WILSON LOOPS & CONFINEMENT

Some gauge theories are confining: colored states cannot be observed, but only neutral bound states can be separated at large distance.

This is the case in QCD: strictly speaking, quarks and ~~glueballs~~ gluons do not exist as asymptotic states (that can be separated at large distances)

perturbative — non-perturbative

$g_{YM} \ll 1$ — Hadrons, glueballs

Quarks + gluons $\simeq \Lambda_{QCD}$ L

E

$$\Lambda_{QCD} \underset{\text{1-loop}}{\simeq} \mu \, e^{-\frac{1}{b_0 g^2(\mu)}}$$

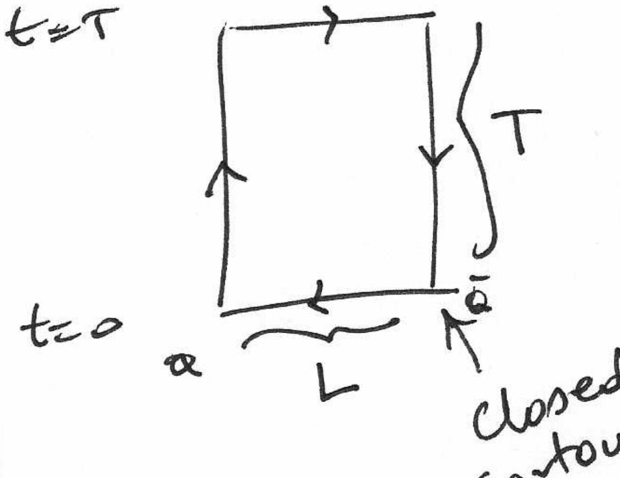
non-perturbative scale

can we do a computation to decide whether a theory confines?

Yes: we have to compute correlators of extended objects

- create a $Q\bar{Q}$ pair at time $t=0$ and Annihilate it at time T, at fixed separation L

coupling of A_μ to a point-like Q



$$W(C) = \mathrm{Tr}\, P\, e^{i\oint_C A_\mu dx^\mu}$$

(in the Euclidean space $\to e^{-\int A_\mu dx^\mu}$)

$$\langle W(C)\rangle = \int \mathcal{D}[A_\mu]\, \mathrm{Tr}\, P\, e^{-\oint A_\mu dx^\mu}\, e^{-S[A_\mu]}$$

$$\equiv e^{-\mathcal{F}[C]} \qquad \mathcal{F} = E(L) \times T$$

$\mathcal{F}[C] \equiv$ Free energy for the Quark following the contour C

<u>confinement</u> : $E(L) \simeq \sigma L$

for large L

$$\Rightarrow \langle W(C) \rangle \sim e^{-\sigma \, \mathrm{Area}(C)}$$

for large contour

this is an "operational" definition of the string tension :

$$\cancel{\frac{1}{A(C)}} \lim_{L \to \infty} \left[-\frac{1}{A(C)} \log W(C) \right] = \sigma$$

(if $E(L)$ grows slower than L, this is zero).

- Does $N=4$ SYM confine?

of course <u>not</u>: it is a CFT, so, no scale Λ_{QCD}. Moreover, the $Q\bar{Q}$ potential for external quarks is fixed by conformal invariance to be $V(L) \sim \frac{1}{L}$,

i.e. the Coulomb potential.

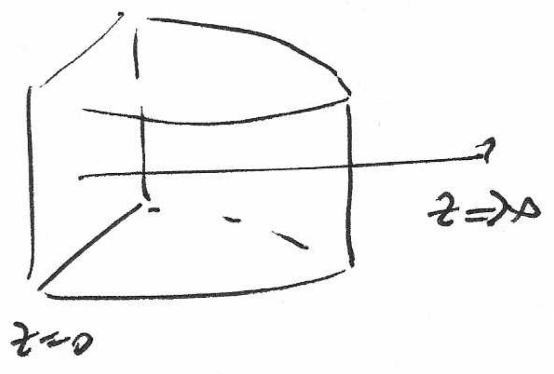
- Can we compute this in AdS/CFT? we need to find an excitation on the string theory side which corresponds to the $W(C)$ operator

$\Rightarrow$ it <u>cannot</u> be a <u>local</u> field in AdS;

it can be a <u>string</u> propagating in the bulk.

- take a contour C on the boundary of AdS

- Pin on it a string-worldsheet, extending in the bulk, and ending on the contour C



the action for this string is the classical Nambu-Goto action for a string in AdS:

$$S_{NG} = \tau \int d\sigma^2 \sqrt{-\gamma_{ind}}$$

$$\tau \equiv \frac{1}{2\pi\alpha'}$$

- in the semiclassical limit ($N \to \infty$):

$$W(C) \sim e^{-S_{NG}|_{min}}$$

Where $S_{NG}|_{min}$ is the NG action evaluated on a <u>surface of minimal area</u> extending into the bulk.

