

# Inverse Elastic Scattering Problem

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# Content

- 1 Motivation and Problem Formulation
- 2 Reverse Time Migration
- 3 Resolution Analysis
- 4 Numerical Test

## Motivation



Figure: Find the support of the unknown obstacle from the knowledge of the scattered waves on a given surface.

## Direct Scattering Problem in the Half Space

We consider elastic wave propagating in the half space with Neumann condition (Traction Free),

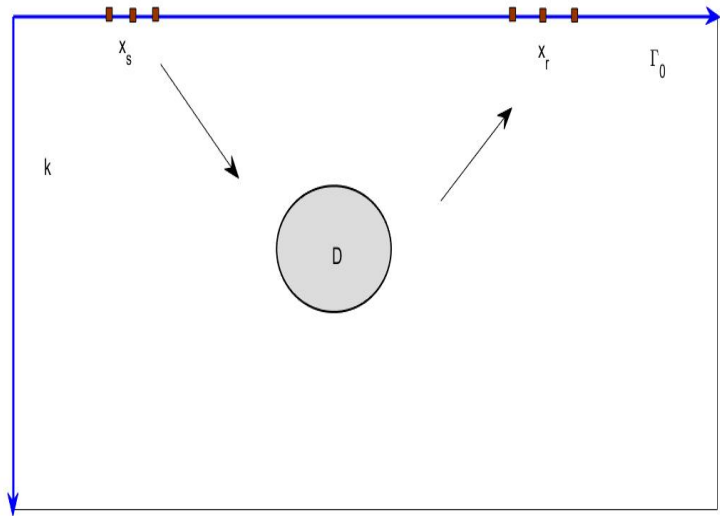
$$\begin{aligned}\nabla \cdot \sigma(u_q) + \rho\omega^2 u_q &= -\delta_{x_s}(x)q \quad \text{in } \mathbb{R}_+^2 \setminus \bar{D} \\ u_q &= 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \sigma(u_q) \cdot e_2 = 0 \quad \text{on } \Gamma_0\end{aligned}$$

together with the constitutive relation (Hookes law)

$$\begin{aligned}\sigma(u) &= 2\mu\varepsilon(u) + \lambda\operatorname{div}u\mathbb{I} \\ \varepsilon(u) &= \frac{1}{2}(\nabla u + (\nabla u)^T)\end{aligned}$$

where  $\omega$  is the circular frequency,  $u(x) \in \mathbb{C}^2$  denotes the displacement fields and  $\sigma(u)$  is the stress tensor. We also need to define the surface traction  $T_x^n(\cdot)$  on the normal direction  $n$ ,

$$T_x^n u(x) := \sigma \cdot n = 2\mu \frac{\partial u}{\partial n} + \lambda n \operatorname{div}u + \mu n \times \operatorname{curl}u$$



## Green Tensor in the Half Space

Green Tensor in the half-space with Neumann boundary :

$$\begin{aligned}\Delta_e \mathbb{N}(x; y) + \omega^2 \mathbb{N}(x, y) &= -\delta_y(x) \mathbb{I} \quad \text{in } \mathbb{R}_+^2, \\ \sigma_x(\mathbb{N}(x, y)) e_2 &= 0 \quad \text{on } x_2 = 0\end{aligned}$$

Green Tensor in the half-space with Dirichlet Boundary

$$\begin{aligned}\Delta_e \mathbb{D}(x, y) + \omega^2 \mathbb{D}(x, y) &= -\delta_y(x) \mathbb{I} \quad \text{in } \mathbb{R}_+^2, \\ \mathbb{D}(x, y) &= 0 \quad \text{on } x_2 = 0\end{aligned}$$

where  $\delta_y(x)$  is the Dirac source at  $y \in \mathbb{R}_+^2$  and  $N(x, y)$ ,  $\mathbb{D}(x, y)$  are  $\mathbb{C}^{2 \times 2}$  matrixes.

**Remark:** we will assume that for  $z \in \mathbb{C}$ ,  $z^{1/2}$  is the analytic branch of  $\sqrt{z}$  such that  $\text{Im}(z^{1/2}) \geq 0$ .

