Géométrie différentielle et théorie de jauge. Howework exam 2020.

I Hairy ball theorem

Recall that if M^n is a compact oriented manifold, then the map $\alpha \mapsto \int_M \alpha$ gives an isomorphism $H^n(M) \simeq \mathbb{R}$. Also, if $f : M \to N$, then the pullback of forms gives a map $f^* : H^k(N) \to H^k(M)$ which is invariant by deformations of f.

If M^n and N^n are compact oriented manifolds and $f : M \to N$, since $H^n(M)$ and $H^n(N)$ are identified to $\mathbb{R}$ by integration, we can identify $f^* : H^n(N) \to H^n(M)$ to a number called the degree of f, and denoted $\deg f$ (actually one can prove that $\deg f \in \mathbb{Z}$).

1. Verify that $\deg f$ is defined by $\int_M f^*\alpha = (\deg f) \int_N \alpha$ for all $\alpha \in \Omega^n N$. Calculate the degree of the map $f : S^1 \to S^1$ given by $f(z) = z^n$.

2. Let ω be the volume form of S^n, and $f(x) = -x$ the antipodal map in S^n. Prove that $f^*\omega = (-1)^{n-1}\omega$. Calculate $\deg f$.

3. Let X be a vector field on S^n. Prove that if n is even, then X has to vanish at some point. (Suppose the contrary, and consider the maps $f_t : S^n \to S^n$ given by $f_t(x) = \cos(t)x + \sin(t)\frac{X(x)}{\|X(x)\|}$ for $t \in [0, \pi]$).

II Spinors on the even dimensional sphere

Let n be **even**. We see $S^n \subset \mathbb{R}^{n+1}$, and consider E the vector space of spinors of $\mathbb{R}^{n+1}$ with its standard scalar product. So $\dim E = 2^{\frac{n}{2}}$. For $x \in \mathbb{R}^{n+1}$, we denote $c(x)$ the Clifford action of x on E, so $c(x) \in \mathfrak{su}(E)$.

If $x \in S^n$ then we have $c(x)^2 = -q(x) = -1$, so we have a decomposition $E = E_x^+ \oplus E_x^-$, where $c(x)$ acts on $E_x^\pm$ by $\pm i$. We have projections $\pi_x^\pm$ from E to $E_x^\pm$ defined by

$$\pi_x^+(a) = \tfrac{1}{2}(a - ic(x)a), \quad \pi_x^-(a) = \tfrac{1}{2}(a + ic(x)a).$$

1. Prove that if $x \in S^n$ and $y \in \mathbb{R}^{n+1}$ satisfy $y \perp x$ then $\pi_x^+ c(y) = c(y)\pi_x^-$ and $\pi_x^- c(y) = c(y)\pi_x^+$. Deduce that $c(y)(E_x^+) \subset E_x^-$ and $c(y)E_x^- \subset E_x^+$.

2. Why have we $\dim E_x^\pm = 2^{\frac{n}{2}-1}$? explain why the collection of the $(E_x^\pm)_{x\in S^n}$ gives two vector bundles over S^n. We still denote these vector bundles by $E^\pm$. Therefore the trivial bundle $\mathscr{E} = S^n \times E$ over S^n is the sum $E^+ \oplus E^-$.

3. On the trivial bundle $\mathscr{E}$ we have the trivial connection d. Therefore we define a connection on E^+ and E^- by projection of d, that is by taking $\nabla^{E^\pm} s(x) = \pi_x^\pm(ds)$, where the section s of E^+ is seen as a section of $\mathscr{E}$. Why is this a connection ? a unitary connection ?

Given a fixed $e \in E$, we define sections $\hat{e}^+$ of E^+ and $\hat{e}^-$ of E^- by $\hat{e}^\pm(x) = \pi_x^\pm(e)$. Prove that for all $X \in T_x S^n$ one has

$$\nabla_X^{E^+} \hat{e}^+ = -\tfrac{i}{2}c(X)\hat{e}^-, \quad \nabla_X^{E^-} \hat{e}^- = \tfrac{i}{2}c(X)\hat{e}^+.$$

4. Deduce that the curvatures of E^+ and E^- are given by the formulas, for $X, Y \in T_x S^n$,

$$F_{X,Y}^{E^+}\hat{e}^+ = -\tfrac{1}{4}[c(X), c(Y)]\hat{e}^+, \quad F_{X,Y}^{E^-}\hat{e}^- = -\tfrac{1}{4}[c(X), c(Y)]\hat{e}^-. \tag{$*$}$$

5. Why are E^+ and E^- the spinor bundles S^+ and S^- of S^n ? can we give another proof of the formulas ($*$) from the curvature of the Levi-Civita connection on the sphere $F_{X,Y}^{S^n} = -X \wedge Y$?

