Géométrie différentielle et théorie de jauge
Solutions to exercises

03/03/2020

4. Chern class of $T\mathbb{C}P^n$. Let L be the tautological bundle, so that $L_x = x \subset \mathbb{C}^{n+1}$ and we can consider at each point x the orthogonal decomposition $\mathbb{C}^{n+1} = L_x \oplus E_x$ which defines a rank n vector bundle E on $\mathbb{C}P^n$.
Since $T\mathbb{C}P^n = \mathrm{Hom}(L, E) = L^* \otimes E$ and $L^* \otimes L = \mathrm{Hom}(L, L) = \mathbb{C}$, we have $T\mathbb{C}P^n \oplus \mathbb{C} = L^* \otimes (E \oplus L) = L^* \otimes \mathbb{C}^{n+1} = L^* \oplus \cdots \oplus L^*$.
From exercise 1, we then have, since $c(T\mathbb{C}P^n \oplus \mathbb{C}) = c(T\mathbb{C}P^n)$,

$$c(T\mathbb{C}P^n) = (1 + c_1(L^*))^{n+1}.$$

In particular $c_1(T\mathbb{C}P^n) = (n+1)c_1(L^*)$. It turns out that $H^2(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}c_1(L^*)$, and that a complex manifold like $\mathbb{C}P^n$ is spin if and only if $c_1(T\mathbb{C}P^n)$ is even, which is the case only if n is odd.

5. $\hat{A}$-genus. It is defined by ($x = \frac{iF}{2\pi}$)

$$\hat{A} = \exp\big(\frac{1}{2} \operatorname{tr} \ln \frac{x/2}{\sinh x/2}\big).$$

The right hand side is an even function of x, therefore has a development $a_0 + a_2 x^2 + a_4 x^4 + \cdots$, and we obtain the $\hat{A}$-genus by substituting $x = \frac{iF}{2\pi}$. Since F is a 2-form, we obtain results only in degrees multiple of 4.
If we have a direct sum $V = V_1 \oplus V_2$ with a direct sum connection $\nabla = \nabla_1 \oplus \nabla_2$ then denote $x_j = \frac{iF^{\nabla_j}}{2\pi}$, we then have

$$x = \begin{pmatrix} x_1 & \\ & x_2 \end{pmatrix}.$$

Therefore

$$\operatorname{tr} \ln \frac{x/2}{\sinh x/2} = \sum_1^2 \operatorname{tr} \ln \frac{x_j/2}{\sinh x_j/2}$$

from which follows, taking exponentials, that $\hat{A}(\nabla_1 \oplus \nabla_2) = \hat{A}(\nabla_1) \wedge \hat{A}(\nabla_2)$.
From $\ln \frac{x/2}{\sinh x/2} = -\frac{x^2}{24} + O(x^4)$ it follows that the two first terms are

$$\hat{A} = 1 + \frac{1}{24 \cdot 8\pi^2} \operatorname{tr}(F \wedge F) = 1 - \frac{1}{24} p_1(\nabla).$$

Note the sign error in the exercise, $p_1(V) = -c_2(V \otimes \mathbb{C}) = -\frac{1}{8\pi^2} \operatorname{tr}(F \wedge F)$.

10/03/2020

2. I think that we had not calculated the action of the chirality operator $\Gamma = i^{\frac{n}{2}} e_1 \cdots e_n$ on $S_\pm$. Recall the construction as $S = \Lambda W$ where $W = \langle e_1 - ie_2, e_3 - ie_4, \ldots \rangle$, and $v = w + \bar{w}$ acts by $c(v) = \sqrt{2}(\epsilon(w) - \iota(\bar{w}))$. Note $w_1 = \frac{1}{\sqrt{2}}(e_1 - ie_2)$, etc. Then one has