Let us do the calculation in $N=4$ SYM, i.e. in AdS_5 (and let us suppose that the string is constant on the S_5 coordinates).

$$ds^2 = u^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2}$$

(we set $l=1$)

the string worldsheet will be parametrized by two coordinates (σ, τ) and it will be embedded on a surface:

$$X^m(\sigma,\tau) \qquad X^m = \{x^\mu, u\}$$

$$S_{NG} = \int_C d\sigma d\tau \sqrt{\det_{(\alpha,\beta)} g_{mn} \partial_\alpha X^m \partial_\beta X^n}$$

consider a configuration which is T-independent, and in which (say) the X_1 changes as a function of u only

→ embedding can be taken as:

$$\begin{cases} t = \tau, & y = z = 0 \\ x = \sigma \\ U = U(\sigma) & \text{(some unknown function)} \end{cases}$$

$$\Rightarrow \quad S \sim \tau \int_{-L/2}^{L/2} dx \int_0^T dt \sqrt{(\partial_x U)^2 + u^4}$$

$$\left[\quad \gamma_{\alpha\beta} = \begin{pmatrix} u^2 & 0 \\ 0 & u^2 + \frac{(\partial_x u)^2}{u^2} \end{pmatrix} \right]$$

$$= \tau T \int_{-L/2}^{L/2} dx \sqrt{(\partial_x U)^2 + u^4}$$

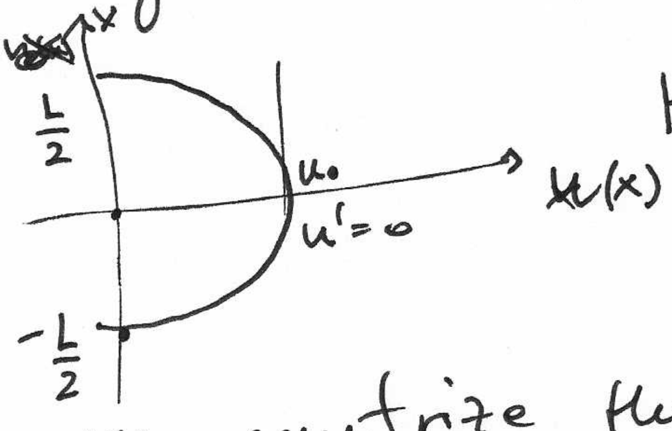
it is a 1-dimensional action for a particle $u(x)$.

the "Hamiltonian" is conserved:

$$H = \frac{\partial L}{\partial(\partial_x u)}\partial_x u - L = \frac{u^4}{\sqrt{(\partial_x u)^2 + u^4}}$$

= const

$\Rightarrow$ we can compute it anywhere, e.g. at the point where $u' = 0$ (the string turns around)



$$H(x=0) = u_0^2$$

$\rightarrow$ we can parametrize the solution by u_0

$\Rightarrow$ rewrite as: $\boxed{u'(x) = u^2\sqrt{\frac{u^4}{u_0^4} - 1}}$

this can be solved: $\int dx = \int \frac{du}{u^2}\frac{1}{\sqrt{\frac{u^4}{u_0^4} - 1}}$

so, x varies between $L/2$ and 0. (on $\frac{1}{2}$ string)

At $x = L/2 \Rightarrow u = \infty$

$$\Rightarrow \frac{L}{2} = \frac{1}{u_0} \underbrace{\int_1^\infty \frac{dy}{y^2\sqrt{y^4-1}}}_{\text{some (finite?) integral}}$$

(where $y = u/u_0$)

$$\rightarrow L \doteq \frac{1}{u_0} \times \text{constant}$$

We can now evaluate the action:

$$S = \tau T \int_{-L/2}^{L/2} \sqrt{u'^2 + u^4}\, dx$$

$$= 2T\tau \int_0^{L/2} \frac{u^4}{u_0^2}\, du \left(\frac{dx}{du}\right)$$

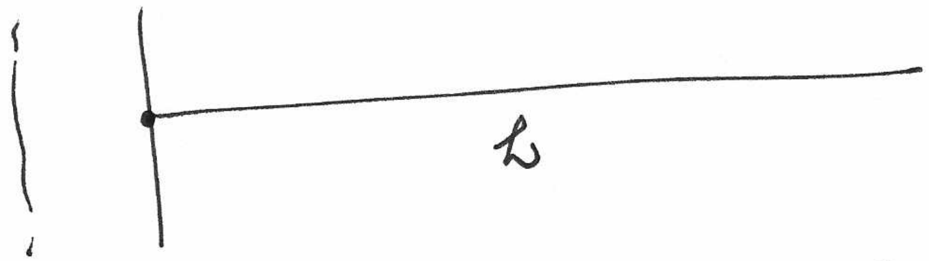
$$= 2T\tau \int_{u_0}^\infty du\, \frac{u^4}{u_0^2} \frac{1}{u^2\sqrt{\frac{u^4}{u_0^4}-1}}$$

