Géométrie différentielle et théorie de jauge
Solutions to exercises

07/01/2020

4. Theorem of submersions. Since $d_x f : \mathbb{R}^n \to \mathbb{R}^k$ is surjective, the kernel $\ker d_x f$ has dimension $n-k$, and we can find a basis of $\mathbb{R}^n$ so that $\ker d_x f$ is generated by the $(n-k)$ last vectors of the basis, that is $\ker d_x f = \{0\} \times \mathbb{R}^{n-k}$. Consider the map $g : U \to \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ defined by $g(y) = (f(y), y^{k+1}, ..., y^n)$. Then, in the decomposition $\mathbb{R}^k \times \mathbb{R}^{n-k}$ of $\mathbb{R}^n$ we can decompose $d_x f : \mathbb{R}^n \to \mathbb{R}^k$ as

$$d_x f = \begin{pmatrix} F & 0 \end{pmatrix}$$

with $F : \mathbb{R}^k \to \mathbb{R}^k$ a linear isomorphism. Therefore

$$d_x g = \begin{pmatrix} F & 0 \\ 0 & Id \end{pmatrix}$$

has $\det d_x g = \det F \neq 0$ so $d_x g$ is a linear isomorphism. In this situation, the Inverse Function Theorem says that there exist open sets $V \ni x$ and $W \ni g(x)$ such that $\phi := g|_V : V \to W$ is a diffeomorphism. Then $\phi^{-1}(x^1, \ldots, x^n)$ is a point $y \in V$ such that $g(y) = x$, in particular $f(y) = (x^1, \ldots, x^k)$. So we obtain

$$f(\phi^{-1}(x)) = (x^1, \ldots, x^k). \tag{$*$}$$

Fix $c = 0$ to simplify. If f is a submersion at any point of $f^{-1}(0)$, then for any $x \in f^{-1}(0)$ we have $d_x f$ surjective so we can apply the previous result: there exists a local diffeomorphism $\phi : V \to W$ such that $f \circ \phi^{-1}$ is the map $(*)$. Then $\phi(V \cap f^{-1}(0)) = W \cap (\{0\} \times \mathbb{R}^{n-k})$, so ϕ is a submanifold chart for $f^{-1}(0)$ at x, and $T_x f^{-1}(0) = (d_x\phi)^{-1}(\{0\} \times \mathbb{R}^{n-k}) = \ker d_x f$.

5. Discrete quotients. Take a small ball B_r around x of radius r in some chart ϕ_x, so the closed ball $\overline{B}_r$ is compact, and therefore the set $\{g \in \Gamma, \overline{B}_r \cap g(\overline{B}_r) \neq \emptyset\}$ is finite, let's say $= \{g_1, \ldots, g_k\}$. Since the action is free, $g_i(x) \neq x$, therefore if we take $0 < \rho < r$ small enough, then $\overline{B}_\rho \cap g_i(\overline{B}_\rho) = \emptyset$ for all $i = 1, ..., k$. So finally $\overline{B}_\rho \cap g_i(\overline{B}_\rho) = \emptyset$ for all $g \in \Gamma$, which means that $p|_{\overline{B}_\rho}$ is injective (no two points in $\overline{B}_\rho$ represent the same point in the quotient). So we can take $U_x = B_\rho$. Therefore, $\phi_x \circ (p|_{U_x})^{-1}$ is a chart on $p(U_x) \subset M/\Gamma$ at the point $p(x)$. To check that we have an atlas, we look at the transition between two such chart, that is $\phi_y \circ (p|_{U_y})^{-1} \circ (\phi_x \circ (p|_{U_x})^{-1})^{-1} = \phi_y \circ q_{yx} \circ \phi_x^{-1}$, where $q_{yx} = p|_{U_y}^{-1} \circ p|_{U_x}$. If $x' \in (p|_{U_x})^{-1}(p(U_x) \cap p(U_y))$, then there is a point $y' \in U_y$ such that $p(x') = p(y')$, which means that there exist a (unique) $g \in \Gamma$ such that $y' = gx'$; then $q_{yx}(x') = y' = gx'$. If we move a bit x' then we have the same, but since the group is discrete, $q_{yx}(x')$ is given by the action of the same element g on x', and therefore q_{yx} is a diffeomorphism. Finally the transition $\phi_y \circ q_{yx} \circ \phi_x^{-1}$ is a diffeomorphism, and we have an atlas on M/Γ.

Remark. In the charts ϕ_x on M and $\phi_x \circ (p|_{U_x})^{-1}$ on M/Γ, the local expression of p is just the identity, therefore p is a local diffeomorphism.

