Scattering Coefficient and Kirchhoff Approximation

1. Reflection of Plane wave (Reflected by x_1 axis)

We assume that a plane P-wave u_p(or S-wave u_s) with incident direction $d_0 = (\sin)t_0, \cos t_0$ at a plane $\Gamma := x \in \mathbb{R}^2 : x_0 = 0$.

1.1. *P-wave*

We denote incident P-wave [1, p172] as

$$u^0 = A_0(\sin t_0, \cos t_0)^T e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)} \tag{1.1}$$

and its stress as

$$\sigma(u^0) = \mathbf{i}k_p A_0(2\mu \sin t_0 \cos t_0, \lambda + 2\mu \cos^2 t_0)^T e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)}$$

The reflected P-wave is represented as

$$u^1 = A_1(\sin t_1, -\cos t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 - x_2 \cos t_1)}$$
$$\sigma(u^1) = \mathbf{i}k_p A_1(-2\mu \sin t_1 \cos t_1, \lambda + 2\mu \cos^2 t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 - x_2 \cos t_1)}$$

and reflected S-wave as

$$u^2 = A_2(\cos t_2, \sin t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$
$$\sigma(u^2) = \mathbf{i}k_s A_2(\mu(\sin^2 t_2 - \cos^2 t_2), -2\mu \sin t_2 \cos t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$

We consider the clamped condition, then the total field on the $x_2 = 0$ vanish:

$$u^0(x_1, 0) + u^1(x_1, 0) + u^2(x_1, 0) = 0$$

for any $x_1 \in \mathbb{R}$. A simple computation show that

$$t_1 = t_0 \qquad \text{and} \qquad \frac{\sin t_2}{\sin t_0} = \frac{k_p}{k_s} := \kappa$$
$$A_0 = \cos(t_0 - t_2) \qquad A_1 = \cos(t_0 + t_2) \qquad A_2 = -\sin 2t_0$$

1.2. *S-wave*

Similarly, we denote incident S-wave as

$$u^0 = A_0(-\cos t_0, \sin t_0)^T e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)} \tag{1.2}$$
$$\sigma(u^0) = \mathbf{i}k_s(\mu(\sin^2 t_0 - \cos^2 t_0), 2\mu \sin t_0 \cos t_0) e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)} \tag{1.3}$$

The reflected P-wave is represented as

$$u^1 = A_1(\sin t_1, -\cos t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 - x_2 \cos t_1)}$$
$$\sigma(u^1) = \mathbf{i}k_p A_1(-2\mu \sin t_1 \cos t_1, \lambda + 2\mu \cos^2 t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 + x_2 \cos t_1)}$$

and reflected S-wave as

$$u^2 = A_2(\cos t_2, \sin t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$
$$\sigma(u^2) = \mathbf{i}k_s A_2(\mu(\sin^2 t_2 - cos^2 t_2), -2\mu \sin t_2 \cos t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$

The result is

$$t_2 = t_0 \qquad \text{and} \qquad \frac{\sin t_1}{\sin t_0} = \frac{k_s}{k_p} = \frac{1}{\kappa}$$
$$A_0 = \cos(t_0 - t_1) \qquad A_1 = \sin 2t_0 \qquad A_2 = \cos(t_0 + t_1)$$

2. Reflection of Plane wave (General Case)

Thus, for the general case, the solution for the scattering of a plane P-wave u_p(or S-wave u_s) with incident direction d_0 at a plane $\Gamma := x \in \mathbb{R}^2 : x \cdot \nu = 0$ through the origin with normal vector ν is described by

$$u = u_p + u_{p,p} + u_{p,s} = A_0 d_0 e^{\mathbf{i}kpx\cdot d_0} + A_1 d_1 e^{\mathbf{i}kpx\cdot d_1} + A_2 d_2^{\perp} e^{\mathbf{i}ksx\cdot d_2} \tag{2.1}$$

$$u = u_s + u_{s,p} + u_{s,s} = A_0 d_0^{\perp} e^{\mathbf{i}ksx\cdot d_0} + A_1 d_1 e^{\mathbf{i}kpx\cdot d_1} + A_2 d_2^{\perp} e^{\mathbf{i}ksx\cdot d_2} \tag{2.2}$$

where $d_i = (d_i^1, d_i^2)^T$ are unit vectors, $d_i^{\perp} = (d_i^2, -d_i^1)^T$ and A_i are corresponding amplitude. For fixed boundary, we have $u = 0$ for $x \in \Gamma$. Let $d_0 = (\sin t_0, \cos t_0)$ and $\nu = (\sin\phi, \cos\phi)^T$, and taking a rotation transformation such that

$$\hat{x} = Sx \quad \text{and} \quad x = S^T\hat{x}$$

where

$$S = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} (\nu^{\perp})^T \\ \nu^T \end{bmatrix}$$

Before proceeding, we give a theorem concerning axis transformation.