## Green Tensor in Frequency Domain after Fourier Transformation

$$\hat{N}(\xi, x_2; y_2) = \hat{G}(\xi, x_2; y_2) - \hat{G}(\xi, x_2; -y_2) + \hat{N}_c(\xi, x_2; y_2)$$

$$\begin{aligned} \hat{N}_c(\xi, x_2; y_2) = &= \frac{\mathbf{i}}{\omega^2 \delta(\xi)} \left\{ A(\xi) e^{\mathbf{i} \mu_s (x_2 + y_2)} + B(\xi) e^{\mathbf{i} \mu_p (x_2 + y_2)} \right. \\ &\left. + C(\xi) e^{\mathbf{i} \mu_s x_2 + \mathbf{i} \mu_p y_2} + D(\xi) e^{\mathbf{i} \mu_p x_2 + \mathbf{i} \mu_s y_2} \right\} \end{aligned}$$

where  $\mathbb{G}(x, y)$  is the fundamental solution of elastic equation and

$$\begin{aligned} A(\xi) &= \begin{pmatrix} \mu_s \beta^2 & -4\xi^3 \mu_s \mu_p \\ -\xi \beta^2 & 4\xi^4 \mu_p \end{pmatrix} & B(\xi) &= \begin{pmatrix} 4\xi^4 \mu_s & \xi \beta^2 \\ 4\xi^3 \mu_s \mu_p & \mu_p \beta^2 \end{pmatrix} \\ C(\xi) &= \begin{pmatrix} 2\xi^2 \mu_s \beta & -2\xi \mu_s \mu_p \beta \\ -2\xi^3 \beta & 2\xi^2 \mu_p \beta \end{pmatrix} & D(\xi) &= \begin{pmatrix} 2\xi^2 \mu_s \beta & 2\xi^3 \beta \\ 2\xi \mu_s \mu_p \beta & 2\xi^2 \mu_p \beta \end{pmatrix} \end{aligned}$$

and  $\mu_\alpha = (k_\alpha^2 - \xi^2)^{1/2}$ ,  $\alpha \in \{s, p\}$ ,  $\beta(\xi) = k_s^2 - 2\xi^2$ ,  $\delta(\xi) = \beta^2 + 4\xi^2 \mu_s \mu_p$ .

# Dirichlet Green Tensor in Frequency Domain after Fourier Transformation

$$\hat{\mathbb{D}}(\xi, x_2; y_2) = \hat{\mathbb{G}}(\xi, x_2; y_2) - \hat{\mathbb{G}}(\xi, x_2; -y_2) + \hat{M}(\xi, x_2; y_2)$$

$$\begin{aligned} \hat{M}(\xi, x_2; y_2) = & \frac{\mathbf{i}}{\omega^2 \gamma(\xi)} \left\{ A(\xi) e^{\mathbf{i} \mu_s (x_2 + y_2)} + B(\xi) e^{\mathbf{i} \mu_p (x_2 + y_2)} \right. \\ & \left. - A(\xi) e^{\mathbf{i} \mu_s x_2 + \mathbf{i} \mu_p y_2} - B(\xi) e^{\mathbf{i} \mu_p x_2 + \mathbf{i} \mu_s y_2} \right\} \end{aligned}$$

where

$$A(\xi) = \begin{pmatrix} \xi^2 \mu_s & -\xi \mu_s \mu_p \\ -\xi^3 & \xi^2 \mu_p \end{pmatrix} \quad B(\xi) = \begin{pmatrix} \xi^2 \mu_s & \xi^3 \\ \xi \mu_s \mu_p & \xi^2 \mu_p \end{pmatrix}$$

and  $\gamma(\xi) = \xi^2 + \mu_s \mu_p$ .



### Lemme 1

Let Lamé constant  $\lambda, \mu \in \mathbb{R}^+$ , then the Rayleigh equation  $\delta(\xi) = 0$  has only two roots denoted by  $\pm k_R$  in complex plane. Moreover,  $k_R > k_s > k_p$ ,  $k_R \in \mathbb{R}$  and  $k_R$  is called Rayleigh wave number.

