Scattering Coefficient and Kirchhoff Approximation

1. Reflection of Plane wave (Reflected by x_2 axis)

We assume that a plane P-wave u_p(or S-wave u_s) with incident direction $d_0 = (\sin)t_0, \cos t_0$ at a plane $\Gamma := x \in \mathbb{R}^2 : x_0 = 0$.

1.1. P-wave

We denote incident P-wave [1, p172] as

$$u^0 = A_0(\sin t_0, \cos t_0)^T e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)} \tag{1.1}$$

and its stress as

$$\sigma(u^0) = \mathbf{i}k_p A_0(2\mu \sin t_0 \cos t_0, \lambda + 2\mu \cos^2 t_0)^T e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)}$$

The reflected P-wave is represented as

$$u^1 = A_1(\sin t_1, -\cos t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 - x_2 \cos t_1)}$$
$$\sigma(u^1) = \mathbf{i}k_p A_1(-2\mu \sin t_1 \cos t_1, \lambda + 2\mu \cos^2 t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 + x_2 \cos t_1)}$$

and reflected S-wave as

$$u^2 = A_2(\cos t_2, \sin t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$
$$\sigma(u^2) = \mathbf{i}k_s A_2(\mu(\sin^2 t_2 - \cos^2 t_2), -2\mu \sin t_2 \cos t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$

We consider the clamped condition, then the total field on the $x_2 = 0$ vanish:

$$u^0(x_1, 0) + u^1(x_1, 0) + u^2(x_1, 0) = 0$$

for any $x_1 \in \mathbb{R}$. A simple computation show that

$$t_1 = t_0 \qquad \text{and} \qquad \frac{\sin t_2}{\sin t_0} = \frac{k_p}{k_s} := \kappa$$
$$A_0 = \cos(t_0 - t_2) \qquad A_1 = \cos(t_0 + t_2) \qquad A_2 = -\sin 2t_0$$

1.2. S-wave

Similarly, we denote incident S-wave as

$$u^0 = A_0(-\cos t_0, \sin t_0)^T e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)} \tag{1.2}$$
$$\sigma(u^0) = \mathbf{i}k_s(\mu(\sin^2 t_0 - \cos^2 t_0), 2\mu \sin t_0 \cos t_0) e^{\mathbf{i}k_p(x_1 \sin t_0 + x_2 \cos t_0)} \tag{1.3}$$

The reflected P-wave is represented as

$$u^1 = A_1(\sin t_1, -\cos t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 - x_2 \cos t_1)}$$
$$\sigma(u^1) = \mathbf{i}k_p A_1(-2\mu \sin t_1 \cos t_1, \lambda + 2\mu \cos^2 t_1)^T e^{\mathbf{i}k_p(x_1 \sin t_1 + x_2 \cos t_1)}$$

and reflected S-wave as

$$u^2 = A_2(\cos t_2, \sin t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$
$$\sigma(u^2) = \mathbf{i}k_s A_2(\mu(\sin^2 t_2 - cos^2 t_2), -2\mu \sin t_2 \cos t_2)^T e^{\mathbf{i}k_s(x_1 \sin t_2 - x_2 \cos t_2)}$$

The result is

$$t_2 = t_0 \qquad \text{and} \qquad \frac{\sin t_1}{\sin t_0} = \frac{k_s}{k_p} = \frac{1}{\kappa}$$
$$A_0 = \cos(t_0 - t_1) \qquad A_1 = \sin 2t_0 \qquad A_2 = \cos(t_0 + t_1)$$

2. Reflection of Plane wave (General Case)

Thus, for the general case, the solution for the scattering of a plane P-wave u_p(or S-wave u_s) with incident direction d_0 at a plane $\Gamma := x \in \mathbb{R}^2 : x \cdot \nu = 0$ through the origin with normal vector ν is described by

$$u = u_p + u_{p,p} + u_{p,s} = A_0 d_0 e^{\mathbf{i}kpx\cdot d_0} + A_1 d_1 e^{\mathbf{i}kpx\cdot d_1} + A_2 d_2^{\perp} e^{\mathbf{i}ksx\cdot d_2} \tag{2.1}$$
$$u = u_s + u_{s,p} + u_{s,s} = A_0 d_0^{\perp} e^{\mathbf{i}ksx\cdot d_0} + A_1 d_1 e^{\mathbf{i}kpx\cdot d_1} + A_2 d_2^{\perp} e^{\mathbf{i}ksx\cdot d_2} \tag{2.2}$$

where $d_i = (d_i^1, d_i^2)^T$ are unit vectors, $d_i^{\perp} = (d_i^2, -d_i^1)^T$ and A_i are corresponding amplitude. For fixed boundary, we have $u = 0$ for $x \in \Gamma$. Let $d_0 = (\sin t_0, \cos t_0)$ and $\nu = (\sin\phi, \cos\phi)^T$, and taking a rotation transformation such that

$$\hat{x} = Sx \quad \text{and} \quad x = S^T\hat{x}$$

where

$$S = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Then we have

$$\hat{d}_i = Sd_i, \quad \hat{\nu} = S\nu = (0,1)^T$$
$$\hat{u}_p(\hat{x}) := Su_p(S^T x) = A_0(\sin(\theta-\phi), \cos(\theta-\phi))^T e^{\mathbf{i}k_p\hat{x}_1\sin(\theta-\phi)+\hat{x}_2\cos(\theta-\phi)}$$

