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}}$$
$$= \frac{\frac{(x_1-z_1)^2}{(x_1-z_1)^2+z_2^2} - \frac{(x_1-y_1)^2}{(x_1-y_1)^2+y_2^2}}{\frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}} + \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}}}$$
$$= \frac{(x_1-z_1)^2y_2^2 - (x_1-y_1)^2z_2^2}{(\frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}} + \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}})((x_1-z_1)^2+z_2^2)((x_1-y_1)^2+y_2^2)}$$
$$\phi''(x_1) = \frac{z_2^2}{((x_1-z_1)^2+z_2^2)^{3/2}} - \frac{y_2^2}{((x_1-y_1)^2+y_2^2)^{3/2}}$$

Using integration by parts, we can obtain

$$\int_{-d}^{d} g(x_1)e^{\mathbf{i}\phi(x_1)}dx_1$$
$$= (\frac{g(d)}{\phi'(d)}e^{\mathbf{i}\phi(d)} - \frac{g(-d)}{\phi'(-d)}e^{\mathbf{i}\phi(-d)}) - \int_{-d}^{d}\frac{g'(x_1)}{\phi'(x_1)} - \frac{g(x_1)\phi''(x_1)}{(\phi'(x_1))^2}dx_1$$

Assume that

$$|y_1| \le c_0 d \quad |z_1| \le c_0 d \quad h \le y_2, z_2 \le c_1 h \quad d \le c_2 h$$

where $0 < c_0 < 1$. Let define $0 < \theta_y, \theta_z < \pi$ such that

$$\cos\theta_y = \frac{x_1-y_1}{((x_1-y_1)^2+y_2^2)^{1/2}}$$
$$\cos\theta_z = \frac{x_1-z_1}{((x_1-z_1)^2+z_2^2)^{1/2}}$$

By mean value theorem and the law of sines, we get

$$
\begin{aligned}
|\phi'(x_1)| &= |\cos\theta_z - \cos\theta_y| = |\sin\theta'||\theta_z - \theta_y| \\
&\geq \frac{h}{(1+c_0)d}|\sin(\theta_z - \theta_y)| \\
&= \frac{h}{(1+c_0)d}\frac{|z-y|}{|x-y|}\sin\theta_{|x-y|} \\
&= \frac{h}{(1+c_0)d}\frac{|z-y|}{|x-z|}\sin\theta_{|x-z|} \\
&\geq \frac{h^2}{(1+c_0)^2d^2}\frac{|z-y|}{|x-y|} \\
\text{or} \qquad &\geq \frac{h^2}{(1+c_0)^2d^2}\frac{|z-y|}{|x-z|}
\end{aligned}
$$

Then we have

$$
\begin{aligned}
|\frac{g(x_1)}{\phi'(x_1)}| &\leq \frac{(1+c_0)^2d^2}{h^2}\frac{1}{|z-y||x-y|^{1/2}|x-z|^{1/2}} \\
&\leq C\frac{d^2}{h^3}\frac{1}{|z-y|}
\end{aligned}
$$

Moreover, by mean value theorem agian, we have

$$
\begin{aligned}
|\phi''(x_1)| &= |\frac{\sin^2\theta_z}{|x-z|} - \frac{\sin^2\theta_y}{|x-y|}| \\
&= |\frac{2\sin\theta'\cos\theta'}{|x-y'|}(\theta_z - \theta_y) - \frac{\sin^2\theta'}{|x-y'|^2}(|x-z| - |x-y|)| \\
&\leq \pi\frac{|\sin(\theta_z-\theta_y)|}{h} + \frac{|z-y|}{h^2} \\
&\leq \pi\frac{|\sin\theta_{|x-z|}||z-y|}{h|x-z|} + \frac{|z-y|}{h^2} \\
&\leq C\frac{|z-y|}{h^2}
\end{aligned}
$$

Now, it is easy to see that

$$
\begin{aligned}
&\Big|\int_{-d}^{d}\frac{g'(x_1)}{\phi'(x_1)} - \frac{g(x_1)\phi''(x_1)}{(\phi'(x_1))^2}dx_1\Big| \\
\leq\ & C\frac{d^3}{h^4}\frac{1}{|z-y|} + C\frac{d^3}{h^3}\frac{1}{|z-y|}\frac{d^2}{h^3}
\end{aligned}
$$

Based on the above analysis, we can obtain

$$
|\int_{-d}^{d} z_2 g(x_1)e^{\mathbf{i}\phi(x_1)}| \leq C((\frac{d}{h})^2 + (\frac{d}{h})^3 + (\frac{d}{h})^5)\frac{1}{|z-y|}
$$

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$$
\begin{aligned}
&\sin\phi_\kappa - \sin(t+\phi) = -2\cos(\frac{\phi_\kappa+\phi+t}{2})\sin(\frac{t+\phi-\phi_\kappa}{2}) \\
&\sin(\frac{t+\phi-\phi_\kappa}{2}) = \sin\frac{t}{2}\cos(\frac{\phi-\phi_\kappa}{2}) + \cos\frac{t}{2}\sin(\frac{\phi-\phi_\kappa}{2})
\end{aligned}
$$

Some think, substituting $t = 2\arcsin s/2$ into following integral

$$\int_0^\infty \chi(t)(\sin\phi_\kappa - \sin(t+\phi))^{1/2} e^{-\mathbf{i}\rho\cos t}$$
$$= \int_0^\infty \chi(t(s))(-s\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-s^2}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2} e^{-\mathbf{i}\rho s^2/2}$$
$$= \int_0^\infty \chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2} t^{-1/2} e^{-\mathbf{i}\rho t/2}$$

Let

$$f(t) = t^{-1/2} e^{-\mathbf{i}\rho t/2}$$
$$g(t) = -\int_t^{t-\mathbf{i}\infty} x^{-1/2} e^{-\mathbf{i}\rho x/2} dx$$
$$= \mathbf{i}\int_0^\infty (t - \mathbf{i}x)^{-1/2} e^{-\mathbf{i}\rho t - \rho x} dx$$

It is easy to see that $g'(t) = f(t)$. Then we have

$$= \int_0^\infty \chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2} t^{-1/2} e^{-\mathbf{i}\rho t/2}$$
$$= \chi(0)(-2\sin(\frac{\phi-\phi_\kappa}{2}))^{1/2} g(0)$$
$$-\int_0^\infty (\chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{1/2})' g(t) dt$$

We get

$$g(x) = \int_0^\infty \chi(t)(-\sqrt{t}\cos(\frac{\phi-\phi_\kappa}{2}) - \sqrt{4-t}\sin(\frac{\phi-\phi_\kappa}{2})^{-1/2} t^{-1/2} (t-\mathbf{i}x)^{-1/2} e^{-\mathbf{i}\rho t} dt$$
$$R(\rho) = \int_0^\infty g(x) e^{-\rho x} dx$$

Because $\chi(t)$ has compact support $(-\delta, \delta)$, we obtain

$$gg(x) = \int_0^\delta (\sqrt{t}\cos(\theta) - \sqrt{4-t}\sin\theta)^{-1/2} t^{-1/2} (t^2 + x^2)^{-1/4} dt$$

where $\theta = \frac{\phi-\phi_\kappa}{2}$. For $x > 0$, Put L(x):

$$\int_0^a \frac{1}{t^{3/4}} \frac{1}{(t^2+x^2)^{1/4}} dt$$
$$= 4\int_0^a \frac{1}{(t^2+x^2)^{1/4}} dt^{1/4}$$
$$= 4\int_0^{a^{1/4}} \frac{1}{(t^8+x^2)^{1/4}} dt$$
$$= 4x^{-1/4}\int_0^{(\frac{a}{x})^{1/4}} \frac{1}{(t^8+1)^{1/4}} dt$$
$$= 4x^{-1/4}\int_0^{(\frac{a}{x})^{1/4}} \frac{1}{(t^8+1)^{1/4}} dt$$
$$\leq 4x^{-1/4}\int_0^\infty \frac{1}{(t^8+1)^{1/4}} dt$$

Back to analysis $gg(x)$, we have

$$
\begin{aligned}
gg(x) &\le \int_0^\delta \Big|\frac{\sqrt{t}+2|\sin\theta|}{t-4\sin^2\theta}\Big|^{1/2} t^{-1/2}(t^2+x^2)^{-1/4}dt \\
&= \int_0^\delta \Big|\frac{1}{\sqrt{t}-2|\sin\theta|}\Big|^{1/2} t^{-1/2}(t^2+x^2)^{-1/4}dt \\
&= 2\int_0^{\sqrt{\delta}} \Big|\frac{1}{t-2|\sin\theta|}\Big|^{1/2}(t^4+x^2)^{-1/4}dt \\
&= 2\int_{-2|\sin\theta|}^{\sqrt{\delta}-2|\sin\theta|} |t|^{-1/2}((t+2|\sin\theta|)^4+x^2)^{-1/4}dt \\
&\le 4\int_0^{\delta^{1/4}}(t^8+x^2)^{-1/4}dt + 4\int_0^{\sqrt{2|\sin\theta|}}((t^2-2|\sin\theta|)^4+x^2)^{-1/4}dt \\
&\le Cx^{-1/4}(1+\int_0^{\sqrt{2|\sin\theta|}}((t^2-2|\sin\theta|)^4/x+x)^{-1/4}dt) \\
&\le Cx^{-1/4}(1+\int_0^{\sqrt{2|\sin\theta|}}(t^2-2|\sin\theta|)^{-1/2}dt) \\
&= Cx^{-1/4}(1+\int_0^1(1-t^2)^{-1/2}dt) \le Cx^{-1/4}
\end{aligned}
$$

Immediately, we can obtain

$$|g(x)| \le Cx^{-1/4}$$

It follows that

$$R(\rho) \le \int_0^\infty x^{-1/4}e^{-\rho x} \le C\rho^{-3/4}$$

## 5. stationary of phase lemma

**Lemma 5.1** *Assume that $0<\kappa:=\sin\phi_\kappa<1, 0<\phi_\kappa<\pi/2$, $0\le\phi\le\pi/2$. Let*

$$f(t,\phi) := F(\sin(t+\phi),\cos(t+\phi),(\kappa^2-\sin^2(t+\phi))^{1/2}) \tag{5.1}$$

*be a complexed function in $C([-\pi/2,\pi/2]\times[0,\pi/2])$. Moreover, its partial derivative with respect to $t$ can be represented as*

$$\frac{\partial f(t,\phi)}{\partial t} = g(t,\phi)(\kappa^2-\sin^2(t+\phi))^{-1/2} \tag{5.2}$$

*where $g(t,\phi)$ is uniformly bounded. Then for any $\rho\ge 1$, we have*

$$
\begin{aligned}
&\Big|I(\rho,\phi) := \int_{-\pi/2}^{\pi/2} f(t)e^{\mathbf{i}\rho\cos t}dt - \Big(\frac{2\pi}{\rho}\Big)^{1/2} f(0)e^{\mathbf{i}\rho-\mathbf{i}\pi/4}\Big| \\
&\le C\frac{1}{\rho^{3/4}}
\end{aligned} \tag{5.3}
$$

**Proof.** Solving the following equation:

$$\kappa^2-\sin^2(t+\phi)=0$$

we have, if $0 < \phi < \pi/2 - \phi_\kappa$,

$$t_1(\phi) = \phi_\kappa - \phi \qquad t_2(\phi) = -\phi_\kappa - \phi$$

and if $\pi/2 - \phi_\kappa \le \phi < \pi/2$,

$$t_1(\phi) = \phi_\kappa - \phi \qquad t_2(\phi) = \pi - \phi_\kappa - \phi$$

Since $|t_2(\phi)| < \phi_\kappa$ or $|t_2(\phi)| < \pi/2 - \phi_\kappa$ , we now define $\delta := \min(\frac{\phi_\kappa}{2}, \frac{\pi/2-\phi_\kappa}{2})$ and it is easy to see that

$$\kappa + \sin(t+\phi) \neq 0 \tag{5.4}$$

$$\cos(\frac{t+\phi+\phi_\kappa}{2}) \neq 0 \tag{5.5}$$

for any $(t, \phi) \in [-\delta, \delta] \times [0, \pi/2]$. Let $\chi_\delta \in C_0^\infty(-\pi/2, \pi/2)$ be the cut-off function with that $0 \le \chi_\delta \le 1$, $\chi_\delta = 1$ in $(-\delta/2, \delta/2)$ and $\chi_\delta = 0$ in $(-\pi/2, \pi/2)\backslash(-\delta, \delta)$. Then we can divide $I$ into two parts such that

$$\begin{aligned} I &= \int_{-\delta}^{\delta} f(t)\chi_\delta(t) e^{\mathbf{i}\rho\cos t} dt + \int_{-\pi/2}^{\pi/2} f(t)(1-\chi_\delta(t)) e^{\mathbf{i}\rho\cos t} dt \\ &=: I_1 + I_2 \end{aligned}$$

Subtitating $t(s) = 2\arcsin s/2$ for t in $I_1$, we can obtain

$$I_1 = \int_{-2\sin\frac{\delta}{2}}^{2\sin\frac{\delta}{2}} f(t(s))\chi_\delta(t(s)) \frac{1}{\sqrt{1-s^2/4}} e^{\mathbf{i}\rho} e^{-\mathbf{i}\rho s^2/2} ds \tag{5.6}$$

$$= \int_0^{2\sin\frac{\delta}{2}} (f(t(s))\chi_\delta(t(s)) + f(-t(s))\chi_\delta(-t(s))) \frac{1}{\sqrt{1-s^2/4}} e^{\mathbf{i}\rho} e^{-\mathbf{i}\rho s^2/2} ds \tag{5.7}$$

$$:= I_{11} + I_{12} \tag{5.8}$$

Taking substitution $s = \sqrt{x}$, we get

$$I_{11} = \frac{1}{2}\int_0^{(2\sin\frac{\delta}{2})^2} f(t(\sqrt{x}))\chi_\delta(t(\sqrt{x})) \frac{1}{\sqrt{1-x/4}} x^{-1/2} e^{\mathbf{i}\rho} e^{-\mathbf{i}\rho x/2} dx$$