Using Cauchy integral theorem, we carry out:

### Formula 1

$$\mathbb{N}(x, y) = \frac{1}{2\pi} \text{P.V} \int_{\mathbb{R}} \hat{\mathbb{N}}(\xi, x_2; y_2) e^{\mathbf{i}(x_1 - y_1)\xi} d\xi - \frac{\mathbf{i}}{2} \frac{\mathbb{N}_\delta(-k_R)}{\delta'(-k_R)} e^{-\mathbf{i}(x_1 - y_1)k_R} + \frac{\mathbf{i}}{2} \frac{\mathbb{N}_\delta(k_R)}{\delta'(k_R)} e^{\mathbf{i}(x_1 - y_1)k_R}$$

where  $\mathbb{N}_\delta(\xi) = \hat{\mathbb{N}}(\xi, x_2; y_2)\delta(\xi)$ .

## Lemma 2

Let Lamé constant  $\lambda, \mu \in \mathbb{C}$  and  $\text{Im}(k_s) \geq 0, \text{Im}(k_p) \geq 0$ , then equation  $\gamma(\xi) = 0$  has no root in complex plane.

## Formula 2

Let  $\mathbb{T}_D(x, y)$  denote the traction of  $\mathbb{D}(x, y)$  in direction  $e_2$  with respect to  $x$  such that  $\mathbb{T}_D(x, y)e_i = \sigma_x(\mathbb{D}(x, y)e_i)e_2$ .

$$\mathbb{T}_D(x, y) = \mathbb{T}(x, y) - \mathbb{T}(x, y') + \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\mathbb{T}}_M(\xi, x_2; y_2) e^{i(x_1 - y_1)\xi} d\xi$$

and

$$\begin{aligned} \hat{\mathbb{T}}_M(\xi, x_2; y_2) = & \frac{\mu}{\omega^2 \gamma(\xi)} \left\{ E(\xi) e^{i\mu_s(x_2 + y_2)} + F(\xi) e^{i\mu_p(x_2 + y_2)} \right. \\ & \left. - E(\xi) e^{i\mu_s x_2 + \mu_p y_2} - F(\xi) e^{i\mu_p x_2 + \mu_s y_2} \right\} \end{aligned}$$

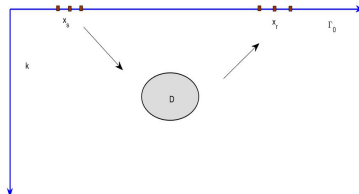
where

$$E(\xi) = \begin{pmatrix} -\xi^2 \beta & \xi \mu_p \beta \\ 2\xi^3 \mu_s & -2\xi^2 \mu_s \mu_p \end{pmatrix} \quad F(\xi) = \begin{pmatrix} -2\xi^2 \mu_s \mu_p & -2\xi^3 \mu_p \\ -\xi \mu_s \beta & -\xi^2 \beta \end{pmatrix}$$

# Inverse Scattering Problem

$$\begin{cases} \nabla \cdot \sigma(u_q) + \rho\omega^2 u_q = -\delta_{x_s}(x)q & \text{in } \mathbb{R}_+^2 \setminus \bar{D} \\ u_q = 0 & \text{on } \Gamma_D \\ \sigma(u_q) \cdot e_2 = 0 & \text{on } \Gamma_0 \end{cases}$$

satisfies the generalized radiation condition



## Direct Problem

To determine the scattering wave field  $u^s(x, x_s) = u(x, x_s) - u^i(x, x_s)$  from the given incident field  $u^i(x, x_s) = \mathbb{N}(x, x_s)$ , the differential equation governing the wave motion and the information of obstacle.

## Inverse problem

To determine the location, size, shape of the obstacle by the measured field  $u^s(x, x_s)$  on  $x_r$

# Algorithms of inverse obstacle problem

## Direct Imaging Method

- Linear Sample MethodFactorization methodPoint source method
- Multiple Signal Classification
- Prestack depth migrationReverse Time Migration

## Feature

Fast computationDifficult mathematics analysis

## Iterative Method

- Differential semblance optimization
- Full waveform inversion
- Recursive linearization algorithm

## Feature

Need prior information, Difficult to convergence, Provide quantitative information

## Reverse Time Migration

- Do not require any priori information of the physical properties of the obstacle such as penetrable or non-penetrable, and for non-penetrable obstacles, the type of boundary conditions on the boundary of the obstacle.
- The previous analysis of the migration method is usually based on the high frequency assumption so that the geometric optics approximation can be used.

