

Conformal Field theory for 2d Statistical Mechanics

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Lecture 1

Critical Phenomena and Scale Invariance

1.1 Phase transitions

1.1.1 First- and second-order transitions

Let us start by considering a sample of material in a definite physical phase, with homogeneous measurable properties: density, elasticity, magnetisation, *etc.* Any measure of these quantities, performed anywhere in the sample, should give the same result. Hence, if the sample has the shape of an $L \times L \times L$ cube, a portion of size $\frac{L}{2} \times \frac{L}{2} \times \frac{L}{2}$ should have the same properties. If we iterate this procedure, we finally reach a scale where all the elementary constituents are correlated, and the above argument is not valid anymore. This scale is called the *correlation length*, and we will denote it by ξ . For example, in a crystal, ξ is usually of the order of a few interatomic spacings. The correlation length depends on the external parameters (pressure, temperature, magnetic field, *etc.*). An *order parameter* is defined as a local quantity which characterises the phases of the system, *i.e.* whose values are different from one phase to another. Examples : the local density of a fluid, the (components of) local magnetisation in a magnet, *etc.*

When one or several external parameters are varied, the properties of the material may change drastically: this is called a *phase transition*. We distinguish two types of

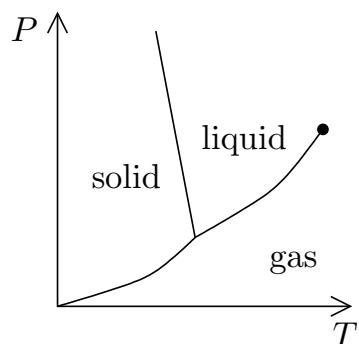


Figure 1.1: The phase diagram of water. The critical point is indicated by a black dot.

phase transitions:

- A phase transition is of first order if the two phases remain different at the transition. An example is the liquid-gas transition of water at 100°C. The order parameter (in the case of water, the local density ρ) is discontinuous at the phase transition. In this case, the two phases coexist at the transition, and an extensive amount of energy (latent heat) must be exchanged with the environment during the transition. This reflects the fact that, when the two phases coexist, the respective spatial domains of each phase have a definite size: *the correlation length ξ remains finite at a first-order transition*. In the example of water, ξ is the average size of liquid droplets which coexist with the gas phase during the transition.
- A phase transition is of second order if the two phases become identical at the transition, and hence all the order parameters are continuous. This is the case for the critical endpoint of the liquid-gas transition of water, at temperature 374°C and pressure 218 atm. When approaching this point along the transition line, the latent heat vanishes. In the vicinity of the critical point, liquid droplets of various scales appear, up to the size of the sample. *The correlation length diverges at a second-order transition*. As a consequence, for a large range of spatial scales, the spatial distribution of physical properties (*e.g.* the local density) is *self-similar* : it remains unchanged as one “zooms” into a subregion. *The system displays scale invariance at a second-order transition*.

Let us make the notion of scale invariance more precise, by discussing order parameters and their correlation functions. Let us consider a system in dimension d , close to a second-order phase transition, and denote the order parameter generically by $S(r)$. From mean-field arguments [neglecting local fluctuations of $S(r)$] one may show that, at short distances, the two-point correlation function $G(r)$ obeys a Laplace equation, and therefore it scales as $G(r) \propto 1/r^{d-2}$. To take spatial fluctuations into account, we should assume that $G(r)$ depends non-trivially on the ratios r/ξ and a/ξ , where a is the microscopic distance:

$$G(r) := \langle S(0)S(r) \rangle = \frac{1}{r^{d-2}} f(r/\xi, a/\xi). \quad (1.1)$$

Now suppose that f vanishes as a power law in its second argument: $f(u, v) \propto_{v \rightarrow 0} v^\eta$, and compute the susceptibility χ , in the regime $a \ll \xi$.

$$\chi = \int d^d r G(r) = \int \frac{d^d r}{r^{d-2}} f(r/\xi, a/\xi) = \xi^2 \int \frac{d^d u}{u^{d-2}} f(u, a/\xi) \propto \xi^{2-\eta} a^\eta. \quad (1.2)$$

The exponent η is called the anomalous dimension.

In this course, we shall concentrate on second-order phase transitions. The central task of the course will be to characterise the various classes of second-order phase transitions by determining scaling exponents such as η , and multipoint correlation functions like $G(r)$.