Theorem 2.1 *Let $u(x) \in \mathbb{C}^2$ and*

$$\Delta_e^x := \begin{pmatrix} (\lambda+2\mu)\frac{\partial^2}{\partial x_1^2} + (\lambda+\mu)\frac{\partial^2}{\partial x_1 \partial x_2} + \mu\frac{\partial^2}{\partial x_2^2} \\ \mu\frac{\partial^2}{\partial x_1^2} + (\lambda+\mu)\frac{\partial^2}{\partial x_1 \partial x_2} + (\lambda+2\mu)\frac{\partial^2}{\partial x_1 2^2} \end{pmatrix}$$

. Assume that u(x) satisfies $\Delta_e^x u(x) + \omega^2 u(x) = 0$, then we have $\Delta_e^{\hat{x}}\hat{u}(\hat{x}) + \omega^2\hat{u}(\hat{x}) = 0$ where $\hat{u}(\hat{x}) := Su(S^T\hat{x})$.

Proof. since

$$\frac{\partial^2}{\partial \hat{x}_1^2} = \cos^2\phi\frac{\partial^2}{\partial x_1^2} - 2\cos\phi\sin\phi\frac{\partial^2}{\partial x_1\partial x_2} + \sin^2\phi\frac{\partial^2}{\partial x_2^2}$$
$$\frac{\partial^2}{\partial \hat{x}_2^2} = \sin^2\phi\frac{\partial^2}{\partial x_1^2} + 2\cos\phi\sin\phi\frac{\partial^2}{\partial x_1\partial x_2} + \cos^2\phi\frac{\partial^2}{\partial x_2^2}$$
$$\frac{\partial^2}{\partial \hat{x}_1\partial\hat{x}_2} = \cos\phi\sin\phi\frac{\partial^2}{\partial x_1^2} + (\cos^2\phi - \sin^2\phi)\frac{\partial^2}{\partial x_1\partial x_2} - \cos\phi\sin\phi\frac{\partial^2}{\partial x_2^2}$$

This completes proof after substituting above equation into $\Delta_e^{\hat{x}}\hat{u}(\hat{x})$. □

Let

$$\hat{d}_i = Sd_i, \quad \hat{\nu} = S\nu = (0,1)^T$$
$$\hat{d}_0 = (\sin(\theta-\phi), \cos(\theta-\phi))^T = ((d_0, \nu^{\perp}), (d_0, \nu))^T$$
$$\hat{u}_p(\hat{x}) := Su_p(S^Tx) = A_0(\sin(\theta-\phi), \cos(\theta-\phi))^T e^{\mathbf{i}k_p\hat{x}_1\sin(\theta-\phi)+\hat{x}_2\cos(\theta-\phi)}$$

Using results in section 1, we have

$$\hat{d}_1 = (\sin(\theta-\phi), -\cos(\theta-\phi))^T = ((d_0, \nu^{\perp}), (d_0, \nu))^T$$
$$\hat{d}_2 = (\sin t_2, -\cos t_2)^T = (\kappa(d_0, \nu^{\perp}), -sgn((d_0, \nu))\sqrt{1-\kappa^2(d_0, \nu^{\perp})^2})^T$$
$$A_0 = \cos(\theta - \phi - t_2) = -\hat{d}_1 \cdot \hat{d}_2 = -d_1 \cdot d_2$$

$$A_1 = \cos(\theta - \phi + t_2) = \hat{d}_0 \cdot \hat{d}_2 = d_0 \cdot d_2$$
$$A_2 = -\sin 2(\theta - \phi) = 2(\hat{d}_0, \hat{\nu})(\hat{d}_0, \hat{\nu}^\perp) = -2(d_0, \nu)(d_0, \nu^\perp)$$

where $\sin t_2 = \kappa \sin(\theta - \phi)$. After a standard computation, we get for P-wave:

$$d_1 = S^T \hat{d}_1 = d_0 - 2\alpha\nu \tag{2.3}$$
$$d_2 = S^T \hat{d}_2 = \kappa d_0 - \beta\nu \tag{2.4}$$
$$A_0 = -\kappa(d_0, \nu)^2 + \kappa(d_0, \nu^\perp)^2 + \beta(d_0, \nu) \tag{2.5}$$
$$A_1 = -\kappa + \beta(d_0, \nu) \tag{2.6}$$
$$A_2 = -2(d_0, \nu)(d_0, \nu^\perp) \tag{2.7}$$

where $\alpha = (d_0, \nu)$, $\beta = \kappa\alpha + sgn(\alpha)\sqrt{\kappa^2\alpha^2 - \kappa^2 + 1}$ and $\kappa = k_p/k_s$. Similarly, for S-wave, we have

$$d_1 = \kappa_1 d_0 - \gamma\nu \tag{2.8}$$
$$d_2 = d_0 - 2\alpha\nu \tag{2.9}$$
$$A_0 = -\kappa_1(d_0, \nu)^2 + \kappa_1(d, \nu^\perp)^2 + \gamma(d, \nu) \tag{2.10}$$
$$A_1 = 2(d_0, \nu)(d_0, \nu^\perp) \tag{2.11}$$
$$A_2 = -\kappa_1 + \gamma(d_0, \nu) \tag{2.12}$$