$$= \frac{2T\tau}{\cancel{u_0}} u_0 \int_1^\infty dy\, \frac{y^2 dy}{\sqrt{y^4-1}}$$

this integral is <u>finite</u> at u_0 (corresponding to $y=1$) but infinite as $y \longrightarrow \infty$.

$$\int^{L} dy \cdot y^0 \sim L$$

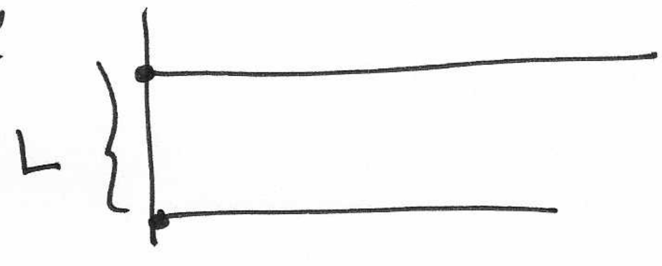
this is because the boundary is a infinite distance away, and a string stretching to the boundary has ∞ length;



this corresponds to the fact that we put quarks of infinite mass

$$M_Q \sim \tau L \xrightarrow[L\to\infty]{} \infty$$

$\rightarrow$ We need to subtract the quark masses = two infinitely long strings not connecting



the action for such strings is just

$$2T\tau \int_0^\infty dy = 2T\tau \int_{u_0}^\infty du + \text{finite}$$

$$\to \quad S^{ren} = 2T \underbrace{u(0)}_{\div L/2^{-1}} \tau \underbrace{\int_1^\infty \left(\frac{y^2}{\sqrt{y^4 - 1}} - 1 \right) dy}_{\text{some finite integral}}$$

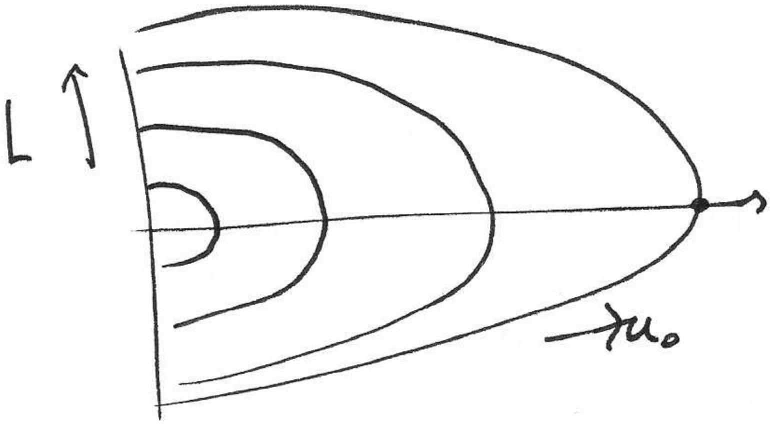
$$\to \quad S \sim \frac{T}{L} \tau \ell^2$$