$$\iota(\bar{w}_j)\epsilon(w_j) w_{i_1} \wedge \cdots \wedge w_{i_k} = \begin{cases} w_{i_1} \wedge \cdots \wedge w_{i_k} & j \notin \{i_1, \ldots, i_k\} \\ 0 & j \in \{i_1, \ldots, i_k\} \end{cases}$$

$$\epsilon(w_j)\iota(\bar{w}_j) w_{i_1} \wedge \cdots \wedge w_{i_k} = \begin{cases} w_{i_1} \wedge \cdots \wedge w_{i_k} & j \in \{i_1, \ldots, i_k\} \\ 0 & j \notin \{i_1, \ldots, i_k\} \end{cases}$$

Then $e_1 e_2 = \frac{1}{2}(w_1 + \bar{w}_1) i (w_1 - \bar{w}_1)$ and therefore

$$c(e_1 e_2) = i(\epsilon(w_1) - \iota(\bar{w}_1))(\epsilon(w_1) + \iota(\bar{w}_1)) = i(\epsilon(w_1)\iota(\bar{w}_1) - \iota(\bar{w}_1)\epsilon(w_1)).$$

We can now calculate $c(e_1 \cdots e_n) = c(e_1 e_2) \cdots c(e_{n-1} e_n)$ acting on $w_{i_1} \wedge \cdots \wedge w_{i_k} \in \Lambda^k W$: each $c(e_{2j-1} e_{2j})$ acts by $\frac{i}{2}$ if $j \in \{i_1, \ldots, i_k\}$, and $-\frac{i}{2}$ if $j \notin \{i_1, \ldots, i_k\}$; therefore the action of Γ is by

$$i^{\frac{n}{2}} i^k (-i)^{\frac{n}{2} - k} = i^n (-1)^{\frac{n}{2} - k} = (-1)^k.$$

Therefore Γ acts by 1 on $S_+ = \Lambda^{even} W$ and -1 on $S_- = \Lambda^{odd} W$.

3. Recall that if V is odd dimensional, the spinors $S = S(V)$ are obtained as one of the two representations $S_\pm(V \oplus \mathbb{R})$, which are equivalent as $Spin(V)$-representations.
If $\dim V = 3$, the spin representation is 2-dimensional, so we get a morphism $Spin_3 \longrightarrow SU_2$. The kernel of ρ would act trivially on both $S_\pm(V \oplus \mathbb{R})$ which is impossible (recall $C\ell(V \oplus \mathbb{R}) \otimes \mathbb{C} \simeq \operatorname{End}(S_+ \oplus S_-)$), therefore ρ is injective.

This implies that the differential $\rho_* : \mathfrak{spin}_3 = \mathfrak{so}_3 \longrightarrow \mathfrak{su}_2$ is injective, and therefore an isomorphism: so ρ is a local diffeomorphism. Since $Spin_3$ is compact, it follows that the image of ρ is both open and closed, and therefore is the whole SU_2 since SU_2 is connected. Therefore $\rho : Spin_3 \longrightarrow SU_2$ is an isomorphism.

4. Triality. The element $-1 \in Spin_8$ acts by -1 on $\Delta_\pm$ but by 1 on the standard representation $\mathbb{R}^8$ of SO_8, since -1 is in the kernel of the morphism $\mathrm{Ad} : Spin_8 \longrightarrow SO_8$.

We know that the chirality element $Spin_8 \ni \Gamma = e_1 \cdots e_8$ acts by ± 1 on $\Delta_\pm$. On the other hand, one calculates easily that $\mathrm{Ad}(e_1e_2)(x_1e_1 + x_2e_2) = e_1e_2(x_1e_1 + x_2e_2)e_2e_1 = -(x_1e_2 + x_2e_2)$; decomposing $\mathbb{R}^8 = \langle e_1, e_2\rangle \oplus \langle e_3, e_4\rangle \oplus \cdots$, it follows that $\mathrm{Ad}(\Gamma) = -1$.

