MASTER ICFP 2019-2020

Lectures Notes on String Theory

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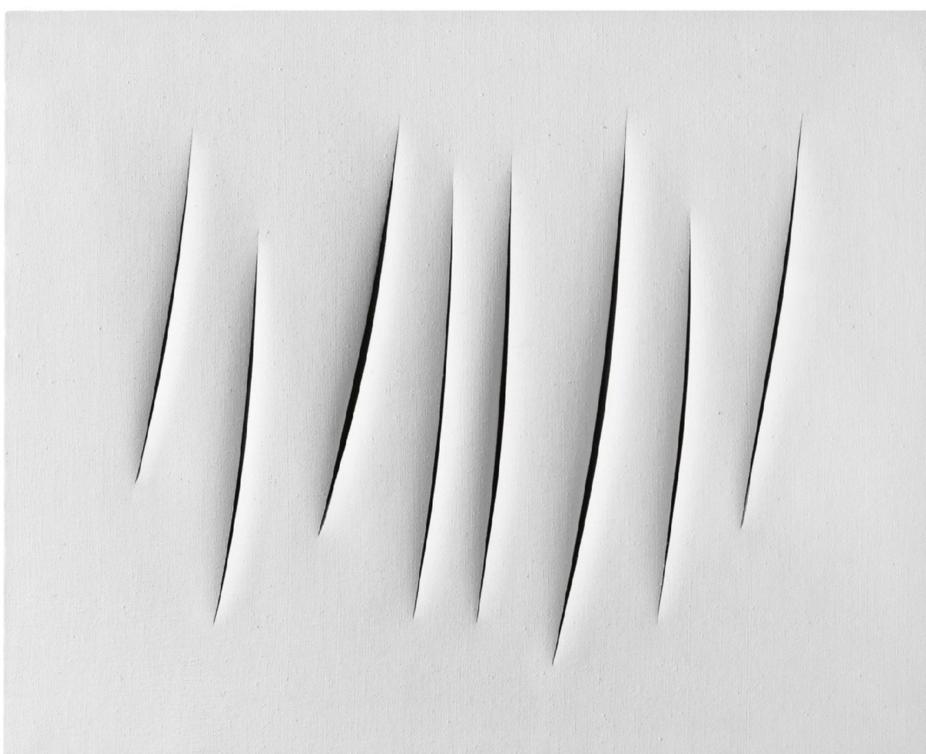

Lucio Fontana, « Concetto Spaziale »

Foreword

These lecture notes are based upon a series of courses given at the master program ICFP from 2018 by the author. Comments and suggestions are welcome. Some references that can complement these notes are

- *Superstring Theory* (Green, Schwarz, Witten) [1,2]: the classic textbook from the eighties, naturally outdated on certain aspects but still an unvaluable reference on many topics including the Green-Schwarz string and compactifications on special holonomy manifolds.

- *String Theory* (Polchinski) [3,4]: the standard textbook, with a very detailed derivation of the Polyakov path integral and strong emphasis on conformal field theory methods.

- *String Theory in a Nutshell* (Kiritsis) [5]: a concise presentation of string and superstring theory which moves quickly to rather advanced topics

- *String Theory and M-Theory: A Modern Introduction* (Becker, Becker, Schwarz) [6]: a good complement to the previous references, with a broad introduction to modern topics as AdS/CFT and flux compactifications.

- *A first course in String theory* (Zwiebach) [7]: an interesting and different approach, making little use of conformal field theory methods, in favor of a less formal approach.

- *Basic Concepts of String Theory* (Blumenhagen, Lüst, Theisen) [8]. As its name does not suggest, this book covers a lot of rather advanced topics about the worldsheet aspects of string theory. It is also rather appropriate for a math-oriented reader.

- The lectures notes of David Tong (http://www.damtp.cam.ac.uk/user/tong/string.html) are rather enjoyable to read, with a good balance between mathematical rigor and physical intuition.

- The very lively online lectures of Shiraz Minwalla: http://theory.tifr.res.in/ minwalla/

Conventions

- The space-time metric is chosen to be of signature $(-, +, \ldots, +)$.

- We work in units $\hbar = \mathsf{c} = 1$

Latest update

March 15, 2020

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Contents

Chapter 1

Introduction

In Novembrer 1994, Joe Polchinski published on the ArXiv repository a preliminary version of his celebrated textbook on String theory, based on lectures given at Les Houches, under the title *"What is string theory?"* [1]. If he were asked the same question today, the answer would probably be rather different as the field has evolved since in various directions, some of them completely unexpected at the time.

One may try to figure out what string theory is about by looking at the program of *Strings 2017*, the last of a series of annual international conferences about string theory that have taken place at least since 1989, all over the world. Among the talks less than half were about string theory proper (*i.e.* the theory you will read about in these notes) while the others pertained to a wide range of topics, such as field theory amplitudes, dualities in field theory, theoretical condensed matter or general relativity.

The actual answer to the question raised by Joe Polchinski, *"What is string theory?"*, may be answered at different levels:

- *litteral:* the quantum theory of one-dimensional relativistic objects that interact by joining and splitting.
- *historical:* before 1974, a candidate theory of strong interactions; after that date, a quantum theory of gravity.
- *practical:* a non-perturbative quantum unified theory of fundamental interactions whose degrees of freedom, in certain perturbative regime, are given by relativistic strings.
- *sociological:* a subset of theoretical physics topics studied by people that define themselves as doing research in string theory.

In these notes, we will provide the construction of a consistent first quantized theory of interacting quantum closed strings. We will show that such theory automatically includes a (perturbative) theory of quantum gravity. We will introduce also open strings that incorporate gauge interactions, and give rise to the concept of D-branes that plays a prominent role in the AdS/CFT correspondence.

Along the way we will introduce some concepts and techniques that are as useful in other areas of theoretical physics as they are in string theory, for instance conformal field theories, BRST quantization of gauge theories or supersymmetry.

1.1 Gravity and quantum field theory

String theory has been investigated by a significant part of the high-energy theory community for more than forty years as it provides a compelling answer – and maybe *the* answer – to the following outstanding question: what is the quantum theory of gravity?

A successful theory of quantum gravity from the theoretical physics viewpoint should at least satisfy the following properties:

1. the theory should reproduce general relativity, in an appropriate classical, low-energy regime;
2. the theory should be renormalizable or better UV-finite in order to have predictive power;
3. it should satisfy the basic requirements for any quantum theory, such as unitarity;
4. it should explain the origin of black hole entropy, possibly the only current prediction for quantum gravity;
5. last but not least, the physical predictions should be compatible with experiments, in particular with the Standard Model of particle physics, astrophysical and cosmological observations.

According to our current understanding, string theory passes successfully the first four tests. Whether string theory reproduces accurately at energies accessible to experiments the known physics of fundamental particles and interaction is still unclear, given that such physics occurs in a regime of the theory that is beyond our current analytical control (to draw an analogy, one cannot reproduce analytically the physics of condensed matter systems from the microscopic quantum mechanical description in terms of atoms). At least it is clear that the main ingredients are there: chiral fermions, non-Abelian gauge interactions and Higgs-like bosons.

The problem with quantizing general relativity

The classical theory of relativistic gravity in four space-time dimensions, or Einstein theory, follows in the absence of matter from the Einstein-Hilbert (EH) action in four space-time dimensions, that takes the form

$$\mathcal{S}_{\text{EH}} = \frac{1}{2\kappa^2}\int_M \mathrm{d}^4x\, \sqrt{-\det g}\,(\mathcal{R}(g) - 2\Lambda)\ , \tag{1.1}$$

where $\mathcal{R}$ is the Ricci scalar associated with the space-time manifold M, endowed with a metric g, and Λ the cosmological constant that has no *a priori* reason to vanish. The coupling constant κ of the theory is related to the Newton's constant through $\kappa = \sqrt{8\pi G}$; by dimensional analysis it has dimension of length. Its inverse defines the Planck mass $M_{\text{PL}} \sim 10^{19}$ GeV.

Quantizing general relativity raises a number of deep conceptual issues, that can be raised even before attempting to make any explicit computation. Some of them are:

- Because of diffeormorphism invariance, there are no *local observables* in general relativity.
- A path-integral formulation of quantum gravity should include, by definition, a sum over space-time geometries. Which geometries should be considered? Should we specify boundary conditions?

- A Hamiltonian formulation of quantum gravity would require a foliation of space-time in terms of space-like hypersurfaces. Generically, such foliation does not exist.
- Classical dynamics of general relativity predicts the formation of event horizons, shielding regions of space-time from the exterior. This challenges the unitarity of the theory, through the *black hole information paradox*.

Quantum gravity with a positive cosmological constant – which seems to be relevant to describe the Universe – raises a number of additional conceptual issues that will be ignored in the rest of the lectures. We will mainly focus on theories with a vanishing cosmological constant; the case of negative cosmological constant will be discussed in the AdS/CFT lectures.

Perturbative QFT for gravity

One may try to ignore these conceptual problems and build a quantum field theory of gravity in the usual way, *i.e.* by defining propagators, vertices, Feynman rules, etc..., from the non-linear EH action, eqn. (1.1) [2].

With vanishing cosmological constant, one considers fluctuations of the metric around a reference Minkowski space-time metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,. \tag{1.2}$$

Linearizing the equations of motion that follows from (1.1), in the absence of sources, we arrive to:

$$\Box\bar{h}_{\mu\nu} - 2\partial^{\rho}\partial_{(\mu}\bar{h}_{\nu)\rho} + \eta^{\mu\nu}\partial^{\rho}\partial^{\sigma}\bar{h}_{\mu\nu} = 0\,, \tag{1.3}$$

where we have defined the "trace-reversed" tensor $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h^{\rho}{}_{\rho}\,\eta_{\mu\nu}$. This theory possesses a gauge invariance that comes from the diffeomorphism invariance of the full theory. The equations of motion are invariant under

$$h_{\mu\nu} \;\mapsto h_{\mu\nu} + \partial_{\mu}\zeta_{\nu} + \partial_{\nu}\zeta_{\mu}\,. \tag{1.4}$$

One can choose to work in a Lorentz gauge, defined by $\partial^{\mu}\bar{h}_{\mu\nu} = 0$, in which case the field equations (1.3) amounts to a wave equation for each component, $\Box\bar{h}_{\mu\nu} = 0$.

The solutions of these equations are naturally plane waves $\bar{h}_{\mu\nu}(x^{\rho}) = h^{0}_{\mu\nu}\exp(ik_{\rho}x^{\rho})$, and the Lorentz gauge condition means that they are transverse. Finally, the residual gauge invariance that remains in the Lorentz gauge, corresponding to vector fields ζ_{μ} satisfying the wave equation $\Box\zeta_{\mu} = 0$, can be fixed by choosing the longitudinal gauge $\bar{h}_{0\mu} = 0$. As a result, the gravitational waves have two independent transverse polarizations. The corresponding quantum theory is a theory of free *gravitons* that are massless bosons of helicity two.

The interactions between gravitons are added by expanding the EH action around the background (1.2) in powers of $h_{\mu\nu}$. In pure gravity one obtains three-graviton and four-graviton vertices, that have a rather complicated form. For instance the four-graviton vertex looks roughly like:

$$G^{\mu_1\nu_1,\dots,\mu_4\nu_4}(k_1,\dots,k_4) = \kappa^2\Big(k_1\cdot k_2\,\eta^{\mu_1\nu_1}\cdots\eta^{\mu_4\nu_4} + k_1^{\mu_3}k_2^{\nu_3}\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2} + \cdots\Big) \tag{1.5}$$

Using these vertices one can define Feynman rules for the quantum field theory of gravitons and compute loop diagrams like the one below.

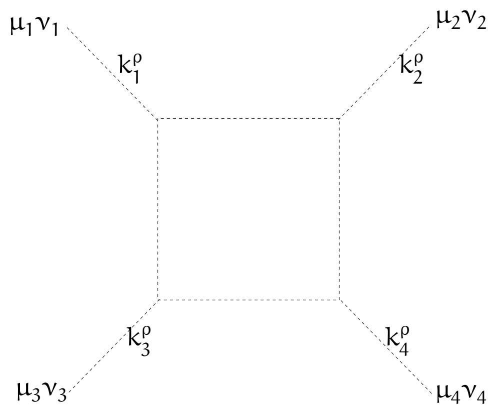

Figure 1.1: *Graviton scattering*

As in most quantum field theories, such loops integrals diverge when the internal momenta propagating in the loop become large, and should be regularized. By dimensional analysis, the regularized loop diagrams will be weighted by positive powers of $(\Lambda_{\rm uv}/M_{\rm PL})$, where $\Lambda_{\rm uv}$ is the ultraviolet cutoff.

In *renormalizable* QFTs as quantum chromodynamics, such high-energy – or ultraviolet – divergences can be absorbed into redefinitions of the couplings and fields of the theory, which leads to theories with predictive power. In contrast, this cannot be done for general relativity, for the simple reason that the coupling constant is dimensionfull (it has the dimension of length). Therefore, the divergences cannot be absorbed by redefining fields and couplings in the original two-derivative action; rather higher derivative terms should be included to do so. General relativity is thus a prominent example of *non-renormalizable* quantum field theory. Still it doesn't mean that such a theory is meaningless in the Wilsonian sense; it can describe the low-energy dynamics, well below the Planck scale $M_{\rm PL}$, of an "ultraviolet" theory of quantum gravity that is not explicitly known. However, as in any non-renormalizable theory, this effective action has little predictive power, as higher loop divergences need to be absorbed in extra couplings that were not present in the original action,[1] but become less important as the energy becomes lower. As we shall see string theory solves the problem in a rather remarkable way, by removing *all* the ultraviolet divergences of the theory.

[1]Strictly speaking, the one-loop divergence of pure GR can be absorbed by field redefinition. This not the case when matter is present, and from two-loops onwards for pure gravity.

1.2 String theory: historical perspective

This is a very sketchy account about the history of string theory; interested reader may consult the book [3] for first-hand testimonies.

String theory as a theory of quantum gravity came almost by accident, after being proposed as a theory of strong interactions. The prehistory of string theory occurred during the sixties. At that time, general relativity was not, with few exceptions, a topic of interest for theoretical physicists, but was rather a playground for mathematicians.

Quantum field theory itself didn't have the central role that it has today in our understanding of fundamental interactions. While quantum electrodynamics was acknowledged as the appropriate description of electromagnetic interactions, most physicists thought that it was an inappropriate tool to solve the big problems of the time, the physics of strong and weak interactions. This was especially true for the strong interactions, as the experiments were finding a growing number of hadronic particles, with large masses and spins. These particles were mostly resonances *i.e.* particles with a finite lifetime. Defining a QFT including all of these resonances did not seem, rightfully, a sensible idea. Quarks did make their appearance in the theoretical physicists' lexicon, however they were thought as mathematical tools rather than as actual elementary particles – the fact that they cannot be observed individually supported this point of view.

A different approach, called the *S-matrix program*, was widely popular back then. The idea was to construct directly the S-matrix of the theory using some general physical principles (unitarity, analyticity,...), as well as some experimental input from the specific theory that was considered, without any reference to a "microscopic" Lagrangian.

One crucial experimental observation was that hadronic resonances could be classified in families along curves in the mass-angular momentum plane (M, J) called *Regge trajectories*:

$$J = \alpha(0) + \alpha' M^2 \tag{1.6}$$

The value of the parameter $\alpha(0)$, or *intercept*, was determining a given family of resonances, while the slope α' was universal – with one exception – and given experimentally by

$$\alpha' \simeq 1\text{GeV}^{-2} \tag{1.7}$$

in natural units.

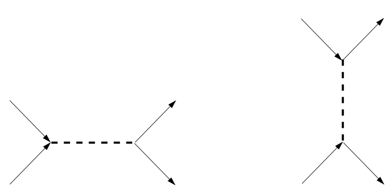

Figure 1.2: *Channel duality: s-channel (left) and t-channel (right)*

Among important requirements imposed upon the S-matrix, was that all the hadrons along the Regge trajectories should appear on the same footing, and both as intermediate particles (resonances) in the s-channel or as virtual exchanged particles in the t-channel, see fig. 2.11; actually either point of view was expected to give a complete description of the scattering process. This *channel duality* property of the S-matrix, together with the other physical constraints, led Gabriele Veneziano to write down, in 1968, an essentially unique solution to the problem for the decay $\omega \to \pi^+ + \pi^0 + \pi^-$ [4]:

$$T = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))} + \Big((s,t) \to (s,u)\Big) + \Big((s,t) \to (t,u)\Big)\,, \tag{1.8}$$

where $\alpha(s) = \alpha_0 + \alpha' s$ describes a Regge trajectory. This amplitude has remarkable properties; it exhibits an infinite number of poles in the s- and t-channels, and its ultraviolet behavior is softer than of any quantum field theory.

This breakthrough was the starting point for lot of activity in the theoretical physics community, and remarkably lot of progress was done without having any microscopic Lagrangian to underlie this physics. For instance the generalization to N-particle S-matrices was obtained, the addition of $SU(N)$ quantum numbers, the analysis of the unitarity of the theory (by looking at the signs of the residues) and even loop amplitudes.

Soon however people discovered strange properties of what was known at the time as the *dual resonance model*. In order to avoid negative norm states, the intercept of the Regge trajectory had to be tuned in such a way that unexpected massless particles of spin 1,2,... appeared in the theory. Embarrassingly, it was also needed that the dimension of space-time was 26! Around the same time it was realized, finally, that the states of the theory were describing the quantized fluctuations of relativistic strings by Nambu, Nielsen and Susskind in 1970.

Another problem was the appearance of a tachyon, *i.e.* an imaginary mass particle, in the spectrum. This was solved soon after, following the work of Neveu, Schwarz [5] and Ramond [6], who introduced fermionic degrees of freedom on the string in 1971 (bringing the space-time dimension to 10) by Gliozzi, Scherk[2] and Olive, who obtained the first consistent *superstring* theories in 1976 [7].

At the same time that these remarkable achievements were obtained, the non-Abelian quantum field theory of the strong interactions, or quantum chromodynamics, was recognized as the valid description of the hadronic world and, together with the electroweak theory, gave to quantum field theory the central role in theoretical high-energy physics that it has today.

It could had been the end of string theory, however, by a remarkable change of perspective, Scherk and Schwarz proposed in 1974 that string theory, instead of a theory of strong interactions, was providing a theory of quantum gravity [8]. From this point of view the annoying massless spin two particule of the dual resonance model was corresponding to the graviton, and they show that it has indeed the correct interactions.

The six extra dimensions of the superstring could be considered in this context as *compact* dimensions, given that the geometry was now dynamical, resurrecting the old idea of

[2]Joël Scherk (1946-1980) was a remarkable French theoretical physicist who made many key contributions to string theory and supergravity in the seventies, and died tragically when he was 33 years old only, leaving an indelible imprint in the field. The library at the LPTENS is dedicated to his memory.

Kaluza [9], and Klein [10] from the twenties. The value of the Regge slope should be radically different from was it was considered before in the hadronic context, in order to account for the observed magnitude of four-dimensional gravity. One was considering

$$\alpha' \simeq 10^{-38}\text{Gev}^{-2}\,, \tag{1.9}$$

or equivalently strings of a size smaller by 19 orders of magnitude than the hadronic string, *i.e.* impossible to resolve directy by current or foreseeable experiments. Despite that string theory was able to fullfill an old dream – quantizing general relativity – research in string theory remained rather confidential before the next turning point of its history,

Between 1984 and 1986, several important discoveries occurred and changed the fate of the theory: the invention of heterotic string [11] (which made easy to incorporate non-Abelian gauge interactions in string theory), the *Green-Schwarz* anomaly cancellation mechanism [12] which strengthened the link between string theory and low-energy supergravity, thereby making the former more convincing, and finally the discovery of Calabi-Yau compactifications [13] and orbifold compactifications [14] which allowed to get at low energies models of particle physics with $\mathcal{N} = 1$ supersymmetry in four dimensions. After that string theory became more mainstream, as many theoretical physicists started to realize that it was a promising way of unifying all fundamental particles and interactions in a consistent quantum theory.

Thirty years and a second revolution after, we haven't yet achieved this goal fully but tremendous progress has been made, the hallmarks being the discoveries of D-branes [15], of strong/weak dualities [16–18], of holographic dualities [19] and of flux compactifications [20, 21] to name a few. We still have a long way to go, and it is certainly worth trying.

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Chapter 2

Bosonic strings: action and path integral

Bosonic string theory, which is the most basic form of string theory, describes the propagation of one-dimensional relativistic extended objects, the *fundamental strings*, and their interactions by joining and splitting.

Quantum field theories of point particles are obtained by starting with a classical action, and quantizing the fluctuations around a given classical solution of the equations of motion. Upon quantization one gets field operators acting on the Fock space of the theory by creating or annihilating particles at a given point in space. An analogous *string field theory* exists, but is still poorly understood. In such theory one should have operators creating a *loop* in space, which is certainly more difficult to describe mathematically.

Rather the practical way to handle string theory is to follow the propagation in time of a single string in a fixed reference space-time. As restrictive as it looks like, this *first-quantized* formalism does not prevent for studying the interactions between strings, computing loop amplitudes and make a large number of predictions. As we will see below this "first-order" formalism can be used for point particles as well, as an alternative to QFT Feynman diagrams that allows to perform perturbative computations; however it misses important aspects as solitons or instantons that can be handled semi-classically from a field theory, and is not suited for all types of computations.

2.1 Relativistic particle in the worldline formalism

We consider a relativistic particle of mass m and charge q in a given d-dimensional space-time $\mathcal{M}$ of metric G and background electromagnetic field. Its dynamics is governed by the action

$$\mathcal{S} = -m \int_{l} ds - q \int_{l} A \,, \tag{2.1}$$

where l is the worldline of the particle, s the proper time and $A(x^{\mu}) = A(x^{\mu})_{\rho} dx^{\rho}$ the gauge potential. Under a gauge transformation, $A \mapsto A + d\Lambda$, the worldline action (2.1) is invariant up to possible boundary terms.

The worldline of the particle in space-time $\mathcal{M}$ corresponds to an embedding map[1]

$$\mathbb{R} \hookrightarrow \mathcal{M} \tag{2.2}$$

$$\tau \mapsto x^{\mu}(\tau) \,, \tag{2.3}$$

where τ is an affine parameter and $\{x^{\mu},\ \mu = 0, \ldots, D-1\}$ a set of coordinates on $\mathcal{M}$. The proper time differential can be expressed as

$$ds^2 = -G_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} d\tau^2 \,, \tag{2.4}$$

therefore the action (2.1) can be rewritten as

$$\mathcal{S} = -m \int d\tau \sqrt{-\dot{x}^{\mu}(\tau)\dot{x}^{\nu}(\tau) G_{\mu\nu}[x^{\rho}(\tau)]} - q \int_{l} A_{\mu}[x^{\mu}(\tau)] dx^{\mu}(\tau) \,. \tag{2.5}$$

[1]Strictly speaking, this is in general valid in an open set of $\mathcal{M}$ where the coordinates $\{x^{\mu},\ \mu = 0, \ldots, D-1\}$ are well-defined.

This action is invariant under diffeomorphisms of the worldline, *i.e.* under any differentiable reparametrization

$$\tau \mapsto \tilde{\tau}(\tau) \tag{2.6}$$

The embedding is now given by definition by the set of differentiable functions $\{\tilde{x}^\mu(\tilde{\tau}) = x^\mu(\tau),\ \mu = 0, \ldots, D-1\}$.

Let us consider the variation of the particle action (2.5) under the infinitesimal change of the path, namely

$$x^\mu \mapsto x^\mu + \delta x^\mu \tag{2.7a}$$

$$G_{\mu\nu} \mapsto G_{\mu\nu} + \partial_\sigma G_{\mu\nu} \delta x^\sigma \,. \tag{2.7b}$$

At first order, one gets

$$\delta S = m \int d\tau \left\{ \frac{G_{\mu\sigma}\dot{x}^\mu \delta\dot{x}^\sigma}{\sqrt{-\dot{x}^\mu \dot{x}^\nu G_{\mu\nu}}} + \frac{\dot{x}^\mu \dot{x}^\nu \partial_\sigma G_{\mu\nu} \delta x^\sigma}{\sqrt{-\dot{x}^\mu \dot{x}^\nu G_{\mu\nu}}} - \frac{q}{m}\left(A_\sigma \delta\dot{x}^\sigma + \partial_\sigma A_\mu \delta x^\rho \dot{x}^\mu\right) \right\} \tag{2.8}$$

After integration by parts of the first and third term, and trading the integral over the affine parmeter τ for the integral over the proper time s, one gets

$$\delta S = m \int ds \left[\frac{d^2 x^\nu}{ds^2} + \Gamma^\nu_{\rho\sigma} \frac{dx^\rho}{ds}\frac{dx^\sigma}{ds} - \frac{q}{m} F^\nu{}_\mu \frac{dx^\mu}{ds} \right] \delta x_\nu \tag{2.9}$$

Not surprisingly, one obtains the relativistic equation of motion of a massless charged particle, *i.e.* the geodesic equation plus the coupling to the electromagnetic field strength $F = dA$.

In order to make more explicit the diffeomorphism invariance of the worldline action, one can introduce an independent *worldline metric* as $ds^2 = -h_{\tau\tau}(\tau) d\tau^2$. In the one-dimensional analogue of the tetrad formalism of general relativity, one defines the *einbein* $e(\tau) = \sqrt{-h_{\tau\tau}}$. The action (2.5) can be then rewritten in a classically equivalent way as:

$$\begin{aligned} \mathcal{S}_e &= -\frac{1}{2}\int d\tau \sqrt{-h_{\tau\tau}} \left(h^{\tau\tau}\partial_\tau x^\mu \partial_\tau x^\nu G_{\mu\nu} + m^2\right) - q\int_{\mathfrak{l}} A_\mu dx^\mu \\ &= \frac{1}{2}\int d\tau \left(\frac{1}{e} G_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - m^2 e\right) - q\int_{\mathfrak{l}} A_\mu dx^\mu \end{aligned} \tag{2.10}$$

where one can see that $e(\tau)$ play the role of a Lagrange multiplier field $e(\tau)$ – *i.e.* a non-dynamical field that enforces a constraint in field space. Its equation of motion is simply $0 = -e^{-2} G_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - m^2$, which, upon replacing e by the solution in the action (2.10), gives back the original action (2.5).

One can view this action as a one-dimensional theory of gravity coupled to a set of free scalar fields $x^\mu(t)$ (there is naturally no curvature term in one dimension). Notice that the coupling of the particle to the electromagnetic four-potential A – the last term in equation (2.10) – is independent of the worldline metric. In this sense this coupling is of *topological* nature. Under diffeorphisms $\tau \mapsto \tilde{\tau}(\tau)$ the einbein transforms according to

$$e(\tau) d\tau = \tilde{e}(\tilde{\tau}) d\tilde{\tau} \tag{2.11}$$

which gives, for an infinitesimal transformation

$$\tau \mapsto \tilde{\tau} = \tau + \epsilon(\tau) \tag{2.12}$$

$$e(\tau) \mapsto \tilde{e}(\tilde{\tau}) = e(\tilde{\tau}) - \frac{d}{d\tilde{\tau}}\big(e(\tilde{\tau})\epsilon(\tilde{\tau})\big)\,. \tag{2.13}$$

As in four-dimensional gravity, this reparametrization invariance is a gauge symmetry, *i.e.* a redundancy in the description of the system that will eventually remove some degrees of freedom from the theory.[2]

Path integral quantization

The quadratic action (2.10) for the relativistic charged particle is a convenient starting point for quantizing the theory through the path integral formalism. It is convenient as well to analytically continue the space-time to Euclidean signature $x^0 \mapsto ix^0$, as well as the worldline time $\tau \mapsto i\tau$. We will consider a particle moving in flat space, *i.e.* with metric $G_{\mu\nu} = \delta_{\mu\nu}$, in the absence of electromagnetic field.

The one-particle vacuum energy in Euclidean space deduced from the wordline action is given by summing over closed paths of the particle, which is given schematically by the functional integral:

$$Z_1 = \int \frac{\mathcal{D}e}{\text{Vol}(\text{diff})} \int_{x(0)=x(1)} \mathcal{D}x \exp\left\{-\frac{1}{2}\int_0^1 d\tau \left(\frac{1}{e}\dot{x}^2 + m^2 e\right)\right\}\,, \tag{2.14}$$

where one has to divide the functional integral over the einbein (or equivalently over the one-dimensional metrics) by the infinite volume of the group of diffeomorphisms of the worldline. This group contains the transformations of the vielbein given infinitesimally by (2.13). On top of this shifts of τ by a constant, $\tau \mapsto \tau + \tau_0$, are diffeomorphisms that are not fixed by the choice of a reference einbein. The volume of this factor of the gauge group is finite and given by T, the invariant length of the closed path of the particle, see eq. (2.16) below. We choose finally the parameter τ to be in the interval $[0, 1]$, and, the path being closed, the einbein is a periodic function: $e(\tau + 1) = e(\tau)$.

Gauge symmetry

To carry the functional integral over the "gauge field" $e(\tau)$ one starts by slicing the field space into gauge orbits, *i.e.* einbeins that are related to each other by a diffeomorphism. The ratio of the integral over the whole field space over the volume of the group of diffeomorphisms is then equivalent to a functional integral over a slice in field space that cuts once each orbit, see figure 2.1 up to the Jacobian of the change of coordinates in field space; this is the Faddeev-Popov method [1] (FP for short).[3]

[2] This redundancy was already explicit in the original description of the theory, eq. (2.1), as one could have chosen the gauge $x^0(\tau) = \tau$ to start with.

[3] This method is rather overkill for dealing with a free particle but will be used again in the case of the string.

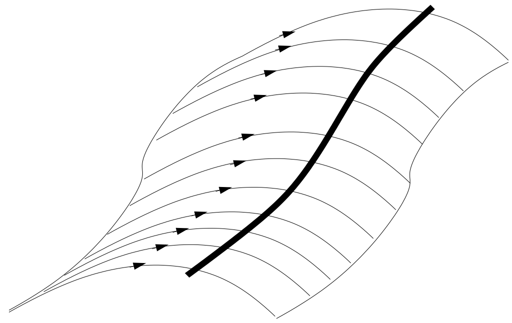

Figure 2.1: *Foliation of the space of gauge fields into gauge orbits. A slice through field space intersecting all orbits once is represented in bold.*

The Faddev-Popov method is the standard way of dealing with path-integral quantization of non-Abelian gauge theories and is presented in most quantum field theory textbooks, see *e.g.* in [2], chapter 12.

To start we pick a gauge choice corresponding to a reference einbein $\hat{e}$. For convenience we may want to take the reference einbein to be $\hat{e} = 1$. This reference einbein $\hat{e}$ generates a gauge orbit, the family $\{\hat{e}_\alpha\}$ of all einbeins obtained from $\hat{e}$ by some diffeomorphism α: $\hat{e} \mapsto \hat{e}_\alpha$. Using eqn. (2.11), starting from an arbitrary einbein e one can reach in principle the reference einbein $\hat{e} = 1$ with a diffeomorphism α that satisfies $\frac{d}{d\tau}\alpha(\tau) = e(\tau)$. Choosing the boundary condition $\alpha(0) = 0$ one has then

$$\alpha(\tau) = \int_0^\tau e(\tau')d\tau' , \tag{2.15}$$

hence it seems that all metrics on the worldline can be brought to the reference metric by a diffeomorphism. However the periodicity of the einbein is not preserved, as $\alpha(1) \neq 1$ for a generic diffeomorphism; this is a global obstruction for all metrics on the closed worldline being diffeomorphic-equivalent. The invariant length of the path is, as its name suggests, invariant under diffeomorphisms:

$$T = \int_0^1 e(\tau)d\tau = \int_0^{\tilde{\tau}(1)} \tilde{e}(\tilde{\tau})d\tilde{\tau} . \tag{2.16}$$

Hence the positive parameter T labels gauge-equivalent classes of metrics over closed worldlines; it is called a *modulus*. If one fixes the integration domain $[0, 1]$ to preserve the periodicity, the reference einbein should be defined accordingly. We choose then our reference einbein, in a class of metrics of invariant length T, as $\hat{e}(T) := T$, such that $\int_0^1 \hat{e}(T)(\tau)d\tau = T$.

In the path integral, one should perform the ordinary integral over all possible values of T, as we integrate over all possible geometries of the worldline. To take care of the translation symmetry of the closed path $\tau \mapsto \tau+\tau_0$, one splits a generic diffeomorphism into a translation and a coordinate transformation orthogonal to it, *i.e.* a differentiable function α such that $\alpha(0) = 0$ (as we have assumed already).

The functional integral measure $\mathcal{D}e$ splits then into a gauge-invariant measure $\mathcal{D}\alpha$ over the gauge group and an integral over the modulus, $\int dT$.

As in ordinary gauge theories like quantum chromodynamics in four space-time dimensions, one introduces then the Faddeev-Popov determinant through the relation:

$$\frac{1}{\Delta_{\text{FP}}(e)} := \int \mathrm{d}T \int \mathcal{D}\alpha\, \delta(e - \hat{e}_\alpha(T))\delta(\alpha(0)) \,, \tag{2.17}$$

The distribution $\delta(e - \hat{e}_\alpha(T))$ will eventually project the integral over the metrics onto an integral over the chosen gauge slice in field space, which is a one-parameter family of diffeomorphism-inequivalent reference einbeins $\hat{e}(T)$ depending of the moduli T of the closed path.

The Faddeev-Popov determinant should be thought of as the Jacobian of the change of coordinates from $\mathcal{D}e$, the integral over all one-dimensional einbeins, to $\mathrm{d}T\,\mathcal{D}\alpha$, the integral over the gauge slice on the one hand, and over the directions orthogonal to it – in other words over the gauge orbits – on the other hand.

Finally, the distribution $\delta(\alpha(0))$ ensures that we consider only diffeomorphisms keeping the origin fixed, as explained above.

One can plug this expression into the path integral (2.14) and integrate readily over the einbein e:

$$\begin{aligned} Z_1 &= \int \frac{\mathcal{D}e\,\mathrm{d}T\,\mathcal{D}\alpha\,\delta(\alpha(0))}{\mathrm{V}\textsc{ol}(\mathrm{diff})}\,\Delta_{\text{FP}}(e)\delta(e - \hat{e}_\alpha(T)) \int \mathcal{D}x \exp\left\{-\frac{1}{2}\int_0^1 \mathrm{d}\tau \left(\frac{1}{e}\dot{x}^2 + m^2 e\right)\right\} \\ &= \int \mathrm{d}T \int \frac{\mathcal{D}\alpha\,\delta(\alpha(0))}{\mathrm{V}\textsc{ol}(\mathrm{diff})}\Delta_{\text{FP}}(\hat{e}_\alpha(T)) \int \mathcal{D}x \exp\left\{-\frac{1}{2}\int_0^1 \mathrm{d}\tau \left(\frac{1}{\hat{e}_\alpha(T)}\dot{x}^2 + m^2 \hat{e}_\alpha(T)\right)\right\} \,. \end{aligned} \tag{2.18}$$

The last expression can be simplified further by noticing that *(i)* the Faddeev-Popov determinant is gauge-invariant (being defined as an average over the gauge group) and and *(ii)* that by trading the functional integral over x^μ by the functional integral over the transformed field x^μ_α under the diffeomorphism α, the integrand of the integral over the gauge group is actually gauge-independent, hence the integral $\int \mathcal{D}\alpha$ factors out and cancels the volume of the gauge group, except the factor T corresponding to the group of translations giving finally:

$$Z_1 = \int_0^\infty \frac{\mathrm{d}T}{T}\, e^{-\frac{m^2 T}{2}} \Delta_{\mathsf{FP}}(T) \int \mathcal{D}x \exp\left\{-\frac{1}{2T}\int \mathrm{d}\tau\, \dot{x}^2\right\} \,. \tag{2.19}$$

The determinant $\Delta_{\mathsf{FP}}(T)$ can be expressed as a functional determinant as follows. Using the infinitesimal expression (2.13) for the gauge transformation of the einbein, together with an infinitesimal variation of the loop modulus $T \mapsto T + \chi$, one can write

$$\delta(T - (T+\chi)_\alpha) = \delta\left(\chi - T\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau}\right) = \int \mathcal{D}\beta\, e^{-2i\pi T \int_0^1 \mathrm{d}\tau\, \beta\left(\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau} - \chi/T\right)} \,, \tag{2.20}$$

where one has introduced a path integral over a Lagrange multiplier field β to implement the desired constraint in field space. Similarly one can write

$$\delta(\epsilon(0)) = \int \mathrm{d}\lambda\, \exp -2i\pi\lambda\epsilon(0) \,. \tag{2.21}$$

To obtain the Faddeev-Popov determinant, rather than its inverse, as a functional integral, one can trade $(\beta, \epsilon, \chi, \lambda)$ for Grassmann variables (b, c, ψ, ρ), *i.e.* fermionic variables, and write, after some rescaling of the fields

$$\Delta_{\text{FP}}(T) = \int \mathcal{D}b\mathcal{D}c d\psi d\rho \; e^{-T\int_0^1 d\tau\, b(\frac{dc}{d\tau} - \psi/T) - \rho c(0)} \,. \tag{2.22}$$

One can perform immediately the integral over the Grassmann variables ψ and ρ, which gives finally

$$\Delta_{\text{FP}}(T) = \int \mathcal{D}b\mathcal{D}c \left(\int_0^1 d\tau\, b \right) \; c(0) \;\; e^{-T\int_0^1 d\tau\, b\frac{dc}{d\tau}} \,. \tag{2.23}$$

In other words, one has inserted into the path integral over (b, c) the mean value of $b(\tau)$ over the worldline, *i.e.* the zero-mode of the field, as well as $c(0)$; both insertions are actually necessary to cancel the integration over the zero-modes of the fields in the path integral as we will see shortly.

Functional determinants

We need now to perform the functional integral over the coordinate fields x^μ. One considers then the path integral

$$\int \mathcal{D}x \; e^{-\frac{1}{2T}\int_0^1 d\tau \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}} \,. \tag{2.24}$$

One expands then x^μ over a complete set of eigenfunctions of the positive-definite operator $-\frac{1}{T}\partial_\tau^2$ satisfying the right boundary conditions. It is convenient to separate the zero-modes, *i.e.* the classical solutions of the equations of motion, from the fluctuations:

$$x^\mu(\tau) = x_0^\mu + q^\mu(\tau) \,, \tag{2.25}$$

where $q(\tau)$ satisfies the Dirichlet boundary conditions $q(0) = q(1) = 0$, such that $x(0) = x(1) = x_0$. The norm in field space for the fluctuations is naturally

$$||q||^2 = \frac{1}{2}\int_0^1 \hat{e}_T(\tau) d\tau\, q^2(\tau) = \frac{T}{2}\int_0^1 d\tau\, q^2(\tau) \,. \tag{2.26}$$

One has then the expansion on an orthonormal basis on eigenfunctions of the differential operator $\mathfrak{D} = -\frac{1}{T}\partial_\tau^2$ with the right boundary conditions:

$$q^\mu(\tau_{\text{E}}) = \sum_{n=1}^{\infty} c_n^\mu \frac{2}{\sqrt{T}} \sin \pi n \tau \,. \tag{2.27}$$

and the measure of integration is

$$\mathcal{D}x = dx_0 \prod_{n=1}^{\infty} dc_n \,. \tag{2.28}$$

The ordinary integral over the zero-mode x_0 gives the (infinite) volume V of space-time, while from the integration over the non-zero modes one gets

$$\left(\int \prod_n dc_n e^{-\frac{\pi^2 n^2}{T^2} c_n^2}\right)^D = \left(\prod_n \frac{\pi n^2}{T^2}\right)^{-D/2} \tag{2.29}$$

This product of eigenvalues is divergent and needs to be regularized.

Functional determinants of the form $\det(\mathfrak{D}) = \prod_{n=1}^{\infty} \lambda_n$ can be evaluated using the *zeta-function regularization*. One assumes that $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$ and defines the spectral zeta-function as

$$\zeta_{\mathfrak{D}}(z) = \sum_{n=1}^{\infty} \lambda_n^{-z}\,, \tag{2.30}$$

which converges provided $\mathfrak{R}(z)$ is large enough. It can be analytically continued to the whole z plane except possibly at a finite set of points. Next we notice that

$$\log\det\mathfrak{D} = \sum_{n=1}^{\infty} \log\lambda_n = -\zeta'_{\mathfrak{D}}(0)\,, \tag{2.31}$$

and define the regularized functional determinant as

$$\prod_n \lambda_n = e^{-\zeta'_{\mathfrak{D}}(0)}\,. \tag{2.32}$$

In the present case one has

$$\zeta_{\mathfrak{D}} = \sum_{n=1}^{\infty} \left(\frac{\pi n^2}{T^2}\right)^{-z} = \left(\frac{T^2}{\pi}\right)^{z} \zeta(2z)\,, \tag{2.33}$$

in terms of the Riemann zeta-function ζ. Since $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\ln 2\pi$, the path integral over $x^\mu(\tau)$ gives finally, dropping the infinite volume factor and after some T-independent rescaling

$$\int \mathcal{D}x\; e^{-\frac{1}{2T}\int_0^1 dt\, \dot{x}^2} = T^{-D/2}\,. \tag{2.34}$$

This result can be obtained – in a perhaps simpler way – by viewing the path integral over a closed loop in Euclidean time as a partition function. One has

$$Z_x = \int \mathcal{D}x\; e^{-\int_0^1 d\tau \frac{1}{2T}\dot{x}^2} = \mathrm{Tr}\left(e^{-\beta H}\right)\ , \quad \beta = 1\,. \tag{2.35}$$

The Hamiltonian $H = \frac{T}{2}p^2$ is the same as a free (non-relativistic) particle of mass $1/T$ in D spatial dimensions, and the computation of the partition function gives simply

$$Z_x = \int \frac{d^D p}{(2\pi)^D} e^{-T\frac{p^2}{2}} = (2\pi T)^{-D/2}\,. \tag{2.36}$$

We now turn to the evaluation of the ghost path integral. We start with the expression of the FP determinant that we have obtained before:

$$\Delta_{\rm FP}(T) = \int \mathcal{D}b\mathcal{D}c \left(\int_0^1 d\tau\, b\right)\, c(0)\ \ e^{-T\int_0^1 d\tau\, b\frac{dc}{d\tau}}\,. \tag{2.37}$$

Because b and c are ghosts, with are dealing with fermionic variables with *periodic* (rather than anti-periodic) boundary conditions, hence having zero-mode. We recall here the rules of integration over Grassmann variables:

$$\int d\theta = 0\ , \quad \int d\theta\,\theta = 1\ , \quad \int d\theta\, f(\theta) = f'(0)\,. \tag{2.38}$$

which implies that, to get a non-zero answer, the integrand should contain the right number of zero-modes to cancel the corresponding zero-mode integration measure. Fortunately, the path integral (2.23 contains the right number of insertions of ghosts zero modes. The integral over the zero-modes yields

$$\int db_0\, dc_0\, b_0 c_0 = 1\,. \tag{2.39}$$

Deriving that the integral over the fluctuations is trivial is a little bit subtle.

It is simpler to move from the Lagrangian formalism to the Hamiltonian formalism and consider this problem from a statistical mechanics point of view. The equations of motion for the b and c classical fields are

$$\dot{b} = \dot{c} = 0 \tag{2.40}$$

hence, with periodic boundary conditions, the classical solutions are just the two zero-modes b_0 and c_0. In the quantum theory, since b can be seen as the canonical momentum conjugate to c, one has the anti-commutator

$$\{b_0, c_0\} = 1\,. \tag{2.41}$$

Since the Hamiltonian vanishes the Hilbert space contains two states of zero energy, $|\pm\rangle$, that satisfy

$$\begin{aligned} b_0|-\rangle &= 0\ , \quad b_0|+\rangle = |-\rangle \\ c_0|+\rangle &= 0\ , \quad c_0|-\rangle = |+\rangle\,. \end{aligned} \tag{2.42}$$

From these relations one learns that b_0c_0 projects onto the ground state $|-\rangle$. Then the path integral on a Euclidean circle of length T is interpreted as a thermal average of b_0c_0 at inverse temperature $\beta = 1$, and one has[4]:

$$\int \mathcal{D}b\mathcal{D}c\, b_0c_0\ e^{-\int_0^1 d\tau\, T\, b\frac{dc}{d\tau}} = \langle -|e^{-H}|-\rangle = 1\,. \tag{2.43}$$

[4]To be precise, the Euclidean fermionic path integral with *periodic* boundary conditions rather than anti-periodic is not exactly the partition function but rather Tr $\left[(-1)^F \exp(-\beta H)\right]$, where $F = b_0c_0$ counts the number of fermionic excitations.

Worldline versus QFT

Collecting all pieces, the path integral computation of the vacuum amplitude gives, up to an overall normalization factor,

$$Z_1 = \int_0^\infty \frac{dT}{T^{1+d/2}} e^{-\frac{m^2 T}{2}} \tag{2.44}$$

This integral is clearly divergent for $T \to 0$, *i.e.* when the particle loop shrinks to zero-size. The QFT picture will give us a more familiar understanding of this divergence.

Let us then consider the one-particle contribution to the vacuum energy of a free massive Klein-Gordon QFT. One has

$$Z_{\text{KG}} = \log \int \mathcal{D}\phi \, e^{-\frac{1}{2}\int d^D x \, \phi(-\nabla^2 + m^2)\phi} = \log \left(\det(-\nabla^2 + m^2) \right)^{-1/2} = -\frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \log(p^2 + m^2) \tag{2.45}$$

We can now move to Schwinger parametrization by using the simple identity

$$\frac{1}{x^a} = \frac{1}{\Gamma(a)} \int_0^\infty \frac{dT}{T^{1-a}} e^{-Tx} \, , \quad a > 0 \, . \tag{2.46}$$

which allows to get the general result

$$\int \frac{d^D p}{(2\pi)^d} \frac{1}{(p^2 + m^2)^a} = \frac{1}{\Gamma(a)} \int_0^\infty \frac{dT}{T^{1-a}} \frac{d^d p}{(2\pi)^D} e^{-T(p^2+m^2)} = \frac{1}{\Gamma(a)} \int_0^\infty \frac{dT}{T^{1-a}} e^{-Tm^2} (4\pi T)^{-D/2} \, . \tag{2.47}$$

At first order in the expansion in the parameter a one gets formally

$$Z_{\text{KG}} = -\frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \log(p^2 + m^2) = \frac{1}{2} \lim_{a \to 0} \int_0^\infty \frac{dT}{T^{1-a}} e^{-Tm^2} (4\pi T)^{-D/2} \, . \tag{2.48}$$

In other words,

$$Z_{\text{KG}} = \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \int_0^\infty \frac{dT}{T} e^{-\frac{T}{2}(p^2+m^2)} = \frac{1}{2} \int_0^\infty \frac{dT}{T} e^{-\frac{m^2}{2}T} (2\pi T)^{-D/2} \, , \tag{2.49}$$

which is the same, up to the numerical normalization factor that we did not compute precisely, the same as the worldline computation (2.44).

As was noticed before, the expressions (2.44,2.49) present a divergence for $T \to 0$; its origin is clear from eqn. (2.48). In the momentum-space expression (2.45), it is understood as the usual UV divergence of the loop integral for $||p|| \to +\infty$. One can compute directly the integral in (2.47) and get:

$$\int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2 + m^2)^a} = (4\pi)^{-D/2} \frac{\Gamma(a - D/2)}{\Gamma(a)} m^{D-2a} \, . \tag{2.50}$$

In the $a \to 0$ limit, matching the $\mathcal{O}(a)$ terms on both sides yields to

$$\int \frac{d^D p}{(2\pi)^d} \log(p^2 + m^2) = (4\pi)^{-D/2} \frac{2}{d} \Gamma(1 - D/2) m^D \, . \tag{2.51}$$

Hence the ultraviolet divergence can be removed by dimensional regularization, as usual.

By analogous computations, one can get the Klein-Gordon propagator by considering open worldlines between two points x and x' in Euclidean space-time. It is also interesting to consider the worldline path integral in the presence of an electromagnetic potential, starting from the more general action (2.10); in this way one can get for instance the photon N-point function in scalar QED, in particular the vacuum polarization ($N = 2$). Choosing a constant electric field, one finds also the famous Schwinger formula for production of charged particles/antiparticle pairs in an electric field [3].

We have learned with from this little exercise two important lessons that will be important in the forthcoming study of relativistic strings:

1. quantum mechanics (or equivalently 0+1 dimensional QFT) on the worldline of a massive relativistic charged particle provides an equivalent formulation of massive scalar QFT in an external electromagnetic field;

2. The UV loop divergences of QFT are mapped in the worldline formalism to closed wordlines shrinking to zero size.

There exists several extensions of this worldline formalism, in order to include spinors, non-Abelian gauge interactions, etc... As this is not the main topic of the lectures we will not comment further but refer the interested reader to [4].

2.2 Relativistic strings

We start our journey in string theory by considering the exact analogue of the relativistic point particle in the worldline formalism for relativistic objects with an extension in a space-like direction – the fundamental strings of string theory. These objects have a mass per unit length, or *tension* $\mathcal{T}$, which is by tradition parametrized as

$$\mathcal{T} = \frac{1}{2\pi\alpha'}\,, \tag{2.52}$$

where the parameter α', the *Regge slope*, has dimensions of length squared. The interested reader may consult section 1.2 to have some idea about where this terminology comes from.

In order to have a finite energy, the strings should have a finite length. It leaves two possibilities, topologically speaking: a loop or an interval, which are called *closed strings* and *open strings* respectively, see figure 2.2. In both cases the position along the string is parametrized by σ.

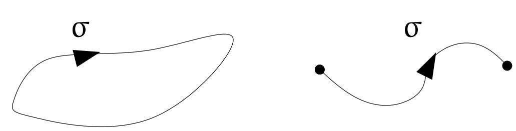

Figure 2.2: *Closed strings (left) and open strings (right).*

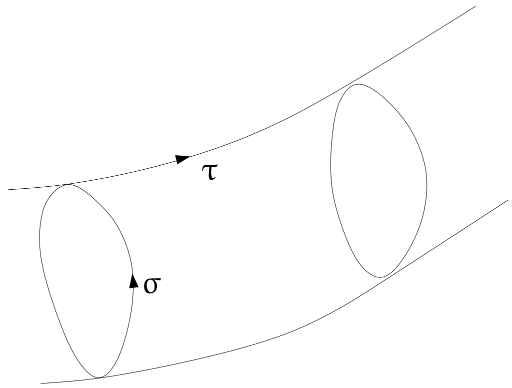

Figure 2.3: *Closed string worldsheet.*

Open strings are a little bit more subtle to handle, as one has to specify what are the boundary conditions at the end of the strings. We will deal mostly in these lectures with closed strings.

Classical closed strings

A propagating relativistic closed string swaps in spacetime a two-dimensional surface $\mathfrak{s}$, or *worldsheet*, in analogy with the worldline of point particles. It has the topology of a cylinder, parametrized by σ for the space-like direction and τ for the time-like direction, see fig. 2.3. For closed strings, the coordinate σ is periodic. We choose the convention

$$\sigma \sim \sigma + 2\pi \, . \tag{2.53}$$

The worldsheet swapped by a relativistic closed string in a space-time $\mathcal{M}$ of metric G defines an embedding map, given in a patch of the manifold $\mathcal{M}$ by

$$\begin{aligned} S^1 \times \mathbb{R} \ &\hookrightarrow \ \mathcal{M} && (2.54) \\ (\sigma, \tau) \ &\mapsto \ x^\mu(\sigma,\tau) \ , \quad x^\mu(\sigma+2\pi,\tau) = x^\mu(\sigma,\tau) \, , && (2.55) \end{aligned}$$

where the set of functions $\{x^\mu(\sigma,\tau), \mu = 0, \ldots, D-1\}$ should be periodic in σ. For convenience we will use the notation $(\sigma^0, \sigma^1) = (\tau, \sigma)$. The codomain of the map, *i.e.* the space-time $\mathcal{M}$ where the string leaves, is usually called the *target space* of the string.

The space-time metric $G_{\mu\nu}(x^\rho)$ of the ambient space-time induces a metric h on the world-sheet parametrized by σ and τ:

$$h_{ij} = G_{\mu\nu} \frac{\partial x^\mu(\sigma^k)}{\partial \sigma^i} \frac{\partial x^\nu(\sigma^k)}{\partial \sigma^j} \, , \tag{2.56}$$

such that the area element on the worldsheet is given by

$$dA = \sqrt{-\det h} \, d\sigma^0 d\sigma^1 \, . \tag{2.57}$$

The negative sign in the square root takes into account that the tangent space of the surface can be split into a time-like and a space-like directions over every point.

In complete analogy with the relativistic particle case, see eq. (2.5), one postulates a string action of the form

$$S_{\text{NG}} = -\frac{1}{2\pi\alpha'}\int d\sigma^0 \int_0^{2\pi} d\sigma^1 \sqrt{-\det\, G_{\mu\nu}[x^\rho(\sigma^k)]\frac{\partial x^\mu(\sigma^k)}{\partial\sigma^i}\frac{\partial x^\nu(\sigma^k)}{\partial\sigma^j}}\,. \tag{2.58}$$

This action is known as the *Nambu-Goto* action [5, 6]. It is invariant under diffeomorphisms of the worldsheet as it should:

$$\sigma^i \mapsto \tilde\sigma^i(\sigma^j) \tag{2.59a}$$

$$d\sigma^0 d\sigma^1 \mapsto (\det \partial_i\tilde\sigma^j)^{-1} d\tilde\sigma^0 d\tilde\sigma^1 \tag{2.59b}$$

$$\frac{\partial x^\mu}{\partial\sigma^i}\frac{\partial x^\nu}{\partial\sigma^j} \mapsto \frac{\partial x^\mu}{\partial\tilde\sigma^l}\frac{\partial x^\nu}{\partial\tilde\sigma^m}\partial_i\tilde\sigma^l\partial_j\tilde\sigma^m\,. \tag{2.59c}$$

In Minkowski space-time ($G_{\mu\nu} = \eta_{\mu\nu}$) the action is also invariant under space-time Poincaré transformations $x^\mu \mapsto \lambda^\mu{}_\nu x^\nu + a^\mu$.

In the point particle case, there was a natural coupling to the electromagnetic potential, *i.e.* to the one form $A(x^\mu) = A(x^\mu)_\rho dx^\rho$. There exists an analogous coupling allowed for the string, but this time to a two-form potential $B(x^\mu) = \frac{1}{2}B_{\rho\sigma}(x^\mu)dx^\mu \wedge dx^\nu$, which is called the *Kalb-Ramond* potential [7]:[5]

$$\begin{aligned} S_{\text{KR}} &= -\frac{1}{2\pi\alpha'}\int_{\mathfrak{S}} B = -\frac{1}{4\pi\alpha'}\int_{\mathfrak{S}} B_{\rho\sigma}[x^\mu(\tau)]\frac{\partial x^\mu}{\partial\sigma^i}\frac{\partial x^\nu}{\partial\sigma^j} d\sigma^i\wedge d\sigma^j \\ &= -\frac{1}{4\pi\alpha'}\int_{\mathfrak{S}} d\sigma^0 d\sigma^1 \epsilon^{ij} B_{\rho\sigma}\frac{\partial x^\mu}{\partial\sigma^i}\frac{\partial x^\nu}{\partial\sigma^j}\,. \end{aligned} \tag{2.60}$$

As its one-dimensional cousin, the Kalb-Ramond coupling is independent of the parametrization of the worldsheet as it does not depend explicitly on the two-dimensional induced metric h. Furthermore, one can naturally associate to the coupling (2.60) a gauge invariance

$$B_{\mu\nu}\, dx^\mu\wedge dx^\nu \;\mapsto B_{\mu\nu}\, dx^\mu\wedge dx^\nu + d(\Lambda_\nu dx^\nu) = B_{\mu\nu}\, dx^\mu dx^\nu + \partial_\mu\Lambda_\nu\, dx^\mu\wedge dx^\nu\,, \tag{2.61}$$

where the parameter of the gauge transformation is a one-form $\Lambda = \Lambda_\mu dx^\mu$. This transformation leaves invariant (2.60) up to boundary terms using Stokes' theorem:

$$\int_{\mathfrak{S}} B \mapsto \int_{\mathfrak{S}} B + \int_{\mathfrak{S}} d\Lambda = \int_{\mathfrak{S}} B + \int_{\partial\mathfrak{S}} \Lambda\,. \tag{2.62}$$

[5]We use the two-dimensional epsilon symbol with non-zero components $\epsilon^{01} = -\epsilon^{10} = 1$. Note that $(-\det\gamma)^{-1/2}\epsilon^{ij}$ is a two-index contravariant antisymmetric tensor.

Polyakov action

The non-linear Nambu-Goto action (2.58) is not a convenient starting point for quantizing the theory. In analogy with the relativistic particle case, we will introduce an independent worldsheet metric γ and consider instead the action known as the *Polyakov action* [8]:

$$S_P = -\frac{1}{4\pi\alpha'}\int_{\mathfrak{s}} d^2\sigma\sqrt{-\det\gamma}\,\gamma^{ij}G_{\mu\nu}\partial_i x^\mu(\sigma^k)\partial_j x^\nu(\sigma^k)\,, \tag{2.63}$$

where the non-dynamical field $\gamma_{ij}(\sigma^k)$ is determined by its equation of motion. The Polyakov action can be understood as the minimal coupling of a two-dimensional metric to a set of scalar fields, hence is automatically invariant under diffeomorphisms of the worldsheet $\sigma^i \mapsto \tilde{\sigma}^i(\sigma^k)$.

The equations of motion for the scalar fields $x^\mu(\sigma^i)$ are easy to determine. Under an arbitrary variation of the fields $x^\mu \mapsto x^\mu + \delta x^\mu$ the variation of the action is

$$\begin{aligned}\delta S_P = -\frac{1}{2\pi\alpha'}\int_{\mathfrak{s}} & d^2\sigma\sqrt{-\det\gamma}\,\gamma^{ij}G_{\mu\nu}\partial_i x^\mu(\sigma^k)\partial_j\delta x^\nu(\sigma^k)\\ &= \frac{1}{2\pi\alpha'}\int_{\mathfrak{s}} d^2\sigma\,\partial_j\Big(\sqrt{-\det\gamma}\,\gamma^{ij}G_{\mu\nu}\partial_i x^\mu(\sigma^k)\Big)\delta x^\nu(\sigma^k)\\ &\qquad + \frac{1}{2\pi\alpha'}\int_{\mathfrak{s}} d^2\sigma\,\partial_j\big(\sqrt{-\det\gamma}\,\gamma^{ij}G_{\mu\nu}\partial_i x^\mu(\sigma^k)\delta x^\nu(\sigma^k)\big)\,.\end{aligned} \tag{2.64}$$

While the third term is a total derivative, and therefore do not play any role on a closed worldsheet which has no boundaries, the second term gives a Laplace equation :

$$\nabla_i\left(G_{\mu\nu}\nabla^i x^\nu(\sigma^k)\right) = 0\,. \tag{2.65}$$

So, whenever the target space is a flat Minkowski space-time, one can choose $G_{\mu\nu}$ to be constant and the fields $x^\mu(\sigma^k)$ are just free massless fields in two-dimensions.

Let us now prove that the equation of motion of the dynamical metric γ in the Polyakov action (2.63) gives back the Nambu-Goto action (2.58). Under an infinitesimal variation $\gamma \mapsto \gamma + \delta\gamma$, one finds that at first order

$$\delta S_P = -\frac{1}{4\pi\alpha'}\int_{\mathfrak{s}} d^2\sigma\sqrt{-\det\gamma}\left[\frac{\delta(\sqrt{-\det\gamma})}{\sqrt{-\det\gamma}}\gamma^{kl}h_{kl} + \delta\gamma^{ij}G_{\mu\nu}\partial_i x^\mu\partial_j x^\nu\right]. \tag{2.66}$$

Given that $\gamma^{ij}\gamma_{jk} = \delta^i{}_k$ one finds at first order that

$$\gamma^{ij}\delta\gamma^{jk} + \gamma_{jk}\delta\gamma_{ij} = 0 \implies \delta\gamma^{jk} = -\gamma^{ij}\gamma^{kl}\delta\gamma_{il} \tag{2.67}$$

and

$$\frac{\delta(\sqrt{-\det\gamma})}{\sqrt{-\det\gamma}} = \frac{1}{2}\delta\ln(-\det\gamma) = \frac{1}{2}\gamma^{ij}\delta\gamma_{ij}\,. \tag{2.68}$$

We obtain then

$$\delta S_P = -\frac{1}{4\pi\alpha'}\int_{\mathfrak{s}} d^2\sigma\sqrt{-\det\gamma}\left[\frac{1}{2}\gamma^{ij}\gamma^{kl}h_{kl} - \gamma^{ik}\gamma^{jl}h_{kl}\right]\delta\gamma_{ij}\,. \tag{2.69}$$

The vanishing of the first order variation leads therefore to

$$\frac{1}{2}\gamma^{ij}\gamma^{kl}h_{kl} = \gamma^{ik}\gamma^{jl}h_{kl} \implies h_{ij} = \frac{1}{2}\gamma^{kl}h_{kl}\gamma_{ij}\,. \tag{2.70}$$

The determinant of this relation gives

$$\det h = \left(\frac{1}{2}\gamma^{kl}h_{kl}\right)^2 \det\gamma\,, \tag{2.71}$$

from which we deduce that

$$\begin{aligned} S_{\mathrm{P}} &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\det\gamma}\,\gamma^{ij}h_{ij} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\det h}\left(\frac{1}{2}\gamma^{kl}h_{kl}\right)^{-1}\gamma^{kl}h_{kl} \\ &= -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-\det h} = S_{\mathrm{NG}} \end{aligned} \tag{2.72}$$

Hence, at least classically, the Nambu-Goto and the Polyakov actions give equivalent dynamics for the relativistic strings.

The Polyakov action (2.63) is certainly not the most general action on the string worldsheet that one can write. First, the coupling to the Kalb-Ramond field, eq. (2.63), is independent of the worldsheet metric hence takes the same form in the Nambu-Goto and Polyakov formalisms. Second, an acute reader may have wondered why, in the Polyakov action, we did not include a "cosmological constant" term

$$-\frac{1}{4\pi}\int d^2\sigma\sqrt{-\det\gamma}\,\Lambda\,, \tag{2.73}$$

that would be analogous to the mass term $\int d\tau\, e(\tau)\, m^2$ in the worldline action (2.10) for the relativistic particle. Such a term would imply that

$$\frac{1}{2}\gamma^{ij}\gamma^{kl}h_{kl} + \frac{1}{2}\alpha'\Lambda\gamma^{ij} = \gamma^{ik}\gamma^{jl}h_{kl}\,. \tag{2.74}$$

Contracting this equation with γ_{ij} gives

$$\gamma^{kl}h_{kl} + \alpha'\Lambda = \gamma^{kl}h_{kl} \tag{2.75}$$

which has no solutions unless $\Lambda = 0$. This is a peculiarity of string actions, which is not shared with actions of particles or higher-dimensional extended objects.[6] We will understand shortly its significance.

There exists a last possible coupling of the relativistic string that has no analogue in the particle case. From the two-dimensional worldsheet metric γ one can construct the Ricci

[6] Indeed for a p-dimensional extended object, $p+1$ being the dimension of its worldvolume, everything is the same except the contraction with γ_{ij} which gives $\frac{p+1}{2}(\gamma^{kl}h_{kl} + \alpha'\Lambda) = \gamma^{kl}h_{kl}$.

scalar $\mathcal{R}[\gamma]$, and write down a last term in the action of the Einstein-Hilbert type (after all we are considering a dynamical worldsheet metric):

$$\chi(\mathfrak{s}) = \frac{1}{4\pi}\int d^2\sigma\sqrt{-\det\gamma}\,\mathcal{R}[\gamma]\,. \tag{2.76}$$

In short, one has traded the problem of quantizing gravity in four dimensions to the problem of quantizing gravity in two dimensions! Einstein gravity is two dimensions is much simpler, as first it has no propagating degrees of freedom because of diffeorphism invariance (standard counting gives -1 degrees of freedom). The Einstein-Hilbert action is actually a topological invariant of the two-dimensional manifold $\mathfrak{s}$, known as the *Gauss-Bonnet term*. In Euclidean space it is equal to the *Euler characteristic* $\chi(\mathfrak{s})$ of the two-dimensional worldsheet. If the worldsheet is an oriented surface without boundaries, it is given by

$$\chi(\mathfrak{s}) = 2 - 2g \tag{2.77}$$

where g is the number of handles, or "holes", of the surface. For the sphere $g = 0$, the torus $g = 1$, etc... We will come back latter to the significance of these topologies.

There exists a generalization of the Einstein-Hilbert term that involves a coupling to a scalar field in space time $\Phi(x^\mu)$ and that is not topological:

$$S_D = \frac{1}{4\pi}\int d^2\sigma\sqrt{-\det\gamma}\Phi[x^\mu(\sigma^i)]\mathcal{R}[\gamma]\,. \tag{2.78}$$

The field $\Phi(x^\mu)$, which plays an important role in string theory, is called the *dilaton*.

To summarize this discussion, the general fundamental string action is given by the sum of (2.63), (2.63) and (2.78), hence a (1+1)-dimensional quantum field theory on the worlsheet given by:

$$\boxed{\mathcal{S} = -\frac{1}{4\pi\alpha'}\int_{\mathfrak{s}} d^2\sigma\left(\sqrt{-\det\gamma}\,\gamma^{ij}G_{\mu\nu} + \epsilon^{ij}B_{\mu\nu}\right)\partial_i x^\mu\partial_j x^\nu - \frac{1}{4\pi}\int d^2\sigma\sqrt{-\det\gamma}\,\Phi[x^\mu(\sigma^i)]\mathcal{R}[\gamma]} \tag{2.79}$$

This action describes the propagation of a single relativistic string in a background specified by a metric G, a Kalb-Ramond field B and a dilaton Φ. The later two have no obvious interpretation at this stage; note that in four dimensions the Kalb-Ramond field is actually equivalent to a real pseudo-scalar field as its field strength $H = dB$ is Hodge-dual to the differential of a scalar field: $\star H = da$.

2.3 Symmetries

We now turn to the path integral quantization of the bosonic string. To start, one has to pay attention to the symmetries of the theory, in particular to the gauge symmetries that need to be carefully taken care of, as in the example of the point particle that we have dealt with in section 2.1. As there we will consider the path integral with an imaginary time coordinate, *i.e.* we will consider an Euclidean worldsheet of coordinates $(\sigma^1, \sigma^2) = (\sigma^1, -i\sigma^0)$ endowed with an Euclidean metric γ. However we will keep the signature of space-time to be $(-, +, \cdots, +)$.[7]

To simplify the discussion, consider the action of a string action with vanishing Kalb-Ramond field and constant dilaton field:

$$\mathcal{S} = \frac{1}{4\pi\alpha'}\int_{\mathfrak{s}} d^2\sigma\sqrt{\det\gamma}\,\gamma^{ij}G_{\mu\nu}\partial_i x^\mu \partial_j x^\nu + \frac{\Phi_0}{4\pi}\int_{\mathfrak{s}} d^2\sigma\sqrt{\det\gamma}\,\mathcal{R}[\gamma]\,. \tag{2.80}$$

The symmetries of the theory splits into worldsheet and target space symmetries. We will start by looking at the latter. One has first *target-space symmetries* of the action (2.80) corresponding to symmetries of space-time.

If the space-time is Minskowki space-time ($g_{\mu\nu} = \eta_{\mu\nu}$) the action is invariant under Poincaré transformations:

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\mu + a^\mu\ , \quad \Lambda \in SO(1, d-1)\,. \tag{2.81}$$

These are *global symmetries* of the two-dimensional field theory.

Under diffeomorphisms in space-time, *i.e.* infinitesimal field redefinitions $\delta x^\mu = r^\mu[x^\rho]$, the Polyakov action (2.63) keeps the same form if we perform the change of space-time metric $\delta G_{\mu\nu} = -2\nabla_{(\mu} r_{\nu)}$ at the same time.

2.3.1 Worldsheet symmetries

The string action (2.80), or its more general version (2.79), is by design invariant under coordinate transformations, or diffeomorphims of the worldsheet,

$$\sigma^i \mapsto \tilde{\sigma}^i(\sigma^k)\,, \tag{2.82a}$$

$$\gamma_{ij} \mapsto \tilde{\gamma}_{ij} = \frac{\partial\sigma^k}{\partial\tilde{\sigma}^i}\frac{\partial\sigma^l}{\partial\tilde{\sigma}^j}\gamma_{k\ell}\,, \tag{2.82b}$$

being a theory of two-dimensional gravity minimally coupled to scalar fields $x^\mu(\sigma)$.

The Polyakov action has an extra local symmetry, which corresponds to *Weyl transformations* of the world-sheet metric, *i.e.* local scale transformations:

$$\gamma_{ij} \ \mapsto \ e^{2\omega(\sigma^i)}\gamma_{ij}\ , \tag{2.83}$$

where $\omega(\sigma^i)$ is an arbitrary differentiable function on the two-dimensional manifold $\mathfrak{s}$. This property comes from the scaling of the determinant of the metric in d dimensions

$$\det\gamma \overset{\text{Weyl}}{\mapsto} e^{2d\omega}\det\gamma\,, \tag{2.84}$$

which, specialized to two dimensions, implies that the action (2.63) invariant under Weyl transformations. The Kalb-Ramond coupling (2.60) is also by definition invariant being independent of the metric.

[7]As we will see the price to pay will be the appearance of negative norm states. The latter can be fortunately removed by using the gauge symmetry of the theory.

The Ricci scalar transforms simply under a Weyl rescaling of the metric. We leave as an exercise to show that, in d dimensions,

$$\gamma \mapsto e^{2\omega}\gamma \tag{2.85a}$$
$$R[\gamma] \mapsto e^{-2\omega}\left(R[\gamma] - 2(d-1)\nabla^2\omega - 2(d-2)(d-1)\partial_a\omega\partial^a\omega\right) . \tag{2.85b}$$

In two dimensions, this expression implies that $\sqrt{\det\gamma}\, R[\gamma]$ transforms as a total derivative, since $\sqrt{\det\gamma}\,\nabla^2\omega = \partial_i(\sqrt{\det\gamma}\,\nabla^i\omega)$. One concludes that, at the classical level, the dilaton action (2.78) is not invariant under Weyl transformations, unless Φ is a constant, in which case it was expected since the two-dimensional Einstein-Hilbert term (2.76) is topological. We will see later on that, in the quantum theory, the story is somewhat altered by the presence of anomalies.

A careful reader would have noticed that the Weyl symmetry is not present in the original Nambu-Goto action (2.58). One can trace back its origin to the equation of motion for the worldsheet metric, eq. (2.70), which is invariant under Weyl transformations. This feature of the Polyakov action is not problematic. The Weyl symmetry is a gauge symmetry, hence does not really correspond to a symmetry but rather to a redundancy of our description of the theory. In the path integral quantization of the theory, this gauge symmetry will need to be taken care of properly, as the diffeomorphism invariance.

Finally, we notice that the cosmological constant term (2.73) that we considered to include in the action is not Weyl invariant, which explains why this term is forbidden in the first place by the gauge symmetries of the problem.

2.3.2 Gauge choice

The Euclidean path integral of the fundamental string is defined by a functional integral over the fields $\{x^\mu\}$ as well as over the two-dimensional metrics γ – *i.e.* over Euclidian worldsheet geometries – moded out by the volume of the gauge group of the theory, made of two-dimensional diffeomorphisms and Weyl transformations.

As we have done for the point particle, we will properly define this path integral by gauge-fixing and introducing the corresponding Faddeev-Popov determinant. To start with, one associates to each two-dimensional metric γ its *gauge orbit*, the set of its images $\{\gamma^\Xi\}$ under gauge transformations $\Xi = (\Sigma, \Omega)$ composed of diffeomorphisms and Weyl rescalings:

$$\Xi \ : \left\{ \begin{array}{lll} \sigma^i & \mapsto & \Sigma^i(\sigma^k) \\ \gamma_{ij} & \mapsto & \exp(2\Omega) \ \times \ \frac{\partial\sigma^k}{\partial\Sigma^i}\frac{\partial\sigma^l}{\partial\Sigma^j}\gamma_{k\ell} \end{array} \right. \tag{2.86}$$

To define properly the gauge-fixing condition, one has then to understand how to classify all metrics over two-dimensional surfaces into equivalence classes under gauge transformations.