6. Calculate the trace of $(\frac{iF^{E^\pm}}{2\pi})^{\frac{n}{2}}$. Given that the volume of S^n is $2^{n+1}\pi^{\frac{n}{2}}\frac{(\frac{n}{2})!}{n!}$, deduce that

$$\int_{S^n} ch(E^+) = (-1)^{\frac{n}{2}}, \quad \int_{S^n} ch(E^-) = (-1)^{\frac{n}{2}+1},$$

where in the integral one takes only the degree n part of $ch(E^\pm)$. Are the bundles $E^\pm$ trivial ?

III BPS monopole

You can admit the result of a calculation to do the questions after the calculation.

Let $E \longrightarrow M$ be a $SU(2)$-bundle on a 3-dimensional oriented Riemannian manifold M. The magnetic monopole equations are equations on a pair (A, ϕ) of a $SU(2)$-connection A on E and a 'Higgs field' $\phi \in \Gamma(\mathfrak{su}(E))$, that is ϕ is a trace-free antiselfadjoint endomorphism of E. They are written as

$$\nabla^A \phi = *F_A, \tag{†}$$

where $* : \Omega^2 \longrightarrow \Omega^1$ is the Hodge operator of M, so both sides of (†) are elements of $\Omega^1(\mathfrak{su}(E))$.
0. Prove that a solution of (†) must satisfy $(\nabla^A)^* \nabla^A \phi = 0$. Deduce that if M is compact, then $F_A = 0$ and $\nabla^A \phi = 0$.
Therefore on a compact manifold, all solutions are trivial. The aim of the exercise is to construct the first nontrivial solution on $\mathbb{R}^3$, which is explicit. We consider $\mathbb{R}^3$ with its standard spinors $S \simeq \mathbb{C}^2$, so $c(\frac{\partial}{\partial x^i}) = \sigma_i \in \mathfrak{su}_2$, with $\sigma_1 = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\sigma_3 = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$.
We denote $r = |x|$. We are looking for a solution of (†) on the spinor bundle $\mathbb{R}^3 \times S$ of the following form:

$$\begin{aligned} \phi(x) &= f(r) c(x), \\ A(x) &= g(r)[c(x), c(dx)], \end{aligned} \tag{‡}$$

where the second equality means that the connection form is given by $A(x)_X = g(r)[c(x), c(X)]$ for $X \in T_x\mathbb{R}^3 = \mathbb{R}^3$.
1. Prove the formulas

$$\nabla^A \phi = (f' - 4rfg) dr \otimes c(x) + (f + 4r^2 fg) c(dx), \quad F^A = \Big(\frac{2g'}{r} - 8g^2\Big)\omega + (2g + 4r^2 g^2)\psi,$$

where $\omega_{X,Y} = c(x)c((X \wedge Y)(x))$, $\psi_{X,Y} = [c(X), c(Y)]$ and $(X \wedge Y)(x) = \langle X, x\rangle Y - \langle Y, x\rangle X$.
2. Prove that $*\omega = r^2 c(dx) - r dr \otimes c(x)$ and $*\psi = 2c(dx)$. Deduce the equations satisfied by f and g. Taking $F = 2rf - \frac{1}{r}$ and $G = 4rg + \frac{1}{r}$, reduce the system to

$$F' = G^2, \quad G' = GF. \tag{§}$$

Deduce in particular the Prasad-Sommerfield monopole on $\mathbb{R}^3$ given by

$$f(r) = \frac{1}{2r}\left(\frac{1}{r} - \frac{1}{\tanh r}\right), \quad g(r) = -\frac{1}{4r}\left(\frac{1}{r} - \frac{1}{\sinh r}\right).$$

3. Prove that when $r \longrightarrow \infty$, the eigenvalues of ϕ on S converge to $\pm\frac{i}{2}$. So near infinity we can decompose $S = S^+ \oplus S^-$ with $S^\pm$ being the eigenspace for the eigenvalue going to $\pm\frac{i}{2}$ at infinity. What is the limit of A on larger and larger spheres going to infinity ? Prove that on these spheres $c_1(S^+) = 1$ (the charge of the monopole).
4. Prove that the other solutions of (§) giving monopoles on $\mathbb{R}^3$ can be deduced from the Prasad-Sommerfield solution by the scaling $(\lambda A(\lambda x), \lambda \phi(\lambda x))$.
5. The group $SU(2)$ acts on $\mathbb{R}^3 \times S$ by $(x, s) \mapsto (\rho(g)x, gs)$, where $\rho : SU(2) \longrightarrow SO(3)$ is the standard morphism from $SU(2) = Spin(3)$ to $SO(3)$. Prove that pairs (A, ϕ) with spherical symmetry (that is, invariant under the action of $SU(2)$), and satisfying $A_{\frac{\partial}{\partial r}} = 0$ (gauge obtained by parallel transport along rays from the origin) must be of the form (‡). Therefore we have produced all solutions with spherical symmetry.