AdS/CFT - TD 4

1

1. Apply $(-\Box_{(z,x)} + m^2)$ to $\phi(x,z)$:

$*$ $$(-\Box + m^2)\phi = (-\Box_{x,z} + m^2)\int d^dx' \, K(z, x-x')\phi_-(x')$$

$$= \int d^dx' \underbrace{(-\Box_{x,z} + m^2) K(z, x-x')}_{=0} \phi_-'(x') = 0$$

$*$ As $z \to 0$:

$$\phi = \int d^dx' \underset{\substack{\downarrow z\to 0 \\ \delta^d(x-x')}}{K(z, x-x')} \phi_-(x') \longrightarrow \int \delta^d(x-x')\phi_-(x')$$

$$= \phi_-(x')$$

2

2. let us compute $\Box K(z,x)$:

$$\Box = \frac{1}{\sqrt{g}}\partial_a \sqrt{g}\, g^{ab}\partial_b = \frac{z^{d+1}}{\ell^{d+1}}$$

$$= \frac{z^{d+1}}{\ell^{d+1}}\left[\partial_z \frac{\ell^{d+1}}{z^{d+1}}\frac{z^2}{\ell^2}\partial_z + \eta^{\mu\nu}\partial_\mu \frac{\ell^{d+1}}{z^{d+1}}\frac{z^2}{\ell^2}\partial_\nu\right]$$

$$= \frac{z^2}{\ell^2}\left[\partial_z^2 - \frac{d-1}{z}\partial_z + \eta^{\mu\nu}\partial_\mu\partial_\nu\right]$$

now: $$\partial_z \frac{z^\Delta}{(x^2+z^2)^\Delta} = \frac{\Delta z^{\Delta-1}(x^2-z^2)}{(x^2+z^2)^{\Delta+1}}$$

$$\partial_z^2 \frac{z^\Delta}{(x^2+z^2)^\Delta} = \frac{\Delta z^{\Delta-2}}{(x^2+z^2)^{\Delta+2}}\Big((\Delta-1)x^4 + (\Delta+1)z^4 - 2(\Delta+2)x^2z^2\Big)$$

$$\eta^{\mu\nu}\partial_\mu\partial_\nu = \Box_x \frac{z^\Delta}{(x^2+z^2)^\Delta} = \frac{2\Delta z^\Delta}{(x^2+z^2)^{\Delta+2}}\Big((2(\Delta+1)-d)x^2 - dz^2\Big)$$

Add everything up:

$$\frac{\ell^2}{z^2} \Box K = \Delta(\Delta - d) \frac{z^{\Delta-2}}{(x^2+z^2)^{\Delta+2}} (x^2+z^2)^2$$

$$= \Delta(\Delta-d) \frac{z^{\Delta-2}}{(x^2+z^2)^{\Delta}}$$

therefore:

$$-\Box K + m^2 K = -\frac{1}{\ell^2}\left(\Delta(\Delta-d) \frac{z^{\Delta}}{(x^2+z^2)^{\Delta}} - \ell^2 m^2 \frac{z^{\Delta}}{(x^2+z^2)^{\Delta}} \right)$$

$$= -\frac{1}{\ell^2} \frac{z^{\Delta}}{(x^2+z^2)^{\Delta}} \Big[\underbrace{\Delta(\Delta-d) - m^2 \ell^2}_{\substack{\parallel \\ 0}} \Big]$$

by definition of Δ_+

$$\Rightarrow \left(-\Box' + m^2\right) K(x,z) = 0$$

Let us compute :

$$\frac{1}{z^{\Delta_-}}\phi(x,z) = \frac{1}{z^{\Delta_-}}\int d^dx' \frac{z^{\Delta_+}}{\left(z^2+(x-x')^2\right)^{\Delta_+}}\phi_-(x')$$

$$= \frac{1}{z^{\Delta_-+\Delta_+}}\int d^dx' \frac{\phi_-(x')}{\left(1+\frac{(x-x')^2}{z^2}\right)^{\Delta_+}}$$

change variables to $x^\mu - x'^\mu = y^\mu \cdot z$

$$= \frac{1}{z^{\Delta_-+\Delta_+}}\int d^dy \cdot z^d \frac{\phi_-(x-zy)}{\left(1+y^2\right)^{\Delta_+}}$$

now, $\Delta_+ + \Delta_- = d$

$$= \int d^dy \frac{\phi_-(x-zy)}{(1+y^2)^{\Delta_+}}$$

$$\xrightarrow[z\to 0]{} \int d^dy \frac{\phi_-(x)}{(1+y^2)^{\Delta_+}} = C^{-1}\phi_-(x)$$

with $C^{-1} = \int d^dy \frac{1}{(1+y^2)^{\Delta_+}}$

therefore $$C \frac{z^{\Delta_+}}{(x^2+z^2)^{\Delta_+}} \xrightarrow[z\to 0]{} \delta^{(d)}(x)$$