1.1.2 Spontaneous symmetry breaking

Consider a material whose internal interactions enjoy some (discrete or continuous) symmetry for any value of the external parameters. We shall illustrate this situation on the example of a magnetic system, described by the canonical ensemble, with temperature T

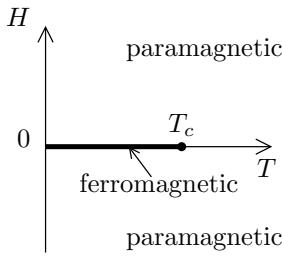
and external magnetic field H . In the canonical ensemble (also called Gibbs measure), the probability of a spin configuration $[S] = \{S(r)\}$ is given by:

$$\mathbb{P}_{T,H}[S] = \frac{1}{Z(T,H)} \times \exp\left(-\frac{\mathcal{H}_{\text{int}}[S] + H \int d^d r S(r)}{k_B T}\right), \quad (1.3)$$

where $\mathcal{H}_{\text{int}}[S]$ is the classical Hamiltonian encoding the interactions between the spins. The external magnetic field couples to the total magnetisation $M = \int d^d r S(r)$. Let us consider the case when this interaction is symmetric under spin reversal:

$$\mathcal{H}_{\text{int}}[S] = \mathcal{H}_{\text{int}}[-S] \quad (1.4)$$

We assume moreover that the interaction is ferromagnetic, *i.e.* it favours the configurations where neighbouring spins are aligned. For any spatial dimension $d > 1$, one observes experimentally or numerically the following phase diagram:



In the $H > 0$ (resp. $H < 0$) half-plane, the system is in a paramagnetic phase, and the total magnetisation M is positive (resp. negative). For $H = 0$ and $T > T_c$, the system is in a disordered phase, with $M = 0$. Along the line $0 < T < T_c$ at $H = 0$, the system is in a ferromagnetic phase, *i.e.* it has a non-zero total magnetisation $\pm M_0$. The latter is a symmetry-broken phase: although the Hamiltonian \mathcal{H}_{int} is symmetric under spin reversal, the state of the system is not symmetric. This ferromagnetic line represents a first-order phase transition between the two paramagnetic phases, because the order parameter M is discontinuous across the transition: it goes from $+M_0$ to $-M_0$ as H changes sign. Note that $M_0 \rightarrow 0$ as $T \rightarrow T_c$ along this line. In contrast, at the critical point $(T = T_c, H = 0)$ the system goes through a second-order phase transition as T is varied.

We have described above the simplest example of symmetry breaking in a spin system, where the symmetry group is \mathbb{Z}_2 . Note that other discrete or continuous symmetry groups may give rise to symmetry-broken phases.

1.1.3 Critical exponents

The above example of magnetic systems is convenient to introduce scaling exponents. We first introduce the reduced temperature and magnetic field:

$$t := \frac{T - T_c}{T_c}, \quad h := \frac{H}{k_B T_c}. \quad (1.5)$$

Let us list the most common critical exponents, defined by the behaviour of the system in the vicinity of the critical point:

- The specific heat $C = \frac{dF}{dT} \propto |t|^{-\alpha}$.
- The spontaneous magnetisation $M_0 = \lim_{H \rightarrow 0^+} M \propto |t|^\beta$ for $t < 0$.
- The zero-field susceptibility $\chi \propto |t|^{-\gamma}$.
- The magnetisation at critical temperature $M \propto |h|^{1/\delta}$.
- The divergence of the correlation length $\xi \propto |t|^{-\nu}$.
- The anomalous dimension at the critical point $G(r) = \langle S(0)S(r) \rangle \propto 1/|r|^{d-2+\eta}$.

1.1.4 Simple lattice models

To describe a phase transition, one considers a discrete model as simple as possible to allow some computations (sometimes up to a complete solution), but with enough ingredients to capture the important features of the transition. Moreover, even when the degrees of freedom are of quantum nature (*e.g.* atomic spins in a crystal), in many cases one may still capture the essential physical features by considering a classical discrete model, *i.e.* by neglecting the quantum fluctuations at ordinary temperatures. Furthermore, in the case of spin systems, one may simplify the model even more by only including the (classical) fluctuations of one component of the spin. This gives rise to the Ising model, with Hamiltonian:

$$\mathcal{H}[S] = - \sum_{i,j} J_{ij} S_i S_j - H \sum_i S_i, \quad (1.6)$$

where $[S]$ denotes a configuration of spins with values $S_i = \pm 1$, living on the sites of some regular lattice, representing the position of nuclei of the crystal. The set of parameters J_{ij} are the coupling constants, which determine the interaction between spins. A standard choice (which is the only one we shall discuss in this course) is to set

$$J_{ij} = \begin{cases} J & \text{if } i \text{ and } j \text{ are adjacent sites,} \\ 0 & \text{otherwise.} \end{cases} \quad (1.7)$$

This defines the nearest-neighbour Ising model:

$$\mathcal{H}_{\text{Ising}}[S] = -J \sum_{\langle i,j \rangle} S_i S_j - H \sum_i S_i, \quad (1.8)$$

where $\langle i,j \rangle$ denotes adjacent sites on the lattice. This model has an internal \mathbb{Z}_2 symmetry (global spin reversal $S_i \mapsto -S_i$). When the lattice is regular, the model also enjoys translation invariance. In the scaling limit, it exhibits the behaviour described in Sec. 1.1.2, with a \mathbb{Z}_2 broken symmetry phase ending at a second-order critical point.