It follows that none of the central elements $1, -1, \Gamma, -\Gamma$ has the same action on the three representations $\mathbb{R}^8$, Δ_+ and Δ_-. Therefore these representations are unequivalent: indeed, if two representations ρ_1, ρ_2 are equivalent, then there exists and isomorphism ϕ such that $\rho_2(g) = \phi \circ \rho_1(g) \circ \phi^{-1}$ for all g, but our central elements go to ± 1 so the value should be the same.

5. Bochner formula. We consider the Clifford module $E = S^* \otimes S \simeq \Lambda T^*M$ and its Dirac operator $D = d + d^*$. One has the Lichnerowicz formula $D^2 = \nabla^*\nabla + \mathscr{R}$, with $\mathscr{R}s = \sum_{i<j} e_ie_jR_{e_i,e_j}s$. In the case of a 1-form α, we can make this very explicit: we write more conveniently $\mathscr{R}s = \frac{1}{2}\sum_{i,j} e_ie_jR_{e_i,e_j}s$ and identify 1-forms with vectors:

$$\begin{aligned}
\mathscr{R}\alpha &= \frac{1}{2}\sum_{i,j} e_ie_jR_{e_i,e_j}\alpha \\
&= \frac{1}{2}\sum_{i,j}(\epsilon(e_i) - \iota(e_i))(\epsilon(e_j) - \iota(e_j))R_{e_i,e_j}\alpha
\end{aligned}$$

by the algebraic Bianchi identity:

$$\begin{aligned}
&= -\frac{1}{2}\sum_{i,j} \epsilon(e_i)\iota(e_j)R_{e_i,e_j}\alpha + \iota(e_i)\epsilon(e_j)R_{e_i,e_j}\alpha \\
&= -\frac{1}{2}\sum_{i,j}\langle R_{e_i,e_j}\alpha, e_j\rangle e^i - \langle R_{e_i,e_j}\alpha, e_i\rangle e^j \\
&= -\sum_{i,j}\langle R_{e_i,e_j}\alpha, e_j\rangle e^i \\
&= \sum_{i,j}\langle R_{e_i,e_j}e_j, \alpha\rangle e^i \\
&= \sum_{i}\langle \mathrm{Ric}\, e_i, \alpha\rangle e^i \\
&= \mathrm{Ric}\,\alpha.
\end{aligned}$$

We therefore obtain the famous Bochner formula for all $\alpha \in \Omega^1M$: since $D^2\alpha = (dd^* + d^*d)\alpha = \Delta\alpha$,

$$\Delta\alpha = \nabla^*\nabla\alpha + \mathrm{Ric}\,\alpha.$$

If M is compact, integrating against α, we obtain

$$(\Delta\alpha, \alpha) = \|\nabla\alpha\|^2 + (\mathrm{Ric}\,\alpha, \alpha).$$

If α is harmonic ($\Delta\alpha = 0$) and $\mathrm{Ric} \geq 0$, then $\nabla\alpha = 0$ and $\mathrm{Ric}\,\alpha = 0$.

- If $\mathrm{Ric} > 0$ at some point, if follows that α vanishes at this point, and therefore everywhere since $\nabla\alpha = 0$ (indeed this in particular implies that $d|\alpha|^2 = 2\langle\alpha, \nabla\alpha\rangle = 0$ so $|\alpha|$ is constant). In particular $H^1(M) = 0$ so a geometric hypothesis (Ric positive) implies a topological conclusion ($b_1(M) = 0$).

- If $\mathrm{Ric} = 0$, then $\nabla\alpha = 0$, which implies that α is determined by its value at one point (suppose we have α' with $\nabla\alpha' = 0$ and α' and α have the same value at one point, then $\nabla(\alpha - \alpha') = 0$ and $\alpha - \alpha'$ is zero at some point, and therefore zero everywhere). So the space of such α's is at most n, so $b_1(M) \leq n$. One can show that equality is achieved only for a flat torus.