3. Use Green's identity:

$$\int_V d^{d+1}x \sqrt{g} \left[\phi(\Box - m^2)\psi - \psi(\Box - m^2)\phi \right]$$

$$= \int_{\partial V} d^d x \sqrt{\gamma} \left(\phi\, n\cdot\nabla\psi - \psi\, n\cdot\nabla\phi \right)$$

where n is the unit normal vector to the boundary ∂V.

or, in coordinates, using the expression of $\Box$:

$$\int d^d x\, dz \frac{\ell^{d-1}}{z^{d-1}} \left[\phi\left(\partial_z^2 - \frac{d-1}{z}\partial_z + \eta^{\mu\nu}\partial_\mu\partial_\nu - \frac{m^2\ell^2}{z^2} \right)\psi \right.$$
$$\left. - \phi \leftrightarrow \psi \right]$$

$$= \int_{z=\epsilon} d^d x \frac{\ell^{d-1}}{z^{d-1}} \left(\phi\partial_z\psi - \psi\partial_z\phi \right)$$

Now take $\phi(z,x) = G(z,x;z',x')$

$\psi(z,x) = K(z;x,x'')$

and substitute: on the left hand side:

$$\Rightarrow (\Box - m^2)\psi = 0$$

$$(\Box - m^2)\phi = \delta^{(d+1)}(z-z',x-x')$$

$\Rightarrow$

$$-K(z';x',x'') = \int_{z=\epsilon} d^dx\, z^{1-d} \times$$

$$\Big[G(z,x;z',x')\partial_z K(z,x,x'') - K(z,x,x'')\partial_z G(z,x;z',x')\Big]_{z=\epsilon}$$

- We know that close to $z \sim 0$, $K \sim z^{\Delta_-}\delta(x)$

$$\Rightarrow \partial_z K(z;x,x'')\Big|_\epsilon \sim \Delta_- z^{\Delta_- -1}\delta(x-x'')$$

- We also know how $G(z,x;z',x')$ behaves as $z \to 0$ and z' is finite: we have taken it to be <u>normalizable</u> as $z \to 0$

$$\Rightarrow G \simeq z^{\Delta_+} f(z';x,x')$$

$$\Rightarrow \partial_z G \sim \Delta_+ \underbrace{\epsilon^{\Delta_+ - 1} f(z'; x, x')}$$

$$\sim \Delta_+ \frac{1}{\epsilon} G(\epsilon, z'; x, x')$$

therefore we have, for $z' \neq 0$:

$$- K(z'; x', x'') =$$

$$= \int d^d x \; \epsilon^{1-d} G(\epsilon, x; z', x') \Delta_- \epsilon^{\Delta_- - 1} \delta^{(d)}(x - x'')$$

$$- \int d^d_x \, \epsilon^{1-d} \epsilon^{\Delta_-} \delta^{(d)}(x - x'') \frac{\Delta_+}{\epsilon} G(\epsilon, z'; x, x')$$

$$(\Delta_- - d \to \Delta_+)$$

$$= \frac{(\Delta_- - \Delta_+)}{\epsilon^{\Delta_+}} G(\epsilon, z'; x'', x')$$

set $x' = 0$ and use symmetry and take $\epsilon \to 0$ (all $\sim$ becomes exact)

$$\rightarrow \boxed{K(z, x) = \lim_{\epsilon \to 0} \frac{\Delta_+ - \Delta_-}{\epsilon^{\Delta_+}} G(z, \epsilon; x)}$$

2] Vector fields in AdS

2.1 Gauge fields

1) the gauge transformation for A_z is

$$A'_z(z,x^\mu) = A_z(\cancel{x},x^\mu) + \partial_z \Lambda(z,x^\mu)$$

$\Rightarrow$ choose a $\Lambda(z,x^\mu) = \Lambda_1(z,x^\mu)$
such that $\partial_z \Lambda_1 = -A_z$

$$\Rightarrow \quad A'_z(z,x^\mu) = 0$$

Any gauge transformation with
$\Lambda = \Lambda_2(x^\mu)$ (independent of z)
does not change $A'_z = 0$.