Various generalisations of the Ising model may be considered. *Clock models* are classical spin models where the degrees of freedom can take N values equally spaced on the unit circle, with an interaction invariant under global spin rotation and spin reversal. The internal symmetry group is then \mathbb{D}_N , the dihedral group. It is convenient to introduce $\omega = e^{2i\pi/N}$, and to use complex notations for the spins $S_i \in \{1, \omega, \dots, \omega^{N-1}\}$. A generic nearest-neighbour interaction can be written as:

$$\mathcal{H}_{\text{clock}} = - \sum_{\langle i,j \rangle} \sum_{k=1}^{N-1} \frac{J_k}{2} (S_i^k S_j^{-k} + S_i^{-k} S_j^k). \quad (1.9)$$

These models have a richer phase diagram than Ising: in particular, there can be more than two ordered phases. For $N = 2$ one recovers the Ising model.

Another way of generalising the Ising model is to consider n -dimensional vector spin variables \mathbf{S}_i living on the sphere $\mathbf{S}_i^2 = 1$, with an interaction invariant under global spin rotation. The internal symmetry group is then $O(n)$, the group of isometric linear transformations. A generic nearest-neighbour Hamiltonian will be of the form:

$$\mathcal{H}_{O(n)} = - \sum_{\langle i,j \rangle} \sum_{k=1}^{\infty} J_k (\mathbf{S}_i \cdot \mathbf{S}_j)^k. \quad (1.10)$$

An important difference with the Ising model is that the internal symmetry group is continuous (and unitary). As a consequence, from the Mermin-Wagner theorem, in two spatial dimensions there cannot be a broken-symmetry phase. The case $n = 2$ is of specific importance: it is usually called the XY model. It exhibits a peculiar type of phase transition, called the BerezinskiiKosterlitzThouless transition. The XY model may be obtained as the $N \rightarrow \infty$ limit of a \mathbb{Z}_N clock model.

1.2 The Renormalisation Group

1.2.1 Block-spin variables

The Renormalisation Group (RG), as applied to problems of Statistical Mechanics, is considered in this course as a conceptual framework for the physical theory of critical phase transitions. The main ideas may be exposed by considering the example of block-spin transformations on the Ising model. We start with the Ising model with spin variables $S_i = \pm 1$ on a regular lattice of mesh size a , with an interaction defined by the Hamiltonian $\mathcal{H}[S]$. We set some rescaling factor $\lambda > 1$ (practically, λ is a positive integer), and we form blocks of $n_b = \lambda^d$ neighbouring variables, which we replace by block-spin variables $S'_\alpha = \pm 1$, using some definite rule. This will result in an Ising model on a lattice of mesh λa , with a new Hamiltonian $\mathcal{H}'[S']$. Let us explain how this new Hamiltonian is defined. Let α denote a block of n_b spins, and $(S_1^{(\alpha)}, \dots, S_{n_b}^{(\alpha)})$ be the set of original variables included in this block. The block-spin rule can be encoded by some function μ :

$$\mu : \begin{cases} \{+1, -1\}^{n_b} & \rightarrow \{+1, -1\} \\ (S_1^{(\alpha)}, \dots, S_{n_b}^{(\alpha)}) & \mapsto S'_\alpha. \end{cases} \quad (1.11)$$

We want the Boltzmann weights on block-spin variables to be formed by the sum over internal degrees of freedom in each block:

$$e^{-\mathcal{H}'[S']} := \sum_{[S]} \prod_{\alpha} \delta[S'_\alpha - \mu(S_1^{(\alpha)}, \dots, S_{n_b}^{(\alpha)})] e^{-\mathcal{H}[S]}. \quad (1.12)$$

Note that we have absorbed the inverse temperature $\beta = 1/k_B T$ into the definition of \mathcal{H} . By construction, the partition function is invariant under this process:

$$Z = \sum_{[S]} e^{-\mathcal{H}[S]} = \sum_{[S']} e^{-\mathcal{H}'[S']}. \quad (1.13)$$

Any correlation function which depends only on block-spin variables is also left invariant:

$$\frac{1}{Z} \sum_{[S]} \mathcal{A}[\mu(S_1^{(\alpha)}, \dots, S_{n_b}^{(\alpha)}), \mu(S_1^{(\beta)}, \dots, S_{n_b}^{(\beta)}), \dots] e^{-\mathcal{H}[S]} = \frac{1}{Z} \sum_{[S']} \mathcal{A}(S'_\alpha, S'_\beta, \dots) e^{-\mathcal{H}'[S']}. \quad (1.14)$$