Using results in section 1, we have

$$\hat{d}_1 = (\sin(\theta-\phi), -\cos(\theta-\phi))^T \qquad \hat{d}_2 = (\sin t_2, -\cos t_2)^T$$
$$A_0 = \cos(\theta-\phi-t_2) = \hat{d}_1\cdot\hat{d}_2 = d_1\cdot d_2$$
$$A_1 = \cos(\theta-\phi+t_2) = \hat{d}_0\cdot\hat{d}_2 = d_0\cdot d_2$$
$$A_2 = -\sin 2(\theta-\phi) = 2(\hat{d}_0, \hat{a}\nu)(\hat{d}_0, \hat{\nu}^{\perp}) = 2(d_0,\nu)(d_0,\nu^{\perp})$$

where $\sin t_2 = \kappa\sin(\theta-\phi)$. After a standard computation, we get for P-wave:

$$d_1 = S^T\hat{d}_1 = d_0 - 2\alpha\nu \tag{2.3}$$
$$d_2 = S^T\hat{d}_2 = \kappa d_0 - \beta\nu \tag{2.4}$$
$$A_0 = -\kappa(d_0,\nu)^2 + \kappa(d,\nu^{\perp})^2 + \beta(d_0,\nu) \tag{2.5}$$
$$A_1 = -\kappa + \beta(d,\nu) \tag{2.6}$$
$$A_2 = -2(d_0,\nu)(d_0,\nu^{\perp}) \tag{2.7}$$

where $\alpha = (d_0,\nu)$, $\beta = \kappa\alpha - \sqrt{\kappa^2\alpha^2 - \kappa^2 + 1}$ and $\kappa = k_p/k_s$. Similarly, for S-wave, we have

$$d_1 = \kappa_1 d_0 - \gamma\nu \tag{2.8}$$
$$d_2 = d_0 - 2\alpha\nu \tag{2.9}$$
$$A_0 = -\kappa_1(d_0,\nu)^2 + \kappa_1(d,\nu^{\perp})^2 + \gamma(d,\nu) \tag{2.10}$$
$$A_1 = 2(d_0,\nu)(d_0,\nu^{\perp}) \tag{2.11}$$
$$A_2 = -\kappa_1 + \gamma(d_0,\nu) \tag{2.12}$$

where $\gamma = \kappa_1\alpha - \sqrt{\kappa_1^2\alpha^2 - \kappa_1^2 + 1}$ and $\kappa_1 = 1/\kappa$. Thus the traction of $u(x)$ on the plane Γ can be obtained. For P-wave

$$\begin{aligned}\sigma(u)\cdot\nu &= [\mathbf{i}k_pA_0(\lambda\nu + 2\mu(d_0,\nu)d_0) + \mathbf{i}k_pA_1(\lambda\nu + 2\mu(d_1,\nu)d_1)\\ &+\mathbf{i}k_sA_2\mu((d_2,\nu)d_2^\perp + (d_2^\perp,\nu)d_2)]e^{\mathbf{i}k_px\cdot d} := \mathbf{i}k_pA_0\hat{\mathbf{R}}_p(x,d,\nu)e^{\mathbf{i}k_px\cdot d} \qquad (2.13)\end{aligned}$$

For S-wave

$$\begin{aligned}\sigma(u)\cdot\nu &= [\mathbf{i}k_sA_0\mu((d_0,\nu)d_0^\perp + (d_0^\perp,\nu)d_0) + \mathbf{i}k_pA_1(\lambda\nu + 2\mu(d_1,\nu)d_1)\\ &+\mathbf{i}k_sA_2\mu((d_2,\nu)d_2^\perp + (d_2^\perp,\nu)d_2)]e^{\mathbf{i}k_sx\cdot d} := \mathbf{i}k_sA_0\hat{\mathbf{R}}_s(x,d,\nu)e^{\mathbf{i}k_sx\cdot d} \qquad (2.14)\end{aligned}$$