Observe that

$$\begin{aligned} \sin\phi_\kappa - \sin(t+\phi) &= -2\cos(\frac{\phi_\kappa+\phi+t}{2})\sin(\frac{t+\phi-\phi_\kappa}{2}) \\ \sin(\frac{t+\phi-\phi_\kappa}{2}) &= \sin\frac{t}{2}\cos(\frac{\phi-\phi_\kappa}{2}) + \cos\frac{t}{2}\sin(\frac{\phi-\phi_\kappa}{2}) \\ &:= \sin\frac{t}{2}\cos\theta + \cos\frac{t}{2}\sin\theta \end{aligned}$$

where $\theta = \frac{\phi-\phi_\kappa}{2}$. By lemma (1.3) and using representation (5.2), inequality (5.4-5.5), it follows that

$$\begin{aligned} &|I_{11} - \frac{1}{2}\sqrt{\frac{2\pi}{\rho}} f(0) e^{\mathbf{i}\rho - \mathbf{i}\frac{\pi}{4}}| \\ \le &\int_0^\infty e^{-\rho y} dy \int_0^{(2\sin\frac{\delta}{2})^2} \Big| \frac{\partial (f(t(\sqrt{x}))\chi_\delta(t(\sqrt{x}))\frac{1}{\sqrt{1-x/4}})}{\partial x} \Big| (x^2+y^2)^{-\frac{1}{4}} dx \end{aligned}$$

$$\leq C\int_0^\infty e^{-\rho y}dy\int_0^{(2\sin\frac{\delta}{2})^2} |\sqrt{x}\cos\theta+\sqrt{4-x}\sin\theta|^{-1/2}x^{-1/2}(x^2+y^2)^{-\frac14}dx$$
$$\leq C\int_0^\infty e^{-\rho y}dy\int_0^{(2\sin\frac{\delta}{2})^2} \frac{(\sqrt{x}|\cos\theta|+\sqrt{4-x}|\sin\theta|)^{1/2}}{|x-4\sin^2\theta|^{1/2}}x^{-1/2}(x^2+y^2)^{-\frac14}dx$$
$$\leq C\int_0^\infty e^{-\rho y}dy\int_0^{(2\sin\frac{\delta}{2})^2} \frac{1}{|\sqrt{x}-2|\sin\theta||^{1/2}}x^{-1/2}(x^2+y^2)^{-\frac14}dx$$
$$\leq C\int_0^\infty e^{-\rho y}dy\int_0^{2\sin\frac{\delta}{2}} \frac{1}{|x-2\sin|\theta||^{1/2}}(x^4+y^2)^{-\frac14}dx$$
$$\leq C\int_0^\infty e^{-\rho y}dy\int_{-2\sin|\theta|}^{2\sin\frac{\delta}{2}-2\sin|\theta|} \frac{1}{|x|^{1/2}}((x+2\sin|\theta|)^4+y^2)^{-\frac14}dx$$
$$\leq C\int_0^\infty e^{-\rho y}dy\int_{-2\sin|\theta|}^{2\sin\frac{\delta}{2}} \frac{1}{|x|^{1/2}}((x+2\sin|\theta|)^4+y^2)^{-\frac14}dx$$
$$\leq C\int_0^\infty e^{-\rho y}dy\bigg(\int_0^{\sqrt{2\sin\frac{\delta}{2}}}(x^8+y^2)^{-\frac14}dx+\int_0^{\sqrt{2\sin|\theta|}}((x^2-2\sin|\theta|)^4+y^2)^{-\frac14}dx\bigg)$$
$$\leq C\int_0^\infty e^{-\rho y}dy\bigg(y^{-\frac14}\int_0^\infty(x^8+1)^{-\frac14}dx+y^{-\frac14}\int_0^{\sqrt{2\sin|\theta|}}(2\sin|\theta|-x^2)^{-\frac12}dx\bigg)$$
$$\leq C\int_0^\infty y^{-\frac14}e^{-\rho y}dy\bigg(\int_0^\infty(x^8+1)^{-\frac14}dx+\int_0^1(1-x^2)^{-\frac12}dx\bigg)\leq C\frac{1}{\rho^{3/4}}$$

Using the same agrument , we can also carry out

$$|I_{12}-\frac12\sqrt{\frac{2\pi}{\rho}}f(0)e^{\mathbf{i}\rho-\mathbf{i}\frac{\pi}{4}}|\leq C\frac{1}{\rho^{3/4}} \tag{5.9}$$

It remains to estimate $I_2$. Note that there exits $m>0$ such that $|\sin t|\geq m$ for any $t\in[-\pi/2,\pi/2]\backslash(-\delta/2,\delta/2)$. Upon integration by parts and representation (5.2) again, we have

$$|I_{12}|\leq C\rho^{-1}(1+\bigg|\int_{[-\pi/2,\pi/2]\backslash(-\delta/2,\delta/2)}\frac{\partial(f(t)(1-\chi_\delta(t)))}{\partial t}\frac{1}{\sin t}dt\bigg|)$$
$$\leq C\rho^{-1}(1+\int_{-\pi/2}^{\pi/2}\bigg|\frac{\partial(f(t)(1-\chi_\delta(t)))}{\partial t}\bigg|dt)$$
$$\leq C\rho^{-1}(1+\int_{-\pi/2}^{\pi/2}|(\kappa^2-\sin^2(t+\phi))^{-1/2}|dt)$$
$$\leq C\rho^{-1}(1+\int_{-\pi/2}^{\pi/2}|(\kappa^2-\sin^2 t)^{-1/2}|dt)$$
$$\leq C\rho^{-1}$$

This completes the proof. □

## 6. cross term of psf, 17.11.15

We need the following slight generalization of Van der Corput lemma for the oscillatory integral [4, P.152].

**Lemma 6.1** *Let $-\infty < a < b < \infty$, and $u$ is a $C^k$ function $u$ in $(a,b)$.*
*1. If $|u'(t)| \geq 1$ for $t \in (a,b)$ and $u'$ is monotone in (a,b), then for any $\phi(t)$ in $(a,b)$ with integrable derivatives*

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 3\lambda^{-1}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

*2. For all $k \geq 2$, if $|u^{(k)}(t)| \geq 1$ for $t \in (a,b)$ , then for any $\phi(t)$ in $(a,b)$ with integrable derivatives*

$$\left|\int_a^b e^{\mathbf{i}\lambda u(t)}\phi(t)dt\right| \leq 12k\lambda^{-1/k}\left[|\phi(b)| + \int_a^b |\phi'(t)|dt\right].$$

**Proof.** The assertion can be proved by extending the Van der Corptut lemma in [4]. Here we omit the details. □

**Lemma 6.2** *For $0 < \kappa < 1$, let $F(\lambda) = \int_0^\kappa f(t)e^{\mathbf{i}\lambda(\sqrt{1-t^2}-\tau\sqrt{\kappa^2-t^2}+\alpha t)}dt$, where $\tau \geq c_0 > 0$ and $\alpha \in \mathbb{R}$, then we have*

$$|F(\lambda)| \leq C(\kappa)\lambda^{-\frac{1}{2N_*}}\left[|f(\kappa)| + \int_0^\kappa |f'(t)|dt\right]$$

*where $N_* = \min\{N|\kappa^{2N-1} < c_0, N \in \mathbb{Z}_+\}$.*

**Proof.** Put $\phi(t) = -\sqrt{1-t^2}$ and $\psi(t,\tau) = \tau\phi(t/\kappa) - \phi(t) + \alpha t$. For easy of notations, we denote the $n$-th partial derivative of $g(t)$ with respect to $t$ by $g^{(n)}(t)$. Then, it is to see that, for $n > 1$

$$\psi^{(n)}(t,\tau) = \frac{\tau}{\kappa^{n-1}}\phi^{(n)}(\frac{t}{\kappa}) - \phi^{(n)}(t)$$

A standard computation show that

$$\phi^{(1)}(t) = \frac{t}{\sqrt{1-t^2}}$$
$$\phi^{(2)}(t) = \frac{1}{(1-t^2)^{3/2}}$$
$$\phi^{(3)}(t) = \frac{3t}{(1-t^2)^{5/2}}$$

Moreover, for $n \geq 3$, we have

$$\phi^{(n)}(t) = \frac{p_n(t)}{(1-t^2)^{n-1/2}} \tag{6.1}$$

where $p_n = \sum_0^{n-2} a_k^n t^k$ is a $(n-2)$-th polynomial such that its coefficients satisfy the following recursion formula:

$$a_{n-1}^{n+1} = (n+1)a_{n-2}^n, \quad a_{n-2}^{n+1} = (n+2)a_{n-3}^n$$
$$a_k^{n+1} = (k+1)a_{k+1}^n + (2n-k)a_{k-1}^n \quad \text{for } 1 \leq k \leq n-3$$
$$a_0^{n+1} = a_1^n$$

Since the polynmial coefficients are all positive, it is obvious that for $n \geq 1$, $\phi^{(n)}(t)$ is a monotone increasing positive function. Using the recursion formula, it follows that

$$\phi^{(n)}(0) = \begin{cases} 0 & \text{n is odd,} \\ (n-1)!!(n-3)!! & \text{n is even.} \end{cases} \tag{6.2}$$

where $(2k-1)!!$ is double factorial and $n > 3$. We are now in the position to proof the inequality. Since $0 < \kappa < 1$, obersev that

$$\psi^{(2N_*+1)}(t,\tau) \geq \frac{\tau}{\kappa^{2N_*}}\phi^{(2N_*+1)}(t) - \phi^{(2N_*+1)}(t) > 0$$

Therefore, $\psi^{(2N_*)}(t,\tau)$ is monotone increasing in $[0,\kappa)$. By (6.2), we get

$$\psi^{(2N_*)}(t,\tau) \geq \psi^{(2N_*)}(0,\tau) \geq \psi^{(2N_*)}(0,c_0) = C(2N_*)(\frac{c_0}{\kappa^{2N_*-1}} - 1) > 0 \tag{6.3}$$

The lemma is now a direct consequence of lemma (6.1). □

## 7. Other exponential decay term: 17.11.16 on G1

The parameterization of hyperbolic curve passing $(\pm 1, 0)$ is:

$$\xi_1 = \pm\sqrt{t^2+1} \qquad \xi_2 = t$$

where $t \in \mathbb{R}$. Substituting $\xi = \xi_1 + \mathbf{i}\xi_2$ into $\mu(\xi) := (1-\xi^2)^{1/2}$ and $\mu_\kappa(\xi) := (\kappa^2 - \xi^2)^{1/2}$, we get

$$\begin{aligned} \operatorname{Im}\mu(\xi) &= \operatorname{Im}(1 - (\xi_1^2 - \xi_2^2 + \mathbf{i}2\xi_1\xi_2))^{1/2} \\ &= \operatorname{Im}(-2t\sqrt{t^2+1}\mathbf{i})^{1/2} = t^{1/2}(t^2+1)^{1/4} \end{aligned}$$

$$\begin{aligned} \operatorname{Im}\mu_\kappa(\xi) &= \operatorname{Im}(\kappa^2 - (\xi_1^2 - \xi_2^2 + \mathbf{i}2\xi_1\xi_2))^{1/2} \\ &= \operatorname{Im}(\kappa^2 - 1 - 2t\sqrt{t^2+1}\mathbf{i})^{1/2} \\ &= \sqrt{\frac{\sqrt{(1-\kappa^2)^2 + 4t(t^2+1)} + 1 - \kappa^2}{2}} \\ &\geq t^{1/2}(t^2+1)^{1/4} \end{aligned}$$

where we only consider the branch, denoted by $\Gamma^+$, in the first quadrant here. For $a > 0, b > 0$, we have

$$|e^{\mathbf{i}\xi a + \mathbf{i}\mu(\xi)b + \mathbf{i}\mu_\kappa(\xi)c}| \leq e^{-ta - t^{1/2}(t^2+1)^{1/4}b - t^{1/2}(t^2+1)^{1/4}c} \leq e^{-t(b+c)}$$

**Lemma 7.1** *For $\xi \in \Gamma_0$, let $f(\xi)$ is a complex valued function in $L^1(\Gamma^+)$ such that $|f(\xi)| \leq C(1+\xi^k)$, $k \in \mathbb{Z}_+$. Then we have*

$$\begin{aligned} &|I(a,b,c) := \int_{\Gamma^+} f(\xi)\ e^{\mathbf{i}\xi a + \mathbf{i}\mu(\xi)b + \mathbf{i}\mu_\kappa(\xi)c}\ d\xi| \\ &\leq C(\frac{1}{b+c} + \frac{1}{(b+c)^k}) \end{aligned}$$

**Proof.**

$$\frac{d\xi(t)}{dt} = \frac{t}{\sqrt{t^2+1}} + \mathbf{i}$$

Substituting $\xi(t)$ into $I(a,b,c)$, we hvae

$$\begin{aligned}|I(a,b,c)| &= \Big|\int_0^\infty |f(\xi(t))\frac{d\xi(t)}{dt} e^{\mathbf{i}\xi(t)a+\mathbf{i}\mu(\xi(t))b+\mathbf{i}\mu_\kappa(\xi(t))c} dt\Big| \\ &\leq C\int_0^\infty (1+t^k)e^{-t(b+c)}dt \\ &\leq C(\frac{1}{b+c} + \frac{1}{(b+c)^k})\end{aligned}$$

□

**Lemma 7.2** *Let $f(\xi)$ is a bounded complex valued function in $L^1((\kappa,1))$. Then we have*

$$|I(a,b) := \int_\kappa^1 |f(\xi)e^{\mathbf{i}\xi a+\mathbf{i}\mu_\kappa(\xi)b}d\xi|$$
$$\leq C\frac{1}{b}$$

**Proof.** It is simple to see that

$$\begin{aligned}|I(a,b)| &\leq C\int_\kappa^1 e^{-b\sqrt{\xi^2-\kappa^2}}d\xi \\ &\leq C\int_0^{\sqrt{1-\kappa^2}} \frac{t}{\sqrt{t^2+\kappa^2}}e^{-bt}dt \\ &\leq C\frac{1}{b}\end{aligned}$$

□

## 8. about principle of arguement

Put

$$\begin{aligned}\delta_\pm(t) = &\ (\kappa-2t^2)^2 \mp \mathbf{i}4t^2\sqrt{1-t^2}\sqrt{t^2-\kappa} \\ &:= f_1(t) \mp \mathbf{i}f_2(t)\end{aligned}$$

where $0<\kappa<1$ and we have

$$\delta'_\pm(t) = f_1'(t) \mp \mathbf{i}f_2'(t)$$

It is easy to see $f_2(1) = f_2(\kappa) = 0$ and $f_1(t) > 0$ for any $\kappa \leq t \leq 1$. Then

$$\int_\kappa^1 \frac{\delta'_+(t)}{\delta_+(t)} - \frac{\delta'_-(t)}{\delta_-(t)}dt$$
$$= 2\mathbf{i}\int_\kappa^1 \mathrm{Im}\,(\frac{\delta'_+(t)}{\delta_+(t)})dt$$

$$
\begin{aligned}
&= 2\mathbf{i}\int_{\kappa}^{1} \mathrm{Im}\, \frac{(f_1'(t) - \mathbf{i} f_2'(t))f_1(t) + \mathbf{i} f_2(t))}{(f_1(t) - \mathbf{i} f_2(t))(f_1(t) + \mathbf{i} f_2(t))} dt \\
&= 2\mathbf{i}\int_{\kappa}^{1} \frac{f_1'(t)f_2(t) - f_1(t)f_2'(t)}{f_1^2(t) + f_2^2(t)} dt \\
&= 2\mathbf{i}\int_{\kappa}^{1} \frac{f_1^2(t)}{f_1^2(t) + f_2^2(t)} \frac{f_1'(t)f_2(t) - f_1(t)f_2'(t)}{f_1^2(t)} dt \\
&= \ -2\mathbf{i}\int_{\kappa}^{1} \frac{f_1^2(t)}{f_1^2(t) + f_2^2(t)} d\frac{f_2(t)}{f_1(t)} \\
&= \ -2\mathbf{i} \arctan \frac{f_2(t)}{f_1(t)}\Big|_{\kappa}^{1} = 0
\end{aligned}
$$

Notic that, the condition only used above are $f_2(1) = f_2(\kappa) = 0$ and $f_1(t) > 0$.