Now consider $\partial^\mu A'_\mu(x,z)$: under
a new transformation :

$$\partial^\mu A''_\mu = \partial^\mu A'_\mu + \partial^\mu \partial_\mu \Lambda_2(z,x^\mu)$$

we can choose one point z_0 and
choose $\Lambda_2(x^\mu) = \Lambda_2(z_0,x^\mu)$ such that

$$\partial^\mu A''_\mu = 0 \quad \text{at } z = z_0$$

As we will se next, if $A_z = 0 \Rightarrow \partial^\mu A_\mu = $ constant in z.

therefore making $\partial^\mu A''_\mu = 0$ at $z = z_0$
ensures that $\partial^\mu A''_\mu = 0$ everywhere.
to do this, choose Λ_2 as a solution
of

$$\Box_4 \Lambda_2 (z_0, x^\mu) = - \partial^\mu A'_\mu (z_0, x^\mu)$$

$\to$ now $A''_z = 0$ and $\partial^\mu A''_\mu = 0$.

2)
$$S' = -\frac{1}{\ell^{d-1}} \int d^dx dz \frac{\ell^{d+1}}{z^{d+1}} \frac{z^2}{\ell^2} \frac{z^2}{\ell^2} \times \frac{1}{4} \times$$

$$\Big[2\eta^{\mu\nu} (\partial_z A_\mu - \partial_\mu A_z)(\partial_z A_\nu - \partial_\nu A_z) +$$

$$+ \eta^{\mu\nu}\eta^{\rho\sigma} (\partial_\mu A_\rho - \partial_\rho A_\mu)(\partial_\nu A_\sigma - \partial_\sigma A_\nu) \Big]$$

$$= - \frac{1}{\ell^{d-1}} \int d^dx dz \frac{\ell^{d-3}}{z^{d-3}} \Big[\frac{1}{2} \partial_z A_\mu \partial_z A^\mu$$

$$+ \cancel{A_z \partial^\mu \partial_z} + \frac{1}{2} \partial^\mu A_z \partial_\mu A_z - \partial_z A_\mu \partial^\mu A_z$$

$$+ \frac{1}{2} \partial_\mu A_\rho \partial^\mu A_\rho - \frac{1}{2} (\partial_\mu A^\mu)(\partial_\rho A^\rho) \Big]$$

(up to some integration by parts).

- Varying the action w.r.t. A_μ gives:

$$\partial_z\, z^{-d+3}\, \partial_z A_\mu - \partial_z z^{-d+3} \partial_\mu A_z$$
$$+ z^{-d+3}\, \partial^\nu F_{\mu\nu} = 0$$

(where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$)

- Varying with respect to A_z gives: (notice A_z does not have terms quadratic in ∂_z):

$$z^{-d+3}\left(\partial^\mu \partial_\mu A_z + \partial_z \partial^\mu A_\mu \right) = 0$$

Now choose the gauge $A_z = 0$
$\Rightarrow$ the second equation tells us that
$\partial_z \partial^\mu A_\mu = 0 \Rightarrow \partial^\mu A_\mu =$ constant in z

As we saw in 1), we can choose
$\partial^\mu A_\mu = 0$ at one point z_0
$\Rightarrow \partial^\mu A_\mu \equiv 0$ everywhere.

Now the equation for A_μ becomes simply: $\left(\partial^\nu F_{\mu\nu} = \partial_\mu \cancel{\partial^\nu A_\nu} - \partial^\nu \partial_\nu A_\mu \right)$

$$\boxed{0 = \partial_z z^{-d+3} \partial_z A_\mu + z^{-d+3} \partial^\rho \partial_\rho A_\mu}$$

3). Near the boundary, try $A_\mu \sim z^\Delta V_\mu(x)$

$$\Rightarrow z^{-d+3} \left[\Delta(\Delta-1) z^{\Delta-2} - (d-3)\Delta z^{\Delta-1} \right] V_\mu$$

$$+ z^{-d+3} (\Box_4 V_\mu) z^\Delta = 0$$

as $z \to 0$ the $\Box_4 V_\mu$ term is subleading, so we need:

$$\Delta(\Delta-1) = (d-3)\Delta$$

$$\Rightarrow \begin{cases} \Delta_- = 0 \\ \Delta_+ = d-2 \end{cases}$$

[A massless scalar would have $\Delta_- = 0$ $\Delta_+ = d$!]

$$A_\mu \underset{z\to 0}{\sim} \underset{\substack{\uparrow \\ \text{leading}}}{V_\mu^{(-)}(x)} + z^{d-2} \underset{\substack{\uparrow \\ \text{subleading}}}{V_\mu^{(+)}(x)}$$