The coarser classification of compact two-dimensional surfaces is according to their *topology*. If we restrict ourselves to oriented surfaces without boundaries, their topology is completely specified by the number of handles in the surface, which is called its *genus* g. Explicitly, surfaces with $g = 0$ have the topology of a sphere, surfaces of genus $g = 1$ have the topology of a torus, etc... As we have no reasons to restrict ourselves to a particular type of

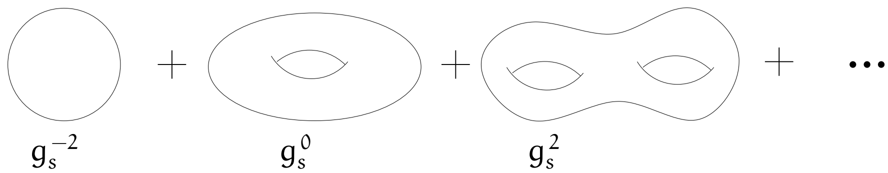

Figure 2.4: *Perturbative expansion of closed string theory.*

surfaces, the Euclidean path integral contains a sum over *topologies* of the worldsheet. For surfaces with fixed topology, the value of the second term of the action (2.80) is fixed:

$$\frac{\Phi_0}{4\pi}\int_{\mathfrak{s}} d^2\sigma\sqrt{\det\gamma}\,\mathcal{R}[\gamma] = \Phi_0\chi(\mathfrak{s}) = \Phi_0(2-2g)\,. \tag{2.87}$$

We have learned something very interesting. Let us define

$$g_s = \exp\Phi_0\,. \tag{2.88}$$

The path integral, in the sector of genus g surfaces, is weighted by the factor g_s^{2-2g}. In other words, the sum over topologies is nothing that the *perturbative expansion*, or loop expansion, of the theory! The parameter g_s is the string coupling constant. This is summarized on figure 2.4.

Having set the topology of the surface by its genus g, one has to find simple representatives in each gauge orbit under the action of diffeomorphisms and Weyl transformations. Locally, as the two-dimensional metric has three independent components, one can use the reparametrization invariance (*i.e.* the two functions $\Sigma^{1,2}(\sigma^k)$) to bring the metric in a conformally flat form:

$$\gamma_{ij}(\sigma_i)\mapsto\exp(2\Omega(\Sigma^i))\delta_{ij}\,. \tag{2.89}$$

This is called the *conformal gauge*. The conformal factor $\exp(2\Omega(\Sigma^i))$ can be naturally offset by a Weyl transformation, leaving a flat Euclidian metric. There could be however a topological obstruction to have a flat metric defined everywhere on the worldsheet (otherwise the Gauss-Bonnet term (2.76) would always vanish).

It will turn out to be convenient to use complex coordinates[8] $w = \sigma^1 + i\sigma^2$ and $\bar{w} = \sigma^1 - i\sigma^2$, and the reference metric

$$ds_{\mathfrak{s}}^2 = 2\,\widehat{\gamma}_{w\bar{w}}\,dw\,d\bar{w} = dw\,d\bar{w}\,. \tag{2.90}$$

In these complex coordinates the integration measure over the worldsheet is

$$\int dw d\bar{w} = 2\int d^2\sigma \tag{2.91}$$

[8] There's actually much more behind this choice than convenience (but I won't develop this aspect in the lectures). We are studying the space of conformal classes of metrics on two-dimensional surfaces which turns out to be the same as the space of **Riemann surfaces**, *i.e.* of complex manifolds of dimension one; an n-dimensional complex manifold is locally equivalent to $\mathbb{C}^n$ and its transition functions are holomorphic.

and the holomorphic and anti-holomorphic derivatives:

$$\partial = \partial_w = \tfrac{1}{2}(\partial_1 - i\partial_2)\ , \quad \bar{\partial} = \partial_{\bar{w}} = \tfrac{1}{2}(\partial_1 + i\partial_2)\,. \tag{2.92}$$

A generic infinitesimal gauge transformation (i.e. an infinitesimal diffeomorphism together with an infinitesimal Weyl transformation) around the flat metric is

$$\delta\gamma_{ij} = 2\delta\omega\delta_{ij} - \delta_{jk}\partial_i\delta\sigma^k - \delta_{ik}\partial_j\delta\sigma^k\,, \tag{2.93}$$

which gives in complex coordinates

$$\delta\gamma_{w\bar{w}} = \delta\omega - \tfrac{1}{2}(\partial\delta w + \bar{\partial}\delta\bar{w})\,, \tag{2.94a}$$

$$\delta\gamma_{ww} = -\partial\delta\bar{w}\,, \tag{2.94b}$$

$$\delta\gamma_{\bar{w}\bar{w}} = -\bar{\partial}\delta w\,, \tag{2.94c}$$

where $\delta\omega$, δw and $\delta\bar{w}$ are arbitrary differentiable functions of w and $\bar{w}$.

Finally, using complex coordinates, the Polyakov action (2.63) in the conformal gauge takes the form

$$\mathcal{S} = \frac{1}{2\pi\alpha'}\int \mathrm{d}w\mathrm{d}\bar{w}\ G_{\mu\nu}\partial x^\mu\bar{\partial}x^\nu\,. \tag{2.95}$$

This theory looks awfully simple in this gauge. Whenever the target space is flat, it seems that string theory reduces to a set of free scalar fields in two dimensions. However, one should not forget that the equations of motion for the worldsheet metric γ should still be satisfied. By definition, the variation w.r.t. the metric of Polyakov action, which is just a theory of two-dimensional gravity coupled to some "matter" fields x^μ, is the stress energy tensor T_{ij}, see chapter 3, eq. (3.17) for more details. Hence in the classical theory of strings we have to enforce the following constraint onto the solutions :

$$T_{ij} = 0\,. \tag{2.96}$$

These constraints, which are known as *Virasoro constraints*, are the analogue of Gauss' law for electromagnetism. Quantizing such constrained theory is a little bit subtle, and can be done in particular using the BRST approach that we will develop in chapter 5.

2.3.3 Residual symmetries and moduli

In the study of the point particle path integral, we have seen two global aspects of the worldline geometry that played an important role: the existence of parameters, or *moduli*, that couldn't be gauged away by the choice of reference metric and the existence of gauge transformations that did not change the one-dimensional metric. In the string theory case these two aspects are still relevant, and needs a little bit more effort to be understood.

The *moduli* are, by definition, given by changes of the metric that are orthogonal to gauge transformations, *i.e.* that cannot be compensated for by a combination of a diffeomorphism and a Weyl transformation. In other words we consider a change of the metric $\delta\gamma_{ij}$ such that

$$\int \mathrm{d}^2\sigma\left(2\delta\omega\delta_{ij} - \partial_i\delta_{jk}\delta\sigma^k - \partial_j\delta_{ik}\delta\sigma^k\right)\delta\gamma^{ij} = 0\,. \tag{2.97}$$

This should hold true for *any* $\delta\omega$ and $\delta\sigma^k$, hence it leads to a pair of independent relations:

$$\mathrm{Tr}\,(\delta\gamma) = 0 \implies \delta\gamma_{\bar{w}w} = 0 \tag{2.98a}$$

$$\partial^i \delta\gamma_{ij} = 0 \implies \begin{cases} \bar{\partial}\delta\gamma_{ww} = 0 \\ \partial\delta\gamma_{\bar{w}\bar{w}} = 0 \end{cases} \tag{2.98b}$$

Solutions of these equations are called *holomorphic quadratic differentials*. The number of independent solutions will give the number of *moduli* of the surface n_μ. In the mathematical literature, the space spanned by these moduli is called the *Teichmüller space*.

A second mismatch between the space of metric and the space of gauge transformations corresponds to combinations of diffeomorphisms and Weyl transformations that leave the metric invariant. From equations (2.94b,2.94c) we learn that they correspond to diffeomorphisms satisfying

$$\partial\delta\bar{w} = \bar{\partial}\delta w = 0\,, \tag{2.99}$$

while the compensating Weyl transformation

$$\delta\omega = \tfrac{1}{2}(\partial\delta w + \bar{\partial}\delta\bar{w}) \tag{2.100}$$

is unambiguously determined by eq. (2.94a).

The solutions of these equations are *holomorphic vector fields*, *i.e.* vectors fields that are defined in any open set by a holomorphic function. The key point here is that we need to find vectors fields that satisfy *globally* this condition on the whole surface, which is a rather strong constraint. These solutions are called the *conformal Killing vectors* (CKV) of the surface; the number of independent CKV will be called n_k.

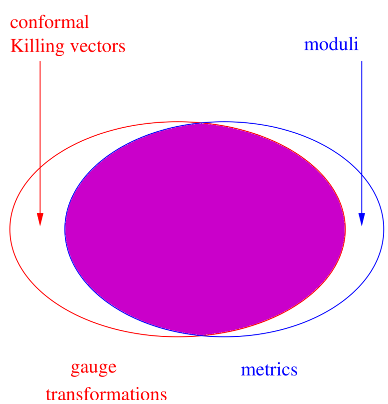

Figure 2.5: Mismatch between integrals over metrics and over the gauge group (diff+Weyl).

The number of moduli n_μ and of conformal Killing vectors n_k for a given surface are not independent but related to each other by the *Riemann-Roch theorem*, in terms the Euler characteristic of the surface, which specifies its topology:

$$n_\mu - n_k = -3\chi(\mathfrak{s}) = 6(g-1)\,. \tag{2.101}$$

The sphere and the two-torus

We will now move away from this rather abstract discussion and derive in detail the moduli and conformal Killing vectors for the two most useful examples, the sphere and the torus.

Genus zero surfaces have the topology of a two-dimensional sphere. The Riemann-Roch theorem (2.101) indicates that $n_\mu - n_k = -6$. A conformally flat metric on the two-dimensional unit sphere is given, in complex coordinates, by[9]

$$ds^2 = \frac{4 dw d\bar{w}}{(1+w\bar{w})^2} \,. \tag{2.102}$$

The coordinates $(w, \bar{w})$ are defined in a patch that excludes the "south pole" of the sphere at $w \to \infty$. A patch including the south pole is covered by the coordinates $(z, \bar{z}) = (1/w, 1/\bar{w})$. The compactification of the complex plane $\bar{\mathbb{C}} = \mathbb{C} \cap \{\infty\}$ is topologically equivalent to the two-sphere; in a way the patch containing the south pole has ben shrunk to the point at infinity.

The sphere has no moduli (in particular the radius can be absorbed by a constant Weyl transformation) and six conformal Killing vectors. Three of them are easy to identify, the generators of the Lie algebra $\mathfrak{so}(3)$. To find all of them, one needs to study the holomorphic vector fields on this manifold. Let us assume that the holomorphic vector field δw admits a holomorphic power series expansion around the north pole $w = 0$:

$$\delta w = c_0 + c_1 w + c_2 w^2 + c_3 w^3 + \cdots \tag{2.103}$$

This holomorphic vector field should be defined everywhere, in particular in the patch around the south pole. Under the coordinate transformation $w \mapsto z = 1/w$, one finds that

$$\delta z = \frac{\partial z}{\partial w} \delta w = -z^2 (c_0 + c_1/z + c_2/z^2 + c_3/z^3 + \cdots) \,, \tag{2.104}$$

hence one gets a globally well-defined holomorphic vector field, in particular at the south pole $z = 0$, provided that $c_n = 0$ for $n \geqslant 3$.

The three complex parameters $\{c_0, c_1, c_2\}$ parametrize generic conformal Killing vectors of the sphere around the identity. Successive actions of the conformal Killing vectors (2.104) define a group by exponentiation, the *conformal Killing group*. One can check – by comparing the multiplication laws – that this group is actually isomorphic to the Möbius group, *i.e.* the group of fractional linear transformations

$$z \mapsto \frac{az+b}{cz+d} \,, \quad a, b, c, d \in \mathbb{C} \,, \quad ad - bc \neq 0 \,. \tag{2.105}$$

Given that the map is invariant under rescalings of the parameters, one can set $ad - bc = 1$ and these transformations define a group isomorphic to $\mathsf{PSL}(2, \mathbb{C})$, the group of complex 2×2 matrices M of determinant one identified under its center $M \mapsto -M$.

[9]Using the coordinate change $w = \tan(\theta/2) e^{i\phi}$ one gets the familiar metric $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$.

Genus one surfaces are topologically equivalent to a two-torus. The Euler characteristic of the two-torus vanishes, hence a two-torus can be endowed with a flat metric $ds^2 = dw d\bar{w}$. The torus has an obvious discrete $\mathbb{Z}_2$ symmetry $w \mapsto -w$ as well as two conformal Killing vectors corresponding to translations along the two one-cycles of the torus. They are described simply by the constant holomorphic vector field $\delta w = c_0$. According to the Riemann-Roch theorem, one expects that the torus has two real moduli.

The torus can be described conveniently as the complex plane quotiented by the discrete identifications

$$w \sim w + 2\pi n u_1 + 2\pi m u_2 \ , \quad n, m \in \mathbb{Z} \, , \tag{2.106}$$

where u_1 and u_2 are complex parameters. By a rescaling of w, accompanied by a constant Weyl transformation, one can get rid of the former hence we consider the quotient

$$w \sim w + 2\pi n + 2\pi m \tau \ , \quad n, m \in \mathbb{Z} \, , \tag{2.107}$$

where we have adopted the standard notation $\tau \in \mathbb{C}$ for the torus modulus. As for the circular worldline in the point particle case, see the discussion above eqn. (2.16), an alternative way to think about the torus is to consider the metric

$$ds^2 = |d\sigma^1 + \tau d\sigma^2|^2 \, , \tag{2.108}$$

with the standard identifications $\sigma^i \sim \sigma^i + 2\pi$. There exists some discrete ambiguity in the identification of the parameter τ. First, the metric (2.108) is invariant under complex conjugation of the parameter τ, hence we can restrict the discussion to $\tau_2 = \Im(\tau) > 0$ (the case $\Im(\tau) = 0$ being degenerate) *i.e.* to the upper half plane $\mathbb{H}$. For a square torus, $\Re(\tau) = 0$, while in general the real part of τ represents the way the circle parametrized by σ^1 is "twisted" before identifying the two enpoints of the cylinder.

The two-torus, being defined as a quotient of the complex plane, is nothing but a two-dimensional lattice, see fig. 2.6. It is obvious that the same lattice is described by replacing

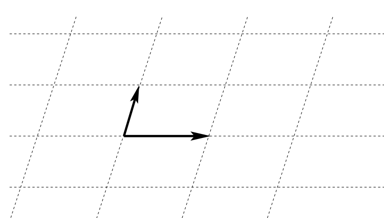

Figure 2.6: *Two-torus as a quotient of the complex plane.*

τ by $\tau + 1$. According to the metric (2.108) it amounts to redefine $\sigma^1 \to \sigma^1 + \sigma^2$, which is compatible with the periodicities of the coordinates.

It it slightly less obvious to realize that another equivalent parametrization of the torus is given by $\tau \mapsto -1/\tau$, if one allows a Weyl rescaling of the metric. Indeed starting from the metric (2.108) one gets

$$\left|d\sigma^1 + \tau d\sigma^2\right|^2 \xmapsto{\tau \mapsto -1/\tau} \left|d\sigma^1 - \tfrac{1}{\tau} d\sigma^2\right|^2 = \frac{1}{|\tau|^2} \left|d\sigma^2 - \tau d\sigma^1\right|^2 \, . \tag{2.109}$$

One sees that, up to a global rescaling of the metric, it amounts to replace $\sigma^1 \to \sigma^2$ and $\sigma^2 \to -\sigma^1$. In other words it exchanges the role of Euclidean worldsheet time and of the space-like coordinate along the string. These two transformations generate the *modular group* $\mathsf{PSL}(2,\mathbb{Z})$, which acts on the modular parameter as

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}\,, \qquad a, b, c, d \in \mathbb{Z} \tag{2.110}$$

This group is indeed the group $\mathsf{SL}(2,\mathbb{Z})$ of 2×2 integer matrices M of determinant one

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, \quad ad - bc = 1\,, \quad a, b, c, d \in \mathbb{Z} \tag{2.111}$$

quotiented by its center, *i.e.* with the identification $M \sim -M$, as replacing (a, b, c, d) by $(-a, -b, -c, -d)$ does not change the action (2.110). To avoid an over-counting in the path integral, we will choose to select a representative of the modular parameter τ into each orbit of the modular group. One can show that every point is the upper half plane $\mathbb{H} > 0$ has a unique antecedent under the modular group in the *fundamental domain* $\mathfrak{F}$, defined by the conditions

$$\mathfrak{F} = \{\tau \in \mathbb{H}\,,\ |\Re(\tau)| \leqslant \tfrac{1}{2}\,,\ |\tau| \geqslant 1\}\,, \tag{2.112}$$

where the boundaries for $\Re(\tau) > 0$ and $\Re(\tau) < 0$ are identified, see fig. (2.7). In other words, the fundamental domain contains a unique point per orbit of $\mathsf{PSL}(2,\mathbb{Z})$.

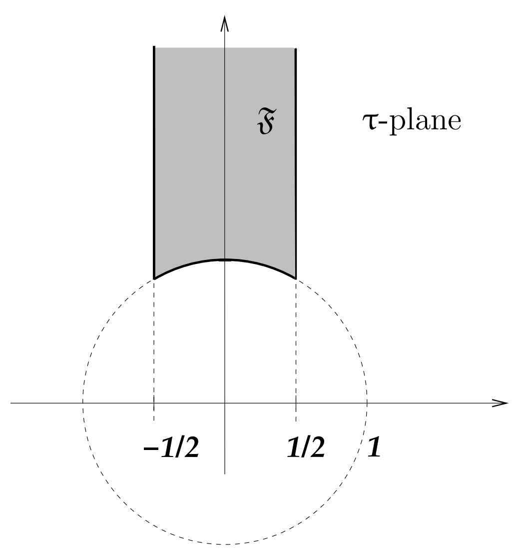

Figure 2.7: *Fundamental domain $\mathfrak{F}$ of the modular group.*

To anticipate a little bit, this technical point will have drastic consequences. Remember that the torus diagram represents the one-loop contribution in the perturbative expansion in string theory, much as a circle represented the one-loop contribution in the point particle case, see section 2.1. We have found there that the UV divergences in QFT were related to circle of perimeter T going to zero size in the worldline formalism. In the string theory case, it would correspond to the limit $\Im(\tau) \to 0$, which is completely excluded from the path integral if we choose to integrate over $\mathfrak{F}$, the fundamental domain! This remarkable feature of string theory, which persists to higher order, indicates that the theory is UV-finite.

2.3.4 Conformal symmetry

Our discussion of conformal Killing vectors – *i.e.* of metric-preserving gauge symmetries – had been *global* as we only focused on holomorphic vector fields (2.99) defined everywhere on the manifold. This was important as we wanted to knew exactly the mismatch between integrating over worldsheet metrics and over diffeomorphisms × Weyl gauge symmetries. There are however important properties of the quantum field theory defined on the worldsheet of the string by the gauge-fixed Polyakov action (2.95) that depend only on the symmetries of the problem in an open set of the complex plane $\mathbb{C}$.

If we allow the transformations (2.99) to be defined only in some patch of the worldsheet, they generate a much bigger group of symmetry. If we consider finite transformations rather than infinitesimal ones, what these equations tell us is that for any holomorphic function f, the change of coordinates

$$w \mapsto f(w) \tag{2.113a}$$
$$\bar{w} \mapsto \bar{f}(\bar{w}) \tag{2.113b}$$
$$dw\, d\bar{w} \mapsto \left|\partial_w f(w)\right|^{-2} dw\, d\bar{w} \tag{2.113c}$$

can be compensated by a Weyl transformation of parameter $\Omega = \log|\partial f|$ such as to leave invariant the flat two-dimensional metric.

This infinite-dimensional symmetry will play a crucial role in the following. The holomorphic coordinate transformations (2.113a,2.113b) preserve the metric up to a local scale transformation as can be seen from eqn. (2.113c), and the set of such transformations corresponds to the *conformal group*, which is indeed of infinite dimension precisely for a space of dimension two.

We will obtain Ward identities associated with these symmetries for the two-dimensional quantum field theory defined on the worldsheet, as such identities do not care about the well-definiteness of the symmetry at the global level. What we will obtain is an infinite number of constraints on this QFT, which is every theoretical physicist dream!

2.4 Polyakov path integral

We have gathered the ingredients to define properly the path integral associated with the Polyakov action (2.95), which is a path integral over two-dimensional scalar fields x^μ and over two-dimensional metrics γ on Euclidean worldsheets.

We have learned first in section 2.3 that the path integral splits into a sum over topologies, *i.e.* over surfaces of different genera g. In each sector we integrate over metrics γ_g of surfaces with a given genus. Accordingly we are considering the formal vacuum amplitude

$$Z_1 = \sum_{g=0}^{\infty} g_s^{2g-2} \int \frac{\mathcal{D}\gamma_g}{\mathrm{Vol}(\text{diff} \times \text{Weyl})} \int \mathcal{D}x \, \exp\Big(- S_{\mathrm{P}}[\gamma_g, x] \Big) \,, \tag{2.114}$$

that receives contributions from single string worldsheets.

As in the point particle case, we have formally divided the integral over the metrics γ_g by the volume of the gauge group $\mathrm{Vol}(\text{diff} \times \text{Weyl})$ in order to take care of the gauge redundancies of the formulation of the theory. As there we will define properly this integral by the Faddeev-Popov method.

Inequivalent gauge orbits of metrics γ_g under diff×Weyl gauge transformations are labeled by a finite set of n_μ parameters m_ℓ, the *moduli*. Along each of these orbits one can choose a reference metric $\widehat{\gamma}_g(m_\ell)$, whose image under a transformation Ξ will be denoted $\widehat{\gamma}_g^{\,\Xi}(m_\ell)$.

We want to trade the integral over the metrics γ_g by the product of an integral over the moduli and an integral over the gauge group, but the latter contains elements that do not change the reference metric, the *conformal Killing vectors*. In the point particle case this extra symmetry was taken care of by fixing the image of the origin under the diffeomorphisms of the worldline. In the present case we have to discuss separately the different topologies, as the outcome will be different. As before we will focus on the two most important cases, the sphere and the torus.

2.4.1 Path integral on the sphere

Surfaces of genus zero have positive curvature, see eqn. (2.76), and are all diff.× Weyl equivalent to the round unit two-sphere, as the latter has no moduli, see section 2.3.

The conformal Killing vectors on the two-sphere form a group isomorphic to $\mathsf{PSL}(2,\mathbb{C})$, the Möbius group, which is a non-compact Lie group of complex dimension three. One could split the integral over diffeomorphisms as an integral over rotations and over diffeomorphisms keeping fixed the origin of the coordinates $w = \bar{w} = 0$ (as we did for the point particle), taking care of the rotation subgroup $\mathsf{SO}(3) \subset \mathsf{PSL}(2,\mathbb{C})$ but not of the full Möbius symmetry.

The vacuum amplitude path integral (2.114) contains the inverse of the volume of the Diff. × Weyl gauge group and therefore retains a factor of $1/\mathrm{Vol}\big(\mathsf{PSL}(2,\mathbb{C})\big)$ which vanishes, as the Möbius group is non-compact. We learn that the vacuum amplitude on the sphere, *i.e.* at tree level, vanishes, as it should be because otherwise it would mean that we don't expand around a vacuum of the theory.[10]

Of course, string theory, as QFT, is not just about computing the vacuum amplitude. Physical observables correspond generically to time-ordered correlation functions of gauge-invariant observables of the theory. Because of diffeomorphism invariance, the observables take the form

$$\mathcal{O}_k = \int \mathrm{d}^2 w \sqrt{\det \gamma}\, \mathcal{V}_k[x^\mu(w,\bar{w})] \,, \tag{2.115}$$

[10] As we will see below, the same conclusion can be obtained by noting that the zero-modes of the ghost path integral measure will not be "saturated" by the appropriate ghost insertions.

where $\mathcal{V}_k$ is some functional of the fields $x^\mu(w, \bar{w})$ that transforms as a scalar under world-sheet diffeomorphisms; other constraints should be imposed on these functionals, and will be discussed later. Then we have to consider path integrals of the form

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{g=0} \sim g_s^{-2} \int \mathcal{D}\gamma_0 \mathcal{D}x \prod_{k=1}^{n} \int d^2 w_k \sqrt{\det \gamma}\, \mathcal{V}_k e^{-S_P[x,\gamma_0]} . \tag{2.116}$$

In this context the gauge-fixing problem that we had for the vacuum amplitude is easy to solve. If the number n of operators is larger or equal to three, the Möbius symmetry is completely fixed by setting the positions $(w_k, \bar{w}_k)$ of three operators to arbitrary fixed values $(\widehat{w}_k, \widehat{\bar{w}}_k)$ instead of integrating over them.

To formulate the problem a bit differently, the integral $\int \mathcal{D}\gamma_0 \prod_{k=1}^{3} d^2 w_k$ over the metrics and the position of three operators covers the whole gauge group, hence can be traded for an integral $\int \mathcal{D}\Xi$ which cancels out completely the volume of the gauge group in the path integral (2.114).

We define then the Faddeev-Popov determinant of string theory on the sphere in terms of the path integral over the gauge group:[11]

$$\frac{1}{\Delta_{\text{FP}}(\gamma_0)} := \int \mathcal{D}\Xi\ \delta(\gamma_0 - \widehat{\gamma}_0^{\Xi}) \prod_{k=1}^{3} \delta\left(w_k - \widehat{w}_k^{\Xi}\right) \delta\left(\bar{w}_k - \widehat{\bar{w}}_k^{\Xi}\right) \tag{2.117}$$

where $\widehat{w}_k$ are arbitrary positions and $\widehat{w}_k^{\Xi}$ their images under gauge transformations (diffeomorphisms of the surface). We obtain for the full path integral at tree-level

$$\begin{aligned}\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{g=0} &= g_s^{-2} \int \frac{\mathcal{D}\gamma_0\, \mathcal{D}\Xi}{\textsc{Vol}(\text{diff} \times \text{Weyl})} \Delta_{\text{FP}}(\gamma_0) \prod_{k=1}^{n} \int d^2 w_k \sqrt{\det \gamma_0} \int \mathcal{D}x \prod_{k=1}^{n} \mathcal{V}_k[x(w_k, \bar{w}_k)] \\ &\qquad \times \exp\Big(- S_P[\gamma_0, x]\Big)\, \delta(\gamma_0 - \widehat{\gamma}_0^{\Xi}) \prod_{k=1}^{3} \delta\left(w_k - \widehat{w}_k\right) \\ &= g_s^{-2} \int \mathcal{D}x \ \exp\Big(- S_P[\widehat{\gamma}_0, x]\Big)\, \Delta_{\text{FP}}(\widehat{\gamma}_0) \prod_{k=1}^{3} \sqrt{\det \widehat{\gamma}_0} \mathcal{V}_k[x(\widehat{w}_k, \widehat{\bar{w}}_k)] \prod_{k=4}^{n} \mathcal{O}_K .\end{aligned} \tag{2.118}$$

By diffeomorphisms and Weyl transformations, one can bring the metric on the sphere to a flat metric, the price to pay being that the south pole is mapped to $|w| \to \infty$. This is not a problem as long as we consider the compactification of the complex plane, $\bar{\mathbb{C}} := \mathbb{C} \cap \infty$, which is topologically a two-sphere.

In order to evaluate the Faddeev-Popov determinant, we first recall that the argument of the distribution $\delta(\gamma_0 - \widehat{\gamma}_0^{\Xi})$ around the reference metric is given by eqns. (2.94). We have

[11] Actually one should write $\Delta_{\text{FP}}(\gamma_0; w_k, \bar{w}_k)$ as it depends also of the position of the fixed local operators but we have chosen not to clutter the equations too much.

then

$$\frac{1}{\Delta_{\text{FP}}(\widehat{\gamma}_0)} = \int \mathcal{D}\delta\omega\mathcal{D}\delta w\mathcal{D}\delta\bar{w}\ \delta\big[\delta\omega - \tfrac{1}{2}(\partial\delta w + \bar{\partial}\delta\bar{w})\big]\ \delta\big[\partial\delta\bar{w}\big]\ \delta\big[\bar{\partial}\delta w\big]$$
$$\prod_{k=1}^{3} \delta\left(\delta w(\widehat{w}_k, \widehat{\bar{w}}_k)\right)\ \delta\left(\delta\bar{w}(\widehat{w}_k, \widehat{\bar{w}}_k)\right)\ . \quad (2.119)$$

All these Dirac distributions are exponentiated by means of a corresponding Lagrange multiplier:

$$\frac{1}{\Delta_{\text{FP}}(\widehat{\gamma}_0)} = \int \mathcal{D}\delta\omega\mathcal{D}\delta w\mathcal{D}\delta\bar{w}\mathcal{D}\eta\ \mathcal{D}\beta\ \mathcal{D}\bar{\beta}\ \prod_{k=1}^{3} d\phi_k d\widetilde{\phi}_k$$
$$e^{2i\pi\int d^2w\,\eta\left(\delta\omega - \frac{1}{2}(\partial\delta w + \bar{\partial}\delta\bar{w})\right)} e^{2i\pi\int d^2w\,\beta\bar{\partial}\delta w} e^{2i\pi\int d^2w\,\bar{\beta}\partial\delta\bar{w}} e^{2i\pi\phi_k\delta w(\widehat{w}_k, \widehat{\bar{w}}_k)} e^{2i\pi\widetilde{\phi}_k\delta\bar{w}(\widehat{w}_k, \widehat{\bar{w}}_k)} \quad (2.120)$$

We eventually need to insert the FP determinant rather than its inverse as for the particle, therefore we substitute for the variables $(\delta\omega, \delta w, \delta\bar{w}, \eta, \beta, \bar{\beta}, \phi_k, \bar{\phi}_k)$, the Grassmann variables $(\kappa, c, \widetilde{c}, \zeta, b, \widetilde{b}, \psi_k, \bar{\psi}_k)$. We can compute immediately the integrals over κ, ζ, ψ_k and $\bar{\psi}_k$ which gives, after a rescaling of the fields, the relatively simple expression:[12]

$$\Delta_{\text{FP}}(\widehat{\gamma}_0) = \int \mathcal{D}b\mathcal{D}\widetilde{b}\mathcal{D}c\mathcal{D}\widetilde{c}\ e^{-\int\frac{d^2w}{2\pi}\,(b\bar{\partial}c + \widetilde{b}\partial\widetilde{c})} \prod_{k=1}^{3} c(\widehat{w}_k)\widetilde{c}(\widehat{\bar{w}}_k) \quad (2.121)$$

To summarize the sphere path integral of string theory is given in its full glory by the expression $(n > 3)$:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{g=0} = g_s^{-2} \int \mathcal{D}x\ \mathcal{D}b\mathcal{D}\widetilde{b}\mathcal{D}c\mathcal{D}\widetilde{c}\ e^{-\int\frac{d^2w}{2\pi\alpha'}\ g_{\mu\nu}\partial x^\mu\bar{\partial}x^\nu}\ e^{-\int\frac{d^2w}{2\pi}\,(b\bar{\partial}c + \widetilde{b}\partial\widetilde{c})}$$
$$\prod_{k=1}^{3} c(\widehat{w}_k)\widetilde{c}(\widehat{\bar{w}}_k)\mathcal{V}_k[x^\mu(\widehat{w}_k, \widehat{\bar{w}}_k)] \prod_{k=4}^{n} \mathcal{O}_k\,, \quad (2.122)$$

where $\{\widehat{w}_k\}$ are arbitrary positions, that we take usually to be $\{0, 1, +\infty\}$.

2.4.2 Path integral on the torus

The second important case is the two-torus, that corresponds to the one-loop amplitude in string theory, and present some differences with the previous case.

Gauge-inequivalent surfaces of genus one correspond to two-tori characterized by one complex parameter τ in the upper half-plane $\mathbb{H} = \{z \in \mathbb{C}, \Im(z) > 0\}$. Because of the ambiguity

[12] If we reinstall the index notation, we start with $\beta_{ij}\partial^i\delta\sigma^j$ in the expansion of the Faddeev-Popov determinant (2.117), which gives a ghost Lagrangian of the form $b_{ij}\partial^i c^j$, and in complex coordinates the components are given below by eqn. (2.130).

in assigning a modular parameter τ to a given two-torus – or equivalently to a given two-dimensional lattice – one restricts further τ to be in the fundamental domain $\mathfrak{F}$ of the modular group, see eqn. (2.112). A two-dimensional metric on the torus of complex modular parameter

$$\tau = \tau_1 + i\tau_2 \, , \quad |\tau_1| \leqslant \tfrac{1}{2} \, , \quad \tau_1^2 + \tau_2^2 \geqslant 1 \tag{2.123}$$

is then given by the flat metric (2.108) with $\sigma^i \sim \sigma^i + 2\pi$.

The two-torus has two real conformal Killing vectors, that are easy to understand, as they correspond to translations along σ^1 and along σ^2. These translations form a group isomorphic to $U(1)^2$, whose volume is $4\pi^2\tau_2$, the area of the torus of metric (2.108). It means that the integral $\int d^2\tau \, \mathcal{D}\Xi$ over the moduli and over the gauge group covers more that the integral $\int \mathcal{D}\gamma_1$ over the metrics on genus one surfaces.

Fixing this extra symmetry is possible, even for the vacuum amplitude in the following way (this is the same method that we used for the point particle vacuum amplitude). A general diffeomorphism is given by two functions $\Sigma^i(\sigma^k)$ that will not generically map the origin of the coordinates $\sigma^1 = \sigma^2 = 0$ to the origin, as $\Sigma^i(0,0) \neq 0$ in general. Because of translation invariance is possible to restrict the path integral to diffeomorphisms preserving the origin, *i.e.* such that $\Sigma^i(0,0) = 0$ or, say differently, for every diffeomorphism Σ^i one can translate back the image of the origin to $\Sigma^i(0,0) = 0$, which will select an element of the translation group . In this way, the spurious gauge freedom will be removed. In addition to these continuous symmetries, the torus metric is naturally invariant under the $\mathbb{Z}_2$ symmetry $\sigma^i \mapsto -\sigma^i$, which doubles the volume of the group of metric-preserving gauge transformations.

These considerations lead to the following expression for the Faddeev-Popov determinant on the two-torus, as an integral over the gauge transformations $\Xi = (\Sigma, \Omega)$:

$$\frac{1}{\Delta_{\text{FP}}(\gamma_1)} := \int_{\mathfrak{F}} d^2\tau \int \mathcal{D}\Xi \; \delta\Big(\gamma_1 - \widehat{\gamma}_1^{\,\Xi}(\tau)\Big)\delta\big(\Sigma^1(0)\big)\delta\big(\Sigma^2(0)\big) \, , \tag{2.124}$$

with the reference metric $\widehat{\gamma}_1(\tau)$ obtained from eqn. (2.108)

$$\widehat{\gamma}_1(\tau) = \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix} . \tag{2.125}$$

Expanding the Dirac distribution near the reference metric gives

$$\begin{aligned} &\delta\Big(\gamma_1 - \widehat{\gamma}_1^{\,\Xi}(\tau)\Big) = \\ &\quad \delta\Big(2\delta\omega - 2\partial_1(\delta\sigma^1 + \tau_1\delta\sigma^2)\Big)\delta\Big(2|\tau|^2\delta\omega - 2\partial_2(|\tau|^2\delta\sigma^2 + \tau_1\delta\sigma^1) + 2(\tau_1\delta\tau_1 + \tau_2\delta\tau_2)\Big) \\ &\qquad \delta\Big(2\delta\omega\tau_1 - \partial_1(|\tau|^2\delta\sigma^2 + \tau_1\delta\sigma^1) - \partial_2(\delta\sigma^1 + \tau_1\delta\sigma^2) + \delta\tau_1\Big) \end{aligned} \tag{2.126}$$

As before we introduce Lagrange multipliers fields. Compared to the sphere case as the reference metric is not diagonal it will be technically slightly more cumbersome. We introduce a symmetric two-index tensor of Lagrange multipliers of components β^{ij} as well as a one-form of components η_i. It gives

$$\frac{1}{\Delta_{\text{FP}}(\widehat{\gamma}_1(\tau))} = \int d^2\delta\tau\, \mathcal{D}\delta\omega\, \mathcal{D}\delta\sigma^i\, \mathcal{D}\beta^{ij} d\eta^i \, \exp\left(2i\pi\eta_i\delta\sigma^i(0)\right)$$
$$\exp 2i\pi\tau_2 \int d^2\sigma \Big\{ \beta^{11}\left(\delta\omega - \partial_1(\delta\sigma^1 + \tau_1\delta\sigma^2)\right) + \beta^{22}\left(|\tau|^2\delta\omega - \partial_2(|\tau|^2\delta\sigma^2 + \tau_1\delta\sigma^1) + (\tau_1\delta\tau_1 + \tau_2\delta\tau_2)\right)$$
$$+ \beta^{12}\left(2\delta\omega\tau_1 - \partial_1(|\tau|^2\delta\sigma^2 + \tau_1\delta\sigma^1) - \partial_2(\delta\sigma^1 + \tau_1\delta\sigma^2) + \delta\tau_1\right)\Big\} \quad (2.127)$$

Integrating over $\delta\omega$ imposes that the tensor β is traceless, *i.e.* that $\beta^{ij}\widehat{\gamma}_1(\tau)_{ij} = 0$. The remaining path integral in $\mathcal{D}\beta$ will be therefore on traceless tensors only. In order to get the FP determinant rather that its inverse, we replace $(\beta^{ij}, \delta\sigma^i, \eta^i, \delta\tau_\ell)$ by Grassmann variables $(b^{ij}, c^i, \psi^i, \kappa_\ell)$ and get after rescaling of the fields

$$\Delta_{\text{FP}}(\widehat{\gamma}_1(\tau)) = \int d^2\kappa\, \mathcal{D}c^i\, \mathcal{D}b^{ij} d\psi_i \, \exp\left(\psi_i c^i(0)\right) \exp\Big\{ -\frac{1}{2\pi}\int d^2\sigma \sqrt{\det \widehat{\gamma}_1(\tau)}\; b^{ij}\partial_i c_j \Big\}$$
$$\times \exp\Big\{ \frac{1}{2\pi}\int d^2\sigma \sqrt{\det \widehat{\gamma}_1(\tau)}\kappa_\ell b^{ij}\partial_{\tau_\ell}\widehat{\gamma}_1(\tau)_{ij}\Big\}, \quad (2.128)$$

which can be further simplified by integrating over κ_ℓ and over ψ^i.

Inserting this result in the Polyakov path integral (2.114) one gets finally the vacuum amplitude as[13]

$$Z_1 = \int_{\mathfrak{F}} \frac{d\tau d\bar{\tau}}{16\pi^2\tau_2} \int \mathcal{D}x^\mu\, \mathcal{D}b^{ij}\, \mathcal{D}c_i \, \exp\Big\{ -\frac{\tau_2}{4\pi}\int d^2\sigma \left(b^{ij}\partial_i c_j + \tfrac{1}{\alpha'} g_{\mu\nu}\widehat{\gamma}_1^{ij}\, \partial_i x^\mu \partial_j x^\nu\right)\Big\}$$
$$c^1(0)c^2(0) \times \frac{\tau_2}{4\pi}\int d^2\sigma\, b^{ij}\partial_\tau \widehat{\gamma}_1(\tau)_{ij} \times \frac{\tau_2}{4\pi}\int d^2\sigma\, b^{ij}\partial_{\bar{\tau}} \widehat{\gamma}_1(\tau)_{ij} \quad (2.129)$$

The dependence in $1/\tau_2$ in the measure of integration over the modulus τ comes from the volume of the group of translations along the torus, since have we have explained we don't integrate over this part of the gauge group. As we will see this factor ensures that the result is invariant under the modular group $\mathsf{PSL}(2,\mathbb{Z})$.

It is finally convenient to come back to complex coordinates $(w,\bar{w}) = (\sigma^1 + \tau\sigma^2, \sigma^1 + \bar{\tau}\sigma^2)$. We write then

$$b_{ww} = b\ , \quad b_{\bar{w}\bar{w}} = \widetilde{b}\ , \quad c^w = c\ , \quad c^{\bar{w}} = \widetilde{c} \quad (2.130)$$

The insertion in $b(w,\bar{w})$ and $\widetilde{b}(w,\bar{w})$ takes in this basis the form

$$\frac{1}{2\pi\tau_2}\int d^2w\, b(w,\bar{w})\; \frac{1}{2\pi\tau_2}\int d^2w\, \widetilde{b}(w,\bar{w})\ , \quad (2.131)$$

[13]The exact normalization of the result could be obtained by checking carefully how the insertions in the path integral are normalized, however since the original path integral is ill-defined, and, since we did several field rescalings while manipulating formally the path integral, the honest way to get the factor right is to ask that the end result is properly normalized if interpreted as a partition function. Concretely if one chooses a fully compact space-time one can request that the vacuum appears with degeneracy one.

As we will see later, one can replace $b(w,\bar{w})$ by its value at any given point, for instance $w=\bar{w}=0$, as the non-zero modes of the field don't contribute to the path integral (2.129). Therefore we can replace $\int d^2w\, b \to 4\pi^2\tau_2 b(0)$, and the same for the $\widetilde{b}$ insertion.

We get the final result for the vacuum amplitude of string theory at one-loop as

$$\boxed{Z_1 = \int_{\mathfrak{F}} \frac{d^2\tau}{4\tau_2} \int \mathcal{D}x^\mu\, \mathcal{D}b\mathcal{D}\widetilde{b}\, \mathcal{D}c\mathcal{D}\widetilde{c}\ \ c(0)\widetilde{c}(0)\, b(0)\widetilde{b}(0) e^{-\int \frac{d^2w}{2\pi}\left(b\bar{\partial}c + \widetilde{b}\partial\widetilde{c} + \frac{1}{\alpha'}g_{\mu\nu}\partial x^\mu \bar{\partial} x^\nu\right)}} \tag{2.132}$$

The dependence of the integrand in the modulus τ is hidden in the periodicity of the complex variable $w \sim w+1 \sim w+\tau$.

If one wants to compute a different observable as an n-point function $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{g=1}$, it is enough to insert the operators $\mathcal{O}_k$ in the path integral above, as no gauge-fixing of the position of some operators is needed in the present case.

This result is close to the sphere path integral (2.122), the differences reflecting the number of moduli and conformal Killing vectors in each case. One can generalize of course this discussion to surfaces of higher genera, but a rigorous presentation would be rather technical.

In this chapter we have assumed that the gauge symmetries of the classical theories, diffeomorphisms and Weyl transformations, were also valid in the quantum theory. As we shall see, the latter may be violated by an *anomaly* that put the theory in danger of being inconsistent. Before proceeding to this computation, we will introduce in the next chapter the powerful methods of two-dimensional conformal field theory.

2.5 Open strings

We will close this chapter with a very brief overview of open strings. Most of the material developed in the previous sections is relevant for open strings, however with some key modifications.

First, the worldsheet of an open string is a strip rather than a cylinder, see fig. 2.8. One has therefore a map :

$$[0,\pi]\times\mathbb{R} \hookrightarrow \mathcal{M} \tag{2.133}$$

$$(\sigma,\tau) \mapsto x^\mu(\sigma,\tau)\,, \tag{2.134}$$

where the set of functions $\{x^\mu(\sigma,\tau), \mu=0,\ldots,D-1\}$ have some boundary conditions at the ends of the interval, $\sigma\in\{0,\pi\}$, that we will specify below.

Let us consider a classical open string in a generic curved space-time. It is governed by the same Polyakov action as the bosonic string :

$$S_P = -\frac{1}{4\pi\alpha'}\int_{\mathfrak{s}} d^2\sigma \sqrt{-\det\gamma}\,\gamma^{ij}G_{\mu\nu}\partial_i x^\mu \partial_j x^\nu\,, \tag{2.135}$$

If one considers the variation of the action w.r.t. the fields x^μ, one picks in the present case

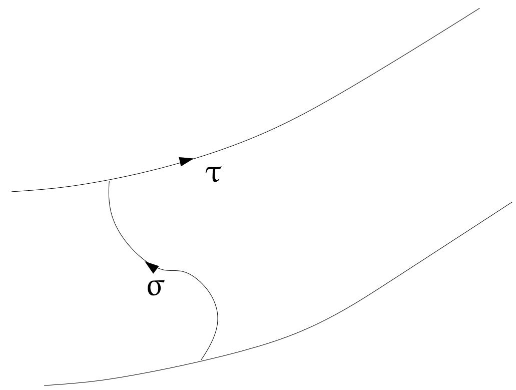

Figure 2.8: *Open string worldsheet.*

a boundary term :

$$\delta S_{\text{P}} = \frac{1}{2\pi\alpha'}\int_{\mathfrak{s}} d^2\sigma\, \partial_j\left(\sqrt{-\det\gamma}\,\gamma^{ij}G_{\mu\nu}\partial_i x^\mu\right)\delta x^\nu - \frac{1}{2\pi\alpha'}\int_{\partial\mathfrak{s}} d\tau\,\sqrt{-\det\gamma}\,\gamma^{0i}G_{\mu\nu}\partial_i x^\mu \delta x^\nu\,. \quad (2.136)$$

While the first term vanishes if the fields x^μ satisfy the equations of motion (2.65), one should impose suitable boundary conditions on the boundary of the worlsheed $\partial\mathfrak{s}$ ($i.e.$ for $\sigma = 0$ and $\sigma = \pi$) in order to cancel the boundary term. The possible choices are :

- **Dirichlet boundary conditions**. They are defined as $\delta x^\nu(\sigma = 0, \pi) = 0$; in this case the endpoints of the string cannot move.
- **Neumann boundary conditions**. They are defined as $\partial_i x^\mu(\sigma = 0, \pi) = 0$; in this case the endpoints of the string are free to move since there are no constraints on δx^ν.

If one wants to retain full Lorentz invariance in the target space-time, it is natural to take the same boundary conditions in all directions, *i.e.* for all fields x^μ. Then the Dirichlet boundary conditions looks rather unnatural, as the endpoints of the strings would be stuck in particular in the time direction. Therefore the natural boundary conditions for the open strings, at this stage, are *Neumann boundary conditions for all fields* x^μ.

As for closed strings, one can consider open strings coupled to more general background fields. The action (2.79) that describes the couplings of strings to a metric $G_{\mu\nu}$, a Kalb-Ramond field $B_{\mu\nu}$ and a dilaton $\Phi(x^\mu)$ receives two new contributions. First, one can write a curvature term on the curve defined by a boundary, known as geodesic curvature[14] which leads to a new boundary couplings to the dilaton :

$$\mathcal{S}_{\text{gc}} = -\frac{1}{2\pi}\int_{\partial\mathfrak{s}} ds\, \Phi(x^\mu)k(s)\,. \quad (2.137)$$

[14] The geodesic curvature of a boundary defined as $k = \pm t^a n_b \nabla_a t^b$, where t^a is a unit vector tangent to the boundary and n^a is a vector orthogonal to it pointing outwards. The plus and minus signs correspond respectively to time-like and space-like boundaries.

On an Eulidean world-sheet and with constant dilaton, this term and the other dilaton coupling combine to the Euler characteristic of a surface with boundaries :

$$\chi(\mathfrak{s}) = \frac{1}{4\pi}\int d^2\sigma\sqrt{\det\gamma}\,\mathcal{R}[\gamma] + \frac{1}{2\pi}\int_{\partial\mathfrak{s}} ds\, k(s)\,. \tag{2.138}$$

For a surface with g handles and b boundaries, the Euler characteristic is given by :

$$\chi(\mathfrak{s}) = 2 - 2g - b\,. \tag{2.139}$$

Compared to the perturbative expansion of a theory of closed strings, see fig. 2.4, the perturbative expansion of a theory containing open strings receive new types of contributions corresponding to surfaces with boundaries, see figure 2.9. Restricting as before to compact surfaces, the first open string contributions are a disk ($\chi = 2 - 0 - 1 = 1$) and a cylinder ($\chi = 2 - 0 - 2 = 0$). As we will see, the disk is associated with tree-level contributions

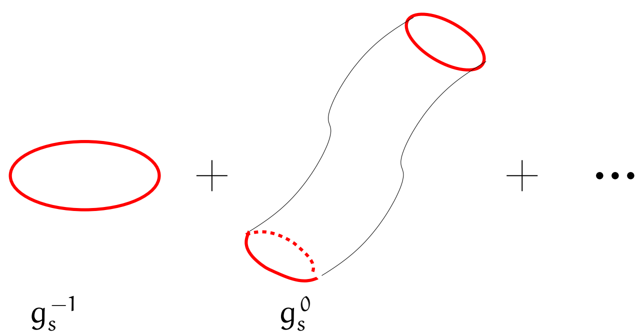

Figure 2.9: *Perturbative expansion for open strings.*

while the cylinder is associated with a one-loop contribution, if one considers the periodic coordinate on the cylinder to be associated with Euclidean time on the worldsheet.

2.5.1 D-branes

D-branes appear quite naturally in string theory if one forgoes Lorentz invariance in the D-dimensional target space-time. One may impose only Lorentz invariance in $p+1$ space-time dimensions, in which case one can choose the following boundary conditions :

- Neumann boundary conditions for x^μ, $\mu = 0, \ldots, p$
- Dirichlet boundary conditions for x^M, $M = p+1, \ldots, D-1$.

Hence, at a given time, the endpoints of the open strings are attached to a space-like sub-manifold of dimension p. This sub-manifold, together with time, can be viewed as the world-volume of a new extended object of string theory, known as a *D*p-brane (where *D* stands, of course, for Dirichlet).[15] In the quantum theory this extended object is not rigid,

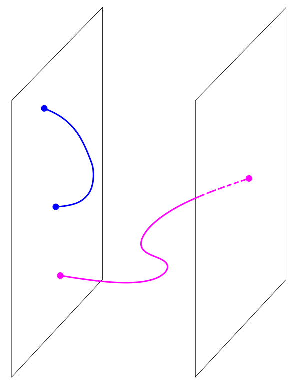

Figure 2.10: *Open string with both ends on the same D-brane (blue) and ending on two different D-branes (magenta).*

but rather acquires dynamical properties. Open strings can have either both ends attached to the same D-brane or each end of a different *D*-brane, see fig. 2.10.

On top of the boundary curvature coupling, see eqn. (2.137), another type of boundary term is allowed for the action of an open string. Let us consider for the moment that the open string has Neumann boundary conditions along all space-like directions, *i.e.* that preserving Lorentz invariance in D-dimensional space-time. Since each boundary of the string sweeps a worldline in space-time, it couples naturally to a one-form, *i.e.* to an Abelian gauge field, as the point-particle example studied in section 2.1. We supplement therefore the Polyakov action of the string by the following contribution :

$$\mathcal{S}_a = \int_{\partial\mathfrak{s}} A_\mu dx^\mu \,. \tag{2.140}$$

If the open string ends on a *D*p-brane (with $p < D-1$), it couples to an electromagnetic fields in $p+1$ dimensions. One can view this gauge field as living on the world-volume of the *D*p-brane. If both ends of the string are on different *D*-branes, then each end couples to a different gauge field, each living on the corresponding worldvolume.[16]

Invariance of the coupling w.r.t. gauge transformations of the gauge fields is immediate, as in the point particle case. There is however another type of gauge invariance that we should consider here. The variation of the Polyakov action under a gauge transformation of the Kalb-Ramond field $B_{\mu\nu}$ was given, see eqn. (2.62), by

$$\int_{\mathfrak{s}} B \mapsto \int_{\mathfrak{s}} B + \int_{\partial\mathfrak{s}} \Lambda_\mu dx^\mu \,. \tag{2.141}$$

On a worldsheet with boundaries, the second term is not zero, and should be canceled against the variation of the Abelian gauge field coupling 2.5.1. Therefore the general transformation

[15] Whenever one chooses Neumann boundary conditions in all directions, one can consider that all space is filled by a *D*-brane.

[16] Whenever the positions of N *D*p-branes coincide in all their transverse dimensions, the gauge symmetry $U(1)^N$ is actually enhanced to $U(N)$ and the gauge theory becomes non-Abelian.

of the gauge field is

$$A_\mu \mapsto A_\mu + \partial_\mu \lambda - \frac{1}{2\pi\alpha'}\Lambda_\mu , \tag{2.142}$$

where the scalar λ is associated with the gauge symmetry of the Maxwell field A_μ and the one-form $\Lambda_\mu dx^\mu$ to the generalized gauge symmetry of the Kalb-Ramond field $B_{\mu\nu}$. Notice that the field strength $F_{\mu\nu}$ of the Maxwell field is not invariant under the latter. Rather, the invariant combination under both gauge symmetries is

$$\mathcal{F}_{\mu\nu} = B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu} , \tag{2.143}$$

which is therefore the natural object that should appear in low-energy effective actions for *D*-branes degrees of freedom.

2.5.2 Path integral

We end this section with a brief discussion about the path integral quantization of open strings. The story is very similar to the case of closed strings, except that the number of moduli and conformal Killing vectors differ for the surfaces with boundaries under consideration. The Riemann-Roch theorem still holds, with now

$$n_\mu - n_k = -3\chi(\mathfrak{s}) = 3(2g + b - 2) . \tag{2.144}$$

The first surface under consideration is the *disk*, with $g = 0$ and $b = 1$ hence $n_\mu - n_k = -3$. As the unit sphere, the disk has no moduli (since a change of radius can be absorbed by a Weyl rescaling). To obtain the conformal Killing vectors, one can obtain the unit disk from the unit sphere using the identification :

$$z \sim 1/\bar{z} . \tag{2.145}$$

This identifies pairwise points inside and outside the unit disk in the complex plane, and the boundary of the disk $|z| = 1$ is left invariant. The conformal Killing vectors of the disk are therefore obtained from those of the sphere that leave the boundary of the disk invariant. It is rather straightforward to see that it consists of a subgroup of the Möbius group $\mathsf{PSL}(2, \mathbb{C})$, see eqn. (2.105), whose elements are of the form

$$z \mapsto e^{i\phi} \frac{z + b}{\bar{b}z + 1} , \quad |b| < 1 , \quad \phi \in \mathbb{R} . \tag{2.146}$$

It contains in particular rotations along the circle ($b = 0$). It is convenient to map the unit disk to the upper half-plane using the conformal mapping

$$z \mapsto w = \frac{z + i}{iz + 1} \tag{2.147}$$

sending the boundary of the disk $|z| = 1$ to the real axis $w \in \mathbb{R}$. In this case the relevant conformal Killing group is the subgroup of Möbius transformations leaving the real axis invariant, *i.e.* the group $\mathsf{PSL}(2, \mathbb{R})$:

$$w \mapsto \frac{aw + b}{bw + d} , \quad ad - bc = 1 , \quad a, b, c, d \in \mathbb{R} , \quad (a, b, c, d) \sim (-a, -b, -c, -d) . \tag{2.148}$$

One can show, conjugating with the map (2.147), that the group of transformations (2.146) is isomorphic to $\mathsf{PSL}(2,\mathbb{R})$.

Since this group is non-compact, the situation is similar to what we have found for the sphere in the closed string sector. The vacuum amplitude vanishes, and the first non-vanishing observable in the open string sector at tree-level is the correlation function of three boundary operators. There is another possibility however, with no direct analogue in the closed string sector. One can consider a one-point function for an operator inside the disk, for instance at the origin. Such configuration preserves only a compact subgroup $\mathsf{U}(1)$ of $\mathsf{PSL}(2,\mathbb{R})$ hence gives a non-zero answer. It represents the coupling between a closed string mode and a D-brane.

The cylinder has no holes and two boundaries and therefore, $n_\mu - n_k = 0$. This surface has a conformal Killing vector, corresponding to translations along the cylinder (*i.e.* in imaginary time if one considers the open string channel), and a single real modulus, which can be taken to be the length of the cylinder. One can therefore parametrize it as follows :

$$w = \sigma^1 + i\sigma^2 \ , \quad \sigma_1 \in [0,\pi] \ , \quad \sigma^2 \sim \sigma^2 + 2\pi t \ , \quad t \in \mathbb{R}_{>0} \, . \tag{2.149}$$

One may wonder what would happen if we consider instead that the coordinate σ^2 along the one-cycle of the cylinder is the space-like coordinate along a closed string and the coordinate σ^1 along the interval is Euclidean time. While it does not make any difference in Euclidean signature, the interpretation is vastly different. Here we consider a process where a closed string is emitted from the vacuum, propagates for some Euclidean time and is then absorbed. The fact that these two viewpoints are associated with the same field theory quantity is known as *channel duality* between open and closed strings, see fig. 2.11. This observation has far-reaching consequences ; in particular, it played a crucial role in the birth of the second string revolution, associated with the discovery of *D*-branes [9].

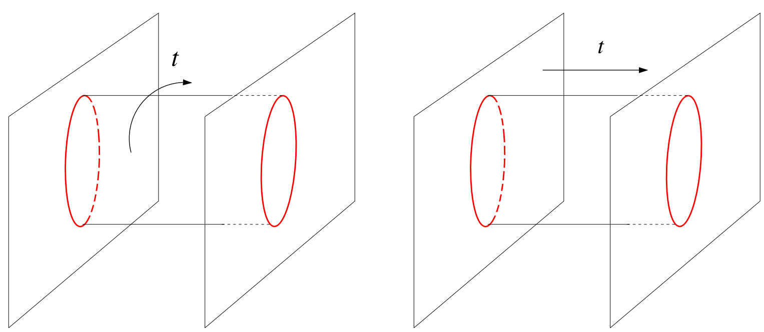

Figure 2.11: *Channel duality: open string channel (left panel) vs. closed string channel (right panel).*

In the D-branes perspective channel duality has indeed the following meaning: the open string viewpoint is interpreted as a one-loop vacuum amplitude for open strings whose end-points are attached to a pair of *D*-branes, while the closed string viewpoint is interpreted as the tree-level emission of closed strings by the first *D*-brane, followed by their absorption by the second *D*-brane.

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Chapter 3

Conformal field theory

Two-dimensional conformally-invariant quantum field theories were introduced by Belavin, Polyakov, and Zamolodchikov in 1984 [1]. These theories play also an essential role in statistical physics, for the description of critical phenomena in two dimensions. The book [2] is probably the most comprehensive book on the subject, and reference [3] is geared towards string applications.

After gauge-fixing of the diffeomorphisms and Weyl transformations to the flat worldsheet metric $\mathrm{d}s^2 = \mathrm{d}w\mathrm{d}\bar{w}$, the Euclidean Polyakov action

$$\mathcal{S} = \frac{1}{4\pi\alpha'} \int_{\mathfrak{s}} \mathrm{d}^2 w \, G_{\mu\nu}[X^\rho] \, \partial X^\mu \bar{\partial} X^\nu \tag{3.1}$$

has an infinite-dimensional residual gauge symmetry that consists in reparametrizations

$$w \mapsto f(w) \, , \quad \bar{w} \mapsto \bar{f}(\bar{w}) \, , \tag{3.2}$$

where f is a holomorphic function, that leaves the metric invariant up to a conformal factor:

$$\mathrm{d}w\mathrm{d}\bar{w} \mapsto \left(\frac{\partial f(w)}{\partial w} \right) \left(\frac{\partial \bar{f}(\bar{w})}{\partial \bar{w}} \right) \mathrm{d}w\mathrm{d}\bar{w} \, , \tag{3.3}$$

and this conformal factor is absorbed by a Weyl transformation such that the reference metric does not change.

As we have seen previously, the topology of the worldsheet restricts severely the transformations of this type that are allowed globally, *i.e.* the conformal Killing vectors of the surface. On the sphere we have found that they corresponded to the Möbius group $\mathsf{PSL}(2, \mathbb{C})$, while on the two-torus the only holomorphic functions periodic around both of the one-cycles are constants. As a result, the transformations (3.2) are not properly speaking local symmetries of string theory itself. In addition we will see shortly that the Weyl symmetry might not hold in the quantum theory due to an anomaly.

At this stage we will be interested in a slightly different problem, the properties of two-dimensional quantum field theories defined on the complex plane[1] $(w, \bar{w})$, or on another manifold with a given fixed metric. We assume that the symmetries of the field theory includes the geometrical transformations defined by eqn. (3.2). Quantum field theories with conformal invariance are called *conformal field theories* (CFTs for short). In two dimensions this symmetry is often powerful enough to solve exactly the theory, without using any perturbative expansion.

It should be stressed that one considers a field theory on a two-dimensional manifold with a *fixed* metric, and the map $w \mapsto f(w)$, $\bar{w} \mapsto \bar{f}(\bar{w})$ is really changing the geometry, in particular the distances between points. In a two-dimensional field theory coupled to gravity, this conformal map can be decomposed as a reparametrization (3.2) and a compensating Weyl invariance.

The relation between two-dimensional CFTs and the Polyakov formulation of string theory is as follows. When a conformal field theory in two-dimensions is coupled to two-dimensional

[1] One often needs to consider the compactification of the complex plane by adding the point at infinity, $\bar{\mathbb{C}} = \mathbb{C} \cap \{\infty\}$, which is related to the two-dimensional sphere (2.102) by a Weyl transformation $\omega = \log(1 + w\bar{w})$.

gravity, it gives rise – at the classical level at least – to a Weyl-invariant theory. Conversely, after gauge-fixing of the diff.×Weyl local symmetry, the string theory action, which is now formulated with respected to a fixed reference metric, is given in full generality by the action of a two-dimensional conformal field theory.

3.1 The conformal group in diverse dimensions

The conformal group in two-dimensions is of infinite dimension, unlike the conformal group in higher dimensions. To understand this we will study the conformal group in arbitrary Euclidean space-time of dimension D.

Conformal transformations are defined as coordinate transformations that preserve the metric up to a conformal factor, *i.e.*

$$x^i \mapsto \tilde{x}^i(x^j) \,, \quad \delta_{ij}dx^i dx^j \mapsto \delta_{ij}\frac{\partial x^i}{\partial \tilde{x}^k}\frac{\partial x^j}{\partial \tilde{x}^\ell}d\tilde{x}^k d\tilde{x}^\ell = \exp\left(2\Omega(\tilde{x}^k)\right)\delta_{ij}d\tilde{x}^i d\tilde{x}^j \,. \tag{3.4}$$

Here the metric is fixed, hence such transformation is really changing the geometry, in particular distances between points. The defining propery of conformal transformations is to preserve angles between vectors of tangent space over any given point.

Let us consider a generic differentiable change of coordinates $x^i = x^i + \delta x^i(x^j)$. At linear order in δx, this change generates a conformal transformation of the D-dimensional metric γ provided that

$$\exists \alpha \in \mathcal{C}^0(\mathbb{R}^D) \,, \quad -(\partial_\ell \delta x_k + \partial_k \delta x_\ell) = \alpha(x^i)\delta_{k\ell} \,. \tag{3.5}$$

Taking the trace of tells us that $-2\partial_i \delta x^i = D\alpha$, hence one has the equation

$$(\partial_\ell \delta x_k + \partial_k \delta x_\ell) - \tfrac{2}{D}(\partial_i \delta x^i)\delta_{k\ell} = 0 \,. \tag{3.6}$$

Acting on this equation with ∂^ℓ finally gives

$$\Box \delta x_k + \left(1 - \tfrac{2}{D}\right)\partial_k(\partial_i \delta x^i) = 0 \,. \tag{3.7}$$

Let us assume a power series expansion:

$$\delta x^i = a^i + m^i{}_j x^j + b^i x^2 + x^i(c_j x^j) + \mathcal{O}(x^3) \tag{3.8}$$

and look for constraints on all the coefficients. While a is unconstrainted, eqn. (3.6) tells us that

$$m_{(ij)} - \tfrac{1}{D}\delta_{ij}\mathrm{Tr}\,(m) = 0 \,, \tag{3.9}$$

hence m splits into a trace part and an antisymmetric part. Equation (3.7) becomes at this order

$$2Db^i + 2c^i + \left(1 - \tfrac{2}{D}\right)(2b^i + (D+1)c^i) = 0 \tag{3.10}$$

which is solved for $c^i = -2b^i$.

One can check that there are no solutions to the problem for higher order terms in (3.8) if $D > 2$. The space of solutions of these equations is then finite and corresponds to:

- translations: $\delta x^i = a^i$ constant
- rotations: $\delta x^i = r^i{}_j x^j$ with r in the vector representation of $\mathfrak{so}(D)$, i.e. the antisymmetric part of m in eqn. (3.8)
- dilatations: $\delta x^i = \lambda x^i$ with λ a non-vanishing constant, i.e. the trace part of m in eqn. (3.8)
- special conformal transformations $\delta x^i = b^i x^2 - 2x^i (b_j x^j)$.

These are actually a set of generators of the Lie algebra $\mathfrak{so}(D+1,1)$, *i.e.* of the Lie Algebra of the Lorentz group in $D+2$ dimensions.

In two dimensions, the conformal transformations are given, in complex coordinates, by $w \mapsto f(w)$, $\bar{w} \mapsto \bar{f}(\bar{w})$ with f holomorphic, as we have already noticed. Naturally this infinite-dimensional group contains as a subgroup the transformations existing in general dimensions. Explicitly:

- translations: $\delta w = a$, $\delta \bar{w} = \bar{a}$
- rotations: $\delta w = i\theta w$, $\delta \bar{w} = -i\theta \bar{w}$ with $\theta \in \mathbb{R}$.
- dilatations: $\delta w = \lambda w$, $\delta \bar{w} = \lambda \bar{w}$ with $\lambda \in \mathbb{R}$.
- special conformal transformations $\delta w = -\bar{b}\, w^2$, $\delta \bar{w} = -b\, \bar{w}^2$, $b \in \mathbb{C}$.

Following the general discussion this generates a Lie group isomorphic to $SO(3,1)$, i.e. the Lorentz group in three dimensions, whose component connected to the identity is isomorphic to the Möbius group $PSL(2,\mathbb{C})$ that appeared already in section 2.3. It was shown there that the Möbius group was the subgroup of two-dimensional conformal transformations that are globally defined on the two-sphere, or equivalently on the compactified complex plane $\bar{\mathbb{C}}$.

3.2 Radial quantization

The Euclidean two-dimensional conformal field theory associated with the worldsheet of a propagating string is naturally associated with a surface with the topology of a cylinder, *i.e.* parametrized by two coordinates (σ_1, σ_2) with $\sigma_1 \sim \sigma_1 + 2\pi$, or equivalently the complex coordinate $w = \sigma^1 + i\sigma^2$ with $w \sim w + 2\pi$. The coordinate σ^2, which runs from $-\infty$ to $+\infty$, is the Euclidean time, obtain through Wick rotation: $\sigma^2 = -i\tau$.

Even outside the string theory context, the natural starting point for canonical quantization of conformal field theories is on the cylinder, in order to avoid infrared problems in the case of an infinite space direction.

States of the quantum field theory are defined on a space-like slice, *i.e.* on a slice of constant σ_2 after Wick rotation to Euclidean space. In particular, an initial state $|\text{in}\rangle$ of the theory is defined on a slice with $\sigma_2 \to -\infty$.

Invariance of the theory under conformal transformation allows to give a different representation of the (Euclidean) time evolution of the QFT and of states of the theory. Let us consider the conformal mapping from the cylinder to the complex plane:

$$w \mapsto z = e^{-iw}\ , \quad \bar{w} \mapsto \bar{z} = e^{i\bar{w}}\ . \tag{3.11}$$

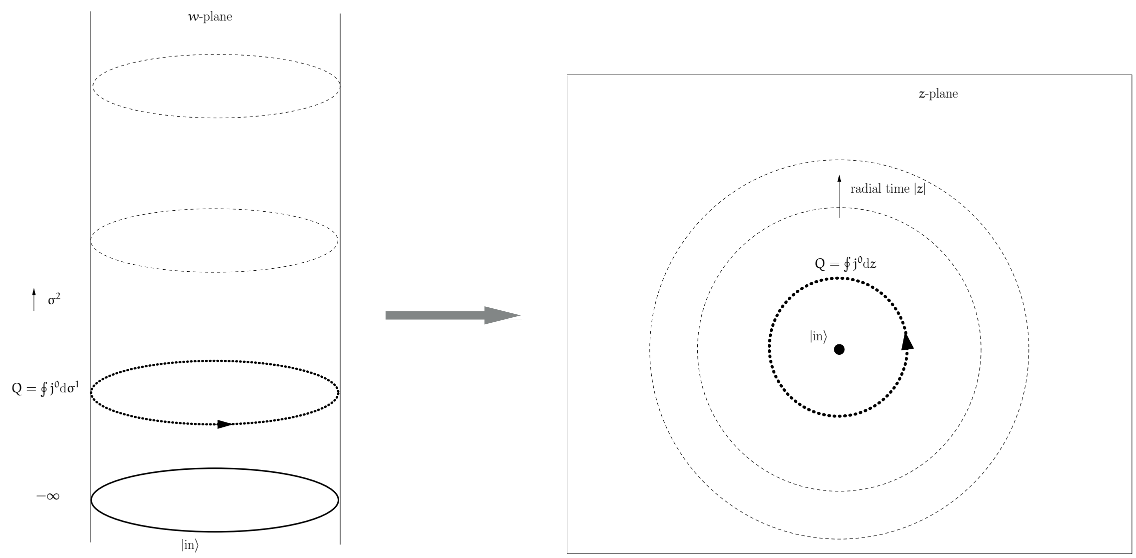

Figure 3.1: *Conformal mapping from the cylinder to the complex plane.*

In this description, $\sigma^2 \to -\infty$ is mapped to the origin $z = 0$ and slices of constant σ^2 are mapped to circles around the origin, see fig. 3.1.

Time evolution correspond to dilatations or, said differently, the Hamiltonian operator corresponds to the dilatation operator. Likewise, time-ordered correlation functions of the QFT become *radial-ordered* correlation functions.

3.2.1 State - operator correspondence

In quantum field theories, states and operators are very different kind of objects.

On the one hand, in the path integral formalism, a state corresponds to a wave functional $\Psi_1[\phi_1(\vec{x}), t_1]$ defined on a whole space-like slice of space-time at a given time t_1, in the present case on the circle parametrized by σ. Such a state will evolve to an other state $\Psi_2[\phi_2(\vec{x}), t_2]$ defined on a space-like slice at time t_2 through:[2]

$$\Psi_2[\phi_2(\vec{x}), t_2] = \int \mathcal{D}\phi_1\, \Psi_1[\phi_1(\vec{x}, t_1)] \int_{\phi(\vec{x},t_1)=\phi_1(\vec{x})}^{\phi(\vec{x},t_2)=\phi_2(\vec{x})} \mathcal{D}\phi\ e^{iS[\phi]}\ . \tag{3.12}$$

[2]A wave functional $\Psi[\phi(\vec{x})]$ in QFT gives the probability amplitude associated with the whole field configuration $\phi(\vec{x})$ on the space-like slice. In quantum mechanics, a state is defined by a wave-function $\psi(q)$ and the analoguous formula is $\psi_2(q_2, t_2) = \int dq_1\, \Psi_1(q_1, t_1) \int_{q(t_1)=q_1}^{q(t_2)=q_2} \mathcal{D}q\ \exp \frac{i}{\hbar} \int_{t_1}^{t_2} dt' \mathcal{L}$.

On the other hand, local operators in quantum field theories are defined as arbitrary *local* expressions constructed from the elementary fields of the theory and their derivatives, *i.e.* local functionals $\mathcal{O}[\phi^i, \partial_\nu \phi^i, \cdots](x^\mu)$.[3]

The conformal mapping between the cylinder and the plane has some surprising consequence. Let us consider an initial state $|\text{in}\rangle$ of the CFT on the cylinder, defined on the circle parametrized by σ^1 in the infinite past $\sigma^2 \to -\infty$. Under the conformal mapping (3.11) it is mapped to the origin of the plane $z = \bar{z} = 0$. This means that the initial state is mapped to a local object at the origin, in other words a *local operator*, inserted at the origin. This is called the *state-operator correspondence*:

$$\widehat{A}|0\rangle \;\longleftrightarrow\; \lim_{z\to 0, \bar{z}\to 0} \mathcal{O}_A(z, \bar{z})\,, \tag{3.13}$$

which says that a state obtained by acting on the vacuum with some operator $\hat{A}$ is equivalent to a local operator $\mathcal{O}_A(z, \bar{z})$ inserted at the origin of the complex plane. In this perspective the vacuum $|0\rangle$ corresponds to the identity operator on the right-hand side.