## 9. Fundamental solution of Elastic wave

$$
\begin{aligned}
G(x;y) &= \frac{1}{\omega^2}(\nabla \times \nabla \cdot (g_s(x;y)\mathbb{I}) - \nabla\nabla g_p(x;y)) && (9.1)\\
&= \frac{1}{\omega^2}(k_s^2 g_s(x,y) + \nabla\nabla(g_s(x;y) - g_p(x;y))) && (9.2)
\end{aligned}
$$

where y is the Dirac source, $g_p(x;y)$ or $g_s(x;y)$ is the fundamental solution of the scalar Helmhotlz equation with wavenumbers $k_p = \omega/c_p$ or $k_s = \omega/cs$.

$$
g_\alpha = \frac{\mathrm{i}}{4} H_0^{(1)}(k_\alpha|x-y|) \tag{9.3}
$$

where $H_0^{(1)}(t)$ is the Hankel function of the first type and order zero. By straight calculation using $H_1^{(1)}(t) = -dH_0^{(1)}(t)/dt$ and $dH_1^{(1)}(t)/dt = H_0^{(1)}(t) - H_1^{(1)}(t)/t$, we have

$$
\begin{aligned}
G_{ij}(x;y) = \frac{\mathbf{i}}{4}\{(&\frac{k_s^2}{\omega^2} H_0^{(1)}(k_s|x-y|) - \frac{1}{\omega^2}\frac{k_s H_1^{(1)}(k_s|x-y| - k_p H_1^{(1)}(k_p|x-y|)}{|x-y|})\delta_{ij} \\
+\frac{1}{\omega^2}[(&\frac{2k_s H_1^{(1)}(k_s|x-y| - 2k_p H_1^{(1)}(k_p|x-y|)}{|x-y|} - (k_s^2 H_0^{(1)}(k_s|x-y|) - k_p^2 H_0^{(1)}(k_p|x-y|))]\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\}
\end{aligned}
$$

The definition of hankal function is $H_k^{(1)}(t) = J_k(t) + \mathbf{i}Y_k(t)$ where

$$
J_k(t) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(k+p)!}(t/2)^{k+2p}
$$

Specially

$$
\begin{aligned}
J_0(t) &= \sum_{p=0}^{\infty} \frac{(-1)^p}{p!p!}(t/2)^{2p} = 1 + \cdots \\
J_1(t) &= \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(1+p)!}(t/2)^{1+2p} = \frac{t}{2} + \cdots
\end{aligned}
$$

and

$$Y_k(t) = \frac{1}{\pi}\{\ln t^2 - 2\ln 2 + 2C_{euler}\}J_k(t) - \frac{1}{\pi}\sum_{p=0}^{k-1}\frac{(k-1-p)!}{p!}(2/t)^{k-2p}$$
$$- \frac{1}{\pi}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!(k+p)!}(t/2)^{k+2p}\{\psi(p+k)+\psi(p)\}$$

Speceilly

$$Y_0(t) = \frac{1}{\pi}\{\ln t^2 - 2\ln 2 + 2C_{euler}\}J_0(t) - \frac{1}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^p}{p!p!}(t/2)^{2p}\{2\psi(p)\}$$
$$Y_1(t) = \frac{1}{\pi}\{\ln t^2 - 2\ln 2 + 2C_{euler}\}J_1(t) - \frac{1}{\pi}\frac{2}{t} - \frac{t}{2\pi}$$
$$-\frac{1}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^p}{p!(1+p)!}(t/2)^{1+2p}\{\psi(p+1)+\psi(p)\}$$

Thus, we have

$$H_0^{(1)}(kr) = 1 + \mathbf{i}\frac{2}{\pi}(C_{euler} + \ln k - \ln 2) + \mathbf{i}\frac{1}{\pi}\ln r^2 + o(kr)$$
$$H_1^{(1)}(kr) = \frac{kr}{2} + \mathbf{i}\frac{1}{\pi}(C_{euler} + \ln k - \ln 2 - \frac{1}{2})kr - \mathbf{i}\frac{1}{\pi}\frac{2}{kr} + \mathbf{i}\frac{1}{\pi}\ln r^2\frac{kr}{2} + o(k^2r^2)$$
$$\frac{\mathbf{i}}{4}H_0^{(1)}(kr) = \frac{\mathbf{i}}{4} - \frac{1}{2\pi}(C_{euler} + \ln k - \ln 2) - \frac{1}{4\pi}\ln r^2 + o(kr)$$
$$\frac{\mathbf{i}}{4}H_1^{(1)}(kr) = \mathbf{i}\frac{kr}{8} - \frac{1}{4\pi}(C_{euler} + \ln k - \ln 2 - \frac{1}{2})kr + \frac{1}{4\pi}\frac{2}{kr} - \frac{1}{4\pi}\ln r^2\frac{kr}{2} + o(k^2r^2)$$

We also need to define the surface traction $T_x^n(\cdot)$ on the normal direction n,

$$T_x^n u(x) := \sigma\cdot n = 2\mu\frac{\partial u}{\partial n} + \lambda n \mathrm{div} u + \mu n \times \mathrm{curl} u$$

where

$$\sigma(u) = \begin{pmatrix} (\lambda+2\mu)\partial u_1/\partial x_1 + \lambda\partial u_2/\partial x_2 & \mu\partial u_1/\partial x_2 + \mu\partial u_2/\partial x_1 \\ \mu\partial u_1/\partial x_2 + \mu\partial u_2/\partial x_1 & (\lambda+2\mu)\partial u_2/\partial(x_2) + \lambda\partial u_1/\partial x_1 \end{pmatrix}$$

A simple computation show that

$$\frac{\partial^3 H_0^{(1)}(k|x-y|)}{\partial x_i^2\partial x_j} = (1+2\delta_i j)(-k^2H_0^{(1)}(kr)\frac{r_j}{r^2} + 2kH_1^{(1)}(kr)\frac{r_j}{r^3})$$
$$+k^3H_1^{(1)}(kr)\frac{r_i^2r_j}{r^3} + 4k^2H_0^{(1)}(kr)\frac{r_i^2r_j}{r^4} - 8kH_1^{(1)}(kr)\frac{r_i^2r_j}{r^5}$$

where $r = |x-y|$ and $r_i = x_i - y_i$.

$$\frac{\mathbf{i}}{4}H_0^{(1)}(kr) = -\frac{1}{2\pi}(\ln\frac{kr}{2} + C_{euler})(1-(\frac{kr}{2})^2+\cdots) + \frac{1}{4\pi}(2(\frac{kr}{2})^2+\cdots) + \frac{\mathbf{i}}{4}(1-(\frac{kr}{2})^2+\cdots)$$
$$= -\frac{1}{2\pi}(\ln\frac{kr}{2})(1+O(r^2)) - \frac{1}{2\pi}C_{euler} + \frac{\mathbf{i}}{4} + O(r^2)$$
$$\frac{\mathbf{i}}{4}H_1^{(1)}(kr) = -\frac{1}{2\pi}(\ln\frac{kr}{2} + C_{euler})(\frac{kr}{2} - \frac{1}{2}(\frac{kr}{2})^3+\cdots) + \frac{1}{4\pi}(\frac{kr}{2}+O(r^3)) + \frac{\mathbf{i}}{4}(\frac{kr}{2} - \frac{1}{2}(\frac{kr}{2})^3+\cdots) + \frac{1}{2\pi}\frac{1}{kr}$$
$$= -\frac{1}{4\pi}(\ln\frac{kr}{2})(kr+O(r^3)) - \frac{kr}{4\pi}C_{euler} + \frac{kr}{8\pi} + \frac{\mathbf{i}kr}{8} + \frac{1}{2\pi}\frac{1}{kr} + O(r^3)$$

$$A(kr) := \frac{\mathbf{i}}{4}(k^2 H_0^{(1)}(kr) - 2kH_1^{(1)}(kr)/r) = \frac{k^2}{4\pi}(\ln\frac{kr}{2})(\frac{kr}{2})^2 - \frac{1}{\pi r^2} - \frac{k^2}{4\pi} + O(r^2)$$

and

$$A_{sp}(r) = A(k_s r) - A(k_p r) = \frac{k_s^2}{4\pi}(\ln\frac{kr}{2})(\frac{kr}{2})^2 - \frac{k_p^2}{4\pi}(\ln\frac{kr}{2})(\frac{kr}{2})^2 - (\frac{k_s^2}{4\pi} - \frac{k_p^2}{4\pi}) + O(r^2)$$

Let $g^{jkk} = \frac{\partial^3 g}{\partial x_j \partial x_k^2}$, $d = g_s - g_p$, thus

$$\begin{aligned}
g^{iij} &= (1+2\delta_{ij})(-A(kr)\frac{r_j}{r^2}) + \frac{\mathbf{i}k^3}{4}H_1^{(1)}(kr)\frac{r_i^2 r_j}{r^3} + 4A(kr)\frac{r_i^2 r_j}{r^4} \\
&= (1+2\delta_{ij})(\frac{1}{\pi r^3} + \frac{k^2}{4\pi r})\frac{r_j}{r} - (\frac{4}{\pi r^3} + \frac{k^2}{2\pi r})\frac{r_i^2 r_j}{r^3} + O(r\ln r) \\
d^{iij} &= (1+2\delta_{ij})(-A_{sp}\frac{r_j}{r^2}) + (\frac{\mathbf{i}k_s^3}{4}H_1^{(1)}(k_s r) - \frac{\mathbf{i}k_p^3}{4}H_1^{(1)}(k_p r))\frac{r_i^2 r_j}{r^3} + 4A_{sp}\frac{r_i^2 r_j}{r^4} \\
&= (1+2\delta_{ij})(\frac{k_s^2}{4\pi r} - \frac{k_p^2}{4\pi r})\frac{r_j}{r} - (\frac{k_s^2}{2\pi r} - \frac{k_p^2}{2\pi r})\frac{r_i^2 r_j}{r^3} + O(r\ln r) \\
&= (1+2\delta_{ij})\frac{(\lambda+\mu)\omega^2}{\mu(\lambda+2\mu)}\frac{1}{4\pi r}\frac{r_j}{r} - \frac{(\lambda+\mu)\omega^2}{\mu(\lambda+2\mu)}\frac{1}{2\pi r}\frac{r_i^2 r_j}{r^3} + O(r\ln r) \\
k^2 g^i &= -\frac{\mathbf{i}}{4}H_1^{(1)}(kr)\frac{kr_i}{r} = -\frac{k^2}{2\pi r}\frac{r_i}{r} + O(r\ln r) \\
d^{iij} + d^{jjj} &= 2\frac{(\lambda+\mu)\omega^2}{\mu(\lambda+2\mu)}\frac{1}{4\pi r}\frac{r_j}{r} + O(r\ln r)
\end{aligned}$$

Then we have [**?**, p43]

$$\begin{aligned}
\sigma(Ge_1)n &= \frac{1}{\omega^2}\begin{pmatrix} (\lambda+2\mu)(k_s^2 g_s^1 + d^{111}) + \lambda d^{122} & \mu(k_s^2 g_s^2 + d^{112}) + \mu d^{112} \\ \mu(k_s^2 g_s^2 + d^{112}) + \mu d^{112} & (\lambda+2\mu)d^{122} + \lambda(k_s^2 g_s^1 + d^{111}) \end{pmatrix} n \\
&= \frac{\mu}{2\pi(\lambda+2\mu)}\left(\begin{pmatrix} \frac{2(\lambda+\mu)r_1^2}{\mu r^2} + 1 \\ \frac{2(\lambda+\mu)r_1 r_2}{\mu r^2} \end{pmatrix}(-\frac{r_1 n_1}{r^2} - \frac{r_2 n_2}{r^2}) - \begin{pmatrix} 0 \\ \frac{r_2 n_1 - r_1 n_2}{r^2} \end{pmatrix}\right) + O(r\ln r)
\end{aligned}$$

and

$$\begin{aligned}
\sigma(Ge_2)n &= \frac{1}{\omega^2}\begin{pmatrix} (\lambda+2\mu)d^{112} + \lambda(k_s^2 g_s^2 + d^{222}) & \mu(k_s^2 g_s^1 + d^{122}) + \mu d^{122} \\ \mu(k_s^2 g_s^1 + d^{122}) + \mu d^{122} & (\lambda+2\mu)(k_s^2 g_s^2 + d^{222}) + \lambda d^{112} \end{pmatrix} n \\
&= \frac{\mu}{2\pi(\lambda+2\mu)}\left(\begin{pmatrix} \frac{2(\lambda+\mu)r_1 r_2}{\mu r^2} \\ \frac{2(\lambda+\mu)r_2^2}{\mu r^2} + 1 \end{pmatrix}(-\frac{r_1 n_1}{r^2} - \frac{r_2 n_2}{r^2}) - \begin{pmatrix} \frac{r_1 n_2 - r_2 n_1}{r^2} \\ 0 \end{pmatrix}\right) + O(r\ln r)
\end{aligned}$$