However, some information is lost during the process, since the internal degrees of freedom of each block are summed over, and replaced by a single spin variable S'_α : the new Hamiltonian is a *coarse-grained* version of the original one, sharing the same internal symmetry group – \mathbb{Z}_2 in the Ising case. Even if the original Hamiltonian has only nearest-neighbour interactions, the new Hamiltonian may include more general interactions. Generically, if we take the original (reduced) Hamiltonian to be of the form:

$$\mathcal{H} = - \sum_{\langle i,j \rangle} K_1 S_i S_j - \sum_{\langle\langle i,j \rangle\rangle} K_2 S_i S_j - \dots \quad (1.15)$$

where $\langle i,j \rangle$ denotes a pair of nearest-neighbour sites, $\langle\langle i,j \rangle\rangle$ a pair of next-nearest neighbours, *etc.* and we denote by $K = (K_1, K_2, \dots)$ the set of coupling constants, then we expect \mathcal{H}' to be of the same form as \mathcal{H} , with different values of the coupling constants $K' = (K'_1, K'_2, \dots) = \mathcal{R}(K)$. The map \mathcal{R} is called the renormalisation group (RG) transformation. When the RG procedure is iterated, the vector K converges to some attractive fixed point of \mathcal{R} , which corresponds to a physical phase of the system. The regions of attraction of different phases are typically separated by transition lines.

1.2.2 Example: Decimation procedure in 1d

Before we make the RG picture more precise, let us perform a particular block-spin procedure, namely the spin decimation, on the one-dimensional case, and compute explicitly its RG transformation. We start with the 1d Ising Hamiltonian:

$$\mathcal{H}[S] = - \sum_j K S_j S_{j+1}. \quad (1.16)$$

As a first step, we write the exact expansion $e^{KS_1 S_2} = \cosh K (1 + x S_1 S_2)$, where $x = \tanh K$, so that the Boltzmann weight reads:

$$e^{-\mathcal{H}[S]} = \cosh^N K \times \prod_j (1 + x S_j S_{j+1}), \quad (1.17)$$

where N is the number of sites of the system. We form blocks of three spins, and we choose the *decimation* rule:

$$\mu(S_1, S_2, S_3) := S_2. \quad (1.18)$$

The rescaling factor is thus $\lambda = 3$. Let α, β be two adjacent spin blocks. The factors of $e^{-\mathcal{H}[S]}$ involving α and β are

$$\cosh^3 K \times (1 + x S'_\alpha S_3^{(\alpha)})(1 + x S_3^{(\alpha)} S_1^{(\beta)})(1 + x S_1^{(\beta)} S'_\beta). \quad (1.19)$$

Expanding this product and summing over $S_3^{(\alpha)}$ and $S_1^{(\beta)}$, we get

$$4 \cosh^3 K (1 + x^3 S'_\alpha S'_\beta) = \frac{4 \cosh^3 K}{\cosh K'} e^{-K' S'_\alpha S'_\beta}, \quad (1.20)$$

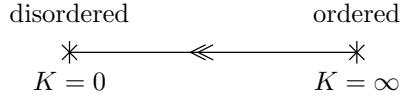
where the new coupling constant is given by:

$$K' = \tanh^{-1}(\tanh^3 K). \quad (1.21)$$

Hence the coarse-grained Hamiltonian is given by

$$\mathcal{H}'[S'] = N \varphi(K) - \sum_\alpha K' S'_\alpha S'_{\alpha+1}, \quad (1.22)$$

where $\varphi(K) = \frac{1}{3} \log(2\sqrt{1 + 3 \cosh^2 2K})$. This constant term only plays a role in the normalisation of the partition function. In terms of the variable $x = \tanh K$, the RG transformation (1.21) corresponds to the map $x \mapsto x' = x^3$. The physical range for this variable is $0 \leq x \leq 1$. Let us analyse the behaviour of the system when the RG is iterated many times. The fixed point $x = 0$ is attractive, and corresponds to $K = 0$, *i.e.* to infinite temperature. The fixed point $x = 1$ is repulsive, and corresponds to $K = +\infty$, *i.e.* zero temperature. This may be summarised in a diagram of RG flow:



The main idea of the scaling hypothesis is that the large-scale behaviour of the system is predicted by this RG flow. In this case, the situation is very simple, and does not involve any phase transition. For any finite value of K , the system remains in the disordered phase governed by the fixed point $K = 0$. The other fixed point, $K = \infty$, is unstable under RG, and represents the totally ordered (ferromagnetic) phase.

The correlation length transforms under an RG iteration as a geometric sequence:

$$\xi(K') = \frac{\xi(K)}{\lambda}, \quad \tanh K' = \tanh^3 K. \quad (1.23)$$

Hence, we may write the correlation length as a function of K along the RG flow:

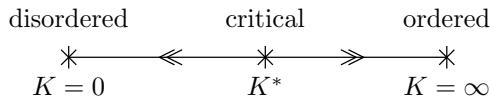
$$\xi(K) = \frac{\text{const}}{|\log \tanh K|^{1/3}} = \frac{\text{const}}{|\log \tanh K|}. \quad (1.24)$$

The correlation length decreases from $\xi = \infty$ to $\xi = 0$ along the RG flow. Close to the repulsive fixed point $K = \infty$, the correlation length diverges as $\xi \propto e^{2K}$.