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4. Interior product. By definition, if $p = |\alpha|$ and $q = |\beta|$, and defining $X_1 := v$,

$$i_v(\alpha \wedge \beta)(X_2, ..., X_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \epsilon(\sigma)\alpha(X_{\sigma(1)}, ..., X_{\sigma(p)})\beta(X_{\sigma(p+1)}, ..., X_{\sigma(p+q)}).$$

We subdivise the sum into two sums, the first one corresponding to the permutations σ such that $\sigma(1) \in \{1, ..., p\}$ and the second one to the permutations such that $\sigma(1) \in \{p+1, ..., p+q\}$. Let us fix $\tau_{i,j}$ to be the transposition which exchanges i and j. For the first sum, consider the permutation $\sigma' = \tau_{1,\sigma(1)} \circ \sigma$, then $\sigma'(1) = 1$ so σ' is actually an element of the group of permutations of the $p+q-1$ elements $\{2, ..., p+q\}$. Now σ is determined by $\sigma(1) \in \{1, ..., p\}$ and $\sigma' \in S_{p+q-1}$, so we can rewrite the corresponding sum as

$$\begin{aligned}
\frac{1}{p!q!} \sum_{i=1}^{p} \sum_{\sigma' \in S_{p+q-1}} & \epsilon(\sigma')\alpha(v, X_{\sigma'(2)}, ..., X_{\sigma'(p)})\beta(X_{\sigma'(p+1)}, ..., X_{\sigma'(p+q)}) \\
&= \frac{1}{(p-1)!q!} \sum_{\sigma' \in S_{p+q-1}} \epsilon(\sigma')\alpha(v, X_{\sigma'(2)}, ..., X_{\sigma'(p)})\beta(X_{\sigma'(p+1)}, ..., X_{\sigma'(p+q)}) \\
&= \Big((i_v\alpha) \wedge \beta\Big)(X_2, ..., X_{p+q}).
\end{aligned}$$

Similarly, in the second sum, σ is determined by $\sigma(1)$ and $\sigma' = \tau_{p+1,\sigma(1)} \circ \sigma$, a permutation such that $\sigma'(p+1) = p+1$, so we can consider σ' as a permutation of the $p+q-1$ elements $\{1, ..., p, p+2, ..., p+q\}$. Then, in the same way, we obtain that the second sum equals

$$\frac{1}{p!(q-1)!}\sum_{\sigma'\in S_{p+q-1}}\epsilon(\sigma')\alpha(X_{\sigma'(1)},\dots,X_{\sigma'(p)})\beta(v,X_{\sigma'(p+2)},\dots,X_{\sigma'(p+q)})=\big(\alpha\wedge(i_v\beta)\big)(X_2,\dots,X_{p+q}).$$

Hence we get $i_v(\alpha\wedge\beta)=(i_v\alpha)\wedge\beta+(-1)^p\alpha\wedge i_v\beta$.
In particular, if $p=1$, then $i_v\alpha=\alpha(v)$ and one has $i_v(\alpha\wedge\beta)=\alpha(v)\beta-\alpha\wedge i_v\beta$. By induction, one gets the general formula $i_v(\alpha_1\wedge\cdots\wedge\alpha_p)=\sum_1^p(-1)^{i-1}\alpha_i(v)\alpha_1\wedge\cdots\wedge\widehat{\alpha_i}\wedge\cdots\wedge\alpha_k$ for 1-forms $\alpha_1,\dots,\alpha_p$.
10. Forms and submersions. Begin by the case of the submersion

$$p(x^1,\dots,x^n)=(x^1,\dots,x^k). \tag{†}$$

For $I=\{i_1<\cdots<i_q\}$ denote $dx^I=dx^{i_1}\wedge\cdots\wedge dx^{i_q}$. If we have a q-form β on $\mathbb{R}^k$, we can write $\beta=\sum_{I\subset\{1,\dots,k\}}\beta_I(x^1,\dots,x^k)dx^I$, therefore we also have $p^*\beta=\sum_{I\subset\{1,\dots,k\}}\beta_I(x^1,\dots,x^k)dx^I$. Hence a form

$$\alpha=\sum_{I\subset\{1,\dots,n\}}\alpha_I(x^1,\dots,x^n)dx^I$$

is of the form $\alpha=p^*\beta$ if and only if:

1. $\alpha_I=0$ if $I\not\subset\{1,\dots,k\}$;
2. each coefficient α_I is a function of $(x^1,\dots,x^k)$ only.