Now Let $u$ be represented as single potential:

$$u = \int_{\partial D} G(x,y)\phi(y)ds(y) \tag{9.4}$$

with Neumann boundary condition

$$T_x u(x) = f(x) \quad \text{on} \quad \partial D \tag{9.5}$$

Then we obtain corresponding integral equation

$$\mathbf{P.V.}\int_{\partial D} T_x G(x,y)\phi(y)ds(y) - \frac{1}{2}\phi(x) = f(x) \tag{9.6}$$

where $x \in \partial D$. We describe the necessary parametrization of the integral equation in the two-dimensional case. We assume that the boundary curve $\partial D$ possesses a regular analytic and $2\pi$-periodic parametric representation of the form

$$x(t) = (x_1(t), x_2(t))$$

in counterclockwise orientation satisfying $|x'(t)| > 0$ for all t. Let [?]

$$\begin{aligned}
T(x,y) &= (T_x(N_1(x,y))n, T_x(N_1(x,y))n)\\
T_0(x,y) &= -\frac{\mu}{2\pi(\lambda+2\mu)}\begin{pmatrix} 0 & \frac{r_1n_2-r_2n_1}{r^2}\\ \frac{r_2n_1-r_1n_2}{r^2} & 0\end{pmatrix}\\
T_1(x,y) &= T(x,y) - T_0(x,y)
\end{aligned}$$

Then by above analysis we have

$$\int_0^{2\pi} T_1(x(t),x(\tau))\phi(x(\tau))|x'(\tau)|dt + \mathbf{P.V.}\int_0^{2\pi} T_0(x(t),x(\tau))\phi(x(\tau))|x'(\tau)|dt - \frac{1}{2}\phi(x(t)) = f(x(t))$$

In particular, using expasion above, we can deduce the diagonal terms:

$$\lim_{\tau\to t}\frac{-r_1n_1 - r_2n_2}{r^2} = \lim_{\tau\to t}\frac{(x_1(\tau)-x_1(t))x_2'(t) - (x_2(\tau)-x_2(t))x_1'(t)}{|x(t)-x(\tau)|^2|x'(t)|} = \frac{x_1''(t)x_2'(t) - x_2''(t)x_1'(t)}{2|x'(t)|^3}$$

and

$$\begin{aligned}
&\lim_{\tau\to t}\frac{(r_1n_2 - r_2n_1)|x'(\tau)|(\tau-t)}{r^2}\\
&= \lim_{\tau\to t}\frac{(x_1(\tau)-x_1(t))x_1'(t) + (x_2(\tau)-x_2(t))x_2'(t))|x'(\tau)|(\tau-t)}{|x(t)-x(\tau)|^2|x'(t)|} = 1
\end{aligned}$$

According to above analysis, it is easy to see that $T_1$ has no singularity. Therefore, the numerical formulartion for $T_1$ requires only straightforward application of simple quadrature formula. we choose an equidistant set of knots $t_j := \pi j/n, j = 0, ..., 2n-1$ and divide $[0, 2\pi)$ into n equivalent interval $I_i = [2j\pi/n, 2(j+1)\pi/n)$ where $[0, 2\pi) = \bigcup_{j=1}^{j=n-1} I_i$. Each $I_i$ is described by 3 nodes of the intrinsic variable $\xi(-1 \le \xi \le 1)$ and the quadratic shape functions are:

$$\begin{aligned}
A_{-1}(\xi) &= \frac{\xi(\xi-1)}{2}\\
A_0(\xi) &= 1-\xi^2\\
A_1(\xi) &= \frac{\xi(\xi+1)}{2}
\end{aligned}$$

let

$$f(x,y) = \frac{(x_1-y_1)n_2^x - (x_2-y_2)n_1^x}{|x-y|^2}$$

and it is easy to see that $(n_x^1, n_x^2) = (x_2'(t), -x_1'(t))/|x'(t)|$. Now, we can repersent the integral in variable $\xi \in [-1, 1]$. For $x \in \partial D$ we have:

$$\begin{aligned}
&\mathbf{P.V.}\int_{\partial D} f(x,y)\phi(y)ds(y)\\
&= \sum_{x\notin I_i}\int_{I_i} f(x,x(t))\phi(x(t))|x'(t)|dt + \mathbf{P.V.}\sum_{x\in I_i}\int_{I_i} f(x,x(t))\phi(x(t))|x'(t)|dt
\end{aligned}$$

if $x = x(2j\pi/n)$, above integral becomes to

$$\sum_{i\neq j,j-1}\frac{\pi}{n}\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^i(\xi)-x_k\frac{2j\pi}{n}))x_k'(\frac{2j\pi}{n})}{|x(\frac{2j\pi}{n})-x^i(\xi)|^2}\phi(x^i(\xi))\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{2j\pi}{n})|}d\xi$$

$$+\ \mathbf{P}.\mathbf{V}.\sum_{i=j,j-1}\frac{\pi}{n}\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^i(\xi)-x_k\frac{2j\pi}{n}))x_k'(\frac{2j\pi}{n})}{|x(\frac{2j\pi}{n})-x^i(\xi)|^2}\phi(x^i(\xi))\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{2j\pi}{n})|}d\xi$$

$$\approx\ \sum_{i\neq j,j-1}\sum_{l=-1}^{1}\phi(x^i(l))\frac{\pi}{n}\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^i(\xi)-x_k(\frac{2j\pi}{n}))x_k'(\frac{2j\pi}{n})}{|x(\frac{2j\pi}{n})-x^i(\xi)|^2}A_l(\xi)\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{2j\pi}{n})|}d\xi$$

$$+\ \phi(x(\frac{2j\pi}{n}))\mathbf{P}.\mathbf{V}.\frac{\pi}{n}\Bigg(\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^j(\xi)-x_k^j(-1))x_k'(\frac{2j\pi}{n})}{|x^j(-1)-x^j(\xi)|^2}A_{-1}(\xi)\frac{|x'(\frac{(2j+1+\xi)\pi}{n})|}{|x'(\frac{2j\pi}{n})|}d\xi$$

$$+\ \int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^{j-1}(\xi)-x_k^{j-1}(1))x_k'(\frac{2j\pi}{n})}{|x^{j-1}(1)-x^{j-1}(\xi)|^2}A_1(\xi)\frac{|x'(\frac{(2j-1+\xi)\pi}{n})|}{|x'(\frac{2j\pi}{n})|}d\xi\Bigg)$$

and if $x = x((2j+1)\pi/n)$, above integral becomes to

$$\sum_{i\neq j,}\frac{\pi}{n}\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^i(\xi)-x_k\frac{2j+1\pi}{n}))x_k'(\frac{2j+1\pi}{n})}{|x(\frac{2j+1\pi}{n})-x^i(\xi)|^2}\phi(x^i(\xi))\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{2j+1\pi}{n})|}d\xi$$

$$+\ \mathbf{P}.\mathbf{V}.\sum_{i=j,j-1}\frac{\pi}{n}\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^i(\xi)-x_k\frac{2j+1\pi}{n}))x_k'(\frac{2j+1\pi}{n})}{|x(\frac{2j+1\pi}{n})-x^i(\xi)|^2}\phi(x^i(\xi))\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{2j+1\pi}{n})|}d\xi$$

$$\approx\ \sum_{i\neq j}\sum_{l=-1}^{1}\phi(x^i(l))\frac{\pi}{n}\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^i(\xi)-x_k(\frac{(2j+1)\pi}{n}))x_k'(\frac{(2j+1)\pi}{n})}{|x(\frac{(2j+1)\pi}{n})-x^i(\xi)|^2}A_l(\xi)\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{(2j+1)\pi}{n})|}d\xi$$

$$+\ \phi(x(\frac{(2j+1)\pi}{n}))\mathbf{P}.\mathbf{V}.\frac{\pi}{n}\Bigg(\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^j(\xi)-x_k^j(0))x_k'(\frac{(2j+1)\pi}{n})}{|x^j(0)-x^j(\xi)|^2}A_0(\xi)\frac{|x'(\frac{((2j+1)+\xi)\pi}{n})|}{|x'(\frac{(2j+1)\pi}{n})|}d\xi$$

where $x^i(\xi) = x((2i+1+\xi)\pi/n)$. For simplicity, we denote a 3-D tensor $M_{jil}$:

$$M_{jil} = \frac{\pi}{n}\int_{-1}^{1}\frac{\sum_{k=1,2}(x_k^i(\xi)-x_k(\frac{j\pi}{n}))x_k'(\frac{j\pi}{n})}{|x(\frac{j\pi}{n})-x^i(\xi)|^2}A_l(\xi)\frac{|x'(\frac{(2i+1+\xi)\pi}{n})|}{|x'(\frac{j\pi}{n})|}d\xi$$

Then, the general Matrix is

$$G_{ji} = \begin{cases} M_{j,i,-1}+M_{j,i-1,1} & j\text{ is even}\\ M_{j,i,0} & j\text{ is odd}\end{cases}$$

## 10. Traction Tenson of Neumann Green Function

$$\hat{\Phi}(\xi,x_2;y_2) = \frac{\mathbf{i}}{2\omega^2}\left[\begin{pmatrix}\mu_s & -\xi\frac{x_2-y_2}{|x_2-y_2|}\\ -\xi\frac{x_2-y_2}{|x_2-y_2|} & \frac{\xi^2}{\mu_s}\end{pmatrix}e^{\mathbf{i}\mu_s|x_2-y_2|}+\begin{pmatrix}\frac{\xi^2}{\mu_p} & \xi\frac{x_2-y_2}{|x_2-y_2|}\\ \xi\frac{x_2-y_2}{|x_2-y_2|} & \mu_p\end{pmatrix}e^{\mathbf{i}\mu_p|x_2-y_2|}\right]$$

$$\sigma(\Phi)(\xi,x_2;y_2)e_1 = \frac{\mu}{2\omega^2}\left[\begin{pmatrix}-2\xi\mu_s & \frac{x_2-y_2}{|x_2-y_2|}2\xi^2\\ -\frac{x_2-y_2}{|x_2-y_2|}\beta & \frac{\xi\beta}{\mu_s}\end{pmatrix}e^{\mathbf{i}\mu_s|x_2-y_2|}+\begin{pmatrix}-\frac{\alpha\xi}{\mu_p} & -\frac{x_2-y_2}{|x_2-y_2|}\alpha\\ -\frac{x_2-y_2}{|x_2-y_2|}2\xi^2 & -2\xi\mu_p\end{pmatrix}e^{\mathbf{i}\mu_p|x_2-y_2|}\right]$$

$$\sigma(\Phi)(\xi,x_2;y_2)e_2=\frac{\mu}{2\omega^2}\Bigg[\begin{pmatrix} -\frac{x_2-y_2}{|x_2-y_2|}\beta & \frac{\xi\beta}{\mu_s} \\ 2\xi\mu_s & -\frac{x_2-y_2}{|x_2-y_2|}2\xi^2 \end{pmatrix}e^{\mathbf{i}\mu_s|x_2-y_2|}+\begin{pmatrix} -\frac{x_2-y_2}{|x_2-y_2|}2\xi^2 & -2\xi\mu_p \\ -\frac{\beta\xi}{\mu_p} & -\frac{x_2-y_2}{|x_2-y_2|}\beta \end{pmatrix}e^{\mathbf{i}\mu_p|x_2-y_2|}\Bigg]$$

where $\alpha(\xi)=k_s^2-2\mu_p^2$, $\beta(\xi)=k_s^2-2\xi^2$.

$$\hat{N}(\xi,x_2;y_2)=\hat{\Phi}(\xi,x_2;y_2)-\hat{\Phi}(\xi,x_2;-y_2)+\hat{N}_c(\xi,x_2;y_2) \tag{10.1}$$

$$\hat{N}_c(\xi,x_2;y_2)=\frac{\mathbf{i}}{\omega^2\delta(\xi)}\Big\{A(\xi)e^{\mathbf{i}\mu_s(x_2+y_2)}+B(\xi)e^{\mathbf{i}\mu_p(x_2+y_2)} \tag{10.2}$$
$$+C(\xi)e^{\mathbf{i}\mu_s x_2+\mu_p y_2}+D(\xi)e^{\mathbf{i}\mu_p x_2+\mu_s y_2}\Big\}$$

where

$$A(\xi)=\begin{pmatrix} \mu_s\beta^2 & -4\xi^3\mu_s\mu_p \\ -\xi\beta^2 & 4\xi_4\mu_p \end{pmatrix} \qquad B(\xi)=\begin{pmatrix} 4\xi^4\mu_s & \xi\beta^2 \\ 4\xi^3\mu_s\mu_p & \mu_p\beta^2 \end{pmatrix}$$
$$C(\xi)=\begin{pmatrix} 2\xi^2\mu_s\beta & -2\xi\mu_s\mu_p\beta \\ -2\xi^3\beta & 2\xi^2\mu_p\beta \end{pmatrix} \quad D(\xi)=\begin{pmatrix} 2\xi^2\mu_s\beta & 2\xi^3\beta \\ 2\xi\mu_s\mu_p\beta & 2\xi^2\mu_p\beta \end{pmatrix}$$