1.2.3 Qualitative RG flow for Ising with $d > 1$

The above 1d example is very special, because, up to a constant term in the Hamiltonian, the interaction remains of the same form after an RG transformation: it only has nearest-neighbour interaction, with a single parameter K . In higher dimensions, RG transformations typically give rise to more complicated interactions, and in most cases they cannot be treated by an exact computation. The basic reason is that a spin S_i at the boundary of a block α may interact with several neighbouring blocks β, γ, \dots . When summing over this S_i , one typically produces interactions between α and its neighbours, but also between, say, β and γ , which may not be adjacent to one another.

In the case of the zero-field Ising model, we assume that the scaling behaviour can still be analysed by looking only at the RG flow for the nearest-neighbour coupling K . The RG flow consistent with the phase diagram described above for the Ising model with dimension $d > 1$ is:



The regions $0 \leq K < K^*$ and $K^* < K \leq \infty$ correspond respectively to the disordered and ferromagnetic phases. The repulsive fixed point is the critical point where a second-order transition occurs. Although we do not have an exact expression for the RG transformation, we can still analyse the divergence of the correlation length in the vicinity of the critical point.

Let us define the RG exponent y as:

$$y := \frac{\log \left| \left[\frac{d\mathcal{R}}{dK} \right]_{K^*} \right|}{\log \lambda}, \quad \text{so that} \quad \left| \left[\frac{d\mathcal{R}}{dK} \right]_{K^*} \right| = \lambda^y. \quad (1.25)$$

Since K^* is a repulsive fixed point, we have $|d\mathcal{R}/dK| > 1$. By construction, the scaling factor always satisfies $\lambda > 1$. Hence, $y > 0$ for a repulsive RG fixed point. When we linearise the RG transformation for $K \simeq K^*$, we get

$$K' = \mathcal{R}(K) = K^* + \frac{d\mathcal{R}}{dK}(K - K^*) \quad \Rightarrow \quad |K' - K^*| = \lambda^y |K - K^*|. \quad (1.26)$$

The correlation length is always rescaled as $\xi(K') = \xi(K)/\lambda$. After p iterations, starting from some value $K^{(0)}$, we get

$$|K^{(p)} - K^*| = \lambda^{py} |K^{(0)} - K^*|, \quad \xi(K^{(p)}) = \frac{K^{(0)}}{\lambda^p}. \quad (1.27)$$

Eliminating p between these equations, for any $K = K^{(p)}$ we get the relation

$$\xi(K) \propto |K - K^*|^{-1/y}. \quad (1.28)$$

Hence, the correlation exponent ν is simply given by the inverse of the RG exponent y :

$$\nu = \frac{1}{y}. \quad (1.29)$$

1.3 Scale invariance

1.3.1 Scaling variables and scaling operators

Let us now consider the general situation of an RG fixed point $K^* = (K_1^*, K_2^*, \dots)$ in the multidimensional space of parameters. In the vicinity of K^* , when we apply the RG transformation $K' = \mathcal{R}(K)$, we get the linear approximation:

$$K'_a - K_a^* = \sum_b \frac{\partial K'_a}{\partial K_b} \Big|_{K^*} (K_b - K_b^*). \quad (1.30)$$

Let $J_{ab} = \partial K'_a / \partial K_b$ be the Jacobian matrix at $K = K^*$, and $\{e_j\}$ the left eigenvectors of J , with eigenvalues $\{\mu_j\}$:

$$\forall j, \quad e_j^t J = \mu_j e_j^t. \quad (1.31)$$

We assume that all the eigenvalues μ_j are positive¹. From these objects, we define the *scaling variables* $\{u_j\}$ and the associated RG exponents $\{y_j\}$:

$$u_j := e_j^t \cdot (K - K^*), \quad y_j := \frac{\log \mu_j}{\log \lambda}. \quad (1.32)$$

¹If some negative eigenvalues μ_j appear, we can apply the RG transform twice, and redefine $J \rightarrow J^2, \lambda \rightarrow \lambda^2$, so that all eigenvalues become positive.

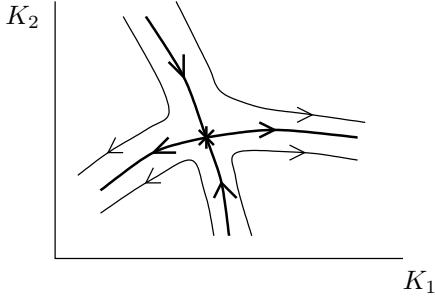


Figure 1.2: An example of RG flow in two-parameter space, in the case of a fixed point with one relevant eigenvalue and one irrelevant eigenvalue of the RG operator \mathcal{R} .