3.2.2 Conserved charges

Let us consider a field theory with a conserved current, $\partial_\mu j^\mu = 0$, and a certain space-like foliation of the ambient space-time. One can define a conserved charge

$$Q = \int j_\mu d\Sigma^\mu \tag{3.14}$$

where $d\Sigma^\mu$ is the surface element over a space-like slice. For a conserved current on the cylindrical worldsheet of the string, it gives

$$Q = \oint j_2 \frac{d\sigma^1}{2\pi}\,. \tag{3.15}$$

Let us map the theory to the plane. A two-dimensional current has components $J_z(z, \bar{z})$ and $J_{\bar{z}}(z, \bar{z})$ in complex coordinates, and current conservation means that $\bar{\partial} J_z + \partial J_{\bar{z}} = 0$.

The integral around the cylinder $\int d\sigma^1 j_2$ becomes an integral over the polar angle, $\int d\theta\, J_r$. In complex coordinates, one obtains

$$Q = \frac{1}{2i\pi}\left(\oint_{\mathcal{C}} J_z dz - \oint_{\mathcal{C}} J_{\bar{z}} d\bar{z}\right)\,, \tag{3.16}$$

where $\mathcal{C}$ denotes a contour encircling the local operator corresponding to the state for which we compute the charge.

In many cases the two components of the current are separately conserved, hence $J := J_z(z)$ is a holomorphic function and $\tilde{J} := \tilde{J}_{\bar{z}}(z)$ a anti-holomorphic one. This will be the case in particular below for the charges associated respectively to holomorphic and anti-holomorphic conformal transformations.

[3]Note that, in a scalar field theory in 4d, a momentum state is given by $|\mathbf{p}\rangle = \hat{a}^\dagger(\mathbf{p})|0\rangle$. However $\hat{a}^\dagger(\mathbf{p}) = -i\int d^3x\, e^{ip\cdot x} \overleftrightarrow{\partial}_0 \hat{\phi}(x)$ is not a local operator, being integrated over a whole space-like slice.

3.3 Conformal invariance and Ward identities

The dynamics of quantum field theories with conformal invariance is severely constrained by this symmetry, especially in two dimensions where this symmetry is infinite dimensional.

3.3.1 Stress-energy tensor

In the classical theory, the stress-energy tensor obeys strong constraints for conformal invariance.

To derive the conservation laws associated with a general conformal field theory in D dimensions, one considers the usual trick of coupling the field theory to a dynamical background metric γ. Then a coordinate change $\delta\sigma^i(\sigma^k)$ accompanied by the change of background metric $\delta\gamma_{ij} = -\nabla_i\delta\sigma_j - \nabla_j\delta\sigma_i$ is a diffeomorphism, under which the theory should be invariant.

The transformation of the action under the infinitesimal change of the background metric is then opposite to the variation under the infinitesimal coordinate transformation $\delta\sigma^i(\sigma^k)$[4]:

$$\begin{aligned} \delta S &= -\frac{1}{2\pi}\int d^Dx\, \delta\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}}\sqrt{-\det\gamma}\,\mathcal{L} = \frac{1}{4\pi}\int d^Dx\,\sqrt{-\det\gamma}\,T^{ij}\delta\gamma_{ij} \\ &= -\frac{1}{2\pi}\int d^Dx\,\sqrt{-\det\gamma}\,T^{ij}\nabla_i\delta\sigma_j \end{aligned} \tag{3.17}$$

where we have introduced the stress-energy tensor

$$T^{ij} = -\frac{2}{\sqrt{-\det\gamma}}\frac{\delta}{\delta\gamma_{ij}}\sqrt{-\det\gamma}\,\mathcal{L}\,. \tag{3.18}$$

Translation invariance of the theory implies then as usual that the stress-energy tensor is covariantly conserved:

$$\nabla^i T_{ij} = 0\,. \tag{3.19}$$

Let consider now invariance under scale transformations, which are special cases of conformal transformations, corresponding to constant rescalings of the metric, $\delta\gamma_{ij} = \delta\lambda\gamma_{ij}$. Equation (3.17) implies that the stress-energy tensor is traceless:

$$T_{ij}\gamma^{ij} = 0\,. \tag{3.20}$$

This is an essential property of conformal field theories.

In two-dimensions, these expressions take a particularly simple form in complex coordinates with a flat background metric:

$$\bar\partial T_{z\star} + \partial T_{\bar z\star} = 0 \tag{3.21}$$

for the former, and

$$T_{z\bar z} = 0 \tag{3.22}$$

[4]We work momentarily in Lorentzian signature for sake of comparison with the computation done in the General relativity course.

for the later. This means that the stress-energy tensor has only two non-vanishing components,

$$\mathsf{T} := \mathsf{T}_{zz} \ , \quad \widetilde{\mathsf{T}} := \mathsf{T}_{\bar{z}\bar{z}} \, , \tag{3.23}$$

and that they are respectively holomorphic and anti-holomorphic functions:

$$\bar{\partial}\mathsf{T}_{zz} = 0 \quad \Longrightarrow \quad \mathsf{T} = \mathsf{T}(z) \, , \tag{3.24a}$$

$$\partial\mathsf{T}_{\bar{z}\bar{z}} = 0 \quad \Longrightarrow \quad \widetilde{\mathsf{T}} = \widetilde{\mathsf{T}}(\bar{z}) \, . \tag{3.24b}$$

We will finally give the form of the Noether currents associated with conformal transformations. Let us consider the infinitesimal transformation

$$z \mapsto z + \rho(z,\bar{z})\varepsilon(z) \ , \quad \bar{z} \mapsto \bar{z} + \bar{\rho}(z,\bar{z})\bar{\varepsilon}(\bar{z}) \, , \tag{3.25}$$

which reduces to a conformal transformation for constant ρ and $\bar{\rho}$. According to eqn. (3.17) the variation of the action of the theory is then

$$\begin{aligned} \delta S &= -\frac{1}{2\pi}\int \mathrm{d}^2 z \left\{ \mathsf{T}(z)\bar{\partial}\big(\rho(z,\bar{z})\varepsilon(z)\big) + \widetilde{\mathsf{T}}(\bar{z})\partial\big(\bar{\rho}(z,\bar{z})\bar{\varepsilon}(\bar{z})\big) \right\} \\ &= -\frac{1}{2\pi}\int \mathrm{d}^2 z \left(\mathsf{T}(z)\varepsilon(z)\bar{\partial}\rho(z,\bar{z}) + \widetilde{\mathsf{T}}(\bar{z})\bar{\varepsilon}(\bar{z})\partial\bar{\rho}(z,\bar{z}) \right) \end{aligned} \tag{3.26}$$

A very powerful point of view in two-dimensional Euclidean QFT is to consider w and $\bar{w}$ are independent variables. This means that we consider the analytic continuation of the Euclidean space $\mathbb{R}^2$ to $\mathbb{C}^2$, and the move from coordinates (σ^1,σ^2) to $(w,\bar{w})$ just as a change of basis in $\mathbb{C}^2$. Naturally at the end of the day one should enforce the reality condition $(\bar{z})^\star = z$.

From this point of view one can consider purely holomorphic conformal transformations, *i.e.* with $\bar{\varepsilon} = 0$. The associated Noether current is then

$$\mathsf{J}_z = \mathsf{T}(z)\varepsilon(z) \ , \quad \mathsf{J}_{\bar{z}} = 0 \tag{3.27}$$

The non-zero component of the current, $\mathsf{J} := \mathsf{J}_z$ is then holomorphic: $\bar{\partial}\mathsf{J} = 0$.

In the same way a purely anti-holomorphic conformal transformation, , *i.e.* with $\varepsilon = 0$, gives the conserved current

$$\tilde{\mathsf{J}}_z = 0 \ , \quad \tilde{\mathsf{J}}_{\bar{z}} = \widetilde{\mathsf{T}}(\bar{z})\bar{\varepsilon}(\bar{z}) \, , \tag{3.28}$$

whose non-zero component $\tilde{\mathsf{J}} := \tilde{\mathsf{J}}_{\bar{z}}$ is anti-holomorphic: $\partial\tilde{\mathsf{J}} = 0$.

As in other quantum field theories, these classical equations lead to functional equations in the quantum theory, known as *Ward identities* that we shall study now.

3.3.2 Ward identities

Let us consider a generic time-ordered N-point correlation function of *local operators* in a two-dimensional conformal field theory. By local operator we mean any operator that can be written as a local expression in the fundamental fields ϕ^i and their derivatives. The corresponding path integral is written as

$$\langle \mathcal{O}_1(z_1,\bar{z}_1)\cdots\mathcal{O}_N(z_N,\bar{z}_N)\rangle = \frac{1}{Z}\int \mathcal{D}\phi^i e^{-S[\phi^i]}\mathcal{O}_1[\phi^i](z_1,\bar{z}_1)\cdots\mathcal{O}_N[\phi^i](z_N,\bar{z}_N) \tag{3.29}$$

We consider a holomorphic conformal transformation with support in a small disc $\mathfrak{d}$ in the complex plane, *i.e.* a transformation (3.25) with ρ non-vanishing only in the neighborhood of some point $(z_0,\bar{z}_0)$ and $\bar{\rho}=0$.

There are two cases to consider: either there are no local operators in this neighborhood, or there is (at least) one inserted at a point inside the disk, see fig. 3.2.

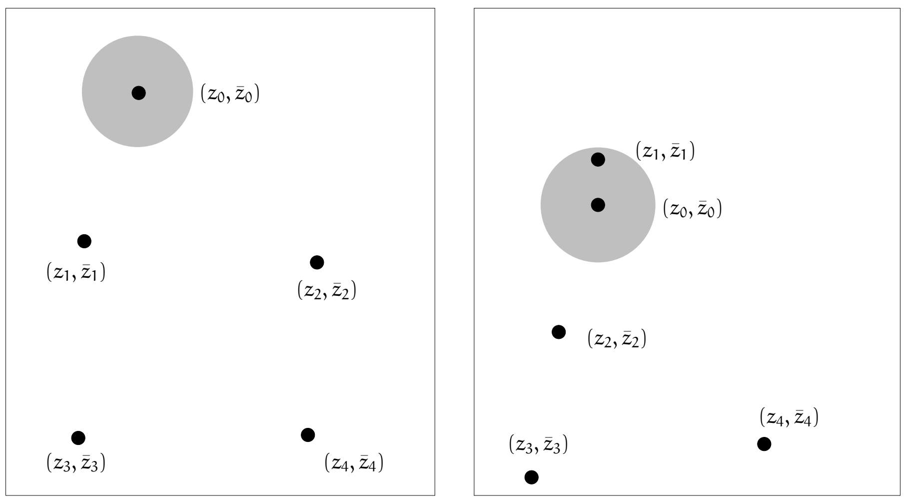

Figure 3.2: *Conformal transformation with support away from local operators (left panel) and including one local operator (right panel).*

Let us start with the former case. By construction the local operators are not affected by the conformal transformation, hence the change in the path integral comes from the measure $\mathcal{D}\phi^i$ and from the action $S[\phi^i]$ only. We will make the assumption that the measure is invariant so the only variation comes from the action.[5] The path integral is then modified at first order as

$$\int \mathcal{D}\phi^i e^{-S[\phi^i]}\mathcal{O}_1\cdots\mathcal{O}_N \;\mapsto\; \int \mathcal{D}\phi^i e^{-S[\phi^i]}\left(1-\frac{1}{2\pi}\int d^2z\, T(z)\varepsilon\bar{\partial}\rho\right)\mathcal{O}_1\cdots\mathcal{O}_N\,. \tag{3.30}$$

The path integral should actually be completely independent of the transformation which should just be thought as some change of variables in the path integral. Therefore one gets

[5] This assumption is very strong and turns out actually to be wrong in many cases; this is the path integral view of QFT anomalies. We will come back to this important issue later on.

the constraint, after integration by parts

$$\int \mathcal{D}\phi^i e^{-S[\phi^i]} \int d^2z\, \rho(z,\bar{z})\, \bar{\partial}\Big(T(z)\varepsilon(z)\Big)\mathcal{O}_1(z_1,\bar{z}_1)\cdots\mathcal{O}_N(z_N,\bar{z}_N) = 0 \tag{3.31}$$

Since this should hold for any choice of ρ, we are led to the condition that[6]

$$\Big\langle \bar{\partial}\Big(T(z)\varepsilon(z)\Big)\mathcal{O}_1(z_1,\bar{z}_1)\cdots\mathcal{O}_N(z_N,\bar{z}_N)\Big\rangle = 0\,. \tag{3.32}$$

This is the quantum version of Noether theorem; naturally there is an analogous formula for anti-holomorphic conformal transformations.

We move now to the latter case. Let assume that the operator $\mathcal{O}(z_1,\bar{z}_1)$ is inserted at a point where $\rho \neq 0$, *i.e.* where the conformal transformation has support. Then on general grounds the operator transforms

$$\mathcal{O}(z_1,\bar{z}_1) \;\mapsto \mathcal{O}(z_1,\bar{z}_1) + \delta\mathcal{O}(z_1,\bar{z}_1)\,. \tag{3.33}$$

We insert this transformation in the path integral and, using similar arguments as before, we get at first order the relation

$$\begin{aligned} -\frac{1}{2\pi}\int d^2z\, \rho(z,\bar{z})\, &\Big\langle \bar{\partial}\Big(T(z)\varepsilon(z)\Big)\mathcal{O}_1(z_1,\bar{z}_1)\cdots\mathcal{O}_N(z_N,\bar{z}_N)\Big\rangle \\ &= \Big\langle \delta\mathcal{O}_1(z_1,\bar{z}_1)\,\mathcal{O}_2(z_2,\bar{z}_2)\cdots\mathcal{O}_N(z_N,\bar{z}_N)\Big\rangle\,. \end{aligned} \tag{3.34}$$

The function ρ being arbitrary, we can take it to be the indicator function of the disk $\mathfrak{d}$ around $(z_0,\bar{z}_0)$. The left-hand side can then be simplified using Stokes' theorem:

$$\int_{\mathfrak{d}} \left(\partial_z J^z + \partial_{\bar{z}} J^{\bar{z}}\right) = -i\oint_{\partial\mathfrak{d}} (J_z dz - J_{\bar{z}} d\bar{z})\,. \tag{3.35}$$

In the present case $J_{\bar{z}} = 0$ as we found before so eqn. (3.34) gives

$$\frac{i}{2\pi}\oint_{\partial\mathfrak{d}} dz \Big\langle T(z)\varepsilon(z)\mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2\cdots\mathcal{O}_N\Big\rangle = \Big\langle \delta\mathcal{O}_1(z_1,\bar{z}_1)\,\mathcal{O}_2\cdots\mathcal{O}_N\Big\rangle\,, \quad (z_1,\bar{z}_1)\in\mathfrak{d}\,. \tag{3.36}$$

This teaches us a very important result: the change of an operator $\delta\mathcal{O}$ under a conformal transformation is given by the residue of its product with the stress-energy tensor, namely

$$\delta\mathcal{O}_1(z_1,\bar{z}_1) = \frac{i}{2\pi}\oint_{\partial\mathfrak{d}} dz\, T(z)\varepsilon(z)\mathcal{O}_1(z_1,\bar{z}_1) = -\mathrm{Res}_{z\to z_1}\,\left(T(z)\varepsilon(z)\,\mathcal{O}_1(z_1,\bar{z}_1)\right) \tag{3.37}$$

The holomorphic function ε being arbitrary, this means that the operator product between $T(z)$ and $\mathcal{O}(z_1,\bar{z}_1)$ has a first order pole when they approach to each other:

$$T(z)\,\mathcal{O}_1(z_1,\bar{z}_1) = \cdots + \frac{\mathrm{Res}\,\left(T(z)\mathcal{O}_1(z_1,\bar{z}_1)\right)}{z-z_1} + \cdots\,. \tag{3.38}$$

[6] Naturally the same holds for the partition function Z in the denominator of (3.29), *i.e.* the zero-point function.

There is naturally a similar story for the anti-holomorphic conformal transformations, which gives in particular:

$$\widetilde{\mathsf{T}}(\bar{z})\,\mathcal{O}_1(z_1,\bar{z}_1) = \cdots + \frac{\mathrm{Res}\left(\widetilde{\mathsf{T}}(\bar{z})\mathcal{O}_1(z_1,\bar{z}_1)\right)}{\bar{z}-\bar{z}_1} + \cdots . \tag{3.39}$$

In a quantum field theory, product of operators make sense when they are time-ordered. In the context of radial quantization of a two-dimensional CFT, time-ordering takes the form of *radial ordering*, namely:

$$\mathcal{R}\left(\mathcal{O}_1(z',\bar{z}')\mathcal{O}_2(z,\bar{z})\right) = \begin{cases} \mathcal{O}_1(z',\bar{z}')\mathcal{O}_2(z,\bar{z}) & |z'|>|z| \\ (-1)^F\mathcal{O}_2(z,\bar{z})\mathcal{O}_1(z',\bar{z}') & |z|>|z'| \end{cases}, \tag{3.40}$$

where $(-)^F = 1$ (resp. $(-)^F = -1$) for bosonic (resp. fermionic) operators. In order to unclutter calculations, radial-ordering will always be implicit in operator equations like (3.38).

Singularities when operators approach each other such as (3.38) are a generic feature of quantum field theories. They have been studied more systematically under the name of *operator product expansions*. The operator product expansion exists in all quantum field theories, and provides the behavior of the theory when two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ come close to each other. The basic idea is that, when the separation between them become infinitesimal, one can expand the product into a sum of local operators. While this is usually an asymptotic expansion only, one can show that in the case of conformal field theories the series converges, with the radius of convergence given by the distance to the nearest operator $\mathcal{O}_{n\neq 1,2}$.

3.4 Primary operators

The variation of a generic local operator $\mathcal{O}(z,\bar{z})$ under a holomorphic conformal transformation $\varepsilon(z)$ is contained in eqn (3.37). We can get some insight by looking at specific simple transformations.

Let us look first at a holomorphic translation defined as $z \mapsto z+a$, $\bar{z}\mapsto\bar{z}$. It acts on any local operator $\mathcal{O}$ as

$$\mathcal{O}(z,\bar{z}) \mapsto \mathcal{O}(z-a,\bar{z}) = \mathcal{O}(z,\bar{z}) - a\partial_z\mathcal{O}(z,\bar{z}) + \cdots \tag{3.41}$$

By identifying both sides of (3.37) one learns that

$$\partial_z\mathcal{O}(z,\bar{z}) = \mathrm{Res}_{z'\to z}\left(\mathsf{T}(z')\mathcal{O}(z,\bar{z})\right) \tag{3.42}$$

or in other words, after doing the same for anti-holomorphic translations:

$$\mathsf{T}(z')\,\mathcal{O}(z,\bar{z}) \overset{z'\to z}{\simeq} \cdots + \frac{\partial\mathcal{O}(z,\bar{z})}{z'-z} + \cdots \tag{3.43a}$$

$$\widetilde{\mathsf{T}}(\bar{z}')\,\mathcal{O}(z,\bar{z}) \overset{z'\to z}{\simeq} \cdots + \frac{\bar{\partial}\mathcal{O}(z,\bar{z})}{\bar{z}'-\bar{z}} + \cdots \tag{3.43b}$$

We look now at infinitesimal scaling transformations, which are by definition the (non-holomorphic) transformations $z \mapsto (1+\delta\lambda)z$, $\bar{z} \mapsto (1+\delta\lambda)\bar{z}$, with real $\delta\lambda$. We consider operators that are eigenstates of the dilatation operator; the corresponding eigenvalue is called the *scaling dimension* Δ of the operator:[7]

$$\mathcal{O} \mapsto \lambda^{-\Delta}\mathcal{O} \tag{3.44}$$

The transformation of an operator $\mathcal{O}_\Delta$ of scaling dimension Δ under an infinitesimal scale transformation is given by:

$$\delta\mathcal{O}_\Delta = -\delta\lambda\big(\Delta\mathcal{O}_\Delta(z,\bar{z}) + z\partial\mathcal{O}_\Delta + \bar{z}\bar{\partial}\mathcal{O}_\Delta\big)\,. \tag{3.45}$$

We consider finally infinitesimal rotations in the complex plane, given by $z \mapsto (1+i\delta\theta)z$, $\bar{z} \mapsto (1-i\delta\theta)\bar{z}$ with real $\delta\theta$. Consider again an operator which is an eigenstate of the rotation operator; its eigenvalue is called the *spin* s of the operator. The transformation law is given by:

$$\delta\mathcal{O}_s = -i\delta\theta\big(s\mathcal{O}_s(z,\bar{z}) + z\partial\mathcal{O}_s - \bar{z}\bar{\partial}\mathcal{O}_s\big)\,. \tag{3.46}$$

In order to use the holomorphic/antiholomorphic splitting of conformal transformations that we have used throughout, it is convenient to combine scalings and rotations into the complex transformations

$$\delta z = (\delta\lambda + i\delta\theta)z =: \delta\alpha\, z\,, \quad \delta\bar{z} = (\delta\lambda - i\delta\theta)\bar{z} =: \delta\bar{\alpha}\, z\,. \tag{3.47}$$

Consider common eigenstates of the dilatation and rotation operators. The eigenvalues are called the *conformal weights* $(h,\bar{h})$ of the state, and are related to the scaling dimension and spin through

$$\Delta = h + \bar{h}\,, \quad s = h - \bar{h}\,. \tag{3.48}$$

We obtain then from the transformations (3.45,3.47) the infinititesimal transformations:

$$\delta_\alpha\mathcal{O}_{h,\bar{h}} = -\delta\alpha\big(h\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + z\partial\mathcal{O}_{h,\bar{h}}\big)\,, \tag{3.49a}$$
$$\delta_{\bar{\alpha}}\mathcal{O}_{h,\bar{h}} = -\delta\bar{\alpha}\big(\bar{h}\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \bar{z}\bar{\partial}\mathcal{O}_{h,\bar{h}}\big)\,. \tag{3.49b}$$

Under a finite scaling transformation and rotation, the operator of conformal weights $(h,\bar{h})$ transforms as

$$\mathcal{O}_{h,\bar{h}} \mapsto \lambda^{-\Delta}e^{-is\theta}\mathcal{O}_{h,\bar{h}} = \big(\lambda e^{i\theta}\big)^{-h}\big(\lambda e^{-i\theta}\big)^{-\bar{h}}\,\mathcal{O}_{h,\bar{h}}\,. \tag{3.50}$$

The residue formula (3.37) will provide then the relation between the (anti)holomorphic scale transformation and the operator product with the stress-energy tensor. One has first

$$\begin{aligned} h\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + z\partial\mathcal{O}_{h,\bar{h}}(z,\bar{z}) &= \mathrm{Res}_{z'\to z}\,\big(z'T(z')\,\mathcal{O}(z,\bar{z})\big) \\ &= \mathrm{Res}_{z'\to z}\,\big((z'-z)T(z')\,\mathcal{O}(z,\bar{z})\big) + z\mathrm{Res}_{z'\to z}\,\big(T(z)\,\mathcal{O}(z,\bar{z})\big) \end{aligned} \tag{3.51}$$

[7] At the classical level Δ is the same as the dimension in units of inverse length coming from dimensional analysis. For instance, a free massless scalar field in dimension D has an action $\frac{1}{4\pi}\int d^D x\,(\partial\phi)^2$, hence ϕ has dimension (length)$^{2-D}$, in other words scaling dimension $\Delta = D-2$.

which gives, using also eqn. (3.42)

$$h\mathcal{O}_{h,\bar{h}}(z,\bar{z}) = \mathrm{Res}_{z'\to z}\ \left((z'-z)T(z')\,\mathcal{O}(z,\bar{z})\right) \tag{3.52}$$

In summary, we have learned that the *operator product expansion* of one of the components of the stress-energy tensor with an operator of conformal weight $(h,\bar{h})$ contains the terms

$$T(z')\mathcal{O}_{h,\bar{h}}(z,\bar{z}) \overset{z'\to z}{\simeq} \cdots + \frac{h}{(z'-z)^2}\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \frac{1}{(z'-z)}\partial\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \cdots \tag{3.53a}$$

$$\widetilde{T}(\bar{z}')\mathcal{O}_{h,\bar{h}}(z,\bar{z}) \overset{z'\to z}{\simeq} \cdots + \frac{\bar{h}}{(\bar{z}'-\bar{z})^2}\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \frac{1}{(\bar{z}'-\bar{z})}\bar{\partial}\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \cdots \tag{3.53b}$$

The operator product with a generic operator of weights $(h,\bar{h})$ contains in principle more singular terms in the Laurent series expansion in powers of $(z-w)$.

Conformal primary operators

A distinguished class of local operators in a two-dimensional conformal field theory are operators for which the terms written in eqns. (3.53) exhaust all the possible singular terms in the expansion, namely

$$T(z')\mathcal{O}_{h,\bar{h}}(z,\bar{z}) \overset{z'\to z}{\simeq} \frac{h}{(z'-z)^2}\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \frac{1}{(z'-z)}\partial\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \text{regular} \tag{3.54a}$$

$$\widetilde{T}(\bar{z}')\mathcal{O}_{h,\bar{h}}(z,\bar{z}) \overset{z'\to z}{\simeq} \frac{\bar{h}}{(\bar{z}'-\bar{z})^2}\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \frac{1}{(\bar{z}'-\bar{z})}\bar{\partial}\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \text{regular} \tag{3.54b}$$

These are called *primary operators*, or primaries for short. Given that the regular terms in the expansions do not contain essential information, we will often remove them from the expressions of the operator product expansions.

Primary operators are interesting because they have particularly simple transformation laws under a generic holomorphic conformal transformation $\delta z = \varepsilon(z)$. Using once again the residue formula (3.37) and the OPE (3.54a) for primary operators one finds that

$$\delta_\varepsilon\mathcal{O} = -\mathrm{Res}_{z'\to z}\left\{\varepsilon(z')\left(\frac{h}{(z'-z)^2}\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \frac{1}{(z'-z)}\partial\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \text{regular}\right)\right\}. \tag{3.55}$$

The infinitesimal conformal transformation $\varepsilon(z)$ is by definition holomorphic in the neighborhood of z and can be Taylor-expanded there:

$$\varepsilon(z') = \varepsilon(z) + (z'-z)\partial\varepsilon(z) + \mathcal{O}\left((z'-z)^2\right) \tag{3.56}$$

Therefore transformation of a primary operator under an infinitesimal generic conformal transformation is given by

$$\delta_\varepsilon\mathcal{O}(z,\bar{z}) = -\varepsilon(z)\partial\mathcal{O}_{h,\bar{h}}(z,\bar{z}) - h\big(\partial\varepsilon(z)\big)\,\mathcal{O}_{h,\bar{h}}(z,\bar{z})\,. \tag{3.57}$$

This transformation law can be generalized to a finite – rather than infinitesimal – transformation, namely $w \mapsto \tilde{w} = f(w)$, as well as to its anti-holomorphic counterpart $\bar{z} \mapsto \tilde{\bar{z}} = \bar{f}(\bar{z})$. One finds then:

$$\mathcal{O}(z,\bar{z}) \mapsto \tilde{\mathcal{O}}\big(\tilde{z},\tilde{\bar{z}}\big) = \left(\frac{\partial f}{\partial z}\right)^{-h} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{-\bar{h}} \mathcal{O}(z,\bar{z})\,. \tag{3.58}$$

Correlation functions of primary operators

Conformal invariance severely constraints the form of correlation functions between primary operators. Let us start with the simplest case, the two-point function:

$$\langle \mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\rangle\,, \tag{3.59}$$

where $\mathcal{O}_1$ (resp. $\mathcal{O}_2$) is a conformal primary of weights $(h_1,\bar{h}_1)$ (resp. $(h_2,\bar{h}_2)$). Under a conformal transformation $z \mapsto y(z)$, $\bar{z} \mapsto \bar{y}(\bar{z})$ one should have

$$\begin{aligned}
\langle \mathcal{O}_1(y_1,\bar{y}_1)\mathcal{O}_2(y_2,\bar{y}_2)\rangle &= \frac{1}{Z}\int \mathcal{D}\phi_i e^{-S[\phi^i]}\mathcal{O}_1(y_1,\bar{y}_1)\mathcal{O}_2[y_2,\bar{y}_2] \\
&= \frac{1}{Z}\int \mathcal{D}\tilde{\phi}_i e^{-S[\tilde{\phi}^i]}\tilde{\mathcal{O}}_1(y_1,\bar{y}_1)\tilde{\mathcal{O}}_2(y_2,\bar{y}_2) \\
= \left(\frac{\partial y}{\partial z}(z_1)\right)^{-h_1} \left(\frac{\partial \bar{y}}{\partial \bar{z}}(\bar{z}_1)\right)^{-\bar{h}_1} &\left(\frac{\partial y}{\partial z}(z_2)\right)^{-h_2} \left(\frac{\partial \bar{y}}{\partial \bar{z}}(\bar{z}_2)\right)^{-\bar{h}_2} \langle \mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\rangle
\end{aligned} \tag{3.60}$$

Invariance under holomorphic and anti-holomorphic translations implies first that

$$\langle \mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\rangle = f(z_1-z_2,\bar{z}_1-\bar{z}_2)\,. \tag{3.61}$$

Next we investigate invariance under holomorphic and anti-holomorphic scale transformations. It leads to the functional identities

$$f\big(\lambda\times(z_1-z_2),\bar{z}_1-\bar{z}_2\big) = \lambda^{-h_1-h_2} f\big(z_1-z_2,\bar{z}_1-\bar{z}_2\big)\,, \tag{3.62a}$$
$$f\big(z_1-z_2,\bar{\lambda}\times(\bar{z}_1-\bar{z}_2)\big) = \bar{\lambda}^{-\bar{h}_1-\bar{h}_2} f\big(z_1-z_2,\bar{z}_1-\bar{z}_2\big)\,. \tag{3.62b}$$

hence f is homogeneous of degree $-(h_1+h_2)$ (resp. of degree $-(\bar{h}_1+\bar{h}_2)$) in z_1-z_2 (resp. in $\bar{z}_1-\bar{z}_2$). In other words,

$$f(z_1-z_2,\bar{z}_1-\bar{z}_2) = \frac{C_{12}}{(z_1-z_2)^{h_1+h_2}(\bar{z}_1-\bar{z}_2)^{\bar{h}_1+\bar{h}_2}}\,, \tag{3.63}$$

where C_{12} is some constant. We now impose invariance under the transformation $z \mapsto -1/z$, which is simpler to handle than the special conformal transformation $z \mapsto z/(1-\bar{b}\,z)$. One gets

$$\frac{1}{(-1/z_1+1/z_2)^{h_1+h_2}} = \big(1/z_1^2\big)^{-h_1}\big(1/z_2^2\big)^{-h_2}\frac{1}{(z_1-z_2)^{h_1+h_2}} \quad\Longrightarrow\quad (z_1z_2)^{h_1+h_2} = z_1^{2h_1}z_2^{2h_2}\,, \tag{3.64}$$

with a similar equation for the anti-holomorphic transformations. Therefore the two-point function can be non-zero only if $h_1 = h_2$ and $\bar{h}_1 = \bar{h}_2$:

$$\boxed{\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle = \frac{C_{12}}{(z_1 - z_2)^{h_1+h_2}(\bar{z}_1 - \bar{z}_2)^{\bar{h}_1+\bar{h}_2}}\delta_{h_1,h_2}\delta_{\bar{h}_1,\bar{h}_2}} . \tag{3.65}$$

Up to a constant, the two-point function is then completely fixed by invariance under the global conformal group $PSL(2,\mathbb{C})$.[8] The set of constants C_{ij} associated with the two-points functions of quasi-primary operators of the same conformal weights define a matrix that can mapped to the identity by choosing an appropriate basis in field space.

There exists a similar story regarding the three-point function of primary operators. Invariance under $PSL(2,\mathbb{C})$ reduces the three-point function computation to a single unknown coefficient:

$$\begin{aligned}\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3)\rangle = C_{123}&\frac{1}{(z_1 - z_2)^{h_1+h_2-h_3}(z_2 - z_3)^{h_2+h_3-h_1}(z_1 - z_3)^{h_3+h_1-h_2}}\\ \times&\frac{1}{(\bar{z}_1 - \bar{z}_2)^{\bar{h}_1+\bar{h}_2-\bar{h}_3}(\bar{z}_2 - \bar{z}_3)^{\bar{h}_2+\bar{h}_3-\bar{h}_1}(\bar{z}_1 - \bar{z}_3)^{\bar{h}_3+\bar{h}_1-\bar{h}_2}} .\end{aligned} \tag{3.66}$$

3.5 The Virasoro Algebra

The operator product expansions (3.53) are valid for all local operators that are eigenstates of the dilatation and rotation operators. By dimensional analysis, the stress-energy tensor T_{ij} given by the definition (3.17) has scaling dimension $\Delta = 2$ in two dimensions. Under a rotation $z \mapsto z' = e^{i\theta}z$ its non-vanishing components transform as $T_{zz} \mapsto T_{z'z'} = e^{-2i\theta}T_{zz}$ and $T_{\bar{z}\bar{z}} \mapsto T_{\bar{z}'\bar{z}'} = e^{2i\theta}T_{\bar{z}\bar{z}}$.

This analysis shows that $T = T_{zz}$ is an operator of conformal weights $(h,\bar{h}) = (2,0)$ while $\widetilde{T} = T_{\bar{z}\bar{z}}$ is an operator of conformal weights $(h,\bar{h}) = (0,2)$, see eq. (3.48). However nothing indicates that these are primary operators. Let us consider then the OPEs

$$T(z')T(z) \overset{z'\to z}{\simeq} \cdots + \frac{2}{(z'-z)^2}T(z) + \frac{1}{(z'-z)}\partial T(z) + \text{reg.} \tag{3.67a}$$

$$\widetilde{T}(\bar{z}')\widetilde{T}(\bar{z}) \overset{z'\to z}{\simeq} \cdots + \frac{2}{(\bar{z}'-\bar{z})^2}\widetilde{T}(\bar{z}) + \frac{1}{(\bar{z}'-\bar{z})}\bar{\partial}\widetilde{T}(\bar{z}) + \text{reg.} \tag{3.67b}$$

3.5.1 The central charge

The missing information in the OPEs (3.67) are the possible terms more singular than $1/(z-w)^2$ in the expansion. The only universal operators that should appear in any conformal field theory are the components of the stress energy tensor and the identity operator which has

[8] Actually all is needed is that the operators transform as (3.58) under the action of $PSL(2,\mathbb{C})$, which is a weaker statement than asking that this equation holds true for any holomorphic function. Such operators are called *quasi-primary* operators.

naturally conformal dimensions $(h, \bar{h}) = (0, 0)$. On general grounds, one should allow terms proportional to the identity on the right-hand side of eqns. (3.17). By dimensional analysis, T being of dimension (2,0), this term should come with the power $(z' - z)^{-4}$ in the operator product expansion:[9]

$$T(z')T(z) \overset{z' \to z}{\simeq} \frac{c}{2(z'-z)^4} + \frac{2}{(z'-z)^2}T(z) + \frac{1}{(z'-z)}\partial T(z) + \text{reg.} \tag{3.68a}$$

$$\widetilde{T}(\bar{z}')\widetilde{T}(\bar{z}) \overset{z' \to z}{\simeq} \frac{\bar{c}}{2(\bar{z}'-\bar{z})^4} + \frac{2}{(\bar{z}'-\bar{z})^2}\widetilde{T}(z,\bar{z}) + \frac{1}{(\bar{z}'-\bar{z})}\bar{\partial}\widetilde{T}(\bar{z}) + \text{reg.} \tag{3.68b}$$

These OPEs depend on two $\mathbb{C}$-numbers c and $\bar{c}$ that are called the *central charges* of the conformal field theory, and encode the failure of the components of the stress-energy tensor to be conformal primaries. They play a central role in the study of conformal field theories, characterizing in particular the number of degrees of freedom.

From the operator product expansion (3.68) one deduces the transformation of T under an arbitrary infinitesimal holomorphic conformal transformation:

$$\begin{aligned}\delta_\varepsilon T(z) &= -\text{Res}_{z' \to z}\left\{\varepsilon(z')\left(\frac{c}{2(z'-z)^4} + \frac{2}{(z'-z)^2}T(z) + \frac{1}{(z'-z)}\partial T(z) + \text{reg.}\right)\right\} \\ &= -\frac{c}{12}\varepsilon'''(z) - 2\varepsilon'(z)T(z) - \varepsilon(z)\partial T(z)\,,\end{aligned} \tag{3.69}$$

and similarly

$$\delta_{\bar{\varepsilon}}\widetilde{T}(\bar{z}) = -\frac{\bar{c}}{12}\bar{\varepsilon}'''(\bar{z}) - 2\bar{\varepsilon}'(z)T(z) - \varepsilon(z)\bar{\partial}\widetilde{T}(\bar{z})\,, \tag{3.70}$$

Notice that for infinitesimal dilatations, rotations and special conformal transformations the terms proportional to the central charges vanish.

For finite transformation $z \mapsto \widehat{z} = f(z)$ and $\bar{z} \mapsto \widehat{\bar{z}} = \bar{f}(\bar{z})$ the components of the stress-tensor transforms as

$$T(z) \mapsto \widehat{T}(\widehat{z}) = \left(\frac{\partial f(z)}{\partial z}\right)^{-2}\left(T(z) + \frac{c}{12}\{f(z), z\}\right)\,, \tag{3.71a}$$

$$\widetilde{T}(\bar{z}) \mapsto \widehat{\widetilde{T}}(\widehat{\bar{z}}) = \left(\frac{\partial \bar{f}(\bar{z})}{\partial \bar{z}}\right)^{-2}\left(\widetilde{T}(\bar{z}) + \frac{\bar{c}}{12}\{\bar{f}(\bar{z}), \bar{z}\}\right)\,, \tag{3.71b}$$

where one defines the *Schwarzian derivative*

$$\{f(z), z\} = \frac{2f'''(z)f'(z) - 3(f''(z))^2}{2(f'(z))^2}\,, \tag{3.72}$$

which is compatible with the composition of successive conformal transformations.

[9] More singular terms would come with operators of negative conformal weights; as we will see later this is forbidden in unitary conformal field theories.

3.5.2 The Virasoro Algebra

The information contained in the operator product expansions (3.68) can be recast in a different way that will turn out to be very useful, as it will allow to study the properties of the CFT using the tools of representation theory.

To start, the components of the stress tensor T and $\widetilde{T}$ are respectively holomorphic and anti-holomorphic functions on the complex plane, hence admit a Laurent series expansion:

$$T = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}} \,, \quad \widetilde{T} = \sum_{n\in\mathbb{Z}} \frac{\widetilde{L}_n}{\bar{z}^{n+2}} \,. \tag{3.73}$$

The coefficients of the expansion can be found by a contour integral

$$L_n = \oint_{\mathcal{C}} \frac{dz}{2i\pi}\, z^{n+1}\, T(z) \tag{3.74a}$$

$$\widetilde{L}_n = -\oint_{\mathcal{C}} \frac{d\bar{z}}{2i\pi}\, \bar{z}^{n+1}\, \widetilde{T}(\bar{z}) \tag{3.74b}$$

where $\mathcal{C}$ is a contour encircling the origin counter-clockwise. The mode L_n corresponds to the conserved charge associated with the conformal transformation $\delta z = z^{n+1}$ following the discussion in subsection 3.2.2. In particular, $L_0 + \widetilde{L}_0$ is the dilatation generator, $L_0 - \widetilde{L}_0$ the rotation generator, while L_{-1} and $\widetilde{L}_{-1}$ generate holomorphic and anti-holomorphic translations respectively.

In the quantum conformal field theory on the plane, we will consider the commutator of two Laurent modes of the stress-energy tensor. In other words, for any state $|\phi\rangle$ in the theory we consider the quantity

$$[L_m, L_n]|\phi\rangle = L_m L_n|\phi\rangle - L_n L_m|\phi\rangle \,. \tag{3.75}$$

The first term corresponds to applying first L_n to $|\phi\rangle$ and then L_m, while the second one corresponds to applying first L_m to $|\phi\rangle$ and then L_n.

Using the state-operator correspondence, the state $|\phi\rangle$ is mapped to an local operator $\mathcal{O}(z,\bar{z})$ that we can put at the origin. We have learned also in section 3.2 that the charge of a state w.r.t. a holomorphic current is computed by a contour integral around the corresponding local operator, see eqn. (3.16).

Let us consider the circular contours $\mathcal{C}$ of radius R and $\mathcal{C}'$ of radius $R' > R$, both around the origin. We have

$$L_m L_n \mathcal{O}(0,0) = \oint_{\mathcal{C}'} \frac{dz_1}{2i\pi} \oint_{\mathcal{C}} \frac{dz_2}{2i\pi}\, z_1^{m+1}\, z_2^{n+1}\, T(z_1)\, T(z_2)\mathcal{O}(0,0) \tag{3.76a}$$

$$L_n L_m \mathcal{O}(0,0) = \oint_{\mathcal{C}'} \frac{dz_2}{2i\pi} \oint_{\mathcal{C}} \frac{dz_1}{2i\pi}\, z_1^{m+1}\, z_2^{n+1}\, T(z_1)\, T(z_2)\mathcal{O}(0,0) \tag{3.76b}$$

were the operator products are implicitely radial-ordered, as explained above. In the following we will remove the operator $\mathcal{O}$ which plays no role in this computation; it is understood that the operators are applied to any local operator in the CFT inserted at the origin.

The only difference between the two equations (3.76) is that in (3.76a) the contour of integration over z_1 is around the contour of integration over z_2, while in (3.76b) it is just the opposite. To compute the commutator, the trick is to consider first the integration over z_1 for fixed z_2, see fig. 3.3. Going from expression (3.76a) to expression (3.76b) amounts to pass the contour of integration over z_1 (solid line) through the locus of the contour of integration over z_2 (dashed line).

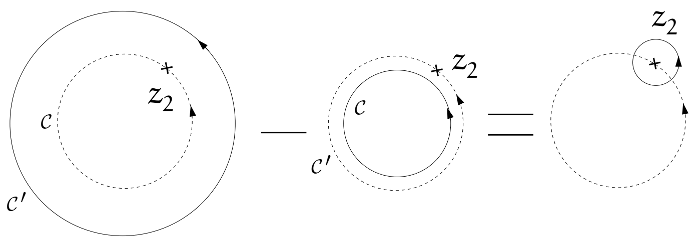

Figure 3.3: *Commutator of Virasoro generators.*

As the operator product $T(z_1)T(z_2)$ has pole, one picks a residue when the contour of integration over z_1 crosses the position z_2 where T is inserted. One has finally to integrate this residue over z_2:

$$[L_m, L_n] = \oint \frac{dz_2}{2i\pi} z_2^{n+1} \operatorname{Res}_{z_1 \to z_2} \left(z_1^{m+1} T(z_1) T(z_2) \right) . \tag{3.77}$$

We now compute the residue using the operator product expansion (3.68a):

$$\begin{aligned} &\operatorname{Res}_{z_1\to z_2}\Big\{ \Big(z_2^{m+1} + (m+1)(z_1-z_2)z_2^m + \tfrac{m(m+1)}{2}(z_1-z_2)^2 z_2^{m-1} + \tfrac{m(m^2-1)}{6} z_2^{m-2}(z_1-z_2)^3 + \cdots \Big) \\ &\qquad \times \Big(\tfrac{c}{2(z_1-z_2)^4} + \tfrac{2}{(z_1-z_2)^2} T(z_2) + \tfrac{1}{(z_1-z_2)} \partial T(z_2) + \text{reg.} \Big) \Big\} \\ &= \tfrac{m(m^2-1)c}{12} z_2^{m-2} + 2(m+1) z_2^m T(z_2) + z_2^{m+1} \partial T(z_2) . \end{aligned} \tag{3.78}$$

So we end up with the following integral

$$\begin{aligned} [L_m, L_n] &= \oint \frac{dz_2}{2i\pi} z_2^{n+1} \left(\tfrac{m(m^2-1)c}{12} z_2^{m-2} + 2(m+1) z_2^m T(z_2) + z_2^{m+1} \partial T(z_2) \right) \\ &= \oint \frac{dz_2}{2i\pi} \left(\tfrac{m(m^2-1)c}{12} z_2^{n+m-1} + 2(m+1) z_2^{m+n+1} T(z_2) - (n+m+2) z_2^{n+m+1} T(z_2) \right) \\ &= \oint \frac{dz_2}{2i\pi} \left(\tfrac{m(m^2-1)c}{12} z_2^{n+m-1} + (m-n) z_2^{m+n+1} T(z_2) \right) , \end{aligned} \tag{3.79}$$

where we have done an integration by parts of the last term in the second step. Now we simply have to express the right-hand side of the final expression in terms of the Laurent coefficients using equation (3.74a) and get the well-known *Virasoro algebra*:

$$\boxed{[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m+n,0}} \tag{3.80}$$

If we did a similar computation, using the equation (3.74b) at the last step, we would find in the same way

$$[\widetilde{L}_m, \widetilde{L}_n] = (m-n)\widetilde{L}_{m+n} + \frac{\bar{c}}{12}m(m^2-1)\delta_{m+n,0}\,. \tag{3.81}$$

The Virasoro algebra (3.80) is very similar two an ordinary Lie algebra, except that it has an infinite number of generators, $\{L_n,\ n \in \mathbb{Z}\}$. Another characteristic feature is the presence of the constant term $\frac{\bar{c}}{12}m(m^2-1)\delta_{m+n,0}$, which commutes with all the generators; such a term is called a *central extension* of the algebra.

Finally a finite sub-algebra of the Virasoro algebra is obtained from the generators $\{L_{-1}, L_0, L_1\}$:

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}\ , \quad [L_{-1}, L_1] = 2L_0\,, \tag{3.82}$$

which is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$; this is nothing but the Lie algebra of the *Möbius group*, the group of globally defined conformal transformations on the sphere $\bar{\mathbb{C}}$, *i.e.* of projective transformations $z \mapsto \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$, already discussed in subsection 2.3.3.

3.5.3 Conformally invariant vacuum and Casimir energy

When a conformal field theory theory is formulated on the cylinder, it is natural to expand the components of the stress energy tensor in terms of its Fourier modes:

$$T_{ww} = -\sum_{n\in\mathbb{Z}} T_n e^{inw} \tag{3.83a}$$

$$T_{\bar{w}\bar{w}} = -\sum_{n\in\mathbb{Z}} \widetilde{T}_n e^{-in\bar{w}} \tag{3.83b}$$

with

$$T_n = -\int \frac{d\sigma^1}{2\pi} e^{-in\sigma_1} T_{ww}(\sigma_1, 0)\,, \tag{3.84a}$$

$$\widetilde{T}_n = -\int \frac{d\sigma^1}{2\pi} e^{in\sigma_1} T_{\bar{w}\bar{w}}(\sigma_1, 0)\,. \tag{3.84b}$$

The conformal mapping (3.11) from the cylinder to the complex plane gives, using equations (3.71) and (3.72):

$$T_{ww}(w) = -z^2 T_{zz}(z) + \frac{c}{24} \tag{3.85a}$$

$$T_{\bar{w}\bar{w}}(\bar{w}) = -\bar{z}^2 T_{\bar{z}\bar{z}}(\bar{z}) + \frac{\bar{c}}{24} \tag{3.85b}$$

Hence the expansion (3.73) on the plane in terms of Laurent coefficients and the expansion (3.83) on the cylinder in terms of Fourier modes are related through

$$T_n = L_n - \frac{c}{24}\delta_{n,0}\ , \quad \widetilde{T}_n = \widetilde{L}_n - \frac{\bar{c}}{24}\delta_{n,0}\,. \tag{3.86}$$

In particular, the Hamiltonian of the conformal field theory on the cylinder is by definition the conserved charge associated with time translations:

$$H = \int \frac{d\sigma^1}{2\pi} T_{\sigma^2\sigma^2} = -\int \frac{d\sigma^1}{2\pi} \left(T_{ww} + T_{\bar{w}\bar{w}} \right) = L_0 + \widetilde{L}_0 - \frac{c+\bar{c}}{24} \,. \tag{3.87}$$

Conformally invariant vacuum

In radial quantization, the natural vacuum state of the conformal field theory on the complex plane, $|0\rangle$, is defined by inserting the identity operator at the origin $z = 0$.

According to the definition (3.73) of the Virasoro generators, in order for the components of the stress-tensor to be non-singular at the origin, one should have

$$L_n|0\rangle = 0 \;, \quad \widetilde{L}_n|0\rangle = 0 \;, \quad \forall n \geqslant -1 \,. \tag{3.88}$$

In particular, the vacuum is invariant under the action of $\{L_{-1}, L_0, L_1\}$ and $\{\widetilde{L}_{-1}, \widetilde{L}_0, \widetilde{L}_1\}$, *i.e.* under global conformal transformations. For this reason, it is called the $PSL(2,\mathbb{C})$-*invariant vacuum*.

If we consider a two-dimensional CFT in the $PSL(2,\mathbb{C})$-invariant vacuum on the plane, equation (3.87) indicates that the corresponding energy of the ground state on the cylinder is given by

$$E_{\text{cyl}} = -\frac{c+\bar{c}}{24} \,, \tag{3.89}$$

which is interpreted as the Casimir energy of the QFT on the cylinder; notice that no regularization of high-energy divergences was needed to obtain this result. This remark provides another interpretation of the central charges of a conformal field theory.

3.5.4 Conformal primaries revisited

Primary operators were defined by the operator product expansion (3.54) with the stress-energy tensor. Let us consider some primary operator $\mathcal{O}_{h,\bar{h}}$. Plugging in the decompositions (3.73) in terms of Laurent modes, one finds that the corresponding state $|\mathcal{O}\rangle$ is annihilated by all positive modes of the Virasoro generators:

$$\forall n > 0 \;, \quad L_n|h,\bar{h}\rangle = 0 \;, \quad \widetilde{L}_n|h,\bar{h}\rangle = 0 \,. \tag{3.90}$$

Hence primary operators play the same role w.r.t. the Virasoro algebra as the highest weight vectors in representation theory of Lie algebras. For this reason they are also referred to as *highest weight states* of the Virasoro algebra.

The other states in the Virasoro representation associated with the highest weight state $|h,\bar{h}\rangle$ are obtained by applying repeatedly the the negative Virasoro modes $\{L_{-n},\, n > 0\}$ (there is a similar story for anti-holomorphic generator). This generates what is known as a *Verma module*, that may or may not be an irreducible representation of the Virasoro algebra depending on the values of h and the central charge c.

The conformal dimension of these *descendant states*, obtained from the highest weight state using lowering operators L_{-n}, is easily computed:

$$L_0 L_{-n}|h,\bar{h}\rangle = nL_{-n}|h,\bar{h}\rangle + L_{-n}L_0|h,\bar{h}\rangle = (h+n)|h,\bar{h}\rangle, \quad n > 0\,. \tag{3.91}$$

I would certainly agree with the reader that would tell me that the primary states should be called *lowest* dimension states rather than *highest* weight states!

3.5.5 Unitarity constraints

One important constraint on sensible quantum field theories is *unitarity*, the conservation of probabilities over time. The time evolution on the cylinder is given by the Hamiltonian density (with Minkowskian signature on the worldsheet):

$$\begin{aligned} \mathcal{H} = -(T_{ww} + T_{\bar{w}\bar{w}}) &= \sum_{n\in\mathbb{Z}} \left(T_n e^{in(\sigma^1+\tau)} + \widetilde{T}_n e^{-in(\sigma^1-\tau)} \right) \\ &= \sum_{n\in\mathbb{Z}} \left(L_n e^{in(\sigma^1+\tau)} + \widetilde{L}_n e^{-in(\sigma^1-\tau)} \right) - \frac{c+\bar{c}}{24}\,. \end{aligned} \tag{3.92}$$

Hence unitarity gives the constraints

$$(L_n)^\dagger = L_{-n} \quad , \qquad \left(\widetilde{L}_n\right)^\dagger = \widetilde{L}_{-n}\,. \tag{3.93}$$

These unitarity relations have two immediate consequences:

- consider a primary state $|h,\bar{h}\rangle$ of conformal dimensions $(h,\bar{h})$ in a unitary CFT.

$$||L_{-1}|h,\bar{h}\rangle||^2 = \langle h,\bar{h}|[L_1,L_{-1}]|h,\bar{h}\rangle = 2\langle h,\bar{h}|L_0|h,\bar{h}\rangle\,, \tag{3.94}$$

  hence in a unitary QFT the conformal dimensions $(h,\bar{h})$ of primary operators – and, consequently, of all descendant states – should be non-negative. The unique state with $h=\bar{h}=0$ in a unitary CFT is the $PSL(2,\mathbb{C})$-invariant vacuum since it satisfies $L_{-1}|0,0\rangle = \widetilde{L}_{-1}|0,0\rangle = 0$ which means in operator language $\partial\mathcal{O}_{0,0} = \bar{\partial}\mathcal{O}(0,0) = 0$.

- $\forall n > 1$, one has

$$||L_{-n}|0\rangle||^2 = \langle 0|[L_n,L_{-n}]|0\rangle = \frac{c}{12}n(n^2-1)\,, \tag{3.95}$$

  and similarly for $\widetilde{L}_n|h,\bar{h}\rangle$. Therefore a unitary CFT has positive central charges: $c \geqslant 0$ and $\bar{c} \geqslant 0$. The only unitary CFT with $c = \bar{c} = 0$ is trivial and contains just the identity operator.

3.6 The Weyl anomaly

The failure of the components $(T,\widetilde{T})$ of the stress-energy tensor to be primary operators w.r.t. generic two-dimensional conformal transformations is a just a characteristic feature of conformal field theories, more precisely as we will see shortly of CFTs defined on a curved (two-dimensional) manifold. In the string theory context however, it leads to a potential disaster.

String theory was defined as a two-dimensional theory of gravity coupled to a set of scalar matter fields $\{x^\mu(\sigma^i)\}$. The theory has an enormous redundancy, since its gauge group corresponds to two-dimensional diffeomorphisms and Weyl transformations. After a suitable gauge-fixing procedure, we have obtained a path integral defined over a gauge slice, *i.e.* with a reference metric depending only on a handful of parameters. All this beautiful construction falls apart if the classical gauge symmetry of the theory is not satisfied at the quantum level.

The potential clash between gauge symmetry and quantum effects is associated with Weyl transformations. Heuristically it is not difficult to understand why. Computations in the two-dimensional quantum field theory we are dealing with have divergences, as in any other QFT, that should be regularized. If one uses dimensional regularization, the invariance under Weyl rescalings is lost, since it depends crucially on being in dimension two, see eqn. (2.84). Using alternatively a Pauli-Villars regularization, one would introduce a scale in the theory breaking explicitly scale invariance. As any sensible regulator breaks the Weyl symmetry, one may wonder if at the end of the computation, which is not covariant w.r.t. the Weyl gauge transformations, the invariance would be miraculously restored. It turns out not to be the case and the Weyl symmetry has potentially an *anomaly* of the gauge symmetry signaling the inconsistency of the theory.

To characterize this anomaly, one considers that the background geometry is fixed to some reference metric and consider whether the classical conservation laws associated with the symmetries of the theory are truly independent of the choice of reference metric. Classically, Weyl-invariance implies that the stress-energy tensor is traceless, see eqn. (3.20). Because of general covariance of the theory, a quantum violation of this condition is very restricted. In the quantum field theory context, one would like to see whether the operator $T^i{}_i$ inserted in an arbitrary correlation function gives zero. The more general parametrization is:

$$\langle T^i{}_i \cdots \rangle = a R[\hat{\gamma}] \langle \cdots \rangle \,, \tag{3.96}$$

where $R[\hat{\gamma}]$ is the Ricci scalar computed for the reference worldsheet metric $\hat{\gamma}$, given that the right-hand side of the equation should be a local expression, and of scaling dimension two; the parameter a is dimensionless.

Since the diffeomorphism symmetry is not anomalous, one can work in the conformal gauge, $ds^2 = \exp(2\omega) dw d\bar{w}$, in which case, the anomaly equation becomes, using eqn. (2.85b),

$$2\gamma^{w\bar{w}} \langle T_{w\bar{w}} \cdots \rangle = -2a e^{-2\omega} \nabla^2 \omega \langle \cdots \rangle \implies \langle T_{w\bar{w}} \cdots \rangle = -\frac{a}{2} \nabla^2 \omega \langle \cdots \rangle \,. \tag{3.97}$$

The conservation of the stress-energy tensor gives $\nabla^{\bar{w}} T_{w\bar{w}} + \nabla^w T_{ww} = 0$. Hence taking the derivative of (3.97) with $\nabla^{\bar{w}}$ one reaches the equation:

$$\langle \nabla^w T_{ww} \cdots \rangle = \frac{a}{2} \nabla^{\bar{w}} \nabla^2 \omega \langle \cdots \rangle \, . \tag{3.98}$$

Let us now consider an infinitesimal Weyl rescaling around the flat metric, *i.e.* $\omega = 1 + \delta\omega(w, \bar{w})$, and compare the variations of the left- and right-hand sides of eqn. (3.98).

In the previous section we have considered the transformation of the components of the stress-tensor of a two-dimensional CFT under an infinitesimal conformal transformation, *i.e.* an infinitesimal transformation $w \mapsto w + \varepsilon(w)$:

$$- \delta T(w) = \underbrace{\frac{c}{12} \partial^3 \varepsilon(w)}_{\text{Weyl}} + \underbrace{2T(w) \partial\varepsilon(w) + \varepsilon(w) \partial T(w)}_{\text{tensor}} \, , \tag{3.99}$$

where the last two terms correspond to the standard tensorial transformation under a change of coordinates, $T^{ij}(x^k) \mapsto \tilde{T}^{ij}(\tilde{x}^k) = (\partial \tilde{x}^i / \partial x^m)(\partial \tilde{x}^j / \partial x^n) T^{mn}(x^k)$; the first term corresponds to the anomalous change of the operator $T(w)$ under the compensating Weyl transformation that brings back the metric to the original one.

Under the infinitesimal compensating Weyl transformation $\delta\omega$ of the background metric, namely $\delta\omega = \frac{1}{2}(\partial\varepsilon + \bar{\partial}\bar{\varepsilon})$ (see eq. 2.100), the stress-energy tensor operator transforms therefore as:

$$\delta_{\delta\omega} T = -\frac{c}{6} \partial_w^2 \delta\omega \, . \tag{3.100}$$

Plugging into eqn. (3.98) one finds, using $\nabla^w = 2\partial_{\bar{w}}$ and $\nabla^2 = 4\partial_w \partial_{\bar{w}}$,

$$- \frac{c}{3} \partial_{\bar{w}} \partial_w^2 \delta\omega = 4a \partial_w^2 \partial_{\bar{w}} \delta\omega \quad \Longrightarrow \quad a = -\frac{c}{12} \, . \tag{3.101}$$

The conclusion of this computation is that

$$\langle T^i{}_i \cdots \rangle = -\frac{c}{12} R[\hat{\gamma}] \langle \cdots \rangle \, . \tag{3.102}$$

This result is not problematic if one considers a conformal field theory outside of the string theory context, in the same way as the axial anomaly of quantum electrodynamics does not imply that the latter is an inconsistent quantum field theory.

However if the two-dimensional conformal field theory at hand corresponds to the world-sheet theory of a string the anomaly (3.102) indicates that *the theory is inconsistent unless* $c = 0$. As we shall see this constraint will have far-reaching consequences.

A careful reader may have noticed that we could do exactly the same computation using $\nabla^w T_{w\bar{w}} + \nabla^{\bar{w}} T_{\bar{w}\bar{w}} = 0$, in which case we will reach the same equation as (3.102) with c replaced by $\bar{c}$. These two equations are not consistent with each other unless $c = \bar{c}$. It turns out that a conformal field theory with $c \neq \bar{c}$ cannot be coupled consistently to two-dimensional gravity; on top of the *Weyl anomaly* discussed here it has also a *gravitational anomaly*.

3.7 Conformal field theory with boundaries

We end up this chapter by saying a few words about conformal field theories on surfaces with boundaries, which is relevant to study open strings. The worldsheet for a propagating open string corresponds to an infinite strip, which can be represented in the complex plane w by

$$0 \leqslant \mathrm{Re}\,(w) \leqslant \pi \,, \quad \mathrm{Im}(z) \in \mathbb{R} \,. \tag{3.103}$$

To perform the analogue of radial quantization, is is convenient to use the conformal mapping from the strip to the upper half-plane[10], see fig. 3.4:

$$w \mapsto z = -e^{-iw} \,, \quad \bar{w} \mapsto \bar{z} = -e^{i\bar{w}} \,. \tag{3.104}$$

Under this map, an initial state $|\mathrm{in}\rangle$ prepared at Euclidean time $\tau \to -\infty$ becomes an operator sitting at the origin, on the real axis $\mathrm{Im}(z) = 0$.

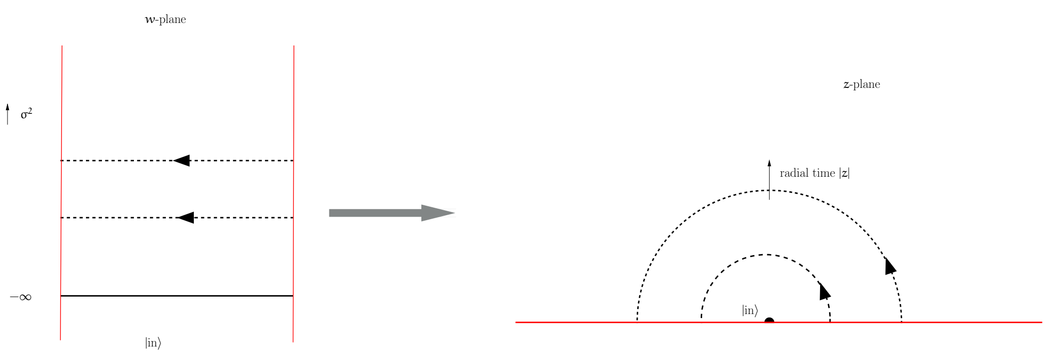

Figure 3.4: *Conformal mapping from the strip to the half-plane.*

Due to translation invariance along the boundary, such operator can be moved anywhere on the real axis. Hence we have learned an important fact: open string states correspond to *boundary operators* in the conformal field theory on the half-plane, *i.e.* operators that are defined on the real axis.

A natural choice of boundary conditions is that there is no flow of momentum across the boundary, *i.e.* that $T_{01} = 0$ on the real axis. In complex coordinates, it means that

$$T_{zz}(z) = T_{\bar{z}\bar{z}}(\bar{z})\Big|_{z=\bar{z}} \,. \tag{3.105}$$

These boundary conditions are called *conformal boundary conditions*, as they preserve one copy of the Virasoro algebra.

To see this, instead of having two currents T_{zz} and $T_{\bar{z}\bar{z}}$ defined on the upper half-plane, is it more convenient to define a unique current on the full complex z plane :

$$T_{zz}(z) := T_{\bar{z}\bar{z}}(\bar{z}') \,, \quad z' = \bar{z} \,, \quad \mathrm{Im}(z) < 0 \,. \tag{3.106}$$

[10] There's a conventional minus sign difference compared to the mapping used in the closed string case, see eqn. (3.11), in order to get the upper half-plane rather than the lower half-plane.

The boundary condition (3.105) ensures naturally that T_{zz} is continuous across the boundary.

Because of the boundary conditions, there exists a unique set of Virasoro generators. Let us define $\mathcal{C}_{1/2}$ as a half-circular contour in the upper half-plane around the origin, and $\mathcal{C}_0$ a closed circular contour around the oriign. The Virasoro generators are defined as :

$$L_n = \frac{1}{2i\pi}\int_{\mathcal{C}_{1/2}} \left(dz\, z^{n+1} T_{zz}(z) - d\bar{z}\, \bar{z}^{n+1} T_{\bar{z}\bar{z}}(\bar{z})\right) = \frac{1}{2i\pi}\oint_{\mathcal{C}_0} dz\, z^{n+1} T_{zz}(z)\,, \tag{3.107}$$

where we have used the definition (3.106) in the second equality. The OPE between T_{zz} and itself is unchanged, since it is a local property of the field theory. Hence we get the same Virasoro algebra (3.80) that we have obtained in the case of a CFT without boundary, however we have a single copy instead of two.

Another way to understand this is the following. The Virasoro generator L_n (resp. $\tilde{L}_n$) was the conserved charge associated to the holomorphic change of coordinates $\delta z = \epsilon(z) = z^{n+1}$ (resp. $\delta\bar{z} = \bar{\epsilon}(\bar{z}) = \bar{z}^{n+1}$). In the present context one should allow only transformations that preserve the boundary, *i.e.* the real axis. They should obey $\epsilon(z) = \bar{\epsilon}(\bar{z})|_{z=\bar{z}}$.

The highest weight representations $|h\rangle_{\rm B}$ of this Virasoro algebra correspond, under the state-operator correspondence, to *boundary primary operators* $\mathcal{O}_{\rm B}(x)$ defined on the real axis.

References

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Chapter 4

Free conformal field theories

The presentation of conformal field theories in two dimensions in chapter 3 was rather formal and abstract. We will look at simple examples of conformal field theories, as free massless scalar fields, that will eventually be the building blocks of bosonic string theory and later on of the superstring theories.

4.1 Free scalar fields

The basic ingredients of the Polyakov action (3.1) in conformal gauge and with a flat Minkowski-space time target space are a set of scalar fields $x^\mu(\sigma^i)$, $\mu \in \{0, 1, \ldots, D-1\}$, governed by the free quadratic action:

$$\mathcal{S} = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \eta_{\mu\nu} \, \delta_{ij} \, \partial^i x^\mu \partial^j x^\nu \, . \tag{4.1}$$

Let us consider a single space-like coordinate, *i.e.* a free two-dimensional massless scalar field with action

$$\mathcal{S} = \frac{1}{4\pi\alpha'} \int d^2\sigma \; \delta_{ij} \, \partial^i x \partial^j x \, . \tag{4.2}$$

The stress-energy tensor deduced from this action is given, following eqn. (3.17), by the expression:

$$T_{ij} = \frac{1}{\alpha'} \partial_i x \partial_j x - \frac{1}{2\alpha'} \delta_{ij} \partial_k x \partial^k x \, , \tag{4.3}$$

In complex coordinates, the action becomes:

$$\mathcal{S} = \frac{1}{2\pi\alpha'} \int d^2 w \, \partial x \bar{\partial} x \, , \tag{4.4}$$

From now on we will consider the conformal field theory on the complex plane. The non-vanishing components of the stress tensor are

$$T(z) = -\frac{1}{\alpha'} \, \partial x \partial x \, , \tag{4.5a}$$

$$\widetilde{T}(\bar{z}) = -\frac{1}{\alpha'} \, \bar{\partial} x \bar{\partial} x \, , \tag{4.5b}$$

with $T_{z\bar{z}} = 0$ because of two-dimensional conformal invariance of the action as stressed in the previous chapter. The equations of motion following from (4.4) are simply

$$\partial \bar{\partial} x(z, \bar{z}) = 0 \, , \tag{4.6}$$

hence x is the sum of a holomorphic and a anti-holomorphic function:[1]

$$x(z, \bar{z}) = x_L(z) + x_R(\bar{z}) \tag{4.7}$$

[1] Whenever the target-space coordinate corresponding to the field $x(z, \bar{z})$ is non-compact, the (anti-) holomorphic functions $x_L(z)$ and $x_R(\bar{z})$ do not define by themselves consistent fields, are they are not univalued on the plane, see eq. (4.31) below.

On the Minkwoskian cylinder, *i.e.* with $z = e^{-i(\tau+\sigma)}$ and $\bar{z} = e^{-i(\tau-\sigma)}$, the first term describes left-moving waves and the second term right-moving ones. For this reason, the Virasoro algebras corresponding to holomorphic and anti-holomorphic conformal transformations are called respectively left-moving and right-moving algebras.

The free scalar action (4.4) has a global symmetry on top of conformal symmetry, the invariance under translations in target space:

$$x \mapsto x + a \,. \tag{4.8}$$

From Noether theorem, and using the equation of motion (4.6), one finds conserved holomorphic and anti-holomorphic currents:

$$J(z) := J_z(z) = \frac{1}{\alpha'}\,\partial x \,, \quad \bar{\partial} J = 0 \tag{4.9a}$$

$$\widetilde{J}(\bar{z}) := J_{\bar{z}}(\bar{z}) = \frac{1}{\alpha'}\,\bar{\partial} x \,, \quad \partial \widetilde{J} = 0 \tag{4.9b}$$

Invariance of the action (4.4) under conformal transformations indicates that x^μ has conformal weights $(0,0)$, hence J (resp. $\widetilde{J}$) has conformal weights $(1,0)$ (resp. conformal weights $(0,1)$).

One can apply the residue formula (3.37), that was established for conformal transformation but holds for any (anti)holomorphic conserved current in a two-dimensional CFT, if one splits the translation of x into a translation of $x_L(z)$ by $a/2$ and a translation of $x_R(\bar{z})$ by $a/2$. It leads to the relation:

$$\begin{aligned} a/2 = \delta x_L &= -\mathrm{Res}_{z'\to z}\left(aJ(z')x(z,\bar{z})\right) \\ &\Longrightarrow \quad a\,\frac{1}{\alpha'}\partial x(z')x(z,\bar{z}) = \cdots - \frac{a}{2(z'-z)} + \text{regular.} \end{aligned} \tag{4.10}$$

which implies that, using the same reasoning for the anti-holomorphic current

$$\partial x(z')x(z,\bar{z}) = -\frac{\alpha'}{2(z'-z)} + \text{reg.} \tag{4.11a}$$

$$\bar{\partial} x(\bar{z}')x(z,\bar{z}) = -\frac{\alpha'}{2(\bar{z}'-\bar{z})} + \text{reg.} \tag{4.11b}$$

Because x has scaling dimensions $(0,0)$, and ∂x (resp. $\bar{\partial} x$) has scaling dimension $(1,0)$ (resp. $(0,1)$) it is actually impossible to get terms more singular than $1/(z'-z)$, as they would be multiplied by operators of negative weights, which are forbidden in unitary conformal field theories. Differenciating once with respect to z one gets the OPE between the currents

$$J(z')J(z) = -\frac{1}{2\alpha'(z'-z)^2} + \text{reg.} \tag{4.12a}$$

$$\widetilde{J}(\bar{z}')\widetilde{J}(\bar{z}) = -\frac{1}{2\alpha'(\bar{z}'-\bar{z})^2} + \text{reg.} \tag{4.12b}$$

Finally, integrating equations (4.11) w.r.t. z and $\bar{z}$, one finds that the OPE between $x(z,\bar{z})$ and itself is given by

$$\boxed{x(z',\bar{z}')x(z,\bar{z}) = -\frac{\alpha'}{2}\log\left(\mu^2(z-z')(\bar{z}-\bar{z}')\right) + \text{reg.}} \tag{4.13}$$

where the parameter μ is an infrared cut-off which is here for dimensional reasons. The singular term on the right-hand side of this equation is actually the (perhaps familiar) position space scalar Green function in two dimensions, that will be computed directly later on, see eqn. (6.9). The present derivation was putting an emphasis on the symmetries of the theory. Unlike Green functions in higher dimensions, this one does not fall off at large distances, which eventually implies the absence of phase transitions for continuous symmetries in two dimensions.

4.1.1 Stress-energy tensor

The classical expressions (4.5) of the components of the stress-energy tensor should actually be regularized in the quantum theory.

