Introduction to AdS/CFT

Final Exam

Due Tuesday April 7 at 5pm

A Simple holographic model for QCD

1 A simple color-confining geometry

Consider the following five-dimensional geometry,

$$ds^2 = \frac{\ell^2}{z^2}\left[dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right], \quad 0 < z < z_0, \tag{1}$$

i.e. a slice of AdS_5 in Poincaré coordinates, cut-off at $z = z_0$. This can be taken as a rough holographic model for color confinement in pure Yang-Mills theory in 4d.

The metric is a solution of pure 5d gravity with a cosmological constant, with action [1]

$$S_{bulk} = -\frac{1}{2\kappa}\int d^5x\sqrt{-g}\,(R - 2\Lambda), \quad \Lambda \equiv -\frac{6}{\ell^2}. \tag{2}$$

(where the overall sign is adapted to the Euclidean signature).

1.1 Wilson Loop

1. Discuss the qualitative features of the geometry, and in particular discuss why one expects to find linear confinement for the color charge.
2. Perform the holographic Wilson loop calculation [2] and compute the confining string tension (assuming the fundamental string tension is $(2\pi\alpha')^{-1}$).

1.2 Deconfinement transition

Now consider the model of the previous section at finite temperature.

1. Discuss which are the competing classical geometries at a given temperature T, and what their features are, especially concerning confinement.
2. Compute the free energy difference between the solutions at the same temperature [3], as a function of T, and discuss the existence of a deconfinement phase transition. What is the value of the critical temperature ?

1. For the sake of simplicity we will ignore the Gibbons-Hawking boundary term, whose correct treatment in this case is quite fishy anyway. This will only change the results by a numerical constant.
2. Warning : the minimal worldsheet may not be differentiable for all values of the boundary separation, but only piece-wise differentiable.
3. Remember that you have to cut-off the boundary at $z = \epsilon$ and add to the action a covariant boundary counterterm to cancel the divergence. Important : the expression of the counterterm action must be fixed, before you evaluate it on-shell on the various solutions.

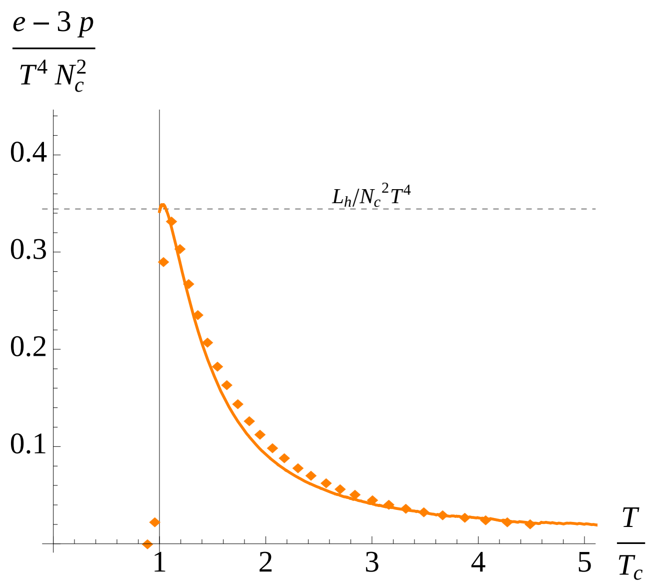

FIGURE 1 – The interaction measure as a function of temperature in $SU(3)$ Yang-Mills theory on the Lattice (from hep-lat/9602007). The squares are the Lattice data (disregard the the solid line, which is a model fit).

3. In finite-T Yang-Mills theory, it is useful to define :

$$\mathcal{A}(T) = \frac{\rho - 3p}{T^4},$$

where ρ is energy density and p is the pressure. This quantity, called (perhaps somewhat inappropriately) *interaction measure*, measures the deviation of the finite-temperature state from a conformal plasma (for which $\mathcal{A} = 0$). Give the expression of $\mathcal{A}$ in the model considered here, and compare it with the result found in lattice QCD, shown in figure 1 Is the high-temperature phase of the holographic model a good model for Yang-Mills theory at finite temperature ? What is missing ?

2 Including other operators

The model of the previous section contains the metric as its only degree of freedom. Therefore, in the dual field theory, the only operator is the stress-energy tensor.

On the other hand, if we consider 4d QCD with gauge group $SU(N_c)$ and N_f quarks in the fundamental representation of the gauge group, there are several others gauge-invariant operators of low dimension Δ. To limit the discussion to operators with $\Delta \leq 4$, these are (the traces are over color) :

— The Yang-Mills Lagrangian density ($\Delta = 4$)

$$TrF_{\mu\nu}F^{\mu\nu}$$

— The Quark bilinears ($\Delta = 3$)

$$Tr\bar{q}^i q_j, \qquad i, j = 1 \ldots N_f,$$

— The $U(N_f)_L \times U(N_f)_R$ flavor currents ($\Delta = 3$) : these are the non-abelian currents associated to the transformations which rotate independently the quark components of left and right chirality,

$$q_{L\,i} \to U_i^j q_{L,j}, \quad q_{R\,i} \to V_i^j q_{R,j}, \quad q_{L,R\,i} = \frac{1 \pm \gamma_5}{2} q_i, \quad U, V \in U(N_f)$$

The abelian part of this symmetry consists in two $U(1)$ groups : the vector part $U(1)_V$ (obtained by setting $U = V$) corresponds to baryon number. The axial part (corresponding to $U = V^{-1}$) is called $U(1)_A$ (and it is anomalous). The respective currents are

$$J_V^\mu = \bar{q}^i \gamma^\mu q_i, \quad J_A^\mu = \bar{q}^i \gamma^\mu \gamma_5 q_i$$

where trace over color is understood and repeated flavor indices are summed over. The non-abelian $SU(N_f)_L \times SU(N_f)_R$ currents are :

$$J^a_{L\,\mu} = \bar{q}^i_L \gamma_\mu (t^a)_i^j q_{L\,j}, \quad J^a_{R\,\mu} = \bar{q}^i_R \gamma_\mu (t^a)_i^j q_{R\,j}. \tag{3}$$

(again, the trace over color indices is suppressed, and repeated indices are summed) where the matrices t^a are generators of $SU(N_f)$ in the fundamental representation, the index $a = 1 \dots N_f^2 - 1$ and $i, j = 1 \dots N_f$.
In the chiral limit (i.e. when the quark mass matrix is identically zero) these transformations are symmetries of the QCD Lagrangian (except $U(1)_A$, which is anomalous, and we can forget it from now on) and the corresponding currents (3) are conserved.

1. Give a list of the fields which one should include in the holographic dual to describe the operators listed above (except the $U(1)_A$), together with their masses. In the case of scalars, what is meaning of the associated source in the field theory ?
2. Write down a bulk action for these fields (include only the quadratic terms and the interactions which are uniquely fixed by symmetries).
3. We assume a large-N limit such that $N_c \to \infty$, N_f fixed (this is also called *quenched flavor* approximation). Argue, by a large-N argument in field theory (e.g. using double-line notation), that the terms in the action describing flavor are suppressed by a factor of N_f/N_c ($\ll 1$) with respect to the ones which are purely associated to color.