The traction of integral part $N_c$:

$$\sigma(N_c(\xi,x_2;y_2))e_1=\frac{-\mu}{\omega^2\delta(\xi)}\Big\{A(\xi)e^{\mathbf{i}\mu_s(x_2+y_2)}+B(\xi)e^{\mathbf{i}\mu_p(x_2+y_2)}$$
$$+C(\xi)e^{\mathbf{i}\mu_s x_2+\mu_p y_2}+D(\xi)e^{\mathbf{i}\mu_p x_2+\mu_s y_2}\Big\}$$

where

$$A(\xi)=\begin{pmatrix} 2\xi\mu_s\beta^2 & -8\mu_s\mu_p\xi^4 \\ \beta^3 & -4\xi^3\mu_p\beta \end{pmatrix} \qquad B(\xi)=\begin{pmatrix} 4\xi^3\mu_s\alpha & \alpha\beta^2 \\ 8\xi^4\mu_s\mu_p & 2\xi\mu_p\beta^2 \end{pmatrix}$$
$$C(\xi)=\begin{pmatrix} 4\xi^3\mu_s\beta & -4\xi^2\mu_s\mu_p\beta \\ 2\xi^2\beta^2 & -2\xi\mu_p\beta^2 \end{pmatrix} \quad D(\xi)=\begin{pmatrix} 2\xi\mu_s\alpha\beta & 2\xi^2\alpha\beta \\ 4\xi^2\mu_s\mu_p\beta & 4\xi^3\mu_p\beta \end{pmatrix}$$

In particular, for $y_2=0$, a more simpler form are deduced:

$$\sigma(N(\xi,x_2;y_2))e_1=\frac{-1}{\delta(\xi)}\Big\{\begin{pmatrix} 2\xi\mu_s\beta & -4\mu_s\mu_p\xi^2 \\ \beta^2 & -2\xi\mu_p\beta \end{pmatrix}e^{\mathbf{i}\mu_s x_2}+\begin{pmatrix} 2\xi\mu_s\alpha & \alpha\beta \\ 4\xi^2\mu_s\mu_p & 2\xi\mu_p\beta \end{pmatrix}e^{\mathbf{i}\mu_p x_2}\Big\}$$

and

$$\sigma(N_c(\xi,x_2;y_2))e_2=\frac{-\mu}{\omega^2\delta(\xi)}\Big\{A(\xi)e^{\mathbf{i}\mu_s(x_2+y_2)}+B(\xi)e^{\mathbf{i}\mu_p(x_2+y_2)}$$
$$+C(\xi)e^{\mathbf{i}\mu_s x_2+\mu_p y_2}+D(\xi)e^{\mathbf{i}\mu_p x_2+\mu_s y_2}\Big\}$$

where

$$A(\xi)=\begin{pmatrix} \beta^3 & -4\xi^3\mu_p\beta \\ -2\xi\mu_s\beta^2 & 8\mu_s\mu_p\xi^4 \end{pmatrix} \qquad B(\xi)=\begin{pmatrix} 8\xi^4\mu_s\mu_p & 2\xi\mu_p\beta^2 \\ 4\xi^3\mu_s\beta & \beta^3 \end{pmatrix}$$
$$C(\xi)=\begin{pmatrix} 2\xi^2\beta^2 & -2\xi\mu_p\beta^2 \\ -4\xi^3\mu_s\beta & 4\xi^2\mu_s\mu_p\beta \end{pmatrix} \quad D(\xi)=\begin{pmatrix} 4\xi^2\mu_s\mu_p\beta & 4\xi^3\mu_p\beta \\ 2\xi\mu_s\beta^2 & 2\xi^2\beta^2 \end{pmatrix}$$

Similarly, for $y_2 = 0$, a more simpler form are deduced:

$$\sigma(N(\xi, x_2; y_2))e_2 = \frac{-1}{\delta(\xi)}\left\{ \begin{pmatrix} \beta^2 & -2\xi\mu_p\beta \\ -2\xi\mu_s\beta & 4\mu_s\mu_p\xi^2 \end{pmatrix} e^{\mathbf{i}\mu_s x_2} + \begin{pmatrix} 4\xi^2\mu_s\mu_p & 2\xi\mu_p\beta \\ 2\xi\mu_s\beta & \beta^2 \end{pmatrix} e^{\mathbf{i}\mu_p x_2} \right\}$$

where $\delta(\xi) = \beta^2 + 4\xi^2\mu_s\mu_p$

## 11. reflection of Plane wave

### *11.1. P-wave*

We denote incident P-wave [1, p172] as

$$u^0 = A_0(\sin t_0, \cos t_0)^T e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)} \tag{11.1}$$

and its stress as

$$\sigma(u^0) = \mathbf{i}k_p A_0(2\mu \sin t_0 \cos t_0, \lambda + 2\mu \cos^2 t_0)^T e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)}$$

The reflected P-wave is represented as

$$u^1 = A_1(\sin t_1, -\cos t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 - x_2 \cos t_1)}$$
$$\sigma(u^1) = \mathbf{i}k_p A_1(-2\mu \sin t_1 \cos t_1, \lambda + 2\mu \cos^2 t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 + x_2 \cos t_1)}$$

and reflected S-wave as

$$u^2 = A_2(\cos t_2, \sin t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$
$$\sigma(u^2) = \mathbf{i}k_s A_2(\mu(\sin^2 t_2 - \cos^2 t_2), -2\mu \sin t_2 \cos t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$

We consider the clamped condition, then the total field on the $x_2 = 0$ vanish:

$$u^0(x_1, 0) + u^1(x_1, 0) + u^2(x_1, 0) = 0$$

for any $x_1 \in \mathbb{R}$. A simple computation show that

$$t_1 = t_0 \qquad \text{and} \qquad \frac{\sin t_2}{\sin t_0} = \frac{k_p}{k_s} := \kappa$$
$$A_0 = \cos(t_0 - t_2) \qquad A_1 = \cos(t_0 + t_2) \qquad A_2 = -\sin 2t_0$$

### *11.2. S-wave*

Similarly, we denote incident S-wave as

$$u^0 = A_0(-\cos t_0, \sin t_0)^T e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)} \tag{11.2}$$
$$\sigma(u^0) = \mathbf{i}k_s(\mu(\sin^2 t_0 - \cos^2 t_0), 2\mu \sin t_0 \cos t_0)e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)} \tag{11.3}$$

The reflected P-wave is represented as

$$u^1 = A_1(\sin t_1, -\cos t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 - x_2 \cos t_1)}$$
$$\sigma(u^1) = \mathbf{i}k_p A_1(-2\mu \sin t_1 \cos t_1, \lambda + 2\mu \cos^2 t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 + x_2 \cos t_1)}$$

and reflected S-wave as

$$u^2 = A_2(\cos t_2, \sin t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$
$$\sigma(u^2) = \mathbf{i}k_s A_2(\mu(\sin^2 t_2 - cos^2 t_2), -2\mu \sin t_2 \cos t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$

The result is

$$t_2 = t_0 \qquad \text{and} \qquad \frac{\sin t_1}{\sin t_0} = \frac{k_s}{k_p} = \frac{1}{\kappa}$$
$$A_0 = \cos(t_0 - t_1) \qquad A_1 = \sin 2t_0 \qquad A_2 = \cos(t_0 + t_1)$$

## 12. scattering relation of elastic wave

The solution for the scattering of a plane P-wave $u_p$(or S-wave $u_s$) with incident direction $d_0$ at a plane $\Gamma := x \in \mathbb{R}^2 : x \cdot \nu = 0$ through the origin with normal vector $\nu$ is described by

$$u = u_p + u_{p,p} + u_{p,s} = A_0 d_0 e^{\mathbf{i}kpx\cdot d} + A_1 d_1 e^{\mathbf{i}kpx\cdot d_1} + A_2 d_2^{\perp} d_0^{\mathbf{i}ksx\cdot d_2} \tag{12.1}$$
$$u = u_s + u_{s,p} + u_{s,s} = A_0 d_0^{\perp} e^{\mathbf{i}ksx\cdot d} + A_1 d_1 e^{\mathbf{i}kpx\cdot d_1} + A_2 d_2^{\perp} d^{\mathbf{i}ksx\cdot d_2} \tag{12.2}$$

where $d_i = (d_i^1, d_i^2)^T$ are unit vectors, $d_i^{\perp} = (d_i^2, -d_i^1)^T$ and $A_i$ are corresponding amplitude. For fixed boundary, we have $u = 0$ for $x \in \Gamma$. After a standard computation, we get for P-wave:

$$d_1 = d_0 - 2\alpha\nu \tag{12.3}$$
$$d_2 = \kappa d_0 - \beta\nu \tag{12.4}$$
$$A_0 = \kappa(d,\nu)^2 - \kappa(d,\nu^{\perp})^2 - \beta(d,\nu) \tag{12.5}$$
$$A_1 = \kappa - \beta(d,\nu) \tag{12.6}$$
$$A_2 = -2(d,\nu)(d,\nu^{\perp}) \tag{12.7}$$

where $\alpha = (d,\nu)$, $\beta = \kappa\alpha - \sqrt{\kappa^2\alpha^2 - \kappa^2 + 1}$ and $\kappa = k_p/k_s$. For S-wave:

$$d_1 = \kappa_1 d_0 - \gamma\nu \tag{12.8}$$
$$d_2 = d_0 - 2\alpha\nu \tag{12.9}$$
$$A_0 = \kappa_1(d,\nu)^2 - \kappa_1(d,\nu^{\perp})^2 - \gamma(d,\nu) \tag{12.10}$$
$$A_1 = 2(d,\nu)(d,\nu^{\perp}) \tag{12.11}$$
$$A_2 = \kappa_1 - \gamma(d,\nu) \tag{12.12}$$

where $\gamma = \kappa_1\alpha - \sqrt{\kappa_1^2\alpha^2 - \kappa_1^2 + 1}$ and $\kappa_1 = 1/\kappa$. Thus the traction of $u(x)$ on the plane $\Gamma$ can be obtained. For P-wave

$$\sigma(u)\cdot\nu = [\mathbf{i}k_p A_0(\lambda\nu + 2\mu(d_0,\nu)d_0) + \mathbf{i}k_p A_1(\lambda\nu + 2\mu(d_1,\nu)d_1)$$
$$+\mathbf{i}k_s A_2\mu((d_2,\nu)d_2^{\perp} + (d_2^{\perp},\nu)d_2)]e^{\mathbf{i}k_p x\cdot d} := \mathbf{i}k_p R f_p(x,d,\nu)e^{\mathbf{i}k_p x\cdot d}$$

For S-wave

$$\sigma(u)\cdot\nu = [\mathbf{i}k_s A_0\mu((d_0,\nu)d_0^{\perp} + (d_0^{\perp},\nu)d_0) + \mathbf{i}k_p A_1(\lambda\nu + 2\mu(d_1,\nu)d_1)$$
$$+\mathbf{i}k_s A_2\mu((d_2,\nu)d_2^{\perp} + (d_2^{\perp},\nu)d_2)]e^{\mathbf{i}k_s x\cdot d} := \mathbf{i}k_s R f_s(x,d,\nu)e^{\mathbf{i}k_s x\cdot d}$$

**Definition 12.1** *For any unit vector $d \in \mathbb{R}^2$, let $u_p^i = de^{\mathbf{i}k_p x\cdot d}$ or $u_s^i = d^{\perp}e^{\mathbf{i}k_s x\cdot d}$ be the incident wave and $u_\alpha^s = u_\alpha^s(x;d)$ be the radiation solution of the Navier equation:*

$$u_\alpha^s + \omega^2 u_\alpha^s = 0 \quad in \quad \mathbb{R}^2\backslash\bar{D} \tag{12.13}$$
$$u_\alpha^s = -u_\alpha^i \quad on \quad \partial D \tag{12.14}$$

*The scattering coecient R(x;d) for $x \in \partial D$ is defined by the relation*

$$\sigma(u_\alpha^s + u_\alpha^i) \cdot \nu = \mathbf{i}k_\alpha R_\alpha(x;d)e^{\mathbf{i}k_\alpha x\cdot d} \quad on \ \ \partial D$$

*where $\alpha = p, s$.*

In the case of high frequency approximation, the scattering coecient can be approximated by

$$R_\alpha(x;d) = \begin{cases} Rf_\alpha(x;d,\nu) & \text{if } \ x \in \partial D_d^- = \{x \in \partial D, \nu(x)\cdot d < 0\}, \\ 0 & \text{if } \ x \in \partial D_d^- = \{x \in \partial D, \nu(x)\cdot d \geq 0\}. \end{cases}$$

## 13. Difference of solution of naviar equation in full-space and half-space,0105

For any $0 < \varepsilon < 1$, we consider the problem

$$\Delta_e u_1^\varepsilon + (1+\mathbf{i}\varepsilon)\omega^2 u_1^\varepsilon = 0 \qquad \text{in } \mathbb{R}_+^2 \backslash \bar{D} \tag{13.1}$$

$$u_1^\varepsilon = g \quad \text{ on } \Gamma_D \tag{13.2}$$

$$\sigma(u_\varepsilon^1)e_2 = 0 \quad \text{ on} \Gamma_0 \tag{13.3}$$

and

$$\Delta_e u_2^\varepsilon + (1+\mathbf{i}\varepsilon)\omega^2 u_2^\varepsilon = 0 \qquad \text{in } \mathbb{R}^2 \backslash \bar{D} \tag{13.4}$$

$$u_2^\varepsilon = g \quad \text{ on } \Gamma_D \tag{13.5}$$

Let $w^\varepsilon(x)$ be the solution of the problem:

$$\Delta_e w^\varepsilon + (1+\mathbf{i}\varepsilon)\omega^2 w^\varepsilon = 0 \qquad \text{in } \mathbb{R}_+^2 \tag{13.6}$$

$$\sigma(w^\varepsilon)e_2 = -\sigma(u_2^\varepsilon)e_2 \quad \text{ on} \Gamma_0 \tag{13.7}$$