Normal-ordered product

We define first the *normal-ordered* product of operators by subtracting the short-distances singularities when the two operators approach each other:

$$:\mathcal{O}_1(z')\mathcal{O}_2(z): \; := \mathcal{O}_1(z')\mathcal{O}_2(z) - \overbracket{\mathcal{O}_1(z')\mathcal{O}_2(z)}\,, \tag{4.14}$$

where $\overbracket{\mathcal{O}_1(z')\mathcal{O}_1(z)}$ means all terms involving negative powers of $(z'-z)$ in the OPE between $\mathcal{O}_1$ and $\mathcal{O}_2$. The relation between this normal ordering and the usual one (annihilation operators to the right) will become more transparent later on. Note that eq. (4.14) is an operator equation that should hold inserted in any correlation function; as such, the first term on the right-hand side should be understood as a time-ordered product (or rather in the present context as a radial-ordered product).

When $\mathcal{O}_1$ and $\mathcal{O}_2$ are at coincident points we define the normal order product as a limit, namely:

$$\boxed{:\mathcal{O}_1\mathcal{O}_2:(z) = \lim_{z'\to z}\left(\mathcal{O}_1(z')\mathcal{O}_2(z) - \overbracket{\mathcal{O}_1(z')\mathcal{O}_1(z)}\right)} \tag{4.15}$$

A convenient way of rewriting this expression uses a contour integral:

$$:\mathcal{O}_1\mathcal{O}_2:(z) = \frac{1}{2i\pi}\int_{\mathcal{C}_z}\frac{dz'}{z'-z}\mathcal{O}_1(z')\mathcal{O}_2(z)\,, \tag{4.16}$$

where $\mathcal{C}_z$ is a contour around z. The integral picks only the regular terms in the OPE of $\mathcal{O}_1$ and $\mathcal{O}_2$ by construction.

Finally we will consider in several computations the OPE between normal-ordered operators. One has the following generalization of *Wick theorem* to arbitrary interacting two-dimensional CFTs:

$$\wick{\c1{\mathcal{O}}_1(z_1) : \c1{\mathcal{O}}_2 \mathcal{O}_3 :}(z_3) = \frac{1}{2i\pi}\int_{\mathcal{C}_{z_3}} \frac{\mathrm{d}z_2}{z_2 - z_3}\left(\wick{\c1{\mathcal{O}}_1(z_1)\c1{\mathcal{O}}_2(z_2)}\mathcal{O}_3(z_3) + \mathcal{O}_2(z_2)\wick{\c1{\mathcal{O}}_1(z_1)\c1{\mathcal{O}}_3(z_3)}\right) \tag{4.17}$$

It can be applied iteratively to deal with more complicated operator products.

4.1.2 OPE of the stress-energy tensor

In terms of the normal product, applied to the bilinear of the current J of self-OPE (4.12), the quantum expression of $T(z)$ reads:

$$T(z) = -\alpha' : JJ : (z)\,, \tag{4.18a}$$

$$\widetilde{T}(\bar{z}) = -\alpha' : \widetilde{J}\widetilde{J} : (\bar{z})\,, \tag{4.18b}$$

with here:

$$: JJ : (z) = \lim_{z' \to z}\left(J(z')J(z) - \wick{\c1 J(z')\c1 J(z)}\right) = \lim_{z' \to z}\left(J(z')J(z) + \frac{\alpha'}{2(z'-z)^2}\right)\,. \tag{4.19}$$

In terms of the fundamental field x and its derivatives the stress energy tensor takes the form:

$$T(z) = -\frac{1}{\alpha'} : \partial x \partial x : (z)\,, \tag{4.20a}$$

$$\widetilde{T}(\bar{z}) = -\frac{1}{\alpha'} : \bar{\partial} x \bar{\partial} x : (\bar{z})\,. \tag{4.20b}$$

Using this definitions (4.20) we will now show that J and $\widetilde{J}$ are conformal primary operators. One needs to compute the OPE:

$$T(z')J(z) = -\frac{1}{(\alpha')^2} : \partial x \partial x : (z')\, \partial x(z) \tag{4.21}$$

The OPE between the composite operator T and J is computed using (4.17). In the present case the computation is rather simple as the self-contraction of J gives just the identity operator. Explicitly we have

$$\begin{aligned}
\wick{\c1 T(z')\c1 J(z)} &= -\frac{1}{(\alpha')^2} \wick{: \partial x \c1{\partial} x : (z')\, \c1{\partial} x(z)} \\
&= -\frac{1}{(\alpha')^2} \wick{\c1{\partial} x(z')\partial x(z')\c1{\partial} x(z)} - \frac{1}{(\alpha')^2} \wick{\partial x(z')\c1{\partial} x(z')\c1{\partial} x(z)} \\
&= -\frac{2}{(\alpha')^2}\left(-\frac{\alpha'}{2(z'-z)^2}\right)\partial x(z') \\
&= \frac{J(z)}{(z'-z)^2} + \frac{\partial J(z)}{(z'-z)}\,,
\end{aligned} \tag{4.22}$$

where we have used a Taylor expansion of the current $J(z')$ around z in the last step,

$$J(z') = J(z) + (z'-z)\partial J(z) + \mathcal{O}\left((z'-z)^2\right) \tag{4.23}$$

In order to determine the central charge of the free scalar conformal field theory, one needs to compute the OPE satisfied by T and $\widetilde{T}$, using the OPE (4.13) of the elementary field and the definitions (4.20). One has to use the Wick contractions:

$$\begin{aligned}\frac{1}{(\alpha')^2}\wick{\c1 T(z')\c1 T(z)} &= \wick{:J\c1 J:(z'):\c1 JJ:(z)} = \wick{:\c1 J(z')J(z')\c1 J(z)J(z):} \\ &+ \wick{:\c1 J(z')J(z')J(z)\c1 J(z):} + \wick{:J(z')\c1 J(z')\c1 J(z)J(z):} + \wick{:J(z')\c1 J(z')J(z)\c1 J(z):} \\ &+ \wick{\c1 J(z')\c2 J(z')\c1 J(z)\c2 J(z)} + \wick{\c1 J(z')\c2 J(z')\c2 J(z)\c1 J(z)}\end{aligned} \tag{4.24}$$

which gives, using at the last step the Taylor expansion,

$$:J(z')J(z): \; = \; :J(z)J(z): + (z'-z):\partial J(z)J(z): + \mathcal{O}\left((z'-z)^2\right) = \; :JJ:(z) + \frac{z'-z}{2}\partial :JJ:(z) + \cdots \tag{4.25}$$

the OPE (3.68) with central charge $c = 1$:

$$T(z')T(z) = \frac{1}{2(z'-z)^4} + \frac{2T(z)}{(z'-z)^2} + \frac{\partial T(z)}{z'-z} + \text{reg.} \tag{4.26}$$

One can do the same exercise with the anti-holomorphic component $\widetilde{T}$, and one reaches the conclusion that the free scalar field conformal field theory has central charges $(c, \bar{c}) = (1, 1)$.

4.1.3 Mode expansions and Virasoro algebra

The currents J and $\widetilde{J}$, being respectively holomorphic and anti-holomorphic, admit naturally on the plane an expansion in terms of Laurent modes, as the stress-energy tensor components. Using standard conventions and normalization, one has

$$J(z) = \frac{1}{\alpha'}\partial x(z) = -\frac{i}{\sqrt{2\alpha'}}\sum_{n\in\mathbb{Z}}\frac{\alpha_n}{z^{n+1}} \tag{4.27a}$$

$$\widetilde{J}(\bar{z}) = \frac{1}{\alpha'}\bar{\partial} x(\bar{z}) = -\frac{i}{\sqrt{2\alpha'}}\sum_{n\in\mathbb{Z}}\frac{\widetilde{\alpha}_n}{\bar{z}^{n+1}} \tag{4.27b}$$

with

$$\alpha_n = i\sqrt{2\alpha'}\oint_{\mathcal{C}_0}\frac{dz}{2i\pi}z^n J(z) \tag{4.28a}$$

$$\widetilde{\alpha}_n = -i\sqrt{2\alpha'}\oint_{\mathcal{C}_0}\frac{d\bar{z}}{2i\pi}\bar{z}^n \widetilde{J}(\bar{z})\,. \tag{4.28b}$$

These coefficients can also be defined from the expansion of $x(z)$ itself. Integrating (4.27) one gets

$$x(z,\bar{z}) = x_c - i\sqrt{\frac{\alpha'}{2}}\left(\alpha_0 \ln z + \widetilde{\alpha}_0 \ln \bar{z}\right) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}^\star} \frac{1}{n}\left(\frac{\alpha_n}{z^n} + \frac{\widetilde{\alpha}_n}{\bar{z}^n}\right). \tag{4.29}$$

If the field x has a non-compact target space, it should be single-valued hence one should impose $\alpha_0 = \widetilde{\alpha}_0$.[2] The interpretation of this coefficient is quite obvious, since $J(z)$ and $\widetilde{J}(\bar{z})$ are the components of the Noether current for space translations. It gives then the momentum conjugate to the zero-mode x_0:[3]

$$p_c = \frac{1}{2\pi}\left(\oint_{\mathcal{C}_0} dz\, J(z) - \oint_{\mathcal{C}_0} d\bar{z}\, \widetilde{J}(\bar{z})\right) = \frac{\alpha_0 + \widetilde{\alpha}_0}{\sqrt{2\alpha'}} = \sqrt{\frac{2}{\alpha'}}\alpha_0 \,. \tag{4.30}$$

We obtain then the final form of the expansion of x:

$$x(z,\bar{z}) = x_c - i\frac{\alpha'}{2}\, p_c \, \ln |z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}^\star} \frac{1}{n}\left(\frac{\alpha_n}{z^n} + \frac{\widetilde{\alpha}_n}{\bar{z}^n}\right). \tag{4.31}$$

Expressed in terms of cylinder coordinates, and after continuation to Minkowski space-time, the meaning of this expansion is perhaps more clear:

$$x(\sigma,\tau) = x_c + \alpha' p_c \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}^\star} \frac{1}{n}\left(\alpha_n e^{-in(\tau+\sigma)} + \widetilde{\alpha}_n e^{-in(\tau-\sigma)}\right). \tag{4.32}$$

In this expression, $x_c + \alpha' p_c \tau$ describes the center-of-mass motion of the string, while the oscillator modes are respectively left-moving and right-moving plane waves propagating on the string worldsheet.

The modes $\{\alpha_n\}$ obey an algebra that is obtained in the same way as the Virasoro algebra was derived before, see around equations (3.76). As there we consider the circular contours $\mathcal{C}$ of radius R and $\mathcal{C}'$ of radius $R' > R$, both around the origin, and define the commutator

$$\begin{aligned}
[\alpha_m, \alpha_n] &= -2\alpha' \left(\oint_{\mathcal{C}'} \frac{dz_1}{2i\pi} \oint_{\mathcal{C}} \frac{dz_2}{2i\pi} - \oint_{\mathcal{C}'} \frac{dz_2}{2i\pi} \oint_{\mathcal{C}} \frac{dz_1}{2i\pi}\right) z_1^m z_2^n J(z_1) J(z_2) \\
&= -2\alpha' \oint \frac{dz_2}{2i\pi} z_2^n \mathrm{Res}_{z_1\to z_2} \left(z_1^m J(z_1) J(z_2)\right) \\
&= -2\alpha' \oint \frac{dz_2}{2i\pi} z_2^n \mathrm{Res}_{z_1\to z_2} \left(-\frac{z_2^m + m(z_1 - z_2) z_2^{m-1}}{2\alpha'(z_1 - z_2)^2}\right).
\end{aligned} \tag{4.33}$$

We obtain then

$$\boxed{[\alpha_m, \alpha_n] = m\delta_{m+n,0} \,,} \tag{4.34}$$

[2] The case of compact x is actually very important for string theory; we will come back to it in due time.

[3] A factor of i in the definition of p_0 was added in order to define a Hermitian operator – in fact the currents J and $\widetilde{J}$ that we have defined are anti-Hermitian, which is the mathematicians' conventions for generators of a Lie algebra.

with a similar algebra for the modes $\{\widetilde{\alpha}_n\}$. On top of this, the zero-modes obey the usual canonical commutation relations:

$$[x_c, p_c] = i\,. \tag{4.35}$$

The infinite-dimensional algebra (4.34) whose origin was translation symmetry in the target space of the field x is called an *affine Lie algebra*, constructed from the (trivial) Lie algebra $\mathfrak{u}(1)$ of the translation group (the latter being isomorphic to $\mathbb{R}$). In general, starting for any classical Lie algebra

$$[j^a, j^b] = f^{ab}{}_c j^c\,, \tag{4.36}$$

one can construct an affine Lie algebra in a similar fashion:

$$[j^a_m, j^b_n] = f^{ab}{}_c j^c_{m+n} + \tfrac{\kappa^{ab}}{2} m \delta_{m+n,0}\,. \tag{4.37}$$

These algebras play an important role for describing string theory in curved space-time, but we won't have time to present this in these lectures.

Given that the components T, $\widetilde{T}$ of the stress-energy tensor are expressed in terms of the currents J, $\widetilde{J}$, see eqn. (4.18), it is clear that the generators $\{L_n\}$ of the Virasoro algebra can be expressed in terms of the generators $\{\alpha_n\}$ of the affine Abelian algebra $\mathfrak{u}(1)$. From the definition (3.74) of the Virasoro generators, one gets

$$L_n = \frac{1}{2}\oint_{\mathcal{C}_0} \frac{dz}{2i\pi} z^{n+1} \sum_{r,s} \alpha_r \alpha_s z^{-r-s-2} = \frac{1}{2}\sum_{r\in\mathbb{Z}} \alpha_r \alpha_{n-r}\,, \quad \forall n \neq 0\,. \tag{4.38}$$

For L_0 more care is needed as α_n and α_{-n} do not commute with each other. Using the expression (4.16) of the normal-ordered product at coincident points, one gets

$$L_0 = -\alpha' \int_{\mathcal{C}_0} \frac{dz}{2i\pi} z \,{:}JJ{:}\,(z) = -\alpha' \int_{\mathcal{C}_0} \frac{dz}{2i\pi} z \oint_{\mathcal{C}_z} \frac{dz'}{2i\pi} \frac{1}{z'-z} J(z')J(z) \tag{4.39}$$

One can employ the same contour manipulation as described on figure 3.3 backwards, and write this expression as

$$L_0 = \frac{1}{2}\sum_{r,s\in\mathbb{Z}} \left(\alpha_r \alpha_s \oint_{\mathcal{C}'} \frac{dz'}{2i\pi} \oint_{\mathcal{C}} \frac{dz}{2i\pi} \frac{z}{z'-z} (z')^{-r-1} z^{-s-1} - \alpha_s \alpha_r \oint_{\mathcal{C}'} \frac{dz}{2i\pi} \oint_{\mathcal{C}} \frac{dz'}{2i\pi} \frac{z}{z'-z} (z')^{-r-1} z^{-s-1} \right) \tag{4.40}$$

where we have taken into account that the operator product should be radial ordered.

In the first term, to perform the integral over z, as by definition $|z| < |z'|$ on $\mathcal{C}$, one can expand

$$\frac{1}{z'-z} = \frac{1}{z'} \sum_{n=0}^{\infty} (z/z')^n\,, \tag{4.41}$$

and compute

$$\oint_{\mathcal{C}} \frac{dz}{2i\pi} \frac{1}{z'-z} z^{-s} (z')^{-r-1} = \sum_{n=0}^{\infty} (z')^{-r-n-2} \oint_{\mathcal{C}} \frac{dz}{2i\pi} z^{n-s} = \begin{cases} (z')^{-r-s-1}\,, & s \geqslant 1 \\ 0\,, & s < 1 \end{cases} \tag{4.42}$$

In the second term, to perform the integral over z', as by definition $|z'| < |z|$ on $\mathcal{C}$, one can expand

$$\frac{1}{z'-z} = -\frac{1}{z}\sum_{n=0}^{\infty}(z'/z)^n\,, \tag{4.43}$$

and compute

$$\oint_{\mathcal{C}} \frac{dz'}{2i\pi}\frac{1}{z'-z}z^{-s}(z')^{-r-1} = -\sum_{n=0}^{\infty} z^{-s-n-1}\oint_{\mathcal{C}}\frac{dz}{2i\pi}(z')^{n-r-1} = \begin{cases} -z^{-r-s-1}\,, & r \geqslant 0 \\ 0\,, & r < 0 \end{cases} \tag{4.44}$$

Putting everything together one has

$$\begin{aligned} L_0 &= \frac{1}{2}\sum_{r\in\mathbb{Z}}\sum_{s=1}^{\infty}\alpha_r\alpha_s\oint\frac{dz'}{2i\pi}(z')^{-r-s-1} + \frac{1}{2}\sum_{r=0}^{\infty}\sum_{s\in\mathbb{Z}}\alpha_s\alpha_r\oint\frac{dz}{2i\pi}z^{-r-s-1} \\ &= \frac{1}{2}\alpha_0^2 + \sum_{r=1}^{\infty}\alpha_{-r}\alpha_r \\ &= \frac{\alpha'}{4}p^2 + \sum_{r=1}^{\infty}\alpha_{-r}\alpha_r \end{aligned} \tag{4.45}$$

Hence one realizes that the normal ordering (4.16) implies the familiar notion of normal ordering, as for the harmonic oscillator (creation operator to the left). In this respect one can write the latter expression as

$$\boxed{L_0 = \frac{\alpha'}{4}p^2 + \frac{1}{2}\sum_{r\neq 0} {}^{\circ}_{\circ}\,\alpha_{-r}\alpha_r\,{}^{\circ}_{\circ}} \tag{4.46}$$

where the normal ordering ${}^{\circ}_{\circ} \cdot {}^{\circ}_{\circ}$ of modes means that the positive modes should be put at the end.

One can check easily that the commutations relations involving (4.38) and (4.45), computed using (4.34) and (4.35), reproduces the Virasoro algebra (3.80).

To summarize, we have obtained a Virasoro algebra from an affine Lie algebra; this construction holds for non-Abelian affine algebras (4.37) as well.

4.1.4 Primary states and descendants

We are now in position to describe the Hilbert space of the conformal field theory of a free scalar in two dimensions. As we have already noticed, a special role is played by primary states, that are analogous to highest weight states in representation theory of Lie algebras. Using the state-operator correspondence, such primary states are mapped to primary local operators on the plane as was already discussed.

As the theory contains two copies of the Virasoro algebra (and of the underlying affine algebra), we will focus the discussion on the holomorphic one.

In terms of the Virasoro generators, a primary state $|h\rangle$ is annihilated by the positive modes and characterized by its L_0 eigenvalue:

$$\forall n > 0 \ , \quad L_n|h\rangle = 0 \ , \quad L_0|h\rangle = h|h\rangle \, . \tag{4.47}$$

Other – generically non-primary – states are obtained by acting with the creation operators. We generate this way a *Verma module*, whose generic state looks like

$$|\psi\rangle = \cdots (L_{-n})^{N_n} \cdots (L_{-2})^{N_2}(L_{-1})^{N_1}|h\rangle \tag{4.48}$$

which is the infinite-dimensional analogue of a representation of a Lie algebra. The quantity $N = \sum_{n=1}^{\infty} nN_n$ is called the *level* of the state.

In general such representation is irreducible. Exceptions to this rule, in $c = 1$ CFTs, occur in the representations generated by primary states of dimension $h = j^2$ with $j \in \frac{1}{2}\mathbb{Z}^+$, which contains a *null vector* at level $2j+1$, *i.e.* a state annihilated by all positive L_n, $n > 0$.

This is true in particular the vacuum state $|0\rangle$, which is by definition the primary state of conformal dimension $h = 0$. Since $||L_{-1}|0\rangle||^2 = \langle 0|[L_1, L_{-1}]|0\rangle = 0$, this state is also annihilated by L_{-1} (this is easy to see in the operator language: $L_{-1}\mathcal{O}_0 \leftrightarrow \partial 1$), therefore $L_{-1}|0\rangle$ is a null vector. Hence the whole corresponding submodule, *i.e.* all states obtained from it by creation operators, should be substracted from the Verma module; the vacuum representation is said to be *degenerate*.

For $j = 1/2$, *i.e.* conformal dimension $h = 1/4$, one can check easily using the Virasoro algebra (3.80) with $c = 1$ that a null vector at level one can be obtained as follows:

$$\forall n > 0 \ , \quad L_n\big(L_{-2}|\tfrac{1}{4}\rangle - (L_{-1})^2|\tfrac{1}{4}\rangle\big) = 0 \, . \tag{4.49}$$

It is easy to see that, as a consequence, $||(L_{-2}-(L_{-1})^2)|\frac{1}{4}\rangle|| = 0$. Likewise for $j = 1$ one finds a null vector at level two, and so on. Conformal field theories with central charges $c < 1$ have a more complicated pattern of null states, that play an essential role in solving the theory algebraically.

It is always more efficient to classify states according to the largest possible symmetry of the theory, which is in this case the affine symmetry generated by translation invariance of the free scalar field action. In the present context one advantage is that the corresponding representation theory is simpler, due to the absence of null vectors. One defines then the highest weight states as follows

$$\forall n > 0 \ , \quad \alpha_n|p\rangle = 0 \ , \quad \alpha_0|p\rangle = \sqrt{\frac{\alpha'}{2}}p|p\rangle \, . \tag{4.50}$$

Using the expression (4.38) for the Virasoro generators in terms of the affine generators, one realizes that $|p\rangle$ is also a Virasoro primary:

$$\forall n > 0, L_n|p\rangle = \frac{1}{2}\sum_{r\in\mathbb{Z}} \alpha_r\alpha_{n-r}|p\rangle = 0 \, , \tag{4.51}$$

since each term in the sum contains at least one positive mode. The conformal dimension of this primary state is given, following (4.45), by

$$L_0|p\rangle = h|p\rangle\ , \quad h = \tfrac{\alpha'}{4}p^2\,. \tag{4.52}$$

A generic state in the module constructed from the highest weight state $|p\rangle$ is

$$|\psi\rangle = \cdots(\alpha_{-n})^{N_n}\cdots(\alpha_{-2})^{N_2}(\alpha_{-1})^{N_1}|p\rangle\,. \tag{4.53}$$

Using the Virasoro algebra one has

$$L_0|\psi\rangle = L_0\cdots(\alpha_{-n})^{N_n}\cdots(\alpha_{-2})^{N_2}(\alpha_{-1})^{N_1}|p\rangle\ = \left(\sum_n nN_n + \tfrac{\alpha'}{4}p^2\right)|\psi\rangle\,, \tag{4.54}$$

giving the dimension of a generic affine descendant state in terms of its momentum and its level

$$N := \sum_n nN_n\,. \tag{4.55}$$

In contrast with the Virasoro symmetry, the Verma module, *i.e.* the set of states of the form (4.53) for given p, contains no null vectors unless one considers the vacuum state $|0\rangle$ since $\alpha_{-1}|0\rangle = 0$.

From the point of view of the Virasoro symmetry, as long as $\frac{\alpha'}{4}p^2 \notin (\mathbb{Z}+\frac{1}{2})^2$, the irreducible affine $u(1)$ representation gives an irreducible Virasoro representation (otherwise, it should be decomposed accordingly).

4.1.5 Operator description

Using the state-operator correspondence, the analysis of the state space of the conformal field theory can be rephrased in terms of operators. Let us consider a primary operator of the left-moving and right-moving affine $u(1)$ algebras. These algebras being infinite-dimensional generalizations of the algebra corresponding to the translation symmetry, it is natural to look for operators corresponding to the primary states in the form of plane waves. These are called *vertex operators*:

$$\mathcal{V}_p(z,\bar z) = {:}e^{ipx}{:}(z,\bar z) = 1 + ipx(z,\bar z) + \frac{(ip)^2}{2!}{:}x^2{:}(z,\bar z) + \cdots \tag{4.56}$$

Let us check that it is indeed a primary operator. One has

$$\alpha_n\mathcal{V}_p(0,0) = i\sqrt{2\alpha'}\oint_{\mathcal{C}_0}\frac{dz}{2i\pi}z^nJ(z){:}e^{ipx}{:}(0,\bar 0)\,, \tag{4.57}$$

with $\mathcal{C}_0$ a contour encircling the origin. One has the OPE:

$$\begin{aligned} J(z){:}e^{ipx}{:}(0,\bar 0) &= \frac{1}{\alpha'}\partial x(z)\sum_n\frac{(ip)^n}{n!}{:}x^n{:}(0) = \frac{1}{\alpha'}\sum_n\frac{(ip)^n}{(n-1)!}\partial x(z)x(0){:}x^{n-1}{:}(0) \\ &= \frac{-ip}{2z}{:}e^{ipx}{:}(0,\bar 0) + \text{reg.} \end{aligned} \tag{4.58}$$

which implies that (4.57) vanishes when $n > 0$.

Equation (4.57) also gives the expression of descendant states, *i.e.* those obtained from the primary states from the action of a negative mode of the current, in terms of normal-ordered products of operators. As only non-singular terms from the OPE between J and $\mathcal{V}_p$ contribute one has simply

$$\begin{aligned}\forall n > 0\,,\ \alpha_{-n}\mathcal{V}_p(0,0) &= i\sqrt{\frac{2}{\alpha'}}\oint_{\mathcal{C}_0}\frac{dz}{2i\pi}\frac{1}{z^n}:\partial x(z)e^{ipx}:(0,\bar{0})\\ &= \sqrt{\frac{2}{\alpha'}}\frac{i}{(n-1)!}:\partial^n x\, e^{ipx}:(0,0)\,.\end{aligned}\tag{4.59}$$

Finally, one can notice that the current itself, J, is obtained from the action of α_{-1} on the vacuum state $|0\rangle$. As we have noticed already, it is a null vector of the vacuum representation, hence a primary state itself. It is consistent with the OPE (4.21), which indicates that J is a conformal primary of dimension $(1,0)$.

One may wonder about the status of the field $x(z,\bar{z})$ itself, which has conformal dimensions $(0,0)$. It does not actually correspond to a conformal operator, as can be realized from the OPE (4.13) which depends on an infrared cutoff. Another way to see this is that inserting $x(z,\bar{z})$ at the origin of the plane corresponds to applying the zero-mode x_c on the vacuum, and $x_c|0\rangle$ is non-normalizable because of the infinite volume of target space. For this reason this state is not bound to satisfy the unitarity constraint (3.94).

4.1.6 Bosonic strings in D dimensions

To close this section, one can repeat the same analysis for the set of D scalar fields corresponding to the embedding of the string in D dimensional Minkowski space-time. One gets the current algebras

$$J^\mu(z)J^\nu(z') = -\frac{\eta^{\mu\nu}}{2\alpha'(z-z')^2}\quad,\qquad \widetilde{J}^\mu(\bar{z})\widetilde{J}^\nu(\bar{z}') = -\frac{\eta^{\mu\nu}}{2\alpha'(\bar{z}-\bar{z}')^2}\,.\tag{4.60}$$

Equivalently one has the commutators

$$[\alpha^\mu_m,\alpha^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu}\ ,\quad [\widetilde{\alpha}^\mu_m,\widetilde{\alpha}^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu}\,.\tag{4.61}$$

The corresponding Virasoro algebras, built from

$$T(z) = -\alpha'\eta_{\mu\nu}:J^\mu J^\nu:(z)\,,\tag{4.62a}$$

$$\widetilde{T}(\bar{z}) = -\alpha'\eta_{\mu\nu}:\widetilde{J}^\mu\widetilde{J}^\nu:(\bar{z})\,,\tag{4.62b}$$

give the central charges $(c,\bar{c}) = (D,D)$. An example of vertex operator, which is a descendant state, is given by:

$$\alpha^\mu_{-1}\widetilde{\alpha}^\nu_{-1}\mathcal{V}_p(0,0) = :\partial x^\mu\partial x^\nu e^{ip\cdot x}:(0,0)\,.\tag{4.63}$$

as we will see, this is the state corresponding to the space-time graviton.

Because the target space has Minkowskian signature, the kinetic term for $x^0(z, \bar{z})$ has the wrong sign. Consequently, states obtained from the action of the corresponding modes α^0_{-n} and $\widetilde{\alpha}^0_{-n}$ have negative norm. While this is a problem at the level of the conformal theory, we will see that all such states are removed from the physical spectrum of the string theory itself.

Note finally that, unlike the generators of translations, the generators of Lorentz transformations and rotations in target space cannot be promoted to affine currents in the conformal field theory; this is a consequence of the previous observation that $x^\mu(z, \bar{z})$ are not themselves conformal operators.

4.1.7 Free bosons on the upper half-plane

In order to describe open string one has to consider the conformal field theory of free bosons on the upper half-plane. Recall that, for any conformal field theory, one copy of the conformal algebra is preserved if one imposes the conformal boundary conditions

$$T_{zz}(z) = T_{\bar{z}\bar{z}}(\bar{z})\Big|_{z=\bar{z}} . \tag{4.64}$$

We will actually impose stronger boundary conditions, that preserve one copy of the affine symmetry associated with the theory. As we have seen above, that the theory of a free boson $x(z, \bar{z})$ on the plane is associated with a pair of holomorphically conserved currents :

$$J(z) = \frac{1}{\alpha'}\partial x \ , \quad \tilde{J}(\bar{z}) = \frac{1}{\alpha'}\bar{\partial} x \, . \tag{4.65}$$

The boundary conditions for these currents on the real axis correspond actually to the Dirichlet and Neumann boundary conditions that were discussed in chapter 3, section 2.5:

- Neumann : $\quad J(z) = \tilde{J}(\bar{z})\Big|_{z=\bar{z}}$ (4.66a)
- Dirichlet : $\quad J(z) = -\tilde{J}(\bar{z})\Big|_{z=\bar{z}}$ (4.66b)

Using eq. (4.18), it is obvious that both affine boundary conditions imply the conformal boundary conditions (4.64).

Neumann boundary conditions. We consider first the mode expansion for a boson with Neumann boundary conditions. Because of eqn. (4.66a). the Fourier modes should by identified as

$$\forall n \neq 0 \ , \quad \alpha_n = \tilde{\alpha}_n \, . \tag{4.67}$$

On top of that, single-valuedness of the field $x(z, \bar{z})$ on the plane (rather, on the doubled half-plane) imposes that $\alpha_0 = \tilde{\alpha}_0$. This zero-mode is related to the momentum in target space, however with a different normalisation as for closed strings as there is only one copy of the current algebra, following the same reasoning as in eq. (3.106) (compare with eq. (4.30)):

$$p_c = \frac{1}{2\pi}\oint_{C_0} \mathrm{d}z \, J = \frac{\alpha_0}{\sqrt{2\alpha'}} \, . \tag{4.68}$$

This momentum is associated with translations $x \mapsto x + a$, which are preserved by Neumann boundary conditions. One can write finaly we the expansion :

$$x(z,\bar{z}) = x_c - i\alpha' p_c \ln|z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}^\star} \frac{\alpha_n}{n}\left(\frac{1}{z^n} + \frac{1}{\bar{z}^n}\right) . \tag{4.69}$$

In order to compute the conformal weight of a given state, it is easy to show that

$$L_0 = \alpha' p_c^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n . \tag{4.70}$$

Dirichlet boundary conditions. We consider now a boson with Dirichlet boundary conditions. First, because of eqn. (4.66b). the Fourier modes should by identified as

$$\forall n \neq 0 , \quad \alpha_n = -\tilde{\alpha}_n . \tag{4.71}$$

Recall that, for an open string attached to a *D*p-brane, Dirichlet boundary conditions correspond to the directions transverse to the brane, along which the endpoints of the strings cannot move. Since there is no translation invariance in those directions, there is no associated conserved charge. Let us consider an open string stretched between a first *D*-brane at $x = X_0$ and a second D-brane at $x = X_1$:

$$x(\sigma = 0, \tau) = X_0 , \quad x(\sigma = \pi, \tau) = X_1 . \tag{4.72}$$

Under the conformal mapping (3.104) from the strip to the upper half-plane, the line $\sigma = 0$ (resp. $\sigma = \pi$) is mapped to $\mathbb{R}_{<0}$ (resp. $\mathbb{R}_{>0}$). Hence, the mode expansion that satisfies the right boundary conditions is

$$x(z,\bar{z}) = X_1 + \frac{X_0 - X_1}{2i\pi}\ln(z/\bar{z}) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}^\star} \frac{\alpha_n}{n}\left(\frac{1}{z^n} - \frac{1}{\bar{z}^n}\right) . \tag{4.73}$$

In this case, since the affine current is expanded as $J(z) = \frac{1}{2i\pi\alpha'}(X_0 - X_1) + \cdots$, one has for the Virasoro zero-mode:

$$L_0 = \frac{1}{\alpha'}\left(\frac{X_1 - X_0}{2\pi}\right)^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n . \tag{4.74}$$

Consider a bosonic string stretched between two parallel *D*p-branes, spanning the directions $x^0, \dots x^p$, and located respectively at X_0^a and X_1^a, $a = p+1, \cdots, D-1$ in their transverse dimensions. The mode expansion is

$$x^\mu(z,\bar{z}) = x_c^\mu - i\alpha' p_c^\mu \ln|z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}^\star} \frac{\alpha_n^\mu}{n}\left(\frac{1}{z^n} + \frac{1}{\bar{z}^n}\right) , \quad \mu = 0, \dots, p \tag{4.75a}$$

$$x^a(z,\bar{z}) = X_1^a + \frac{X_0^a - X_1^a}{2i\pi}\ln(z/\bar{z}) + +i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}^\star} \frac{\alpha_n^a}{n}\left(\frac{1}{z^n} - \frac{1}{\bar{z}^n}\right) , \quad a = p+1, \dots, D-1 \tag{4.75b}$$

4.2 Free fermions

The next conformal field theory that we consider is the theory of free massless fermions on a two-dimensional Minkowskian worldsheet.

The smallest spinorial representation of the two-dimensional spin group $\mathrm{Spin}(1,1)$ is Majorana-Weyl, *i.e.* a real one-component spinor. In the Majorana basis, we choose the two-dimensional gamma matrices to be

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \gamma^2 = \gamma^0\gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{4.76}$$

The two-dimensional action for a Majorana fermion is of the form

$$\mathcal{S} = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma\, \bar{\psi}\gamma^\alpha\partial_\alpha\psi\,, \tag{4.77}$$

with $\bar{\psi} = \psi^\dagger i\gamma^0 = \psi^\mathsf{T} i\gamma^0$ using the Majorana condition.

The two-component Majorana spinor ψ can be decomposed into a pair of Majorana-Weyl one-component spinors $\psi_\pm$ of definite chirality:

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \ , \quad \gamma^2 \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} = \begin{pmatrix} -\psi_- \\ \psi_+ \end{pmatrix} . \tag{4.78}$$

Accordingly the two-dimensional action (4.77) splits into two independent chiral actions for the Majorana-Weyl spinors:

$$\mathcal{S} = \frac{i}{\pi}\int \mathrm{d}^2\sigma\big(\psi_-\partial_+\psi_- + \psi_+\partial_-\psi_+\big)\,, \tag{4.79}$$

the light-cone derivatives being defined as $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$. The equations of motion being $\partial_\pm\psi_\mp = 0$, ψ_+ (resp. ψ_-) is a left-moving (resp. right-moving) fermion.

We now move to Euclidean signature and two complex coordinates, as we did for the free scalar field. We consider the following action for a set of M left-moving and M right-moving real fermions:

$$\mathcal{S} = \frac{1}{4\pi}\int \mathrm{d}^2 z\left(\delta_{ij}\psi^i\bar{\partial}\psi^j + \delta_{ij}\widetilde{\psi}^i\partial\widetilde{\psi}^j\right)\,, \tag{4.80}$$

where ψ^i and $\tilde{\psi}^i$ are two independent real Grassmann variables. The equations of motion are simply

$$\bar{\partial}\psi^i = 0 \implies \psi^i = \psi^i(z) \tag{4.81a}$$

$$\partial\widetilde{\psi}^i = 0 \implies \widetilde{\psi}^i = \widetilde{\psi}^i(\bar{z}) \tag{4.81b}$$

4.2.1 Symmetries and current algebra

The action (4.80) is invariant under independent $\mathsf{SO}(M)$ chiral rotations for the left-moving and right-moving fermions:

$$\psi^i \mapsto R^i{}_j \psi^i\,, \quad \widetilde{\psi}^i \mapsto \tilde{R}^i{}_j \widetilde{\psi}^i\,, \quad R, \tilde{R} \in \mathsf{SO}(M)\,. \tag{4.82}$$

The corresponding holomorphic and anti-holomorphic Noether currents are:

$$J^{ij}(z) = -:\psi^i\psi^j:(z)\,, \quad \widetilde{J}^{ij}(z) = -:\widetilde{\psi}^i\widetilde{\psi}^j:(\bar{z})\,, \quad i<j \tag{4.83}$$

and have respectively conformal weights $(1,0)$ and $(0,1)$. We consider an infinitesimal (left-moving) $\mathsf{SO}(M)$ transformation:

$$\delta\psi^k = \alpha_{ij} T_{ijk\ell}\, \psi^\ell\,, \tag{4.84}$$

where the matrices $\mathcal{T}_{ijk\ell}$ correspond to the vector representation of $\mathsf{SO}(M)$:

$$\mathcal{T}_{ijk\ell} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}\,. \tag{4.85}$$

Using the residue formula (3.37) one gets

$$\alpha_{ij} T_{ijk\ell}\, \psi^\ell(z) = -\alpha_{ij}\mathrm{Res}_{z'\to z}\left(J^{ij}(z')\psi^k(z)\right)\,, \tag{4.86}$$

from which we deduce the OPE

$$J^{ij}(z')\psi^k(z) = -\frac{\mathcal{T}_{ijk\ell}\psi^\ell(z)}{z'-z} + \text{reg.} \tag{4.87}$$

This OPE, which expresses the fact that the set of left-moving fermions $\{\psi^i(z)\}$ are primaries of the affine $\mathfrak{so}(M)$ algebra in the vector representation, can be easily shown to derive from the fundamental OPE:

$$\boxed{\psi^i\psi^j = \frac{\delta^{ij}}{z-w} + \text{reg.}} \tag{4.88}$$

We have indeed (not forgetting that the $\psi^i(z)$s are anticommuting Grassmann variables):

$$\begin{aligned} J^{ij}(z')\psi^k(z) &= :\psi^i\psi^j:(z')\,\psi^k(z) = \psi^i(z')\psi^j(z')\psi^k(z) + \psi^i(z')\psi^j(z')\psi^k(z) \\ &= \frac{1}{z'-z}\left(-\delta^{ik}\psi^j(z) + \delta^{jk}\psi^i(z)\right) \\ &= -\frac{1}{z'-z}\left(\delta^{ik}\delta^{j\ell} - \delta^{jk}\delta^{i\ell}\right)\psi^\ell(z)\,. \end{aligned} \tag{4.89}$$

The OPE between two currents can also be computed and gives, after some slightly tedious algebra:

$$J^{ij}(z')J^{k\ell}(z) = -\frac{\kappa^{ij\,k\ell}}{(z'-z)^2} + \frac{f^{ij\,k\ell}{}_{mn}}{z'-z}J^{mn}(z) + \text{reg.} \tag{4.90}$$

with the Killing form and the structure constants of the Lie algebra $\mathfrak{so}(M)$ (in a basis where the long roots have square length two):

$$\kappa^{ij\,k\ell} = \delta^{ik}\delta^{j\ell} - \delta^{i\ell}\delta^{jk}\,, \tag{4.91}$$

$$\begin{aligned} f^{ij\,k\ell}{}_{mn} = \tfrac{1}{2}(\delta^{ik}\delta^{\ell n} - \delta^{i\ell}\delta^{kn})\delta^{jm} + \tfrac{1}{2}(\delta^{j\ell}\delta^{kn} - \delta^{jk}\delta^{\ell n})\delta^{im} \\ - \tfrac{1}{2}(\delta^{ik}\delta^{\ell m} - \delta^{i\ell}\delta^{km})\delta^{jn}) - \tfrac{1}{2}(\delta^{j\ell}\delta^{km} - \delta^{jk}\delta^{\ell m})\delta^{in})\,. \end{aligned} \tag{4.92}$$

A similar result is obtained with the currents $\widetilde{J}^{ij}(\bar{z})$ constructed from the right-moving fermions $\widetilde{\psi}(\bar{z})$.

Finally, as anticipated in the previous section, from the left and right-moving $SO(M)$ affine currents one can write the left and right-moving components of the stress-energy tensor. They can be obtained from two different perspectives. First, one can just use the stress-energy tensor obtained from the action (4.80) and get:

$$T(z) = -\frac{1}{2}\sum_{i=1}^{M} :\psi^i \partial\psi^i:(z)\,, \quad \widetilde{T}(\bar{z}) = -\frac{1}{2}\sum_{i=1}^{M} :\widetilde{\psi}^i \bar{\partial}\widetilde{\psi}^i:(\bar{z})\,. \tag{4.93}$$

Second, one can obtain directly T and $\tilde{T}$ from the second Casimir operator of the Lie algebra; this is the *Sugawara construction*. Explicitely one has

$$T(z) = -\frac{1}{2(M-1)}\sum_{i<j} :J^{ij}J^{ij}:(z)\,, \quad \widetilde{T}(\bar{z}) = -\frac{1}{2(M-1)}\sum_{i<j} :\widetilde{J}^{ij}\widetilde{J}^{ij}:(\bar{z})\,. \tag{4.94}$$

To show this, one can look at little bit closer at the OPE between the currents $J^{ij}(z')$ and $J^{ij}(z)$. The term giving the first-order pole in (4.90) is

$$\begin{aligned} &:\psi^i(z')\psi^j(z')\psi^i(z)\psi^j(z): + :\psi^i(z')\psi^j(z')\psi^i(z)\psi^j(z): \\ &\qquad + :\psi^i(z')\psi^j(z')\psi^i(z)\psi^j(z): + :\psi^i(z')\psi^j(z')\psi^i(z)\psi^j(z): \\ &= \frac{1}{z'-z}\Big(- :\psi^j(z')\psi^j(z): + \delta^{ij} :\psi^j(z')\psi^i(z): + \delta^{ij} :\psi^i(z')\psi^j(z): - :\psi^i(z')\psi^i(z): \Big)\,. \end{aligned} \tag{4.95}$$

The terms participating in the stress-energy tensor (4.94) are then obtained from the Taylor expansion of $\psi^i(z')$ and $\psi^j(z')$ at first order. Summing over $1 \leqslant i < j \leqslant M$, the terms in δ^{ij} drops, and we are left with

$$\sum_{i<j} :(\psi^j\partial\psi^j + \psi^i\partial\psi^i): = (M-1)\sum_{i=1}^{M} :\psi^i\partial\psi^i:\,, \tag{4.96}$$

which gives eventually the stress-energy tensor (4.93).

Using either form of the stress-energy tensor, the corresponding OPEs are given by the familiar expressions:

$$T(z')T(z) = \frac{M}{4(z'-z)^4} + \frac{2T(z)}{(z-z')^2} + \frac{\partial T(z)}{z-z'} + \text{reg.}\,, \tag{4.97a}$$

$$\widetilde{T}(\bar{z}')\widetilde{T}(\bar{z}) = \frac{M}{4(\bar{z}'-\bar{z})^4} + \frac{2\widetilde{T}(\bar{z})}{(\bar{z}-\bar{z}')^2} + \frac{\partial \widetilde{T}(\bar{z})}{\bar{z}-\bar{z}'} + \text{reg.}\,, \tag{4.97b}$$

which show that the M left-moving fermions $\psi^i(z)$ gives a holomorphic CFT with central charges $(c,\bar{c}) = (M/2, 0)$, while the M right-moving fermions $\widetilde{\psi}^i(\bar{z})$ gives a anti-holomorphic CFT with central charges $(c,\bar{c}) = (0, M/2)$.

Finally one can check that *(i)* the operators ψ^i and $\widetilde{\psi}^i$ are respectively conformal primaries of dimensions $(1/2, 0)$ and $(0, 1/2)$, and *(ii)* that J^{ij} and $\widetilde{J}^{ij}$ are conformal primaries of dimensions $(1,0)$ and $(0,1)$. One has first:

$$\begin{aligned}
T(z')\psi^i(z) &= -\frac{1}{2}\sum_{j=1}^{M} :\psi^j\partial\psi^j:(z')\psi^i(z) \\
&= -\frac{1}{2}\sum_{i=1}^{M} \Big(\psi^j(z')\partial\psi^j(z')\psi^i(z) + \psi^j(z')\partial\psi^j(z')\psi^i(z)\Big) \\
&= -\frac{1}{2}\sum_{j=1}^{M} \Big(-\frac{\delta^{ij}}{z'-z}\partial\psi^j(z') - \frac{\delta^{ij}}{(z'-z)^2}\psi^j(z')\Big) \\
&= \frac{1}{2(z'-z)^2}\psi^i(z) + \frac{1}{z'-z}\partial\psi^i(z) + \text{reg.}
\end{aligned} \tag{4.98}$$

And then, using eqn. (4.17):

$$\begin{aligned}
T(z_1)J^{ij}(z_3) &= -T(z_1):\psi^i\psi^j:(z_3) \\
&= -\frac{1}{2i\pi}\oint_{\mathcal{C}_{z_3}} \frac{dz_2}{z_2-z_3}\Big(T(z_1)\psi^i(z_2)\psi^j(z_3) + T(z_1)\psi^i(z_2)\psi^j(z_3)\Big) \\
&= -\frac{1}{2i\pi}\oint_{\mathcal{C}_{z_3}} \frac{dz_2}{z_2-z_3}\left\{\Big(\frac{\psi^i(z_2)}{2(z_1-z_2)^2} + \frac{\partial\psi^i(z_2)}{z_1-z_2}\Big)\psi^j(z_3) + \psi^i(z_2)\Big(\frac{\psi^j(z_3)}{2(z_1-z_3)^2} + \frac{\partial\psi^j(z_3)}{z_1-z_3}\Big)\right\}
\end{aligned} \tag{4.99}$$

In this expression one needs to compute the remaining OPE's between the fermions, and collecting the regular terms in the limit $z_2 \to z_3$ in the integrand gives finally

$$\begin{aligned}
T(z_1)J^{ij}(z_3) &= \frac{:\psi^i\psi^j:(z_3)}{(z_1-z_3)^2} + \frac{\partial:\psi^i\psi^j:(z_3)}{(z_1-z_3)^2} + \text{reg.} \\
&= \frac{J^{ij}(z_3)}{(z_1-z_3)^2} + \frac{\partial J^{ij}(z_3)}{(z_1-z_3)^2} + \text{reg.}
\end{aligned} \tag{4.100}$$

4.2.2 Modes expansion

We start by decomposing the fermionic fields ψ^i and $\widetilde{\psi}^i$ on the cylinder in Fourier modes. We start by choosing anti-periodic boundary conditions, namely

$$\psi^i(\sigma^1 + 2\pi, \sigma^2) = -\psi^i(\sigma^1, \sigma^2) \tag{4.101}$$

These are called the *Neveu-Schwarz* boundary conditions in the string theory context. In the statistical physics context, if σ^1 is the Euclidean time, these are the natural boundary conditions in the computation of the partition function. We get from these boundary conditions the expansion in Fourier modes:

$$\psi^j_{\text{cyl.}}(\sigma^1, 0) = e^{-\frac{i\pi}{4}} \sum_{n\in\mathbb{Z}+1/2} e^{in\sigma^1} \psi^j_n \,. \tag{4.102}$$

The operator $\psi^j(z)$ being a conformal primary of dimension $(1/2, 0)$, the conformal mapping from the cylinder to the plane gives from eqn. (3.58):

$$\psi^j_{\text{cyl.}}(w) = (-iz)^{1/2} \psi^j_{\text{plane}}(z)\,, \tag{4.103}$$

One obtains then an expansion of $\psi^i(z)$ and of $\widetilde{\psi}^i(\bar{z})$ on the plane in Laurent modes:

$$\psi^i(z) = \sum_{n\in\mathbb{Z}+1/2} \frac{\psi^i_n}{z^{n+1/2}} \,, \quad \widetilde{\psi}^i(\bar{z}) = \sum_{n\in\mathbb{Z}+1/2} \frac{\widetilde{\psi}^i_n}{\bar{z}^{n+1/2}} \,. \tag{4.104}$$

The shift of one-half in the exponent of z is compensated by the one-half shift in n, such that at the end the operators $\psi^i(z)$ and $\widetilde{\psi}^i(\bar{z})$ are single-valued on the complex plane. The modes are obtained from

$$\psi^i_n = \oint \frac{dz}{2i\pi} z^{n-1/2} \psi^i(z) \tag{4.105}$$

and obey an anti-commuting algebra which is obtained in the same way as before, see eqn. (4.34):

$$\{\psi^i_m, \psi^j_n\} = \oint \frac{dz_2}{2i\pi} z_2^{n-1/2} \text{Res}_{z_1\to z_2} \left(z_1^{m-1/2} \psi^i(z_1)\psi^j(z_2) \right) = \oint \frac{dz_2}{2i\pi} z_2^{n+m-1} \delta^{ij} \tag{4.106}$$

Hence

$$\boxed{\{\psi^i_m, \psi^j_n\} = \delta_{m+n,0}\, \delta^{ij}} \tag{4.107}$$

Next one defines the Laurent modes of the currents J^{ij} and $\widetilde{J}^{ij}$ in the same way as we did in the previous section. One has

$$J^{ij}(z) = \sum_{n\in\mathbb{Z}} \frac{J^{ij}_n}{z^{n+1}} \,, \tag{4.108}$$

with

$$J^{ij}_n = \oint \frac{dz}{2i\pi} z^n J^{ij} = \oint \frac{dz}{2i\pi} z^n :\psi^i\psi^j:(z) = \oint \frac{dz}{2i\pi} \oint \frac{dz'}{2i\pi} \frac{z^n}{z'-z} \psi^i(z')\psi^j(z) \tag{4.109}$$

At this stage one can use the same contour manipulation as in eqn. (4.40), taking into account the anticommuting nature of the Grassmann variables, and write this expression as:

$$J_n^{ij} = \sum_{r,s\in\mathbb{Z}+1/2} \Big(\psi_r^i \psi_s^j \oint_{\mathcal{C}'} \frac{dz'}{2i\pi} \oint_{\mathcal{C}} \frac{dz}{2i\pi} \frac{z^n}{z'-z} (z')^{-r-1} z^{-s-1} + \psi_s^j \psi_r^i \oint_{\mathcal{C}'} \frac{dz}{2i\pi} \oint_{\mathcal{C}} \frac{dz'}{2i\pi} \frac{z^n}{z'-z} (z')^{-r-1} z^{-s-1} \Big) \tag{4.110}$$

After the same steps as there, one finds that

$$J_n^{ij} = -\sum_{r\in\mathbb{N}+1/2} \psi_{n-r}^i \psi_r^j + \sum_{r\in-\mathbb{N}-1/2} \psi_r^j \psi_{n-r}^i = -\sum_{r\in\mathbb{Z}+1/2} {}^\circ_\circ \psi_{n-r}^i \psi_r^j {}^\circ_\circ \tag{4.111}$$

These modes obey the commutation relations

$$[J_m^{ij}, j_n^{k\ell}] = -\kappa^{ij\,k\ell} m \delta_{m+n,0} + f^{ij\,k\ell}{}_{mn} J_{m+n}^{k\ell} \,, \tag{4.112}$$

consistently with the currents OPE (4.90).

Finally the modes of the Virasoro algebra can be found using exactly the same methods. One has the expansion

$$L_n = \frac{1}{2} \sum_{r\in\mathbb{Z}+1/2} (r - n/2) {}^\circ_\circ \sum_i \psi_{n-r}^i \psi_r^i {}^\circ_\circ \,, \tag{4.113}$$

in terms of the fermionic modes, or

$$L_n = \frac{1}{2(M-1)} \sum_{r\in\mathbb{Z}} {}^\circ_\circ \sum_{i<j} J_{n-r}^{ij} J_r^{ij} {}^\circ_\circ \,, \tag{4.114}$$

in terms of the modes of the current $J^{ij}(z)$.

4.2.3 Representation theory

As for free bosons, the Hilbert space of the conformal field theory of free fermions is analyzed in terms of representation theory of the corresponding affine Lie algebra of generators $\{J_n^{ij}\}$. An alternative description is to use the representation theory of the algebra of the modes $\{\psi^i\}$ themselves.

As in the previous examples we will define primary states of the affine algebra that are annihilated by the positive modes $\{J_n^{ij}, n > 0\}$, that will also turn out to be conformal primary states.

The affine algebra (4.112) contains a closed sub-algebra of zero-modes, called the *horizontal subalgebra* coinciding with the Lie algebra $\mathfrak{so}(M)$ itself:

$$[J_0^{ij}, J_0^{kl}] = f^{ij\,kl}{}_{mn} J_0^{mn} \,. \tag{4.115}$$

In this respect, one can classify the affine primary and descendant states in terms of representations of the horizontal sub-algebra.

The affine primary states are also primary states with respect to the Virasoro algebra. To see this and to compute their conformal dimension, we consider affine primary operators ϕ^α in a representation R of the Lie algebra,

$$J^{ij}(z')\phi^\alpha(z) = -\frac{R(\mathcal{T}^{ij})^\alpha{}_\beta \phi^\beta}{z'-z} + \text{reg.} \tag{4.116}$$

where the matrix $R(\mathcal{T}^{ij})$ corresponds to the generator $\mathcal{T}^{ij}$ of the Lie algebra in the representation R. In terms of Laurent modes, one has

$$J_0|\alpha\rangle = -R(\mathcal{T}^{ij})^\alpha{}_\beta|\beta\rangle \ , \quad \forall n > 0 \, , \ J_n|\alpha\rangle = 0 \, , \tag{4.117}$$

where $|\alpha\rangle$ is the state obtained from acting with $\phi^\alpha(z)$ on the vacuum.

Next we check that $|\alpha\rangle$ is annihilated by all positive Virasoro modes. From equation (4.114) one finds

$$\forall n > 0 \ , \quad L_n|\alpha\rangle = \frac{1}{2(M-1)} \sum_{r\in\mathbb{Z}} {}^\circ_\circ \sum_{i<j} J^{ij}_{n-r} J^{ij}_r {}^\circ_\circ |\alpha\rangle = 0 \tag{4.118}$$

and

$$L_0|\alpha\rangle = \frac{1}{2(M-1)}\big(J^{ij}_0 J^{ij}_0 + \cdots\big)|\alpha\rangle = \frac{1}{2(M-1)} R(\mathcal{T}^{ij})^\alpha{}_\beta R(\mathcal{T}^{ij})^\beta{}_\gamma|\gamma\rangle. \tag{4.119}$$

We recognize in the last expression the quadratic Casimir of the representation R:

$$\sum_{i<j} R(\mathcal{T}^{ij})^\alpha{}_\beta R(\mathcal{T}^{ij})^\beta{}_\gamma = \delta^\alpha_\gamma c_R \, , \tag{4.120}$$

hence the conformal dimension of this primary operator reads

$$h = \frac{c_R}{2(M-1)} \, . \tag{4.121}$$

Trivial representation

We associate to the trivial representation of the horizontal $\mathfrak{so}(M)$ sub-algebra of the affine algebra (4.112) the *Neveu-Schwarz vacuum*, which is the state $|0\rangle_{\text{NS}}$ annihilated by all positive modes of the fermionic field:

$$\forall\, i \ , \quad \forall r \in \mathbb{N} + \tfrac{1}{2} \ , \quad \psi^i_r|0\rangle_{\text{NS}} = 0 \, . \tag{4.122}$$

It is trivial to see that this state is indeed an affine primary in the trivial representation of the horizontal sub-algebra:

$$\forall\, i < j \ , \quad \forall n \geqslant 0 \ , \quad J^{ij}_n|0\rangle_{\text{NS}} = -\sum_{r\in\mathbb{Z}+1/2} {}^\circ_\circ \psi^i_{n-r}\psi^j_r {}^\circ_\circ |0\rangle_{\text{NS}} = 0 \, , \tag{4.123}$$

and a Virasoro primary of conformal dimension zero:

$$\forall\, n \geqslant 0 \ , \quad L_n|0\rangle_{\text{NS}} = \frac{1}{2}\sum_{r\in\mathbb{Z}+1/2} (r-n/2)\, {}^{\circ}_{\circ}\, \psi^i_{n-r}\psi^i_r \,{}^{\circ}_{\circ}\, |0\rangle_{\text{NS}} = 0\,. \tag{4.124}$$

Descendant states in the trivial representations are obtained by acting with the negative modes of the currents J^{ij}. A generic state is of the form

$$|\psi\rangle = \cdots \underbrace{J^{i^2_1 j^2_1}_{-2} \cdots J^{i^2_{N_2} j^2_{N_2}}_{-2}}_{N_2 \ \text{terms}} \underbrace{J^{i^1_1 j^1_1}_{-1} \cdots J^{i^1_{N_1} j^i_{N_1}}_{-1}}_{N_1 \ \text{terms}} |0\rangle_{\text{NS}}\,, \tag{4.125}$$

where the modes of the currents can be expressed in terms of fermionic modes using (4.111).

Such generic state has a complicated multi-index structure, which corresponds generically to a reducible representation of the horizontal $\mathfrak{so}(M)$ subalgebra (4.115). The dimension of such a state is given by its level,

$$h = N := \sum_n n N_n\,. \tag{4.126}$$

Vector representation

We next consider affine primary states transforming in the vector representation of the horizontal sub-algebra. As the fundamental fermionic field $\psi^{(}z)$ itself transforms as a vector of $SO(M)$, see eq. (4.82), one considers the following set of states

$$|i\rangle := \psi^i_{-1/2}|0\rangle \ , \quad i \in \{1,\dots,M\}\,. \tag{4.127}$$

which correspond, using the state-operator correspondence, to the fields $\psi^i(z)$.

These states transform as expected in the vector representation of the horizontal sub-algebra J^{ij}_0. Explicitely one has:

$$\begin{aligned} J^{ij}_0|k\rangle &= -\sum_{r\in\mathbb{Z}+1/2} {}^{\circ}_{\circ}\, \psi^i_{-r}\psi^j_r \,{}^{\circ}_{\circ}\, \psi^k_{-1/2}|0\rangle_{\text{NS}} = \psi^i_{-1/2}\{\psi^j_{1/2},\psi^k_{-1/2}\}|0\rangle_{\text{NS}} - \psi^j_{-1/2}\{\psi^i_{1/2},\psi^k_{-1/2}\}|0\rangle_{\text{NS}} \\ &= \big(\delta^{jk}\psi^i_{-1/2} - \delta^{ik}\psi^j_{-1/2}\big)|0\rangle_{\text{NS}} = \Big(\delta^{jk}\delta^{i\ell} - \delta^{ik}\delta^{j\ell}\Big)\psi^\ell_{-1/2}|0\rangle_{\text{NS}}\,. \end{aligned} \tag{4.128}$$

They are annihilated by all the positive modes of the currents. For instance

$$J^{ij}_1|k\rangle = -\sum_{r\in\mathbb{Z}+1/2} {}^{\circ}_{\circ}\, \psi^i_{1-r}\psi^j_r \,{}^{\circ}_{\circ}\, \psi^k_{-1/2}|0\rangle_{\text{NS}} = \psi^i_{1/2}\{\psi^j_{1/2},\psi^k_{-1/2}\}|0\rangle_{\text{NS}} + \cdots = 0 \tag{4.129}$$

Finally for the same reasons the states $|i\rangle$ are annihilated by all the positive Virasoro modes:

$$\forall n > 0 \ , \quad L_n\psi^i_{-1/2}|0\rangle_{\text{NS}} = \tfrac{1}{2}\sum_{r\in\mathbb{Z}+1/2} (r-n/2)\, {}^{\circ}_{\circ}\, \psi^j_{m-r}\psi^j_r \,{}^{\circ}_{\circ}\, \psi^i_{-1/2}|0\rangle_{\text{NS}} = 0\,, \tag{4.130}$$

and the conformal dimension is $h = 1/2$ as expected from eqn. (4.98):

$$\begin{aligned} L_0 \psi^i_{-1/2}|0\rangle_{\text{NS}} &= \frac{1}{2} \sum_{r \in \mathbb{Z}+1/2} r \,{}^\circ_\circ\, \psi^j_{-r}\psi^j_r \,{}^\circ_\circ\, \psi^i_{-1/2}|0\rangle_{\text{NS}} \\ &= \tfrac{1}{4}\psi^j_{-1/2}\{\psi^j_{1/2}, \psi^i_{-1/2}\}|0\rangle_{\text{NS}} + \tfrac{1}{4}\psi^j_{-1/2}\{\psi^j_{1/2}, \psi^i_{-1/2}\}|0\rangle_{\text{NS}} \\ &= \tfrac{1}{2}\psi^i_{-1/2}|0\rangle_{\text{NS}} \end{aligned} \tag{4.131}$$

Descendant states in the vector representation are obtained by acting with the negative modes of the currents J^{ij}. A generic state is of the form

$$|\psi\rangle = \cdots \underbrace{J^{i^2_1 j^2_1}_{-2} \cdots J^{i^2_{N_2} j^2_{N_2}}_{-2}}_{N_2 \text{ terms}} \underbrace{J^{i^1_1 j^1_1}_{-1} \cdots J^{i^1_{N_1} j^i_{N_1}}_{-1}}_{N_1 \text{ terms}} \psi^i_{-1/2}|0\rangle_{\text{NS}} \,. \tag{4.132}$$

Its conformal dimension is given by:

$$h = N + 1/2 \,, \tag{4.133}$$

with the level N defined in eqn. (4.126).

Spinorial representations

We focus here to the case where the number of fermions is even, namely $M = 2k$, $k \in \mathbb{N}$, which is the most relevant for string theory applications. $\mathrm{Spin}(2k)$, the double-cover of $SO(2k)$, admits two spinorial representations of dimensions 2^{k-1} each; we will refer to them as the spinor and conjugate spinor respectively.

We introduce an affine primary operator $S_A(z)$ in a (reducible) Dirac representation that is characterized by its OPE with the currents J^{ij}:

$$J^{ij}(z')S_A(z) = \frac{1}{z'-z}\frac{1}{4}\left[\gamma^i, \gamma^j\right]_A^{\ B} S_B(z) + \text{reg.} \,, \tag{4.134}$$

with $\{\gamma^i,\, i = 1, \ldots, 2k\}$ the gamma matrices in $2k$ Euclidean dimensions. This definition follows from the general OPE (4.116), as the rotation generators in the reducible Dirac representation are given by $-\frac{1}{4}[\gamma^i, \gamma^j]$. Such operator is called a *spin field*.

Given the expression (4.83) for the currents in terms of the ψ^i, the OPE (4.134) between the currents and the spin field actually comes from a more fundamental OPE between the fermionic fields ψ^i and the spin fields S_A:

$$\psi^i(z')S_A(z) = \frac{1}{\sqrt{2(z'-z)}}(\gamma^i)_A^{\ B} S^B(z) + \mathcal{O}(z^{1/2}) \,. \tag{4.135}$$

To show that this OPE implies the OPE (4.134) is actually not trivial, as one cannot use the Wick theorem (4.17) because of the branch cuts in the complex plane created by the spin field. Rather one should use a technique known as *bosonization*.

In the same way as a Dirac spinor can be decomposed into a pair of Weyl spinors transforming in a irreducible representation of the $SO(2k)$, one can split the spin field $S_A(z)$ into $S_\alpha(z)$ and $S^{\dot\alpha}$ of positive and negative chirality respectively. Chiral spinors are distinguished by their eigenvalue with respect to the matrix $\gamma^{2k+1} = \gamma^1 \cdots \gamma^{2k}$, which anti-commutes with all Dirac matrices and commutes with the rotation generators $\frac{1}{4}[\gamma^i, \gamma^j]$. We define formally an operator Γ with the same property, which anti-commutes with all the modes of the operator ψ^i:

$$\forall r \,, \quad \forall i \,, \quad \{\Gamma, \psi^i_r\} = 0 \,. \tag{4.136}$$

The operator Γ is usually written in terms of the *Fermion number* operator F as

$$\Gamma = e^{i\pi F} \,, \tag{4.137}$$

where F, which counts the number of fermionic operators, is only defined mod two by this relation.

By inserting one of the spin fields $S_\alpha(z)$ and $S^{\dot\alpha}$ corresponding to the two irreducible spinorial representations of $SO(2k)$ at the origin of the complex plane, one defines states called the *Ramond vacua* of the theory through the state-operator correspondence:

$$S_\alpha(0) \leftrightarrow |\alpha\rangle_{\mathrm{R}} \,, \quad S^{\dot\alpha}(0) \leftrightarrow |\dot\alpha\rangle_{\mathrm{R}} \,. \tag{4.138}$$

These two vacua are distinguished by their eigenvalue under Γ:

$$\Gamma|\alpha\rangle_{\mathrm{R}} = |\alpha\rangle_{\mathrm{R}} \,, \quad \Gamma|\dot\alpha\rangle_{\mathrm{R}} = -|\dot\alpha\rangle_{\mathrm{R}} \tag{4.139}$$

The Hilbert spaces built upon these Ramond vacua are called the *Ramond sector* of the CFT.

From equation (4.121) the dimension of conformal primaries in the spinor or conjugate spinor representations, hence of the two Ramond vacua, are given by

$$h = \frac{c_R}{2(N-1)} = \frac{N}{16} \,. \tag{4.140}$$

The Laurent modes ψ^i_n of the fermionic operators ψ^i on the plane were defined in the Neveu-Schwarz vacuum of the theory by (4.105), with $n \in \mathbb{Z}+\frac{1}{2}$. It is clear that such definition cannot make sense for the action of fermionic modes on a Ramond vacuum, because of the branch cut in the fundamental OPE (4.135). In the Ramond sector one should actually define the modes in the same way but for $n \in \mathbb{Z}$:

$$\forall n \in \mathbb{Z} \,, \quad \psi^i_n = \oint \frac{dz}{2i\pi} z^{n-1/2} \psi^i(z) \,, \tag{4.141}$$

where implicitly these operators act on states in a Ramond sector of the free-fermion CFT; the commutation relations between these modes is the same as (4.107) as the steps of the derivation are similar.

Unlike in the Neveu-Schwarz sector, the fermionic operators have zero modes $\{\psi^i_0\}$. From eqn. (4.107) these zero modes obey the anti-commutation relations

$$\{\psi^i_0, \psi^j_0\} = \delta^{ij} \,, \tag{4.142}$$

which is the same, up to a normalization factor, as the Clifford algebra $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$.

The modes $\{\psi^i_n,\, n \in \mathbb{Z}\}$ are by definition associated with the decomposition of the operators $\psi^i(z)$ in the Ramond sectors as:

$$\psi^i(z) = \sum_{n\in\mathbb{Z}} \frac{\psi^i_n}{z^{n+1/2}}\,. \tag{4.143}$$

It means that two-dimensional fermions in the Ramond sector, on the complex plane, satisfy

$$\psi_{\mathrm{R}}(e^{2i\pi} z) = -\psi_{\mathrm{R}}(z)\,. \tag{4.144}$$

Moving to the cylinder, using the conformal transformation (4.103), one finds that the Ramond sector corresponds to periodic boundary conditions along the spatial circle:

$$\psi_{\mathrm{R}}(\sigma_1 + 2\pi, \sigma_2) = \psi_{\mathrm{R}}(\sigma_1, \sigma_2)\,. \tag{4.145}$$

In summary we have

$$\underline{\text{cylinder:}}\ \left\{ \begin{array}{ll} \psi_{\mathrm{NS}}(w+2\pi) & = -\psi_{\mathrm{NS}}(w) \\ \psi_{\mathrm{R}}(w+2\pi) & = \psi_{\mathrm{R}}(w) \end{array}\right. \quad \xleftrightarrow{\ z = e^{-iw}\ } \quad \underline{\text{plane:}}\ \left\{ \begin{array}{ll} \psi_{\mathrm{NS}}(e^{2i\pi}z) & = \psi_{\mathrm{NS}}(z) \\ \psi_{\mathrm{R}}(e^{2i\pi}z) & = -\psi_{\mathrm{R}}(z) \end{array}\right. \tag{4.146}$$

4.3 The ghost CFT

The last free conformal field theory that we will present in this chapter is the conformal field theory of reparametrization ghosts, which was introduced in section 2.4. We consider in conformal gauge a two-dimensional field theory with the following action in conformal gauge:

$$\mathcal{S} = \frac{1}{2\pi}\int \mathrm{d}^2 z\, \left(b_{zz}\partial^z c^z + b_{\bar z\bar z}\partial^{\bar z} c^{\bar z}\right) = \frac{1}{2\pi}\int \mathrm{d}^2 z\, \left(b\bar\partial c + \tilde b \partial \tilde c\right) \tag{4.147}$$

where all the fields are Grassmann variables.

The ghost action is by construction Weyl-invariant, which means that the ghost fields b_{zz}, $b_{\bar z\bar z}$, c^z and $c^{\bar z}$ are invariant under Weyl transformations (since no factors of the metric or its inverse appear in (4.147)). Consider now a conformal transformation, made of a holomorphic coordinate change $z \mapsto f(z)$, $\bar z \mapsto \bar f(\bar z)$ and a compensating Weyl transformation. The ghost fields being invariant under the compensating Weyl transformation, conformal invariance of the ghost action (4.147) implies that their conformal weights are determined solely from their tensor structure, namely:

$$h_b = 2\,,\ h_c = -1 \quad \text{and} \quad \tilde h_{\tilde b} = 2\,,\ \tilde h_{\tilde c} = -1\,. \tag{4.148}$$

This theory has fields of negative weights hence is not unitary, which is not a problem for an unphysical ghost action.[4] We could have noticed it earlier, as we consider fields with integer spins but fermionic statistics, hence violating the spin-statistics theorem.

[4]Note that if we "forget" about the tensor structure of the action (4.147), conformal invariance does not fix completely the conformal dimensions of the fields. One has in fact a family of free CFTs with $h_b + h_c = \tilde h_{\tilde b} + \tilde h_{\tilde c} = 1$. The special case $h_b = h_c = \tilde h_{\tilde b} = \tilde h_{\tilde c} = 1/2$ corresponds to a Dirac fermion, *i.e.* two left-moving and two-right moving Majorana-Weyl fermions, in which case naturally the theory is unitary.

We focus on the (b,c) ghost system from now (the $(\widetilde{b},\widetilde{c})$ system being of the same nature), defined by the action

$$\frac{1}{2\pi}\int d^2z\, b\bar{\partial}c \tag{4.149}$$

The equations of motion are simply

$$\bar{\partial}b = 0\ , \quad \bar{\partial}c = 0\,, \tag{4.150}$$

Indicating that the ghost fields $b(z)$ and $c(z)$ are holomorphic. The action (4.149) has an obvious classical rotational symmetry, at the infinitesimal level

$$c \mapsto (1+i\delta\alpha)c\ , \quad b \mapsto (1-i\delta\alpha)b\,. \tag{4.151}$$

The corresponding conserved Noether current is holomorphic:[5]

$$j_g(z) = -b(z)c(z)\ , \quad \bar{\partial}j_g(z) = 0\,. \tag{4.152}$$

We deduce from our favorite residue theorem (3.37) that

$$\delta\alpha\, c(z_1) = \mathrm{Res}_{z_2\to z_1}(\delta\alpha\, j_g(z_2)c(z_1))\ , \quad \delta\alpha\, b(z_1) = -\mathrm{Res}_{z_2\to z_1}(\delta\alpha\, j_g(z_2)b(z_1))\,. \tag{4.153}$$

We have then the following OPE

$$j_g(z_2)c(z_1) = -:bc:(z_2)c(z_1) = \frac{c(z_1)}{z_2-z_1} + \text{reg.}\ , \tag{4.154}$$

higher-order poles being forbidden by conformal invariance if we don't allow for operators of dimension more negative than the dimension of c. In the same way, one finds that the OPE

$$j_g(z_2)b(z_1) = :bc:(z_2)b(z_1) = -\frac{b(z_1)}{z_2-z_1} + \text{reg.}\ . \tag{4.155}$$

These two OPE can be deduced from the following OPE between the fundamental fields b and c:

$$\boxed{b(z_1)c(z_2) = c(z_1)b(z_2) = \frac{1}{z_1-z_2} + \text{reg.}} \tag{4.156}$$

taking into account that b and c anti-commute with each other.

