M2 ICFP 2019/2020

Introduction to AdS/CFT TD n°2

Anti-de Sitter spacetime

Coordinate systems for AdS_{d+1}

Throughout, the AdS radius is denoted by ℓ.

Embedding in $R^{2,d}$:

$$-X_0^2+\sum_{i=1}^{d} X_i^2 - X_{d+1}^2 = -\ell^2$$

Poincaré coordinates :

$$X^\mu = \ell\frac{x^\mu}{z}, \quad X^d = \frac{\ell}{2z}(1-x^2-z^2), \quad X^{d+1} = \frac{\ell}{2z}(1+x^2+z^2)$$

where $\mu = 0\ldots d-1$, $x^2 \equiv x^\mu x_\mu$.

$$ds^2 = \frac{\ell^2}{z^2}\left[dz^2+\eta_{\mu\nu}dx^\mu dx^\nu\right] \qquad \mu,\nu = 0\ldots d-1, \qquad 0<z<+\infty$$

Global coordinates :

$$X^0 = \ell\cosh\rho\cos\tau, \quad X^{d+1} = \ell\cosh\rho\sin\tau, \quad X^i = \ell\sinh\rho\Omega^i$$

where Ω^i parametrize S^{d-1}, i.e. $\sum_i \Omega_i^2 = 1$

$$ds^2 = \ell^2\left[-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_{d-1}^2\right]$$

where $d\Omega_{d-1}^2$ is the round metric on the unit $(d-1)$-sphere.

1. Coordinates

Check explicitly that the above coordinate systems do parametrize AdS_{d+1} (or portions of it) and that they lead to the corresponding metrics.

2. Dilatation generator

Given the identification of the conformal generators with the Lorentz generators of $R^{2,d}$,

$$J_{AB} = -i\left(X_A\partial_B - X_B\partial_A\right),$$

and in particular the dilatation generator, $D = iJ_{0,d+1}$, write down the vector field in AdS which generates the dilatation isometry, both in global and in Poincaré coordinates. What does the dilatation generator do, in global coordinates ?

3. Einstein's equation

Show that Anti-de Sitter spacetime AdS_{d+1} is a solution to Einstein's equation with a negative cosmological constant,

$$R_{ab} - \frac{1}{2} g_{ab} R = -\Lambda g_{ab}, \qquad \Lambda < 0$$

and find the relation between Λ and the AdS radius ℓ
[Hint : the calculation is easiest in Poincaré coordinates]

4. Geodesics

4.1 Null Geodesics

1. What are the null geodesics in AdS in Poincaré coordinates ? Show that a massless particles starting at $x^\mu = 0, z = z_0$ reaches the boundary $z = 0$ in a finite time, but it takes it an infinite amount of time to cross the Poincaré horizon.
2. Despite the previous result, show that the spacetime is *geodesically complete* when going towards the boundary, i.e. one can extend arbitrarily the affine parameter of null geodesics (if this were not the case, the spacetime would either have to be singular at the boundary, ot it could be extended by glueing another region past the boundary, which would then be a coordinate singularity of the metric.)
3. Find the radial (i.e. constant on the $(d-2)$-sphere) null geodesics in global coordinates. How long (in global time) does a massless particle take to travel to the boundary, starting from the origin $r = 0$? Supposing that the boundary is perfectly reflecting, what kind of the motion does the massless particle undergo ?

4.2 Timelike Geodesics

1. Write the timelike geodesic for a massive particles at rest at the origin ($\rho = 0$) of global AdS. What does trajectory correspond to in the embedding space $R^{2,d}$?
2. Find timelike radial geodesics in global coordinates, and show that a massive particle in AdS has a periodic radial motion but it gets reflected before reaching the boundary. What is its period ? *[Hint : to find the geodesics, start from the particle at rest and perform a Lorentz boost in the (X^1, X^{d+1}) plane in the embedding space. This is an isometry of AdS, so it maps geodesics to geodesics]*