In the vicinity of the critical point K^* , the scaling variables transform under an RG iteration of factor λ as:

$$u'_j = \lambda^{y_j} u_j. \quad (1.33)$$

The physical properties of the critical point depend on the signs of the exponents y_j :

- If $y_j > 0$, the scaling variable u_j is *relevant*. A small deviation of u_j from the critical value $u_j^* = 0$ drives the system away from the fixed point.
- If $y_j < 0$, the scaling variable u_j is *irrelevant*. A small deviation of u_j from zero is destroyed under the RG, and leaves the system at its critical point.
- If $y_j = 0$, the scaling variable u_j is *marginal*. More detailed analysis is required to predict the phase diagram. In typical situations, a critical line of fixed points parameterised by u_j passes through K^* .

The general structure of the RG flow in the vicinity of K^* can be inferred simply from the signs of the y_j 's: see Fig. 1.2.

Consider the set of parameters $\{K_a\}$ of the Hamiltonian, and denote $\{S_a\}$ the associated operators. Let $\{f_j\}$ be the right eigenvectors of the matrix (J_{ab}) defined in Sec. 1.3.1, so that the bases $\{e_j\}$ and $\{f_j\}$ are dual to one another. We can write the coupling terms as

$$\sum_a (K_a - K_a^*) S_a = \sum_j u_j \Phi_j, \quad \text{where} \quad \Phi_j := \sum_a (f_j)_a S_a. \quad (1.34)$$

The operators Φ_j are called *scaling operators*.

1.3.2 Scaling functions, critical exponents

Let us first discuss the transformation of the free energy density under an RG iteration. Under the RG transformation $K \rightarrow K'$, the configuration energy may vary by an extensive amount $N\varphi(K)$, where N is the number of sites of the system:

$$\mathcal{H}_K[S] \rightarrow \widetilde{\mathcal{H}}_{K'}[S'] = \mathcal{H}_{K'}[S'] + N\varphi(K). \quad (1.35)$$

From the conservation of the partition function

$$Z(K) = \sum_{[S]} e^{-\mathcal{H}_K[S]} = \sum_{[S']} e^{-\tilde{\mathcal{H}}_{K'}[S']} , \quad (1.36)$$

we get the inhomogeneous relation for the free energy density $f(K) = -[\log Z(K)]/N$:

$$f(K) = \lambda^{-d} f(K') + \varphi(K) . \quad (1.37)$$

Since $\varphi(K)$ must be analytic around K^* , one can construct a regular solution f_{reg} of (1.37), say by writing the power series expansion around K^* . Thus the physical free energy has the form

$$f(K) = f_{\text{reg}}(K) + f_{\text{sing}}(K) , \quad \text{with} \quad f_{\text{sing}}(K) = \lambda^{-d} f_{\text{sing}}(K') . \quad (1.38)$$

Consider the situation where only two scaling variables are relevant, and all other variables are irrelevant. In the case of a magnetic system, suppose we may identify, at linear order, these variables as the reduced temperature and magnetic field:

$$u_t \propto \frac{T - T_c}{T_c} := t , \quad u_h \propto \frac{H}{k_b T_c} := h . \quad (1.39)$$

The singular free energy $f_{\text{sing}}(u_t, u_h)$ then satisfies:

$$f_{\text{sing}}(t, h) = \lambda^{-d} f_{\text{sing}}(\lambda^{y_t} t, \lambda^{y_h} h) , \quad (1.40)$$

where λ is the scaling factor of an RG iteration. After p iterations, we get

$$f_{\text{sing}}(t, h) = \lambda^{-dp} f_{\text{sing}}(\lambda^{py_t} t, \lambda^{py_h} h) . \quad (1.41)$$

We suppose that, close enough to the fixed point, this sequence is interpolated by an RG flow line $(\mu^{y_t} t, \mu^{y_h} h)$ parameterised by the real variable μ :

$$\forall \mu > 0 , \quad f_{\text{sing}}(t, h) = \mu^{-d} f_{\text{sing}}(\mu^{y_t} t, \mu^{y_h} h) . \quad (1.42)$$

We want to use this relation, to infer a scaling form for f_{sing} . First, let us fix some arbitrary value t_0 . Let (t, h) be some point close to the fixed point. This point sits on the RG flow line $(\mu^{y_t} t, \mu^{y_h} h)$. For $\mu = (t/t_0)^{-1/y_t}$, we get the point $(t_0, h(t/t_0)^{-y_h/y_t})$ on the same flow line. We then define the variable:

$$u(t, h) := h/t^{y_h/y_t} .$$

Note that this variable is invariant along the RG flow line. We introduce the *scaling function*:

$$\Phi(u) := t_0^{-d/y_t} f_{\text{sing}}(t_0, t_0^{y_h/y_t} u) .$$

From (1.42), we get:

$$f_{\text{sing}}(t, h) = t^{d/y_t} \times \Phi\left(\frac{h}{t^{y_h/y_t}}\right) . \quad (1.43)$$