4.3.1 Stress-energy tensor

We next turn to the stress-energy tensor and the central charge of this CFT. The operator $T(z)$ has to be such that b and c are conformal primary operators of conformal dimensions $h_b = 2$ and $h_c = -1$ respectively. Starting from the elementary fields the more general ansatz compatible with the symmetries of the problem is:

$$T(z) = \lambda_1 :b\partial c:(z) + \lambda_2 :\partial b c:(z)\,. \tag{4.157}$$

[5]There is an anomaly of the ghost current on a curved worldsheet, which plays no role for the moment.

We now compute the OPE

$$\begin{aligned} T(z')c(z) &= -\lambda_1 \partial c(z') b(z') c(z) - \lambda_2 c(z') \partial_{z'} b(z') c(z) + \text{reg.} \\ &= -\lambda_1 \frac{\partial c(z)}{z'-z} + \lambda_2 \frac{c(z) + (z'-z)\partial c(z)}{(z'-z)^2} + \text{reg.} \\ &= \lambda_2 \frac{c(z)}{(z'-z)^2} + (\lambda_2 - \lambda_1) \frac{\partial c(z)}{(z'-z)} + \text{reg.} \end{aligned} \tag{4.158}$$

Hence $\lambda_2 = -1$, $\lambda_1 = -2$ which gives

$$T(z) = -2 :\! b \partial c \!: (z) - :\! \partial b c \!: (z)\,. \tag{4.159}$$

One checks that, as expected from the conformal weight of the primary field $b(z)$:

$$\begin{aligned} T(z')b(z) &= -2 :\! b\partial c \!: (z') b(z) - :\! \partial b c \!: (z') b(z) + \text{reg.} \\ &= -2 b(z') \partial_{z'} c(z') b(z) - \partial b(z') c(z') b(z) + \text{reg.} \\ &= 2 \frac{b(z) + (z'-z)\partial b(z)}{(z'-z)^2} - \frac{\partial b(z)}{z'-z} + \text{reg.} \\ &= \frac{2b(z)}{(z'-z)^2} + \frac{\partial b(z)}{z'-z} + \text{reg.} \end{aligned} \tag{4.160}$$

The central charge of the ghost CFT is computed from the self-OPE of $T(z)$. One has

$$\begin{aligned} T(z')T(z) =& \big(-2 :\! b\partial c \!: (z') - :\! \partial b c \!: (z')\big)\big(-2 :\! b\partial c \!: (z) - :\! \partial b c \!: (z)\big) = \\ & 4\Big(:\! b(z')\partial c(z') b(z) \partial c(z) \!: + :\! b(z')\partial c(z') b(z) \partial c(z) \!: + b(z')\partial c(z') b(z) \partial c(z) \Big) \\ & + \Big(:\! \partial b(z') c(z') \partial b(z) c(z) \!: + :\! \partial b(z') c(z') \partial b(z) c(z) \!: + \partial b(z') c(z') \partial b(z) c(z) \Big) \\ & +2\Big(:\! \partial b(z') c(z') b(z) \partial c(z) \!: + :\! \partial b(z') c(z') b(z) \partial c(z) \!: + \partial b(z') c(z') b(z) \partial c(z) \Big) \\ & +2\Big(:\! b(z') \partial c(z') \partial b(z) c(z) \!: + :\! b(z') \partial c(z') \partial b(z) c(z) \!: + b(z') \partial c(z') \partial b(z) c(z) \Big) \end{aligned} \tag{4.161}$$

which gives, after Taylor-expanding the remaining fields around z,

$$T(z')T(z) = \frac{-13}{(z'-z)^4} + \frac{2T(z)}{(z'-z)^2} + \frac{\partial T(z)}{z'-z} + \text{reg.} \tag{4.162}$$

Therefore the central charge of the ghost CFT is $c_g = -26$. Naturally the computation is the same for anti-holomorphic ghosts and we conclude that $\tilde{c}_g = -26$.

4.3.2 Ghost background charge

Finally we compute the OPE between the stress tensor T and the ghost current j_g. One finds that, surprinsingly, j_g is not a conformal primary:

$$
\begin{aligned}
T(z')j_g(z) =&\Big(2 :b\partial c:(z') + :\partial bc:(z')\Big) :bc:(z) \\
=&2\Big(:b(z')\partial c(z')b(z)c(z): + :b(z')\partial c(z')b(z)c(z): + b(z')\partial c(z')b(z)c(z)\Big) \\
&+ :\partial b(z')c(z')b(z)c(z): + :\partial b(z')c(z')b(z)c(z): + \partial b(z')c(z')b(z)c(z) \\
=&-\frac{3}{(z'-z)^3} + \frac{j_g(z)}{(z'-z)^2} + \frac{\partial j_g(z)}{z'-z} + \text{reg.}
\end{aligned} \tag{4.163}
$$

It means that, using eqn. (3.37)) once again, under an infinitesimal holomorphic transformation $\delta z = \varepsilon(z)$, the current transforms as:

$$
\delta_\varepsilon j_g = \tfrac{3}{2}\partial^2\varepsilon(z) - j_g(z)\partial\varepsilon(z) - \varepsilon(z)\partial j_g(z)\,. \tag{4.164}
$$

For a finite conformal transformation $z \mapsto z' = f(z)$, this infinitesimal change exponentiates to

$$
j'_g(z') = (\partial_z f)^{-1} j_g(z) + \frac{3\partial^2 f(z)}{(\partial f(z))^2}\,. \tag{4.165}
$$

The current j_g can be expanded in Laurent modes,

$$
j_g = \sum_{n\in\mathbb{Z}} \frac{j_n}{z^{n+1}}\,, \tag{4.166}
$$

and the OPE (4.163) implies the commuations relations:

$$
[L_m, j_n] = -n j_{m+n} - \tfrac{3}{2} m(m+1)\delta_{m+n,0}\,. \tag{4.167}
$$

The central extension implies in particular that

$$
[L_1, j_{-1}] = j_0 - 3 \tag{4.168a}
$$
$$
[j_1, L_{-1}] = j_0\,. \tag{4.168b}
$$

Since the left-hand side of (4.168a) and the left-hand side of (4.168b) are Hermitian conjugates to each other, we reach the conclusion

$$
j_0^\dagger = j_0 + Q_b\,, \quad Q_b = -3\,. \tag{4.169}
$$

This has an important consequence for the computation of correlation functions. Let us consider a generic correlation function $\langle \mathcal{O}_1 \cdots \mathcal{O}_n\rangle$ of operators $\mathcal{O}_i$ of ghost charge Q_i under the current (4.152), computed in the ghost vacuum $|0\rangle_{gh}$. Equation (4.169) means that the out vacuum ${}_{\rm gh}\langle 0|$ as a charge of Q_b compared to the in vacuum $|0\rangle_{\rm gh}$, hence the correlation function can be non-vanishing only if the charge conservation condition is modified to

$$\sum_{i=1}^{n} Q_i = Q_b \,. \tag{4.170}$$

The charge Q_b is called a *background charge*, as it is independent of the operator insertions in the path integral.

The presence of the background charge for the ghosts is related to an anomaly of the ghost current conservation on curved worldsheets. One can show that

$$\langle \partial_i j^i_g \cdots \rangle = \frac{Q_b}{2\pi} R[\gamma] \langle \cdots \rangle \,, \tag{4.171}$$

whose right-hand side integrates to $Q_b \chi(\mathfrak{s}) = 6(g-1)$.

The integrated anomaly gives the index counting the difference between the number of b zero modes and of c zero modes, much as the integrated axial anomaly gives the index of the Dirac operator of a charged fermion. These ghost zero-modes were already discussed in the derivation of the Polyakov path integral, see subsection 2.3.3. As we discussed there, the Riemann-Roch theorem (2.101) relates the difference between the number of moduli and of conformal Killing vectors to the Euler characteristic of the surface, and each moduli (resp. each conformal Killing vector) results in an insertion of a zero-mode of b and $\widetilde{b}$ (resp. of c and $\widetilde{c}$) in the Polyakov path integral. This reasoning tells us that

$$\#(b \text{ and } \widetilde{b} \text{ zero-modes}) - \#(c \text{ and } \widetilde{c} \text{ zero-modes}) = -3\chi(\mathfrak{s}) = 6(g-1) \,. \tag{4.172}$$

In the present case we consider the conformal field theory on the compactified complex plane $\bar{\mathbb{C}} = \mathbb{C} \cap \{\infty\}$, which is conformal to a two-sphere, hence $g = 0$ and this counting is compatible with eqn. (4.169).

In a sense, the curvature of the two-sphere after the conformal mapping to the compactified complex plane has support only at ∞, *i.e.* at the south pole. Remember that, in radial quantization, the *in vacuum* was mapped to the origin of the plane. In the same way, the *out vacuum* is mapped to ∞ which which is where the curvature, hence the background charge of the ghost current, is localized. This explains the origin of the relation (4.169).

4.3.3 Mode expansions and Hilbert space

Despite their fermionic nature, the ghost fields have integer spins hence have periodic boundary conditions on the cylinder. After the conformal mapping to the complex plane, the Fourier series expansion becomes a decomposition in Laurent modes with integer powers:

$$b(z) = \sum_{n\in\mathbb{Z}} \frac{b_n}{z^{n+2}} \,, \quad c(z) = \sum_{n\in\mathbb{Z}} \frac{c_n}{z^{n-1}} \,, \tag{4.173}$$

The OPE (4.156) results in the following non-trivial anti-commutation relations:

$$\boxed{\{b_m, c_n\} = \delta_{m+n,0} \,.} \tag{4.174}$$

In particular the ghost CFT has zero modes b_0, c_0 which obey the algebra

$$\{b_0, c_0\} = 1\,. \tag{4.175}$$

We build the Hilbert space of the ghost CFT from ground states that are annihilated by all the positive modes of the ghost operators, $\{(b_n,\, c_n),\, n \in \mathbb{Z}_{>0}\}$. Because of the zero-mode algebra (4.175) we have two such ground states that satisfy the relations:

$$\begin{aligned} b_0|-\rangle_{\text{GH}} &= 0\,, \quad c_0|-\rangle_{\text{GH}} = |+\rangle_{\text{GH}} \\ c_0|+\rangle_{\text{GH}} &= 0\,, \quad b_0|+\rangle_{\text{GH}} = |-\rangle_{\text{GH}}\,. \end{aligned} \tag{4.176}$$

It is natural to select the vacuum state in the ghost sector as

$$|0\rangle_{\text{gh}} = |-\rangle\,, \tag{4.177}$$

which is annihilated by the ghost zero-mode b_0. Indeed the path integral of the Polyakov action on the sphere, see eqn. (2.122) has insertions of zero-modes $(c_0, \widetilde{c}_0)$ but not of zero-modes $(b_0, \widetilde{b}_0)$, given that the sphere has no moduli. We will have a more accurate explanation in the next chapter.

The ghost Hilbert space is then obtained by acting upon the vacuum state $|0\rangle_{gh}$ with c_0 and the creation operators $\{(b_{-n},\, c_{-n}),\, n \in \mathbb{Z}_{>0}\}$. The description of the Hilbert space of the anti-holomorphic ghosts $(\widetilde{b}, \widetilde{c})$ is exactly the same.

The stress energy tensor (4.159) can be decomposed in terms of the modes of the ghost fields by the methods used before for the scalar and fermionic CFTs. For $n \neq 0$ there are no normal ordering ambiguities and one can write

$$\begin{aligned} \forall n \neq 0\,,\ L_n &= \oint \frac{dz}{2i\pi} z^{n+1} \sum_{m,r\in\mathbb{Z}} \Big(2(r-1) z^{-m-2} z^{-r} \,{}^{\circ}_{\circ}\, b_m c_r \,{}^{\circ}_{\circ} + (m+2) z^{-m-3} z^{-r+1} \,{}^{\circ}_{\circ}\, b_m c_r \,{}^{\circ}_{\circ} \Big) \\ &= \sum_{m\in\mathbb{Z}} (2n-m) \,{}^{\circ}_{\circ}\, b_m c_{n-m} \,{}^{\circ}_{\circ}\,. \end{aligned} \tag{4.178}$$

For L_0 the normal ordering of the positive and negative modes could lead to a constant term, exactly as for the Hamiltonian of the ordinary harmonic oscillator. Fixing this ambiguity is important since it gives the conformal dimension of the vacuum. One can use the same contour manipulations as in section 4.1, or instead use the commutation relation

$$L_0|-\rangle_{\text{GH}} = \frac{1}{2}[L_1, L_{-1}]|-\rangle_{\text{GH}} = \frac{1}{2}(-2b_0 c_1 b_{-1} c_0 + \cdots)|-\rangle_{\text{GH}} = -b_o\{c_1, b_{-1}\}|+\rangle_{\text{GH}} = -|-\rangle_{\text{GH}}\,. \tag{4.179}$$

Hence

$$L_0 = -\sum_{m\in\mathbb{Z}} m \,{}^{\circ}_{\circ}\, b_m c_{-m} \,{}^{\circ}_{\circ} - 1\,. \tag{4.180}$$

One important fact for later usage is that the ghost vacuum has conformal dimension minus one.

Chapter 5

The string spectrum

In this chapter we consider bosonic string theory in Minkowski space-time, *i.e.* the Polyakov path integral (2.114) describing the propagation of a single bosonic string in a flat target space-time with metric $G_{\mu\nu} = \eta_{\mu\nu}$, together with a vanishing Kalb-Ramond field and constant dilaton.

We will leverage on the techniques developed in the previous chapters to obtain the first important predictions of string theory, that its spectrum contains the graviton, thereby providing a finite quantized theory of gravity, and that the dimension of space-time is fixed to a *critical dimension* because of the Weyl anomaly derived in section 3.6. We will find also that the bosonic string theory that we have dealt with up to now has a fatal flaw, the presence of an instable mode or *tachyon*, that will be taken care of in the next chapter by introducing supersymmetry on the two-dimensional worldsheet.

5.1 String theory and the dimension of space-time

In classical field theories, the dimension of space-time is a parameter of the theory, which is not fixed *a priori*. In quantum field theories, the situation is a little bit different as the renormalizability of the theory depends on the dimensionality of space-time; for instance gauge theories in more than four dimensions are non-renormalizable – which is not a problem if one views the theory only as a low-energy effective theory with a UV completion making sense at high energies.

In string theory the question of the dimension of space-time arises already at tree-level, *i.e.* at the level of the consistency of the Polyakov path-integral (2.114) on the sphere. This path integral only makes sense if all the gauge symmetries of the two-dimensional quantum field theory are non-anomalous.

5.1.1 Critical dimension of space-time

In section 3.6 we have explained that the Weyl symmetry, which is part of the gauge symmetry of the Polyakov action, is actually potentially anomalous; the anomaly was given by eqn. (3.102), and was proportional to the central charge of the worldsheet conformal field theory.

Let us consider bosonic string theory in a Minwkoskian target-space $\mathbb{R}^{1,D-1}$. The underlying two-dimensional conformal field theory is the product of two free theories:

- the conformal field theory for D free scalar fields, see eqn. (4.1). Following the analysis in section 4.1, its left and right central charges are $(c, \bar{c}) = (D, D)$.

- the conformal field theory of reparametrization ghosts (b, c) and $(\widetilde{b}, \widetilde{c})$, see eqn. (4.147). Following the analysis in section 4.3, its left and right central charges are $(c, \bar{c}) = (-26, -26)$.

Therefore the cancellation of the Weyl anomaly is only possible if $D = 26$. In other words:

$$\boxed{\text{Bosonic string theory in Minkowski space-time is only well-defined in 26 dimensions}}$$

This is a rather bold statement, which is unavoidable is one asks that the quantum theory living on the worldsheet of the string is a consistent one. There are different attitudes towards this result that we can summarize as follows:

1. This is a problem. All experimental facts points towards a four-dimensional space-time.

2. This is a feature of the theory, and we should live with it. As we shall see, if the extra dimensions are small enough, there is no blatant contradiction with experiments.

3. This is a desirable feature, as it introduces some flexibility in an otherwise quite rigid theory, in order to match experimental facts.

One may notice that strictly speaking this result is valid in Minkowski space-time with constant dilaton, and in fact it could be evaded by considering space-times with a strong space-like dilaton gradient, see eqn. (5.4c) below; this is a *non-critical string*. However such space-time cannot be used as a starting point for realistic physics and exhibits a strong-coupling singularity.

Finally, some of you may have heard that string theory predicts a ten-dimensional or eleven-dimensional space-time. For the former, we need to introduce superstring theory, which will wait till the next chapter. For the latter, the route is much longer and this is probably out of sight for this year.

5.1.2 Beyond Minkowski: string theory in curved space-time

So far the analysis of bosonic string theory was concerned with string propagation in a Minkowski space-time. In two-dimensional terms, the "matter part" of the Polyakov action corresponds then to a set of $D = 26$ free scalar field, one having a kinetic term with the wrong sign.

One could actually be more general, and consider a string moving in a generic curved space-time. This is actually given by eqn. (2.79), that we reproduce here for convenience:

$$\mathcal{S} = \frac{1}{4\pi\alpha'} \int_{\mathfrak{s}} d^2\sigma \left(\sqrt{\det \gamma}\, \gamma^{ij} G_{\mu\nu}[x^\rho] + \epsilon^{ij} B_{\mu\nu}[x^\rho] \right) \partial_i x^\mu \partial_j x^\nu + \frac{1}{4\pi} \int d^2\sigma \sqrt{\det \gamma}\, \Phi[x^\mu] \mathcal{R}[\gamma] \,. \quad (5.1)$$

This is an interacting QFT, as the background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ are functionals of the fields $x^\rho(\sigma^i)$.

This action defines a consistent theory if, at the very least, it defines a two-dimensional conformal field theory with $(c, \bar{c}) = (26, 26)$. At the classical level, as we have explained in detail in chapter 2, the first two terms of the action (5.1) are invariant under two-dimensional diffeomorphisms and Weyl transformations, while the last term is only Weyl-invariant for constant dilaton, $\Phi[x^\rho] = \Phi_0$.

At the quantum level, the theory is interacting and should be renormalized. This type of quantum field theory is known as a *non-linear sigma-model*, and has many applications outside of string theory. To view this one can decompose the fields as $x^\mu(\sigma^i) = x_0^\mu + \delta x^\mu(\sigma^i)$, expand the action (5.1) in powers of δx^μ and renormalize the interacting theory.

The computation of the Weyl anomaly in section 3.6 was assuming that the worldsheet two-dimensional QFT was conformal. In the present situation we don't know yet if the conformal symmetry is preserved at the quantum level, and the variation of the renormalized theory under an infinitesimal Weyl transformation is expressed, using the corresponding Ward identity, as:

$$-\tfrac{1}{2\pi}\int d^2\sigma\sqrt{\det\gamma}\,\langle\delta\omega T^a{}_a\cdots\rangle = \delta_{\delta\omega}\langle\cdots\rangle \tag{5.2}$$

It has been shown [1] by computing the one-loop diagrams in this interacting quantum field theory that the trace of the stress-energy tensor obeys the operator equation (which as usual should hold in any correlation function)

$$T^a{}_a = -\tfrac{1}{2\alpha'}\left(\gamma^{ij}\beta^G_{\mu\nu} + i\epsilon^{ij}\beta^B_{\mu\nu}\right)\partial_i x^\mu\partial_j x^\nu - \tfrac{1}{2}\beta^\Phi R[\gamma], \tag{5.3}$$

with coefficients

$$\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi - \tfrac{\alpha'}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + \mathcal{O}(\alpha'^2)\,, \tag{5.4a}$$
$$\beta^B_{\mu\nu} = -\tfrac{\alpha'}{2}\nabla^\rho H_{\rho\mu\nu} + \alpha'\nabla^\rho H_{\rho\mu\nu} + \mathcal{O}(\alpha'^2)\,, \tag{5.4b}$$
$$\beta^\Phi = \tfrac{1}{6}(D-26) - \tfrac{\alpha'}{2}\Delta\Phi + \alpha'\nabla_\rho\Phi\nabla^\rho\Phi - \tfrac{\alpha'}{24}H_{\mu\nu\rho}H^{\mu\nu\rho} + \mathcal{O}(\alpha'^2)\,. \tag{5.4c}$$

The expansion here should be thought as a weak-field expansion from the target space point of view. For instance if space-time has Ricci curvature of order $1/L^2$, the expansion is in powers of α'/L^2, *i.e.* a weak-curvature expansion.

The condition for Weyl invariance of string theory in the background specified by (G, B, Φ) is therefore given by the cancellation of all these coefficients:

$$\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0\,. \tag{5.5}$$

One can view these conditions as the equations of motion coming from the following D-dimensional action:

$$\mathcal{S} = \frac{1}{2\kappa^2}\int d^Dx\,\sqrt{-\det G}e^{-2\Phi}\left[-\frac{2(D-26)}{3\alpha'} + R[G] - \tfrac{1}{2}H_{\mu\nu\rho}H^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi\right] \tag{5.6}$$

i.e. general relativity coupled to the Kalb-Ramond two-form and the dilaton Φ, with a potential term for the dilaton proportionnal to the flat space Weyl anomaly. One can actually view the action (5.6) as the low-energy effective action for bosonic string theory with background fields (G, B, Φ). There is an important caveat though: the bosonic string includes also a tachyon field that couldn't be ignored in such description; however in the superstring theories the tachyon is gone and the supersymmetric analogue of (5.6) will be a faithful effective action, from which the low-energy physics can be reproduced.

We have seen in section 2.3, eqn. (2.88), that (the exponential of) a constant dilaton $\Phi[x^\rho] = \Phi_0$ played the role of the coupling constant in D dimension. Here we see that the dilaton, which can be a non-trivial function of the space-time coordinates, can be thought of a "dynamical" coupling constant.

The equations (5.4) tell us that one could in principle define bosonic strings away from twenty-six dimensions and even directly in four dimensions, however the price to pay is, for instance, a very large gradient of the dilaton field which would induce a strong violation of Lorentz invariance. One simple example is to take a Minkowski space-time with

$$\Phi[x^\mu] = V_\mu x^\mu \ , \quad V^\mu V_\mu = \frac{26 - D}{\alpha'} \, . \tag{5.7}$$

Besides the anisotropic nature of space-time for $D < 26$ due to the space-like gradient, such background has a singularity for the dilaton hence for the coupling constant.

The coefficients β in eqn. (5.4) are actually, as their name suggests, the beta-functions of the corresponding coupling constants in the two-dimensional QFT. Demanding the vanishing of the beta-function means that the theory is at a fixed point of the renormalization flow. This implies scale invariance of the theory, and it can be shown that, under mild assumptions, it implies conformal invariance as well [2].

To be more precise, for a two-dimensional non-linear sigma-model on the plane, a solution of the equations $\beta^G_{\mu\nu} = 0$ and $\beta^B_{\mu\nu} = 0$, without the dilaton terms, provide a quantum field theory which is scale-invariant – hence conformal invariant – at one (worldsheet) loop. When such theory is put on a curved manifold, it couples to the background metric, and one needs in particular to include the dilaton coupling to the two-dimensional curvature in the renormalization of the theory. Then the vanishing of the conformal anomaly demands also that $\beta^\Phi = 0$. This equation is rather interesting, since it indicates that the one-loop conformal anomaly is balanced by the variation of a classically anomalous term, since the dilaton coupling is not classically Weyl invariant unless Φ is constant, see the discussion below equation (2.85b).

5.2 BRST quantization

At this stage, we have understood that bosonic string theory in Minkowski space-time is described by the two-dimensional action

$$\mathcal{S} = \frac{1}{4\pi\alpha'} \int d^2z \, \eta_{\mu\nu} \, \partial x^\mu \bar{\partial} x^\nu + \frac{1}{2\pi} \int d^2z \left(b\bar{\partial}c + \widetilde{b}\partial\widetilde{c} \right) \, . \tag{5.8}$$

We have also described in chapter 4 the Hilbert space of each of these two conformal field theories – the free scalar CFT and the ghost CFT. Clearly the physical spectrum of string theory cannot be simply the tensor product of these two Hilbert spaces. Both conformal field theories are non-unitary, the former because of the field $x^0(z, \bar{z})$ with the wrong-sign kinetic term and the latter because it violates the spin-statistic theorem.

The key point is that the Polyakov action (2.63) corresponds actually to a constrained system; even after gauge fixing one has to impose the equations of motion for the gauge field (the metric), namely that

$$T_{ij} = 0 \, . \tag{5.9}$$

An elegant way of quantizing constrained systems is to use the *BRST formalism* (for Becchi, Rouet, Stora and Tyutin), that provides a selection rule from a "remnant" global symmetry in the gauge-fixed action for the matter and for the ghosts, named the BRST symmetry.

5.2.1 Basics of BRST quantization

Let us start with an overview of BRST quantization at a very general level. Consider a field theory with a gauge symmetry acting on the fields ϕ^i (including the gauge field themselves), obeying the algebra

$$[\delta_u, \delta_v] = f^w{}_{uv}\delta_w \,. \tag{5.10}$$

The path integral over the gauge fields is transformed into an integral gauge-inequivalent configurations by foliating the field space into gauge orbits and introducing gauge-fixing conditions

$$f_\ell[\phi^i] = 0 \tag{5.11}$$

As described in detail in chapter 2, this is done by introducing the Faddeev-Popov determinant, defined through a functional integral over gauge transformations, parametrized by Λ:

$$1 = \Delta_{\rm FP} \int \mathcal{D}\Lambda \; \delta\left(f_\ell^\Lambda[\phi^i]\right) = \Delta_{\rm FP} \int \mathcal{D}\Lambda\mathcal{D}B^\ell \; e^{i\int B^\ell f_\ell^\Lambda[\phi^i]} \,, \tag{5.12}$$

where we have introduced a path integral representation of the Dirac distribution using bosonic Lagrange multipliers B^ℓ. The gauge-fixed path integral is given by

$$\int \mathcal{D}\phi\mathcal{D}B\mathcal{D}b\mathcal{D}c \; \exp\left(-S[\phi] + i\int B_\ell f^\ell[\phi] - \int b_\ell c^u \delta_u f^\ell[\phi]\right) \,. \tag{5.13}$$

Here the fermionic ghosts (b_ℓ, c^u) provide a path integral representation of the Jacobian in field space and the last term contains the variation $\delta_u f^\ell[\phi]$ of the gauge-fixing condition under an infinitesimal gauge transformation.

BRST symmetry

The gauge-fixed path integral has a remnant global symmetry, which acts on the different fields as

$$\delta_{\rm B}\phi^i = -i\epsilon\, c^u \delta_u \phi^i \tag{5.14a}$$
$$\delta_{\rm B} B_\ell = 0 \tag{5.14b}$$
$$\delta_{\rm B} b_\ell = \epsilon B_\ell \tag{5.14c}$$
$$\delta_{\rm B} c^u = i\epsilon \tfrac{1}{2} f^u{}_{vw} c^v c^w \,. \tag{5.14d}$$

The first line, eqn. (5.14a), can be understood as a gauge transformation of parameter ϵc^u. Since the ghost fields are fermions, the parameter ϵ of the BRST transformation is a Grassmann variable.

Gauge invariance of the original action $S[\phi]$ guarantees its invariance under the transformation (5.14a). The second term in eqn. (5.13) transforms as

$$\delta_{\text{B}} i \int B_\ell f^\ell[\phi] = i \int B_\ell \frac{\delta f^\ell}{\delta \phi^i} \delta_{\text{B}} \phi^i = \epsilon \int B_\ell c^u \delta_u f^\ell[\phi] \,. \tag{5.15}$$

Finally the last term transforms as

$$\begin{aligned}
\delta_{\text{B}} \int b_\ell c^u \delta_u f^\ell[\phi] &= \epsilon \int B_\ell c^u \delta_u f^\ell[\phi] - i\epsilon \int b_\ell \tfrac{1}{2} f^u{}_{vw} c^v c^w \delta_u f^\ell[\phi] - i\epsilon \int b_\ell c^u c^v \delta_v \delta_u f^\ell[\phi] \\
&= \epsilon \int B_\ell c^u \delta_u f^\ell[\phi] - i\epsilon \int b_\ell \tfrac{1}{2} f^u{}_{vw} c^v c^w \delta_u f^\ell[\phi] + i\epsilon \int b_\ell c^u c^v \tfrac{1}{2} [\delta_u, \delta_v] f^\ell[\phi] \\
&= \epsilon \int B_\ell c^u \delta_u f^\ell[\phi] \,,
\end{aligned} \tag{5.16}$$

which cancels precisely (5.15). To this symmetry we associate a fermionic Noether current j_{B}, the *BRST current*, such that

$$\delta \mathcal{S} = \frac{1}{2\pi} \int d^D x \, j_{\text{B}}^i \, \partial_i \epsilon \,, \tag{5.17}$$

and a corresponding BRST charge

$$Q_{\text{B}} = \int_\Sigma d\Sigma_i \, j_{\text{B}}^i \,. \tag{5.18}$$

A crucial property of this symmetry is its nilpotence, *i.e.* that two successive transformations cancel each other. One finds first that

$$\begin{aligned}
&\delta_\eta \delta_\epsilon \phi^i = i\epsilon \, \delta_\eta (c^u \delta_u \phi^i) = \epsilon\eta \left(\tfrac{1}{2} f^u{}_{vw} c^v c^w \delta_u \phi^i - c^u c^v \delta_u \delta_v \phi^i \right) = 0 \\
&\delta_\eta \delta_\epsilon B_\ell = 0 \\
&\delta_\eta \delta_\epsilon b_\ell = -\epsilon \delta_\eta B_\ell = 0 \\
&\delta_\eta \delta_\epsilon c^u = -i\epsilon \tfrac{1}{2} f^u{}_{vw} \delta_\eta (c^v c^w) = \tfrac{1}{2} \epsilon\eta f^u{}_{vw} f^v{}_{xy} c^x c^y c^w = 0 \,,
\end{aligned} \tag{5.19}$$

where in the last line we have used the Jacobi identity satisfied by the generators of the Lie algebra:

$$f^u{}_{vw} f^v{}_{xy} - f^u{}_{vx} f^v{}_{wy} - f^u{}_{vy} f^v{}_{xw} = 0 \,. \tag{5.20}$$

Equivalently, the operator Q_{B} associated with the BRST charge (5.18) satisfies

$$Q_{\text{B}}^2 = 0 \,. \tag{5.21}$$

Finally another important property of the BRST symmetry is the following relation:

$$\delta_\epsilon \left(i \int b_\ell f^\ell[\phi] \right) = \epsilon \left\{ - \int b_\ell c^u \delta_u f^\ell[\phi^i] + i \int B_\ell f^\ell[\phi^i] \right\} \,, \tag{5.22}$$

i.e. giving exactly the gauge-fixing term and the ghost action in (5.13). This is of course another way of checking the BRST-invariance of the action, using the nilpotence of the BRST transformation.

BRST constraints

The properties of the BRST symmetry provide crucial information about the space of physical states of the theory. Let us consider the path integral giving the transition amplitude between an initial state at $t = t_0$ given by the wave functional $\Psi_0[\theta_0(\sigma)]$ and the final state at $t = t_1$ given by the wave functional $\Psi_1[\theta_1(\sigma)]$ (here by θ we denote collectively the fields, including the ghosts and by σ the spatial coordinates):

$$\langle \Psi_1(t_1)|\Psi_0(t_0)\rangle = \int \mathcal{D}\theta_1 \int \mathcal{D}\theta_0 \, \Psi_1[\theta_1(\sigma)]^\star \Psi_0[\theta_0(\sigma)] \int_{\theta(\sigma,t_0)=\theta_0(\sigma)}^{\theta(\sigma,t_1)=\theta_1(\sigma)} \mathcal{D}\theta \, e^{-S[\theta]} \, . \tag{5.23}$$

The path integral should be invariant under a change of the gauge-fixing conditions, $f^\ell[\phi^i] \mapsto f^\ell[\phi^i] + \delta f^\ell[\phi^i]$. This implies, using equation (5.22), that:

$$\forall \delta f^\ell \ , \quad \int \mathcal{D}\theta_1 \int \mathcal{D}\theta_0 \, \Psi_1[\theta_1(\sigma)]^\star \Psi_0[\theta_0(\sigma)] \int_{\theta(\sigma,t_0)=\theta_0(\sigma)}^{\theta(\sigma,t_1)=\theta_1(\sigma)} \mathcal{D}\theta \, e^{-S[\theta]} \delta_{\text{B}} \left(\int b_\ell \delta f^\ell[\phi] \right) = 0 \, . \tag{5.24}$$

Or, in Hamiltonian formalism,

$$\forall \delta f^\ell \ , \quad \langle \Psi_1 | \delta_B \left(b_\ell \delta f^\ell[\phi] \right) |\Psi_0\rangle = 0 \, , \tag{5.25}$$

for every pair of physical states. The action of the BRST transformation on the operator $b_\ell \delta f^\ell[\phi]$ can be expressed as the anti-commutator with the fermonic BRST charge (5.18), so we have the condition

$$\forall \delta f^\ell \ , \quad \langle \Psi_1 | \left\{ Q_b, b_\ell \delta f^\ell \right\} |\Psi_0\rangle = 0 \, . \tag{5.26}$$

Assuming that the operator Q_{B} is self-adjoint (it follows from the reality of the underlying gauge transformation), one finds that, since this equality should hold for any δf^ℓ, the BRST charge annihilates every physical state:

$$\boxed{\forall |\Psi\rangle \in \mathcal{H}_{\text{PHYS}} \ , \quad Q_{\text{B}}|\Psi\rangle = 0 \, .} \tag{5.27}$$

We consider now a state $|\Psi_s\rangle$ obtained from the action of the BRST charge on some state $|\varphi\rangle$:

$$|\Psi_s\rangle = Q_{\text{B}}|\varphi\rangle \, , \tag{5.28}$$

which is annihilated by Q_{B} by nilpotence of the BRST operator. This state is orthogonal to all physical states, and orthogonal to itself:

$$\forall |\Psi\rangle \, , \ \langle \Psi|\Psi_s\rangle = 0 \ , \quad \langle \Psi_s|\Psi_s\rangle = 0 \, . \tag{5.29}$$

This means that if $|\Psi\rangle$ is a physical state, the state

$$|\Psi\rangle + |\Psi_s\rangle = |\Psi\rangle + Q_{\text{B}}|\varphi\rangle \tag{5.30}$$

is also a physical state, which has the same overlaps with all physical states as the original state $|\Psi\rangle$, hence is indistinguishable from the former. Thus one can endow the space states with an equivalence relation:

$$\boxed{|\Psi\rangle \sim |\Psi'\rangle \ \Leftrightarrow \ \exists |\varphi\rangle \ , \quad |\Psi'\rangle = |\Psi\rangle + Q_{\text{B}}|\varphi\rangle \, ,} \tag{5.31}$$

and the physical state space of the theory is given by the space of equivalence classes.

Ghost number, grading of state space and BRST cohomology

One defines the *ghost number* as the additive charge such that c^u has charge one, b_ℓ charge minus one, ϕ^i and B_ℓ are neutral.

This ghost charge assignment induces a global Abelian symmetry of the action in (5.13), hence one define a conserved ghost number Noether current and a corresponding ghost charge; we have already discussed such current in the CFT context in section 4.3, see around equation (4.152).

Consistency of the BRST transformations (5.14) indicate that we should assign to the parameter ϵ a ghost charge of minus one. Hence the BRST charge operator Q_B has itself a ghost charge of plus one. In other words we have a *grading* of the space of states $\mathcal{H}$ according to the value q_g of the ghost charge, as it admits the decomposition $\mathcal{H} = \oplus_{q_g} \mathcal{H}_{q_g}$.

The operator Q_B sends elements of $\mathcal{H}_{q_g}$ to elements of $\mathcal{H}_{q_g+1}$, defining the following sequence of maps:

$$\cdots \xrightarrow{Q_B} \mathcal{H}_{q_g} \xrightarrow{Q_B} \mathcal{H}_{q_g+1} \xrightarrow{Q_B} \mathcal{H}_{q_g+2} \xrightarrow{Q_B} \cdots \tag{5.32}$$

Since the BRST charge is a nilpotent operator, *i.e.* $Q_B^2 = 0$, the image of every map $Q_B^{q_g}$ defined by:

$$\begin{aligned} Q_B^{q_g} \,:\, \mathcal{H}_{q_g} &\rightarrow \mathcal{H}_{q_g+1} \\ |\varphi\rangle_{q_g} &\mapsto Q_B|\varphi\rangle_{q_g} \end{aligned} \tag{5.33}$$

is in the kernel of the map $Q_B^{q_g+1}$. In this case, a sequence of maps like (5.32) is called an *exact sequence*.

The component of charge q_g of the space of physical state is then defined, following the discussion in the previous paragraphs, as [1]

$$\mathcal{H}^{q_g}_{\text{PHYS}} = \frac{\text{Ker}(Q_B^{q_g})}{\text{Im}(Q_B^{q_g-1})} \,. \tag{5.34}$$

and the full Hilbert space of physical states is defined by

$$\mathcal{H}_{\text{PHYS}} = \bigoplus_{q_g} \mathcal{H}^{q_g}_{\text{PHYS}} \,. \tag{5.35}$$

To phrase the same statement differently, the space of physical states is defined as the quotient of the space of Q_B-*closed states*, which is the space of states annihilated by the BRST charge:

$$|\Psi\rangle \in \mathcal{H}_{\text{CLOSED}} \;\Leftrightarrow\; Q_B|\Psi\rangle = 0 \tag{5.36}$$

[1]This structure is similar to the de Rahm cohomology of a n-dimensional differentiable manifold, where the space of differential forms Ω is graded according to their degree as $\Omega = \oplus_{k=0}^{n}\Omega_k$, and the exterior derivative d, which is nilpotent, induces a sequence of maps $d_k : \Omega_k \rightarrow \Omega_{k+1}$, $\omega_k \mapsto d\omega_k$.

by the space of Q_B-*exact states*, that are obtained by the action of the BRST charge on another state:

$$|\Psi\rangle \in \mathcal{H}_{\text{EXACT}} \;\Leftrightarrow\; \exists|\varphi\rangle\ , \quad |\Psi\rangle = Q_B|\varphi\rangle \tag{5.37}$$

So

$$\boxed{\mathcal{H}_{\text{PHYS}} = \frac{\mathcal{H}_{\text{CLOSED}}}{\mathcal{H}_{\text{EXACT}}}} \tag{5.38}$$

5.2.2 BRST symmetry of the point-particle action

We start by illustrating the ideas behind BRST quantization with the simpler theory of a point particle already discussed in section 2.1.

In this case the gauge symmetry corresponds to diffeomorphisms of the worldline $\tau \mapsto \tilde{\tau} = \tau + \alpha(\tau)$, under which the einbein transforms as

$$e(\tau) \mapsto \tilde{e}(\tilde{\tau}) = \frac{\partial\tau}{\partial\tilde{\tau}}e(\tau) = e(\tilde{\tau}) - \frac{d}{d\tilde{\tau}}\Big(\alpha(\tilde{\tau})\, e(\tilde{\tau})\Big). \tag{5.39}$$

A basis for these infinitesimal diffeormorphisms is given by

$$\delta_{\tau_1}\tau = \delta(\tau - \tau_1)\,, \tag{5.40}$$

as a general infinitesimal diffeormorphism can be written as

$$\delta\tau = \alpha(\tau) = \int d\tau_1 \delta(\tau - \tau_1)\alpha(\tau_1)\,, \tag{5.41}$$

and the transformations of the fields in this basis are given by

$$\delta_{\tau_1} x^\mu(\tau) = -\delta(\tau - \tau_1)\dot{x}^\mu(\tau) \tag{5.42a}$$

$$\delta_{\tau_1} e(\tau) = -\frac{d}{d\tau}\Big(\delta(\tau - \tau_1)e(\tau)\Big)\,. \tag{5.42b}$$

We consider then the gauge fixing condition

$$F(x, e; T) := T - e(\tau)\,. \tag{5.43}$$

After introducing the Fadeev-Popov determinant, one considers then the action, see around eqn. (2.22):

$$\mathcal{S} = \int_0^1 d\tau\, \left(\frac{1}{2e}\dot{x}^\mu\dot{x}_\mu + \frac{m^2 e}{2} - iB(e - T) - e\dot{b}c\right)\,. \tag{5.44}$$

Unlike there, we have exponentiated the gauge-fixing constraint using the antighost Lagrange multiplier field B. The last term is understood as

$$\begin{aligned}\int d\tau\, b(\tau) \int d\tau_1\, c(\tau_1)\, \delta_{\tau_1}F(\tau) &= \int d\tau\, b(\tau) \int d\tau_1 c(\tau_1)\frac{d}{d\tau}\Big(\delta(\tau - \tau_1)e(\tau)\Big) \\ &= -\int d\tau\, \left(\frac{d}{d\tau}b(\tau)\right) \int d\tau_1 c(\tau_1)\delta(\tau - \tau_1)e(\tau) = -\int d\tau\, e(\tau)\dot{b}(\tau)c(\tau)\,,\end{aligned} \tag{5.45}$$

where we have performed an integration by parts to reach the second line.

The algebra of gauge transformations acting on a field $x^\mu(\tau)$ is given in the present case by:

$$[\delta_{\tau_1}, \delta_{\tau_2}]x^\mu(\tau) = -\Big(\delta(\tau-\tau_2)\frac{d}{d\tau}\delta(\tau-\tau_1) - \delta(\tau-\tau_1)\frac{d}{d\tau}\delta(\tau-\tau_2)\Big)\dot{x}^\mu(\tau)\,. \tag{5.46}$$

The infinitesimal BRST transformations, given by the general formulæ (5.14), are defined in the present context as:

$$\delta_\epsilon x^\mu = i\epsilon\, c\, \dot{x}^\mu\,, \tag{5.47a}$$
$$\delta_\epsilon B = 0 \tag{5.47b}$$
$$\delta_\epsilon b = \epsilon B\,, \tag{5.47c}$$
$$\delta_\epsilon c = i\epsilon\ c\dot{c}\,. \tag{5.47d}$$

For the last transformation we have used

$$\delta_\epsilon c(\tau) = \frac{i\epsilon}{2}\int d\tau_1 \int d\tau_2 \Big(\delta(\tau-\tau_2)\frac{d}{d\tau}\delta(\tau-\tau_1) - \delta(\tau-\tau_1)\frac{d}{d\tau}\delta(\tau-\tau_2)\Big)c(\tau_1)c(\tau_2) = i\epsilon c(\tau)\dot{c}(\tau)\,. \tag{5.48}$$

It is possible to reach a ”reduced” form of the BRST transformations by integrating out the (neutral) field B to reach the gauge-fixed theory and replacing it in the transformation (5.47) using the equation of motion for the einbein e,

$$B = -i\frac{\delta}{\delta e}\Big(\frac{1}{2e}\dot{x}^\mu\dot{x}_\mu + \frac{m^2 e}{2} - e\dot{b}c\Big)\Big|_{e=T}\,. \tag{5.49}$$

The right-hand side of this equation is nothing but the stress-tensor of the worldline theory. One finds then that the gauge-fixed action

$$\mathcal{S} = \int_0^1 d\tau\ \Big(\frac{1}{2T}\dot{x}^\mu\dot{x}_\mu + \frac{m^2 T}{2} - T\dot{b}c\Big) \tag{5.50}$$

is invariant under the infinitesimal fermionic symmetry,

$$\delta_\epsilon x^\mu = i\epsilon\, c\, \dot{x}^\mu\,, \tag{5.51a}$$
$$\delta_\epsilon b = i\epsilon\, \tfrac{1}{T}\left(-\tfrac{1}{2T}\dot{x}^\mu\dot{x}_\mu + \tfrac{m^2 T}{2} - T\,\dot{b}c\right)\,, \tag{5.51b}$$
$$\delta_\epsilon c = i\epsilon\ c\dot{c}\,, \tag{5.51c}$$

which is nilpotent as expected. However it is now true only on-shell, *i.e.* using the equations of motion.

5.2.3 BRST symmetry of bosonic string theory

We consider now the BRST symmetry associated with the gauge-fixed Polyakov path integral of bosonic string theory. It will be similar in many respects to the point particle case and will heavily use the methods of conformal field theories.

Here the gauge symmetry corresponds to diffeomorphisms of the worldsheet and Weyl transformations, and the "reduced" BRST transformations can be expressed as follows:

$$\delta_\epsilon x^\mu = i\epsilon\big(c\partial x^\mu + \tilde{c}\bar{\partial} x^\mu\big), \tag{5.52a}$$

$$\delta_\epsilon b = i\epsilon\Big(-\frac{1}{\alpha'}:\partial x^\mu \partial x_\mu: -2:b\partial c: - :\partial b c:\Big) = i\epsilon(T^x + T^g)\,, \tag{5.52b}$$

$$\delta_\epsilon \widetilde{b} = i\epsilon\Big(-\frac{1}{\alpha'}:\bar{\partial} x^\mu \bar{\partial} x_\mu: -2:\widetilde{b}\bar{\partial}\widetilde{c}: - :\bar{\partial}\widetilde{b}\widetilde{c}:\Big) = i\epsilon(\tilde{T}^x + \tilde{T}^g)\,, \tag{5.52c}$$

$$\delta_\epsilon c = i\epsilon c\partial c \tag{5.52d}$$

$$\delta_\epsilon \widetilde{c} = i\epsilon\widetilde{c}\bar{\partial}\widetilde{c}\,. \tag{5.52e}$$

As in the point-particle case, the transformation of the ghosts b and $\widetilde{b}$ is given respectively by the holomorphic and anti-holomorphic components of the stress-energy tensor, as the fields B and $\tilde{B}$ have been eliminated using the equation of motion for the two-dimensional metric. Note that the ghost corresponding to the Weyl symmetry was just a Lagrange multiplier field and has been already integrated out, see eqn. (2.120) and below (which results in the tensor b^{ab} being traceless).

To the BRST symmetry one associates a conserved fermionic Noether current, whose components in complex coordinates are separately conserved, being respectively holomorphic and anti-holomorphic. They are of the form

$$j_{\rm B} = cT^x + \tfrac{1}{2}:cT^g: + \lambda\partial^2 c\,, \tag{5.53a}$$

$$\widetilde{j}_b = \widetilde{c}\widetilde{T}^x + \tfrac{1}{2}:\widetilde{c}\widetilde{T}^g: + \lambda\bar{\partial}^2\widetilde{c}\,, \tag{5.53b}$$

where T^x, $\widetilde{T}^x$ and T^g, $\widetilde{T}^g$ denote respectively the stress-energy tensors of the x fields and of the ghost fields. We have indeed the OPEs:

$$j_{\rm B}(z)x^\mu(0,0) = \big(c(z)T^x(z) + \cdots\big)x^\mu(0,0) = \frac{c(0)\partial x^\mu(0,0)}{z} + \text{reg.} \tag{5.54a}$$

$$\begin{aligned} j_{\rm B}(z)b(0) &= \big(c(z)T^x(z) + \tfrac{1}{2}:cT^g:(z) + \lambda\partial^2 c(z)\big)b(0) = \cdots + \frac{1}{z}\Big(T^x(0) + \frac{1}{2}T^g(0)\Big) \\ &\quad + \frac{1}{2}\big(c(0) + z\partial c(0) + \cdots\big)\Big(\frac{2b(0)}{z^2} + \frac{\partial b(0)}{z} + \text{reg.}\Big) \end{aligned} \tag{5.54b}$$

$$j_{\rm B}(z)c(0) = \tfrac{1}{2}:cT^g:(z)c(0) = \frac{1}{2}\big(c(0) + z\partial c(0) + \cdots\big)\Big(\frac{-c(0)}{z^2} + \frac{\partial c(0)}{z} + \text{reg.}\Big)\,, \tag{5.54c}$$

and their anti-holomorphic counterparts, from which we recover the action of the BRST charge, hence the transformations (5.52), by the contour integral $Q_{\rm B} = \frac{1}{2i\pi}\oint(\mathrm{d}z\, j_{\rm B} - \mathrm{d}\bar{z}\,\tilde{j}_{\rm B})$.

Notice that the last term in the holomorphic BRST current (5.53a) is a total derivative, that did not contribute to the BRST charge as computed above. It is however necessary to add this improvement term, for a certain value of λ, in order for the current $j_{\rm B}$ to transform as a tensor of dimension $(1,0)$ at it should.

To see this, one starts by computing the following operator product expansions, by taking derivatives of the fundamental OPE (4.156):

$$\begin{aligned}
T^g(z)\partial^2 c(0) &= :(-2b\partial c - \partial b c):(z)\,\partial^2 c(0) = 2\partial c(z)\frac{2}{z^3} - c(z)\frac{6}{z^4} + \text{reg.}\\
&= \frac{-6}{z^4}c(0) + \frac{4-6}{z^3}\partial c(0) + \frac{4-3}{z^2}\partial^2 c(0) + \frac{2-1}{z}\partial^3 c(0) + \text{reg.}\\
&= \frac{-6}{z^4}c(0) + \frac{-2}{z^3}\partial c(0) + \frac{1}{z^2}\partial^2 c(0) + \frac{1}{z}\partial^3 c(0) + \text{reg.}
\end{aligned} \tag{5.55}$$

Noticing that $:cT^g: \; = \; :c(-2b\partial c - \partial b c): \; = 2:bc\partial c:$, one computes the following OPE:

$$\begin{aligned}
T^g(z):cT^g(0): \; = \; &:T^g(z)c(0)T^g(0): + :T^g(z)c(0)T^g(0):\\
&- 4:b\partial c:(z):bc\partial c:(0) - 4:b\partial c:(z):bc\partial c:(0)\\
&- 2:\partial b c:(z):bc\partial c:(0) - 2:\partial b c:(z):bc\partial c:(0)\\
= \; &:\Big(-\frac{c(0)}{z^2} + \frac{\partial c(0)}{z}\Big)T^g(0): + :\Big(\frac{-26}{2z^4} + \frac{2T^g(0)}{z^2} + \frac{\partial T^g(0)}{z}\Big)c(0):\\
&- \frac{8}{z^4}c(0) + \frac{6}{z^3}\partial c(0)
\end{aligned} \tag{5.56}$$

One gets finally the following result for the OPE between the full stress-tensor and the BRST current:

$$\begin{aligned}
T(z)j_{\mathrm{B}}(0) &= \big(T^x(z) + T^g(z)\big)\big(cT^x(0) + \tfrac{1}{2}:cT^g: + \lambda\partial^2 c(0)\big)\\
&= c(0)\Big(\frac{D}{2z^4} + \frac{2T^x(0)}{z^2} + \frac{\partial T^x(0)}{z}\Big) + \Big(-\frac{c(0)}{z^2} + \frac{\partial c(0)}{z}\Big)T^x(0)\\
&\quad + \frac{1}{2}c(0)\Big(\frac{-26}{2z^4} + \frac{2T^g(0)}{z^2} + \frac{\partial T^g(0)}{z}\Big) + \frac{1}{2}\Big(-\frac{c(0)}{z^2} + \frac{\partial c(0)}{z}\Big)T^g(0)\\
&\quad + \frac{1}{2}\Big(-\frac{8c(0)}{z^4} + \frac{6\partial c(0)}{z^3}\Big) + \lambda\Big(\frac{-6c(0)}{z^4} + \frac{-2\partial c(0)}{z^3} + \frac{\partial^2 c(0)}{z^2} + \frac{\partial^3 c(0)}{z}\Big)\\
&= \frac{j_{\mathrm{B}}(0)}{z^2} + \frac{\partial j_{\mathrm{B}}(0)}{z} + \frac{c(0)}{z^4}(D/2 - 6\lambda - 4) + \frac{\partial c(0)}{z^3}(3 - 2\lambda)\,.
\end{aligned} \tag{5.57}$$

The term in $\partial c(0)$ signals that j_{B} does not transform as a tensor, hence one should set $\lambda = 3/2$. One sees furthermore that the term in $1/z^4$ is absent for $D = 26$, *i.e.* when the string propagates in 26 dimensions as was already noticed by other methods.

To understand better the significance of the last condition in the context of BRST quantization, one can compute the OPE of the ghost current with itself. Using similar methods as for the previous computation, after some slightly tedious algebra one gets the result

$$j_{\mathrm{B}}(z)j_{\mathrm{B}}(0) = \frac{18-D}{2z^3}:c\partial c:(0) + \frac{18-D}{4z^2}:c\partial^2 c:(0) + \frac{26-D}{12z}:c\partial c:(0)\,. \tag{5.58}$$

Given that the holomorphic BRST charge of a local operator $\mathcal{O}$ is given, following the general result (3.16), by the contour integral

$$Q_B = \oint_{\mathcal{C}_0} \frac{dz}{2i\pi}\, j_B(z)\mathcal{O}(0)\,, \tag{5.59}$$

The simple pole in equation (5.58) indicates that the BRST current is transformed under a BRST transformation. In other words, the BRST operator is not nilpotent unless $D = 26$:

$$\{Q_B, Q_B\} = 0 \quad \text{only if} \quad D = 26\,. \tag{5.60}$$

Naturally, the same computation can be done using the anti-holomorphic BRST current $\widetilde{\jmath}_g$, leading to the same conclusion for the anti-holmorphic BRST charge $\widetilde{Q}_B$.

More generally, the free action for D scalars on the string worldsheet can be replaced by another more general conformal field theory of central charges $(c, \bar{c})$, and the condition for BRST invariance becomes $c = \bar{c} = 26$.

5.2.4 BRST cohomology of the bosonic string

We are now ready – at least! – to provide the description of the physical states of bosonic string theory. Let us consider the local operator corresponding to a physical state. It splits generically into a matter and ghost parts, as the CFT on the worldsheet is the tensor product of the two factors:

$$\mathcal{V}(z,\bar{z}) = \mathcal{V}^g(z,\bar{z})\mathcal{V}^x(z,\bar{z})\,. \tag{5.61}$$

We have seen in the discussion about the path integral on the sphere that an *unintegrated* operator for the matter CFT should be multiplied by an insertion of the c and $\widetilde{c}$ ghosts at the operator position. One considers therefore that the physical states correspond in their unintegrated version to local operators the form

$$\mathcal{V}_U(z,\bar{z}) = c(z)\widetilde{c}(\bar{z})\mathcal{V}^x(z,\bar{z})\,. \tag{5.62}$$

According to the general discussion in subsection 5.2.1, a physical state should be first BRST closed. Let us assume that the matter operator $\mathcal{V}^x(z,\bar{z})$ has conformal dimensions $(h, \bar{h})$, but is not necessarily a primary field. One has

$$T(z)\mathcal{V}^x(0) = \cdots + \frac{\mathcal{W}(0)}{z^3} + \frac{h\mathcal{V}^x(0)}{z^2} + \frac{\partial\mathcal{V}^x(0)}{z} + \text{reg.} \tag{5.63}$$

where $\mathcal{W}(0)$ is some local operator. Ignoring the possible effect of more singular terms, one considers the OPE

$$\begin{aligned}
j_B(z)\mathcal{V}_U(0) &= \Big(c(z)T^x(z) + \frac{1}{2} :cT^g:(z)\Big) c(0)\tilde{c}(0)\mathcal{V}^x(0,0) \\
&= \Big(c(0) + z\partial c(0) + \frac{1}{2}z^2\partial^2 c(0)\Big) c(0)\tilde{c}(0)\Big(\overline{T^x(z)\mathcal{V}^x}(0,0) + \text{reg.}\Big) \\
&\quad + \frac{1}{2}\Big(c(0) + z\partial c(0)\Big)\Big(\overline{T^g(z)c}(0) + \text{reg.}\Big)\tilde{c}(0)V^x(0,0) \\
&= c(0)\tilde{c}(0)\Big(\frac{1}{z^2}\partial c(0)\mathcal{W}(0) + \frac{1}{z}\big(\partial^2 c(0)\mathcal{W}(0) + (h-1)\partial c(0)\,\mathcal{V}^x(0,0)\big)\Big) + \text{reg.}
\end{aligned} \tag{5.64}$$

The first order pole provides the charge of the state under a BRST transformation, hence, $\mathcal{W}$ should vanish and h should be equal to one. Generalizing easily this computation to more singular terms in the $T^x\mathcal{V}^x$ OPE, and doing the same computation for the anti-holomorphic BRST current $\tilde{j}_B$, we have learned that an unintegrated physical operator should be:

1. a conformal primary operator,
2. an operator of conformal dimensions $h = \bar{h} = 1$.

We have also to check that it is not BRST-exact, *i.e.* not obtained as a BRST transformation of another local operator; this will be done shortly.

The *integrated* operator is obtained by replacing the ghost insertion $c(z)\tilde{c}(\bar{z})$ by the integral over the worldsheet. In this case one can use

$$[Q_B, \mathcal{V}^x] = \oint \frac{dz}{2i\pi} j_B(z)\mathcal{V}^x(0,\bar{0}) = \oint \frac{dz}{2i\pi} c(z)\left(\frac{h\mathcal{V}^x(0,\bar{0})}{z^2} + \frac{\partial\mathcal{V}^x(0,\bar{0})}{z}\right) = h\partial c\mathcal{V}^x + c\partial\mathcal{V}^x\,, \tag{5.65}$$

hence it is a total derivative if $h = 1$ and the BRST variation integrates to zero. Note that if $\mathcal{V}^x$ is non-primary operator it will bring extra terms in $\partial^n c$, $n > 1$ and the statement will not be true.

For a two-dimensional CFT, an integrated primary operator of conformal dimensions $(1,1)$ is called a *marginal deformation*. If one starts with a conformal field theory with action $\mathcal{S}$, the quantum field theory with action

$$\mathcal{S}_\lambda = \mathcal{S} + \lambda \int d^2z\, \mathcal{O}_{1,1}(z,\bar{z}) \tag{5.66}$$

is also a conformal field theory. The parameter λ defines a family of conformal field theories, or said differently a one-parameter family of renormalization group fixed points. In general the parameters that characterize locally families of conformal field theories are viewed as coordinates on a manifold called the *moduli space* of the conformal field theory.

Physical states

We will now discuss the BRST constraints from the point of view of physical states, rather than from the point of view of local operators. We first expand the BRST charge in term of modes, using the results of chapter 4. One has for the holomorphic BRST charge:

$$\begin{aligned}
Q_B &= \oint \frac{dz}{2i\pi} j_B(z) = \oint \frac{dz}{2i\pi}\Big(c(z)T^x(z) + \frac{1}{2} :cT^g:(z)\Big) \\
&= \sum_n c_n L^x_{-n} + \sum_{n,m} \frac{m-n}{2} \,{}^\circ_\circ\, c_m c_n b_{-m-n} \,{}^\circ_\circ\, + \lambda c_0 \,,
\end{aligned} \tag{5.67}$$

where the normal ordering constant λ, coming from terms like $c_0 c_n b_{-n}$ in the expansion (5.67), is determined as follows. On the one hand one has the OPE

$$\begin{aligned}
j_B(z)b(0) &= \Big(c(z)T^x(z) + :bc\partial c:(z) + \frac{3}{2}\partial^2 c\Big) b(0) \\
&= \frac{T^x(z)}{z} - \frac{:b\partial c:(z)}{z} - \frac{:bc:(z)}{z^2} + \frac{3}{z^3} + \text{reg.} \\
&= \frac{3}{z^3} + \frac{j^g(0)}{z^2} + + \frac{1}{z}\left(T^x(0) + T^g(0)\right) + \text{reg.}\,,
\end{aligned} \tag{5.68}$$

where the ghost current was defined in eqn. (4.152), which implies that:

$$\{Q_B, b_0\} = L^x_0 + L^g_0 \,. \tag{5.69}$$

On the other hand one has from eqn. (5.67) the anticommutator (remember that c_0 is viewed as a creation operator):

$$\begin{aligned}
\{Q_B, b_0\} &= \{c_0, b_0\}L^x_0 + \sum_{m\in\mathbb{Z}} \frac{m}{2}\{ {}^\circ_\circ\, c_m c_0 b_{-m} \,{}^\circ_\circ, b_0\} - \sum_{n\in\mathbb{Z}} \frac{n}{2}\{ {}^\circ_\circ\, c_0 c_n b_{-n} \,{}^\circ_\circ, b_0\} + \lambda\{c_0, b_0\} \\
&= L^x_0 - \frac{1}{2}\sum_{m=1}^{\infty} m\{b_{-m}c_0 c_m, b_0\} + \frac{1}{2}\sum_{m=-\infty}^{-1} m\{c_m c_0 b_{-m}, b_0\} \\
&\quad - \frac{1}{2}\sum_{n=1}^{\infty} n\{b_{-n}c_0 c_n, b_0\} + \frac{1}{2}\sum_{n=-\infty}^{-1} n\{c_n c_0 b_{-n}, b_0\} + \lambda \\
&= L^x_0 - \sum_{m=1}^{\infty} m b_{-m} c_m + \sum_{m=-\infty}^{-1} m c_m b_{-m} + \lambda \\
&= L^x_0 + L^g_0 + \lambda + 1\,,
\end{aligned} \tag{5.70}$$

where we have used the expression (4.180) for L^g_0. Therefore one should set $\lambda = -1$.

In a similar fashion one can determine the expansion of the anti-holomorphic BRST charge $\widetilde{Q}_B$ as follows:

$$\widetilde{Q}_B = \sum_n \widetilde{c}_n \widetilde{L}^x_{-n} + \sum_{n,m} \frac{m-n}{2} \,{}^\circ_\circ\, \widetilde{c}_m \widetilde{c}_n \widetilde{b}_{-m-n} \,{}^\circ_\circ\, - \widetilde{c}_0 \,. \tag{5.71}$$

Let us characterize now the physical states of the bosonic closed string theory. A state $|\Psi\rangle$ in the Hilbert space of the CFT for the scalars $x^\mu(z,\bar{z})$ and for the ghosts $(b,c,\tilde{b},\tilde{c})$ should be annihilated by the BRST charge defined above:

$$\left(Q_{\text{B}}+\widetilde{Q}_{\text{B}}\right)|\Psi\rangle = 0\,, \tag{5.72}$$

Since the Hilbert space of a CFT splits into its holomorphic and anti-holomorphic components, one will see that one can impose equivalently the constraints

$$Q_{\text{B}}|\Psi\rangle = \widetilde{Q}_{\text{B}}|\Psi\rangle = 0\,. \tag{5.73}$$

Physical states, being in the BRST cohomology, are furthermore identified by the equivalence relation

$$|\Psi\rangle \sim |\Psi\rangle + \left(Q_{\text{B}}+\widetilde{Q}_{\text{B}}\right)|\chi\rangle\,, \tag{5.74}$$

where $|\chi\rangle$ is any (non-physical) state. Usually $\left(Q_{\text{B}}+\widetilde{Q}_{\text{B}}\right)|\chi\rangle$ is called a *spurious state*.

There is an extra condition that should be imposed on the physical states in order to obtain a sensible spectrum. Remember that the ghost CFT has a two-fold degenerate ground state, see eqn. (4.176). If we allow both ground states one would obtain two families of physical states. It turns out that one choice is sensible and not the other. We impose that the physical states obey the condition known as the *Siegel gauge*:

$$b_0|\Psi\rangle = \tilde{b}_0|\Psi\rangle = 0\,. \tag{5.75}$$

Since $b_0|-\rangle_{\text{GH}} = 0$, it means that the physical spectrum of the string is built from the ghost vacuum $|-\rangle_{\text{GH}} \otimes \widetilde{|-\rangle}_{\text{GH}}$ of the tensor product of the holomorphic and anti-holomorphic ghost CFTs.

Notice finally that the physical states are characterized by the simple relations

$$(L_0^x + L_0^g)|\psi\rangle = \{Q_{\text{B}}, b_0\}|\Psi\rangle = 0\;,\quad (\widetilde{L}_0^x + \widetilde{L}_0^g)|\Psi\rangle = \{\widetilde{Q}_{\text{B}}, \tilde{b}_0\}|\Psi\rangle = 0\,. \tag{5.76}$$

In other words, the left and right conformal dimensions of the physical states (including the ghost contribution) should both vanish. Naturally, since the underlying CFT is not unitary, is does not mean that the physical spectrum contains just the vacuum.

5.3 The closed string spectrum

We have now gathered all the ingredients to construct the full spectrum of bosonic string theory. We will mostly consider the closed string sector. In order to proceed, let us first recapitulate what are the Hilbert spaces of the CFTs on the worldsheet of the string.

First, the Hilbert space of CFT for the free scalar fields $\{x^\mu(z,\bar{z}), \mu = 0,\dots,25\}$, is built from the primary states $|p^\mu\rangle$, of conformal dimensions

$$L_0^x|p\rangle = \widetilde{L}_0^x|p\rangle = \frac{\alpha'}{4}p_\mu p^\mu|p\rangle\,. \tag{5.77}$$

The other states generically, non-primary, are then obtained from those by the creation operators:

$$|\Phi\rangle = \cdots (\widetilde{\alpha}^{\mu_2}_{-2})^{\widetilde{N}^{\mu_2}_2} (\alpha^{\nu_2}_{-2})^{N^{\nu_2}_2} (\widetilde{\alpha}^{\mu_1}_{-1})^{\widetilde{N}^{\mu_1}_1} (\alpha^{\nu_1}_{-1})^{N^{\nu_1}_1} |p\rangle \tag{5.78}$$

The left and right conformal dimension of such a state are

$$L^x_0|\Phi\rangle = \Big(\frac{\alpha'}{4}p^2 + N^x\Big)|\Phi\rangle \ , \quad N^x = \sum_{\mu_r, r} r N^{\mu_r}_r \tag{5.79a}$$

$$\widetilde{L}^x_0|\Phi\rangle = \Big(\frac{\alpha'}{4}p^2 + \widetilde{N}^x\Big)|\Phi\rangle \ , \quad \widetilde{N} = \sum_{\nu_r, r} r \widetilde{N}^{\mu_r}_r \tag{5.79b}$$

The integers N^x and $\widetilde{N}^x$ are called the left and right levels of the states; they specify the "depth" of the state in the representation of the current algebra of highest weight state $|p\rangle$.

Second, the Hilbert space for the holomorphic and anti-holomorphic ghost CFTs in the Siegel gauge (5.75) is built out the vacuum $|-\rangle_{\rm GH} \otimes \widetilde{|-\rangle}_{\rm GH}$ by acting with the creation operators c_{-n} and b_{-n}, for $n > 0$ (acting with the zero-mode c_0 would violate the Siegel gauge condition). Therefore a generic state is of the form

$$|\Phi\rangle_{\rm GH} = \Big(\cdots (b_{-1})^{N_1} (c_{-1})^{M_1} |-\rangle_{\rm GH}\Big) \otimes \Big(\cdots (\widetilde{b}_{-1})^{\widetilde{N}_1} (\widetilde{c}_{-1})^{\widetilde{M}_1} \widetilde{|-\rangle}_{\rm GH}\Big) . \tag{5.80}$$

The left and right conformal dimension of such a state are

$$L^g_0|\Phi\rangle_{\rm GH} = \Big(N^g - 1\Big)|\Phi\rangle_{\rm GH} \ , \quad N^g = \sum_r r(N_r + M_r) \tag{5.81a}$$

$$\widetilde{L}^g_0|\Phi\rangle_{\rm GH} = \Big(\widetilde{N}^g - 1\Big)|\Phi\rangle_{\rm GH} \ , \quad \widetilde{N}^g = \sum_i r(\widetilde{N}_r + \widetilde{M}_r) \tag{5.81b}$$

The integers N^g and $\widetilde{N}^g$ are the left and right ghost levels of the states. Notice that, since the ghosts are fermionic variables, the integers N_r, M_r, $\widetilde{N}_r$ and $\widetilde{M}_r$ are either zero or one.

In the following, we will work out the spectrum level by level, according to the total left and right levels N and $\bar{N}$ of the full conformal field theory for the coordinate fields and for the ghosts. This makes sense as the BRST charge, as can be seen by the mode expansion (5.67), does not mix terms of different levels. Since we have seen that the physical states satisfy $L_0|\Psi\rangle = \widetilde{L}_0|\Psi\rangle = 0$, the left and right levels need to be the same.[2]

5.3.1 The tachyon

Let us start by looking at the physical states at level zero, *i.e.* with neither oscillators for the coordinates fields nor for the ghost fields. The unique such state is of the form

$$|p^\mu\rangle_{\rm T} = |p^\mu\rangle \otimes |-\rangle_{\rm GH} \otimes \widetilde{|-\rangle}_{\rm GH} . \tag{5.82}$$

[2]This is true because we consider states built out of the primary state $|p^\mu\rangle$ for the matter part, which has the same left and right conformal weights. It may not be true, as we will see later, if some of the dimensions of space-time are compact.

We have

$$Q_B|p^\mu\rangle_T = \Big(\sum_n c_n L^x_{-n} + \sum_{n,m}\frac{m-n}{2} \,{}^\circ_\circ\, c_m c_n b_{-m-n} \,{}^\circ_\circ\, - c_0\Big)|p^\mu\rangle \otimes |-\rangle_{GH} \otimes \widetilde{|-\rangle}_{GH}$$

$$= L^x_0|p^\mu\rangle \otimes \big(c_0|-\rangle_{GH}\big) \otimes \widetilde{|-\rangle}_{GH} - |p^\mu\rangle \otimes \big(c_0|-\rangle_{GH}\big) \otimes \widetilde{|-\rangle}_{GH} \tag{5.83}$$

$$= \left(\frac{\alpha'}{4}p^2 - 1\right)|p^\mu\rangle \otimes |+\rangle_{GH} \otimes \widetilde{|-\rangle}_{GH}\,. \tag{5.84}$$

Likewise we find that

$$\widetilde{Q}_B|p^\mu\rangle_T = \left(\frac{\alpha'}{4}p^2 - 1\right)|p^\mu\rangle \otimes |-\rangle_{GH} \otimes \widetilde{|+\rangle}_{GH}\,. \tag{5.85}$$

Therefore the BRST constraint (5.72) is satisfied provided that the state satisfies:

$$\frac{\alpha'}{4}p^2 = 1\,. \tag{5.86}$$

Using the mass-shell condition $p^2 + m^2 = 0$, one gets that

$$m^2 = -\frac{4}{\alpha'}\,. \tag{5.87}$$

From the low-energy perspective, this state of string theory behaves like a scalar particle with imaginary mass. Such particle, which is called a *tachyon*, indicates an instability of the field theory, as we expand around a local maximum of the potential.

This is a *very* severe problem of bosonic string theory, and by itself it indicates that the theory is ill-defined, at least in the perturbative, first-quantized approach that we are using. Fortunately, slightly more sophisticated theories, called *superstring theories*, can get rid successfully of this instability. In this perspective, bosonic string theory is a "toy model" that allows to become familiar with the tools used in the more sophisticated construction.

5.3.2 The graviton and other massless states

We consider now the physical states at level $N = \bar{N} = 1$. The more general ansatz for such physical state is of the form

$$|\Psi^1\rangle = \Big(e_{\mu\nu}\alpha^\mu_{-1}\widetilde{\alpha}^\nu_{-1} + \zeta_\mu\alpha^\mu_{-1}\widetilde{b}_{-1} + \upsilon_\mu\alpha^\mu_{-1}\widetilde{c}_{-1} + \widetilde{\zeta}_\mu\widetilde{\alpha}^\mu_{-1}b_{-1} + \widetilde{\upsilon}_\mu\widetilde{\alpha}^\mu_{-1}c_{-1}$$
$$+ \lambda_1 b_{-1}\widetilde{b}_{-1} + \lambda_2 c_{-1}\widetilde{c}_{-1} + \lambda_3 b_{-1}\widetilde{c}_{-1} + \lambda_4 c_{-1}\widetilde{b}_{-1}\Big)|p^\mu\rangle_T\,. \tag{5.88}$$

We now impose that such a state is annihilated by the BRST charge. As before one can split the discussion between the action of the left BRST charge on the left-moving states and of the right BRST charge on the right-moving states.

The only terms from the BRST charge contributing to the computation at this level are given by

$$Q_B = c_0(L^x_0 - 1) + c_{-1}L^x_1 + c_1 L^x_{-1} - b_{-1}c_0c_1 - c_{-1}c_0b_1 + \cdots \tag{5.89}$$

Technically, the first term will enforce that the conformal dimension of the state in the matter CFT is $h = 1$, the second term will enforce that it is a primary state while the third term will provide the equivalence relations between matter states of different polarizations that differ by a spurious state.