where $\gamma = \kappa_1\alpha + sgn(\alpha)\sqrt{\kappa_1^2\alpha^2 - \kappa_1^2 + 1}$ and $\kappa_1 = 1/\kappa$. Thus the traction of $u(x)$ on the plane Γ can be obtained. For P-wave

$$\begin{aligned}\sigma(u) \cdot \nu = [&\mathbf{i}k_p A_0(\lambda\nu + 2\mu(d_0, \nu)d_0) + \mathbf{i}k_p A_1(\lambda\nu + 2\mu(d_1, \nu)d_1) \\ &+\mathbf{i}k_s A_2 \mu((d_2, \nu)d_2^\perp + (d_2^\perp, \nu)d_2)]e^{\mathbf{i}k_p x\cdot d_0} := \mathbf{i}k_p A_0 \hat{\mathbf{R}}_p(x, d_0, \nu)e^{\mathbf{i}k_p x\cdot d_0}\end{aligned} \tag{2.13}$$

For S-wave

$$\begin{aligned}\sigma(u) \cdot \nu = [&\mathbf{i}k_s A_0 \mu((d_0, \nu)d_0^\perp + (d_0^\perp, \nu)d_0) + \mathbf{i}k_p A_1(\lambda\nu + 2\mu(d_1, \nu)d_1) \\ &+\mathbf{i}k_s A_2 \mu((d_2, \nu)d_2^\perp + (d_2^\perp, \nu)d_2)]e^{\mathbf{i}k_s x\cdot d_0} := \mathbf{i}k_s A_0 \hat{\mathbf{R}}_s(x, d_0, \nu)e^{\mathbf{i}k_s x\cdot d_0}\end{aligned} \tag{2.14}$$

Definition 2.1 *For any unit vector $d \in \mathbb{R}^2$, let $u_p^i = de^{\mathbf{i}k_p x\cdot d}$ or $u_s^i = d^\perp e^{\mathbf{i}k_s x\cdot d}$ be the incident wave and $u_\alpha^s = u_\alpha^s(x; d)$ be the radiation solution of the Navier equation:*

$$u_\alpha^s + \omega^2 u_\alpha^s = 0 \quad in \quad \mathbb{R}^2\backslash\bar{D} \tag{2.15}$$
$$\Delta_e u_\alpha^s = -u_\alpha^i \quad on \quad \partial D \tag{2.16}$$

The scattering coefficient $\mathbf{R}_\alpha(x; d)$ for $x \in \partial D$ is defined by the relation

$$\sigma(u_\alpha^s + u_\alpha^i) \cdot \nu = \mathbf{i}k_\alpha \mathbf{R}_\alpha(x; d)e^{\mathbf{i}k_\alpha x\cdot d} \quad on \quad \partial D$$

where $\alpha = p, s$.

A convex object D locally may be cosidered at each point x as a plane with normal $\nu(x)$. Then the scattering coefficient can be approximated(Kirchhoff approximation) by

$$\mathbf{R}_\alpha(x; d) \approx \begin{cases} \hat{\mathbf{R}}_\alpha(x; d, \nu(x)) & \text{if } \ x \in \partial D_d^- = \{x \in \partial D, \nu(x) \cdot d < 0\}, \\ 0 & \text{if } \ x \in \partial D_d^- = \{x \in \partial D, \nu(x) \cdot d \geq 0\}. \end{cases}$$

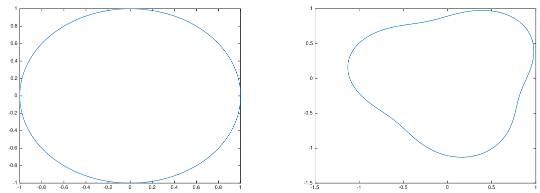

Figure 1.

3. Numerical examples

In this section we present several numerical examples to show the effectiveness of Kirchhoff approximation. To synthesize the real scattering coefficient we compute the solution $\sigma(u^s_\alpha + u^i_\alpha)\cdot\nu$ of the scattering problems by representing the ansatz solution as the single layer potential with the Green tensor $\mathbb{G}(x,y)$ as the kernel

$$u^s(x) = \int_{\Gamma_D} -\mathbb{G}(y,x)^T\sigma(u^s(y)+u^i(y))\nu ds(y) = -u^i(x) \quad \text{on} \;\; x\in\Gamma_D$$

and discretizing the integral equation by standard Nyström methods [2]. Let $\mathbf{R}_\alpha(x;d) = (\mathbf{R}^1_\alpha(x;d), \mathbf{R}^2_\alpha(x;d))^T$, then we have

$$\mathbf{R}^j_\alpha(x;d) = \frac{\sigma(u^s(y)+u^i(y))\nu\cdot e_j}{\mathbf{i}k_\alpha e^{\mathbf{i}k_\alpha x\cdot d}} \tag{3.1}$$

and we can compute $\hat{\mathbf{R}}_\alpha(x;d) = (\hat{\mathbf{R}}^1_\alpha(x;d), \hat{\mathbf{R}}^2_\alpha(x;d))^T$ by (2.13) and (2.14). In all our numerical examples we choose Lamé constant $\lambda = 1/2$, $\mu = 1/4$ and