As a consequence of the scaling form (1.43), we can compute the specific heat exponent α as follows:

$$C = \frac{\partial^2 f_{\text{sing}}}{\partial t^2}(t, 0) \propto t^{d/y_t - 2} \quad \Rightarrow \quad \alpha = 2 - d/y_t. \quad (1.44)$$

Similar arguments yield the spontaneous magnetisation and susceptibility exponents in terms of y_t, y_h :

$$\beta = \frac{d - y_h}{y_t}, \quad (1.45)$$

$$\gamma = \frac{2y_h - d}{y_t}. \quad (1.46)$$

1.3.3 Scaling of correlation functions

Let us now study how correlation functions transform under an RG iteration. We first look at the correlation functions of spin operators in a magnetic system, on the domain $\mathcal{D} \cap a\mathbb{Z}^d$. It is convenient to introduce a non-uniform magnetic field $h(r)$, and to assume that the spatial variations of this field are small enough, so that it transforms locally as a uniform field under RG:

$$h(r) \rightarrow h'(r') = \lambda^{y_h} h(r). \quad (1.47)$$

The n -point correlation function reads

$$G_{K,a}(r_1, \dots, r_n) = \langle S(r_1) \dots S(r_n) \rangle_{K, \mathcal{D} \cap a\mathbb{Z}^d} = \frac{\partial^n \log Z_{K,a}[h]}{\partial h(r_1) \dots \partial h(r_n)}. \quad (1.48)$$

The partition function

$$Z_{K,a}[h] = \sum_{[S]} e^{-\mathcal{H}_{K,a}[S] + \sum_r h(r) S(r)} \quad (1.49)$$

is preserved by the RG transformation, and thus we have

$$G_{K',\lambda a}(r'_1, \dots, r'_n) = \frac{\partial^n \log Z_{K',\lambda a}[h']}{\partial h'(r'_1) \dots \partial h'(r'_n)} = \frac{\partial^n \log Z_{K,a}[h]}{\partial h'(r'_1) \dots \partial h'(r'_n)}. \quad (1.50)$$

In each block labelled by the position $r'_j = r_j/\lambda$, there are λ^d original spins, and we assume that they all couple equally to the local magnetic field $h'(r'_j)$. Hence, each derivative $\partial/\partial h'(r'_j)$ acts as $\lambda^d \times \lambda^{-y_h} \times \partial/\partial h(r_j)$ on $Z_{K,a}[h]$. As a result, we obtain the identity:

$$G_{K',\lambda a}(r_1/\lambda, \dots, r_n/\lambda) = \lambda^{nx_h} G_{K,a}(r_1, \dots, r_n), \quad x_h := d - y_h. \quad (1.51)$$

In particular, at the critical point $K = K^*$, we get $G_{K,a} \propto a^{nx_h}$. To get a finite correlation function as $a \rightarrow 0$, we introduce the renormalised scaling operator for physical coordinates $r \in \mathbb{R}^d$:

$$s(r) := \text{const} \times \lim_{a \rightarrow 0} [a^{-x_h} S(r/a)], \quad (1.52)$$

where the constant is chosen so that $\langle s(r)s(0) \rangle_{\mathbb{R}^d} \sim 1/r^{2x_h}$ as $r \rightarrow \infty$. From the RG relation (1.51), and using the fact that the domain $\mathcal{D} \cap \lambda a\mathbb{Z}^d$ is trivially equivalent to $\mathcal{D}/\lambda \cap a\mathbb{Z}^d$, we get the scale covariance relation:

$$\langle s(r_1/\lambda) \dots s(r_n/\lambda) \rangle_{K^*, \mathcal{D}/\lambda} = \lambda^{nx_h} \langle s(r_1) \dots s(r_n) \rangle_{K^*, \mathcal{D}}. \quad (1.53)$$

From a straightforward generalisation of the above argument, the critical correlation functions have the *scale covariance* property at the critical point:

$$\langle \phi_1(r_1/\lambda) \dots \phi_n(r_n/\lambda) \rangle_{K^*, \mathcal{D}/\lambda} = \left(\prod_{j=1}^n \lambda^{x_j} \right) \langle \phi_1(r_1) \dots \phi_n(r_n) \rangle_{K^*, \mathcal{D}}, \quad (1.54)$$

where the scaling dimensions are given by

$$x_j = d - y_j, \quad (1.55)$$

and the renormalised scaling operators $\phi_j(r)$ are given by

$$\phi_j(r) := \text{const} \times \lim_{a \rightarrow 0} [a^{-x_j} \Phi_j(r/a)], \quad (1.56)$$

where the constant is chosen so that $\langle \phi_j(r) \phi_j(0) \rangle_{\mathbb{R}^d} \sim 1/r^{2x_j}$ as $r \rightarrow \infty$.