Then $u_1^\varepsilon - u_2^\varepsilon - w^\varepsilon$ satisfies (13.1),(13.3) with the boundary condition $u_1^\varepsilon - u_2^\varepsilon - w^\varepsilon = -w^\varepsilon$ on $\Gamma_D$. Thus by the limiting absorption principle, we have

$$\|T_x^\nu(u_1^\varepsilon - u_2^\varepsilon)\|_{H^{-1/2}(\Gamma_D)} \leq C(\|w^\varepsilon\|_{H^{1/2}(\Gamma_D)} + |T_x^\nu(w^\varepsilon)\|_{H^{-1/2}(\Gamma_D)}) \tag{13.8}$$

$$\leq C \max_{x\in D}(|w^\varepsilon(x)| + d_D|\nabla w^\varepsilon(x|) \tag{13.9}$$

where C is independant of $\varepsilon, \omega$. By the integral representation formula we have for any $z \in \Gamma_0$

$$u_2^\varepsilon(z) = \int_{\Gamma_D} (T_y^\nu \Phi^\varepsilon(y,z))^T u_2^\varepsilon(y) - \Phi^\varepsilon(z,y)(T_y^\nu u_2^\varepsilon(y))ds(y) \tag{13.10}$$

which yields by using the integral representation again that for $x \in D$

$$w^\varepsilon(x) = \int_{\Gamma_0} N^\varepsilon(x,z)(T_z^{e_2} u_2^\varepsilon(z))ds(z) \tag{13.11}$$

$$= \int_{\Gamma_D} ds(y) \int_{\Gamma_0} N^\varepsilon(x,z)(T_z^{e_2}((T_y^\nu \Phi^\varepsilon(y,z))^T))ds(z) \tag{13.12}$$

$$- \int_{\Gamma_D} v^\varepsilon(x,y)(T_y^\nu u_2^\varepsilon(y))ds(y) \tag{13.13}$$

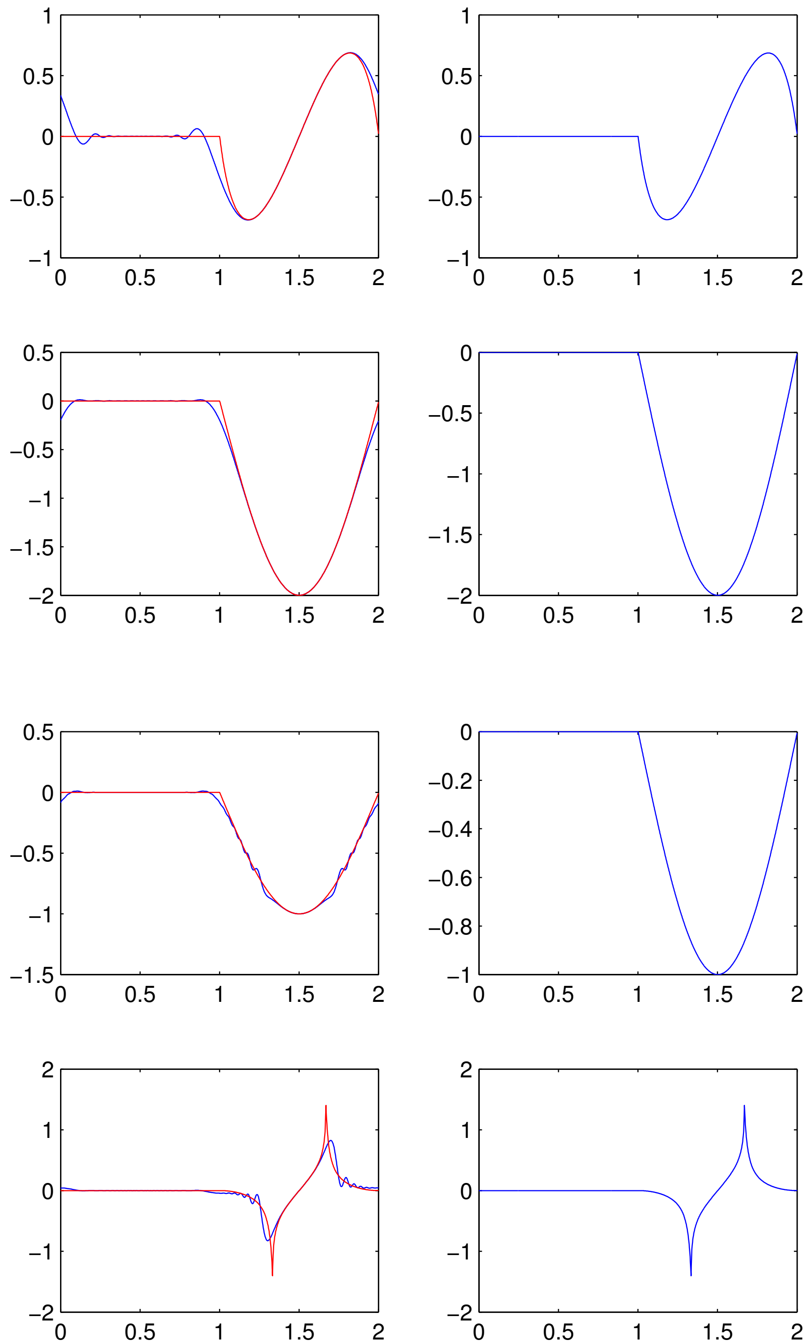



**Figure 1.** $\theta = 0\pi$

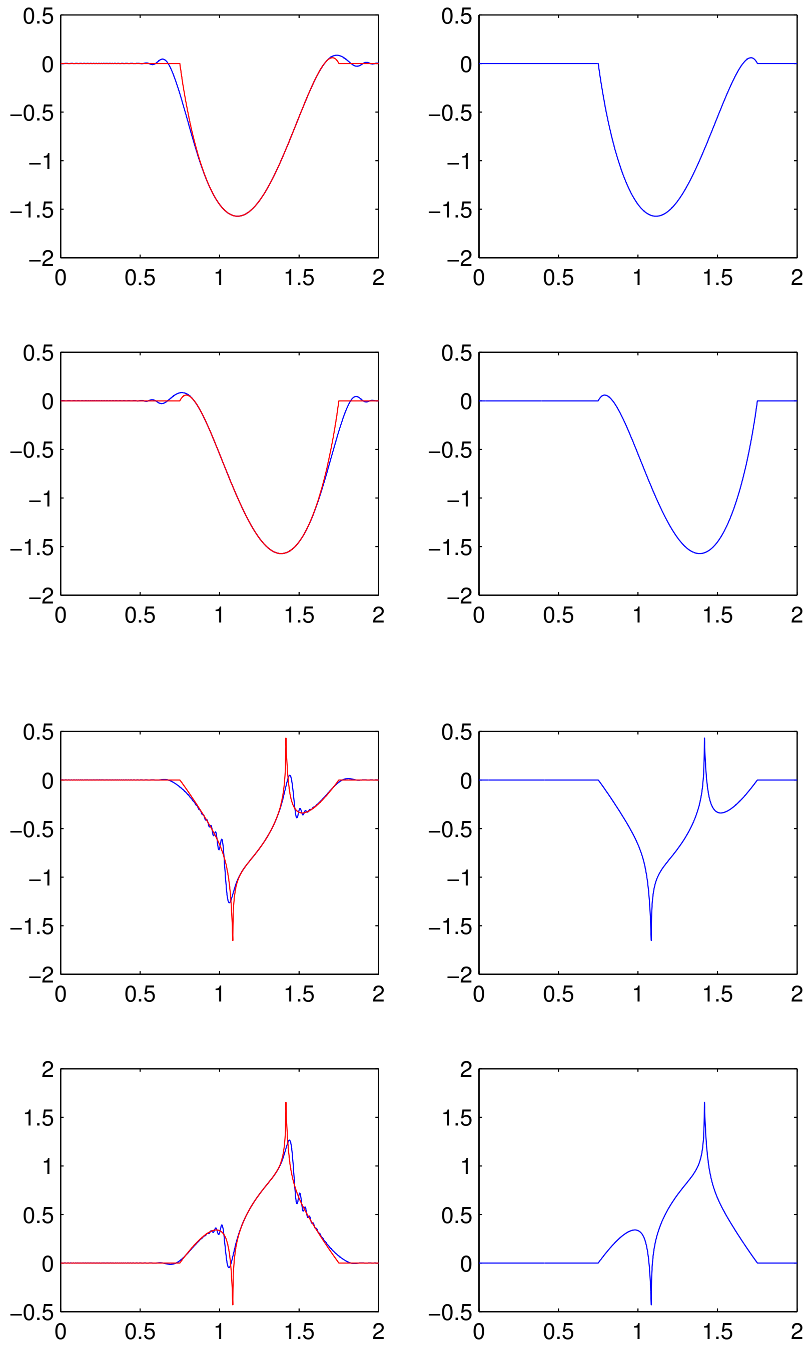



**Figure 2.** $\theta = pi/4$

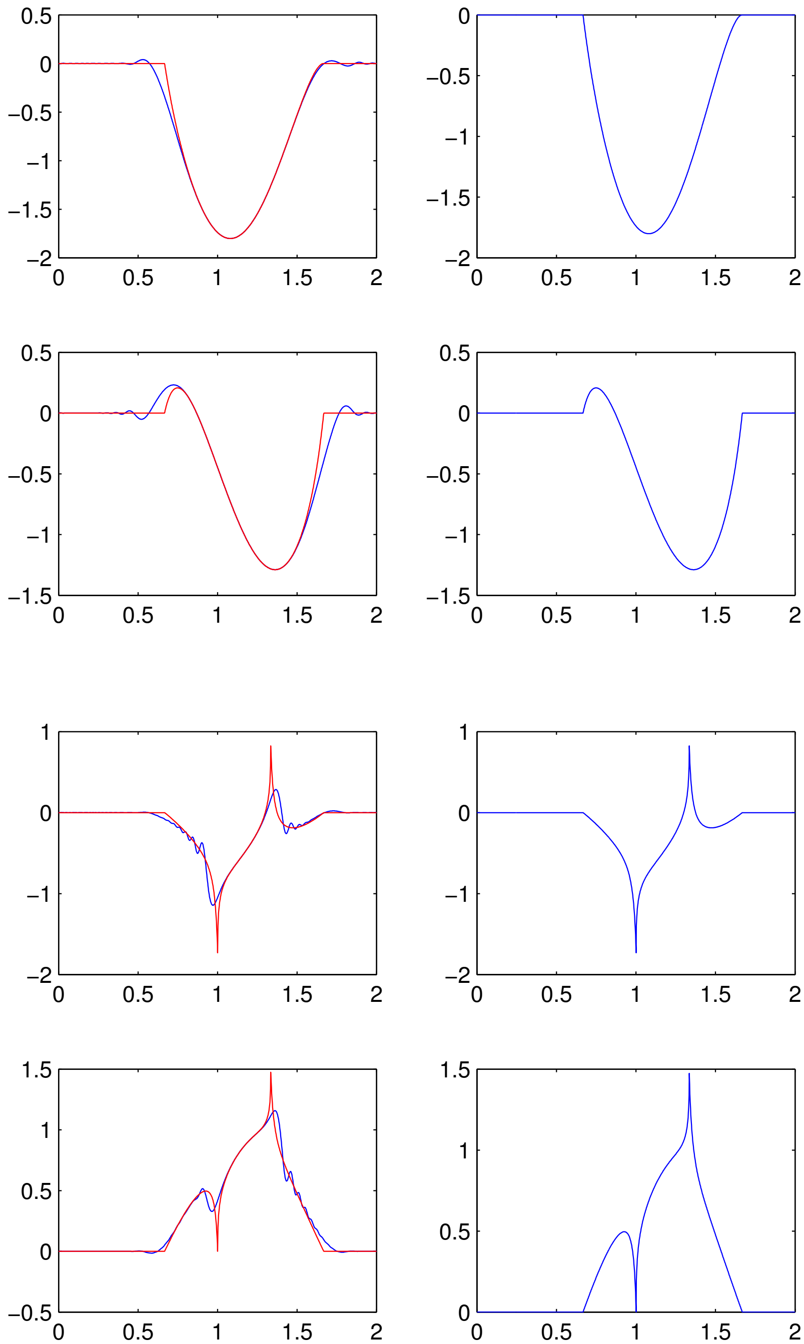



**Figure 3.** $\theta = \pi/3$

$$= \int_{\Gamma_D} ds(y) \int_{\Gamma_0} N^\varepsilon(x,z)(T_z^{e_2}(\Phi^\varepsilon(y,z)^T(T_y^\nu)^T) ds(z) \quad (13.14)$$

$$- \int_{\Gamma_D} v^\varepsilon(x,y)(T_y^\nu u_2^\varepsilon(y)) ds(y) \quad (13.15)$$

$$= \int_{\Gamma_D} ds(y) \int_{\Gamma_0} N^\varepsilon(x,z)(T_y^\nu(T_z^{e_2}\Phi^\varepsilon(z,y))^T)^T ds(z) \quad (13.16)$$

$$- \int_{\Gamma_D} v^\varepsilon(x,y)(T_y^\nu u_2^\varepsilon(y)) ds(y) \quad (13.17)$$

$$= \int_{\Gamma_D} (T_y^\nu(v^\varepsilon(x,y))^T)^T u_2^\varepsilon(y) - v^\varepsilon(x,y)(T_y^\nu u_2^\varepsilon(y)) ds(y) \quad (13.18)$$

where

$$v^\varepsilon(x,y) = \int_{\Gamma_0} N^\varepsilon(x,z)(T_z^{e_2}\Phi^\varepsilon(z,y)) ds(z) \quad (13.19)$$

Since $\|T_x^\nu(u_2^\varepsilon)\|_{H^{-1/2}(\Gamma_D)} \leq C\|g\|_{H^{1/2}(\Gamma_D)}$, we obtain

$$|w^\varepsilon(x)| \leq C\|g\|_{H^{1/2}(\Gamma_D)} \max_{x\in D}(|v^\varepsilon(x,y)| + d_D|\nabla_y v^\varepsilon(x,y)|) \quad (13.20)$$

and

$$|\nabla w^\varepsilon(x)| \leq C\|g\|_{H^{1/2}(\Gamma_D)} \max_{x\in D}(|\nabla_x v^\varepsilon(x,y)| + d_D|\nabla_x\nabla_y v^\varepsilon(x,y)|) \quad (13.21)$$