$$u^i_p = (\cos t, \sin t)^T e^{\mathbf{i}k_p(x_1\cos t + x_2\sin t)}$$
$$u^i_s = (\sin t, -\cos t)^T e^{\mathbf{i}k_s(x_1\cos t + x_2\sin t)}$$

where $t\in[0,2\pi]$.. The boundaries of the obstacles used in our numerical experiments are parameterized as follows,

$$\text{Circle:} \quad x_1 = \cos(\theta), \quad x_2 = \sin(\theta);$$
$$\text{Pear:} \quad \rho = 0.5(2+0.3\cos(3\theta)), x_1 = \sin\frac{\pi}{4}\rho(\cos\theta - \sin\theta), x_2 = \sin\frac{\pi}{4}\rho(\cos\theta+\sin\theta)$$

where $\theta\in[0,2\pi]$ and depicted as figure 1.

In the following examples, figure 2-9, the angular frequency $\omega = \pi, 2\pi, 4\pi, 8\pi$.

References

[1] Achenbach J 1980 *Wave Propagation in Elastic Solids* (North-Holland)

[2] Colton D and Kress R 1998 *Inverse Acoustic and Electromagnetic Scattering Problems* (Heidelberg: Springer)

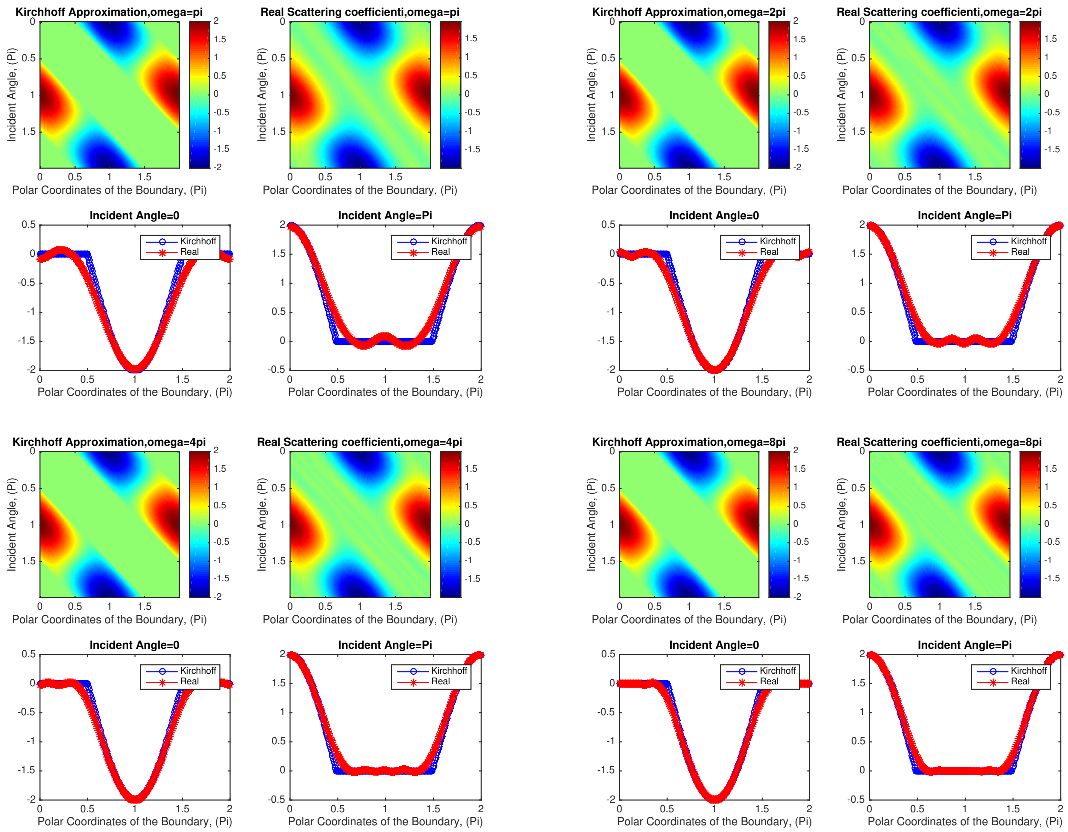

Figure 2. $\mathbf{R}_p^1$ and $\hat{\mathbf{R}}_p^1$ for circle

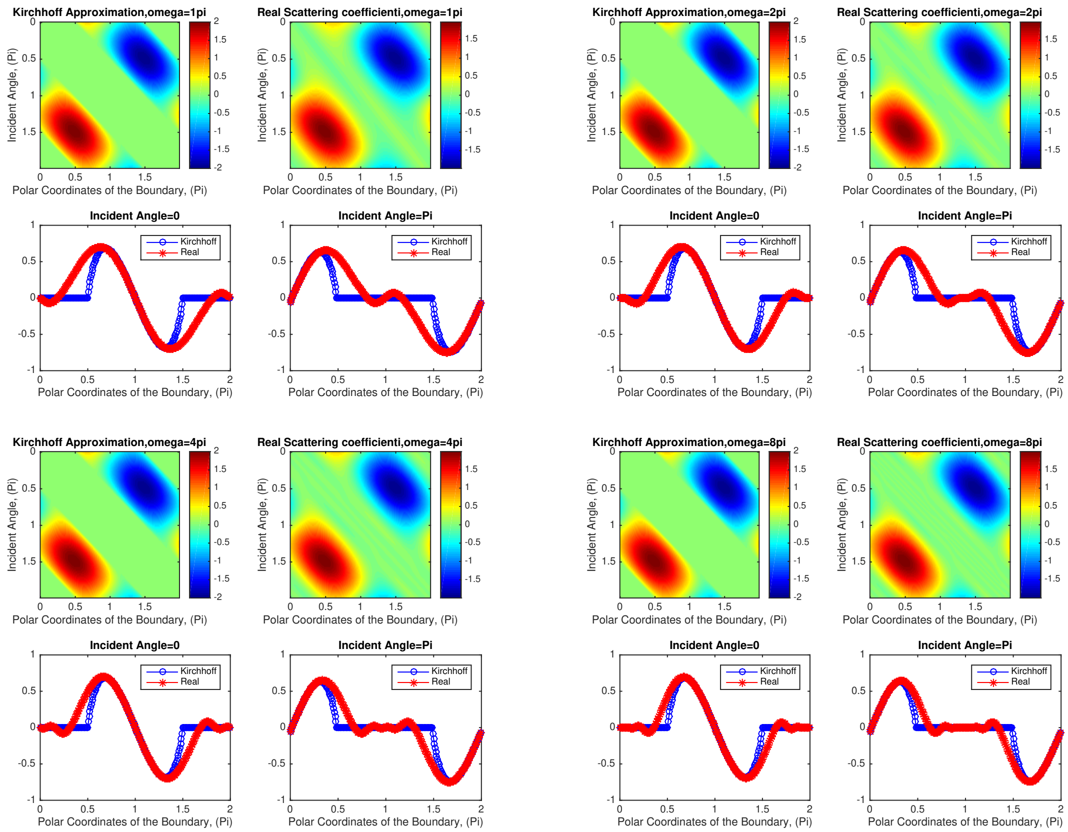