Definition 2.1 *For any unit vector $d \in \mathbb{R}^2$, let $u_p^i = de^{\mathbf{i}k_px\cdot d}$ or $u_s^i = d^\perp e^{\mathbf{i}k_sx\cdot d}$ be the incident wave and $u_\alpha^s = u_\alpha^s(x;d)$ be the radiation solution of the Navier equation:*

$$u_\alpha^s + \omega^2 u_\alpha^s = 0 \quad in \quad \mathbb{R}^2\backslash\bar{D} \qquad (2.15)$$

$$\Delta_e u_\alpha^s = -u_\alpha^i \quad on \quad \partial D \qquad (2.16)$$

The scattering coefficient $\mathbf{R}_\alpha(x;d)$ for $x \in \partial D$ is defined by the relation

$$\sigma(u_\alpha^s + u_\alpha^i)\cdot\nu = \mathbf{i}k_\alpha\mathbf{R}_\alpha(x;d)e^{\mathbf{i}k_\alpha x\cdot d} \quad on \quad \partial D$$

where $\alpha = p, s$.

A convex object D locally may be cosidered at each point x as a plane with normal $\nu(x)$. Then the scattering coefficient can be approximated(Kirchhoff approximation) by

$$\mathbf{R}_\alpha(x;d) \approx \begin{cases}\hat{\mathbf{R}}_\alpha(x;d,\nu(x)) & \text{if} \quad x \in \partial D_d^- = \{x \in \partial D, \nu(x)\cdot d < 0\},\\ 0 & \text{if} \quad x \in \partial D_d^- = \{x \in \partial D, \nu(x)\cdot d \geq 0\}.\end{cases}$$

3. Numerical examples

In this section we present several numerical examples to show the effectiveness of Kirchhoff approximation. To synthesize the real scattering coefficient we compute the solution $\sigma(u_\alpha^s + u_\alpha^i)\cdot\nu$ of the scattering problems by representing the ansatz solution as the single layer potential with the Green tensor $\mathbb{G}(x,y)$ as the kernel

$$u^s(x) = \int_{\Gamma_D} -\mathbb{G}(y,x)^T\sigma(u^s(y) + u^i(y))\nu ds(y) = -u^i(x) \qquad \text{on} \quad x \in \Gamma_D$$

and discretizing the integral equation by standard Nyström methods [2]. Let $\mathbf{R}_\alpha(x;d) = (\mathbf{R}_\alpha^1(x;d), \mathbf{R}_\alpha^2(x;d))^T$, then we have

$$\mathbf{R}_\alpha^j(x;d) = \frac{\sigma(u^s(y) + u^i(y))\nu\cdot e_j}{\mathbf{i}k_\alpha e^{\mathbf{i}k_\alpha x\cdot d}} \qquad (3.1)$$

and we can compute $\hat{\mathbf{R}}_\alpha(x;d) = (\hat{\mathbf{R}}_\alpha^1(x;d), \hat{\mathbf{R}}_\alpha^2(x;d))^T$ by (2.13) and (2.14). In all our numerical examples we choose Lamé constant $\lambda = 1/2$, $\mu = 1/4$ and

$$u_p^i = (\cos t, \sin t)^T e^{\mathbf{i}k_p(x_1\cos t + x_2\sin t)}$$

$$u_s^i = (\sin t, -\cos t)^T e^{\mathbf{i}k_s(x_1\cos t + x_2\sin t)}$$

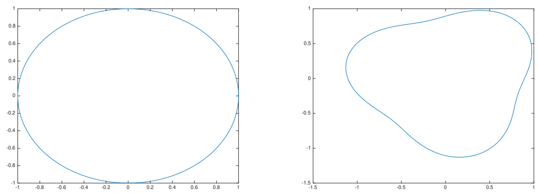

Figure 1.