Notice that no contributions from the cubic terms in the ghost oscillators arise, since those terms, of the form $c_{-1}c_0b_1$ or $b_{-1}c_0c_1$, give zero because of the Siegel gauge condition (5.75).

To see this, we will act with the BRST charge (5.89) on the various terms in (5.88) and provide the condition for BRST closeness. One has first:

$$\begin{aligned} Q_B\alpha^\mu_{-1}|p\rangle \otimes |-\rangle_{GH} &= (L^x_0 - 1)\alpha^\mu_{-1}|p\rangle \otimes c_0|-\rangle_{GH} + L^x_1\alpha^\mu_{-1}|p\rangle \otimes c_{-1}|-\rangle_{GH} \\ &= \frac{\alpha'}{4}p^2\alpha^\mu_{-1}|p\rangle \otimes |+\rangle_{GH} + \sqrt{\frac{\alpha'}{2}}p^\mu|p\rangle \otimes c_{-1}|-\rangle_{GH}\,, \end{aligned} \tag{5.90}$$

where we have used that

$$L_1\alpha^\mu_{-1}|p\rangle = \delta_{\nu\rho}\alpha^\nu_0\alpha^\rho_1\alpha^\mu_{-1}|p\rangle = \delta_{\nu\rho}\alpha^\nu_0[\alpha^\rho_1, \alpha^\mu_{-1}]|p\rangle = \sqrt{\frac{\alpha'}{2}}p^\mu|p\rangle \tag{5.91}$$

Next we have

$$\begin{aligned} Q_B|p\rangle \otimes b_{-1}|-\rangle_{GH} &= \big(c_0(L^x_0 - 1) + c_{-1}L^x_1 + c_1L^x_{-1} - b_{-1}c_0c_1\big)|p\rangle \otimes b_{-1}|-\rangle_{GH} \\ &= -\big(\frac{\alpha'}{4}p^2 - 1\big)|p\rangle \otimes b_{-1}|+\rangle_{GH} + \sqrt{\frac{\alpha'}{2}}p_\mu\alpha^\mu_{-1}|p\rangle \otimes |-\rangle_{GH} - |p\rangle \otimes b_{-1}|+\rangle_{GH} \\ &= -\frac{\alpha'}{4}p^2|p\rangle \otimes b_{-1}|+\rangle_{GH} + \sqrt{\frac{\alpha'}{2}}p_\mu\alpha^\mu_{-1}|p\rangle \otimes |-\rangle_{GH} \end{aligned} \tag{5.92}$$

and finally

$$Q_B|p\rangle\otimes c_{-1}|-\rangle_{GH} = -\big(\frac{\alpha' p^2}{4} - 1\big)|p\rangle\otimes c_{-1}|+\rangle_{GH} - |p\rangle\otimes c_{-1}|+\rangle_{GH} = -\frac{\alpha' p^2}{4}|p\rangle\otimes c_{-1}|+\rangle_{GH} \tag{5.93}$$

One needs first to remove all the terms constructed from the ghost vacuum $|+\rangle_{GH} = c_0|-\rangle_{GH}$. This gives the space-time mass-shell condition

$$\frac{\alpha'}{4}m^2 = \frac{\alpha'}{4}p^2 = 0\,, \tag{5.94}$$

hence level-one states of string theory can be interpreted at low energies as massless particles.

The vanishing of the contribution corresponding to the second term in (5.90) imposes then the condition

$$p^\mu e_{\mu\nu} = 0\,, \quad p^\mu v_\mu = 0 \tag{5.95}$$

while contributions corresponding to the second term in (5.92) vanish if all the terms in b_{-1} are absent in (5.88), namely

$$\widetilde{\zeta}_\mu = \lambda_1 = \lambda_3 = 0\,. \tag{5.96}$$

In a similar way, the anti-holomorphic charge $\widetilde{Q}_B$ annihilate the state (5.88) provided that:

$$p^\nu e_{\mu\nu} = 0\,, \quad p^\mu \widetilde{\upsilon}_\mu = 0 \tag{5.97}$$

and

$$\zeta_\mu = \lambda_4 = 0\,. \tag{5.98}$$

In other words we are left with

$$|\Psi^1\rangle = \left(e_{\mu\nu}\alpha^\mu_{-1}\widetilde{\alpha}^\nu_{-1} + \upsilon_\mu \alpha^\mu_{-1}\widetilde{c}_{-1} + \widetilde{\upsilon}_\mu \widetilde{\alpha}^\mu_{-1} c_{-1} + \lambda_2 c_{-1}\widetilde{c}_{-1}\right)|p^\mu\rangle_T\,,$$
$$p^\nu e_{\mu\nu} = p^\mu e_{\mu\nu} = p^\mu \upsilon_\mu = p^\mu \widetilde{\upsilon}_\mu = 0\,. \tag{5.99}$$

The spurious states at level one correspond to all the states than can be obtained from the generic ansatz (5.88), without imposing of course the physical state constraints. First, given that (with the momentum on-shell)

$$Q_B \hat{e}_{\mu\nu}\alpha^\mu_{-1}\widetilde{\alpha}^\nu_{-1}|p^\mu\rangle_T = \sqrt{\frac{\alpha'}{2}}\hat{e}_{\mu\nu}p^\mu c_{-1}\widetilde{\alpha}^\mu_{-1}|p^\mu\rangle_T\,, \tag{5.100}$$

the terms in υ_μ and $\widetilde{\upsilon}_\mu$ in eqn. (5.99) can be gauged away. Next we notice that

$$Q_B \hat{\upsilon}_\mu \alpha^\mu_{-1}\widetilde{c}_{-1}|p^\mu\rangle_T = \sqrt{\frac{\alpha'}{2}}\,(p_\mu \hat{\upsilon}^\mu)\, c_{-1}\widetilde{c}_{-1}|p\rangle_T\,, \tag{5.101}$$

therefore one can set $\lambda_2 = 0$ in (5.99), given that the term in $c_{-1}\widetilde{c}_{-1}$ is spurious. Finally:

$$\left(Q_B + \widetilde{Q}_B\right)\left(\kappa_\mu \alpha^\mu_{-1}\widetilde{b}_{-1} + \widetilde{\kappa}_\mu \widetilde{\alpha}^\mu_{-1} b_{-1}\right)|p^\mu\rangle_T =$$
$$\sqrt{\frac{\alpha'}{2}}\left(\kappa^\mu p_\mu c_{-1}\widetilde{b}_{-1} + (\kappa_\mu \alpha^\mu_{-1})(p_\nu \widetilde{\alpha}^\nu_{-1}) + \widetilde{\kappa}^\mu p_\mu \widetilde{c}_{-1} b_{-1} + (\widetilde{\kappa}_\mu \widetilde{\alpha}^\mu_{-1})(p_\nu \alpha^\nu_{-1})|p^\mu\rangle_T\right) \tag{5.102}$$

Hence, provided that

$$\kappa^\mu p_\mu = \widetilde{\kappa}^\mu p_\mu = 0\,, \tag{5.103}$$

one should make the identifications

$$e_{\mu\nu} \sim e_{\mu\nu} + \kappa_\mu p_\nu + \widetilde{\kappa}_\nu p_\mu\,. \tag{5.104}$$

To summarize:

The physical states of the bosonic closed string theory at level one are given by the BRST invariant states

$$|\Psi^1\rangle = e_{\mu\nu}\alpha^\mu_{-1}\widetilde{\alpha}^\nu_{-1}|p^\mu\rangle \otimes |-\rangle_{GH} \otimes \widetilde{|-\rangle}_{GH}\,, \tag{5.105}$$

subject to the conditions

$$p_\mu p^\mu = 0\,, \tag{5.106a}$$
$$p^\mu e_{\mu\nu} = p^\nu e_{\mu\nu} = 0\,, \tag{5.106b}$$
$$e_{\mu\nu} \sim e_{\mu\nu} + a_\mu p_\nu + b_\nu p_\mu\,, \quad a_\mu p^\mu = b_\nu p^\nu = 0\,. \tag{5.106c}$$

These states correspond to massless particles in space-time with a gauge symmetry expressed by the equivalence relations.

In the local operator formalism, these massless physical states correspond to the following local operators

$$e_{\mu\nu} :\partial x^{\mu}\bar{\partial}x^{\nu}e^{ip_{\mu}x^{\mu}}: (z,\bar{z})\,. \tag{5.107}$$

To get the space-time particle interpretation of these states, one needs to decompose the tensor of physical polarizations, *i.e.* those satisfying the transversality constraints (5.106b), into irreducible representations of the Poincaré group. The symmetric traceless part will correspond to the graviton $\delta G_{\mu\nu}$, the antisymmetric part to the fluctuations of the Kalb-Ramond tensor $\delta B_{\mu\nu}$ and and the trace part to the fluctuations of the dilaton $\delta\Phi$. To disentangle the trace part from the symmetric part in a covariant way, one needs to introduce a vector $\bar{p}^{\mu}$ such that

$$\bar{p}^{\mu}\bar{p}_{\mu} = 0\ , \quad p^{\mu}\bar{p}_{\mu} = 1\,. \tag{5.108}$$

One has then

$$e^{G}_{\mu\nu} = e_{(\mu\nu)} - \frac{1}{D-2}e^{\rho}{}_{\rho}\left(\eta_{\mu\nu} - p_{\mu}\bar{p}_{\nu} - \bar{p}_{\mu}p_{\nu}\right) \tag{5.109a}$$

$$e^{B}_{\mu\nu} = e_{[\mu\nu]} \tag{5.109b}$$

$$e^{\Phi}_{\mu\nu} = \frac{1}{D-2}e^{\rho}{}_{\rho}\left(\eta_{\mu\nu} - p_{\mu}\bar{p}_{\nu} - \bar{p}_{\mu}p_{\nu}\right)\,. \tag{5.109c}$$

The identifications induced from the BRST cohomology,

$$e^{G}_{\mu\nu} \sim e^{G}_{\mu\nu} + p_{(\mu}v_{\nu)} \tag{5.110a}$$

$$e^{B}_{\mu\nu} \sim e^{B}_{\mu\nu} + p_{[\mu}w_{\nu]} \tag{5.110b}$$

correspond respectively to the Fourier transform of diffeomorphism invariance in space-time and to the gauge symmetry associated with the Kalb-Ramond form:

$$h_{\mu\nu} \sim h_{\mu\nu} + \partial_{(\mu}\zeta_{\nu)} \tag{5.111a}$$

$$b_{\mu\nu} \sim b_{\mu\nu} + \partial_{[\mu}\lambda_{\nu]} \tag{5.111b}$$

Note that the gauge parameters ζ_{ν} and λ_{ν} are transverse in the present context, which means that string theory has automatically chosen a generalized Lorentz gauge.

5.4 Open string spectrum

We discuss briefly the case of open strings. In this case, there is a single set of modes for all the fields (scalar fields and ghosts), and one has to consider the cohomology of the unique BRST charge Q_B.

We describe first the Hilbert space of CFT for the free scalar fields $\{x^{\mu}(z,\bar{z}), \mu = 0,\dots,25\}$. For definiteness we consider that the space-time where the strings live contains a set of N parallel Dp-branes, located at positions X^{a}_{i} with $i = 1,\dots,N$ and $a = p+1,\dots,25$.

Therefore the spectrum splits into sectors (ij) associated with open strings starting on the brane number i and finishing on the brane number j. In each sector one has a primary state $|p^\mu\rangle_{(ij)}$, of conformal dimension (see eq. 4.74):

$$L_0^x|p\rangle_{(ij)} = \alpha' \left(p_\mu p^\mu + \left(\frac{X_i^a - X_j^a}{2\pi\alpha'} \right)^2 \right) |p\rangle_{(ij)} \,. \tag{5.112}$$

with a momentum vector p_μ, $\mu = 0, \ldots, p$ having only components in the Neumann directions. This conformal weight is a function of the distance between the branes,

$$d_{ij} := \sqrt{\sum_a (X_i^a - X_j^a)^2} \,. \tag{5.113}$$

The other states are then obtained from the ground state by acting with the single set of creation operators:

$$|\Phi\rangle_{(ij)} = \cdots (\alpha_{-2}^{\mu_2})^{N_2^{\mu_2}} (\alpha_{-1}^{\mu_1})^{N_1^{\mu_1}} |p\rangle_{(ij)} \tag{5.114}$$

The conformal dimension of such a state is given by:

$$L_0^x|\Phi\rangle_{(ij)} = \left(\alpha' p^2 + \frac{d_{ij}^2}{4\pi^2\alpha'} + N^x \right) |\Phi\rangle_{(ij)} \tag{5.115}$$

Second, the Hilbert space for the holomorphic ghost CFT in the Siegel gauge is built out the vacuum $|-\rangle_{\text{GH}}$ by acting with the creation operators c_{-n} and b_{-n}, for $n > 0$. A generic state is of the form

$$|\Phi\rangle_{\text{GH}} = \cdots \left(b_{-1}\right)^{N_1} \left(c_{-1}\right)^{M_1} |-\rangle_{\text{GH}} \tag{5.116}$$

The conformal dimension of such a state is

$$L_0^g|\Phi\rangle_{\text{GH}} = \left(N^g - 1\right)|\Phi\rangle_{\text{GH}} \,, \quad N^g = \sum_r r(N_r + M_r) \,. \tag{5.117}$$

5.4.1 Open string tachyons

We start by considering the physical states at level zero in the sector (ij). They are of the form

$$|p^\mu\rangle_{(ij)} \otimes |-\rangle_{\text{GH}} \,. \tag{5.118}$$

It is convenient to use an alternate basis by introducing set of N^2 Hermitian matrices of dimensions $N \times N$, that we will denote by $\mathfrak{L}^r$, normalized as

$$\text{Tr}\,(\mathfrak{L}^r \mathfrak{L}^s) = \delta^{rs} \,. \tag{5.119}$$

A generic state at level zero can be expanded on this basis as

$$|\Psi^0\rangle = \sum_{r=0}^{N^2} \psi_r \mathfrak{L}^r_{ij} |p^\mu\rangle_{(ij)} \otimes |-\rangle_{\text{GH}} \,. \tag{5.120}$$

In the present context, these matrices are known as *Chan-Paton factors.* $N \times N$ Hermitian matrices are associated with the Lie Algebra $\mathfrak{u}(N)$; the physical significance of this remark will be explained below.

Let us come back to the study of the BRST cohomology of the open string. We have first to compute:

$$\begin{aligned} Q_B|p^\mu\rangle_{(ij)} &= \Big(\sum_n c_n L^x_{-n} + \sum_{n,m} \frac{m-n}{2} {}^\circ_\circ c_m c_n b_{-m-n} {}^\circ_\circ - c_0\Big)|p^\mu\rangle_{(ij)} \otimes |-\rangle_{GH} \\ &= L^x_0|p^\mu\rangle_{(ij)} \otimes \big(c_0|-\rangle_{GH}\big) \otimes \widetilde{|-\rangle}_{GH} - |p^\mu\rangle_{(ij)} \otimes \big(c_0|-\rangle_{GH}\big) \qquad (5.121) \\ &= \left(\alpha' p^2 + \frac{d_{ij}^2}{4\pi^2\alpha'} - 1\right)|p^\mu\rangle_{(ij)} \otimes |+\rangle_{GH}\,. \qquad (5.122) \end{aligned}$$

Therefore the BRST constraint (5.72) is fullfilled provided that one satisfies the mass-shell condition:

$$m_{ij}^2 = \left(\frac{d_{ij}}{2\pi\alpha'}\right)^2 - \frac{1}{\alpha'}\,. \qquad (5.123)$$

Thus the states at level zero can be recast into a matrix-valued scalar field, with a mass matrix m_{ij} whose square is satisfies the condition (5.123). The matrix m^2_{ij} has always some negative eigenvalues, corresponding to open string tachyons.

If we consider more specifically the ground state of an open string stretched between two *D*p-branes, we get a tachyon provided that the two *D*-branes are close enough. The critical distance d_c is given by

$$\frac{d_c^2}{(2\pi\alpha')^2} - \frac{1}{\alpha'} = 0 \implies d_c = 2\pi\sqrt{\alpha'}\,. \qquad (5.124)$$

Whenever an open string starts and ends on the same *D*-brane, the result is of course always a tachyon, on mass squared $m^2 = -1/\alpha'$. Unlike the closed string tachyon (which exists in bosonic strings, and not in consistent superstring theories) one can have open string tachyon in a superstring theories if one considers unstable configurations of *D*-branes.

5.4.2 Gauge bosons

We consider then the physical states at level one in the sector (ij). Following the discussion about the closed string states, one starts with a generic state of the form We consider now the physical states at level $N = \bar{N} = 1$. The more general ansatz for such physical state is of the form

$$|\Psi^1\rangle_{(ij)} = \Big(u^{(ij)}_\mu \alpha^\mu_{-1} + f^{(ij)}_a \alpha^a_{-1} + \lambda_1 b_{-1} + \lambda_2 c_{-1}\Big)|p^\mu\rangle_{(ij)} \otimes |-\rangle_{GH}\,. \qquad (5.125)$$

Compared to the case of closed string theory, the Lorentz group $O(1,25)$ is broken to $O(1,p)\times O(25-p)$ in the presence of *D*p-branes hence we split the oscillators part accordingly. From this perspective the first and second term in the expansion (5.125) can be considered as corresponding to different physical states.

We now impose that the state (5.125) ia annihilated by the BRST charge. As before one considers first:

$$\begin{aligned}
Q_{\mathrm{B}}\alpha_{-1}^{\mu}|p\rangle_{(ij)}\otimes|-\rangle_{\mathrm{GH}} &= (L_0^x-1)\alpha_{-1}^{\mu}|p\rangle_{(ij)}\otimes c_0|-\rangle_{\mathrm{GH}} + L_1^x\alpha_{-1}^{\mu}|p\rangle_{(ij)}\otimes c_{-1}|-\rangle_{\mathrm{GH}} \\
&= \left(\alpha' p^2+\frac{d_{ij}^2}{4\pi^2\alpha'}\right)\alpha_{-1}^{\mu}|p\rangle_{(ij)}\otimes|+\rangle_{\mathrm{GH}} + \sqrt{2\alpha'}p^{\mu}|p\rangle_{(ij)}\otimes c_{-1}|-\rangle_{\mathrm{GH}}\,,
\end{aligned} \tag{5.126}$$

and similarly

$$\begin{aligned}
Q_{\mathrm{B}}\alpha_{-1}^{a}|p\rangle_{(ij)}\otimes|-\rangle_{\mathrm{GH}} &= (L_0^x-1)\alpha_{-1}^{a}|p\rangle_{(ij)}\otimes c_0|-\rangle_{\mathrm{GH}} + L_1^x\alpha_{-1}^{a}|p\rangle_{(ij)}\otimes c_{-1}|-\rangle_{\mathrm{GH}} \\
&= \left(\alpha' p^2+\frac{d_{ij}^2}{4\pi^2\alpha'}\right)\alpha_{-1}^{a}|p\rangle\otimes|+\rangle_{\mathrm{GH}}
\end{aligned} \tag{5.127}$$

The second term vanishes as there is no conserved spatial momentum in a direction transverse to the *D*-branes.[3]. Next we have

$$\begin{aligned}
Q_{\mathrm{B}}|p\rangle_{(ij)}\otimes b_{-1}|-\rangle_{\mathrm{GH}} &= \big(c_0(L_0^x-1)+c_{-1}L_1^x+c_1L_{-1}^x-b_{-1}c_0c_1\big)|p\rangle_{(ij)}\otimes b_{-1}|-\rangle_{\mathrm{GH}} \\
&= -\left(\alpha' p^2+\frac{d_{ij}^2}{4\pi^2\alpha'}-1\right)|p\rangle_{(ij)}\otimes b_{-1}|+\rangle_{\mathrm{GH}} \\
&\qquad + \sqrt{2\alpha'}p_{\mu}\alpha_{-1}^{\mu}|p\rangle_{(ij)}\otimes|-\rangle_{\mathrm{GH}} - |p\rangle_{(ij)}\otimes b_{-1}|+\rangle_{\mathrm{GH}} \\
&= -\left(\alpha' p^2+\frac{d_{ij}^2}{4\pi^2\alpha'}\right)|p\rangle_{(ij)}\otimes b_{-1}|+\rangle_{\mathrm{GH}} + \sqrt{2\alpha'}p_{\mu}\alpha_{-1}^{\mu}|p\rangle_{(ij)}\otimes|-\rangle_{\mathrm{GH}}
\end{aligned} \tag{5.128}$$

and finally

$$\begin{aligned}
Q_{\mathrm{B}}|p\rangle_{(ij)}\otimes c_{-1}|-\rangle_{\mathrm{GH}} &= -\left(\alpha' p^2+\frac{d_{ij}^2}{4\pi^2\alpha'}-1\right)|p\rangle_{ij}\otimes c_{-1}|+\rangle_{\mathrm{GH}} \\
&\quad + |p\rangle_{(ij)}\otimes c_{-1}|+\rangle_{\mathrm{GH}} = -\left(\alpha' p^2+\frac{d_{ij}^2}{4\pi^2\alpha'}\right)|p\rangle_{(ij)}\otimes c_{-1}|+\rangle_{\mathrm{GH}}
\end{aligned} \tag{5.129}$$

One needs first to remove all the terms constructed from the ghost vacuum $|+\rangle_{\mathrm{GH}} = c_0|-\rangle_{\mathrm{GH}}$. This gives the space-time mass-shell condition

$$m^2=\left(\frac{d_{ij}}{2\pi\alpha'}\right)^2. \tag{5.130}$$

[3] Another way to see this is that, for a boson with Dirichlet b.c., $\alpha_0=\sqrt{\frac{2}{\alpha'}}\frac{1}{2\pi}\oint\left(dz\,\partial x^a-d\bar z\,\bar\partial x^a\right)=0$ using the expansion (4.73)

hence level-one states of open string theory are massive if the string is stretched between two separated *D*-branes, or massless if they stretch between two coincident *D*-branes or have both ends on the same *D*-brane.

The vanishing of the contribution corresponding to the second term in (5.126) imposes then the condition

$$p^\mu u_\mu^{(ij)} = 0 \tag{5.131}$$

while contributions corresponding to the second term in (5.128) vanish if all the terms in b_{-1} are absent in (5.88), namely $\lambda_1 = 0$. In other words we are left with

$$|\Psi^1\rangle_{(ij)} = \left(u_\mu^{(ij)}\alpha_{-1}^\mu + f_a\alpha_{-1}^a + \lambda_2 c_{-1}\right)|p^\mu\rangle_{(ij)} \otimes |-\rangle_{\text{GH}}\ , \quad p^\mu u_\mu^{(ij)} = 0\,. \tag{5.132}$$

To obtain the spurious states at level one sees first that:

$$Q_{\text{B}}\hat{u}_\mu\alpha_{-1}^\mu|p^\mu\rangle_{(ij)} \otimes |-\rangle_{\text{GH}} = \sqrt{2\alpha'}\hat{u}_\mu p^\mu|p^\mu\rangle_{(ij)} \otimes c_{-1}|-\rangle_{\text{GH}} \tag{5.133}$$

hence one can set $\lambda_2 = 0$ in (5.132). Then we have:

$$Q_{\text{B}}b_{-1}|p^\mu\rangle_{(ij)} \otimes b_{-1}|-\rangle_{\text{GH}} = \sqrt{2\alpha'}p_\mu\alpha_{-1}^\mu|p^\mu\rangle_{(ij)} \otimes |-\rangle_{\text{GH}} \tag{5.134}$$

Hence one should make the identifications

$$u_\mu^{(ij)} \sim u_\mu^{(ij)} + p_\mu\lambda^{(ij)}\,. \tag{5.135}$$

If one uses the Chan-Paton basis, the physical states in the open string sector string theory at level one are given by two types of BRST invariant states

$$|A\rangle := \sum_r u_\mu^r\Big(\sum_{i,j}\mathfrak{T}_{ij}^r\alpha_{-1}^\mu|p^\mu\rangle_{(ij)} \otimes |-\rangle_{\text{GH}}\Big) \tag{5.136a}$$

$$|\Phi\rangle := \sum_r f_a^r\Big(\sum_{i,j}\mathfrak{T}_{ij}^r\alpha_{-1}^a|p^\mu\rangle_{(ij)} \otimes |-\rangle_{\text{GH}}\Big) \tag{5.136b}$$

subject to the conditions

$$p_\mu p^\mu = \left(\frac{d_{ij}}{2\pi\alpha'}\right)^2, \tag{5.137a}$$

$$p^\mu u_\mu^r = 0\,, \tag{5.137b}$$

$$u_\mu^r \sim u_\mu^r + p_\mu\lambda^r\,. \tag{5.137c}$$

To understand the physical significance of these physical states, we consider first open strings sectors with both endpoints on the same *D*p-brane. It gives two types of massless states:

- $|A^{(ii)}\rangle = u_\mu^{(ii)}\alpha_{-1}^\mu|p^\mu\rangle_{(ii)} \otimes |-\rangle_{\text{GH}}$ corresponds to a $U(1)$ gauge boson in $p+1$ dimensions. The equivalence relation $u_\mu^{(ii)} \sim u_\mu + p_\mu\lambda^{(ii)}$ expresses gauge invariance.

- $\alpha^a_{-1}|p^\mu\rangle_{(ii)} \otimes |-\rangle_{\rm GH}$ correspond to $25-p$ massless scalar fields. These scalar fields, being massless, can acquire a vacuum expectation value. These expectation values correspond to the center-of-mass position of the *D*p-brane in the corresponding transverse dimensions.

Consider now a stack of N coincident *D*p-branes. In this case all open string sectors give massless states at level one that can be interpreted as follows:

- $|A^{(ij)}\rangle = u^{(ij)}_\mu \alpha^\mu_{-1}|p^\mu\rangle_{(ij)} \otimes |-\rangle_{\rm GH}$ with $i,j = 1,\dots,N$ gives a set of $N \times N = N^2$ gauge fields. While the Cartan states $|A^{(ii)}\rangle$ gives, following the previous discussion, a $U(1)^N$ gauge symmetry, it gets enhanced thanks to the "off-diagonal" states $|A^{(ij)}\rangle$ with $i \neq j$ to a non-Abelian $U(N)$ gauge symmetry. In the Chan-Paton basis, one has the gauge field $|A\rangle = \sum_r u^\mu_r \sum_{i,j} \mathfrak{L}^r_{ij} \alpha^\mu_{-1}|p^\mu\rangle_{(ij)} \otimes |-\rangle_{\rm GH}$ where the matrices $\mathfrak{L}^r_{ij}$ are the generators of the $\mathfrak{u}(N)$ Lie algebra.

- The $(25-p) \times N^2$ scalar fields $\alpha^a_{-1}|p^\mu\rangle_{(ij)} \otimes |-\rangle_{\rm GH}$ arrange themselves as $25-p$ matrix-valued scalar fields transforming in the adjoint representation of $U(N)$.

While the previous identifications rely essentially on the counting of states, one can check that scattering amplitudes are consistent with these properties (see chapter 6).

Finally let's see what happens when we consider a set of N *D*p-branes that are separated in their transverse directions, rather than stacked on top of each other. In this case, for a generic configuration, while the diagonal gauge bosons $|A^{(ii)}\rangle$ remains massless, the off-diagonal gauge bosons $|A^{(ij)}\rangle$ with $i \neq j$ acquire a mass proportionnal to the distance between the *D*p-branes on which they end:

$$m_{ij} = \frac{d_{ij}}{2\pi\alpha'} \,. \tag{5.138}$$

Since the transverse positions of the *D*p-branes correspond to the vaccum expectation values of the scalar fields $\alpha^a_{-1}|p^\mu\rangle_{(ii)} \otimes |-\rangle_{\rm GH}$, there is a very neat interpretation of this phenomenon from the low-energy view point: the *Higgs mechanism*. In the present context, one considers a $U(N)$ gauge theory in $p+1$ dimensions with an adjoint Higgs field (without potential at tree-level). For a generic expectation value of the Higgs field, the gauge symmetry is broken down to $U(1)^N$.

To summarize, introducing D-branes allow to introduce non-Abelian gauge theory in string theory, and even better in four space-time dimensions if one chooses a stack of *D*p-branes. Notice however that, in the perspective of model building, the gravitationnal sector of the theory remains 26-dimensional.

5.5 Physical degrees of freedom and light-cone gauge

A more physical way of working out the string spectrum is to identify first properly the physical degrees of freedom. We will start by discussing the familiar example of a massless vector field which arises in open string theory. We consider to simplify the discussion that we have a single *D*25-brane. As we have discussed before the vector field corresponds to the physical state

$$|\Psi\rangle = u_\mu \alpha_{-1}^\mu |p\rangle \otimes |-\rangle_{\text{GH}}\,, \tag{5.139}$$

whose polarization vector u_μ satisfies the transversality condition

$$p^\mu u_\mu = 0\,, \tag{5.140}$$

coming from the BRST constraint $L_1|\Psi\rangle = 0$. It can be equivalently rephrased as

$$p_\nu \alpha_1^\nu |\Psi\rangle = 0\,, \tag{5.141}$$

which is similar to the familiar physical state condition for the Maxwell field in the Gupta-Bleuler formalism of QED, $\partial_\mu A^\mu|\Psi\rangle = 0$.

One can choose, without loss of generality, $p^\mu = (\omega, \omega, 0, \ldots, 0)$, and choose accordingly the polarization basis as

$$u^{\text{G}} = \tfrac{1}{\sqrt{2}}(1, 1, 0 \cdots, 0)\,, \tag{5.142a}$$
$$u^{\text{L}} = \tfrac{1}{\sqrt{2}}(-1, 1, 0, \cdots, 0)\,, \tag{5.142b}$$
$$u^i = (0, 0, 0 \cdots 0, \underbrace{1}_{i+2}, 0 \cdots 0) \tag{5.142c}$$

and expand the oscillator modes

$$\alpha_{n\,\mu} = \alpha_n^{\text{L}} u_\mu^{\text{G}} + \alpha_n^{\text{G}} u_\mu^{\text{L}} + \alpha_n^i u_\mu^i\,. \tag{5.143}$$

In this basis the Gupta-Bleuler condition (5.141) is simply

$$\alpha_1^{\text{G}}|\Psi\rangle = 0\,, \tag{5.144}$$

which imposes, because of the commutation relation

$$[\alpha_1^{\text{G}}, \alpha_{-1}^{\text{L}}] = 1\,, \tag{5.145}$$

that the physical state has no longitudinal component, *i.e.* no α_{-1}^{L} oscillator mode. The spurious states, exactly as in the closed string computation (5.102), are obtained as

$$Q_{\text{B}} b_{-1}|p\rangle \otimes |-\rangle_{\text{GH}} = \sqrt{\frac{\alpha'}{2}}\, p_\mu \alpha_{-1}^\mu |p\rangle \otimes |-\rangle_{\text{GH}}\,, \tag{5.146}$$

hence $\alpha_{-1}^{\text{G}}|p\rangle \otimes |-\rangle_{\text{GH}}$ is a spurious state. The remaining polarizations give the physical states,

$$|\Psi\rangle = u_i \alpha_{-1}^i |p\rangle \otimes |-\rangle_{\text{GH}}\,, \tag{5.147}$$

which contain only transverse excitations.

To summarize, at the level of the polarization vector the transversality condition $p_\mu u^\mu = 0$ allows to do the decomposition,

$$u^\mu = a p^\mu + u_T^\mu \,, \quad u_T^\mu = (0, 0, \mathbf{u}^T)\,. \tag{5.148}$$

where the longitudinal component is null (since $p^2 = 0$). Because of gauge invariance,

$$u_\mu \sim u_\mu + \lambda p_\mu \,, \tag{5.149}$$

the longitudinal polarizations can be gauged away, leaving only the physical transverse polarizations, that transform in the vector representation of the little group $SO(D-2)$.

We consider now the massless closed string states. The polarization tensor $e_{\mu\nu}$ obeys the transversality condition $p^\mu e_{\mu\nu} = p^\nu e_{\mu\nu} = 0$ hence can be decomposed in the same way as above:

$$e_{\mu\nu} = a_\mu p_\nu + b_\nu p_\mu + e^T_{\mu\nu} \,, \quad p^\mu a_\mu = p^\mu b_\mu = 0\,, \tag{5.150}$$

where the transverse polarization tensor is given by

$$e^T_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_{ij} \end{pmatrix} \,, \; i, j = 2, \ldots 26\,. \tag{5.151}$$

As before, the unphysical longitudinal polarizations are completely removed by the gauge invariance.

One can them decompose the transverse polarization tensor into irreducible representations of the space-time little group $SO(D-2)$, with $D = 26$, namely into its symmetric traceless, antisymmetric and trace part, following (5.109):

$$e^T_{ij} = \underbrace{\left(e_{(ij)} - \tfrac{1}{D-2} e^i{}_i \delta_{ij}\right)}_{s_{ij}} + \underbrace{e_{[ij]}}_{a_{ij}} + \underbrace{\tfrac{1}{D-2} e^i{}_i \delta_{ij}}_{d}\,. \tag{5.152}$$

where s_{ij}, a_{ij} and d give respectively the physical polarizations of the graviton, Kalb-Ramond particle and dilaton.

So, after all this hard work, we have seen that the closed string theory spectrum contains a state that corresponds to a massless spin-two particle, with the appropriate gauge symmetry to remove the unphysical polarizations. By general arguments a sensible effective theory for this spin-two field cannot be anything else than Einstein gravity (coupled to the dilaton and Kalb-Ramond fields). Actual computations of S-matrix elements from string theory confirm explicitly that this result is correct.

If we reconsider the generic Polaykov-Kalb–Ramond-dilaton action (5.1) from this perspective one gets some very interesting result. This action is defined with respect to backgrounds for the metric, B-field and dilaton, which can be viewed as *coherent states* of strings states. Indeed, the exponentiated integrated vertex operator

$$\int d^2 z \, e_{(\mu\nu)} e^{ip\cdot x} \partial x^\mu \bar{\partial} x^\nu \tag{5.153}$$

can be viewed as a deformation of the Polyakov action around a Minkowskian background, due to a change of the metric by a plane wave perturbation.

Light-cone quantization

A more direct approach to string theory quantization, called light-cone quantization, gives directly the spectrum of physical states, by using the residual gauge freedom of the string path integral in the flat gauge (conformal transformations), which allow to eliminate all the oscillator modes of $x^+(\sigma^i) = x^0(\sigma^i) + x^1(\sigma^i)$, by choosing worldsheet coordinates $\hat{\sigma}^i$ such that

$$x^+\left(\widehat{\sigma}^i\right) = x_c^+ + p^+\widehat{\tau}\,. \tag{5.154}$$

It can be shown that the Virasoro constraints, *i.e.* the equation of motion $T_{ij} = 0$ for the worldsheet metric, allows then to eliminate all oscillator modes from $x^-(\sigma^i) = x^0(\sigma^i) - x^1(\sigma^i)$ as well, leaving only the physical transverse oscillator modes $\{\alpha_n^i, i = 2, \ldots 25\}$.

However this formalism is less suited for dealing with interactions and loop corrections, since we are loosing the freedom to use conformal transformations. We refer the reader to the general string theory textbooks for more details.

5.6 General structure of the string spectrum

The structure of the physical spectrum found at level zero and one persist at higher levels. All the other states are massive, and their mass is quantized in units of the inverse of the string length $1/\sqrt{\alpha'}$ which means that these states are extremely massive and out of reach of any foreseeable experiments. They are however necessary for the consistency of the theory, and participate indirectly to the low-energy physics through the loop corrections. We discuss here only the closed string spectrum, the open string spectrum following the same pattern.

Following the previous pattern, a physical level n state (both left and right) will be constructed by taking a level n state of the matter CFT tensored with the $|-\rangle_{\text{GH}}$ vacuum of the ghost CFT:

$$|\Psi_n\rangle = \left(e_{\mu\nu}\alpha^\mu_{-n}\widetilde{\alpha}^\nu_{-n} + f_{\mu\nu\rho}\alpha^\mu_{-n+1}\alpha^\nu_{-1}\widetilde{\alpha}^\rho_{-n} + \cdots\right)|p^\mu\rangle \otimes |-\rangle_{\text{GH}} \otimes \widetilde{|-\rangle}_{\text{GH}}\,, \tag{5.155}$$

depending on a large number of polarization tensors. The relevant part of the BRST charge is

$$Q_{\text{B}} = c_0(L_0^x - 1) + \widetilde{c}_0(\widetilde{L}_0^x - 1) + \sum_{k=1}^{\infty}\left(c_{-k}L_k^x + \widetilde{c}_{-k}\widetilde{L}_k^x\right)\,, \tag{5.156}$$

such that the first terms give the conditions

$$(L_0^x - 1)|\Psi_n\rangle = 0\,,\ (\widetilde{L}_0^x - 1)|\Psi_n\rangle = 0 \;\Leftrightarrow\; \boxed{m^2 = \frac{4}{\alpha'}(n-1)} \tag{5.157}$$

These two constraints are usually reorganized as

$$(L_0 + \widetilde{L}_0 - 2)|\Psi\rangle = 0\,, \tag{5.158a}$$

$$(L_0 - \widetilde{L}_0)|\Psi\rangle = 0\,. \tag{5.158b}$$

The first one is usually refered to as the *mass-shell condition*, while the second one the *level-matching condition*, which is the only relation between left- and right-moving excitations.

The constraints from the other terms in the BRST charge (5.156) indicate that the state in the x^μ matter CFT should be a primary state, namely

$$\forall k > 0\,, \quad L_k^x|\Psi_k\rangle = 0\,,\ \widetilde{L}_k^x|\Psi_k\rangle = 0 \tag{5.159}$$

which sets constraints on the polarization tensors. It is worthwhile to remind that the matter CFT is non-unitary because it contains one boson $x^0(z,\bar{z})$ with wrong sign kinetic term, hence one can construct generically non-trivial primary states at every oscillator level. Given that these particles are massive, the physical polarizations of the states should be decomposed into irreducible representations of the little group $SO(D-1)$, with $D = 26$.

Next we move to the spurious states. Owing to our previous experience, apart from the spurious states that allows to remove the terms with ghost oscillators in the ansatz (5.155), the interesting ones are obtained from terms of the form

$$|\xi\rangle = b_{-r}|\xi_r^x\rangle \otimes |-\rangle_{\text{GH}} \otimes \widetilde{|-\rangle}_{\text{GH}}\,, \tag{5.160}$$

where $|\xi_r^x\rangle$ is a state in the x^μ CFT at left and right levels $(n-r,n)$, satisfying the condition

$$(L_0^x - 1 + r)\,|\xi_r^x\rangle = 0\,, \tag{5.161}$$

and similar states with $\widetilde{b}_{-n}$ oscillators. We have indeed

$$Q_{\text{B}}|\xi\rangle = c_r L_{-r}^x b_{-r}|\Psi^x\rangle \otimes |-\rangle_{\text{GH}} \otimes \widetilde{|-\rangle}_{\text{GH}} = L_{-r}^x|\xi_r^x\rangle \otimes |-\rangle_{\text{GH}} \otimes \widetilde{|-\rangle}_{\text{GH}}\,, \tag{5.162}$$

showing that this state is a (left) descendant state at level n, which satisfies the same mass-shell condition as physical states:

$$(L_0^x - 1)L_{-r}^x|\xi_r^x\rangle = 0\,. \tag{5.163}$$

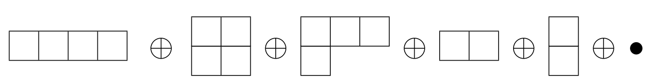
These spurious states are orthogonal to all physical states. Indeed,

$$\langle\Psi_m|Q_{\text{B}}|\xi\rangle = \langle\Psi_m|L_{-r}|\xi_r^x\rangle = \langle\xi^x|L_r|\Psi^x\rangle^\star = 0\,, \tag{5.164}$$

as any physical state is a conformal primary state in the matter CFT.

Example: level two physical states

The physical states at level two have all mass squared $m^2 = \frac{4}{\alpha'}$, and can be organized into the following representations of the little group $SO(25)$:



Moving to the more massive, higher level physical states gives increasingly large $SO(25)$ representations. The number of physical states actually grows tremendously with the level. In general, in a CFT with central charge c the asymptotic density of states is given by [3]:

$$\rho(N) \overset{N\gg 1}{\sim} e^{2\pi\sqrt{\frac{c}{6}(N-c/24)}}\,, \tag{5.165}$$

where in the present context $c = 24$, accounting for the oscillator modes of the 24 transverse coordinates. A striking consequence is the existence of a limiting temperature, the *Hagedorn temperature*, as it implies that, for a string in the canonical ensemble, the partition function

$$Z(E) = \int dE\, \rho(E) e^{-\beta E} \tag{5.166}$$

diverges above a certain temperature.

No-ghost theorem

The spectrum of the bosonic string is built out of the tachyon vacuum by acting by the two set of oscillator modes $\{\alpha^{\mu}_{-n}, \widetilde{\alpha}^{\nu}_{-n}\}$. Because of the commutation relation (4.61), the modes built from oscillators of the time-like coordinate field $x^0(z, \bar{z})$ can potentially be of negative norm. Indeed, for instance

$$\left|\left|\alpha^0_{-n}|p^\mu\rangle\right|\right|^2 = \langle p^\mu|\alpha^0_n \alpha^0_{-n}|p^\mu\rangle = -\langle p^\mu|p^\mu\rangle\,. \tag{5.167}$$

At the massless level, we have already encountered this problem, and it was solved in the same way as it is solved for the quantum theory of a massless vector field; the unphysical negative-norm longitudinal polarizations were removed using gauge invariance of the theory, see eqn. (5.144) and below.

An important consistency check of string theory is to prove that this feature persists at all string levels. The simplest way to prove this statement is to move to the light-cone gauge, discussed briefly at the end of subsection 5.3.2, since in that case one keeps only the transverse oscillators that only create positive norm states from the vacuum.

In the context of the BRST formalism, it is possible to prove as well that all the physical states have positive norm. The proof of this statement, misnamed *no-ghost theorem* (as ghost here means negative norm states and not Faddeev-Popov ghosts!), can be found in [4] and will not be reproduced here since it is rather technical. The statement is:

The inner product on the BRST cohomology of bosonic string theory is positive definite.

References

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Chapter 6

String interactions

In this chapter we will explain how to compute the observables of string theory, in particular the tree-level S-matrix elements and the vacuum amplitude at one-loop in string perturbation theory.

6.1 The string S-matrix

In a quantum field theory, the observables are given by N-point correlation functions of the fields, namely

$$G(x_1, x_2, \ldots, x_n) = \langle \phi(x_1) \cdots \phi(x_2) \rangle = \left\langle 0 \left| T\big(\hat{\phi}(x_1) \cdots \hat{\phi}(x_n)\big) \right| 0 \right\rangle . \tag{6.1}$$

Transforming this position space expression into momentum space, these quantity are clearly *off-shell*, as they are defined outside of the hypersurface $p^2 + m^2 = 0$.

In string theory, the situation is a little bit more complicated. First we know that there are no local observables in a quantum theory of gravity, because diffeomorphisms are gauge symmetries. Second we know that the local operators corresponding to string physical states need to be conformal primaries of dimensions $(1, 1)$, which forces them to be on-shell. For this reason the well-defined observables in string theory are S-matrix elements, *i.e.* on-shell correlation functions of asymptotic states. This prescription solves also the first issue as the asymptotic states, being defined on the boundary of space-time, are well-defined in quantum gravity. All we have said applies to asymptotically Minkowski space-times; in asymptotically Anti-de-Sitter space-times, the observables are a little bit different (and in asymptotically de Sitter space-times, one does not know precisely what happens).

In this lecture we will consider only *closed string observables*. At tree-level, a N-particle contribution to the S-matrix is obtained by gluing together N semi-infinite cylindrical world-sheets into a single surface, see an example on figure 6.1. For a one-loop diagram, one should add a handle to the "blob" in the center, two handles for a two-loops diagram and so on.

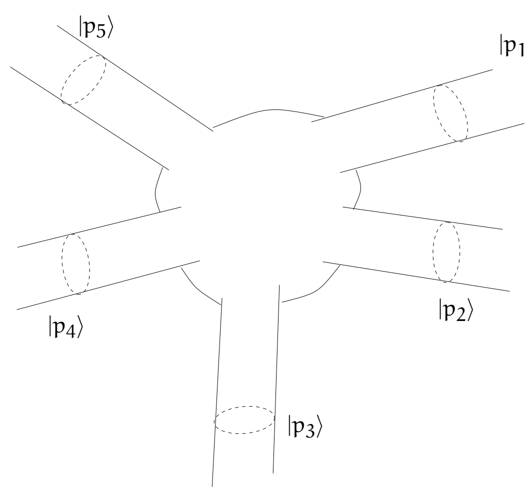

Figure 6.1: S-matrix element for tree-level five-tachyon scattering

This looks quite complicated to evaluate, but fortunately one can use the conformal symmetry of the theory to map this complicated surface to a much simpler one. We have seen in chapter 4 how to map a cylinder to a plane, and how an initial state at minus infinity was mapped to a local operator at the origin. By the same reasoning a semi-infinite cylinder, stretching say from $\sigma_2 \to -\infty$ to $\sigma_2 = 0$, is mapped to a disc and an asymptotic state is mapped to a local operator at the origin of the disc. We can imagine defining a conformal transformation doing the same here for of the semi-infinite cylinders, and we will end up with a sphere with N punctures, and a local operator inserted at each of the punctures, see figure 6.2.

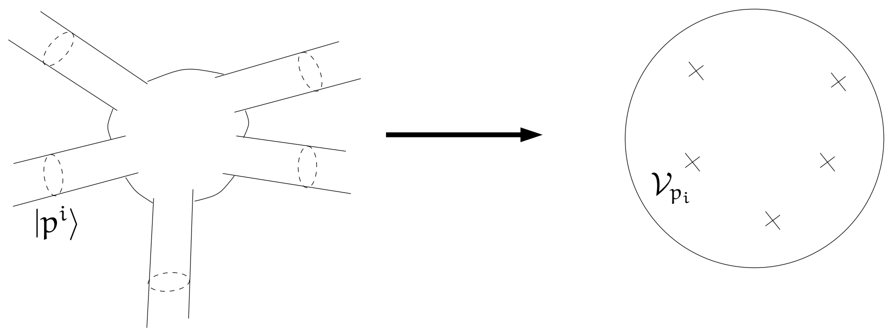

Figure 6.2: Conformal mapping of S-matrix elements

Because of the residual gauge symmetry in the gauge-fixed path integral due to the conformal killing vectors on a sphere, see subsection 2.4.1, three of the N local operators corresponding to the string physical states participating into the scattering are at arbitrary fixed positions, while the rest are integrated over the sphere. For convenience one can represent the two-sphere by the compactified complex plane $\bar{\mathbb{C}} = \mathbb{C} \cap \infty$, with the north pole at the origin and the south pole at infinity. The arbitrary positions are usually chosen to be $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$, respectively the north pole, a point on the equator and the south pole.

Finally, there exists a natural normalization of the string physical operators that we didn't discuss yet. In the loop expansion of string perturbation theory, see fig. 2.4, each extra handle on the worldsheet is coming with an extra factor of g_s^2. Since a handle is associated with virtual string loop in the diagram, it is natural to add a factor of g_s for each external state of the S-matrix.

6.2 Four-tachyon tree-level scattering

To illustrate the computation of S-matrix elements, we consider the example of four-tachyon tree-level scattering. While not physically terribly exciting, this amplitude is historically relevant – it appeared before string theory itself, in the context of dual resonance models – and simpler than the four-graviton scattering, which has a complicated index structure.

Following all the previous discussion, one considers the following expression:

$$S(p_1,p_2,p_3,p_4) = \frac{(g_s)^4}{g_s^2}\left\langle c(0)\tilde{c}(0)\mathcal{V}_{p_1}(0)\, c(1)\tilde{c}(1)\mathcal{V}_{p_2}(1)\, c(\infty)\tilde{c}(\infty)\mathcal{V}_{p_3}(\infty)\int d^2z\, \mathcal{V}_{p_4}(z,\bar{z})\right\rangle_{S^2} \tag{6.2}$$

with the on-shell vertex operators

$$\mathcal{V}_p(z,\bar{z}) = \,:\! e^{ip_\mu x^\mu}\!:\!(z,\bar{z})\ , \quad \tfrac{\alpha'}{4}p^2 = 1\,. \tag{6.3}$$

The CFT correlation function that has to be compute in (6.2) clearly splits into a "matter" part for the fields x^μ and a ghost part for the fields c and $\tilde{c}$. Let us start with the former.

6.2.1 Scalar field correlation functions

We are interested in computing, in a two-dimensional CFT of free scalar fields x^μ with action (4.1), the N-point function of vertex operators[1]

$$G^N(z_i,\bar{z}_i) = \left\langle \prod_{i=1}^N :\! e^{ip_i^\mu x^\mu(z_i,\bar{z}_i)}\!: \right\rangle\,. \tag{6.4}$$

Because we are dealing with a free Gaussian quantum field theory, one can couple the theory to an external current J^μ and write

$$\langle e^{i\int d^2z\, J_\mu(z,\bar{z})x^\mu(z,\bar{z})}\rangle = e^{-\frac{1}{2}\int d^2z\int d^2z' J^\mu(z)J_\mu(z')G(z-z',\bar{z}-\bar{z}')}\,, \tag{6.5}$$

where G is the Green function of the theory, satisfying the equation

$$-\frac{1}{2\pi\alpha'}\partial\bar{\partial}G(z',\bar{z}) = \delta(z)\delta(\bar{z})\,, \tag{6.6}$$

Recall that in d dimensions the scalar propagator for a massive scalar field is computed easily using the Schwinger parametrization, see (2.47) and below:

$$\begin{aligned}\int\frac{d^dp}{(2\pi)^d}\frac{e^{ip\cdot x}}{p^2+\mu^2} &= \int\frac{d^dp}{(2\pi)^d}e^{ip\cdot x}\int_0^\infty dt\, e^{-t(p^2+\mu^2)}\\ &= \frac{1}{(2\pi)^d}\int_0^\infty dt\, e^{-t\mu^2}\int d^dp\, e^{-tp^2+ip\cdot x}\\ &= \frac{1}{(4\pi)^{d/2}}\int_0^\infty\frac{dt}{t^{d/2}}\, e^{-t\mu^2-\frac{x^2}{4t}}\\ &= \frac{1}{(2\pi)^{d/2}}\mu^{d-2}(\mu|x|)^{1-d/2}K_{1-d/2}(\mu|x|)\,,\end{aligned} \tag{6.7}$$

[1] Notice that this is an N-point correlation function from the two-dimensional CFT perspective, which does not contradict the statement made before about correlation functions in string theory!

where K_n is a modified Bessel function of the second kind. Taking $d = 2$, and expanding the result for small μ one can use:

$$K_0(\mu|x|) = -\log\mu|x| + \mathcal{O}(1)\,. \tag{6.8}$$

Going back to complex coordinates and adding the normalization factor, the solution is naturally the two-point function already discussed below (4.13):

$$G(z', \bar{z}) = -\frac{\alpha'}{2}\log\mu^2(z'-z)(\bar{z}'-\bar{z})\,. \tag{6.9}$$

The infrared cutoff μ should disappear from correlation functions of vertex operators. This cutoff is necessary as the fields x^μ have zero-modes in the massless limit.

Fo computing the N-point function (6.4) of scalar vertex operators, one takes the source $J^\mu(z, \bar{z})$ to be

$$J^\mu(z,\bar{z}) = \sum_{\ell=1}^{N} p_\ell^\mu \delta(z - z_\ell)\delta(\bar{z} - \bar{z}_\ell) \tag{6.10}$$

and equation (6.5) gives

$$G^N(z_i, \bar{z}_i) = \exp\left(-\frac{1}{2}\int d^2z d^2z' \sum_{\ell,r} p_\ell \cdot p_r \delta^2(z - z_\ell)\delta^2(z' - z_r) G(z - z', \bar{z} - \bar{z}')\right). \tag{6.11}$$

In the sum the term involving the IR cutoff μ is

$$\exp\left(\frac{\alpha'}{2}\log\mu\sum_{\ell,r} p_\ell \cdot p_r\right)\,. \tag{6.12}$$

Remember that the physical correlation functions should be independent of the infrared cutoff; it imposes the condition

$$\sum_\ell p_\ell^\mu = 0 \tag{6.13}$$

which can be viewed as a charge conservation condition for the affine symmetry of the massless scalar CFT.

Finally, one has to remove from (6.11) the terms with $\ell = r$ that were already taken care of by the normal ordering of the vertex operators. Inserting the expression (6.9) for the Green function one obtains the expression, up to an overall normalization

$$\boxed{G_\mu^N(z_i, \bar{z}_i) = \left\langle \prod_{i=1}^{N} :e^{ip_i^\mu x^\mu(z_i,\bar{z}_i)}: \right\rangle = \delta^{26}\Big(\sum_k p_k^\mu\Big) \prod_{\ell<r} |z_\ell - z_r|^{\alpha' p_\ell\cdot p_r}} \tag{6.14}$$

6.2.2 Ghost correlation function

In the ghost sector of the two-dimensional CFT on the sphere, the S-matrx element (6.2) contains the three-point function

$$G_g^3(z_i,\bar z_i) = \left\langle \sum_{\ell=1}^{3} c(z_\ell)\tilde c(\bar z_\ell) \right\rangle , \tag{6.15}$$

which splits naturally into holomorphic and anti-holomorphic contributions, since the CFT factorizes into the holomorphic CFT for (b,c) and the anti-holomorphic CFT for $(\tilde b,\tilde c)$.

Instead of evaluating the path integral, the value of this correlation function can be determined using uniquely holomorphy arguments. The function G_g^3 should be holomorphic in each of its six variables, and should vanish when two c ghost are at coincident points, since these are anticommuting variables. One has then

$$G_g^3(z_i,\bar z_i) = (z_2-z_1)(z_3-z_1)(z_3-z_2)(\bar z_2-\bar z_1)(\bar z_3-\bar z_1)(\bar z_3-\bar z_2)F(z_1,z_2,z_3)\tilde F(\bar z_1,\bar z_2,\bar z_3) \tag{6.16}$$

with F (resp. $\tilde F$) holomorphic (resp. anti-holomorphic) in its arguments. Let us consider the limit $z_1\to\infty$. On the one-hand the expression (6.16) behaves like

$$G_g^3 \overset{|z_1|\gg 1}{\simeq} z_1^2 F(z_1,z_2,z_3)\,. \tag{6.17}$$

On the other hand, $c(z)$ is a conformal primary field of dimension $h=-1$ therefore, under the transformation $u=1/z$, one gets $c^z(z) = -z^2 c^u(1/z)$, hence the function $G_g^3(z_i,\bar z_i)$ cannot grow faster than z_1^2 when $z_1\to\infty$. It shows that the function F in (6.16) is actually independent of z_1. With a similar reasoning for the other variables one finds that, up to an overall normalization constant

$$G_g^3(z_i,\bar z_i) = \left\langle \sum_{\ell=1}^{3} c(z_\ell)\tilde c(\bar z_\ell) \right\rangle = (z_2-z_1)(z_3-z_1)(z_3-z_2)(\bar z_2-\bar z_1)(\bar z_3-\bar z_1)(\bar z_3-\bar z_2)\,. \tag{6.18}$$

6.2.3 The Virasoro-Shapiro amplitude

We now put together the matter and ghost contributions to the four-tachyon tree-level amplitude in order to get our first observable from string theory. We start with

$$\begin{aligned} S(p_1,p_2,p_3,p_4) = g_s^2\,\delta^D\Big(\sum_k p_k^\mu\Big)\ & |z_2-z_1|^{\alpha' p_1\cdot p_2+2}\,|z_3-z_1|^{\alpha' p_1\cdot p_3+2}\,|z_3-z_2|^{\alpha' p_3\cdot p_2+2} \\ & \times \int d^2z\,|z-z_1|^{\alpha' p_4\cdot p_1}\,|z-z_2|^{\alpha' p_4\cdot p_2}\,|z-z_3|^{\alpha' p_4\cdot p_3} \ , \end{aligned} \tag{6.19}$$

We set the unintegrated vertex operators to the arbitrary positions $z_1=0$, $z_2=1$ and $z_3=\infty$ as explained. The leading term in z_3 reads $|z_3|^{\alpha' p_3\cdot(p_1+p_2+p_4)+4}$ and is equal to one, using momentum conservation $p_1+p_2+p_3+p_4=0$ and the mass-shell condition $p_\ell^2 = 4/\alpha'$.

To restore the symmetry between the momenta p_ℓ, one introduces the usual Mandelstam kinetic invariants

$$s = -(p_1 + p_2)^2 \tag{6.20a}$$
$$t = -(p_1 + p_3)^2 \tag{6.20b}$$
$$u = -(p_1 + p_4)^2 \,, \tag{6.20c}$$

which are constrained by the mass-shell condition as

$$s + t + u = \sum_{\ell=1}^{4} m_\ell^2 = -\frac{16}{\alpha'} \,. \tag{6.21}$$

One ends up then with the following non-trivial integral (since, as the momenta are on shell, we have for instance $u = -p_1^2 - p_4^2 - 2p_1 \cdot p_4 = -8/\alpha' - 2p_1 \cdot p_4$):

$$I(p_\ell) = \int d^2z \; |z|^{\alpha' p_4 \cdot p_1} \, |z-1|^{\alpha' p_4 \cdot p_2} = \int d^2z \, |z|^{-\alpha' u/2-4} |z-1|^{-\alpha' t/2-4} \tag{6.22}$$

Computing this integral need a little bit of work. We use first the integral representation (similar to the Schwinger representation)

$$|z|^{2a-2} = \frac{1}{\Gamma(1-a)} \int_0^\infty df \, f^{-a} e^{-f|z|^2} \tag{6.23}$$

and compute (with $z = x + iy$)

$$\begin{aligned} I(a,b) = \int d^2z \, |z|^{2a-2} |1-z|^{2b-2} &= \frac{1}{\Gamma(1-a)\Gamma(1-b)} \int df \, f^{-a} \int dg \, g^{-b} \int d^2z \, e^{-f|z|^2 - g|1-z|^2} \\ &= \frac{2}{\Gamma(1-a)\Gamma(1-b)} \int df \, f^{-a} \int dg \, g^{-b} \int dx dy \, e^{-fx^2 - fy^2 - g(x-1)^2 - gy^2} \\ &= \frac{2\pi}{\Gamma(1-a)\Gamma(1-b)} \int df \, f^{-a} \int dg \, g^{-b} (f+g)^{-1} e^{-\frac{fg}{f+g}} \,, \end{aligned} \tag{6.24}$$

where we performed the Gaussian integrals over x and y. We consider now the change of variables $f = \alpha\beta$ and $g = (1-\beta)\alpha$, which gives (the Jacobian of the transformation being simply α):

$$I(a,b) = \frac{2\pi}{\Gamma(1-a)\Gamma(1-b)} \int_0^\infty d\alpha \int_0^1 d\beta \, \alpha^{-a-b} \beta^{-a} (1-\beta)^{-b} e^{-\alpha\beta(1-\beta)} \tag{6.25}$$

The integral over α is easy to compute as one recognizes a gamma-function:

$$\int_0^\infty d\alpha \, \alpha^{-a-b} e^{-\alpha\beta(1-\beta)} = \beta^{a+b-1} (1-\beta)^{a+b-1} \Gamma(1-a-b) \tag{6.26}$$

and one has finally

$$I(a,b) = 2\pi \frac{\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)} \int_0^1 d\beta\, \beta^{b-1}(1-\beta)^{a-1} = 2\pi \frac{\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)} B(a,b)\,. \quad (6.27)$$

Using the property of the beta-function

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,, \quad (6.28)$$

one gets finally, introducing the variable $c = 1 - a - b$,

$$I(a,b) = 2\pi \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)}\,. \quad (6.29)$$

It is now time to come back to our original problem and give the expression of the tachyon four-point function. Substituting the value of the integral in equation (6.19) one gets (with the identifications $a = -\alpha' u/4 - 1$, $b = -\alpha' t/4 - 1$, $c = -\alpha' s/4 - 1$):

$$\boxed{S(p_1,p_2,p_3,p_4) \sim g_s^2\, \delta^D\Big(\sum_k p_k^\mu\Big) \frac{\Gamma(-1-\alpha' s/4)\Gamma(-1-\alpha' t/4)\Gamma(-1-\alpha' u/4)}{\Gamma(2+\alpha' s/4)\Gamma(2+\alpha' t/4)\Gamma(2+\alpha' u/4)}} \quad (6.30)$$

This amplitude, which was known long before string theory becomes an actual theory of relativistic strings, is known as the Virasoro-Shapiro amplitude. It has some remarkable properties that we will examine briefly.

6.2.4 Properties of the string S-matrix

Because of the gamma-functions in the numerator of (6.30), the amplitude has many poles for the Mandelstam variables. First, if we keep fixed t, one finds that the amplitude has a series of poles for

$$s = \frac{4}{\alpha'}(n-1)\,, \quad n \in \mathbb{N}\,. \quad (6.31)$$

These correspond precisely to the physical states of the string spectrum: $n = 0$ for the tachyon, $n = 1$ for the massless fields (including the graviton) and $n \geqslant 2$ for the tower of massive string states. Near any of these simple poles, the amplitude behaves like $S \sim 1/(s - m_n^2)$, corresponding to the exchange of a particle of mass squared $m_n^2 = 4(n-1)/\alpha'$. It means that the string amplitude (6.30) can be interpreted in field theory as an infinite sum over tree-level Feynman diagrams given by the residues at the poles in the s-channel. Each of this diagram is interpreted as $1 + 2 \to 3 + 4$ process through an intermediate unstable particle (resonance) corresponding to each of the string physical states, see figure 6.3. In other words, all string states appear as resonances in the four-tachyon scattering amplitude.

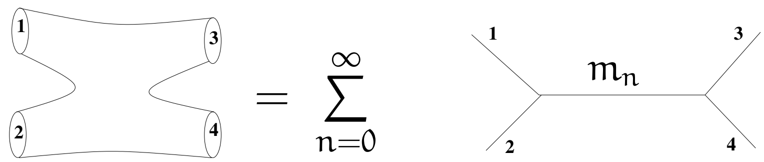

Figure 6.3: *s-channel decomposition of the four-tachyon tree-level amplitude.*

The amplitude (6.30) has another series of poles for the variable t that provides another decomposition of the same S-matrix element. They occur for

$$t = \frac{4}{\alpha'}(n-1)\ , \quad n \in \mathbb{N}\, . \tag{6.32}$$

In this case the simple pole in $S \sim 1/(t - m_n^2)$ corresponds to a Feynman diagram with two particles exchanging a virtual particle of mass m_n (emitted by tachyon 1 and absorbed by tachyon 2). It means that there is an equally valid decomposition of the amplitude obtained by summing over all the residues, see figure 6.4. Again, each of these virtual particles correspond to a state from the string physical spectrum.

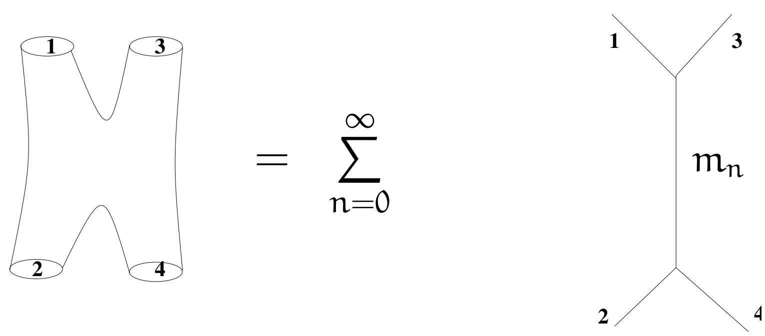

Figure 6.4: *t-channel decomposition of the four-tachyon tree-level amplitude.*

The fact that each of these expansions is equally valid is a manifestation of *channel duality* of the string amplitudes. It means that we can *either* use the s-channel decomposition or the t-channel decomposition to have a complete description of the scattering process. Of course similar statements can be made w.r.t. the u-channel decomposition of the amplitude.

Another interesting property of the amplitude (6.30) appears in the high energy limit. More precisely, we look at the regime

$$s \to \infty\ , \quad t \to \infty\ , \quad s/t \text{ fixed} \tag{6.33}$$

which corresponds to high-energy scattering with a fixed scattering angle between the incoming tachyons 1 and 2. In this limit one has

$$S(p_1, p_2, p_3, p_4) \overset{s\to\infty}{\simeq} g_s^2\, \delta^D\Big(\sum_k p_k^\mu\Big) e^{-\frac{\alpha'}{2}(s \log s + t \log t + u \log u)} \tag{6.34}$$

The exponential decay of the amplitude at high energies makes the UV behavior of string theory much softer than any quantum field theory (which gives only power-like decay), and is due to delicate cancellations between the contributions of the infinite number of string states.

6.3 One-loop partition function

String worldsheets with the topology of a torus correspond to one-loop contributions to the string perturbative expansion. We will focus here on the one-loop amplitude, which can be interpreted in Euclidean time as the statistical partition function of the theory. Crucially, the partition function needs to be invariant under the modular group $PSL(2,\mathbb{Z})$ of the two-torus. When one considers a string theory compactified to four-dimensions, one replaces the free scalar CFT associated with the extra dimension with a more general CFT, and modular invariance is a stringent consistency test of the validity of the construction.

The starting point, including the contribution from the ghosts and the integral over the modulus τ of the torus, is given by eqn. (2.132) that we reproduce for convenience here:

$$Z_1 = \int_{\mathfrak{F}} \frac{d^2\tau}{4\tau_2} \int \mathcal{D}x^\mu \, \mathcal{D}b\mathcal{D}\widetilde{b} \, \mathcal{D}c\mathcal{D}\widetilde{c} \;\; c(0)\widetilde{c}(0)\, b(0)\widetilde{b}(0)\, e^{-\int \frac{d^2w}{2\pi} \left(b\bar{\partial}c+\widetilde{b}\partial\widetilde{c}+\frac{1}{\alpha'}g_{\mu\nu}\partial x^\mu\bar{\partial}x^\nu\right)} \tag{6.35}$$

Naturally the integrand splits into a vacuum amplitude for the x^μ fields and a particular four-point function for the ghost fields. We will examine separately each of these contributions.

6.3.1 Path integral of the free-scalar CFT

We consider the path integral of a single free, massless scalar field $X(\sigma^1,\sigma^2)$ on a two-dimensional torus. For this derivation it is convenient to work with a metric

$$ds^2 = \frac{1}{\tau_2}\left|d\sigma^1 + \tau d\sigma^2\right|^2 \tag{6.36}$$

together with the standard periodicities $\sigma^i \sim \sigma^i + 2\pi$ (rather than the canonical complex flat metric $dzd\bar{z}$ with periodicities 2π and $2\pi\tau$). the two-dimensional Euclidean action for the scalar field is

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\det\gamma}\gamma^{ij}\partial_i X\partial_j X = \frac{1}{4\pi\alpha'\tau_2}\int d^2\sigma\,|\tau\partial_1 X - \partial_2 X|^2 \,. \tag{6.37}$$

After integration by parts it becomes

$$S = \frac{1}{4\pi}\int d^2\sigma\, X(\sigma^i)\,\Box\, X(\sigma_i)\,, \tag{6.38}$$

with the Laplacian

$$\Box = -\frac{1}{\alpha'\tau_2}|\tau\partial_1 - \partial_2|^2 \,. \tag{6.39}$$

To compute the path integral, one has first to split the field X into the classical solutions of the equations of motion and the fluctuations. The classical solutions, which are the saddle points in the path integral, are just the constant modes. We splits then

$$X(\sigma^1,\sigma^2) = X_c + \phi(\sigma^1,\sigma^2)\,, \tag{6.40}$$

and expand the fluctuations $\phi(\sigma^1,\sigma^2)$ in eigenmodes of the Laplacian:

$$\Box\,\varphi_\ell = \lambda_\ell \varphi_\ell\,. \tag{6.41}$$

The solution to this eigenvalue problem is

$$\varphi_{m,n} = \frac{1}{\sqrt{\pi}}\, e^{i(m\sigma^1+n\sigma^2)}\ , \quad \lambda_{m,n} = \frac{1}{\alpha'\tau_2}|m\tau - n|^2\ , \quad m,n\in\mathbb{Z}\,. \tag{6.42}$$

These eigenfunctions form an orthonormal basis, as

$$\frac{1}{4\pi}\int d^2\sigma\ \varphi^\star_{m,n}\varphi_{m',n'} = \delta_{m,m'}\delta_{n,n'}\,. \tag{6.43}$$

Let us expand ϕ on this basis, omitting the zero-mode that was already taken apart in (6.40):

$$\phi = \sum_{(m,n)\neq(0,0)} \alpha_{m,n}\varphi_{m,n}\ , \quad (\alpha_{m,n})^\star = \alpha_{-m,-n} \tag{6.44}$$

The action for the field X becomes then

$$S[X] = \sum_{(m,n)\neq(0,0)} \lambda_{m,n}\,|\alpha_{m,n}|^2\ . \tag{6.45}$$

Since the eigen-modes were properly normalized, the path integral can be written as

$$Z = \int \mathcal{D}X\, e^{-S[X]} = \int dX_c \prod_{(m,n)\neq(0,0)} \frac{d\alpha_{m,n}}{2\pi} e^{-\sum_{(m,n)\neq(0,0)}\lambda_{m,n}|\alpha_{m,n}|^2} = V_X\Big(\prod_{(m,n)\neq(0,0)}\lambda_{m,n}\Big)^{-1/2}\,. \tag{6.46}$$

In this expression, V_X denotes the volume of the target space associated with the field X, which is here an infinite line. It gives an infrared divergence, associated in the whole string theory computation to the volume of space-time (one can divide by this factor and consider the vacuum amplitude per unit volume instead).

Zeta-function regularized determinant

The infinite product of the eigenvalues of the Laplacian appearing in (6.46) is divergent and evaluated using zeta-function regularization, as was done for the point-particle one-loop vacuum amplitude in subsection 2.1. We recall that we define first the spectral zeta-function as (with $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \lambda_n \leqslant \cdots$)

$$\zeta_D(z) = \sum_{n=1}^{\infty} \lambda_n^{-z}\,, \tag{6.47}$$

and, noticing that

$$\log\det D = \sum_{n=1}^{\infty} \log\lambda_n = -\zeta_D'(0)\,, \tag{6.48}$$

we define the regularized functional determinant as

$$\prod_n \lambda_n = e^{-\zeta_D'(0)}\,. \tag{6.49}$$

Zeta-function regularization gives us in particular the identities (by writing each time ζ_D in terms of the usual Riemann zeta-function, $\zeta(z) = \sum_n n^{-z}$):

$$\prod_{n=1}^{\infty} a = e^{-\frac{d}{dz}(a^{-z}\zeta(0))|_{z=0}} = a^{\zeta(0)} = a^{-1/2} \tag{6.50a}$$

$$\prod_{n=-\infty}^{\infty} a = a^{2\zeta(0)+1} = 1 \tag{6.50b}$$

$$\prod_{n=1}^{\infty} n^{\alpha} = e^{-\frac{d}{dz}\zeta(\alpha z)|_{z=0}} = e^{-\alpha\zeta'(0)} = (2\pi)^{\alpha/2} \tag{6.50c}$$

$$\prod_{n\in\mathbb{Z}}(a+n) = a\prod_{n=1}^{\infty}(a+n)(a-n) = a\prod_{n=1}^{\infty}(-n^2)(1-a^2/n^2) = 2\pi i a\frac{\sin\pi a}{\pi a} = 2i\sin\pi a\,. \tag{6.50d}$$

using $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\log 2\pi$. Another useful identity is

$$\prod_{n=1}^{\infty} a^{-n} = e^{-\log a\,\zeta(-1)} = a^{1/12}\,, \tag{6.51}$$

where we used $\zeta_D(z) = \sum_n a^{nz}$, $\zeta_D'(0) = \zeta(-1)\log a$ and $\zeta(-1) = -1/12$.