We have $i_{\frac{\partial}{\partial x^\ell}}\alpha=\sum_{I\ni\ell}\epsilon(\ell,I)\alpha_I dx^{I\setminus\ell}$, where $\epsilon(\ell,I)=(-1)^{j-1}$, where $I=\{i_1<\cdots<i_q\}$ and $\ell=i_j$ for some j. Therefore the first condition is equivalent to $i_{\frac{\partial}{\partial x^\ell}}\alpha=0$ for all $\ell>k$.
Given the first condition, if $\ell>k$ we have $i_{\frac{\partial}{\partial x^\ell}}d\alpha=\sum_I\frac{\partial\alpha_I}{\partial x^\ell}dx^\ell\wedge dx^I$, therefore the second condition is equivalent to $i_{\frac{\partial}{\partial x^\ell}}d\alpha=0$ for any $\ell>k$.
Finally, observe that $\ker d_xp=\langle\frac{\partial}{\partial x^{k+1}},\dots,\frac{\partial}{\partial x^n}\rangle$, so we have proved the result in that case.
Now let us deal with the case of a general submersion p. By the theorem of submersions, p is locally of the form (†) and therefore if a form $\alpha\in\Omega^kM$ is of the form $\alpha=p^*\beta$ one must have $i_v\alpha=0$ and $i_vd\alpha=0$ for all $v\in\ker dp$. Conversely, if $i_v\alpha=0$ and $i_vd\alpha=0$ for all $v\in\ker dp$ then we can cover M by open sets U on which p is of the form (†), and we obtain forms β_U defined on $p(U)$ such that $\alpha|_U=p^*\beta_U$. The problem is now to check that any β_U and β_V coincide on the intersection $p(U)\cap p(V)$, that is to prove that β_z does not depend on the point $x\in p^{-1}(z)$ used to define it by $p^*\beta_z=\alpha_x$ (as we have seen, this defines uniquely β_z). Suppose $z=p(x)=p(y)$: by hypothesis, the fiber $p^{-1}(z)$ is connected, therefore there exists a path $c:[0,1]\to p^{-1}(z)$ such that $c(0)=x$ and $c(1)=y$. At each point $c(t)$ we have a unique $\beta_t\in\Lambda^kT_z^*N$ such that $p^*\beta_t=\alpha_{c(t)}$. From what we have just seen, β_t is locally constant, so $\beta_0=\beta_1$ and therefore β_z does not depend on the choice of the point $x\in p^{-1}(z)$.
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4. Cartan formula. Since for all t one has $\phi_t^*d\alpha=d\phi_t^*\alpha$, taking the derivative at $t=0$ one has $\mathscr{L}_Xd\alpha=d\mathscr{L}_X\alpha$. Similarly, since ϕ_t^* is an algebra morphism on the algebra of forms, one has $\phi_t^*(\alpha\wedge\beta)=(\phi_t^*\alpha)\wedge(\phi_t^*\beta)$ and therefore the derivative gives $\mathscr{L}_X(\alpha\wedge\beta)=(\mathscr{L}_X\alpha)\wedge\beta+\alpha\wedge(\mathscr{L}_X\beta)$, which means that $\mathscr{L}_X$ is a derivation of the algebra ΩM.
We can begin by proving the Cartan formula in special cases:

- for a 0-form, that is a function f, we have $\phi_t^*f=f\circ\phi_t$ and therefore $\mathscr{L}_Xf=\frac{d}{dt}|_{t=0}f\circ\phi_t=df(\frac{d\phi_t}{dt}|_{t=0})=df(X)$; therefore $\mathscr{L}_Xf=X\cdot f=i_Xdf$, so the formula is proved in this case;
- for a 1-form, which is a sum of terms fdg, with f and g functions, we calculate

$$\mathscr{L}_X(fdg)=(\mathscr{L}_Xf)dg+f\mathscr{L}_Xdg=(i_Xdf)dg+fd\mathscr{L}_Xg=(i_Xdf)dg+fd(i_Xdg),$$
$$(i_Xd+di_X)(fdg)=i_X(df\wedge dg)+d(fi_Xdg)=(i_Xdf)dg-(i_Xdg)df+(i_Xdg)df+fd(i_Xdg),$$

so we have indeed $\mathscr{L}_X(fdg)=(i_Xd+di_X)(fdg)$.

In general, we observe that $D=i_Xd+di_X$ is also a derivation of ΩM, that is satisfies $D(\alpha\wedge\beta)=(D\alpha)\wedge\beta+\alpha\wedge(D\beta)$ (do the calculation, this follows from the fact that both d and i_X satisfy the identity $d(\alpha\wedge\beta)=(d\alpha)\wedge\beta+(-1)^{|\alpha|}\alpha\wedge d\beta$). We can write locally a form as $\alpha=\alpha_1\wedge\cdots\wedge\alpha_k$ where the α_i's are 1-forms, and using that $\mathscr{L}_X\alpha_i=D\alpha_i$ for the 1-forms α_i and the fact that $\mathscr{L}_X$ and D are both derivations of ΩM, we obtain that $\mathscr{L}_X\alpha=D\alpha$.