By (13.8) and letting $\varepsilon \to 0^+$, we have

$$\|T_x^\nu(u_1 - u_2)\|_{H^{-1/2}(\Gamma_D)} \leq C\|g\|_{H^{1/2}(\Gamma_D)} \max_{x\in D} \lim_{\varepsilon\to 0^+} (|v^\varepsilon(x,y)| \quad (13.22)$$

$$+ d_D|\nabla_y v^\varepsilon(x,y)| + d_D|\nabla_x v^\varepsilon(x,y)| + d_D^2|\nabla_x\nabla_y v^\varepsilon(x,y)|) \quad (13.23)$$

where $u_1$ is the scattering solution in the half-space and $u_2$ in the full-space. Now, it turns to estimate $v^\varepsilon(x,y)$. Applying the Fourier transformation to the first horizontal variable of $N^\varepsilon(z,x)$ and$T_z^{e_2}\Phi^\varepsilon(z,y)$, we have

$$\mathcal{F}[N^\varepsilon](\xi,0;x) = \frac{\mathbf{i}}{\mu\delta(\xi)}\left[\begin{pmatrix} 2\xi^2\mu_s & -2\xi\mu_s\mu_p \\ -\xi\beta & \mu_p\beta \end{pmatrix} e^{\mathbf{i}\mu_p x_2} + \begin{pmatrix} \mu_s\beta & \xi\beta \\ 2\xi\mu_s\mu_p & 2\xi^2\mu_p \end{pmatrix} e^{\mathbf{i}\mu_s x_2}\right] e^{-\mathbf{i}\xi x_1}$$

$$\mathcal{F}[T_z^{e_2}\Phi^\varepsilon](\xi,0;y) = \frac{\mu}{2\omega^2}\left[\begin{pmatrix} 2\xi^2 & -2\xi\mu_p \\ -\frac{\beta\xi}{\mu_p} & \beta \end{pmatrix} e^{\mathbf{i}\mu_p y_2} + \begin{pmatrix} \beta & \frac{\xi\beta}{\mu_s} \\ 2\xi\mu_s & 2\xi^2 \end{pmatrix} e^{\mathbf{i}\mu_s y_2}\right] e^{-\mathbf{i}\xi y_1}$$

Using Parseval identity combined with above two formula, we have

$$\lim_{\varepsilon\to 0^+} v^\varepsilon(x,y) = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} \mathcal{F}[N^\varepsilon](\xi,0;x)^T \mathcal{F}[T_z^{e_2}\Phi^\varepsilon](-\xi,0;y) d\xi$$

**Lemma 13.1** *For any $x, y \in D$, let*

$$p(x,y) = \lim_{\varepsilon\to 0^+} p^\varepsilon(x,y) := \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} \frac{f(\mu_p^\varepsilon, \mu_s^\varepsilon, \xi)}{\delta^\varepsilon(\xi)} e^{\mathbf{i}\mu_\alpha^\varepsilon x_2 + \mathbf{i}\mu_\beta^\varepsilon y_2 + \mathbf{i}\xi(y_1 - x_1)} d\xi$$

*where $f(a,b,c)$ is a homogeneous fifth order polynomial with repect to a,b,c and $\alpha = s,p$, $\beta = s,p$.Then there exists a constant $C > 0$ only depandent on $\kappa$ such that*

$$|p(x,y)| + k_s^{-1}|\nabla_x p(x,y)| + k_s^{-1}|\nabla_y p(x,y)| + k_s^{-2}|\nabla_x\nabla_y p(x,y)| \leq C((k_s h)^{-1/2} + e^{-\sqrt{k_R^2 - k_s^2}h})$$

*uniformly for $x, y \in D$.*

**Proof.** Without loss of generality, we assume $k_\alpha \le k_\beta$. Then we can divide $p(x,y)$ into two parts:

$$\begin{aligned} p(x,y) =& \lim_{\varepsilon\to 0^+}\int_{I_1}+\int_{I_2}\frac{f(\mu_p^\varepsilon,\mu_s^\varepsilon,\xi)}{(k_\alpha^\varepsilon)^2\delta^\varepsilon(\xi)}e^{\mathbf{i}\mu_\alpha^\varepsilon x_2+\mathbf{i}\mu_\beta^\varepsilon y_2+\mathbf{i}\xi(y_1-x_1)}d\xi \\ =& \int_{I_1}\frac{f(\mu_p,\mu_s,\xi)}{k_\alpha^2\delta(\xi)}e^{\mathbf{i}\mu_\alpha x_2+\mathbf{i}\mu_\beta y_2+\mathbf{i}\xi(y_1-x_1)}d\xi \\ &+ \lim_{\varepsilon\to 0^+}\int_{I_2}\frac{f(\mu_p^\varepsilon,\mu_s^\varepsilon,\xi)}{(k_\alpha^\varepsilon)^2\delta^\varepsilon(\xi)}e^{\mathbf{i}\mu_\alpha^\varepsilon x_2+\mathbf{i}\mu_\beta^\varepsilon y_2+\mathbf{i}\xi(y_1-x_1)}d\xi \\ =& p_1(x,y)+p_2(x,y) \end{aligned}$$

where $I_1=(-k_\alpha,k_\alpha)$, $I_2=(-2k_R+k_\alpha,k_\alpha)\cup(k_\alpha,2k_R-k_\alpha)$ and $I_2=R\backslash[-k_\alpha,k_\alpha]$. Substituting $\xi=k_\alpha t$ into $p_1(x,y)$, we get

$$p_1(x,y)=\int_{-1}^{1}\frac{f(\mu_p(k_\alpha t),\mu_s(k_\alpha t),k_\alpha t)}{k_\alpha\delta(k_\alpha t)}e^{\mathbf{i}k_\alpha x_2(\sqrt{1-t^2}+\tau\sqrt{\varsigma^2-t^2}+\gamma t)}dt$$

where $\tau=y_2/x_2$, $\varsigma=k_\beta/k_\alpha$ and $\gamma=(y_1-x_1)/x_2$. It is easy to see that the phase function $\phi(t)=\sqrt{1-t^2}+\tau\sqrt{\varsigma^2-t^2}+\gamma t$ satifies $|\phi''(t)|\ge 1/(1-t^2)^{3/2}\ge 1$ for $t\in(-1,1)$. Then we can obtain $|p_1(x,y)|\le C1/(k_sh)^{1/2}$ by lemma 6.1.

For $p_2(x,y)$, by changing the integration path and using same argument as in the proof of estimate for psf, we can easily obtain:

$$|p_2(x,y)|\le C(\frac{1}{k_sh}+e^{-\sqrt{k_R^2-k_s^2}}h)$$

This completes the proof of the esitmate for $|p(x,y)|$. The other estimates can be proved by a similar argument. We omit the details □

**Lemma 13.2** *For any $x,y\in D$, let*

$$p(x,y)=\lim_{\varepsilon\to 0^+}p^\varepsilon(x,y):=\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}\frac{f(\mu_p^\varepsilon,\mu_s^\varepsilon,\xi)}{\delta^\varepsilon(\xi)}e^{\mathbf{i}\mu_\alpha^\varepsilon x_2+\mathbf{i}\mu_\beta^\varepsilon y_2+\mathbf{i}\xi(y_1-x_1)}d\xi$$

*where $f(a,b,c)$ is a homogeneous fifth order polynomial with repect to a,b,c and $\alpha=s,p$, $\beta=s,p$.Then there exists a constant $C>0$ only dependent on $\kappa$ such that*

$$|p(x,y)|+k_s^{-1}|\nabla_x p(x,y)|+k_s^{-1}|\nabla_y p(x,y)|+k_s^{-2}|\nabla_x\nabla_y p(x,y)|\le C(1+k_sd_D)((k_sh)^{-1/2}+e^{-\sqrt{k_R^2-k_s^2}h})$$

*uniformly for $x,y\in D$.*

**Proof.** Without loss of generality, we assume $k_\alpha\le k_\beta$. Then we can divide $p(x,y)$ into four parts:

$$\begin{aligned} p(x,y) =& \lim_{\varepsilon\to 0^+}\int_{I_1}+\int_{I_2}+\int_{I_3}\frac{f(\mu_p^\varepsilon,\mu_s^\varepsilon,\xi)}{(k_\alpha^\varepsilon)^2\delta^\varepsilon(\xi)}e^{\mathbf{i}\mu_\alpha^\varepsilon x_2+\mathbf{i}\mu_\beta^\varepsilon y_2+\mathbf{i}\xi(y_1-x_1)}d\xi \\ =& \int_{I_1}+\mathrm{PV}\int_{I_2}+\int_{I_3}\frac{f(\mu_p,\mu_s,\xi)}{k_\alpha^2\delta(\xi)}e^{\mathbf{i}\mu_\alpha x_2+\mathbf{i}\mu_\beta y_2+\mathbf{i}\xi(y_1-x_1)}d\xi \\ &+\mathbf{i}\pi(\frac{f(\mu_p(k_R),\mu_s(k_R),k_R)}{k_\alpha^2\delta'(k_R)}e^{\mathbf{i}\mu_\alpha(k_R)x_2+\mathbf{i}\mu_\beta(k_R)y_2+\mathbf{i}k_R(y_1-x_1)} \end{aligned}$$

$$
\begin{aligned}
&- \frac{f(\mu_p(k_R),\mu_s(k_R),-k_R)}{k_\alpha^2\delta'(-k_R)}e^{\mathbf{i}\mu_\alpha(k_R)x_2+\mathbf{i}\mu_\beta(k_R)y_2-\mathbf{i}k_R(y_1-x_1)})\\
: \quad &= p_1(x,y)+p_2(x,y)+p_3(x,y)+p_4(x,y)
\end{aligned}
$$

where $I_1=(-k_\alpha,k_\alpha)$, $I_2=(-2k_R+k_\alpha,k_\alpha)\cup(k_\alpha,2k_R-k_\alpha)$ and $I_2=R\backslash[-2k_R+k_\alpha,2k_R-k_\alpha]$. Substituting $\xi=k_\alpha t$ into $p_1(x,y)$, we get

$$
p_1(x,y)=\int_{-1}^{1}\frac{f(\mu_p(k_\alpha t),\mu_s(k_\alpha t),k_\alpha t)}{k_\alpha\delta(k_\alpha t)}e^{\mathbf{i}k_\alpha x_2(\sqrt{1-t^2}+\tau\sqrt{\varsigma^2-t^2}+\gamma t)}dt
$$

where $\tau=y_2/x_2$, $\varsigma=k_\beta/k_\alpha$ and $\gamma=(y_1-x_1)/x_2$. It is easy to see that the phase function $\phi(t)=\sqrt{1-t^2}+\tau\sqrt{\varsigma^2-t^2}+\gamma t$ satifies $|\phi''(t)|\geq 1/(1-t^2)^{3/2}\geq 1$ for $t\in(-1,1)$. Then we can obtain $|p_1(x,y)|\leq C1/(k_sh)^{1/2}$ by lemma 6.1.

Let

$$
g_\pm(\xi)=\frac{f(\mu_p,\mu_s,\xi)(\xi\pm k_R)}{\delta(\xi)}e^{\mathbf{i}\mu_\alpha x_2+\mathbf{i}\mu_\beta y_2+\mathbf{i}\xi(y_1-x_1)}d\xi
$$

Then by the definition of cauchy principle value, we have

$$
p_2(x,y)=\int_{-2k_R+k_\alpha}^{-k_\alpha}\frac{g_-(\xi)-g_-(-k_R)}{x+k_R}d\xi+\int_{k_\alpha}^{2k_R-k_\alpha}\frac{g_+(\xi)-g_+(k_R)}{x-k_R}d\xi
$$

□

## 14. Some comment about phaseless imaging, 2018.01.24

## 15. RTM phaseless: elastic; 03.15

The RTM imaging function studied in [2] for reconstructing extended targets is

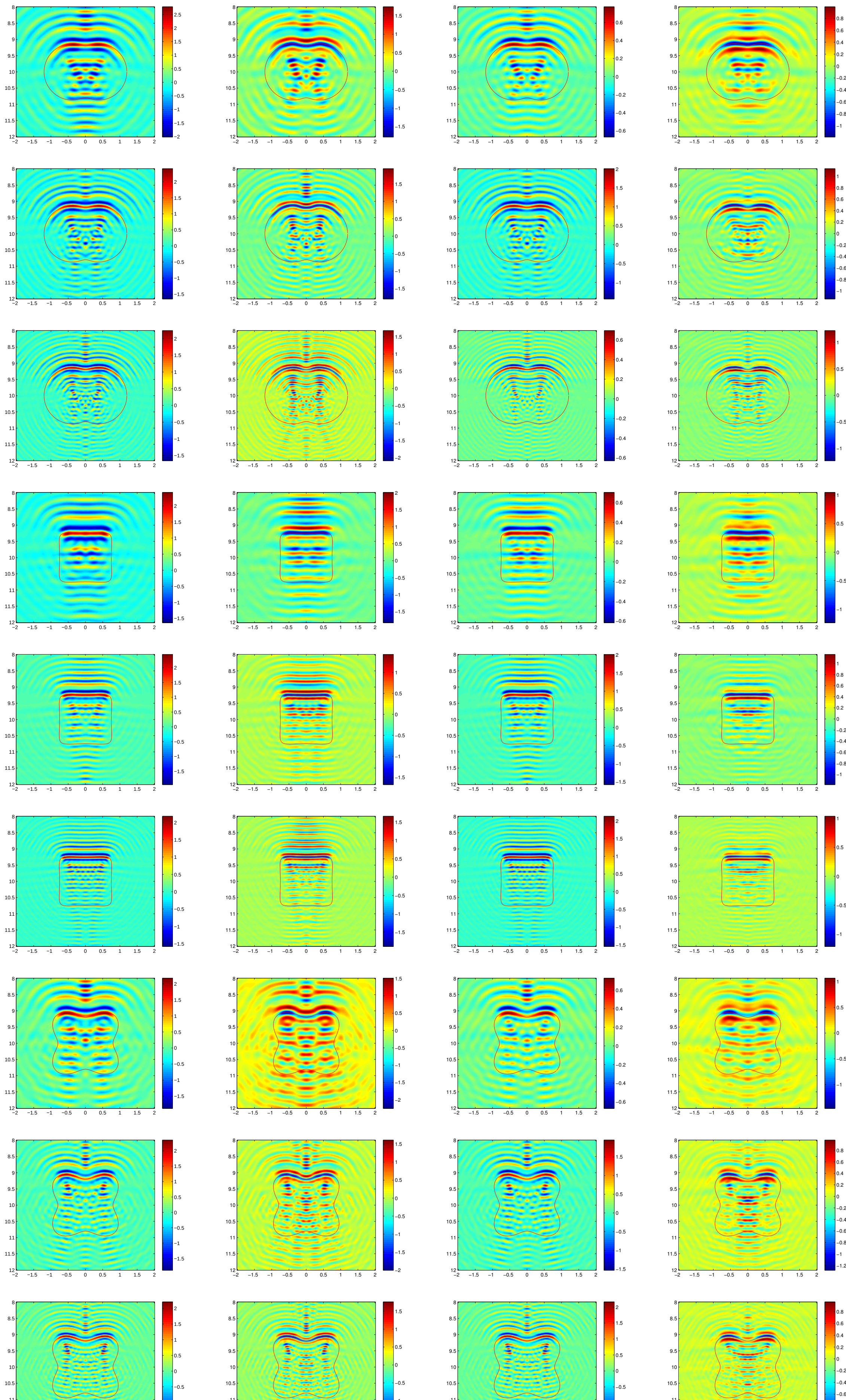
$$
\begin{aligned}
I_1(z)=-\omega^2\mathrm{Im}\sum_{q=e_1,e_2}\int_{\Gamma_s}\int_{\Gamma_r}&\Big(c_pG_p(z,x_rs)q+c_sG_s(z,x_s)q\Big)\\
\cdot\Big(c_pG_p(z,x_r)+c_sG_s(z,x_r)\Big)&\overline{u_q^s(x_r,x_s)}ds(x_r)ds(x_s)
\end{aligned}
$$