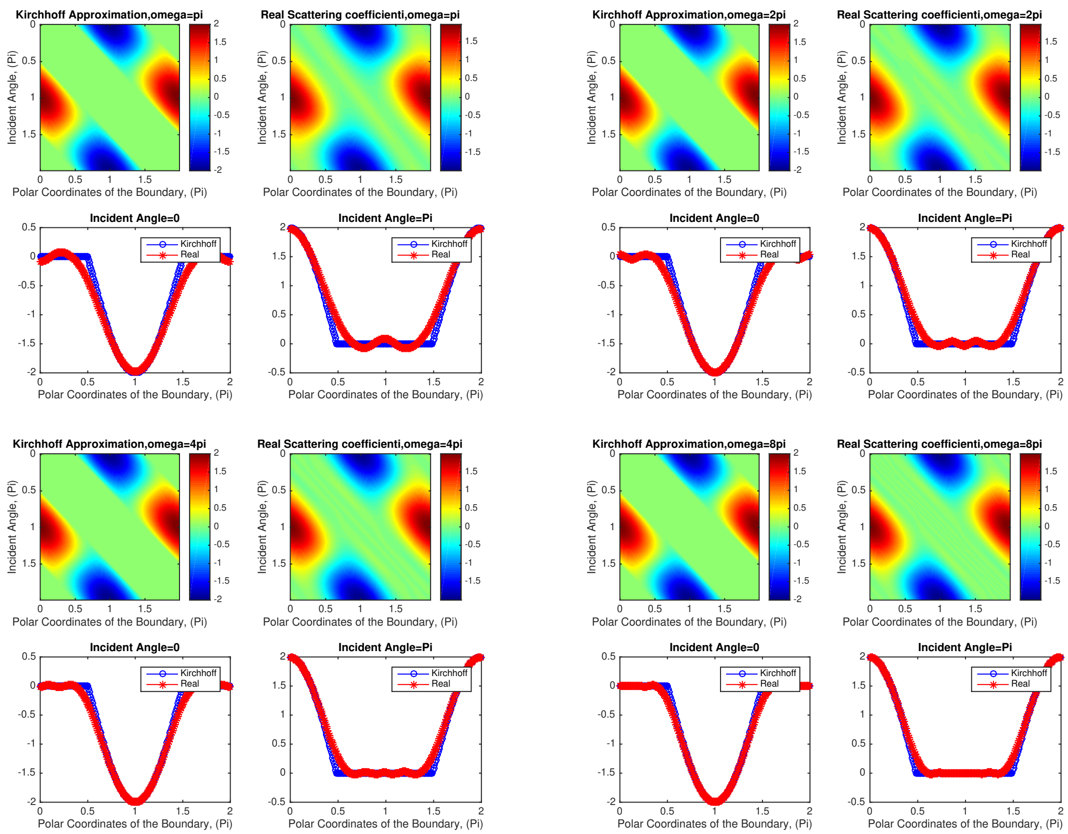

Figure 2. $\mathbf{R}_p^1$ and $\hat{\mathbf{R}}_p^1$ for circle

where $t \in [0, 2\pi]$.. The boundaries of the obstacles used in our numerical experiments are parameterized as follows,

$$\text{Circle:} \quad x_1 = \cos(\theta), \quad x_2 = \sin(\theta);$$

$$\text{Pear:} \quad \rho = 0.5(2 + 0.3\cos(3\theta)), x_1 = \sin\frac{\pi}{4}\rho(\cos\theta - \sin\theta), x_2 = \sin\frac{\pi}{4}\rho(\cos\theta + \sin\theta)$$

where $\theta \in [0, 2\pi]$ and depicted as figure 1.

In the following examples, figure 2-9, the angular frequency $\omega = \pi, 2\pi, 4\pi, 8\pi$.

References

[1] Achenbach J 1980 *Wave Propagation in Elastic Solids* (North-Holland)

