Géométrie différentielle et théorie de jauge, 03/03/2020

Characteristic classes

1. Prove that the curvature of the dual bundle E^* is $F^{E^*} = -(F^E)^t$, that is $(F^{E^*}\alpha)_{X,Y} = -\alpha \circ F^E_{X,Y}$ for $\alpha \in E^*$ and two tangent vectors X, Y. Calculate the Chern classes of the dual bundle E^* in terms of that of E.

2. Let L_1,...,L_k be line bundles. Prove that

$$c(L_1 \oplus \cdots \oplus L_k) = (1 + c_1(L_1)) \wedge \cdots \wedge (1 + c_1(L_k)).$$

3. Let (E, ∇) be a $SU(N)$-complex vector bundle with connection. In that case the connection matrices and the curvature are traceless, so $c_2(\nabla) = \frac{1}{8\pi^2}\operatorname{tr}(F^\nabla \wedge F^\nabla)$. If we have another connection $\nabla_1 = \nabla + a$, then denote $\nabla_t = \nabla + ta$. Prove that $F^{\nabla_t} = F^\nabla + td^\nabla a + t^2 a \wedge a$. Deduce that $c_2(\nabla_1) - c_2(\nabla) = d\beta$, with

$$\beta = \frac{1}{8\pi^2}\operatorname{tr}(2a \wedge F^\nabla + a \wedge d^\nabla a + \frac{2}{3} a \wedge a \wedge a).$$

($\operatorname{tr}(a \wedge da + \frac{2}{3} a \wedge a \wedge a)$ is the Chern-Simons form).

4. Let $L \to \mathbb{C}P^n$ be the tautological complex line bundle. We define a rank n vector bundle E on $\mathbb{C}P^n$ by taking $E_x = L_x^\perp$ for some fixed Hermitian product on $\mathbb{C}^{n+1}$. Therefore $L \oplus E \simeq \mathbb{C}^{n+1}$, the trivial $\mathbb{C}^{n+1}$ vector bundle over $\mathbb{C}P^n$.

If $u \in \operatorname{Hom}(L_x, E_x)$, then we can define a complex line $G(u) =$ graph of $u = \{A + u(A), A \in L_x\} \subset \mathbb{C}^{n+1}$. The map $u \mapsto G(u) \in \mathbb{C}P^n$ is a diffeomorphism of $\operatorname{Hom}(L_x, E_x)$ with an open set around x in $\mathbb{C}P^n$. In particular its tangent map at the origin gives an isomorphism of vector bundles $\operatorname{Hom}(L, E) \simeq T\mathbb{C}P^n$.

Deduce that $T\mathbb{C}P^n \oplus \mathbb{C} \simeq L^* \otimes \mathbb{C}^{n+1} = L^* \oplus \cdots \oplus L^*$. Deduce $c(T\mathbb{C}P^n)$ in terms of $c_1(L^*)$. (Note that $L^* = \mathscr{O}(1)$ is a generator of $H^2(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}$).

5. The $\hat{A}$-genus of a real vector bundle V is defined by

$$\hat{A}(\nabla) = \det{}^{\frac{1}{2}}\Big(\frac{x/2}{\sinh x/2}\Big), \quad \text{where } x = \frac{iF}{2\pi},$$

which is the same as

$$\hat{A}(\nabla) = \exp\Big(\frac{1}{2}\operatorname{tr}\ln\frac{x/2}{\sinh x/2}\Big).$$

Show that $\hat{A}$ is nonzero only in degrees multiple of 4, and that

$$\hat{A}(\nabla_1 \oplus \nabla_2) = \hat{A}(\nabla_1) \wedge \hat{A}(\nabla_2).$$

Calculate the first terms :

$$\hat{A} = 1 - \frac{1}{24}p_1(\nabla) + \cdots, \quad \text{where } p_1(\nabla) = \frac{1}{8\pi^2}\operatorname{tr}(F \wedge F).$$

When one takes $V = TM$, this is the $\hat{A}$-genus of M. For a 4-manifold, it is a number: p_1 (first Pontryagin number) is an integer but $\hat{A}$ is a priori not an integer.