# Reverse Time Migration Mathematics Framework

## Acoustic, electromagnetic, elastic wave in the Full space

- ① Chen J, Chen Z, Huang G. *Reverse time migration for extended obstacles: acoustic waves [J]*. Inverse Problems. 2013, 29(8):645-648
- ② Chen J, Chen Z, Huang G. *Reverse time migration for extended obstacles: Electromagnetic waves [J]*. Scientia Sinica, 2015, 29(8):085005.
- ③ Chen Z, Huang G. *Reverse time migration for extended obstacles: Elastic waves (in Chinese) [J]*. Science China Mathematics, 2015, 45(8):1103-1114.

## Planar acoustic waveguide

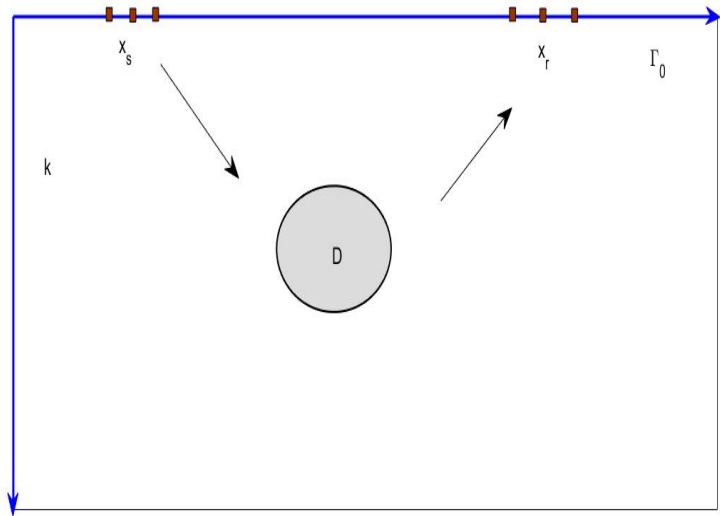
- ① Chen Z M, Huang G H. *Reverse time migration for reconstructing extended obstacles in planar acoustic waveguides [J]*. Science China Mathematics, 2015, 58(9):1811-1834.

## Acoustic in the Half space

- ① Chen Z, Huang G. *Reverse time migration for reconstructing extended obstacles in the half space [J]*. Inverse Problems, 2015, 31(5):055007 (19pp).

## Phaseless Algorithm

- ① Chen Z, Huang G. *Phaseless Imaging by Reverse Time Migration: Acoustic Waves [J]*. Numerical Mathematics Theory Methods and Applications, 2017, 10(1):1-21.



# RTM Algorithm

Given the data  $u_k^s(x_r, x_s)$ ,  $k = 1, 2$  which is the measurement of the scattered field at  $x_r$  when the source is emitted at  $x_s$  along the polarized direction  $e_k$ ,  $s = 1, \dots, N_s$  and  $r = 1, \dots, N_r$ .

$$I_d(z) = \text{Im} \sum_{k=1}^2 \left\{ \frac{|\Gamma_0^d|}{N_s} \sum_{s=1}^{N_s} \sum_{i=1}^2 [\sigma_{x_s}(\mathbb{D}(x_s, z) e_i) e_2 \cdot e_k] [v_k(z, x_s) \cdot e_i] \right\}. \quad z \in \Omega \quad (1)$$

where  $v_k(z, x_s)$  satisfy the following scattering elastic equation:

$$\begin{aligned} \Delta_e v_k(z, x_s) + \omega^2 v_k(z, x_s) &= 0 \quad \text{in } \mathbb{R}_+^2 \\ v_k(z, x_s) &= \frac{|\Gamma_0^d|}{N_r} \sum_{r=1}^{N_r} \overline{u_k^s(x_r, x_s)} \delta_{x_r}(z) \quad \text{on } \Gamma_0 \end{aligned}$$

By letting  $N_s, N_r \rightarrow \infty$ , we know that (1) can be viewed as an approximation of the following continuous integral:

$$\hat{I}_d(z) = \text{Im} \sum_{q=e_1, e_2} \int_{\Gamma_0^d} \int_{\Gamma_0^d} [\mathbb{T}_D(x_s, z)^T q] [\mathbb{T}_D(x_r, z)^T \overline{u_q^s(x_r, x_s)}] ds(x_r) ds(x_s).$$

where  $z \in \Omega$ .