In the present case one needs to perform some manipulation on the determinant of the Laplace operator on the two-torus. Let us denote by $\det{}'\Box$ the determinant of the Laplace operator $\Box$ with the zero-modes omitted. One has

$$\begin{aligned}
\det{}'\Box &= \prod_{(m,n)\neq(0,0)} \frac{1}{\alpha'\tau_2}(n-m\tau)(n-m\bar\tau) \\
&= \alpha'\tau_2 \prod_{n\neq 0} n^2 \prod_{m\neq 0, n\in\mathbb{Z}} (n-m\tau)(n-m\bar\tau) \\
&= \alpha'\tau_2 (2\pi)^2 \prod_{m>0, n\in\mathbb{Z}} (n-m\tau)(n+m\tau)(n-m\bar\tau)(n+m\bar\tau) \\
&= 4\pi^2\alpha'\tau_2 \prod_{m>0} \left(e^{-i\pi m\tau} - e^{i\pi m\tau}\right)^2 \left(e^{-i\pi m\bar\tau} - e^{i\pi m\bar\tau}\right)^2
\end{aligned} \tag{6.52}$$

We introduce now the complex variable

$$q = e^{2i\pi\tau} \tag{6.53}$$

and obtain finally

$$\begin{aligned}\det{}'\square &= 4\pi^2\alpha'\tau_2 \prod_{m>0} (q\bar{q})^{-m} (1-q^m)^2 (1-\bar{q}^m)^2 \\ &= 4\pi^2\alpha'\tau_2 (q\bar{q})^{1/12} \prod_{m>0} (1-q^m)^2 (1-\bar{q}^m)^2 \end{aligned} \tag{6.54}$$

The result can be expressed in terms of the *Dedekind eta function*

$$\eta(\tau) = q^{1/24} \prod_{n=0}^{\infty} (1-q^n) \tag{6.55}$$

such that the regularized determinant is given by

$$\det{}'\square = 4\pi^2\alpha'\tau_2\eta^2(\tau)\bar{\eta}^2(\bar{\tau})\,. \tag{6.56}$$

Putting everything together, the path integral for the free boson on the two-torus becomes, at the end of this tedious computation

$$\boxed{Z_x(\tau,\bar{\tau}) = \frac{V_x}{(4\pi^2\alpha')^{1/2}} \frac{1}{\sqrt{\tau_2}} \frac{1}{\eta(\tau)\bar{\eta}(\bar{\tau})}}\,. \tag{6.57}$$

The Dedekind eta function is widely used by mathematicians in number theory and has some wonderful properties. In particular, it transforms in a nice way under the modular group $\mathsf{PSL}(2,\mathbb{Z})$. For the two generators one has

$$\eta(\tau+1) = e^{i\pi/12}\eta(\tau)\ , \quad \eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau) \tag{6.58}$$

from which we deduce that

$$Z_x(\tau+1,\bar{\tau}+1) = \frac{V_x}{(4\pi^2\alpha')^{1/2}} \frac{1}{\sqrt{\tau_2}} \frac{1}{\eta(\tau+1)\overline{\eta(\tau+1)}} = Z_x(\tau,\bar{\tau}) \tag{6.59a}$$

$$Z_x(-1/\tau,-1/\bar{\tau}) = \frac{V_x}{(4\pi^2\alpha')^{1/2}} \frac{1}{\sqrt{\Im(-1/\tau)}} \frac{1}{\eta(-1/\tau)\overline{\eta(-1/\tau)}} = Z_x(\tau,\bar{\tau})\,, \tag{6.59b}$$

hence the partition function is invariant under the action of the full modular group as it should.

Hamiltonian perspective

The partition function is somewhat easier to compute from a Hamiltonian perspective. Recall that the Hamiltonian of a CFT is given by

$$H = L_0 + \widetilde{L}_0 - \frac{c + \bar{c}}{24} \tag{6.60}$$

while the rotations around the cylinder are generated by

$$R = L_0 - \widetilde{L}_0 \,. \tag{6.61}$$

The coordinate σ_2 of the torus has the interpretation of an Euclidean (worldsheet) time, and because of the periodicity $\sigma_2 \sim \sigma_2 + 2\pi$ one considers actually a two-dimensional QFT at finite inverse temperature $\beta = 2\pi\tau_2$. For a generic torus, the real part τ_1 of the modular parameter is non-vanishing an correspond to a "twist" of the torus along the space-like compact coordinate σ^1 before gluing both ends of the cylinder along σ^2. Since the eigenvalue of $R = L_0 - \bar{L}_0$ for a given state corresponds to its spin, it will pick a corresponding phase in the partition function. In other words, one can view $\chi = 2\pi\tau_1$ as a "chemical potential" for the generator of rotations around the worldsheet.

From these considerations one can reinterpret the path integral over the field $x(\sigma^i)$ on a two-dimensional toroidal surface of modulus τ as the statistical partition function:

$$Z_x = \mathrm{Tr}\,\left(e^{-\beta H} e^{i\chi R}\right) = \mathrm{Tr}\,\left(e^{-2\pi\tau_2\left(L_0 + \widetilde{L}_0 - \frac{c+\bar{c}}{24}\right)} e^{2i\pi\tau_1 (L_0 - \widetilde{L_0})}\right)\,, \tag{6.62}$$

leading to the important result:

$$Z_x = \mathrm{Tr}\,\left(q^{L_0 - \frac{c}{24}} \bar{q}^{\widetilde{L}_0 - \frac{\bar{c}}{24}}\right)\,. \tag{6.63}$$

In the present case, $c = \bar{c} = 1$, but this expression holds for any conformal field theory on a two-torus.

This expression can be easily evaluated by enumerating all the states of the Hilbert space of the CFT. Recall that the spectrum of the theory is

$$h = \frac{\alpha' p^2}{4} + N\,, \quad N = \sum_{r=1}^{\infty} r N_r\,, \tag{6.64a}$$

$$\widetilde{h} = \frac{\alpha' p^2}{4} + \widetilde{N}\,, \quad \widetilde{N} = \sum_{r=1}^{\infty} r \widetilde{N}_r\,, \tag{6.64b}$$

corresponding to states of the form

$$|\Psi\rangle = \cdots (\alpha_{-2})^{N_2} (\widetilde{\alpha}_{-2})^{\widetilde{N}_2} (\alpha_{-1})^{N_1} (\widetilde{\alpha}_{-1})^{\widetilde{N}_1} |p\rangle\,. \tag{6.65}$$

In the partition function (6.63) one can first perform the Gaussian integral over the momentum p. It gives simply

$$\frac{V_x}{2\pi}\int dp\, (q\bar{q})^{\alpha' p^2/4} = \frac{V_x}{2\pi}\int dp\, e^{-\pi\tau_2\alpha' p^2} = \frac{V_x}{2\pi}\frac{1}{\sqrt{\alpha'\tau_2}}\,, \tag{6.66}$$

where we have included the usual density of momentum modes in a volume V_x.

Next one can consider the contributions from the action of the first level left-moving oscillator, *i.e.* of $(\alpha_{-1})^{N_1}$ for all $N_1 \in \mathbb{N}$, on the highest weight states $|p\rangle$. Since α_{-1} shifts the eigenvalue of L_0 by one unit, these terms contribute to the partition function as

$$1 + q + q^2 + \cdots = \frac{1}{1-q}\,, \tag{6.67}$$

where the second term comes from the contribution of $\alpha_{-1}|p\rangle$ to the trace, and so on. In the same way one computes the contribution from the action of $(\alpha_{-2})^{N_2}$ for all $N_2 \in \mathbb{N}$. Since α_{-2} shifts the eigenvalue of L_0 by two units these terms give

$$1 + q^2 + q^4 + \cdots = \frac{1}{1-q^2}\,. \tag{6.68}$$

One can iterate this reasoning to the other oscillators, and add the contribution from the right-moving modes, leading finally to the partition function

$$\begin{align} Z_x(\tau,\bar{\tau}) &= \frac{V_x}{2\pi}\frac{1}{\sqrt{\alpha'\tau_2}}(q\bar{q})^{-1/24}\prod_{n=1}^{\infty}\frac{1}{1-q^n}\prod_{n'=1}^{\infty}\frac{1}{1-\bar{q}^{n'}} \tag{6.69}\\ &= \frac{V_x}{\sqrt{4\pi^2\alpha'\tau_2}}\frac{1}{\eta\bar{\eta}}\,. \tag{6.70}\end{align}$$

This is precisely the same as the path-integral result (6.57).

Partition function for the x^μ fields of string theory

From the partition function of a single free massless scalar field (6.57) it is easy to get the result for the fields $\{x^\mu, \mu = 0, \ldots, 25\}$ of bosonic string theory.

One should be careful however about the field x^0 which has a wrong-sign kinetic term. One way to deal with this issue formally is to replace $\alpha' \mapsto \alpha'\epsilon$ in the computation of the path integral for this field, and take the analytic continuation $\epsilon \to -1$ at the end of the computation. In this way one gets

$$Z_{x^\mu}(\tau,\bar{\tau}) = \frac{iV_{26}}{(4\pi^2\alpha'\tau_2)^{13}}\frac{1}{\eta^{26}(\tau)\bar{\eta}^{26}(\bar{\tau})}\,. \tag{6.71}$$

6.3.2 Path integral of the ghost CFT

We consider now the path integral over the ghost fields

$$Z_g = \int \mathcal{D}b\mathcal{D}\widetilde{b}\mathcal{D}c\mathcal{D}\widetilde{c}e^{-S_g}b(0)\widetilde{b}(0)c(0)\widetilde{c}(0)\,. \tag{6.72}$$

It is more convenient to do directly the computation in the Hamiltonian formalism, which had been successful in giving the result for the x^μ fields rather quickly.

Fermionic path integrals

We need to recall a property of path integrals over fermonic fields. Let us consider a simple two-state system, the zero-mode part of the (b, c) ghost system, with

$$\begin{aligned} \{b_0, c_0\} &= 1 \\ b_0|-\rangle = 0\ ,&\quad c_0|-\rangle = |+\rangle \\ c_0|+\rangle = 0\ ,&\quad b_0|+\rangle = |-\rangle\,. \end{aligned} \tag{6.73}$$

The eigenstates of the lowering operator b_0 can be expanded as

$$|\psi\rangle = |-\rangle + |+\rangle\psi\ ,\quad b_0|\psi\rangle = |-\rangle\psi = \big(|-\rangle + \psi|+\rangle\big)\psi = |\psi\rangle\psi\,, \tag{6.74}$$

using that ψ is a Grassmann variable. We define the conjugate states to satisfy the orthogonality condition

$$\langle\psi|\psi'\rangle = \psi - \psi'\,. \tag{6.75}$$

The right-hand side is the Grassmann Dirac distribution since, for any test function f,

$$\int d\psi(\psi - \psi')f(\psi) = f(\psi') \tag{6.76}$$

as one can see expanding f as $f(\psi) = f_0 + f_1\psi$ and using

$$\int d\psi\,\psi'\psi = -\Big(\int d\psi\,\psi\Big)\psi' = -\psi'\,. \tag{6.77}$$

Expanding $|\psi'\rangle = |-\rangle + |+\rangle\psi'$ one finds from (6.75) the components

$$\langle\psi|-\rangle = \psi\ ,\quad \langle\psi|+\rangle = -1\,. \tag{6.78}$$

Using

$$\langle\psi|\big(b_0|\psi'\rangle\big) = \langle\psi|\psi'\rangle\psi' = \psi\psi' = \big(\langle\psi|b_0\big)|\psi'\rangle\,, \tag{6.79}$$

One should have that

$$\langle\psi|b_0 = -\psi\langle\psi| \tag{6.80}$$

Let us consider now an arbitrary operator A, whose components are

$$A|\pm\rangle = |-\rangle A_{-\pm} + |+\rangle A_{+\pm}\,. \tag{6.81}$$

We compute then the expectation value

$$\begin{aligned} \int d\psi\,\langle\psi|A|\psi\rangle &= \int d\psi\,\langle\psi|\Big(|-\rangle A_{--} + |+\rangle A_{+-} + |-\rangle A_{-+}\psi + |+\rangle A_{++}\psi\Big) \\ &= \int d\psi\,\Big\{\psi\big(A_{--} + A_{-+}\psi\big) - \big(A_{+-} + A_{++}\psi\big)\Big\} \\ &= A_{--} - A_{++}\,. \end{aligned} \tag{6.82}$$

To interpret this result, let us define the fermion number parity operator $(-)^F$, where F counts the number of fermionic excitations mod two. We have

$$(-)^F|\pm\rangle = \mp|\pm\rangle\ , \quad \left\{(-)^F, b_0\right\} = \left\{(-)^F, c_0\right\} = 0 \tag{6.83}$$

Then one can interpret (6.82) in terms of a trace

$$\int d\psi\, \langle\psi|A|\psi\rangle = \text{Tr}\ \left((-)^F A\right) = A_{--} - A_{++}\,. \tag{6.84}$$

It implies that, for the path integral, if one identifies periodically the field $\psi(t)$ along the Euclidean time circle, one does not get the partition function of the theory. Using eqn. (6.84) for the evolution operator in Euclidean time, one gets instead:

$$\int d\psi\, \langle\psi, T|\psi, 0\rangle = \text{Tr}\ \left((-)^F e^{-TH}\right)\ . \tag{6.85}$$

The partition function would correspond instead to a path integral with *antiperiodic* boundary conditions along Euclidean time. More details can be found in appendix A of [1].

In the present case, the ghost path integral is necessarily over periodic field configurations, as they were obtained by changing the grassmanity of the bosonic variables appearing in the Faddeev-Popov determinant.

Trace over the ghost Hilbert space

The CFT for the ghosts factorizes into holomorphic and anti-holomorphic parts. Let us start with the former contribution. Expanding the insertions of $b(0)$ and $c(0)$ in (6.72) into modes, one has to compute terms of the form

$$\text{Tr}\ \left(q^{L_0-\frac{c}{24}}(-)^F b_n c_m\right) \tag{6.86}$$

with $c = -26$. The space of states is constructed from the two vacua $|-\rangle$ and $c_0|-\rangle = |+\rangle$, by acting with the fermionic creation operators b_{-n} and c_{-n}.

Since $(-)^F|-\rangle = |-\rangle$ and $(-)^F|+\rangle = -|+\rangle$, whenever $(n, m) \neq (0, 0)$ the trace (6.86) vanishes, as the terms from the states constructed out of the vacuum $|-\rangle$ have opposite sign as similar terms constructed from $|+\rangle$.

The exception to the rule is for $m = n = 0$, since we have then $b_0c_0|-\rangle = |-\rangle$, $b_0c_0|+\rangle = 0$, which gives a projector onto the ground state $|-\rangle$. We have finally

$$\text{Tr}\ \left(q^{L_0+\frac{13}{12}}(-)^F b_0c_0\right) = \sum_{N_r,M_r} \langle-|\Big((b_{-1}^\dagger)^{M_1}(c_{-1}^\dagger)^{N_1}\cdots\Big)q^{L_0+\frac{13}{12}}(-)^F\Big(\cdots(c_{-1})^{N_1}(b_{-1})^{M_1}\Big)|-\rangle \tag{6.87}$$

where (remembering that the ghost vacuum as conformal dimension minus one):

$$L_0\Big(\cdots(c_{-1})^{N_1}(b_{-1})^{M_1}\Big)|-\rangle = \left(\sum_r r(N_r + M_r) - 1\right)\Big(\cdots(c_{-1})^{N_1}(b_{-1})^{M_1}\Big)|-\rangle\,. \tag{6.88}$$

Because of the $(-)^F$ insertion, which anti-commutes with all oscillators b_{-n} and c_{-n}, the contribution to the trace (6.87) of a given mode b_{-r} contributes to the trace (6.87) as $1-q^r$, consistently with the Fermi-Dirac statistics. Adding the contributions from all b_{-n} and c_{-n}, we obtain then

$$\mathrm{Tr}\,\left(q^{L_0+\frac{13}{12}}(-)^F b_0 c_0\right) = q^{\frac{13}{12}-1}\left(\prod_{r=1}^{\infty}(1-q^r)\right)^2 = q^{\frac{1}{12}}\left(q^{-\frac{1}{24}}\eta(\tau)\right)^2 . \tag{6.89}$$

In summary, the trace over the ghost Hilbert space give the following results

$$\mathrm{Tr}\,\left(q^{L_0-\frac{c}{24}}(-)^F\right) = 0\,, \tag{6.90a}$$

$$\mathrm{Tr}\,\left(q^{L_0-\frac{c}{24}}(-)^F b_n c_m\right) = 0\,, \quad \forall (m,n)\neq(0,0)\,, \tag{6.90b}$$

$$\mathrm{Tr}\,\left(q^{L_0-\frac{c}{24}}(-)^F b_0 c_0\right) = \eta^2(\tau)\,. \tag{6.90c}$$

In the path integral perspective, the trace (6.90a) vanishes as one does not saturate the zero-modes present in the integration measure. The vanishing of (6.90b) justifies that, as was claimed above eqn. (2.132), only the ghost zero-modes contribute to the insertions appearing in the Polyakov path integral on the sphere.

Putting together the left- and right-moving contributions from the holomorphic and anti-holomorphic ghosts, we have obtained finally that the ghost past integral (6.72) on the two-torus gives

$$Z_g = \int \mathcal{D}b\mathcal{D}\widetilde{b}\mathcal{D}c\mathcal{D}\widetilde{c}e^{-S_g}c(0)\widetilde{c}(0)b(0)\widetilde{b}(0) = \eta^2\bar{\eta}^2\,. \tag{6.91}$$

6.3.3 One-loop vacuum amplitude in bosonic string theory

The one-loop amplitude is given by integrating over the modulus of the two-torus the product of the contributions from the matter part and the ghost part. Putting everything together, one gets:

$$Z_1 = \int_{\mathfrak{F}} \frac{d^2\tau}{4\tau_2}\frac{iV_{26}}{(4\pi^2\alpha'\tau_2)^{13}}\frac{1}{\eta^{26}(\tau)\bar{\eta}^{26}(\bar{\tau})}\eta^2(\tau)\bar{\eta}^2(\bar{\tau}) \tag{6.92}$$

One sees that the ghost contribution cancels the oscillators from two of the scalar fields x^μ. One can interpret this result by considering that the contributions from the light-cone coordinates $x^\pm(\sigma^i) = x^0(\sigma^i) \pm x^1(\sigma^i)$ are removed by the gauge symmetry, leaving only the degrees of freedom from the transverse fields $\{x^i(\sigma),\ i = 2,\cdots 25\}$.

The final result for the one-loop amplitude of bosonic string theory is:

$$\boxed{Z_1 = \frac{iV_{26}}{(4\pi^2\alpha')^{13}}\int_{\mathfrak{F}}\frac{d^2\tau}{4\tau_2^2}\frac{1}{\left(\sqrt{\tau_2}\eta(\tau)\bar{\eta}(\bar{\tau})\right)^{24}}} \tag{6.93}$$

An important consistency check of this result is invariance under the modular group. One observes that $\sqrt{\tau_2}\eta(\tau)\bar{\eta}(\bar{\tau})$ is modular invariant by itself, as follows from equation (6.58). It is easy to check as well that $d^2\tau/\tau_2^2$ is a modular-invariant measure over the moduli space of the two-torus.

6.3.4 Interpretation of the one-loop vacuum amplitude

The content of the one-loop amplitude of string theory can be understood easily if one compares the result (6.93) with the analogue massive point-particle expression given by eqn. (2.44). Generalizing the latter to a theory having several (non-interacting) massive bosonic particle species, and taking the dimension of space-time to be 26, the QFT parition function we take as a reference point is

$$Z_1 = \int_0^\infty \frac{dT}{T^{14}} \sum_i e^{-\frac{T}{2}m_i^2} . \tag{6.94}$$

The string partition function (6.93) is integrated over the fundamental domain $\mathfrak{F}$ of the modular group, which excludes the UV singularity at $|\tau| \to 0$. In order to compare to the QFT result, which does include the singularity, we will trade the integration over $\mathfrak{F}$ – which is hard to perform anyway – we will consider the integration over a simpler domain, the half-strip

$$\mathfrak{S} = \{\tau \in \mathbb{H} ,\ |\Re(\tau)| \leqslant \tfrac{1}{2}\} . \tag{6.95}$$

We have observed that the contributions of the ghosts modes cancelled the contribution of the oscillators of two of the coordinate fields x^μ in (6.93), leaving only the transverse oscillators. Let us then introduce the transverse levels $N^\perp$ and $\widetilde{N}^\perp$ corresponding to the action of the transverse modes $\{\alpha^i_{-n}, \widetilde{\alpha}^i_{-n}, i = 2, \ldots, 25\}$ on the tachyon vacuum $|p^\mu\rangle$ and write $D(N^\perp, \widetilde{N}^\perp)$ the degeneracy of states with given transverse levels.

One can simplify then the "fake" string partition function integrated over the modified domain $\mathfrak{S}$ as follows:

$$\begin{aligned}
\hat{Z}_1 &= \frac{iV_{26}}{(4\pi^2\alpha')^{13}} \int_{\mathfrak{S}} \frac{d\tau_1 d\tau_2}{4\tau_2^2} \frac{1}{\left(\sqrt{\tau_2}\eta(\tau)\bar{\eta}(\bar{\tau})\right)^{24}} \\
&= \frac{iV_{26}}{(4\pi^2\alpha')^{13}} \int_{\mathfrak{S}} \frac{d\tau_1 d\tau_2}{4\tau_2^{14}} \sum_{N^\perp, \widetilde{N}^\perp} D(N^\perp, \widetilde{N}^\perp) q^{N^\perp - 1} \bar{q}^{\widetilde{N}^\perp - 1} \\
&= \frac{iV_{26}}{(4\pi^2\alpha')^{13}} \sum_{N^\perp, \widetilde{N}^\perp} D(N^\perp, \widetilde{N}^\perp) \left(\int_{-1/2}^{1/2} d\tau_1 e^{2i\pi\tau_1(N^\perp - \widetilde{N}^\perp)} \right) \int_0^\infty \frac{d\tau_2}{4\tau_2^{14}} e^{-2\pi\tau_2(N^\perp + \widetilde{N}^\perp - 2)} .
\end{aligned} \tag{6.96}$$

The integral over τ_1 enforces the level-matching condition $N^\perp = \widetilde{N}^\perp$. Next we recall that the mass spectrum of the bosonic string is given by

$$m^2_{N^\perp} = \frac{4}{\alpha'}(N^\perp - 1) , \tag{6.97}$$

and we arrive finally to the expression

$$\hat{Z}_1 = \frac{iV_{26}}{(4\pi^2\alpha')^{13}} \sum_{N^\perp} D(N^\perp) \int_0^\infty \frac{d\tau_2}{4\tau_2^{14}} e^{-\pi\tau_2\alpha' m^2_{N^\perp}} \tag{6.98}$$

which is precisely of the same type as the QFT expression (6.94).

One concludes that, away from the dangerous UV region $\tau \to 0$, the string theory partition function behaves exactly as the partition function for a QFT with an infinite tower of massive particles. Unlike the QFT expression though, the actual partition function (6.93) has no ultraviolet singularities as the integration is restricted to the fundamental domain $\mathfrak{F}$, which avoids the region $|\tau| < 1$.

In the case of the bosonic string, one has to worry however about the contribution of the tachyon which runs into the loop. The dangerous region in this case is the IR limit $\tau_2 \to +\infty$, which is dominated by the lightest states circulating into the loop. One uses the expansion

$$\eta(\tau) \overset{\tau_2 \to \infty}{\sim} q^{1/24}(1 - q + \mathcal{O}(q^2)) \tag{6.99}$$

to see that the integrand of the partition function behaves like

$$Z_1 \sim \frac{iV_{26}}{(4\pi^2\alpha')^{13}} \int^{\infty} \frac{d\tau_2}{4\tau_2^{14}} \left(e^{4\pi\tau_2} + 24^2 + \mathcal{O}(e^{-4\pi\tau_2})\right) \tag{6.100}$$

The dominant term is coming from the tachyon and diverges, while the second one, which is convergent, corresponds to the contribution of the massless states. Given that the tachyon has a negative mass squared, this divergence is easy to understand from the QFT perspective. Fortunately in the superstring theories, that we are about to consider, this problem is absent as there are no tachyons in the physical spectrum.

References

[1] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string.* Cambridge University Press, 2007.

Chapter 7

Introduction to superstring theories

Bosonic string theory is certainly a beautiful theory, providing the first known UV-finite quantum theory of gravity, with a well-defined classical limit. It has however two major drawbacks. The first is that it does not incorporates excitations corresponding to space-time fermions, which are necessary to describe the real world and, more annoyingly, its is an unstable theory as the lowest lying state corresponds to a space-time tachyon.

These two problems can be fortunately overcome, by adding an extra ingredient to the construction. The presence of the tachyon, at the technical level, was due to the shift of the ghost Hamiltonian by minus one. This normal ordering constant is similar in spirit to the normal ordering constant of an harmonic oscillator. As in the latter case, if can be offset if one adds degrees of freedom with a similar Hamiltonian but opposite statistics.

Since the (b, c) ghosts of bosonic string theory were coming from the gauge-fixing of a local symmetry of the string theory action, adding to them "partners" with opposite statistics – *i.e.* Bose-Einstein statics – has some profound implication since they should be associated as well to the gauge-fixing of a local symmetry, this time of fermionic nature. As we shall see below, superstring theories arise by considering a string action with *local supersymmetry* in two dimensions.

In the same way that bosonic string theory, which is a theory of two-dimensional gravity coupled to matter (the coordinates fields), describes quantum gravity in space-time, the superstring theories that are two-dimensional supergravity theory coupled to matter, will turn out to describe quantum supergravity in space-time.

7.1 Two-dimensional local supersymmetry

The Polyakov action (2.63) describes a set of two-dimensional scalar fields coupled to two-dimensional gravity. After gauge-fixing, one obtains a ghost system (b_{ij}, c^i) of bosonic statistics. In order to get a theory with *local* fermionic symmetry, one introduces two-dimensional local supersymmetry.

Since we will deal with fermions in curved two-dimensional space-time, we introduce a two-dimensional local frame, or zweibein e^i_a, with, as usual

$$\eta^{ab} e^i_a e^j_b = \gamma^{ij} \ , \quad \gamma_{ij} e^i_a e^j_b = \eta_{ab} \, , \tag{7.1}$$

and the inverse zweibein e^a_i. In the following $(a, b, \dots)$ corresponds to orthonormal frame indices, for tensors transforming under local Lorentz transformations, and $(i, j, \dots)$ to coordinate indices, for tensors transforming under diffeomorphisms. To the usual gamma matrices Γ^a with Lorentz indices, which satisfy the algebra

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab} \, , \tag{7.2}$$

we can define gamma matrices with coordinate indices

$$\Gamma^i = e^i_a \Gamma^a \ , \quad \{\Gamma^i, \Gamma^j\} = 2\gamma^{ij} \, . \tag{7.3}$$

The field content of the Polyakov action is a two dimensional metric γ, or equivalently a two-dimensional zweibein e^i_a, together with a set of scalar fields $x^\mu(\sigma^i)$, $\mu = 0, \dots, D-1$, which transform as vectors under the target-space Lorentz group. In the supersymmetric theory, one adds fermionic partners to these bosonic fields, namely:

- to the bosonic scalars $x^\mu(\sigma^i)$ one associates two-dimensional Majorana fermions $\psi^\mu(\sigma^i)$ (in terms of the space-time Lorentz group, they transform as vectors)
- to the zweibein e^i_a one associates a gravitino $\zeta_a(z,\bar{z})$, which is, in two-dimensional terms, a Majorana fermion with an extra vector index.

The supersymmetric generalization of the Polyakov action, in a Minkowski *target* space-time $\eta_{\mu\nu}$ with no background Kalb-Ramond field, is given by the following action:

$$\mathcal{S} = -\frac{1}{8\pi}\int d^2\sigma\sqrt{-\det\gamma}\Big(\tfrac{2}{\alpha'}\gamma^{ij}\partial_i x^\mu\partial_j x_\mu + 2\bar{\psi}^\mu\Gamma^i\partial_i\psi_\mu\eta_{\mu\nu} - \bar{\zeta}_i\Gamma^j\Gamma^i\psi^\mu\big(\sqrt{\tfrac{2}{\alpha'}}\partial_j x_\mu - \frac{1}{4}\bar{\zeta}_j\psi_\mu\big)\Big)\,. \tag{7.4}$$

This can be generalized to arbitary background fields $(G_{\mu\nu}, B_{\mu\nu})$ but leads to quite complicated couplings. Note that there is no kinetic term for the gravitino, since the latter would need to be of the form $\bar{\zeta}_a\Gamma^{abc}\partial_b\zeta_c$ and Γ^{abc} vanishes in two-dimensions; this mirrors the fact that the kinetic term for the two-dimensional metric, or Einstein term, is topological in two dimensions.

7.1.1 Gauge symmetries

The action (7.4), being written in a completely covariant way with respect to the two-dimensional metric γ, is invariant under **diffeomorphisms** by construction.

One can check that it is invariant as well under **local supersymmetric transformations**, of Grassmann parameter $\kappa(\sigma^i)$, which is a two-dimensional Majorana fermion:

$$\delta_\kappa e^a_i = \tfrac{1}{2}\bar{\kappa}\gamma^a\zeta_i\,, \tag{7.5a}$$
$$\delta_\kappa\zeta_i = 2D_i\kappa\,, \tag{7.5b}$$
$$\delta_\kappa x^\mu = \sqrt{\tfrac{\alpha'}{2}}\bar{\kappa}\psi^\mu\,, \tag{7.5c}$$
$$\delta_\kappa\psi^\mu = \tfrac{1}{2}\Gamma^i\left(\sqrt{\tfrac{2}{\alpha'}}\,\partial_i x^\mu - \tfrac{1}{2}\bar{\zeta}_i\psi^\mu\right)\kappa \tag{7.5d}$$

The gravitino variation involves a covariant derivative that we give for completeness:

$$D_i\kappa = \partial_i\kappa - \tfrac{1}{2}\left(\tfrac{-1}{\sqrt{-\det\gamma}}e_{i\,a}\Gamma^{jk}\partial_j e^a_k + \tfrac{1}{4}\bar{\zeta}_i\Gamma^2\Gamma^j\zeta_j\right)\Gamma^2\kappa\,. \tag{7.6}$$

The term involving the zweibein gives the usual spin connection of fermions coupled to a two-dimensional metric and the second term is a torsion that arises in a gravitino background.

In the bosonic string, an extra local symmetry, the **Weyl symmetry**, played a crucial role. It is a symmetry of the super-Polyakov action as well, with transformations

$$\delta_\lambda x^\mu = 0 \tag{7.7a}$$

$$\delta_\lambda \psi^\mu = -\frac{1}{2}\lambda\psi^\mu \tag{7.7b}$$

$$\delta_\lambda e_i^a = \lambda e_i^a \tag{7.7c}$$

$$\delta_\lambda \zeta_i = \tfrac{1}{2}\lambda\zeta_i \tag{7.7d}$$

which is consistent in particular with the scaling dimension of the fermionic fields ψ^μ.

The action has an extra symmetry, of Grassmann parameter η, which can be seen as a fermionic partner of the Weyl symmetry. The **super-Weyl symmetry** acts only on the gravitino, as:

$$\delta_\eta \zeta_i = \Gamma_i \eta\,. \tag{7.8}$$

using the gamma-matrix identity in two dimensions

$$\Gamma_a\Gamma^b\Gamma^a = \Gamma_a(2\eta^{ab} - \Gamma^a\Gamma^b) = 0\,. \tag{7.9}$$

7.1.2 Gauge fixing

As in the case of the bosonic string, one can take advantage of this gauge freedom to simplify drastically the action. We will start by considering the gauge transformations that cannot be anomalous in this setting, local supersymmetry and diffeomorphism invariance.

One can first gauge away some components of the gravitino with a local symmetry transformation. Since ζ_i has a vector index, it cannot be absorbed completely by κ however. We start by decomposing the gravitino into its "traceless" (helicity $\pm 3/2$) and "trace" (helicity $\pm 1/2$) parts:

$$\zeta_i = \chi_i + \tfrac{1}{2}\Gamma_i\lambda\,, \tag{7.10}$$

with $\Gamma^i\chi_i = 0$ and $\lambda = \Gamma^i\zeta_i$. Explicitely one has

$$\chi_i = \left(h_{ij} - \tfrac{1}{2}\Gamma_i\Gamma_j\right)\zeta^j = \tfrac{1}{2}\Gamma^j\Gamma_i\zeta_j\,. \tag{7.11}$$

whose contraction with Γ^i indeed vanishes using (7.9). Using the identity (7.9), one can then find a spinor ξ such that, locally, $\chi_i = \Gamma^j\Gamma_i D_j\xi$.

Next the supersymmetry transformation of the gravitino $\delta_\kappa\zeta_i = 2D_i\kappa$ can be decomposed in the same way:

$$2D_i\kappa = \Gamma^j\Gamma_i D_j\kappa + \Gamma_i\Gamma^j\Gamma_j\kappa. \tag{7.12}$$

which indicates that one can absorb ξ by a local supersymmetric transformation. Using now diffeomorphism invariance to bring the two-dimensional metric to a conformally flat metric, one reaches finally the **superconformal gauge**:

$$\gamma_{ij} = e^{2\omega}\eta_{ij}\,, \quad \zeta_i = \tfrac{1}{2}\Gamma_i\lambda\,. \tag{7.13}$$

If the remaining local symmetries (Weyl and super-Weyl) are non-anomalous, one can simplify the theory further by taking

$$\gamma_{ij} = \eta_{ij} \; , \quad \zeta_i = 0 \, . \tag{7.14}$$

As in the case of the bosonic string, there is a slight mismatch between the space of possible geometries (here it is the moduli space of super-Riemann surfaces) and the space of gauge configurations. Conformal Killing vectors were gauge transformations leaving invariant the metric. In the same way, a **conformal Killing spinor** is defined as a globally defined two-dimensional spinor ρ that satisfies the equation

$$\Gamma^j \Gamma_i D_j \rho = 0 \, . \tag{7.15}$$

In other words, ρ is in the kernel of the projection operator onto traceless vector-spinors:

$$\rho \in \mathrm{Ker}\, \Pi \; , \quad \Pi(\rho)_i = \tfrac{1}{2} \Gamma^j \Gamma_i D_j \rho_j \tag{7.16}$$

The superconformal gauge is invariant under a local supersymmetry transformation with a conformal Killing spinor.

Moduli were defined as parameters of the metric that could not be gauged away, since they were orthogonal to gauge transformations. They were found by asking that the variation of metric moduli are orthogonal to all gauge transformations, see eqn. (2.97). In the same way, one defines **supermoduli** as the variations $\delta\zeta_i$ of the gravitino that are orthogonal to the traceless gauge transformations $\Pi(\kappa)$. In other words,

$$\delta\zeta_i \in \mathrm{Ker}\, \Pi^\dagger \, . \tag{7.17}$$

As for the bosonic moduli, they are related to the genus of the surface by

$$\#(\text{supermoduli}) - \#(\text{conformal Killing spinors}) = \dim \mathrm{Ker}\, \Pi^\dagger - \dim \mathrm{Ker}\, \Pi = 2g - 2 \, . \tag{7.18}$$

When there are vertex operators inserted in the superstring path integral the surface is punctured and this formula is modified.

On the **two-dimensional sphere**, the conformal Killing vectors generates the group $\mathsf{PSL}(2, \mathbb{C})$ as we have seen already, while there are two conformal Killing spinors whose explicit description will not be needed here.

7.1.3 Supersymmetric action in the superconformal gauge

In the superconformal gauge, the super-Polyakov action (7.4) simplifies dramatically. After rescaling the fermionic fields as $\psi \mapsto e^{-\omega}\psi$, one has

$$\mathcal{S} = -\frac{1}{4\pi} \int d^2\sigma \left(\frac{1}{\alpha'} \eta^{ij} \partial_i x^\mu \partial_j x_\mu + \bar{\psi}^\mu \Gamma^i \partial_i \psi_\mu \right) \, . \tag{7.19}$$

which is the theory of D free bosons and D free Majorana fermions. In Euclidian space and complex coordinates one gets the *fermionic string* action:

$$\boxed{\mathcal{S} = \frac{1}{4\pi} \int d^2 z \left(\frac{2}{\alpha'} \partial x^\mu \bar{\partial} x_\mu + \psi^\mu \bar{\partial} \psi_\mu + \widetilde{\psi}^\mu \partial \widetilde{\psi}_\mu \right)} \tag{7.20}$$

As we know from chapter 4, this action is invariant under conformal transformations $z \mapsto z+\iota(z)$ with $\bar\partial\iota = 0$, which is a residual symmetry after gauge fixing of the diffeomorphism and Weyl invariance. In contrast with the discussion about the moduli and conformal Killing vectors, we don't ask at this stage that this transformation is well-defined globally.

In the same fashion the action is invariant under supersymmetric transformations (7.5) of parameter κ such that the variation of the gaugino, eqn. (7.5b), can be offset by a super-Weyl transformation (7.8). By definition, this is a transformation such that $\Pi(\kappa) = 0$, using the definition (7.16). In complex coordinates, splitting the spinor κ into its components that we denote $(\kappa, \widetilde{\kappa})$, one gets the simple expressions:

$$\delta_{\kappa,\bar\kappa} x^\mu = -\sqrt{\tfrac{\alpha'}{2}}\left(\kappa(z)\psi^\mu + \bar\kappa(\bar z)\widetilde{\psi}^\mu\right) \tag{7.21a}$$

$$\delta_\kappa \psi^\mu = -\sqrt{\tfrac{2}{\alpha'}}\kappa(z)\partial x^\mu \tag{7.21b}$$

$$\delta_{\bar\kappa}\widetilde{\psi}^\mu = -\sqrt{\tfrac{2}{\alpha'}}\bar\kappa(\bar z)\bar\partial x^\mu \tag{7.21c}$$

$$\tag{7.21d}$$

where $\bar\partial\kappa = \partial\bar\kappa = 0$. This defines **super-conformal transformations**. Super-conformal and conformal transformation are not independent, since for instance

$$[\delta_\kappa, \delta_{\kappa'}] = \delta_\iota\ , \quad \iota(z) = -2\kappa(z)\kappa'(z) \tag{7.22}$$

where the right-hand side is an infinitesimal conformal transformation of parameter ι. Likewise, the commutator of a conformal transformation with a superconformal transformation gives another superconformal transformation. A quantum field theory invariant under superconformal transformations – and, as a consequence, under conformal transformations – is called a *superconformal field theory*, or SCFT.

Particular cases of superconformal transformations are global supersymmetric transformations, corresponding to constant κ and $\widetilde{\kappa}$.

Superspace formulation

One can conveniently repackage the fermionic string action by introducing the notion of **superspace**. The two-dimensional Euclidian superspace is obtained by adding to the usual coordinates $(z, \bar z)$ Grassmann coordinates $(\theta, \bar\theta)$. The global supersymmetric transformations are generated by

$$\mathcal{Q} = \partial_\theta - \theta\partial_z\ , \quad \bar{\mathcal{Q}} = \partial_{\bar\theta} - \bar\theta\partial_{\bar z} \tag{7.23}$$

which satisfy the algebra

$$\{\mathcal{Q}, \mathcal{Q}\} = -2\partial_z\ , \quad \{\bar{\mathcal{Q}}, \bar{\mathcal{Q}}\} = -2\bar\partial_z\ , \quad \{\mathcal{Q}, \bar{\mathcal{Q}}\} = 0\,. \tag{7.24}$$

Covariant derivatives with respect to the fermionic coordinates are defined such that they commute with the supersymmetry generators (7.23)

$$D_\theta = \partial_\theta + \theta\partial_z\ , \quad D_{\bar\theta} = \partial_{\bar\theta} + \bar\theta\partial_{\bar z} \tag{7.25}$$

which satisfy the algebra

$$\{D_\theta, D_\theta\} = 2\partial_z \ , \quad \{D_{\bar\theta}, D_{\bar\theta}\} = 2\bar\partial_z \ , \quad \{D_\theta, D_{\bar\theta}\} = 0 \, . \tag{7.26}$$

Since they commute with the supersymmetry generators, they can be used to write supersymmetric actions.

One introduces superfields that are local functions of $(z, \bar z, \theta, \bar\theta)$. Because of the Grassmann nature of the odd coordinates, their Taylor expansion is finite. In the present case let us define

$$\mathbb{X}^\mu(z, \bar z, \theta, \bar\theta) = \sqrt{\tfrac{2}{\alpha'}} x^\mu(z, \bar z) - \theta\psi^\mu - \bar\theta\widetilde{\psi}^\mu + \theta\bar\theta F^\mu \, , \tag{7.27}$$

where, if x^μ is a bosonic field, ψ^μ and $\widetilde{\psi}^\mu$ should be fermionic fields. F^μ is an auxiliary field that vanishes upon imposing the equations of motion as we will see shortly.

The action of the supercharges on the superfields $\mathbb{X}^\mu$ give the global supersymmetry transformations:

$$\kappa \mathcal{Q}\, \mathbb{X}^\mu = -\kappa\psi^\mu + \theta\kappa\sqrt{\tfrac{2}{\alpha'}}\partial x^\mu + \kappa\bar\theta F^\mu + \kappa\theta\bar\theta\bar\partial\widetilde{\psi}^\mu \tag{7.28a}$$

$$\bar\kappa \bar{\mathcal{Q}}\, \mathbb{X}^\mu = -\bar\kappa\widetilde{\psi}^\mu - \bar\kappa\theta F^\mu + \bar\theta\bar\kappa\sqrt{\tfrac{2}{\alpha'}}\bar\partial x^\mu - \bar\kappa\theta\bar\theta\partial\psi^\mu \tag{7.28b}$$

upon imposing the equations of motion $F^\mu = 0$ and $\partial\widetilde{\psi}^\mu = \bar\partial\psi^\mu = 0$. A supersymmetric action is then obtained as[1]

$$\mathcal{S} = \frac{1}{4\pi}\int d^2z d^2\theta \, D_\theta \mathbb{X}^\mu D_\theta \mathbb{X}_\mu \tag{7.29}$$

$$= \frac{1}{4\pi}\int d^2z \left(\frac{2}{\alpha'}\partial x^\mu \bar\partial x^\mu + \psi^\mu\bar\partial\psi^\mu + \widetilde{\psi}^\mu\partial\widetilde{\psi}^\mu + F_\mu F^\mu\right) \, . \tag{7.30}$$

The equation of motion of F is $F^\mu = 0$ hence the auxiliary field vanishes on-shell as expected.

7.1.4 Ghosts and superghosts

In the bosonic string path integral, gauge-fixing of the diffeomorphism invariance introduced, through the Faddeev-Popov determinants, the fermionic ghosts system (b_{ij}, c^j). Because the Weyl transformations involve no derivatives of the gauge parameter, the corresponding ghost was just a Lagrange multiplier imposing that b_{ij} was traceless.

In the present case, gauge-fixing of local supersymmetric transformations leads in the same way to a ghost system (β_i, γ), where γ comes from the infinitesimal supersymmetry variation κ and β_i is the field introduced to enforce the gauge-fixing constraint in field space.

Since κ is a Grassmann variable, and that in evaluating the Faddeev-Popov determinant the statistics of the ghosts is flipped w.r.t the gauge parameter, (β_i, γ) are two-dimensional

[1]The integration measure over the Grassmann coordinates is defined such that $\int d^2\theta \, \theta\bar\theta = 1$.

bosons. In the same way as integrating over Weyl transformations imposes ${b^i}_i = 0$, integrating over super-Weyl symmetries in the FP determinant amounts to integrate over a Lagrange multiplier imposing that β_i satisfy the "traceless" condition

$$\Gamma^i \beta_i = 0\,, \tag{7.31}$$

leaving only two degrees of freedom.

Instead of deriving the action for the (β, γ) ghost system from the Faddeev-Popov determinant, we will take advantage of the global supersymmetry of the gauge-fixed action to obtain it from a supersymmetrization of the (b, c) action. In superspace one starts with the superfields

$$\mathbb{B} = \beta + \theta b\,, \tag{7.32a}$$

$$\widetilde{\mathbb{B}} = \widetilde{\beta} + \bar{\theta}\widetilde{b}\,, \tag{7.32b}$$

$$\mathbb{C} = c + \theta\gamma\,, \tag{7.32c}$$

$$\widetilde{\mathbb{C}} = \widetilde{c} + \bar{\theta}\widetilde{\gamma}\,. \tag{7.32d}$$

And write the following supersymmetric action

$$\mathcal{S}_g = -\frac{1}{2\pi}\int d^2z \int d^2\theta \left(\mathbb{B} D_{\bar{\theta}} \mathbb{C} - \widetilde{\mathbb{B}} D_\theta \widetilde{\mathbb{C}}\right) = \frac{1}{2\pi}\int d^2z \left(b\bar{\partial}c + \tilde{b}\partial\tilde{c} + \beta\bar{\partial}\gamma + \widetilde{\beta}\partial\widetilde{\gamma}\right)\,.$$

Hence the (β, γ) super-ghost CFT is defined by a first-order action similar to the (b, c) ghost CFT. There are however two important differences:

1. The fields β and γ follow Bose-Einstein statistics.
2. $(\beta, \widetilde{\beta})$ being the components of a "traceless" spinor-vector field β_i, their conformal dimension are respectively $(3/2, 0)$ and $(0, 3/2)$. By conformal invariance of the action (7.33), γ and $\widetilde{\gamma}$ have respectively conformal dimensions $(-1/2, 0)$ and $(0, -1/2)$.[2]

The superconformal transformations between the ghost and superghost fields are deduced from

$$\kappa Q \mathbb{B} = \kappa b + \theta \kappa \partial \beta\,, \tag{7.33a}$$

$$\kappa Q \mathbb{C} = \kappa c + \theta \kappa \partial \gamma\,, \tag{7.33b}$$

$$\bar{\kappa}\bar{Q}\widetilde{\mathbb{B}} = \bar{\kappa}\widetilde{b} + \bar{\theta}\bar{\kappa}\bar{\partial}\widetilde{\beta}\,, \tag{7.33c}$$

$$\bar{\kappa}\bar{Q}\widetilde{\mathbb{C}} = \bar{\kappa}\widetilde{c} + \bar{\theta}\bar{\kappa}\bar{\partial}\widetilde{\gamma}\,. \tag{7.33d}$$

The conformal theory of the (β, γ) system can be studied in close analogy with the (b, c) system. The classical equations of motion are the same:

$$\bar{\partial}\beta = 0\,, \quad \bar{\partial}\gamma = 0\,. \tag{7.34}$$

[2]Another way to obtain this is to notice that the definition (7.27) implies that θ (resp. $\bar{\theta}$) has conformal weights $(-1/2, 0)$ (resp. $(0, -1/2)$).

There exists also a superghost rotation symmetry, $\beta \mapsto e^{-i\alpha}\beta$, $\gamma \mapsto e^{i\alpha}\gamma$, to which we associate a holomorphic Noether current

$$j_{\text{SG}} = -:\beta\gamma:(z)\ , \quad \bar{\partial} j_{\text{NG}} = 0\,. \tag{7.35}$$

From the associated Ward identities (3.37) one finds the OPEs

$$j_{\text{SG}}(z_1)\beta(z_2) = -\frac{\beta(z_2)}{z_1 - z_2} + \text{reg.}\ , \quad j_{\text{SG}}(z_1)\gamma(z_2) = -\frac{\gamma(z_2)}{z_1 - z_2} + \text{reg.}\,, \tag{7.36}$$

coming from the fundamental OPE

$$\beta(z_1)\gamma(z_2) = -\gamma(z_1)\beta(z_2) = \frac{1}{z_1 - z_2} + \text{reg.} \tag{7.37}$$

Notice the negative sign in the $\gamma\beta$ OPE due to the bosonic statistics, compared to the bc OPE in equation (4.156).

The more general ansatz for the stress-tensor, compatible with conformal dimensions $(2, 0)$, is of the form

$$T(z) = u:\beta\partial\gamma: + v:\partial\beta\gamma:\,. \tag{7.38}$$

The coefficients u and v are then fixed by requiring that β and γ are primary fields of respective holomorphic conformal weights 3/2 and -1/2. The OPE with γ gives (remembering that these are commuting variables)

$$T(z_1)\gamma(z_2) = \frac{u\partial\gamma(z_2)}{z_1 - z_2} - \frac{v\gamma(z_1)}{(z_1 - z_2)^2} + \text{reg.} = -\frac{v\gamma(z_2)}{(z_1 - z_2)^2} + \frac{(u - v)\partial\gamma(z_2)}{z_1 - z_2} + \text{reg.} \tag{7.39}$$

This sets $u = 3/2$ and $v = 1/2$. One can check the OPE with β as well:

$$T(z_1)\beta(z_2) = \left(\frac{3}{2}:\beta\partial\gamma:(z_1) + \frac{1}{2}:\partial\beta\gamma:(z_1)\right)\beta(z_2) = \frac{3}{2}\frac{\beta(z_2)}{(z_1 - z_2)^2} + \frac{\partial\beta(z_2)}{z_1 - z_2} + \text{reg.} \tag{7.40}$$

Finally the central charge of the superghost CFT is deduced from the self-OPE of the stress tensor, more precisely from the higher order pole which is given by the fully contracted term:

$$\begin{aligned}
\frac{c_{\text{SG}}}{2(z_1 - z_2)^4} =& \frac{9}{4}:\beta\partial\gamma:(z_1):\beta\partial\gamma:(z_2) + \frac{3}{4}:\beta\partial\gamma:(z_1):\partial\beta\gamma:(z_2)\\
&+ \frac{3}{4}:\partial\beta\gamma:(z_1):\beta\partial\gamma:(z_2) + \frac{1}{4}:\partial\beta\gamma:(z_1):\partial\beta\gamma:(z_2)\\
=& \frac{11}{2(z_1 - z_2)^4}
\end{aligned} \tag{7.41}$$

Hence the central charges of the (β, γ) superghost CFT are $(c, \bar{c}) = (11, 0)$. Similarly the central charges of the $(\tilde{\beta}, \tilde{\gamma})$ CFT are $(c, \bar{c}) = (0, 11)$.

7.1.5 Critial dimension of the fermionic string

The local supersymmetric action (7.4) includes the Weyl symmetries in its gauge symmetry group, hence the considerations for cancellation of the Weyl anomaly, discussed in subsection 3.6 apply also here.

The total central charge of the worldsheet conformal field theory, which is made of the tensor product of the x^μ, ψ^μ, (b, c) and (β, γ) theories should vanish to cancel the anomaly.

One gets the following condition on the dimensionality of Minkowski space-time $\mathbb{R}^{1,D-1}$ for the fermionic string:

$$c^x + c^\psi + c_{\text{G}} + c_{\text{SG}} = \left(1 + \tfrac{1}{2}\right) D - 26 + 11 = 0 \implies D = 10\,. \tag{7.42}$$

The same considerations apply naturally for the right-movers, as we consider a left-right symmetric theory. Therefore the consistent **superstring theories** that we will build out of the fermionic string will be defined in a **ten-dimensional space-time**.

This is closer to the real world that the twenty-six dimensional space-time of the bosonic string, but there are still six dimensions that should be compact in order to avoid direct clash with experiments.

7.2 Superconformal symmetry

In chapter 3, using that the variation of the action (2.66) with respect to the two-dimensional metric gave the stress-energy tensor of the CFT, we obtained that the infinitesimal generators of the conformal transformations $(\epsilon(z), \bar\epsilon(\bar z))$ were the components of the stress-tensor $(T_{zz}, T_{\bar z,\bar z})$.

In the same way, the variation of the supergravity action (7.4) with respect to a gravitino variation gives the *supercurrent* of the theory. In the superconformal gauge it takes the simple form

$$G^i = \frac{\delta\mathcal{L}}{\delta\bar{\bar\zeta}_i}\Big|_{\bar\zeta_i=0} = \frac{1}{4}\sqrt{\frac{2}{\alpha'}}\Gamma^j\Gamma^i\psi^\mu\partial_j x_\mu\,. \tag{7.43}$$

It has only two components since $\Gamma_i G^i = 0$, using again (7.9).

7.2.1 Superconformal field theory

In Euclidian space and moving to complex coordinates, the two independent components $G(z)$ and $\widetilde{G}(\bar z)$, which are separately conserved, are given by

$$G(z) = i\sqrt{\frac{2}{\alpha'}}\psi^\mu\partial x_\mu\,, \quad \bar\partial G = 0 \tag{7.44a}$$

$$\widetilde{G}(\bar z) = i\sqrt{\frac{2}{\alpha'}}\widetilde{\psi}^\mu\bar\partial x_\mu\,, \quad \partial\widetilde{G} = 0\,. \tag{7.44b}$$

One gets Ward identities for the superconformal transformations similar to the conformal one, see eqn. (3.37)

$$\delta_\kappa \psi^\mu(z) = -\sqrt{\tfrac{2}{\alpha'}}\kappa(z)\partial x^\mu(z) = i\mathrm{Res}_{z'\to z}\kappa(z')G(z')\psi^\mu(z)\,, \tag{7.45}$$

an a similar one for the anti-holomorphic superconformal transformation.

From the commutator (7.22) we know that the supercurrent and the stress-energy tensor of the superconformal field theory should have non-trivial OPE with each other. These OPE will encode an infinite-dimensional algebra, the superconformal algebra.

Let us consider the free theory (7.20) in D dimensions. One computes first the OPE of a supercurrent G with itself:

$$\begin{aligned}
G(z_1)G(z_2) &= -\frac{2}{\alpha'} :\psi^\mu\partial x_\mu:(z_1):\psi^\nu\partial x_\nu:(z_2)\\
&= -\frac{2}{\alpha'}\left(\frac{\eta^{\mu\nu}}{z_1-z_2} + :\psi^\mu(z_1)\psi^\nu(z_2):\right)\left(\frac{-\alpha'}{2}\eta_{\mu\nu}\frac{1}{(z_1-z_2)^2} + :\partial x_\mu(z_1)\partial x_\mu(z_2):\right)\\
&= \frac{\eta_{\mu\nu}\eta^{\mu\nu}}{(z_1-z_2)^3} + \frac{1}{z_1-z_2}\left(-\frac{2}{\alpha'}:\partial x_\mu\partial x_\mu:(z_2) + \eta_{\mu\nu}:\partial\psi^\mu\psi^\nu:(z_2)\right) + \text{reg.}\\
&= \frac{D}{(z_1-z_2)^3} + \frac{2}{z_1-z_2}\left(T^x + T^\psi\right) + \text{reg.}
\end{aligned} \tag{7.46}$$

where we have used (see chapter 3)

$$T^x(z) = -\frac{1}{\alpha'}:\partial x^\mu\partial x_\mu:(z)\,, \tag{7.47a}$$

$$T^\psi(z) = -\frac{1}{2}:\psi^\mu\partial\psi_\mu:(z)\,. \tag{7.47b}$$

Next the OPE between the stress-energy tensor and the supercurrent gives

$$\begin{aligned}
T(z_1)G(z_2) &= \left(-\frac{1}{\alpha'}:\partial x^\mu\partial x_\mu:(z_1) - \frac{1}{2}:\psi^\mu\partial\psi_\mu:(z_1)\right) i\sqrt{\frac{2}{\alpha'}}\psi^\mu\partial x_\mu(z_2)\\
&= i\sqrt{\frac{2}{\alpha'}}\frac{\partial x^\mu(z_1)\psi^\mu(z_2)}{(z_1-z_2)^2} + \frac{i}{\sqrt{2\alpha'}}\frac{\partial\psi^\mu(z_1)\partial x_\mu(z_2)}{z_1-z_2} + \frac{i}{\sqrt{2\alpha'}}\frac{\psi^\mu(z_1)\partial x_\mu(z_2)}{(z_1-z_2)^2}\\
&= \frac{3}{2}\frac{1}{(z_1-z_2)^2} i\sqrt{\frac{2}{\alpha'}}\psi^\mu\partial x_\mu(z_2) + \frac{1}{z_1-z_2}\partial_{z_2}\left(i\sqrt{\frac{2}{\alpha'}}\psi^\mu\partial x_\mu(z_2)\right)\\
&= \frac{3}{2(z_1-z_2)^2}G(z_2) + \frac{1}{z_1-z_2}\partial G(z_2)\,.
\end{aligned} \tag{7.48}$$

In other words, G is a conformal primary of conformal dimension $h = 3/2$.

Noticing that the CFT of D scalars ψ^μ and D fermions ψ^μ has central charge $c = 3D/2$, one can infer the general superconformal OPE, which is given by:

$$\boxed{\begin{aligned} T(z_1)T(z_2) &= \frac{c}{2(z_1-z_2)^4} + \frac{2T(z_2)}{(z_1-z_2)^2} + \frac{\partial T(z_2)}{z_1-z_2} + \text{reg.} && \text{(7.49a)} \\ G(z_1)G(z_2) &= \frac{2c}{3(z_1-z_2)^3} + \frac{2T(z_2)}{z_1-z_2} + \text{reg.} && \text{(7.49b)} \\ T(z_1)G(z_2) &= \frac{3G(z_2)}{2(z_1-z_2)^2} + \frac{\partial G(z_2)}{z_1-z_2} + \text{reg.} && \text{(7.49c)} \end{aligned}}$$

7.2.2 Representations of the superconformal algebra

In order to obtain the infinite-dimensional algebra coming from the OPEs (7.49) one expands first T and G in Laurent modes. We know already that

$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}\,, \quad [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\,. \tag{7.50}$$

The supercurrent G is a composite operator of the fermionic fields $\psi^\mu(z)$ and the bosonic fields ∂x^μ. Remember that the Laurent expansion of the former is

$$\psi^\mu(z) = \sum_{r\in\mathbb{Z}+1/2} \frac{\psi^\mu_r}{z^{r+1/2}}\,, \quad \text{Neveu-Schwarz sector} \tag{7.51a}$$

$$\psi^\mu(z) = \sum_{r\in\mathbb{Z}} \frac{\psi^\mu_r}{z^{r+1/2}}\,, \quad \text{Ramond sector} \tag{7.51b}$$

Hence the supercurrent G will have half-integer moded Laurent modes in the Neveu-Schwarz sector (NS) and integer moded ones in the Ramond (R) sector. Accordingly we expand the supercurrent as

$$G(z) = \sum_{r\in\mathbb{Z}+\frac{1-a}{2}} \frac{G_r}{z^{r+3/2}}\,, \tag{7.52}$$

with $a = 0$ (resp. $a = 1$) in the Neveu-Schwarz (resp. Ramond) sector. Since the modes ψ^μ_r and α^μ_n commute with each other, there are no ordering ambiguities in the expansion and we find that in the present case

$$G_r = \sum_{n\in\mathbb{Z}} \eta_{\mu\nu}\alpha^\mu_n \psi^\nu_{r-n}\,. \tag{7.53}$$

The (anti)commutation relations of the superconformal algebra can be deduced from the OPE (7.49), or equivalently from the commutation relations of the α^μ_n, see eqn. (4.34), and of the ψ^μ_r, see eqn. (4.107). Either way, one finds

$$\boxed{\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \tfrac{c}{12}m(m^2-1)\delta_{m+n,0} && \text{(7.54a)} \\ \{G_r, G_s\} &= 2L_{r+s} + \tfrac{c}{12}(4r^2-1)\delta_{r+s,0} && \text{(7.54b)} \\ [L_m, G_r] &= \left(\tfrac{m}{2} - r\right) G_{m+r}\,. && \text{(7.54c)} \end{aligned}}$$

Representation theory of the superconformal algebra has the same kind of structure as representation theory of the Virasoro algebra. The *superconformal primary states* are annihilated by the positive modes of the currents, however the presence of zero-modes in the Ramond sector needs to be taken into account carefully.

Finally, it is worthwile to mention that the Ramond boundary conditions preserve two-dimensional supersymmetry, as they relate bosons to fermions of identical periodicities, while the Neveu-Schwarz boundary conditions break two-dimensional supersymmetry at the global level, since they relate periodic bosons to anti-periodic fermions.

Neveu-Schwarz sector

In the Neveu-Schwarz sector, the modes of the supercurrent $\{G_r, r \in \mathbb{Z} + 1/2\}$ are all half-integers so we define a superconformal primary as

$$L_m|\Psi\rangle = G_{n+1/2}|\Psi\rangle = 0 \ , \quad \forall m, n \in \mathbb{N} \, . \tag{7.55}$$

Descendant states are obtained then by acting with the negative modes L_{-n} and $G_{-n+1/2}$, $n > 0$.

In the present context, the primary states of momentum p^μ are given by the tensor product of the conformal primaries $|p^\mu\rangle$ for the x^μ CFT and the NS vacuum $|0\rangle_{\rm NS}$ for the ψ^μ CFT:

$$|p^\mu\rangle_{\rm NS} = |p^\mu\rangle \otimes |0\rangle_{\rm NS} \, . \tag{7.56}$$

Using equations (4.46) and (4.121) its conformal dimensions is given by

$$L_0|p^\mu\rangle_{\rm NS} = \left(L_0^x + L_0^\psi\right)|p^\mu\rangle \otimes |0\rangle_{\rm NS} = \frac{\alpha' p^2}{4}|p^\mu\rangle)\mathrm{NS} \, . \tag{7.57}$$

The excited states of the lowest conformal dimension arise at level $N = 1/2$. They correspond to

$$\psi^\rho_{-1/2}|p^\mu\rangle_{\rm NS} = |p^\mu\rangle \otimes \psi^\rho_{-1/2}|0\rangle_{\rm NS} \ , \quad L_0\psi^\rho_{-1/2}|p^\mu\rangle_{\rm NS} = \left(\frac{\alpha' p^2}{4} + \frac{1}{2}\right)|p^\mu\rangle_{\rm NS} \, . \tag{7.58}$$

These states transform in the vector representation of $SO(1, D-1)$, as explained in subsection 4.2.3. These states are conformal primaries, *i.e.* annihilated by all positive Virasoro modes.

However for generic p^μ they are descendant states w.r.t. the modes of the supercurrent. Explicitly one has, for a mode with polarization ν_ρ, forgetting about the right-moving sector at this stage,

$$\begin{aligned} G_{1/2}\nu_\rho\psi^\rho_{-1/2}|p^\mu\rangle_{\rm NS} &= \eta_{\mu\nu}\left(\alpha_0^\mu\psi^\nu_{1/2} + \cdots\right)|p^\mu\rangle \otimes \psi^\rho_{-1/2}|0\rangle_{\rm NS} \\ &= \sqrt{\tfrac{\alpha'}{2}}\nu_\mu p^\mu |p^\mu\rangle \otimes \psi^\rho_{-1/2}|0\rangle_{\rm NS} \, . \end{aligned} \tag{7.59}$$

We will be interested by states of the form (7.58) that are superconformal primaries. From this computation it occurs for $\nu_\mu p^\mu = 0$.

As we will see, all higher-level states, including those constructed with the action of α^μ_{-1}, are massive in superstring theory, hence will not be discussed in detail.

We will only make the following important remark. In subsection 4.2.3 we considered on the one hand states in the trivial representation of the current algebra, obtained by acting with J^{ij}_{-n} on the NS vacuum, and on the other hand the states obtained by acting on the vector representation $\psi^i_{-1/2}|0\rangle_{\rm NS}$ with J^{ij}_{-n}. From the point of view of the superconformal algebra, it amounts to split representation in two, keeping states which are even under $(-)^F$ in the first case and odd in the second case.

Ramond sector

In the Ramond sector, there exists a zero mode G_0 of the supercurrent. From the commutation relation (7.54b) one finds that

$$G_0^2 = L_0 - \frac{c}{24}\,, \tag{7.60}$$

hence in the Ramond sector of a unitary CFT, all the conformal dimensions of operators obey the bound

$$h \geqslant \frac{c}{24}\,, \tag{7.61}$$

which is saturated by the *Ramond ground states* with $h = c/24$. For a theory of N free bosons and fermions, this is essentially the content of eqn. (4.140).

Using the results of chapter 4.2, the fermionic zero-modes give an algebra isomorphic to the Clifford algebra in $1+9$ dimensions:

$$\{\psi^\mu_0, \psi^\nu_0\} = \eta^{\mu\nu}\,, \tag{7.62}$$

and the Ramond ground state of the ψ^μ CFT corresponds to a (reducible) Majorana representation of this algebra.

For $\mathrm{Spin}(1,9)$ this is a 32-dimensional representation. It decomposes into a pair of irreducible Majorana-Weyl representations of dimensions 16,

$$\mathbf{32} = \mathbf{16} + \mathbf{16}'\,, \tag{7.63}$$

which are denoted spinor and conjugate spinor representations respectively (see chapter 4.2). The corresponding states, $|\alpha\rangle_{\rm R}$ and $|\dot\alpha\rangle_{\rm R}$, are distinguished by the eigenvalue of the chirality operator $(-1)^F$ which anticommutes with all fermionic operators:

$$(-1)^F|\alpha\rangle_{\rm R} = |\alpha\rangle_{\rm R}\,, \quad (-1)^F|\dot\alpha\rangle_{\rm R} = -|\dot\alpha\rangle_{\rm R}\,. \tag{7.64}$$

We decompose accordingly the ten-dimensional gamma-matrices as

$$(\Gamma^\mu)_a{}^b = \begin{pmatrix} 0 & \Gamma^\mu_{\alpha}{}^{\dot\beta} \\ \Gamma^\mu_{\dot\alpha}{}^{\beta} & 0 \end{pmatrix} \tag{7.65}$$

In the fermonic string context, the lowest-dimension states are therefore obtained as the tensor product of the momentum eigenstates $|p^\mu\rangle$ with the Ramond ground states:

$$|p^\mu;\alpha\rangle = |p^\mu\rangle \otimes |\alpha\rangle_{\text{R}}\ , \quad |p^\mu;\dot\alpha\rangle = |p^\mu\rangle \otimes |\dot\alpha\rangle_{\text{R}}\,. \tag{7.66}$$

These states are by construction conformal primaries. It is easy to check as well that they are annihilated by G_1 and all other positive modes of the supercurrent. All the Ramond sector excited states are massive, hence will not be considered further.

7.2.3 Superghosts mode expansion

The (β,γ) superghosts are spinors (with Bose-Einstein statistics) so they can have also either periodic and anti-periodic boundary conditions on the cylinder. Because the field γ came originally from the supersymmetry parameter κ, consistency of the transformations (7.5) require that they obey the same boundary conditions as the fields ψ^μ:

- In the Neveu-Schwarz sector of the ψ^μ fields, (β,γ) will have *anti-periodic* boundary conditions on the cylinder
- In the Ramond sector of the ψ^μ fields, (β,γ) will have *periodic* boundary conditions on the cylinder

The superghosts are therefore expanded as

$$\beta = \sum_{n\in\mathbb{Z}+\frac{1-a}{2}} \frac{\beta_n}{z^{n+3/2}}\ , \quad \gamma = \sum_{n\in\mathbb{Z}+\frac{1-a}{2}} \frac{\gamma_n}{z^{n-1/2}}\,, \tag{7.67}$$

and the modes obey the algebra

$$[\beta_m,\gamma_m] = \delta_{m+n,0}\,. \tag{7.68}$$

We consider the superconformal field theory made of the ghosts (b,c) and the superghosts (β,γ) first in the **Neveu-Schwarz sector**. The modes of the superconformal algebra are

$$\forall n\neq 0\ , \quad L_n = L_n^g + L_n^{sg} = \sum_{m\in\mathbb{Z}} (2n-m)\,{}^\circ_\circ\, b_m c_{n-m}\,{}^\circ_\circ + \sum_{m\in\mathbb{Z}+\frac{1}{2}} \left(\tfrac{3}{2}n - m\right)\,{}^\circ_\circ\, \beta_m\gamma_{n-m}\,{}^\circ_\circ\ , \tag{7.69}$$

where the first term was computed in (4.178) and the second follows from a similar calculation. As usual there exists a normal ordering ambiguity for the zero-mode L_0. We have computed already the normal ordering constant coming from the (b,c) ghosts, and the contribution from the (β,γ) superghosts follows from a similar reasoning.

The superghost vacuum $|-\rangle^{sg}_{\text{NS}}$ is annihilated by all the positive modes of the ghost and superghost fields (similar statements hold for the anti-holomorphic superghosts):

$$\forall n\geqslant 0\,,\ \beta_{n+1/2}|-\rangle^{sg}_{\text{NS}} = 0\ , \quad \gamma_{n+1/2}|-\rangle^{sg}_{\text{NS}} = 0\,, \tag{7.70a}$$

$$\forall n\geqslant 0\,,\ b_n|-\rangle^{sg}_{\text{NS}} = 0\ , \quad \forall n>0\,,\ c_n|-\rangle^{sg}_{\text{NS}} = 0\,. \tag{7.70b}$$

We have the following relation:

$$2L_0^{sg}|-\rangle_{\rm NS} = L_1 L_{-1}|-\rangle_{\rm NS} = \left(\beta_{1/2}\gamma_{1/2} + \cdots\right)\left(-\beta_{-1/2}\gamma_{-1/2} + \cdots\right)|-\rangle_{\rm NS} = |-\rangle_{\rm NS}\,. \tag{7.71}$$

Adding the normal ordering constant of the ghosts (b, c) that we have already computed, one finds that in the Neveu-Schwarz sector

$$L_0 = L_0^g + L_0^{sg} = -\sum_{m\in\mathbb{Z}} m \, {}^\circ_\circ b_m c_{-m} {}^\circ_\circ - \sum_{m\in\mathbb{Z}+\frac{1}{2}} m \, {}^\circ_\circ \beta_m \gamma_{-m} {}^\circ_\circ - \frac{1}{2}\,. \tag{7.72}$$

The Neveu-Schwarz vacuum of the ghost superconformal field theory, which has conformal dimension $h = -1/2$, is annihilated by all positive modes of T and G:

$$\forall n > 0\,, \quad L_n|-\rangle_{\rm NS} = 0\,, \quad G_{n-1/2}|-\rangle_{\rm NS} = 0\,. \tag{7.73}$$

We now move to the **Ramond sector** of the ghost SCFT, where the mode expansion of the Virasoro generators in now

$$\forall n \neq 0\,, \quad L_n = L_n^g + L_n^{sg} = \sum_{m\in\mathbb{Z}} (2n-m)\, {}^\circ_\circ b_m c_{n-m} {}^\circ_\circ + \sum_{m\in\mathbb{Z}} \left(\tfrac{3}{2}n - m\right) {}^\circ_\circ \beta_m \gamma_{n-m} {}^\circ_\circ\,. \tag{7.74}$$

While nothing changes for the (b, c) ghosts the (β, γ) superghosts acquire zero-modes (β_0, γ_0). The Ramond superghost vacuum will be defined by

$$\forall n \geqslant 0\,,\ \beta_n|-\rangle_{\rm R} = 0\,, \quad \forall n > 0\,,\ \gamma_n|-\rangle_{\rm R} = 0\,, \tag{7.75a}$$

$$\forall n \geqslant 0\,,\ b_n|-\rangle_{\rm R} = 0\,, \quad \forall n > 0\,,\ c_n|-\rangle_{\rm R} = 0\,. \tag{7.75b}$$

Again the normal ordering constant is obtained from

$$2L_0^{sg}|-\rangle_{\rm R} = L_1 L_{-1}|-\rangle_{\rm R} = \left(\frac{3}{2}\beta_0\gamma_1 + \cdots\right)\left(-\frac{1}{2}\beta_{-1}\gamma_0 + \cdots\right)|-\rangle_{\rm R} = -\frac{3}{4}|-\rangle_{\rm R} \tag{7.76}$$

Hence we get in the Ramond sector

$$L_0 = L_0^g + L_0^{sg} = -\sum_{m\in\mathbb{Z}} m \, {}^\circ_\circ b_m c_{-m} {}^\circ_\circ - \sum_{m\in\mathbb{Z}} m \, {}^\circ_\circ \beta_m \gamma_{-m} {}^\circ_\circ - \frac{5}{8}\,. \tag{7.77}$$

Note that, following (4.140), this is exactly the opposite of the Ramond ground state conformal dimension for the ψ^μ CFT, in the critical dimension $D = 10$. So, whenever we impose the supersymmetric Ramond boundary conditions for the fermions, it seems that we have achieved our goal of having string ground states with zero mass. For the (supersymmetry-breaking) Neveu-Schwarz conditions however, this is not quite the case, as the zero-point energy was just shifted from -1 in the bosonic string to $-1/2$ in the fermionic string. We will see shortly how to get rid of this potential tachyonic state.