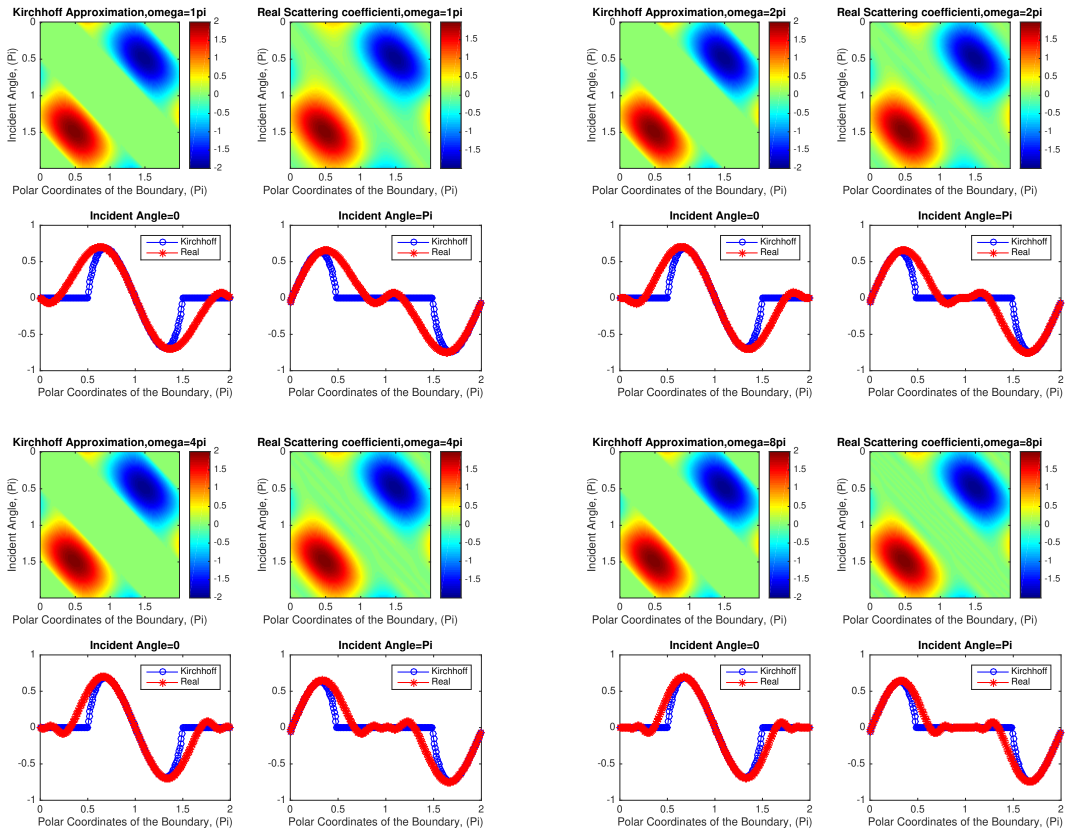

Figure 3. $\mathbf{R}_p^2$ and $\hat{\mathbf{R}}_p^2$ for circle

[2] Colton D and Kress R 1998 *Inverse Acoustic and Electromagnetic Scattering Problems* (Heidelberg: Springer)