4) A_μ is a vector, so under a coordinate transformation it transforms as:

$$\tilde{A}_a(\tilde{x}) = \frac{\partial x^b}{\partial \tilde{x}^a} A_b(x)$$

So take a scaling: $\tilde{x}^a = \lambda x^a$

$$\Rightarrow \tilde{A}_\mu(\lambda z, \lambda x^\mu) = \lambda^{-1} A_\mu(x)$$

$$\begin{array}{ccc} \| & & \| \\ (\lambda z)^\Delta \tilde{V}_\mu(\lambda x^\mu) & = & \lambda^{-1} z^\Delta V_\mu(x^\mu) \end{array}$$

$$\rightarrow \tilde{V}_\mu(\lambda x^\mu) = \lambda^{-1-\Delta} V_\mu$$

$\Rightarrow V_\mu^{(-)}$ has weight 1,

$V_\mu^{(+)}$ has weight $d-1$

$\rightarrow$ the dimension of the operator J^μ dual to A_μ is $d-1$

$\Rightarrow J^\mu$ is a conserved current in the QFT side

5) Conservation of J^μ in the boundary theory follows from gauge-invariance: the coupling on the boundary is:

$$\int d^d x \; V_\mu^{(-)}(x) J^\mu(x)$$

Under a <u>Bulk</u> gauge transformation:

$$A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda$$

take Λ independent of z. if we expand A_μ close to the boundary, we have:

$$A_\mu \simeq V_\mu^{(-)} + \ldots \rightarrow V_\mu^{(-)} + \partial_\mu \Lambda + \ldots$$

$\Rightarrow$ $V_\mu^{(-)}(x)$ has the same gauge transformations, under which:

$$\int V_\mu^{(-)} J^\mu \longrightarrow \int (V_\mu^{(-)} + \partial_\mu \Lambda) J^\mu$$

So under a gauge transformation:

$$\overset{+}{Z}_1 [V_\mu^{(-)}] = \left\langle e^{i \int V_\mu^{(-)} J^\mu} \right\rangle_{CFT}$$

$$\longrightarrow \left\langle e^{\int V_\mu^{(-)} J^\mu - \int \Lambda \partial^\mu J_\mu} \right\rangle_{CFT}$$

here we have done an i.b.p.

But $\overset{+}{Z}_1 [V_\mu^{(-)}] = Z_{Grav} [V_\mu^{(-)}]$

$\left(\overset{+}{Z}_{1,grav} = \int_{\substack{A_\mu \to V_\mu \\ z \to 0}} \mathcal{D}[A] e^{i S[A]} \right)$

This side is <u>gauge invariant</u>, so it must not change under $V_\mu \to V_\mu + \partial_\mu \Lambda$

$$\Rightarrow \boxed{\partial^\mu J_\mu = 0}$$

2.2) Massive vector.

1) Adding the term $\frac{m^2}{2} A_a A^a$ adds a new term to the field equations, which now are:

$$\bullet\ \partial_z z^{-d+3} \partial_z A_\mu - \partial_z z^{-d+3} \partial_\mu A_z - z^{-d+3} \partial^\nu F_{\mu\nu} - \ell^2 m^2 z^{-d+1} A_\mu = 0$$

$$\bullet\ z^{-d+3} \left(\partial^\mu \partial_\mu A_z - \partial_z \partial^\mu A_\mu \right) - \ell^2 m^2 z^{-d+1} A_z = 0$$

Now consider only the transverse components: A_μ such that $\partial^\mu A_\mu = 0$, $A_z = 0$

they solve the equation:

$$\partial_z z^{-d+3} \partial_z A_\mu - \cancel{\partial_z} z^{-d+3} \partial^\nu \partial_\nu A_\mu - m^2 \ell^2 z^{-d+1} A_\mu$$

~~assume, the $\Box A_\mu$ term is subleading w.r.t. the mass term~~

Expanding and multiplying by z^{d+3}:

$$0 = \partial_z^2 A_\mu - \frac{(d-3)}{z}\partial_z A_\mu + \Box A_\mu - \frac{m^2 l^2}{z^2} A_\mu$$

2) As $z \to 0$, the $\Box A_\mu$ term is subleading w.r.t. the mass term, so we can ignore it.

Set $A_\mu \simeq z^\Delta V_\mu(x)$

$$\Rightarrow 0 = \left[\Delta(\Delta-1) z^{\Delta-2} - (d-3)\Delta z^{\Delta-2} - m^2 l^2 z^{\Delta-2}\right]$$

$$\Rightarrow \boxed{\Delta(\Delta - d + 2) = m^2 l^2}$$

(like for a scalar field, but with the substitution $d \to d-2$)

$$\Rightarrow \boxed{\Delta_\pm = \frac{d-2}{2} \pm \frac{1}{2}\sqrt{(d-2)^2 + 4m^2 l^2}}$$

As before, the <u>conformal</u> <u>dimension</u> has to be increased by 1 due to the vector index: $\Delta_0 = \frac{d}{2} + \frac{1}{2}\sqrt{(d-2)^2 + 4m^2 l^2}$