7.3 BRST quantization

As for the bosonic string, after gauge-fixing of the local symmetries, there exists a remnant global symmetry whose cohomology is identified with the physical spectrum, the BRST symmetry.

7.3.1 BRST current

Following the general logic sketched in section 5.2 the BRST current will receive an extra contribution, which is essentially the superghost γ multiplied by the variation of the action (7.4) under local supersymmetry transformations, which is proportional to the supercurrent. An explicit computation would give then the holomorphic and anti-holomorphic currents:

$$j_{\mathrm{B}}(z) = cT^{x,\psi} + \gamma G^{x,\psi} + \frac{1}{2}\left(cT^{g,sg} + \gamma G^{g,sg}\right), \tag{7.78a}$$

$$\widetilde{j}_{\mathrm{B}}(\bar{z}) = \widetilde{c}\widetilde{T}^{x,\psi} + \widetilde{\gamma}\widetilde{G}^{x,\psi} + \frac{1}{2}\left(\widetilde{c}\widetilde{T}^{g,sg} + \widetilde{\gamma}\widetilde{G}^{g,sg}\right), \tag{7.78b}$$

up to total derivative terms, as in (5.53a), that do not contribute to the charge. A local operator $\mathcal{O}(z,\bar{z})$ would then correspond, through the state-operator correspondence, to a physical state of the fermionic string theory if

$$(Q_{\mathrm{B}} + \widetilde{Q}_{\mathrm{B}})\mathcal{O}(0,0) = \frac{1}{2i\pi}\oint_{\mathcal{C}_0}(j_{\mathrm{B}}dz - \widetilde{j}d\bar{z})\,\mathcal{O}(0,0) = 0\,, \tag{7.79}$$

up to spurious states.

One can check easily that the OPE between the current (7.78a) with itself has a single pole unless the matter CFT satifies $c = 15$, *i.e.* provided that the dimension of space-time is ten. Naturally the self-OPE of (7.78b) leads to the same conclusion.

7.3.2 Mode expansion of the BRST current

To work with states rather than with local operators, we need to expand the BRST currents in terms of modes of the various elementary fields of the theory. For the holomorphic current for instance, one finds that

$$\begin{aligned} Q_{\mathrm{B}} = \sum_{n\in\mathbb{Z}} c_{-n}(L^x_n + L^\psi_n) + \sum_{r\in\mathbb{Z}+\frac{1-a}{2}} \gamma_{-r}G^{x,\psi}_r - \sum_{m,n\in\mathbb{Z}} \tfrac{n-m}{2} \,{}^\circ_\circ\, b_{-m-n}c_m c_n \,{}^\circ_\circ \\ + \sum_{m\in\mathbb{Z},r\in\mathbb{Z}+\frac{1-a}{2}} \left(\tfrac{2r-m}{2}\,{}^\circ_\circ\,\beta_{-m-r}c_m\gamma_r\,{}^\circ_\circ - {}^\circ_\circ\, b_{-m}\gamma_{m-r}\gamma_r\,{}^\circ_\circ\right) + \lambda_a c_0 \end{aligned} \tag{7.80}$$

where the normal ordering constant, which depends on the sector, was computed before:

$$\lambda_0 = -\frac{1}{2}\,, \quad \lambda_1 = -\frac{5}{8}\,. \tag{7.81}$$

7.3.3 BRST cohomology

Physical states in the fermionic string will be given in terms of the BRST cohomology, *i.e.* by states annihilated by the sum of (7.80) and the corresponding anti-holomorphic charge, modulo exact states that can be described as $(Q_B + \widetilde{Q}_B)|\chi\rangle$.

In the context of the bosonic string, in order to obtain a consistent spectrum from the space-time point of view, one had to impose the Siegel gauge (5.75). Likewise we will impose here the following constraints on all physical states:

$$b_0|\Psi\rangle = \widetilde{b}_0 = 0 \qquad \text{Neveu-Schwarz sector} \tag{7.82a}$$

$$b_0|\Psi\rangle = \widetilde{b}_0|\Psi\rangle = \beta_0|\Psi\rangle = \widetilde{\beta}_0|\Psi\rangle = 0 \qquad \text{Ramond sector} \tag{7.82b}$$

The BRST charge (7.80) does not change the level of a state that it is applied to, hence we can study the spectrum level by level. The analysis is complicated by the fact that one can choose between the the Neveu-Schwarz and Ramond boundary conditions separately for the left-movers and right-movers, hence we have four different sectors to consider: (NS, NS), (NS, R), (R, NS), (R, R).

NS-NS sector

In this sector both the left-moving and right-moving fermions obey the Neveu-Schwarz boundary conditions. At level zero in this sector one has a unique ground state

$$|p^\mu\rangle_{\text{NS-NS}} = |p^\mu\rangle^x \otimes |0\rangle^\psi_{\text{NS}} \otimes \widetilde{|0\rangle}^{\widetilde{\psi}}_{\text{NS}} \otimes |-\rangle^{sg}_{\text{NS}} \otimes \widetilde{|-\rangle}^{\widetilde{sg}}_{\text{NS}}\,, \tag{7.83}$$

i.e. the tensor product of the conformal primary of momentum p^μ for the x^μ CFT, the NS vacuum for the ψ^μ CFT, and the superghost vacuum in the NS sector defined by eqns. (7.70) and the similar vacua for the right-moving fermions, ghosts and superghosts. From a space-time perspective, this state is a scalar field.

This state is annihilated by all the oscillator modes in the BRST charge (7.80), and the only condition that remains to be imposed is:

$$c_0(L_0^x + L_0^\psi - \tfrac{1}{2})|p^\mu\rangle_{\text{NS}} = 0 \implies \frac{\alpha' m^2}{4} = -\frac{\alpha' p^2}{4} = -\frac{1}{2}\,. \tag{7.84}$$

the right-moving component of the BRST charge gives naturally exactly the same result. So it seems that the progress obtained by moving from the bosonic string to the superstring was fairly modest; instead of a tachyon of mass squared $m^2 = -4/\alpha$, we have a tachyon of mass squared $m^2 = -2/\alpha'$. The cancellation of the zero-point energies of the ghosts and superghosts does not work in the NS sector as these fields have different boundary conditions. Don't dispair yet, we will be able to solve this problem in due time.

At the next level, imposing level-matching conditions, one can build several states with levels $N = \bar{N} = 1/2$. The general ansatz is actually similar to (5.88), with the appropriate replacements ($\alpha^\mu_{-1} \to \psi^\mu_{-1/2}$, $b_{-1} \to \beta_{-1/2}$, etc...) and the analysis works in the same way. The physical state is built out of the term without superghost modes, namely

$$|\Psi_{1/2}\rangle = e_{\mu\nu}\psi^\mu_{-1/2}\widetilde{\psi}^\mu_{-1/2}|p^\rho\rangle_{ns-ns}\,. \tag{7.85}$$

Different terms coming from the BRST charge (7.80) are in different states of the ghost and superghost CFT hence should be cancelled separately. One has first

$$c_0(L_0^x + L_0^\psi - \tfrac{1}{2})|\Psi_{1/2}\rangle = 0 \quad \Longrightarrow \quad \frac{\alpha' m^2}{4} = -\frac{\alpha' p^2}{4} = 0\,. \tag{7.86}$$

One has then

$$c_{-1}L_1^\psi|\Psi_{1/2}\rangle = \tfrac{1}{2}c_{-1}\left(\sum_{r\in\mathbb{Z}+1/2}(r-1/2)\,{}^\circ_\circ\sum_i \psi^\sigma_{1-r}\psi_{r\,\sigma}\,{}^\circ_\circ\,\psi^\mu_{-1/2}\right)e_{\mu\nu}\widetilde{\psi}^\nu_{-1/2}|p^\rho\rangle_{ns-ns} = 0\,, \tag{7.87}$$

consistently with the fact that $\psi^\mu_{-1/2}|0\rangle_{\rm NS}$ are the conformal primary operators in the vector representation of $SO(1,9)$. The last term to consider is

$$\gamma_{-1/2}G^{\psi,x}_{1/2}|\Psi_{1/2}\rangle = \gamma_{-1/2}\left(\eta_{\sigma\tau}\alpha^\sigma_0\psi^\tau_{1/2}\psi^\mu_{-1/2}\right)e_{\mu\nu}\widetilde{\psi}^\nu_{-1/2}|p^\rho\rangle_{ns-ns} = 0\,, \tag{7.88}$$

implying, using eqn. (4.30) and the anti-commutator (4.107) that (adding the similar constraint from the anti-holomorphic component of the BRST charge):

$$p^\mu e_{\mu\nu} = p^\nu e_{\mu\nu} = 0\,. \tag{7.89}$$

Finally one has to find those of the spurious states that give equivalence relations among the states (7.85) with different polarizations. One has that

$$(Q_{\rm B} + \widetilde{Q}_{\rm B})\beta_{-1/2}\nu_\mu\widetilde{\psi}^\mu_{-1/2}|p^\mu\rangle_{ns-ns} = \nu_\mu\left(G^{x,\psi}_{-1/2}\widetilde{\psi}^\mu_{-1/2} + \beta_{-1/2}\widetilde{G}^{x,\psi}_{1/2}\widetilde{\psi}^\mu_{-1/2}\right)|p^\mu\rangle_{ns-ns} \tag{7.90}$$

Next using that

$$G^{x,\psi}_{-1/2}|p^\mu\rangle\otimes|0\rangle_{\rm NS} = \eta_{\mu\nu}\alpha^\mu_0\psi^\nu_{-1/2}|p^\mu\rangle\otimes|0\rangle_{\rm NS} = \sqrt{\tfrac{\alpha'}{2}}\,p_\mu\psi^\mu_{-1/2}|p^\mu\rangle\otimes|0\rangle_{\rm NS}\,, \tag{7.91}$$

and

$$G^{x,\psi}_{1/2}\psi^\mu_{-1/2}|p^\rho\rangle\otimes|0\rangle_{\rm NS} = \eta_{\sigma\tau}\alpha^\sigma_0\psi^\tau_{1/2}\psi^\mu_{-1/2}|p^\rho\rangle\otimes|0\rangle_{\rm NS} = \sqrt{\tfrac{\alpha'}{2}}\,p^\mu|p^\rho\rangle\otimes|0\rangle_{\rm NS}\,, \tag{7.92}$$

from which we deduce, using also the anti-holomorphic analogous equation, the equivalence relations

$$e_{\mu\nu} \sim e_{\mu\nu} + a_\mu p_\nu + b_\nu p_\mu\,, \quad a_\mu p^\mu = b_\nu p^\nu = 0\,. \tag{7.93}$$

All these conditions give naturally the fluctuations of the metric, of the Kalb-Ramond field and of the dilaton exactly as in the bosonic strings.

R-NS sector

We consider a sector of string states such that the left-moving fermions ψ^μ have periodic boundary conditions on the cylinder, while the right-moving fermions $\widetilde{\psi}^\mu$ have anti-periodic boundary conditions. As was said before, the left and right superghosts should follow the same pattern.

The right-moving part of the theory are described exactly in the same way as before, so we will concentrate on the left-moving part which is in the Ramond sector. At level zero we consider that both the superconformal field theory (x^μ, ψ^μ) and the superconformal field theory $(b, c; \beta, \gamma)$ are in their Ramond ground state. The conformal dimension of the ground state in the tensor product of the two superconformal field theories

$$|p_\mu\rangle \otimes |a\rangle_{\text{R}} \otimes |-\rangle_{\text{R}} \tag{7.94}$$

is, using (7.77) and in $D = 10$ spacetime,

$$h = \frac{\alpha'}{4}p^2 + \frac{D}{16} - \frac{5}{8} = \frac{\alpha'}{4}p^2 \,. \tag{7.95}$$

as we said already, the reducible representation $|a\rangle_{\text{R}}$ of the Clifford algebra reduces in a pair of irreducible representations $|\alpha\rangle_{\text{R}}$ and $|\dot{\alpha}\rangle_{\text{R}}$ of opposite chiralities.

By the level matching constraint $h = \bar{h}$ the states cannot be paired with the NS ground state of the right but rather with the first excited state. We are therefore looking for physical states of the form

$$|\Psi_s\rangle = |p^\rho\rangle \otimes |\alpha\rangle_{\text{R}} u_\alpha{}^\mu \otimes (\psi_\mu)_{-1/2}\widetilde{|0\rangle}_{\text{NS}} \otimes |-\rangle_{\text{R}}^{sg} \otimes \widetilde{|-\rangle}_{\text{NS}}^{\widetilde{sg}} \,. \tag{7.96a}$$

The BRST constraint $(Q_{\text{B}} + \widetilde{Q}_{\text{B}})|\Psi_s\rangle = 0$ gives first the terms

$$c_0(L_0^x + L_0^\psi - 5/8)|\Psi_1\rangle + \widetilde{c}_0(\widetilde{L}_0^x + \widetilde{L}_0^\psi - 1/2)|\Psi_1\rangle = 0 \implies m^2 = -p^2 = 0\,, \tag{7.97}$$

hence these states are massless. They transform as vector-spinors of the space-time Lorentz group. Next we have

$$\gamma_0 G_0^{\psi,x}|\Psi_s\rangle = 0 \tag{7.98}$$

Using $G_0 = \eta_{\rho\sigma}\alpha_0^\rho\psi_0^\sigma$, one finds that

$$p_\rho \Gamma^\rho{}_{\dot{\alpha}}{}^{\alpha}\, u_\alpha^\mu = 0 \tag{7.99}$$

which is essentially the same as the Dirac equation. Next we have

$$\widetilde{\gamma}_{-1/2}\widetilde{G}_{1/2}^{\psi,x}|\Psi_a\rangle = 0 \tag{7.100}$$

which gives as before

$$p_\mu u_\alpha^\mu = 0 \tag{7.101}$$

We now look at the spurious states. On the left, being already in the ground state there is no freedom left. On the right one can do the same as in the NS-NS sector, so consider a state

$$(Q_B + \widetilde{Q}_B)|p^\rho\rangle \otimes |\alpha\rangle_R v_\alpha \otimes \widetilde{\beta}_{-1/2}\widetilde{|0\rangle}_{NS} \otimes |-\rangle_R^{sg} \otimes \widetilde{|-\rangle}_{NS}^{\widetilde{sg}} . \tag{7.102}$$

which gives the equivalence relation

$$u^\mu_\alpha \sim u^\mu_\alpha + p^\mu v_\alpha \ , \quad p_\rho \Gamma^\rho{}_{\dot\alpha}{}^\alpha v_\alpha = 0 \ ,. \tag{7.103}$$

To summarize, one gets a massless particle whose polarization is a vector-spinor u^μ_α that satisfies:

$$\boxed{p_\rho \Gamma^\rho{}_{\dot\alpha}{}^\alpha u^\mu_\alpha = 0 \ , \quad p_\mu u^\mu_\alpha = 0 \ , \quad u^\mu_\alpha \sim u^\mu_\alpha + p^\mu v_\alpha \ , \quad p_\rho \Gamma^\rho{}_{\dot\alpha}{}^\alpha v_\alpha = 0 .} \tag{7.104}$$

In terms of irreducible representations of the Lorentz group, this vector-spinor decomposes first into a gamma-trace part

$$\lambda_{\dot\alpha} = (\Gamma_\mu)_{\dot\alpha}{}^\beta u^\mu_\beta, \tag{7.105}$$

which is a spinor, which is called a *dilatino*. Then, the gamma-traceless part of the vector-spinor,

$$\zeta^\mu_\alpha = u^\mu_\alpha - \frac{1}{10}(\Gamma^\mu)_\alpha{}^{\dot\beta}\lambda_{\dot\beta} \ , \tag{7.106}$$

corresponds to a *gravitino* in space-time, which is a massless particle of helicity 3/2, the gauge field associated with local supersymmetric transformations. The equivalence relation (7.103), that leaves the dilatino invariant by construction, corresponds to the associated gauge transformations, the local supersymmetry transformations in ten dimensions.

The R-NS sector contains another gravitino $\zeta^\mu_{\dot\alpha}$ and another dilatino λ_α, built on the conjugate spinor representation $|\dot\alpha\rangle_R$, with opposite chirality.

There exists naturally a sector with anti-periodic boundary conditions on the left and periodic boundary conditions on the right, the NS-R sector. The field content one gets at the massless level is exactly the same: a pair of gravitini $(\hat\zeta^\mu_\alpha, \hat\zeta^\mu_{\dot\alpha})$ and a pair of dilatini $(\hat\lambda_{\dot\alpha}, \hat\lambda_\alpha)$ transforming in conjugate spinorial representations of the space-time Lorentz group.

R-R sector

The massless states in the Ramond-Ramond sector are built out of the left and right Ramond ground states. Given that there are two choices of Lorentz irreducible representations on each side (spinor or conjugate spinor), one has overall four possibilities.

In terms of group theory, one has to decompose the tensor products of spinorial representations $\mathbf{16} \otimes \mathbf{16}$, $\mathbf{16} \otimes \mathbf{16}'$, $\mathbf{16}' \otimes \mathbf{16}$ and $\mathbf{16}' \otimes \mathbf{16}'$ into irreducible representations of the Lorentz group each of them corresponding to a massless field in space-time. Obviously it is enough to consider the first two combinations, the last two giving the same decompositions.

Using the properties of the Gamma-matrix algebra, one can show that, in ten dimensions, the smallest spinorial irreducible representation of the Lorentz group consists in Majorana-Weyl fermions, since one can impose the Majorana condition both for ψ and $\Gamma^{11}\psi$ at the same time, where $\Gamma^{11} = \Gamma^0 \cdots \Gamma^9$ is the chirality matrix in ten dimensions, which satisfies $(\Gamma^{11})^2 = 1$.

In order to construct bilinear combinations of spinors that transform under Lorentz transformations as tensors of irreducible representations, one defines first the totally antisymmetric product of Gamma matrices:

$$\Gamma^{\mu_1\,\mu_2\cdots\mu_p} = \frac{1}{p!}\sum_{\sigma\in\mathcal{S}_p} \mathrm{sgn}(\sigma)\Gamma^{\mu_{\sigma(1)}\cdots\mu_{\sigma(p)}}\,. \tag{7.107}$$

Whenever p is odd, $\Gamma^{\mu_1\,\mu_2\cdots\mu_p}$ maps a spinor to a conjugate spinor, while when p is even it maps a spinor to a spinor. Hence we have the following index structure:

$$\left(\Gamma^{\mu_1\cdots\mu_p}\right)_a{}^b = \begin{pmatrix} 0 & \left(\Gamma^{\mu_1\cdots\mu_p}\right)_\alpha{}^{\dot\beta} \\ \left(\Gamma^{\mu_1\cdots\mu_p}\right)_{\dot\alpha}{}^{\beta} & 0 \end{pmatrix},\ p \text{ odd}\,, \tag{7.108a}$$

$$\left(\Gamma^{\mu_1\cdots\mu_p}\right)_a{}^b = \begin{pmatrix} \left(\Gamma^{\mu_1\cdots\mu_p}\right)_\alpha{}^{\beta} & 0 \\ 0 & \left(\Gamma^{\mu_1\cdots\mu_p}\right)_{\dot\alpha}{}^{\dot\beta} \end{pmatrix},\ p \text{ even}\,. \tag{7.108b}$$

$$\tag{7.108c}$$

The charge conjugation matrix, defined by $C\Gamma^\mu C^{-1} = -(\Gamma^\mu)^T$, is used to raise spinorial indices (and is such that the Majorana conjugate of a spinor ψ is $\bar\psi = \psi^T C$). It has the following block-diagonal form in ten dimensions:

$$C^{ab} = \begin{pmatrix} 0 & C^{\alpha\dot\beta} \\ C^{\dot\alpha\beta} & 0 \end{pmatrix} \tag{7.109}$$

while C^{-1} can be similarly used to lower indices. Using the properties (7.108) and (7.109) gives

$$\left(\Gamma^{\mu_1\cdots\mu_p}C^{-1}\right)_{ab} = \begin{pmatrix} \left(\Gamma^{\mu_1\cdots\mu_p}\right)_{\alpha\beta} & 0 \\ 0 & \left(\Gamma^{\mu_1\cdots\mu_p}\right)_{\dot\alpha\dot\beta} \end{pmatrix},\ p \text{ odd}\,, \tag{7.110a}$$

$$\left(\Gamma^{\mu_1\cdots\mu_p}C^{-1}\right)_{ab} = \begin{pmatrix} 0 & \left(\Gamma^{\mu_1\cdots\mu_p}\right)_{\alpha\dot\beta} \\ \left(\bar\Gamma^{\mu_1\cdots\mu_p}\right)_{\dot\alpha\beta} & 0 \end{pmatrix},\ p \text{ even}\,. \tag{7.110b}$$

$$\tag{7.110c}$$

One deduces from eqn. (7.110) how to decompose spinor bilinears in ten dimensions into totally antisymmetric tensors. One has first

$$\zeta_\alpha\chi_\beta = \sum_{p\ \text{odd}} F_{\mu_1\cdots\mu_p}\left(\gamma^{\mu_1\cdots\mu_p}C^{-1}\right)_{\alpha\beta} \tag{7.111}$$

and then

$$\zeta_\alpha\chi_{\dot\beta} = \sum_{p\ \text{even}} F_{\mu_1\cdots\mu_p}\left(\gamma^{\mu_1\cdots\mu_p}C^{-1}\right)_{\alpha\dot\beta}\,. \tag{7.112}$$

Hence the product of representations $\mathbf{16}\otimes\mathbf{16}$ gives p-forms with p odd and $\mathbf{16}\otimes\mathbf{16}'$ gives p-forms with p even.

To find the number of independent terms in the sums (7.111) and (7.112), one notices the following gamma-matrix identity:

$$\Gamma^{\mu_1\cdots\mu_p}\Gamma^{11} = \frac{1}{(d-p)!}(-1)^{\frac{p(p+1)}{2}+1}\epsilon^{\mu_1\cdots\mu_d}\Gamma_{\mu_{p+1}\cdots\mu_d}\,, \tag{7.113}$$

which follows from the definition of Γ^{11}. Using this property, the sum over p odd in (7.111) can be restricted to only three independent terms, $p \in \{1,3,5\}$, and the last term gives a five-form satisfying a *self-duality* property:

$$F_{\mu_1\cdots\mu_5} = (\star F)_{\mu_1\cdots\mu_5} = \frac{1}{5!}\epsilon_{\mu_1\cdots\mu_{10}}F^{\mu_5\cdots\mu_{10}}\,. \tag{7.114}$$

In the same way, the sum over p even in (7.112) can be restricted to $p \in \{0,2,4\}$. In summary one can write

$\mathbf{16}\otimes\mathbf{16}$	$[1]+[3]+[5]_+$
$\mathbf{16}'\otimes\mathbf{16}'$	$[1]+[3]+[5]_+$
$\mathbf{16}\otimes\mathbf{16}'$	$[0]+[2]+[4]$
$\mathbf{16}'\otimes\mathbf{16}$	$[0]+[2]+[4]$

(7.115)

where $[p]$ means an antisymmetric p-form representation, and $[5]_+$ means that the corresponding tensor obeys the self-duality property (7.114).

Finally the physical states conditions on the Ramond ground states is obtained as follows. Notice first that acting with G_0, as in (7.98), amounts to multiply the spinorial polarization by $p_\mu\Gamma^\mu$. Hence one has the following condition, for instance for $\mathbf{16}\times\mathbf{16}$:

$$p_\rho(\Gamma^\rho)_{\dot\alpha}{}^{\beta}\,u_\beta v_\gamma = u_\alpha v_\beta p_\rho(\Gamma^\rho)_{\dot\gamma}{}^{\beta} = 0\,. \tag{7.116}$$

Using the identities

$$\Gamma^\rho\Gamma^{\mu_1\cdots\mu_p} = \Gamma^{\rho\mu_1\cdots\mu_p} - \tfrac{1}{(p-1)!}\eta^{\rho[\mu_1}\Gamma^{\mu_2\cdots\mu_p]} \tag{7.117a}$$

$$\Gamma^{\mu_1\cdots\mu_p}\Gamma^\rho = \Gamma^{\mu_1\cdots\mu_p\rho} - \tfrac{1}{(p-1)!}\eta^{\rho[\mu_p}\Gamma^{\mu_2\cdots\mu_{p-1}]} \tag{7.117b}$$

(7.117c)

in the decomposition (7.111), and $C\Gamma^\mu C^{-1} = -(\Gamma^\mu)^T$, leads to

$$p^{[\rho}F^{\mu_1\cdots\mu_p]} = 0 \quad ,\, p_\rho F^{\rho\mu_2\cdots\mu_p} = 0\,. \tag{7.118}$$

One obtains therefore the Bianchi identities

$$dF = 0\,, \quad d\star F = 0\,. \tag{7.119}$$

In other words, each differential p-form F is the field strength of a $(p-1)$-form C, *i.e.*

$$F = dC\,, \tag{7.120}$$

with an associated Abelian gauge symmetry $C \sim C + d\Lambda$ where Λ is a $(p-1)$-form, called a R-R form.

The fact that the physical states of the string correspond to the field strengths F of the R-R forms rather than to the potential themselves indicate that the fundamental strings are not charged under the corresponding gauge symmetries.

7.4 Type II superstring theories

The full set of physical states constructed from the Neveu-Schwarz and Ramond vacua for the left- and right-moving sectors does not give rise to a satisfactory physical theory as contains a tachyon in the NS-NS sector.

The solution to this problem is to restrict the space of states to a consistent subset that does not contain the tachyon. This procedure, known as the *Gliozzi-Olive-Scherk projection* (GSO projection) succeeds to do this, while guaranteeing supersymmetry in space-time, as suggested by the existence of space-time gravitini in the massless sector.

7.4.1 Generalized fermion number

In the ψ^μ conformal field theory, we have defined the operator $(-)^F$, the generalization of the chirality matrix of the Clifford algebra, that anticommutes with all fermionic modes and as a consequence with the modes of the supercurrent:

$$\{(-)^F, \psi^\mu_n\} = 0 \ , \quad \{(-)^F, G_n\} = 0 \ , \quad \forall n \in \mathbb{Z} + \frac{1-a}{2} \,, \tag{7.121}$$

This operator separates states in representations of the superconformal algebra in two halves. Its eigenvalue is equal to one for states with even fermion number (*i.e.* with an even number of fermionic creation operators) and to minus one for states with odd fermion number (*i.e.* with an odd number of fermionic creation operators).

The full projection on the space of physical states should be compatible with the BRST symmetry, therefore the generalization of the operator $(-)^F$ to the full theory, noted $(-1)^{\mathcal{F}}$, should commute with the BRST currents (7.78). We therefore define the action of this operator in the ghost and superghost system as:

- The operator $(-1)^{\mathcal{F}}$ commutes with all modes of the (b,c) fields; note that the operator we are defining here is different from the ghost number operator discussed in section 6.3.2.

- The operator $(-1)^{\mathcal{F}}$ anticommutes with all modes of the (β,γ) fields, which have therefore odd charge w.r.t. $\mathcal{F}$. A proper way to define the action of $(-1)^{\mathcal{F}}$ of the (β,γ) CFT is to identify $\mathcal{F}$ with the charge under the superghost current (7.35) mod two.

As Polchinski remarked in his textbook, for this reason $\mathcal{F}$ should be called a *spinor number* rather than a fermion number, as it cares more about spin than about statistics.

Neveu-Schwarz sector

The charge of the Neveu-Schwarz vacuum can be deduced from the charge associated with the superghost current. Inserting the mode expansion (7.67) into the superghost number current (7.35), and using the same technique as in eqn. (4.40) and below it, one finds

$$Q_{sg} = \oint \frac{dz}{2i\pi} j_{sg}(z) = -\sum_{r\in\mathbb{N}+\frac{3}{2}} \beta_{-r}\gamma_r - \sum_{r\in\mathbb{N}-\frac{1}{2}} \gamma_{-r}\beta_r \,. \tag{7.122}$$

One has therefore

$$Q_{sg}|-\rangle_{\text{NS}} = -\gamma_{1/2}\beta_{-1/2}|-\rangle_{\text{NS}} = |-\rangle_{\text{NS}} \tag{7.123}$$

hence

$$(-1)^{\mathcal{F}}|-\rangle_{\text{NS}} = -|-\rangle_{\text{NS}} \,. \tag{7.124}$$

We define in the same way an operator $(-)^{\widetilde{\mathcal{F}}}$ acting on the states of the right-moving fermions and superghosts, which satisfies

$$(-1)^{\widetilde{\mathcal{F}}}\widetilde{|-\rangle}_{\text{NS}} = -\widetilde{|-\rangle}_{\text{NS}} \,. \tag{7.125}$$

One can then split the superconformal representation built by acting with the oscillators on the $|-\rangle_{\text{NS}}$ vacuum in two halves, with odd and even fermion number respectively, denoted NS_- and NS_+. Since all states in the sector NS_+ (resp. NS_-) have the same $(-)^{\mathcal{F}}$ eigenvalue, they should be obtained from the NS vacuum with an odd (resp. even) number of fermionic creation operators $\psi^{\mu}_{-n-1/2}$. These two sectors correspond respectively to the vector and trivial representation of the $SO(1,9)$ current algebra as discussed in subsection 4.2.3.

As a consequence of this construction, the conformal dimensions of states in these two sectors will have a different modding

$$h \in \frac{\alpha'}{4}p^2 + \mathbb{N} \quad , \quad \mathsf{NS}_+ \text{ sector,} \tag{7.126a}$$

$$h \in \frac{\alpha'}{4}p^2 + \mathbb{N} - \frac{1}{2} \, , \quad \mathsf{NS}_- \text{ sector.} \tag{7.126b}$$

Ramond sector

For the Ramond ground states, since the operator $(-1)^{\mathcal{F}}$ generalizes the chirality matrix – which anti-commutes with all Gamma-matrices hence all zero modes ψ^{μ}_0 – to all oscillators modes ψ^{μ}_n, all is needed is that the ground states of the combined ψ^{μ} and (β,γ) theories of opposite space-time chiralities have opposite eigenvalues:

$$(-)^{\mathcal{F}}|\alpha\rangle_{\text{R}} \otimes |-\rangle^{sg}_{\text{R}} = |\alpha\rangle_{\text{R}} \otimes |-\rangle^{sg}_{\text{R}} \,, \tag{7.127a}$$

$$(-)^{\mathcal{F}}|\dot{\alpha}\rangle_{\text{R}} \otimes |-\rangle^{sg}_{\text{R}} = -|\dot{\alpha}\rangle_{\text{R}} \otimes |-\rangle^{sg}_{\text{R}} \,. \tag{7.127b}$$

Hence the ground states in the **16** of $SO(1,9)$ have even fermion number and the ground states in the **16′** of $SO(1,9)$ have odd fermion number. Obviously taking the opposite convention for both the left- and right-movers at the same time would not change the physical content of the theory.

In the same way as before, beyond the ground state, the whole superconformal representation built out of the Ramond vacuum splits into two sectors of opposite $(-1)^{\mathcal{F}}$ eigenvalue, denoted by R_+ and R_- respectively.

Contrary to what happens with Neveu-Schwarz boundary conditions, the conformal dimensions in the sectors R_+ and R_- have the same modding:

$$h \in \frac{\alpha'}{4}p^2 + \mathbb{N}\ , \quad \mathsf{R}_+ \text{ sector,} \tag{7.128a}$$

$$h \in \frac{\alpha'}{4}p^2 + \mathbb{N}\ , \quad \mathsf{R}_- \text{ sector.} \tag{7.128b}$$

Therefore one could choose one for the left-movers and the other one for the right-movers without spoiling the level-matching condition.

7.4.2 Type IIA and IIB superstring theories

Before defining consistent space-time theories from the fermionic string, let us first summarize what we have found so far in this chapter. In the various sectors of the theory (NS-NS, R-NS, NS-R and R-R) we have obtained the following BRST-invariant and level-matched states at the lowest levels that we list together with their properties (omitting the ghost/superghost part for clarity):

mass^2	state	Space-time	$(-)^{\mathcal{F}}$	$(-)^{\widetilde{\mathcal{F}}}$
$-2/\alpha'$	$\lvert p^\mu\rangle_{\text{NS-NS}}$	tachyon	-1	-1
0	$\psi^\mu_{-1/2}\widetilde{\psi}^\nu_{-1/2}\lvert p^\mu\rangle_{\text{NS-NS}}$	$g_{\mu\nu}$, $b_{\mu\nu}$, Φ	1	1
0	$\lvert\alpha;p^\mu\rangle_{\text{R}}\otimes\widetilde{\psi}^\mu_{-1/2}\widetilde{\lvert p^\mu\rangle}_{\text{NS}}$	ζ^μ_α, $\lambda_{\dot\alpha}$	1	1
0	$\lvert\dot\alpha;p^\mu\rangle_{\text{R}}\otimes\widetilde{\psi}^\mu_{-1/2}\widetilde{\lvert p^\mu\rangle}_{\text{NS}}$	$\zeta^\mu_{\dot\alpha}$, λ_α	-1	1
0	$\psi^\mu_{-1/2}\lvert p^\mu\rangle_{\text{NS}}\otimes\widetilde{\lvert\alpha;p^\mu\rangle}_{\text{R}}$	$\widehat{\zeta}^\mu_\alpha$, $\widehat{\lambda}_{\dot\alpha}$	1	1
0	$\psi^\mu_{-1/2}\lvert p^\mu\rangle_{\text{NS}}\otimes\widetilde{\lvert\dot\alpha;p^\mu\rangle}_{\text{R}}$	$\widehat{\zeta}^\mu_{\dot\alpha}$, $\widehat{\lambda}_\alpha$	1	-1
0	$\lvert\alpha;p^\mu\rangle_{\text{R}}\otimes\widetilde{\lvert\beta;p^\mu\rangle}_{\text{R}}$	$[1]+[3]+[5]_+$	1	1
0	$\lvert\dot\beta;p^\mu\rangle_{\text{R}}\otimes\widetilde{\lvert\dot\beta;p^\mu\rangle}_{\text{R}}$	$[1]+[3]+[5]_+$	-1	-1
0	$\lvert\alpha;p^\mu\rangle_{\text{R}}\otimes\widetilde{\lvert\dot\beta;p^\mu\rangle}_{\text{R}}$	$[0]+[2]+[4]$	1	-1
0	$\lvert\dot\alpha;p^\mu\rangle_{\text{R}}\otimes\widetilde{\lvert\beta;p^\mu\rangle}_{\text{R}}$	$[0]+[2]+[4]$	-1	1

(7.129)

In order to get rid of the tachyon vacuum (7.83), one can try to project onto states with $(-1)^{\mathcal{F}}=1$ in the left Neveu-Schwarz sector and $(-1)^{\widetilde{\mathcal{F}}}=1$ in the right Neveu-Schwarz sector, *i.e.* to consider the sector $(\mathsf{NS}_+,\mathsf{NS}_+)$. Performing this projection in the NS-NS sector alone is not a consistent choice however.

A consistent choice of projection in the various sectors of the theory should satisfy at least the following constraints:

- The level matching condition should give a non-empty result; this rules out the sectors $(\mathsf{NS}_-,\mathsf{NS}_+)$ and $(\mathsf{NS}_+,\mathsf{NS}_-)$

- States that are projected out should not appear in the operator product expansion of the remaining ones
- All local operators that are kept in the spectrum should have OPEs between themselves without branch cuts, with only integer powers of $(z' - z)$ and no half-integer ones
- The one-loop vacuum amplitude should be modular invariant, *i.e.* the integrand should be invariant under the modular group of the torus $\mathsf{PSL}(2,\mathbb{Z})$.

All these constraints, as well as higher-loop modular invariance, have a common set of solutions, using a projection known as *GSO projection* that keeps only half of the states that were described so far.

We will first present these solutions, before showing that modular invariance is satisfied for them. In both of the consistent theories, the projection into the NS sectors should be accompanied with a projection into the R sectors.

Type IIB superstring theory

The first consistent superstring theory is the IIB superstring theory, which consists in considering states in the following sectors:

Type IIB: (7.130)

sector	massless states
$(\mathrm{NS}_+, \mathrm{NS}_+)$	$g_{\mu\nu},\ b_{\mu\nu},\ \Phi$
$(\mathrm{R}_+, \mathrm{NS}_+)$	$\zeta^\mu_\alpha, \lambda_{\dot\alpha}$
$(\mathrm{NS}_+, \mathrm{R}_+)$	$\hat\zeta^\mu_\alpha, \hat\lambda_{\dot\alpha}$
$(\mathrm{R}_+, \mathrm{R}_+)$	$[1] + [3] + [5]_+$

The most interesting part of this massless spectrum is the existence of a pair of gravitini ζ^μ_α and $\hat\zeta^\mu_\alpha$, which have the same spinor chirality. These states indicate that the space-time theory is actually invariant under local space-time supersymmetry, more explicitely under $\mathcal{N} = (2,0)$ supersymmetry since there exists two local supersymmetric transformations parametrized with spinors of identical chirality.

The low-energy dynamics of the massless fields of bosonic string theory was captured by a space-time action (5.6) for the gravition, B-field and dilaton. This result was coming from asking conformal invariance of the two-dimensional worldsheet theory, and could also be tested by taking the low-energy limit of the S-matrix elements between massless fields.

In the same way, the low-energy dynamics of type IIB string theory is captured by a ten-dimensional theory with local supersymmetry and diffeomorphism invariance, in other words a *supergravity* action for the massless fields, unvariant under two local supersymmmetries of identical chirality, as it should. This supergravity is known as *type IIB supergravity* in ten dimensions.

Type IIA superstring theory

The second consistent superstring theory is the IIA superstring theory, which consists in considering states in the following sectors:

Type IIA: (7.131)

sector	massless states
$(\mathsf{NS}_+, \mathsf{NS}_+)$	$g_{\mu\nu}$, $b_{\mu\nu}$, Φ
$(\mathsf{R}_+, \mathsf{NS}_+)$	$\zeta^\mu_\alpha, \lambda_{\dot\alpha}$
$(\mathsf{NS}_+, \mathsf{R}_-)$	$\hat\zeta^\mu_{\dot\alpha}, \hat\lambda_\alpha$
$(\mathsf{R}_+, \mathsf{R}_-)$	$[0] + [2] + [4]$

In this case, one has a pair of gravitini ζ^μ_α and $\hat\zeta^\mu_{\dot\alpha}$, which have the opposite spinor chirality. These states indicate that the space-time theory is actually invariant under local space-time supersymmetry, more explicitely under $\mathcal{N} = (1, 1)$ supersymmetry since there exists two local supersymmetry transformations parametrized with spinors of opposite chiralities. The low energy dynamics of the theory is captured by *type IIA supergravity* in ten dimensions.

Light-cone gauge and physical degrees of freedom

As for the bosonic string, one can solve for the transversality constraints on the massless physical states. A similar conclusion is obtained by quantizing the theory in the light-cone gauge, instead of using the covariant BRST formalism chosen in these notes. From both point of views, the modes from the fields $\psi_0 \pm \psi^1$ are removed and physical states are classified in terms of representations of the little group $SO(8)$.

The relevant representations are the vector $\mathbf{8}_v$, the spinor $\mathbf{8}_s$ and the conjugate spinor $\mathbf{8}_c$. Notice that these three representations have the same dimension, a property known as triality. The relevant tensor product of representations for the type IIB string are then:

Type IIB: (7.132)

sector	$SO(8)$ representations
$(\mathsf{NS}_+, \mathsf{NS}_+)$	$\mathbf{8}_v \otimes \mathbf{8}_v = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$
$(\mathsf{R}_+, \mathsf{NS}_+)$	$\mathbf{8}_s \otimes \mathbf{8}_v = \mathbf{8}_c + \mathbf{56}_s$
$(\mathsf{NS}_+, \mathsf{R}_+)$	$\mathbf{8}_v \otimes \mathbf{8}_s = \mathbf{8}_c + \mathbf{56}_s$
$(\mathsf{R}_+, \mathsf{R}_+)$	$\mathbf{8}_s \otimes \mathbf{8}_s = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+$

Here $\mathbf{35}_+$ corresponds to the self-dual 4-form representation, $\mathbf{35}$ to the symmetric traceless rank two tensor representation and $\mathbf{56}_s$ to the gamma-traceless vector-spinor of positive chirality. One can do the same for type IIA and get:

Type IIA: (7.133)

sector	$SO(8)$ representations
$(\mathsf{NS}_+, \mathsf{NS}_+)$	$\mathbf{8}_v \otimes \mathbf{8}_v = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$
$(\mathsf{R}_+, \mathsf{NS}_+)$	$\mathbf{8}_s \otimes \mathbf{8}_v = \mathbf{8}_c + \mathbf{56}_s$
$(\mathsf{NS}_+, \mathsf{R}_-)$	$\mathbf{8}_v \otimes \mathbf{8}_c = \mathbf{8}_s + \mathbf{56}_c$
$(\mathsf{R}_+, \mathsf{R}_-)$	$\mathbf{8}_s \otimes \mathbf{8}_c = \mathbf{8}_v \oplus \mathbf{56}_t$

where $\mathbf{56}_t$ refers to the antisymmetric 3-form and $\mathbf{56}_c$ to the gamma-traceless vector-spinor of negative chirality.

7.5 One-loop vacuum amplitudes

A strong consistency constraint for the superstring theories that we constructed, type IIA and type IIB, is provided by modular invariance of the one-loop vacuum amplitude.

The integrand of the modular integral over the torus modulus is the product of the contribution from the x^μ and (b,c) ghosts, that we computed earlier, giving (for $d=10$)

$$Z^{x,g}(\tau,\bar\tau) = \frac{iV_{10}}{(4\pi^2\alpha')^5}\frac{1}{\left(\sqrt{\tau_2}\eta(\tau)\bar\eta(\bar\tau)\right)^8} \tag{7.134}$$

and the contribution from the fermionic fields ψ^μ and from the (β,γ) superghosts that we consider below.

The one-loop amplitude for both the fermions and the superghosts split into sectors according to the boundary conditions for the fields. The Euclidean two-torus is parameterized by a pair of coordinates σ^1 and σ^2 with periodicities $(\sigma^1,\sigma^2)\sim(\sigma^1,\sigma^2)+2\pi(n+m\tau_1,m\tau_2)$. Along both of these one-cycles, one can specify whether the fields are periodic and anti-periodic.

The periodicity along the space cycle parametrized by σ^1 was discussed already. It corresponds to the choice of Ramond (periodic) or Neveu-Schwarz (anti-periodic) boundary conditions:

$$\psi^\mu(\sigma_1+2\pi,\sigma_2) = (-1)^{1-a}\psi^\mu(\sigma^1,\sigma^2)\,, \tag{7.135}$$

with $a=0$ (resp. $a=1$) in the NS (resp. R) sector.

The periodicty along the Euclidean time direction was also indirectly considered in subsection 6.3.2. We have seen that, *for fermionic fields* the path integral with periodic boundary conditions was corresponding to the trace of $(-1)^F\exp-\beta\hat H$, while the partition function itself, or trace over $\exp-\beta\hat H$, was given by the path integral with anti-periodic boundary conditions. In other words, one considers the periodicity

$$\psi^\mu(\sigma_1+2\pi\tau_1,\sigma_2+2\pi\tau_2) = (-1)^{1-b}\psi^\mu(\sigma^1,\sigma^2)\,, \tag{7.136}$$

for a path integral corresponding in the Hamiltonian formalism to an insertion of $(-1)^{bF}$ in the trace.

7.5.1 Zero-modes

At a more abstract level, when one defines two-dimensional fermions on a genus zero surface, one needs to specify the *spin structure*, which is a four-fold choice of periodicities along both one-cycles: (A,A), (A,P), (P,A) and (P,P) where A stands for anti-periodic and P stands for periodic. All the first three case are called even spin structures while the last one is called an odd spin structure (even or odd refers to the number of zero modes of the Dirac operator mod two).

In terms of super-space coordinates (z,θ), a two-torus is defined by the following identifications

$$(z,\theta)\sim(z+2\pi,\epsilon\theta)\sim(z+2\pi\tau,\epsilon'\theta)\ ,\quad \epsilon,\epsilon'\in\{-1,1\}. \tag{7.137}$$

where ϵ and ϵ' define the spin structure.

An important aspect of the computation of the one-loop vacuum amplitude for the bosonic string was the presence of zero-modes for the ghosts fields, associated with the moduli and the conformal Killing vectors on the two-torus. In the present case, there cannot be any fermionic zero mode for the even spin structures, because of the anti-periodicity along at least a one-cycle, while for the odd spin structure there exists a single zero-mode, which is a constant spinor. Accordingly, there exists, for odd spin structure, a single fermionic supermodulus ρ on the two-torus, which can be used to define a more general periodicity

$$(z,\theta) \sim (z+2\pi,\theta) \sim (z+2\pi\tau+\theta\rho,\theta+\rho) \tag{7.138}$$

as well as a superconformal Killing spinor, which corresponds to a spinorial shift ξ:

$$(z,\theta) \mapsto (z+\theta\xi,\theta+\xi)\,. \tag{7.139}$$

This moduli and superconformal Killing spinor cannot exist for even spin structures, as $\theta\rho$ and $\theta\xi$ would not have then the right periodicity properties for a bosonic coordinate.

7.5.2 Partition functions in the NS sector

Let us consider first the partition function for a single chiral fermion $\psi(z)$. We define the following partition functions for each spin structure:

$$Z\left[{0 \atop b}\right] = \mathrm{Tr}_{\mathrm{NS}}\left[(-1)^{bF} q^{L_0-1/48}\right]\,, \tag{7.140a}$$
$$Z\left[{1 \atop b}\right] = \mathrm{Tr}_{\mathrm{R}}\left[(-1)^{bF} q^{L_0-1/48}\right]\,. \tag{7.140b}$$

We start by the partition function with (A,A) boundary conditions, *i.e* $\mathrm{Tr}_{\mathrm{NS}} e^{L_0-1/48}$, which is easy to compute. A general state is of the form

$$\cdots \left(\psi_{-3/2}\right)^{N_2} \left(\psi_{-1/2}\right)^{N_1} |0\rangle_{\mathrm{NS}} \tag{7.146}$$

where $N_i \in \{0,1\}$. It gives the following result

$$Z\left[{0 \atop 0}\right](\tau) = q^{-1/48}\left(1+q^{1/2}\right)\left(1+q^{3/2}\right)\cdots = q^{-1/48}\prod_{n=0}^{\infty}\left(1+q^{n+1/2}\right)\,. \tag{7.147}$$

In terms of the theta-functions (7.141) and the Dedekind eta-function (6.55) it can be written as

$$Z\left[{0 \atop 0}\right](\tau) = \sqrt{\frac{\vartheta\left[{0 \atop 0}\right](\tau,0)}{\eta(\tau)}}\,. \tag{7.148}$$

Next we consider the partition function with (A,P) boundary conditions, which means a trace in the NS sector with $(-)^F$ inserted inside the trace. We still consider states of the

The **Jacobi theta-function** with characteristics is an analytic function of two complex variables $\tau \in \mathbb{H}$ and $\upsilon \in \mathbb{C}$, depending on two parameters $\alpha, \beta \in \mathbb{R}$ and defined as

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau,\upsilon) = e^{-i\pi a(\upsilon-\frac{b}{2})} q^{\frac{a^2}{8}} \prod_{n=1}^{\infty} \left(1 + e^{-i\pi b} e^{2i\pi\upsilon} q^{n-\frac{1+a}{2}}\right) \left(1 + e^{+i\pi b} e^{-2i\pi\upsilon} q^{n-\frac{1-a}{2}}\right) (1-q^n) \tag{7.141}$$

From its definition it is obvious that

$$\vartheta\begin{bmatrix}1\\1\end{bmatrix}(\tau,0) = 0\,. \tag{7.142}$$

The theta function obeys the modular transformations

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau+1,\upsilon) = e^{-\frac{i\pi}{4}a(a-2)} \vartheta\begin{bmatrix}a\\a+b-1\end{bmatrix}(\tau,\upsilon)\,, \tag{7.143a}$$

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(-1/\tau,\upsilon/\tau) = \sqrt{-i\tau}\, e^{\frac{i\pi}{2}ab+\frac{i\pi}{\tau}\upsilon^2} \vartheta\begin{bmatrix}b\\-a\end{bmatrix}(\tau,\upsilon)\,. \tag{7.143b}$$

It has also the periodicity properties

$$\vartheta\begin{bmatrix}a+2m\\b+2n\end{bmatrix}(\tau,\upsilon) = e^{i\pi n a}\, \vartheta\begin{bmatrix}a+2m\\b+2n\end{bmatrix}(\tau,\upsilon)\,, \quad m,n \in \mathbb{Z}\,. \tag{7.144}$$

Finally it satifies the completely non-trivial *Jacobi abstruse identity*

$$\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4(\tau,\upsilon) - \vartheta\begin{bmatrix}0\\1\end{bmatrix}^4(\tau,\upsilon) - \vartheta\begin{bmatrix}1\\0\end{bmatrix}^4(\tau,\upsilon) = -\vartheta\begin{bmatrix}1\\1\end{bmatrix}^4(\tau,\upsilon)\,. \tag{7.145}$$

We often use the notation $\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau) := \vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau,0)$.

form (7.146), however one gets a minus sign for each oscillator mode present. Therefore one gets

$$Z\begin{bmatrix}0\\1\end{bmatrix}(\tau) = q^{-1/48}\left(1-q^{1/2}\right)\left(1-q^{3/2}\right)\cdots = q^{-1/48}\prod_{n=0}^{\infty}\left(1-q^{n+1/2}\right)\,, \tag{7.149}$$

which can be written as

$$Z\begin{bmatrix}0\\1\end{bmatrix}(\tau) = \sqrt{\frac{\vartheta\begin{bmatrix}0\\1\end{bmatrix}(\tau)}{\eta(\tau)}}\,. \tag{7.150}$$

A partition function in the NS_+ sector is then defined as

$$\mathrm{Tr}_{\text{NS}}\left(\frac{1+(-1)^F}{2}q^{L_0-c/24}\right)\,. \tag{7.151}$$

Then we move to the (β,γ) ghosts, which are, in the NS sector, bosons with anti-periodic boundary conditions. We consider the trace with a $(-)^{b\mathcal{F}}$ insertion, where $b \in \{0,1\}$. General states are of the form

$$\cdots(\beta_{-3/2})^{M_2}(\gamma_{-3/2})^{N_2}(\beta_{-1/2})^{M_1}(\gamma_{-1/2})^{N_1}|-\rangle_{\text{NS}}\,. \tag{7.152}$$

The NS vacuum contributes to the partition function as

$$q^{1/2-c/24}(-1)^b = q^{1/24}(-1)^b\,, \tag{7.153}$$

as the vacuum as superghost charge one and dimension one-half.

Since the (β,γ) fields are bosons, the path integral with anti-periodic boundary conditions ($b=1$) along Euclidean time corresponds to the trace with a $(-)^{\mathcal{F}}$ insertion, while periodic boundary conditions ($b=0$) corresponds to the trace without a $(-)^{\mathcal{F}}$ insertion. Here $\mathcal{F}$ is identified with the superghost number mod two, hence counts the number of superghost oscillator modes mod two.

We consider then, for instance, the contribution from $\gamma_{-1/2}$ oscillators to the partition function with $(-1)^{b\mathcal{F}}$ inserted. One gets

$$1+(-1)^{b+1}q^{1/2}+q+(-1)^{b+1}q^{3/2}-\cdots = \frac{1}{1+(-1)^b q^{1/2}}\,. \tag{7.154}$$

Doing the same for all oscillators of β and γ give s

$$\begin{aligned} Z_{sg}\left[{0 \atop b}\right] &= (-1)^b q^{1/24}\left(\frac{1}{1+(-1)^b q^{1/2}}\right)^2\left(\frac{1}{1+(-1)^b q^{3/2}}\right)^2\cdots \\ &= (-1)^b q^{1/24}\prod_{n=1}^{\infty}\left(\frac{1}{1+(-1)^b q^{n-1/2}}\right)^2. \end{aligned} \tag{7.155}$$

In terms of the theta-functions, this is

$$Z_{sg}\left[{0 \atop b}\right] = (-1)^b\frac{\eta(\tau)}{\vartheta\left[{0 \atop b}\right](\tau)}\,. \tag{7.156}$$

We can give describe the combined contribution to the partition function of the $D=10$ fermions and (β,γ) ghosts in the NS_+ sector. Putting together the contribution (7.150), (7.147) and (7.156) one obtains

$$\begin{aligned} Z^{\psi,sg}_{NS_+}(\tau) &= \frac{1}{2}\left\{\left(\frac{\vartheta\left[{0 \atop 0}\right](\tau)}{\eta(\tau)}\right)^5\frac{\eta(\tau)}{\vartheta\left[{0 \atop b}\right](\tau)} - \left(\frac{\vartheta\left[{0 \atop 1}\right](\tau)}{\eta(\tau)}\right)^5\frac{\eta(\tau)}{\vartheta\left[{0 \atop 1}\right](\tau)}\right\} \\ &= \frac{1}{2}\left(\frac{\vartheta\left[{0 \atop 0}\right](\tau)}{\eta(\tau)}\right)^4 - \frac{1}{2}\left(\frac{\vartheta\left[{0 \atop 1}\right](\tau)}{\eta(\tau)}\right)^4. \end{aligned} \tag{7.157}$$

We remark that this is exactly the same as the contribution of eight fermions, with the projection $\frac{1}{2}(1-(-1)^F)$. This can be understood as follows. The superghosts (β,γ) are actually removing the contribution of two towers of ψ^μ oscillators, much as the (b,c) ghosts remove two towers of x^μ oscillators. If we have used the light-cone gauge, we would have found the same result as only transverse oscillators remain, the light-cone oscillators along $\psi^0\pm\psi^1$ being removed by the gauge choice.

What remains after the projection is the tower of oscillators built out of the affine primary states $\{\psi^i_{-1/2}|0\rangle_{\rm NS},\, i = 2,\ldots,10\}$, containing only states with $N \in \mathbb{N}+1/2$. This is precisely the character in the vector representation of the affine $SO(8)$ algebra, as discussed in subsection 4.2.3. Indeed in this context one acts on the affine primary in the vector representations with modes of the currents J^{ij}, which are integer-modded.

7.5.3 Partition functions in the R sector

Let us consider the partition function in the Ramond sector, *i.e.* with periodic boundary conditions along the space circle. We start by considering directly $d = 10$ fermions with PA spin structure, which is

$$Z\begin{bmatrix}1\\0\end{bmatrix}(\tau) = \mathrm{Tr}_{\rm R}\, q^{L_0-5/24}\,. \tag{7.158}$$

Since we haven't performed yet the GSO projection, the ground state $|a\rangle_{\rm R}$ is in the reducible Majorana representation of the ten-dimensional Clifford algebra, which has dimension $2^5 = 32$; it has conformal dimension $h = 5/8$ (see eq. (4.140). The contribution from the ground states to the partition function, taking into account their degeneracy is therefore $32q^{5/8-5/24} = (2q^{1/12})^5$.

One way to analyze this result is to consider that all the Ramond ground states are obtained from the highest weight state in the spinorial representation by acting with the lowering operators $(-\psi_0^0+\psi_0^1)$, $(\psi_0^2 - i\psi_0^3)$, etc...

Adding the contributions from all the oscillators ψ^μ_{-n} with $n \geqslant 1$ is easy, and we get the result

$$Z_\psi\begin{bmatrix}1\\0\end{bmatrix}(\tau) = (2q^{1/12})^5\Big((1+q)(1+q^2)\cdots\Big)^{10} = \left(2q^{1/12}\prod_{n=1}^{\infty}(1+q^n)^2\right)^5\,. \tag{7.159}$$

In terms of theta-functions, this is

$$Z_\psi\begin{bmatrix}1\\0\end{bmatrix}(\tau) = \left(\frac{\vartheta\begin{bmatrix}1\\0\end{bmatrix}(\tau)}{\eta(\tau)}\right)^5\,. \tag{7.160}$$

For the set of ten fermions with PP spin structure, the result of the computation is deceptively simple. One is interested in computing:

$$Z_\psi\begin{bmatrix}1\\1\end{bmatrix}(\tau) = \mathrm{Tr}_{\rm R}(-1)^F q^{L_0-5/24}\,. \tag{7.161}$$

Remember that the operator $(-1)^F$ was an extension of the chirality matrix Γ^{11} to the full Hilbert space of the fermionic CFT. If we focus on the 32-dimensional ground state, we know that for each ground state of positive chirality there exists a ground state with opposite chirality. Hence the two contributions cancel each other and we are left with

$$Z_\psi\begin{bmatrix}1\\1\end{bmatrix}(\tau) = 0\,. \tag{7.162}$$

Another way to see this is that the explicit computation would give

$$Z_\psi\left[{}^1_1\right](\tau) = \left(\frac{\vartheta\left[{}^1_1\right](\tau)}{\eta(\tau)}\right)^5 = 0\,, \tag{7.163}$$

following eqn. (7.142).

Finally, we have to compute the contribution of the superghosts to the Ramond sector partition function. We focus on the PA spin structure, since the PP spin structure will not contribute to the full partition function at the end. The superghosts being bosons, anti-periodicity in Euclidean time means that we compute the trace with $(-1)^{\mathcal{F}}$ inserted.

The superghost Ramond vacuum $|-\rangle_{\text{R}}$ is defined in eq. (7.75a) and has conformal dimension $h = -3/8$, hence contributes to the trace as $q^{-11/24+3/8} = q^{-1/12}$. The problem in the present case is there there is an infinite degeneracy of the ground states, as any state of the form

$$(\gamma_0)^{N_0}|-\rangle_{\text{R}} \tag{7.164}$$

will have the same energy as the ground state. Taking into account the $(-1)^{\mathcal{F}}$ insertion, the contribution from all these states of conformal dimension $h = 3/8$ needs to be regularized, giving the result

$$\begin{aligned} q^{-1/12}\left(1 - 1 + 1 - 1 + \cdots\right) &= \lim_{\epsilon\to 0^+} q^{-1/12}\left(1 - q^\epsilon + q^{2\epsilon} - q^{3\epsilon} + \cdots\right) \\ &= \lim_{\epsilon\to 0^+} q^{-1/12}\frac{1}{1+q^\epsilon} \\ &= q^{-1/12}\,\frac{1}{2}\,. \end{aligned} \tag{7.165}$$

For the reader that doubts of this result, a more convincing argument will be given later.

From there we act with the various oscillators as usual. Because of the $(-1)^{\mathcal{F}}$ insertion, the action of γ_{-1} for instance gives the factor

$$1 - q + q^2 + \cdots = \frac{1}{1+q} \tag{7.166}$$

Adding all the contributions from the action of the oscillators β_{-n} and γ_{-n} with $n \in \mathbb{Z}_{>0}$, one reaches the result

$$Z_{sg}\left[{}^1_0\right](\tau) = \frac{1}{2q^{1/12}\prod_{n=1}^\infty(1+q^n)^2} \tag{7.167}$$

In terms of theta-functions, one obtains then

$$Z_{sg}\left[{}^1_0\right](\tau) = \frac{\eta(\tau)}{\vartheta\left[{}^1_0\right](\tau)}\,. \tag{7.168}$$

To conclude, the full contribution from the fermions and the superghosts (β,γ) in the R_+ and R_- sectors are the same, as the term with PP boundary conditions gives a vanishing result from eq. (7.142). One gets then

$$Z^{\psi,sg}_{\mathrm{R}_\pm}(\tau) = \frac{1}{2} Z_\psi \begin{bmatrix}1\\0\end{bmatrix}(\tau) Z_{sg} \begin{bmatrix}1\\0\end{bmatrix}(\tau) \tag{7.169}$$

$$= \frac{1}{2}\left(\frac{\vartheta\begin{bmatrix}1\\0\end{bmatrix}(\tau)}{\eta(\tau)}\right)^4 . \tag{7.170}$$

As before this result can be understood from the point of view of representation theory of the $SO(8)$ affine algebra. $Z_{\mathrm{R}_\pm}$ corresponds respectively to the characters of the spinor and conjugate spinor representations of $SO(8)$, which are both eight-dimensional, which is reflected in the degeneracy of the ground state in (7.169).

In order to discriminate the characters for the spinor and conjugate spinor representations, it is possible, from the point of view of representation theory, to consider a character for a non-trivial group element in the Cartan sugroup of $SO(8)$ rather than the identity. It amounts to give non-zero values to the υ argument in each theta-function. One can write then:

$$Z_{\mathrm{R}_\pm}(\tau, \upsilon_\ell) = \frac{1}{2\eta(\tau)^4}\left\{\prod_{\ell=1}^{4}\vartheta\begin{bmatrix}1\\0\end{bmatrix}(\tau,\upsilon_\ell) \pm \prod_{\ell=1}^{4}\vartheta\begin{bmatrix}1\\1\end{bmatrix}(\tau,\upsilon_\ell)\right\} . \tag{7.171}$$

7.5.4 Partition functions of type II string theories

The full partition function of type II string theories are obtained by adding the contributions from the various left and right sectors that we have considered before. For the type IIB superstring theory, we need to consider schematically

$$Z_{\mathrm{IIB}} = Z_{\mathrm{NS}_+,\mathrm{NS}_+} - Z_{\mathrm{R}_+,\mathrm{NS}_+} - Z_{\mathrm{NS}_+,\mathrm{R}_+} + Z_{\mathrm{R}_+,\mathrm{R}_+} . \tag{7.172}$$

Minus signs have be added to the second and third terms to respect spin-statistics: space-time fermions running into the loop should contribute to the partition function with a negative sign.

For the type IIA superstring theory, we have to reverse the GSO projection in the right Ramond sector and get:

$$Z_{\mathrm{IIA}} = Z_{\mathrm{NS}_+,\mathrm{NS}_+} - Z_{\mathrm{R}_+,\mathrm{NS}_+} - Z_{\mathrm{NS}_+,\mathrm{R}_-} + Z_{\mathrm{R}_+,\mathrm{R}_-} . \tag{7.173}$$

The fermion and superghost contribution to the partition function completely factorizes into its holomorphic and anti-holomorphic parts, given that the associated field theories are chiral. One has thus the following structure:

$$Z_{\mathrm{IIB}}(\tau,\bar\tau) = Z^{x,g}(\tau,\bar\tau)\Big(Z^{\psi,sg}_{\mathrm{NS}_+}(\tau) - Z^{\psi,sg}_{\mathrm{R}_+}(\tau)\Big)\Big(Z^{\tilde\psi,\tilde{sg}}_{\mathrm{NS}_+}(\bar\tau) - Z^{\tilde\psi,\tilde{sg}}_{\mathrm{R}_+}(\bar\tau)\Big) . \tag{7.174a}$$

$$Z_{\mathrm{IIA}}(\tau,\bar\tau) = Z^{x,g}(\tau,\bar\tau)\Big(Z^{\psi,sg}_{\mathrm{NS}_+}(\tau) - Z^{\psi,sg}_{\mathrm{R}_+}(\tau)\Big)\Big(Z^{\tilde\psi,\tilde{sg}}_{\mathrm{NS}_+}(\bar\tau) - Z^{\tilde\psi,\tilde{sg}}_{\mathrm{R}_-}(\bar\tau)\Big) . \tag{7.174b}$$

Gathering then the contributions (7.134), (7.157) and (7.169) one finds the following important result for the unintegrated partition function of type II superstring theories:

$$\boxed{Z_{\text{II}}(\tau,\bar\tau) = \frac{iV_{10}}{4(4\pi^2\alpha')^5}\frac{1}{\tau_2^4|\eta(\tau)|^{24}}\left|\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4(\tau) - \vartheta\begin{bmatrix}0\\1\end{bmatrix}^4(\tau) - \vartheta\begin{bmatrix}1\\0\end{bmatrix}^4(\tau)\right|^2} \tag{7.175}$$

Because the $R_\pm$ sectors give the same contribution, this result applies both to type IIB and type IIA superstring theories. The partition function proper is then given as for the bosonic string by

$$\mathcal{Z}_{\text{II}} = \int_{\mathcal{F}} \frac{d^2\tau}{4\tau_2} Z_{\text{II}}(\tau,\bar\tau)\,. \tag{7.176}$$

This beautiful formula encapsulates the full spectrum of the type IIA and type IIB superstring theories, but the outcome of the computation is actually completely trivial. The Jacobi abstruse identity (7.145) tells us that

$$Z_{\text{II}}(\tau,\bar\tau) = 0\,. \tag{7.177}$$

This result has actually a profound meaning. It indicates that both type IIA and type IIB superstring theories have a completely supersymmetric spectrum in space-time, as the contribution from the infinite number of space-time bosons compensates precisely the contribution from the infinite number of space-time fermions and the one-loop vacuum energy of the theory is exactly zero.

We have seen already that the massless degrees of freedom of type IIA and type IIB superstring theories describe at low energies supergravity theories with local supersymmetry, and this result indicates that this property holds for the whole tower of massive string states.

While the vanishing of the one-loop amplitude is the relevant physical result for the superstring theory themselves, one may want to consider a derived quantity from the one-loop partition function that allows to keep track unambiguously of the various states propagating into the loop.

For this one can consider, as in (7.171), characters for non-trivial $SO(8)$ group elements in the Cartan, for instance the diagonal generator. We have then for the type IIB superstring theory, following the pattern (7.172)

$$Z_{\text{IIB}}(\tau,\bar\tau,\upsilon,\bar\upsilon) = \frac{iV_{10}}{4(4\pi^2\alpha')^5}\frac{1}{\tau_2^4|\eta(\tau)|^{24}}\left|\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4(\tau,\upsilon) - \vartheta\begin{bmatrix}0\\1\end{bmatrix}^4(\tau,\upsilon) - \vartheta\begin{bmatrix}1\\0\end{bmatrix}^4(\tau,\upsilon) - \vartheta\begin{bmatrix}1\\1\end{bmatrix}^4(\tau,\upsilon)\right|^2 \tag{7.178}$$

The type IIA superstring theory, which follows the pattern (7.172), gives the result:

$$\begin{aligned} Z_{\text{IIA}}(\tau,\bar\tau,\upsilon,\bar\upsilon) = \frac{iV_{10}}{4(4\pi^2\alpha')^5}\frac{1}{\tau_2^4|\eta(\tau)|^{24}} &\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4(\tau,\upsilon) - \vartheta\begin{bmatrix}0\\1\end{bmatrix}^4(\tau,\upsilon) - \vartheta\begin{bmatrix}1\\0\end{bmatrix}^4(\tau,\upsilon) - \vartheta\begin{bmatrix}1\\1\end{bmatrix}^4(\tau,\upsilon)\right) \\ \times &\left(\bar\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4(\bar\tau,\bar\upsilon) - \bar\vartheta\begin{bmatrix}0\\1\end{bmatrix}^4(\bar\tau,\bar\upsilon) - \bar\vartheta\begin{bmatrix}1\\0\end{bmatrix}^4(\bar\tau,\bar\upsilon) + \bar\vartheta\begin{bmatrix}1\\1\end{bmatrix}^4(\bar\tau,\bar\upsilon)\right) \end{aligned} \tag{7.179}$$

Compared to the type IIB result, we have replaced the R_+ contribution with R_- in the anti-holomorphic sector.

7.5.5 Modular invariance of the superstring vacuum amplitude

An important consistency check of the validity of the construction outlined in this chapter is to show that the one-loop vacuum amplitude of the superstring theories satisfies modular invariance. While this was done for the bosonic string, hence for the fields x^μ and (b, c), we have to understand what happens for fields ψ^μ and (β, γ). This is especially necessary as we have considered a non-trivial separate GSO projection for the left-moving and right-moving degrees of freedom.

Modular $\tau \mapsto -1/\tau$ transformation

We start by examining the behavior under the transformation $\tau \mapsto -1/\tau$. Using equations (7.143a) and (6.58) one has the following map:

$$\frac{\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4(-1/\tau, \upsilon/\tau)}{\eta^4(-1/\tau)} = e^{\frac{4i\pi\upsilon^2}{\tau}} \frac{\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4(\tau, \upsilon)}{\eta^4(\tau)} \tag{7.180a}$$

$$\frac{\vartheta\begin{bmatrix}0\\1\end{bmatrix}^4(-1/\tau, \upsilon/\tau)}{\eta^4(-1/\tau)} = e^{\frac{4i\pi\upsilon^2}{\tau}} \frac{\vartheta\begin{bmatrix}1\\0\end{bmatrix}^4(\tau, \upsilon)}{\eta^4(\tau)} \tag{7.180b}$$

$$\frac{\vartheta\begin{bmatrix}1\\0\end{bmatrix}^4(-1/\tau, \upsilon/\tau)}{\eta^4(-1/\tau)} = e^{\frac{4i\pi\upsilon^2}{\tau}} \frac{\vartheta\begin{bmatrix}0\\1\end{bmatrix}^4(\tau, \upsilon)}{\eta^4(\tau)} \tag{7.180c}$$

$$\frac{\vartheta\begin{bmatrix}1\\1\end{bmatrix}^4(-1/\tau, \upsilon/\tau)}{\eta^4(-1/\tau)} = e^{\frac{4i\pi\upsilon^2}{\tau}} \frac{\vartheta\begin{bmatrix}1\\1\end{bmatrix}^4(\tau, \upsilon)}{\eta^4(\tau)} \tag{7.180d}$$

Setting $\upsilon = 0$, it implies that the combination of these factors that appears in the partition function (7.175) is invariant under $\tau \mapsto -1/\tau$ as it should.

The relations (7.180) can be interpreted as follows. The transformation $\tau \mapsto -1/\tau$ interchanges the one-cycles of the two-torus, hence the spin structures. While the AA (first equation) and PP (last equation) spin structures are invariant, the AP and PA spin structures are exchanged. This is one way to see that a hypothetic string theory with only the (NS_+, NS_+) sector would inconsistent, as the Ramond sector would automatically appear from modular transformations.

One sees also that the minus sign in front of the R-NS and NS-R contributions in (7.172), that were justified by spin-statistics in space-time, are actually forced upon us by modular invariance.

Modular $\tau \mapsto \tau + 1$ transformation

The transformation $\tau \mapsto \tau + 1$ will also mix the different spin structures, as it corresponds to replacing the identification along Euclidean time $z \sim z + \tau$ with $z \sim z + (\tau + 1)$, thereby adding an extra twist of 2π along the spatial circle. Hence an AA spin structure will become an AP spin structure, and an AP spin structure will become an AA spin structure. The explicit computation gives (we removed the $1/\eta^4$ factor giving a common phase):

$$\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4(\tau+1,\upsilon) = \vartheta\begin{bmatrix}0\\1\end{bmatrix}^4(\tau,\upsilon) \tag{7.181a}$$

$$\vartheta\begin{bmatrix}0\\1\end{bmatrix}^4(\tau+1,\upsilon) = \vartheta\begin{bmatrix}0\\0\end{bmatrix}^4(\tau,\upsilon) \tag{7.181b}$$

$$\vartheta\begin{bmatrix}1\\0\end{bmatrix}^4(\tau+1,\upsilon) = -\vartheta\begin{bmatrix}1\\0\end{bmatrix}^4(\tau,\upsilon) \tag{7.181c}$$

$$\vartheta\begin{bmatrix}1\\1\end{bmatrix}^4(\tau+1,\upsilon) = -\vartheta\begin{bmatrix}1\\1\end{bmatrix}^4(\tau,\upsilon) \tag{7.181d}$$

Hence

$$\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4 - \vartheta\begin{bmatrix}0\\1\end{bmatrix}^4 - \vartheta\begin{bmatrix}1\\0\end{bmatrix}^4 \pm \vartheta\begin{bmatrix}1\\1\end{bmatrix}^4 \mapsto -\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4 - \vartheta\begin{bmatrix}0\\1\end{bmatrix}^4 - \vartheta\begin{bmatrix}1\\0\end{bmatrix}^4 \pm \vartheta\begin{bmatrix}1\\1\end{bmatrix}^4\right) \tag{7.182}$$

and combining the holomorphic and anti-holomorphic contributions eliminates the extra minus sign.

This completes the proof of the invariance of type IIA and type IIB superstring one-loop amplitudes under $\mathsf{PSL}(2,\mathbb{Z})$, the modular group of the two-torus.