## Point Spread Function

The point spread function measures the resolution to find a point source. Given the  $\mathbb{N}(x, y)$  on  $\Gamma_0^d = \{(x_1, x_2)^T \in \Gamma_0, x_1 \in (-d, d)\}$  with the source  $y \in \mathbb{R}_+^2$ , we define the PSF as the back-propagated field  $\mathbb{J}_d(x, y)$ .

$$\Delta_e \mathbb{J}_d(x, y) + \omega^2 \mathbb{J}_d(x, y) = -\delta_y(x) \mathbb{I} \quad \text{in } \mathbb{R}_+^2, \quad \mathbb{J}_d(x, y) = \mathbb{N}(x, y) \chi_{(-d, d)} \quad \text{on } \Gamma_0$$

By integral representation

$$\begin{aligned} \mathbb{J}_d^{ij}(z, y) : &= e_i \cdot \mathbb{J}_d(z, y) e_j \\ &= \int_{\Gamma_0^d} \sigma_x(\mathbb{D}(x, z) e_i) e_2 \cdot \overline{\mathbb{N}(x, y)} e_j ds(x) \\ &= \int_{-d}^d \sigma_x(\mathbb{D}(x_1, 0; z_1, z_2) e_i) e_2 \cdot \overline{\mathbb{N}(x_1, 0; y_1, y_2)} e_j dx_1 \end{aligned}$$



## Numerical test: PSF

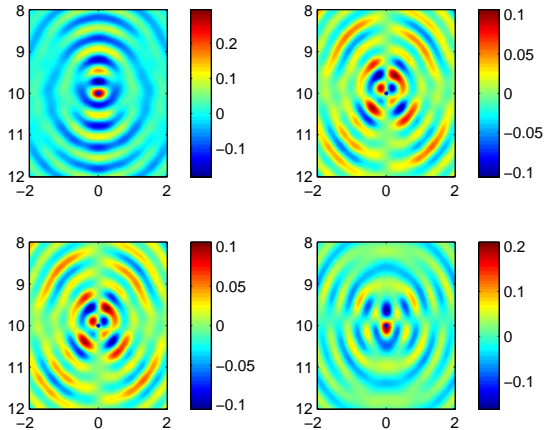


Figure: The figures on the diagonal line show that psf have peaks on the point  $x=y$

# Analysis of PSF

## Hypothesis

We assume the obstacle  $D \subset \Omega$  and there exist constants  $0 < c_1 < 1, c_2 > 0, c_3 > 0$  such that

$$h < d, \quad |x_1| \leq c_1 d, \quad |x_1 - y_1| \leq c_2 h, \quad |x_2| \leq c_3 h \quad \forall x, y \in \Omega$$

## Theorem 1

Let  $k_s h > 1$ . For any  $z, y \in \Omega$ , PSF can be represented by  $J_d(z, y) = \mathbb{F}(z, y) + \mathbb{R}(z, y)$  and it satisfy that

$$\begin{aligned} |\mathbb{R}_{ij}(z, y)| + k_s^{-1} |\nabla_y \mathbb{R}_{ij}(z, y)| &\leq \frac{C}{\mu} \left( \frac{1}{(k_s h)^{\frac{1}{2n^*}}} + k_s h e^{-k_s h \sqrt{\kappa_R^2 - 1}} \right) \\ &\quad + \frac{C}{\mu} \left( \left( \frac{h}{d} \right)^2 + \frac{(k_s h)^{1/2}}{e^{k_s h \sqrt{\kappa_R^2 - 1}}} \left( \frac{h}{d} \right)^{1/2} \right) \end{aligned}$$

uniformly for  $z, y \in \Omega$ . Here  $\kappa_R := k_R/k_s$  and the constant  $C$  may dependent on  $k_s d_D$  and  $\kappa := k_p/k_s$ , but is independent of  $k_s, k_p, h, d_D$ .