Figure 3. $\mathbf{R}_p^2$ and $\hat{\mathbf{R}}_p^2$ for circle

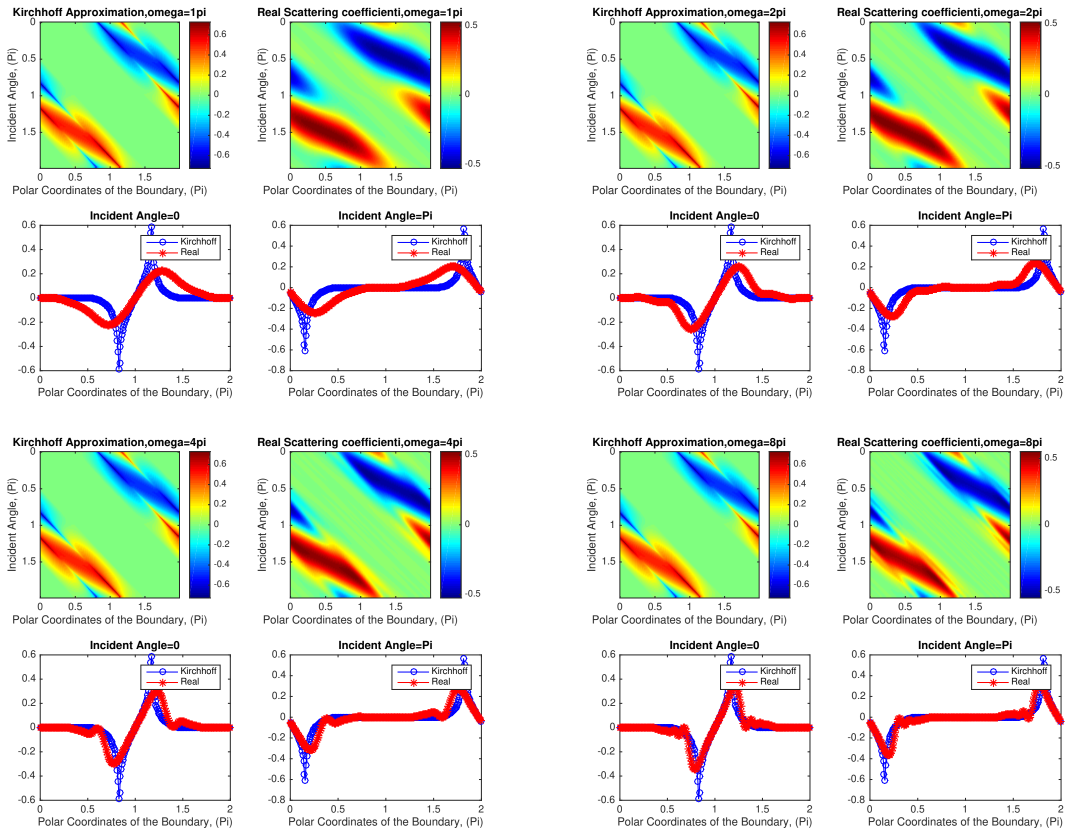

Figure 4. $\mathbf{R}_s^1$ and $\hat{\mathbf{R}}_s^1$ for circle

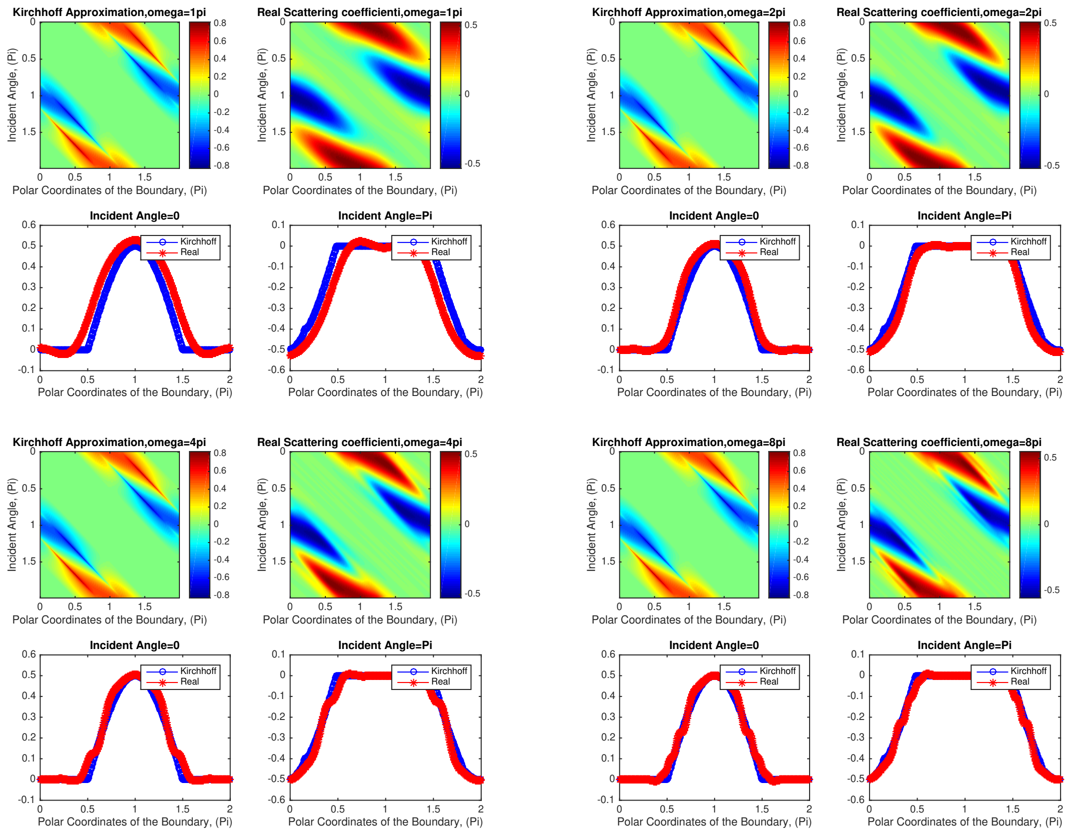

Figure 5. $\mathbf{R}_s^2$ and $\hat{\mathbf{R}}_s^2$ for circle

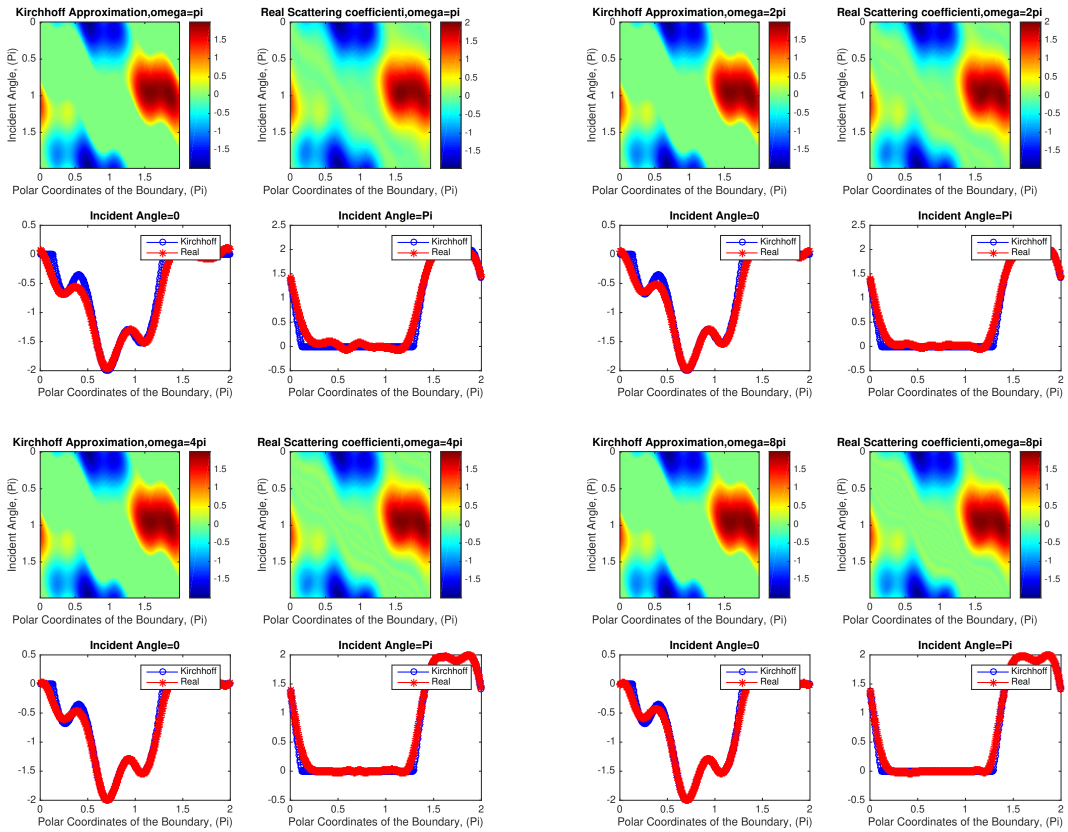

Figure 6. $\mathbf{R}_p^1$ and $\hat{\mathbf{R}}_p^1$ for pear