$\uparrow$ radius of ℓ_{AdS}

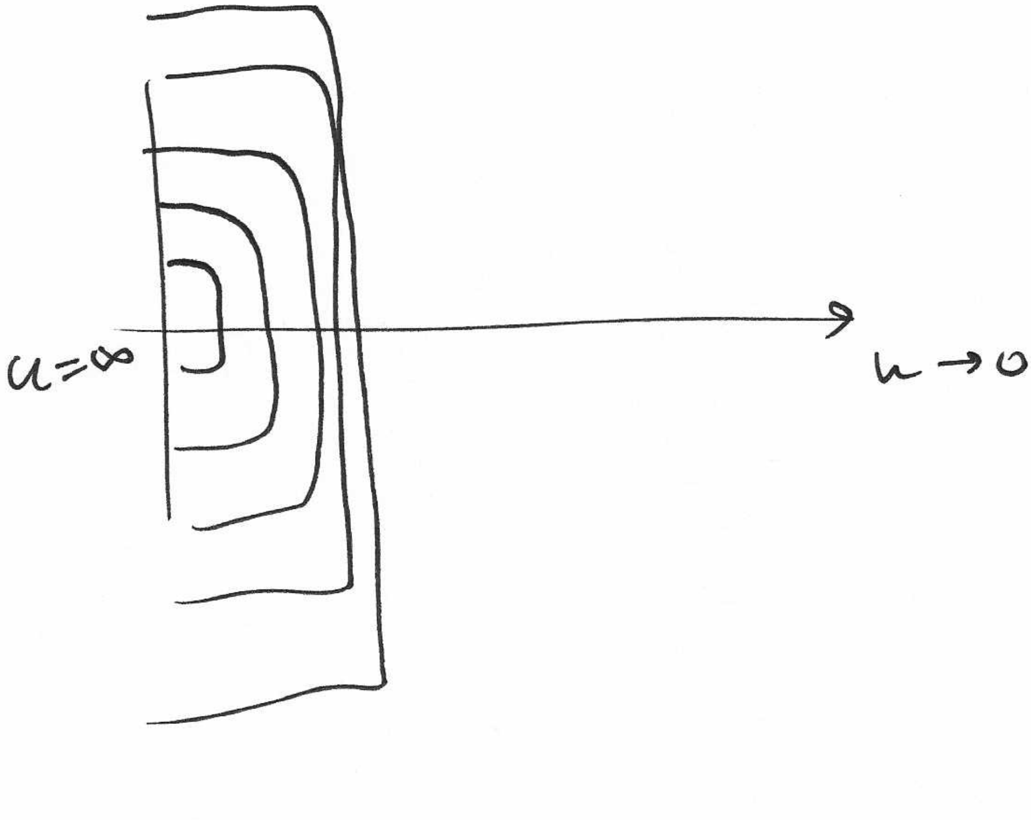
$$\Rightarrow \quad W(L) \sim e^{-\text{const } T/L}$$

and $\quad V(L) \sim \frac{T\ell^2}{L}$ Coulomb potential

Also, as $L \to \infty$, $u_0 \to 0$ (towards the center of AdS)



In order to get an <u>area</u> <u>law</u>
($S \sim T\underline{\underline{L}}$) we need to
prevent the string from falling down
into the center:



A simple confining geometry

Consider a Black hole in AdS_6 (in Poincaré coordinates, i.e. a planar Black hole):

$$ds^2 = \frac{\ell^2}{z^2}\left[\frac{dz^2}{f(z)} - f(z)\,dt^2 + \sum_{i=1}^{3} dx_i^2 + dy^2\right]$$

$$f(z) = 1 - \frac{z^5}{z_0^5} \qquad 0 < z < z_0$$

Now do a double Wick rotation:

$$t \longrightarrow i\tau \,, \quad y \longrightarrow it$$

this becomes:

$$ds^2 = \frac{\ell^2}{t^2}\left[\frac{dz^2}{f(z)} + f(z)\,\underbrace{d\tau^2}_{\substack{\text{Euclidean time} \\ \Rightarrow \text{must be} \\ \text{on a circle} \sim \beta}} + \underbrace{d\vec{x}^2 - dt^2}_{\text{4-dim Minkowski}}\right]$$

("AdS Soliton")