# Main Term of PSF

Based on the above argument, we know that  $\mathbb{R}(z, y)$  becomes small when  $z, y$  move away from  $\Gamma_0$  and  $d \gg h$ . Our goal is to show  $\mathbb{F}(z, y)$  has the similar decay to the elastic fundamental solution  $\text{Im } \Phi(z, y)$  as  $|z - y| \rightarrow \infty$ .

## Theorem 2

For any  $z, y \in \mathbb{R}_+^2$ , when  $z = y$

$$|\text{Im } \mathbb{F}_{ii}(z, y)| \geq \frac{1}{4(\lambda + 2\mu)}, \quad i = 1, 2$$

$$\text{Im } \mathbb{F}_{12}(z, y) = \text{Im } \mathbb{F}_{21}(z, y) = 0$$

and for  $z \neq y$

$$|\mathbb{F}_{ij}(z, y)| \leq \frac{C}{\mu} [(k_s |z - y|)^{-1/2} + (k_s |z - y|^{-1})]$$

where constant  $C$  is only dependent on  $\kappa := k_p/k_s$ .

Now, We turn to study the resolution of the function  $\hat{I}_d(z)$ . To do this, we first show the difference between the half space scattering solution and the full space scattered solution is small if the scatterer is far away from the boundary  $\Gamma_0$ .

### Theorem 3

Let  $g \in H^{1/2}(\Gamma_D)$  and  $\mathbf{u}_1, \mathbf{u}_2$  be the scattering solution of following problems:

$$\begin{aligned}\Delta_e \mathbf{u}_1 + \omega^2 \mathbf{u}_1 &= 0 && \text{in } \mathbb{R}_+^2 \setminus \bar{D} \\ \mathbf{u}_1 &= g && \text{on } \Gamma_D \\ \sigma(\mathbf{u}_1) e_2 &= 0 && \text{on } \Gamma_0\end{aligned}$$

and

$$\begin{aligned}\Delta_e \mathbf{u}_2 + \omega^2 \mathbf{u}_2 &= 0 && \text{in } \mathbb{R}^2 \setminus \bar{D} \\ \mathbf{u}_2 &= g && \text{on } \Gamma_D\end{aligned}$$

Then there exists a constant  $C$  independent of  $k_s, k_p$ , such that

$$\|\sigma_x(\mathbf{u}_1 - \mathbf{u}_2)\nu\|_{H^{-1/2}(\Gamma_D)} \leq \frac{C}{\mu} (1 + \|T_f\|)(1 + \|T_h\|)(1 + k_s d_D)^2 \epsilon_1(k_s h) \|g\|_{H^{1/2}(\Gamma_D)}$$

# Resolution Analysis

## Theorem 4

For any  $z \in \Omega$ , let  $\Psi(y, z) \in \mathbb{C}^{2 \times 2}$  be the radiation solution of the problem:

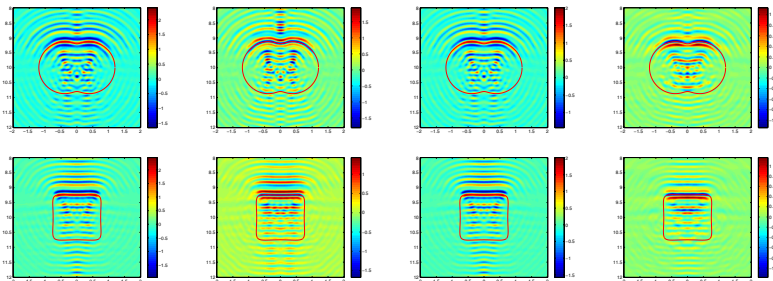
$$\begin{aligned} \Delta_e \Psi(y, z) e_i + \omega^2 \Psi e_i &= 0 \quad \text{in } \mathbb{R}_+^2 \setminus \bar{D} \quad i = 1, 2 \\ \Psi(y, z) &= -\overline{\mathbb{F}(z, y)} \quad \text{on } \Gamma_D \end{aligned}$$