For vector $x=(x_1,x_2)^T$, we introduce tow unit vectors $\hat{x}=x/|x|:=(\hat{x}_1,\hat{x}_2)^T$ and $\tilde{x}=(-\hat{x}_2,\hat{x}_1)$. We define $A(x)=\hat{x}\hat{x}^T$ and $B(x)=\tilde{x}\tilde{x}^T$

$$
\begin{aligned}
I_2(z)=-\omega^2\mathrm{Im}\sum_{q=e_1,e_2}\int_{\Gamma_s}\int_{\Gamma_r}&\Big(k_pg_p(z,x_rs)A(x_s)q+k_sg_s(z,x_s)B(x_s)q\Big)\\
\cdot\Big(k_pg_p(z,x_r)A(x_r)+k_sg_s(z,x_r)B(x_r)\Big)&\overline{u_q^s(x_r,x_s)}ds(x_r)ds(x_s)
\end{aligned}
$$

or

$$
\begin{aligned}
I_2(z)=-\omega^2\mathrm{Im}\sum_{q=e_1,e_2}\int_{\Gamma_s}\int_{\Gamma_r}&\Big(c_pG_p(z,x_rs)q+c_sG_s(z,x_s)q\Big)\\
\cdot\Big(k_pg_p(z,x_r)A(x_r)+k_sg_s(z,x_r)B(x_r)\Big)&\overline{u_q^s(x_r,x_s)}ds(x_r)ds(x_s)
\end{aligned}
$$



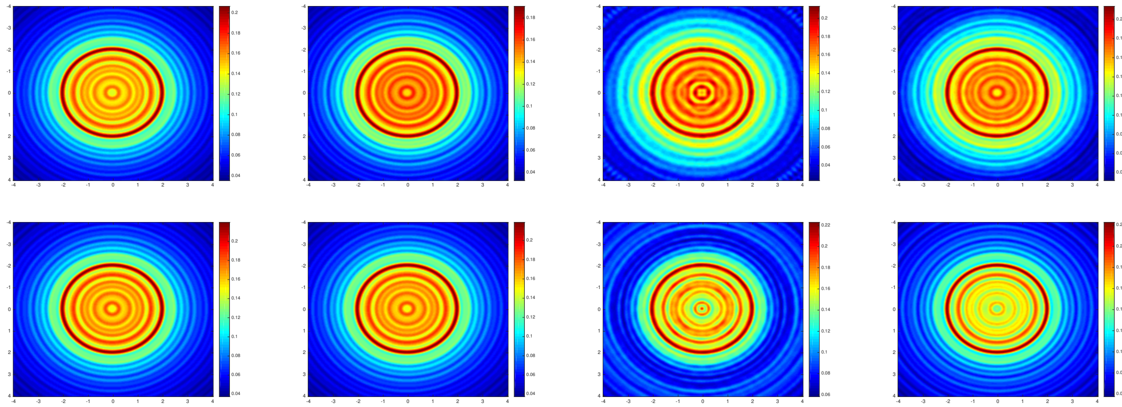


**Figure 5.** Circle;From left to right: vector imaging, scalar imaging, phaseless imaging128, phaseless imaging512; From up to down: R=10, R=100

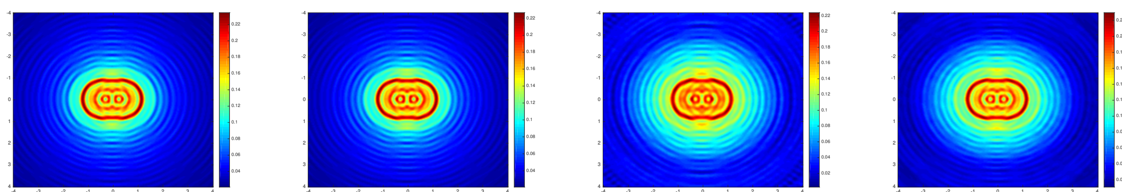


**Figure 6.** Peanut;From left to right: vector imaging, scalar imaging, phaseless imaging128, phaseless imaging512;

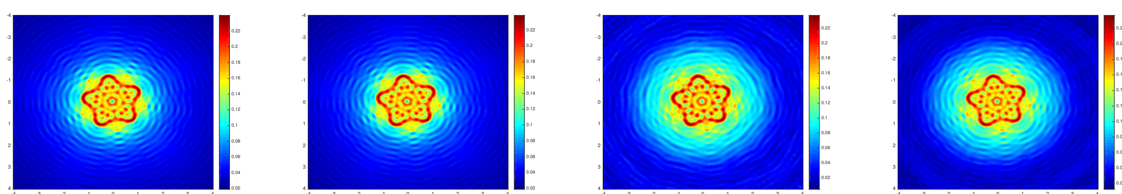


**Figure 7.** Peanut;From left to right: vector imaging, scalar imaging, phaseless imaging128, phaseless imaging512;

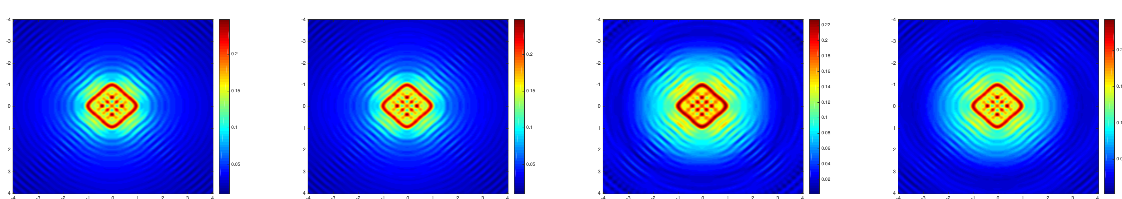


**Figure 8.** Peanut;From left to right: vector imaging, scalar imaging, phaseless imaging128, phaseless imaging512;

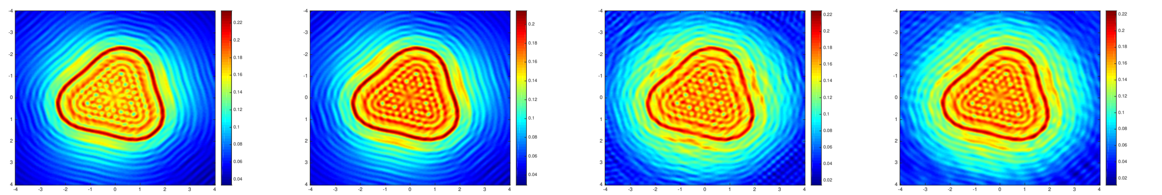


**Figure 9.** Peanut;From left to right: vector imaging, scalar imaging, phaseless imaging128, phaseless imaging512;

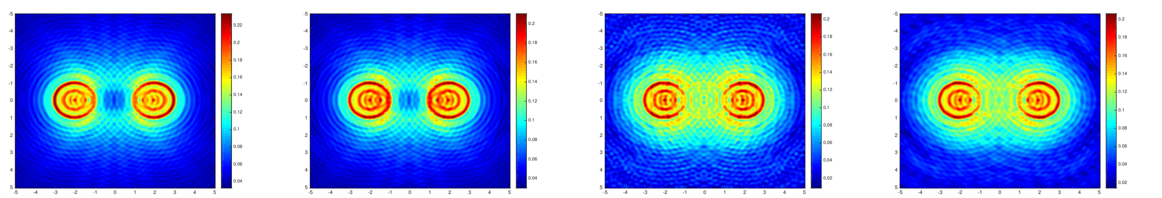


**Figure 10.** Circle;From left to right: vector imaging, scalar imaging, phaseless imaging128, phaseless imaging512; From up to down: R=10, R=100

and

$$I_3(z) = -\omega^2 \mathrm{Im} \sum_{q=e_1,e_2} \int_{\Gamma_s} \int_{\Gamma_r} \Big( k_p g_p(z, x_r s) A(x_s) q + k_s g_s(z, x_s) B(x_s) q \Big)$$

$$\cdot \Big( k_p g_p(z, x_r) \hat{x_r} D_p(x_r, x_s) + k_s g_s(z, x_r) \tilde{x_r} D_s(x_r, x_s) \Big) ds(x_r) ds(x_s)$$

or

$$I_3(z) = -\omega^2 \mathrm{Im} \sum_{q=e_1,e_2} \int_{\Gamma_s} \int_{\Gamma_r} \Big( c_p G_p(z, x_r s) q + c_s G_s(z, x_s) q \Big)$$

$$\cdot \Big( k_p g_p(z, x_r) \hat{x_r} D_p(x_r, x_s) + k_s g_s(z, x_r) \tilde{x_r} D_s(x_r, x_s) \Big) ds(x_r) ds(x_s)$$

where

$$D_p(x_r, x_s) = \frac{|\hat{x_r}^T u_q(x_r, x_s)|^2 - |\hat{x_r}^T u_q^i(x_r, x_s))|^2}{\hat{x_r}^T u_q^i(x_r, x_s))}$$

$$D_s(x_r, x_s) = \frac{|\tilde{x_r}^T u_q(x_r, x_s)|^2 - |\tilde{x_r}^T u_q^i(x_r, x_s))|^2}{\tilde{x_r}^T u_q^i(x_r, x_s))}$$

Conjecture

$$|I_1(z) - I_2(z)| \leq C \frac{1}{k_p R_s}, \quad |I_2(z) - I_3(z)| \leq C \frac{1}{k_p R_s}$$

**Lemma 15.1** *We have*

$$k_p \int_{|x|=R} g_p(z, x) A(x) \overline{G(x, y)} ds(x) = \mathrm{Im}\, G_p(z, y) + W_p(y, z)$$

$$k_s \int_{|x|=R} g_s(z, x) B(x) \overline{G(x, y)} ds(x) = \mathrm{Im}\, G_s(z, y) + W_s(y, z)$$

*where* $|W_\alpha^{ij}(z, y)| + k_\alpha^{-1} |\nabla_z W_\alpha^{ij}(z, y)| \leq C_\alpha R^{-1}$ *for some constanc* $C_\alpha$ *depending on* $k_\alpha |z|, k_\alpha |y|$, $\alpha \in \{p, s\}$.

**Proof.** We first recall the following estimate for the first Hankel function in [5, p.197], for any $t > 0$, we have

$$H_0^{(1)}(t) = \left(\frac{2}{\pi t}\right)^{1/2} e^{\mathbf{i}(t-\pi/4)} + R_0(t), \quad H_1^{(1)}(t) = \left(\frac{2}{\pi t}\right)^{1/2} e^{\mathbf{i}(t-3\pi/4)} + R_1(t),$$

where $|R_j(t)| \leq Ct^{-3/2}$, $j = 0, 1$, for some constant $C > 0$ independent of $t$. By the defination of Green Tensor, we have

$$G_p(x,y) = \frac{\mathbf{i}}{\sqrt{8\pi}(\lambda+2\mu)} A(x-y) \frac{1}{(k_p|x-y|)^{1/2}} e^{\mathbf{i}k_p|x-y|-\mathbf{i}\frac{\pi}{4}} + O(\frac{1}{(k_p|x-y|)^{3/2}})$$
$$G_s(x,y) = \frac{\mathbf{i}}{\sqrt{8\pi}\mu} B(x-y) \frac{1}{(k_s|x-y|)^{1/2}} e^{\mathbf{i}k_p|x-y|-\mathbf{i}\frac{\pi}{4}} + O(\frac{1}{(k_s|x-y|)^{3/2}})$$

Some simple manipulation yields:

$$|A(x-y) - A(x)| \leq C_1/|x|, \quad |B(x-y) - B(x)| \leq C_2/|x|$$
$$|\frac{1}{|x-y|} - \frac{1}{|x|}| \leq C_3/|x|^2, \quad ||x-y| - (|x| - \hat{x}\cdot y)| \leq C_4/|x|$$

where $C_i$, i=1,2,3,4 depend on $|y|$. □

Now we turn to the analysisi of the imaging function $I_3(z)$. We first observe that:

$$D_p(x_r, x_s) = \hat{x_r}^T \overline{u_q^s} + \frac{|\hat{x_r}^T u_q^s(x_r,x_s)|^2}{\hat{x_r}^T u_q^i(x_r,x_s))} + \frac{(\hat{x_r}^T u_q^s(x_r,x_s))(\hat{x_r}^T \overline{u_q^i(x_r,x_s)})}{\hat{x_r}^T u_q^i(x_r,x_s))}$$
$$D_s(x_r, x_s) = \tilde{x_r}^T \overline{u_q^s} + \frac{|\tilde{x_r}^T u_q^s(x_r,x_s)|^2}{\tilde{x_r}^T u_q^i(x_r,x_s))} + \frac{(\tilde{x_r}^T u_q^s(x_r,x_s))(\tilde{x_r}^T \overline{u_q^i(x_r,x_s)})}{\tilde{x_r}^T u_q^i(x_r,x_s))}$$

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