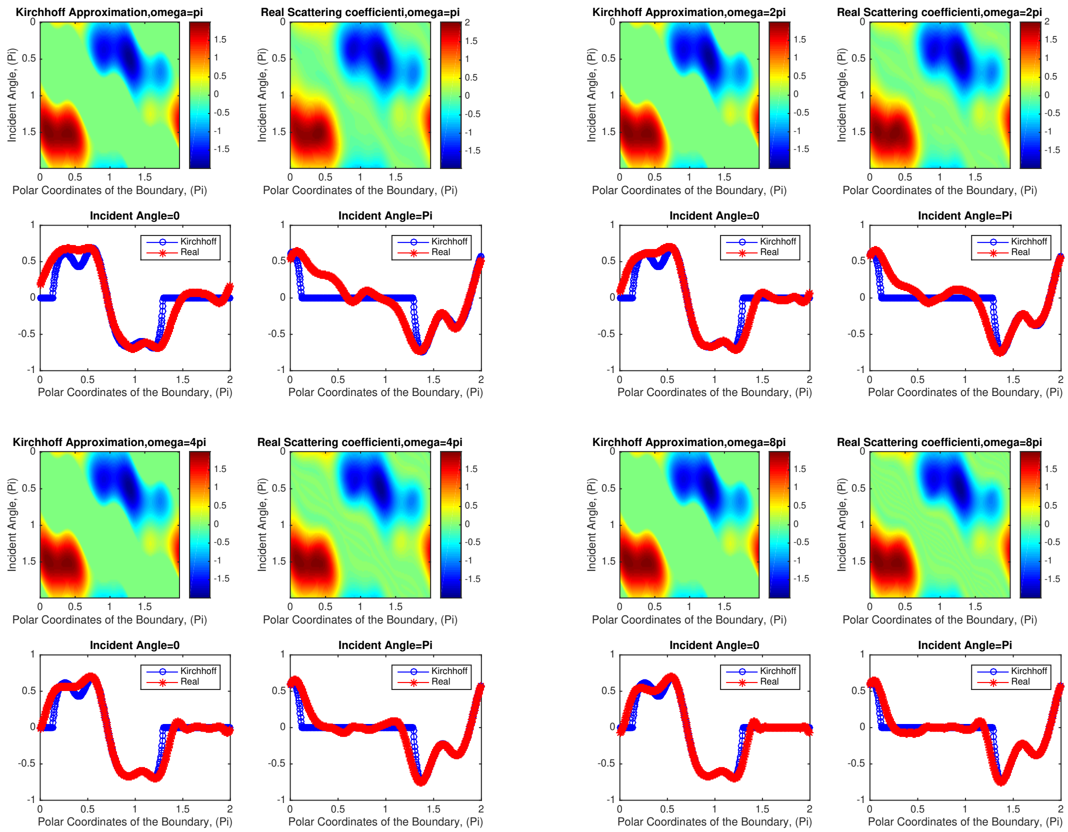

Figure 7. $\mathbf{R}_p^2$ and $\hat{\mathbf{R}}_p^2$ for pear

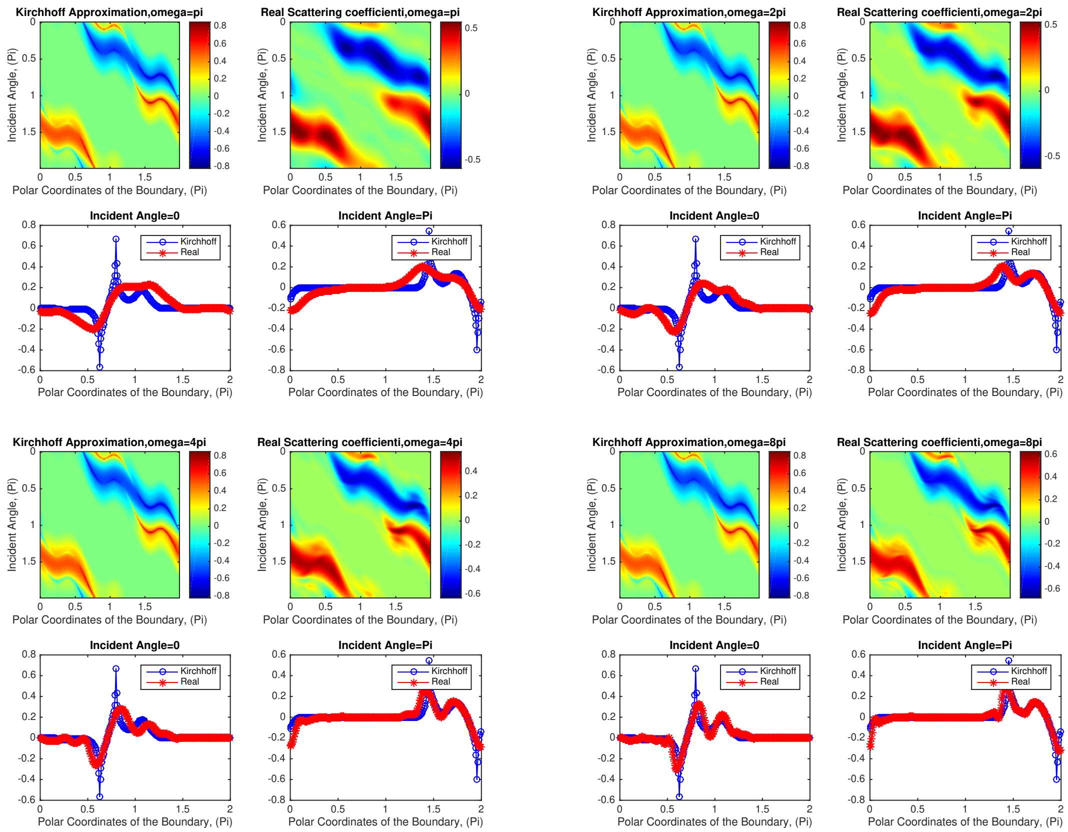

Figure 8. $\mathbf{R}_s^1$ and $\hat{\mathbf{R}}_s^1$ for pear

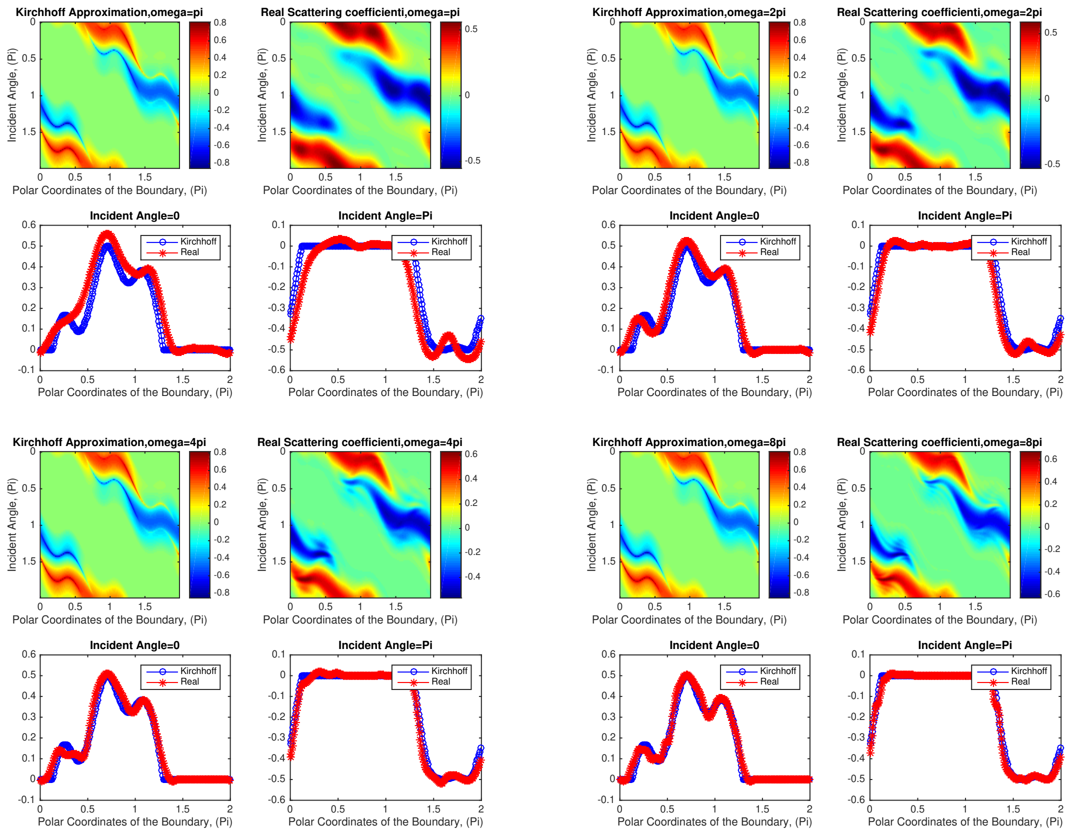

Figure 9. $\mathbf{R}_s^2$ and $\hat{\mathbf{R}}_s^2$ for pear