Then, we have

$$\hat{I}_d(z) = \text{Im} \int_{\Gamma_D} \sum_{i=1}^2 [\sigma_y(\overline{\mathbb{F}(z, y)} + \Psi(y, z)) e_i] \cdot \overline{\mathbb{F}(z, y)} e_i] ds(y) + \mathbb{W}_{\hat{f}}(z) \quad (2)$$

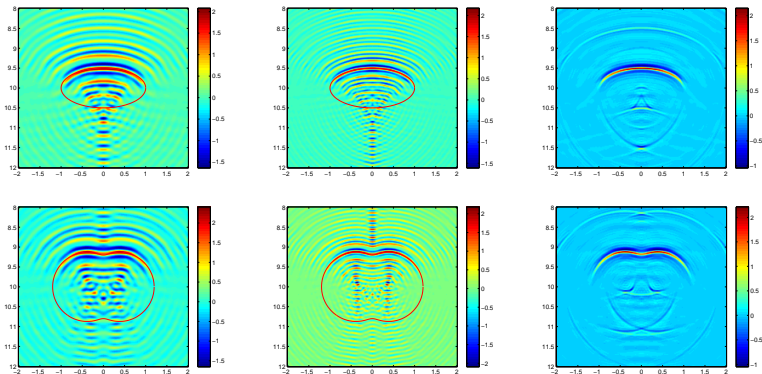
where  $|\mathbb{W}_{\hat{f}}(z)| \leq \frac{C}{\mu} (1 + \|T_f\|)(1 + \|T_h\|)(1 + k_s d_D)^4 (\epsilon_1(k_s h) + \epsilon_2(k_s h, h/d))$   
 uniformly for  $z$  in  $\Omega$ .

## Numerical Test: Different Boundary Condition



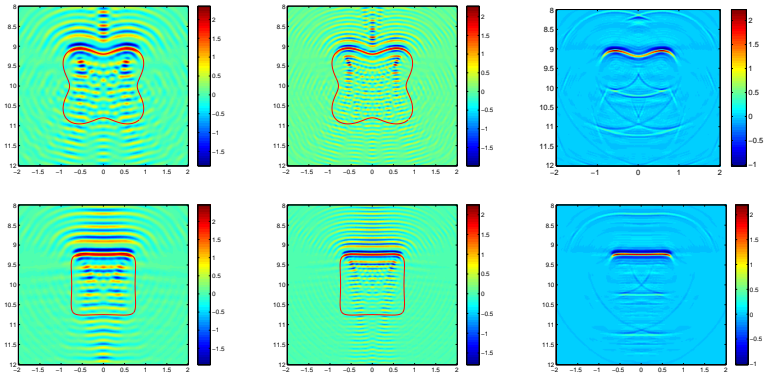
**Figure:** Example 1: From left to right: imaging results of a Dirichlet, a Neumann, a Robin boundary with impedance  $\eta(x) = 1$ , and a penetrable obstacle with diffractive index  $n(x) = 0.25$

## Numerical Test: Different Sharp



**Figure:** Example 2: Imaging results of clamped obstacles with different shapes from top to below. The left row is imaged with single frequency data where  $\omega = 3\pi$ , The middle row is imaged with single frequency data where  $\omega = 5\pi$  and The left row is imaged with multi frequency data

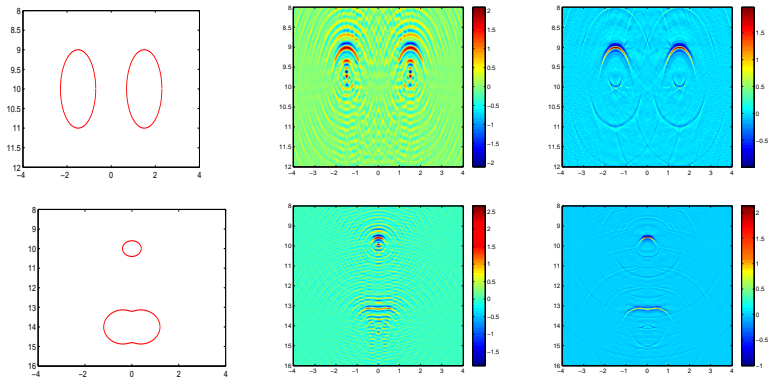
## Numerical Test: Different Sharp



**Figure:** Example 3: Imaging results of clamped obstacles with different shapes from top to below. The left row is imaged with single frequency data where  $\omega = 3\pi$ , The middle row is imaged with single frequency data where  $\omega = 5\pi$  and The left row is imaged with multi frequency data