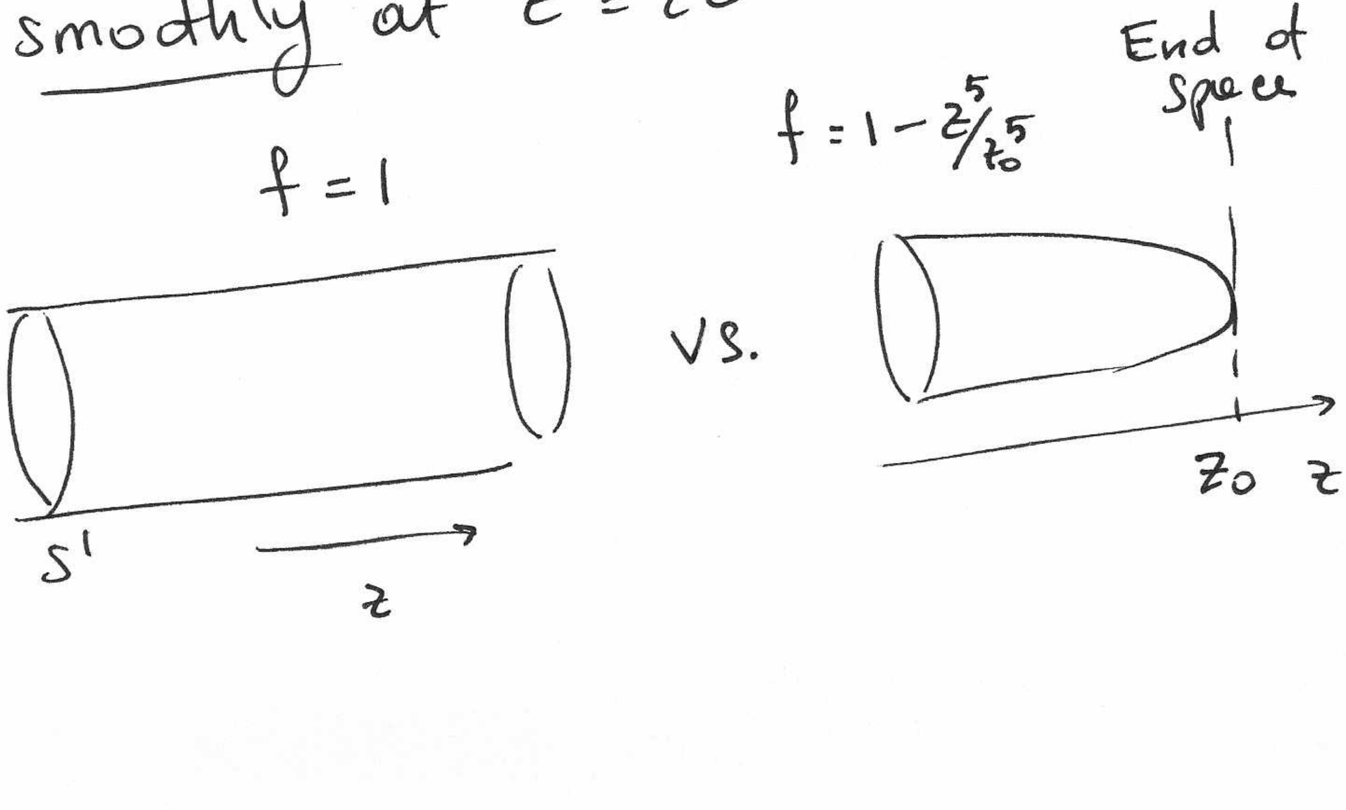
the asymptotic metric close to the boundary ($f \to 1$) is

$$ds^2 \longrightarrow \frac{\ell^2}{z^2}\Big[dz^2 + \underbrace{\eta_{\mu\nu}dx^\mu dx^\nu + d\tau^2}_{\mathbb{R}^{1,3} \times S^1_{\beta_0}}\Big]$$

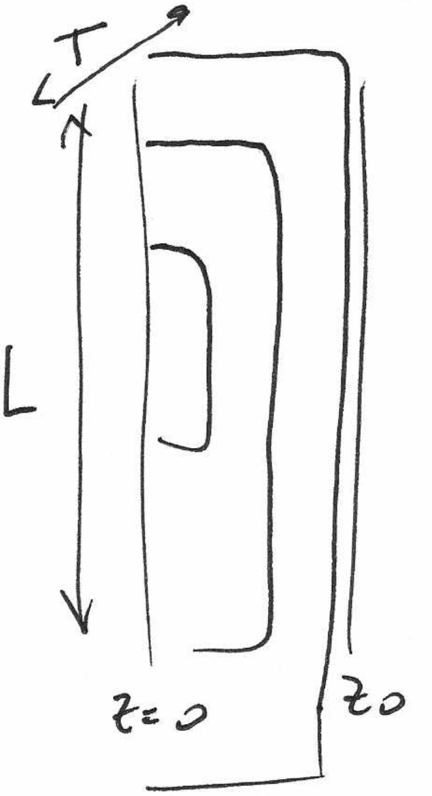
size $= \frac{4\pi}{5} z_0 \equiv \beta_0$

$\Rightarrow$ At large distances, $L \gg \beta_0$, the theory looks 4D (we can dimensionally reduce on the S^1)

But now the ~~b~~bulk geometry ends smoothly at $z = z_0$:



Now, if we attach a fundamental string to the boundary, it *cannot* extend to arbitrarily large z :



As L increases, most of the NG action will come from the region $\sim z_0$, where the string is straight :

Fundamental string tension

$$S_{NG} \sim \tau T L \sqrt{g_{xx} g_{tt}} \sim \underbrace{T L}_{\text{Area}} \frac{\tau \ell^{2}}{z_0^{2}}$$

$$\rightarrow \boxed{\sigma_c = \frac{\tau \ell^2}{z_0^2}}$$

[However there is a problem: the QCD scale ($\Lambda_{QCD}^2 \sim \sigma_c$) is higher than the UV cutoff, $1/z_0^2$]