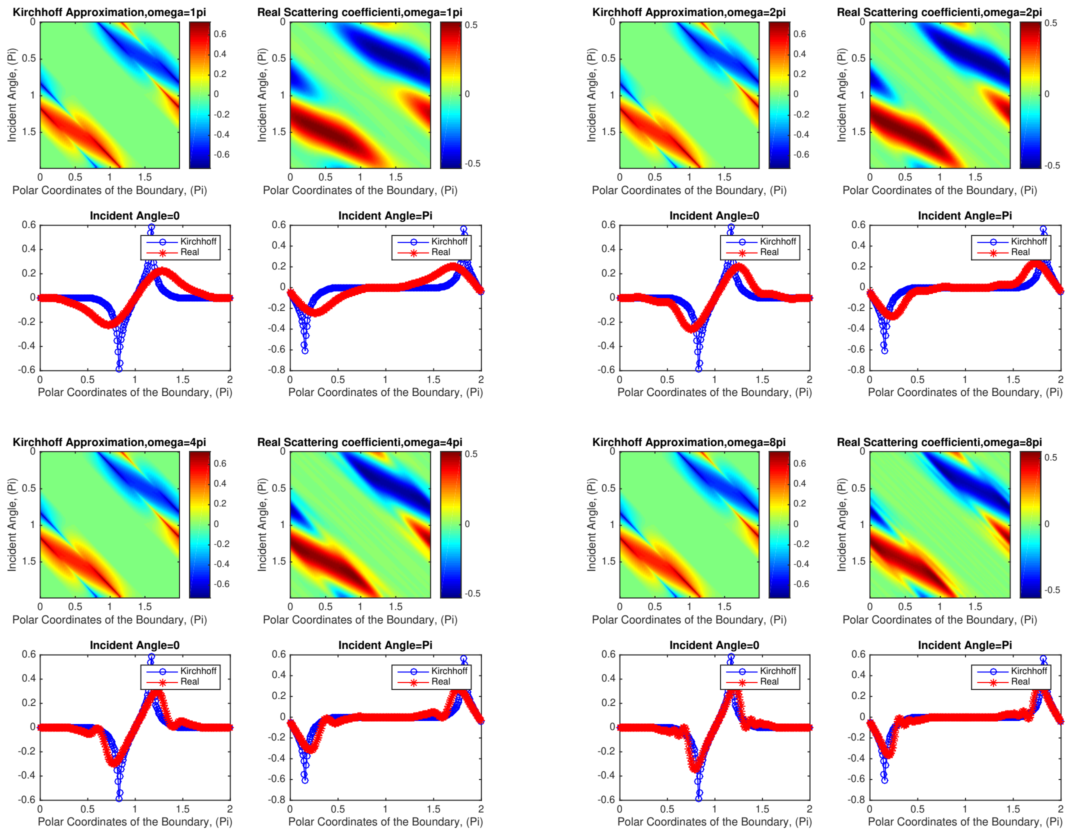

Figure 4. $\mathbf{R}_s^1$ and $\hat{\mathbf{R}}_s^1$ for circle

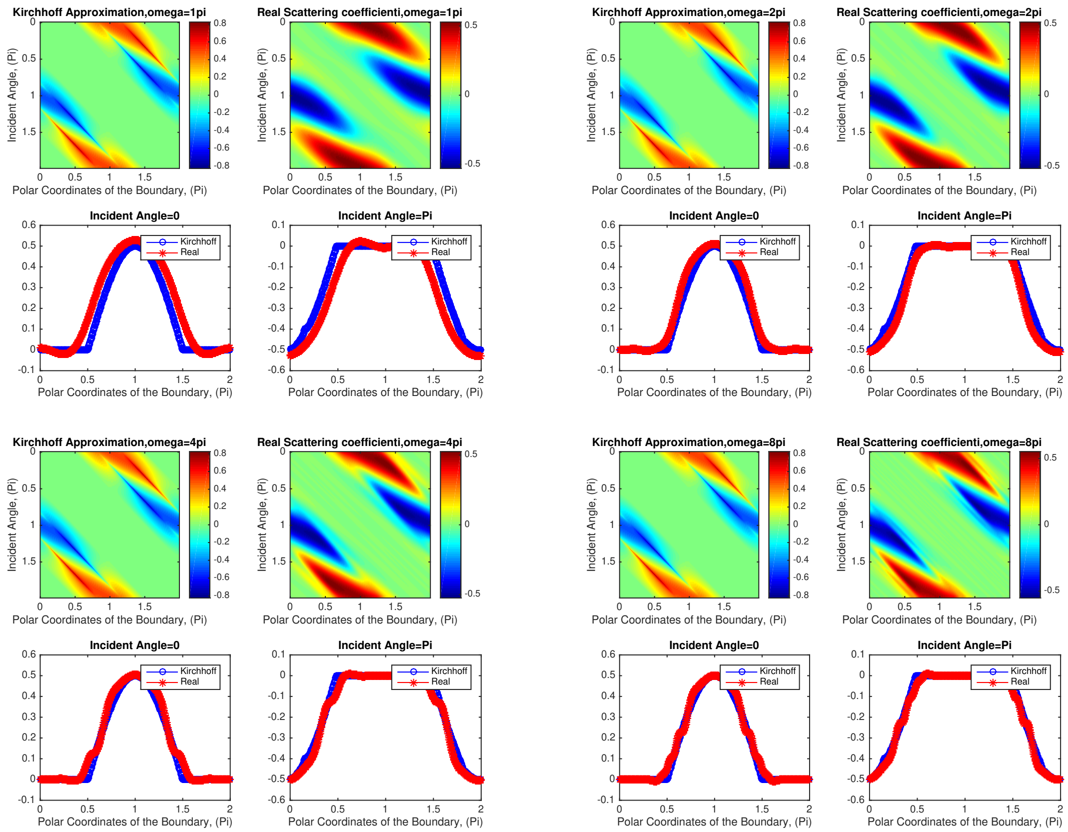

Figure 5. $\mathbf{R}_s^2$ and $\hat{\mathbf{R}}_s^2$ for circle