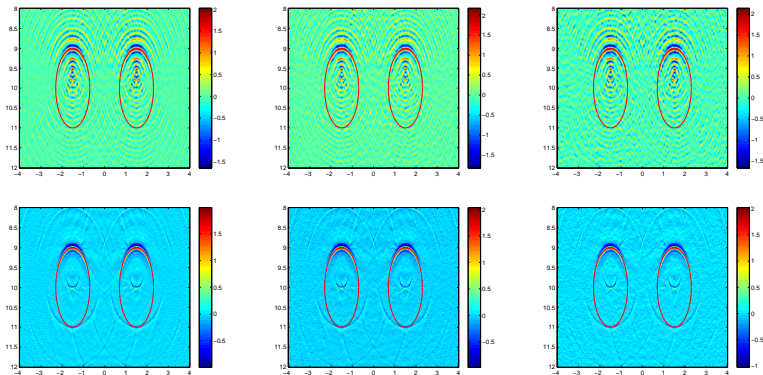


## Numerical Test: Two Obstacles



**Figure:** Example 4: From left to right, true obstacle model with two circles. the imaging result with single frequency data where  $\omega = 3\pi$ , the imaging result with multiple frequency data.

## Numerical Test: additive Gaussian noise



**Figure:** Example 5: Imaging results of a clamped obstacle with noise levels  $\mu = 0.2; 0.3; 0.4$  (from left to right). The top row is imaged with single frequency data where  $\omega = 4\pi$ , and the bottom row is imaged with multi-frequency data.

## Physical interpretation: high frequency

Let  $y(s)$  be the arc length parametrization of the boundry  $\Gamma_D$  and  $y_{\pm}(\eta_{\theta}) = y(s_{\pm})$  be the points such that  $\nu(y(s_{\pm})) = \pm \eta_{\theta}$  and  $\eta(y)$  be Gauss curvature. In the case of  $\omega \gg 1$ , by stationary phase theorem and Kirchhoff approximation, the imaging function for the clamped obstacle is

$$\begin{aligned} \hat{I}_d(z) &\approx \sqrt{8\pi k_p} \operatorname{Im} \operatorname{tr} \int_0^{\pi} ((\lambda + 2\mu) A_p(\theta) \eta_{\theta} e^{\mathbf{i} k_p (y_-(\eta_{\theta}) - z) \cdot \eta_{\theta} - \mathbf{i} \frac{\pi}{4}})^T \frac{\overline{F(z, y_-(\eta_{\theta}))}}{\sqrt{\vartheta(y_-(\eta_{\theta}))}} d\theta \\ &+ \sqrt{8\pi k_s} \operatorname{Im} \operatorname{tr} \int_0^{\pi} (\mu A_s(\theta) \eta_{\theta}^{\perp} e^{\mathbf{i} k_s (y_-(\eta_{\theta}) - z) \cdot \eta_{\theta} - \mathbf{i} \frac{\pi}{4}})^T \frac{\overline{F(z, y_-(\eta_{\theta}))}}{\sqrt{\vartheta(y_-(\eta_{\theta}))}} d\theta \end{aligned}$$

Now for  $z$  in the part of  $\Gamma_D$  which is back to  $\Gamma_0$ , ie  $\nu(z) \cdot \eta_{\theta} > 0$  for any  $\theta \in [0, \pi]$ , we know that  $z$  and  $y_-(\eta_{\theta})$  are far away and thus  $\hat{I}_d(z) \approx 0$ . By above formula, we can explain that one cannot image the back part of the obstacle with only the data collected on  $\Gamma_0$ . This confirmed in our numerical examples.

Our work:

- New form and asymptotic analysis of Green tensor in the half space..
- Regularity estimate of direction elastic wave equation in the Half-space.
- Mathematics analysis of point spread function.
- Resolution analysis of Reverse Time Migration without any geometric optics approximation .
- Scattered wave data simulation and numerical test of RTM.

Thank you !