Géométrie différentielle et théorie de jauge, 14/01/2020

1. In $\mathbb{R}^2 \setminus \{0\}$, express the vector fields $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. What is $[\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}]$?

2. Observe that in local coordinates $[X, Y](x) = d_x Y(X(x)) - d_x X(Y(x))$. Is there an intrinsic operation $X \cdot Y$ on vector fields given by the formula in local coordinates $X \cdot Y = dY(X)$, that is $(X \cdot Y)^j = X^i \frac{\partial Y^j}{\partial x^i}$?

3. Consider the map $X : S^3 \to \mathbb{R}^4$ defined by $X(x^0, x^1, x^2, x^3) = (-x^1, x^0, -x^3, x^2)$. Prove that X is actually a vector field on S^3. Prove that its trajectories are the fibers of the Hopf fibration $S^3 \to \mathbb{C}P^1$.

4. Let $v \in \mathbb{R}^n$, and define the **interior product** by v, $i_v : \Lambda^p(\mathbb{R}^n)^* \longrightarrow \Lambda^{p-1}(\mathbb{R}^n)^*$, by the formula

$$(i_v\alpha)(X_1, \ldots, X_{p-1}) = \alpha(v, X_1, \ldots, X_{p-1}).$$

Prove that

$$i_v(\alpha \wedge \beta) = (i_v\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \wedge i_v\beta,$$

and, if $\alpha_1, \ldots, \alpha_p$ are 1-forms,

$$i_v(\alpha_1 \wedge \cdots \wedge \alpha_p) = \sum_1^p (-1)^{i-1}\alpha_i(v)\alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \alpha_p.$$

Calculate the volume form $i_{\vec{n}}(dx^1 \wedge \cdots \wedge dx^n)$ of $S^{n-1} \subset \mathbb{R}^n$.

5. Calculate the volume form of S^2 in angular coordinates (ϕ, θ), where ϕ is the angle with the vertical axis, and θ the angle in the horizontal plane. Calculate the area of S^2.

6. Use Stokes formula and the inequality $\int_0^L x^2(s)ds \leq \frac{L^2}{(2\pi)^2}\int_0^L \dot{x}^2(s)ds$ for any L-periodic function such that $\int_0^L x(s)ds = 0$ (this inequality follows from decomposition in Fourier series), to prove the isoperimetric inequality for a smooth domain $D \subset \mathbb{R}^2$:

$$\text{area}(D) \leq \frac{1}{2\pi}\text{length}(\partial D)^2.$$

7. Show that the projective space $\mathbb{R}P^n$ is orientable if and only if n is even (use the projection $p : S^n \to \mathbb{R}P^n$ and use $\tau^* p^* \omega = p^* \omega$, where τ is the antipodal map of S^n). Show that the Möbius band $(S^1 \times \mathbb{R})/\{(z, x) \mapsto (-z, -x)\}$ is not orientable.

8. Let X be a vector field generating the flow of diffeomorphism $(\phi_t)_{t\in\mathbb{R}}$, defined by $\phi_0(x) = x$ and $\frac{d}{dt}\phi_t(x) = X(\phi_t(x))$. For a p-form α define the p-form $\mathscr{L}_X\alpha = \frac{d}{dt}\Big|_{t=0}\phi_t^*\alpha$. Prove that $\mathscr{L}_X \circ d = d \circ \mathscr{L}_X$, $\mathscr{L}_X(\alpha \wedge \beta) = (\mathscr{L}_X\alpha) \wedge \beta + \alpha \wedge \mathscr{L}_X\beta$. Deduce the **Cartan formula**

$$\mathscr{L}_X\alpha = i_X d\alpha + d(i_X\alpha).$$

9. Let M^n be an oriented compact manifold with boundary ∂M. Prove that there is no retraction of M on its boundary, that is no smooth map $r : M \to \partial M$ such that $r(x) = x$ if $x \in \partial M$. (Apply Stokes theorem to $r^*\omega$, where ω is a $(n-1)$-form on ∂M of nonzero integral).

10. Let $p : M \to N$ be a smooth submersion with connected fibers. A tangent vector $v \in T_xM$ is vertical if $d_xp(v) = 0$. Let α be a p-form on M. Prove that there exists a p-form β on N such that $\alpha = p^*\beta$ if and only if for any vertical vector v one has $i_v\alpha = 0$ and $i_v d\alpha = 0$. (Begin by the case of the submersion $(x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^k)$).