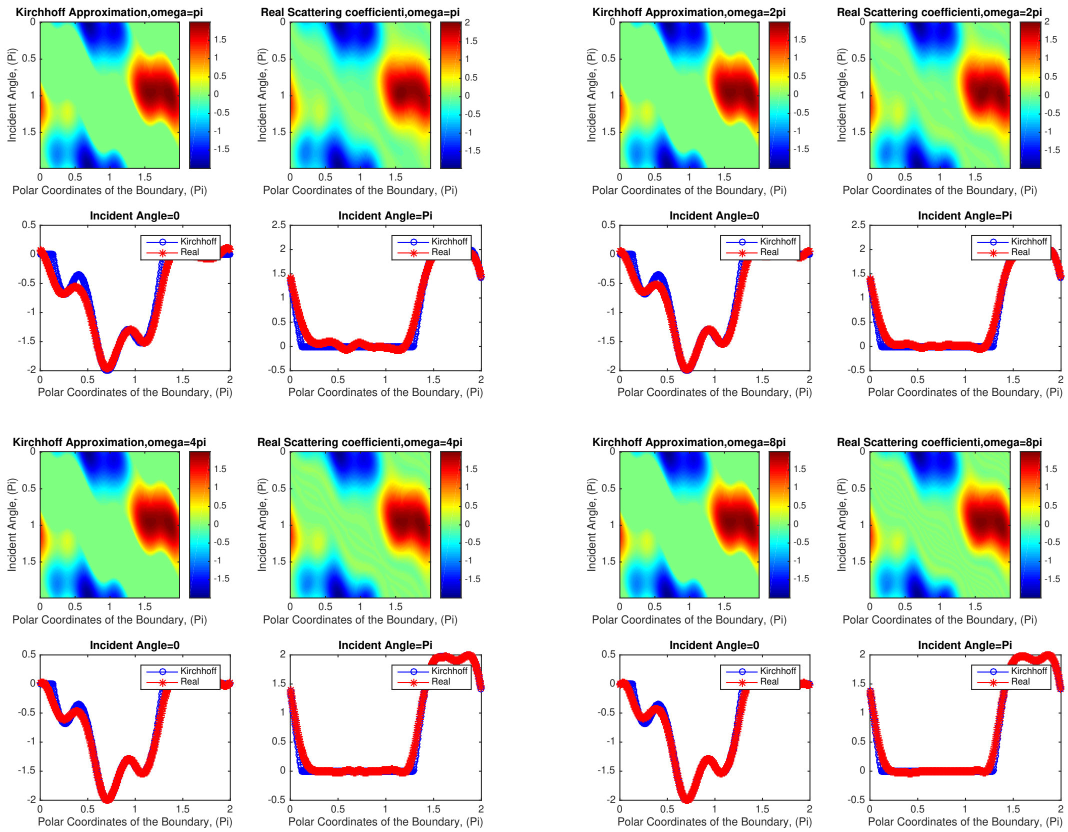

Figure 6. $\mathbf{R}_p^1$ and $\hat{\mathbf{R}}_p^1$ for pear

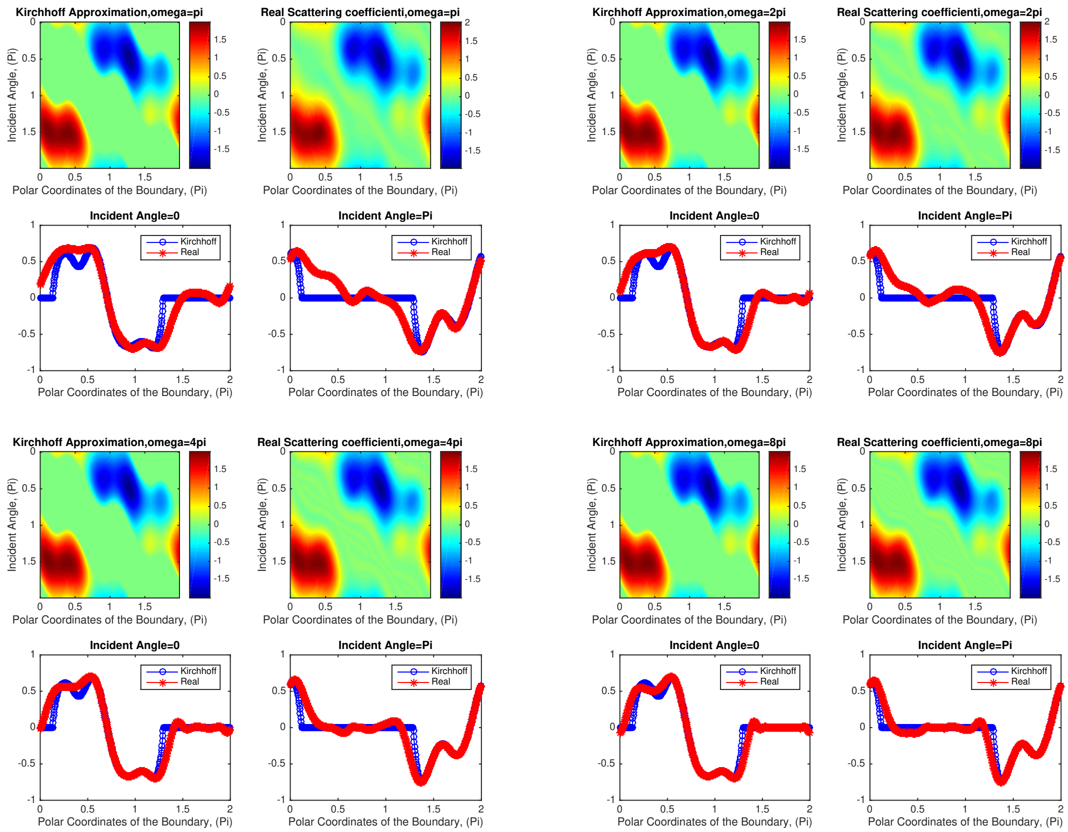

Figure 7. $\mathbf{R}_p^2$ and $\hat{\mathbf{R}}_p^2$ for pear

