Géométrie différentielle et théorie de jauge, 04/02/2020

De Rham cohomology, Hodge theory

1. Show that if $i : Y^k \hookrightarrow M$ is an oriented compact submanifold of M, then $\alpha \mapsto \int_Y i^*\alpha$ defines a linear form on $H^k(M)$.

2. A theorem of De Rham says that if $\alpha \in \Omega^k M$ is closed and satisfies $\int_Y \alpha = 0$ for all k-dimensional compact oriented submanifolds $Y \subset M$, then $[\alpha] = 0$. Using this theorem, give a proof that $H^k(S^n) = 0$ for $0 < k < n$. (Use the fact that $H^k(\mathbb{R}^n) = 0$ for $k > 0$).

3. Show that $H^1(\mathbb{R}^2 \setminus \mathbb{Z})$ is infinite dimensional: look at what happens when we integrate along small circles along each point of $\mathbb{Z}$.

4. Recall the projection $p : S^n \longrightarrow \mathbb{R}P^n$ is a local diffeomorphism (the differential is everywhere an isomorphism). Note $\tau(x) = -x$ the antipodal map of S^n, so $\mathbb{R}P^n = S^n/\tau$.

Given $\alpha \in \Omega^k(\mathbb{R}P^n)$, prove that $\tilde{\alpha} = p^*\alpha$ satisfies $\tau^*\tilde{\alpha} = \tilde{\alpha}$. Conversely, prove that if a form $\tilde{\alpha} \in \Omega^k(S^n)$ satisfies $\tau^*\tilde{\alpha} = \tilde{\alpha}$ then there exists $\alpha \in \Omega^k(\mathbb{R}P^n)$ such that $\tilde{\alpha} = p^*\alpha$.

Prove that if $\alpha \in \Omega^k(\mathbb{R}P^n)$ satisfies $p^*\alpha = d\tilde{\beta}$ then there exists $\beta \in \Omega^{k-1}(\mathbb{R}P^n)$ such that $\alpha = d\beta$ (consider the average $\frac{1}{2}(\tilde{\beta} + \tau^*\tilde{\beta})$ of $\tilde{\beta}$ under τ).

Deduce that $H^k(\mathbb{R}P^n) = \{c \in H^k(S^n), [\tau^*]c = c\}$ and calculate $H^k(\mathbb{R}P^n)$.

5. Let X be a vector field on M, generating a flow (ϕ_t) of diffeomorphisms. Let α be a closed form on M. Using $\phi_{t+t'} = \phi_t \circ \phi_{t'}$, prove that

$$\frac{d}{dt}\phi_t^*\alpha = \phi_t^*\mathscr{L}_X\alpha.$$

Deduce from the Cartan formula that the cohomology class $[\phi_t^*\alpha]$ is constant.

Prove that if a *connected* Lie group G acts on M, then the action of G on $H(M)$ is trivial.

6. Prove the formulas given in the lecture :

$$*^2 = (-1)^{p(n-p)} \text{ on } \Omega^p, \qquad *(\alpha_i e^i) = (-1)^{j-1}\sqrt{\det(g_{ij})}g^{ij}\alpha_i e^1 \wedge \cdots \wedge \widehat{e^j} \wedge \cdots \wedge e^n.$$

7. (i) that on the torus $\mathbb{T}^n$ or on $\mathbb{R}^n$, for a form $\alpha = \alpha_I dx^I$, one has

$$d^*\alpha = -\sum_I \sum_{i\in I} \epsilon(\{i\}, I \setminus \{i\}) \frac{\partial \alpha_I}{\partial x^i} dx^{I\setminus\{i\}}, \qquad \Delta\alpha = (\Delta\alpha_I)dx^I.$$

(To simplify do the calculation only for a form of the type $f dx^1 \wedge \cdots \wedge dx^p$). In particular, for a 1-form, one has $d^*\alpha = -\frac{\partial \alpha_i}{\partial x^i} = -\mathrm{div}(\alpha)$.

(ii) Deduce that harmonic forms on the torus have constant coefficients, and calculate the cohomology of the torus.

8. Let M^n be a compact oriented Riemannian manifold. Show that if $\alpha \in \Omega^k M$ and $\beta \in \Omega^{k-1}M$, then $\|\alpha + d\beta\|^2 = \|\alpha\|^2 + 2(d^*\alpha, \beta) + \|d\beta\|^2$.

Fix a cohomology class $c \in H^k(M)$. Deduce that the minimum of $\{\|\alpha\|^2, d\alpha = 0, [\alpha] = c\}$ is attained exactly once, for the harmonic representative of c.

9. On a compact oriented Riemannian manifold M^n, the Laplacian Δ on forms can be diagonalized: there are eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$ with $\lambda_k \longrightarrow \infty$ with corresponding unit eigenforms α_k which is a Hilbert basis of L^2-forms: any form α can be decomposed as $\alpha = \sum_1^\infty \langle \alpha, \alpha_k \rangle \alpha_k$.

Consider p-forms on a Riemannian product $M \times N$, the bundle of p-forms decomposes as $\Lambda^p T^*(M \times N) = \oplus_0^p \Lambda^i T^*M \otimes \Lambda^{p-i}T^*N$. Show that $\Delta^{M\times N} = \Delta^M \otimes 1_N + 1_M \otimes \Delta^N$, and from the above spectral decomposition deduce the Künneth formula

$$H^p(M \times N) = \oplus_0^p H^i(M) \otimes H^{p-i}(N).$$