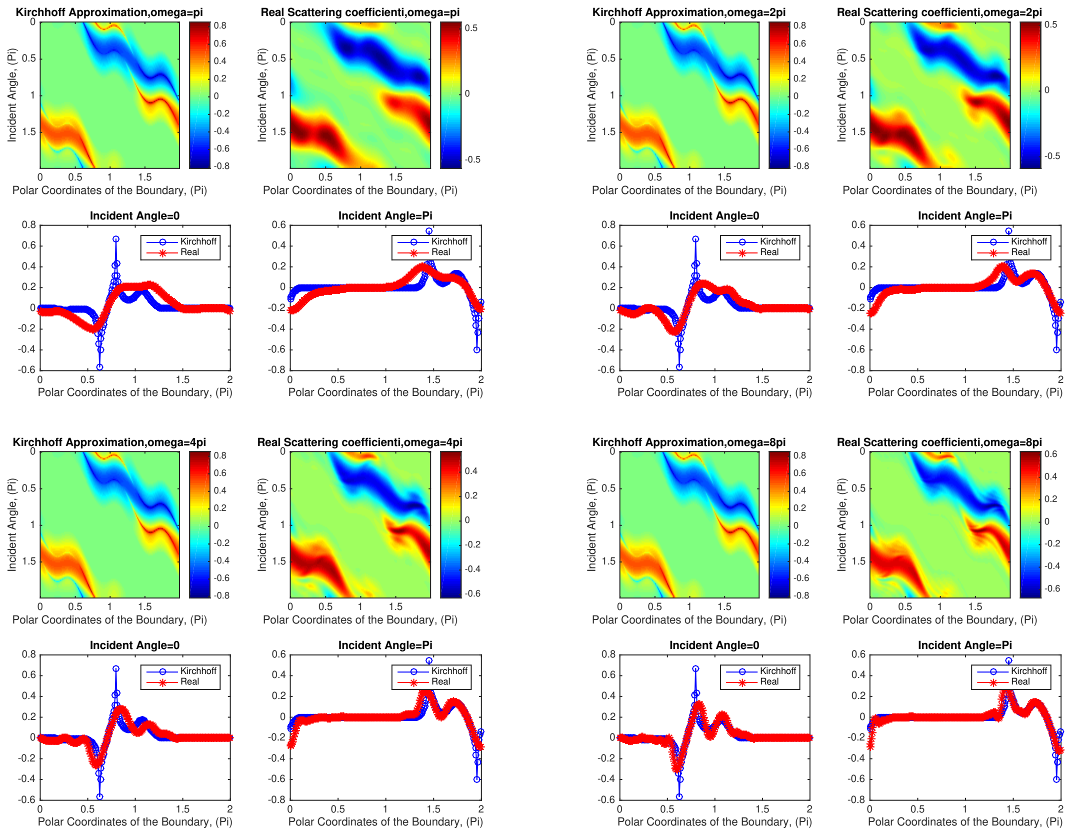

Figure 8. $\mathbf{R}_s^1$ and $\hat{\mathbf{R}}_s^1$ for pear

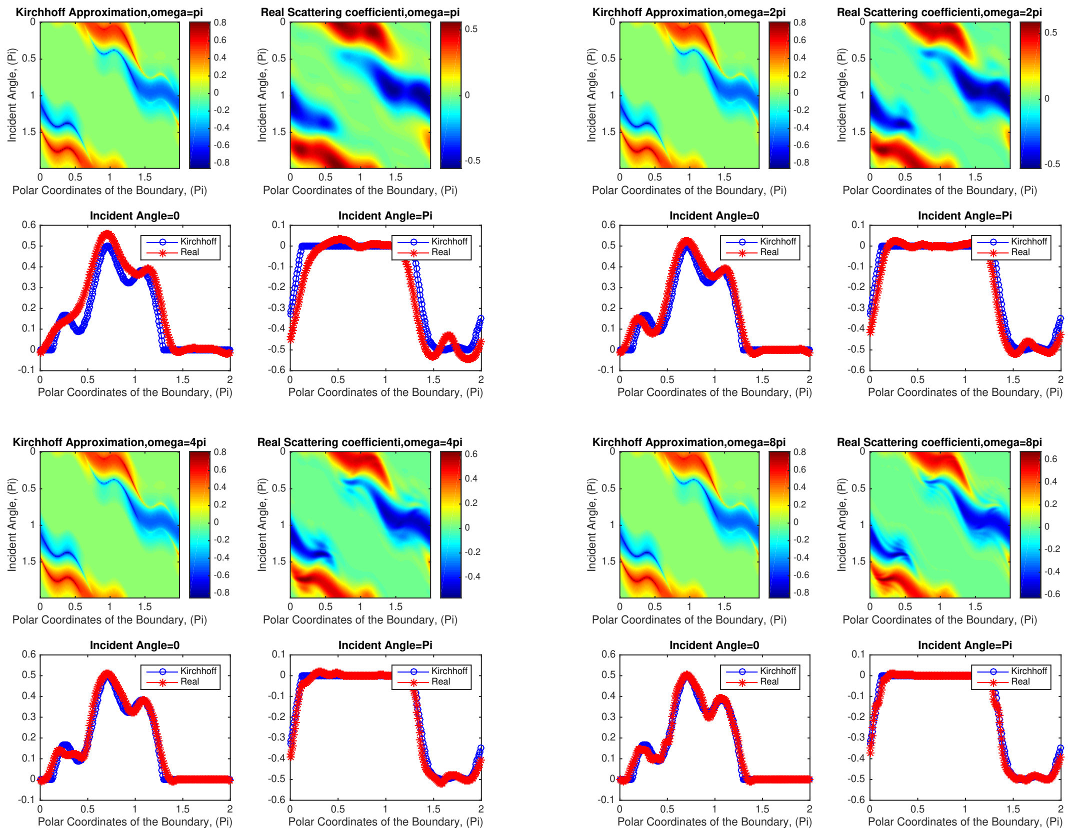

Figure 9. $\mathbf{R}_s^2$ and $\hat{\mathbf{R}}_s^2$ for pear