In the vicinity of the critical point, if all relevant parameters are set to their critical value except t , the identity (1.51) imposes a scaling form for G :

$$G(r_1, \dots, r_n | t) = \xi^{-nx_h} \Psi_S(r_1/\xi, \dots, r_n/\xi), \quad \xi \propto t^{-\nu}, \quad (1.57)$$

where we have used $\nu = 1/y_t$. Similarly, if a single relevant parameter t is slightly different from its critical value $t^* = 0$, we have the scaling form:

$$\langle \phi_1(r_1) \dots \phi_n(r_n) \rangle_t = \left(\prod_{j=1}^n \xi^{-x_j} \right) \Psi(r_1/\xi, \dots, r_n/\xi), \quad \xi \propto t^{-\nu}. \quad (1.58)$$

Lecture 2

Conformal transformations

2.1 Conformal symmetry at the critical point

2.1.1 Continuum limit

We consider a lattice model on a d -dimensional lattice \mathcal{L}_a of mesh a , and we specify how to define its correlations in the limit of small lattice mesh, at the critical point, or in vicinity of this critical point. In both cases, we consider the model on a (finite or infinite) domain \mathcal{D} of the lattice. The continuum limit corresponds to letting $a \rightarrow 0$, while keeping the domain \mathcal{D} fixed. Similarly, correlation functions in the continuum limit are defined by keeping the positions (r_1, \dots, r_n) fixed in \mathcal{D} , and choosing the closest approximations $(a\hat{r}_1, \dots, a\hat{r}_n)$ of these points on the lattice of mesh $a \rightarrow 0$.

At the critical point K^* , the correlation length $\xi \rightarrow \infty$, and the two-point functions on the full space $\mathcal{L}_a = a\mathbb{Z}^d$ obey power laws

$$\langle S(r_1)S(r_2) \rangle_{K^*} = \frac{\text{const}}{|r_1 - r_2|^{2x_h}}. \quad (2.1)$$

At this stage, it is important to stress that the multiplicative constant in (2.1) depends on the lattice mesh a . Actually, the location of operators as described in the equation above are expressed in lattice units. To clarify the argument, let us denote by \hat{r}_j the integer coordinates of a lattice point, and $r_j = a\hat{r}_j$ the corresponding physical location. The correct interpretation for (2.1) is

$$\langle \widehat{S}(\hat{r}_1)\widehat{S}(\hat{r}_2) \rangle_{\mathbb{Z}^d} = \frac{A}{|\hat{r}_1 - \hat{r}_2|^{2x_h}} = \frac{A \times a^{2x_h}}{|r_1 - r_2|^{2x_h}}, \quad (2.2)$$

where we have denoted by \widehat{S} the lattice spin operators. In the limit $a \rightarrow 0$ with (r_1, r_2) fixed, the points (\hat{r}_1, \hat{r}_2) become infinitely separated on the lattice, and the two-point function vanishes. In order to obtain a finite result, we need to introduce a normalisation for the operator S in the continuum limit:

$$S(r) := A^{-1/2} \times a^{-x_h} \widehat{S}(r/a), \quad (2.3)$$

so that

$$\langle S(r_1)S(r_2) \rangle_{\mathbb{R}^d} = \frac{1}{|r_1 - r_2|^{2x_h}}. \quad (2.4)$$

For n -point correlation functions of general operators, we apply the similar definition

$$\phi_j(r) := A_j^{-1/2} \times \lim_{a \rightarrow 0} [a^{-x_j} \widehat{\phi}_j(r/a)] , \quad (2.5)$$

where $\widehat{\phi}_j(\widehat{r})$ is a lattice operator, $\phi_j(r)$ is its continuum counterpart, and A_j is the amplitude of the lattice two-point function $\langle \widehat{\phi}_j \widehat{\phi}_j \rangle$. The scale covariance property (1.54) for the $\widehat{\phi}_j$'s ensures that the continuum correlation function $\langle \phi_1(r_1) \dots \phi_n(r_n) \rangle_{\mathcal{D}}$ remains finite as $a \rightarrow 0$.

The procedure to define the continuum limit in the *vicinity* of the critical point – often called the scaling limit – is a little bit more subtle. If the parameters are set to $K = (K_1, K_2, \dots)$ distinct from the RG fixed point K^* , and if at least one of the scaling variables, say u_t , is relevant, then the RG flow diverges far from K^* : after p iterations, $u_t \mapsto \lambda^{py_t} u_t$. The proper definition of the scaling limit in this situation, is to let $a \rightarrow 0$ and $u_t \rightarrow 0$, while keeping the ratio $m = u_t^{1/y_t} / a$ fixed, for every relevant variable u_t . The quantity m is called the *mass scale*, and has the dimension of an inverse length. The correlation length $\xi \propto m^{-1}$ is thus kept fixed and finite in this scaling limit.

2.1.2 Conformal mappings

In the previous section, we have seen from RG arguments that, at the critical point, correlation functions of scaling fields are expected to be covariant under scale transformations: see (1.54). The basic assumption of CFT is that this property extends to more general space transformations: the conformal maps, *i.e.* the diffeomorphisms which locally preserve angles.

Let us describe more precisely these conformal maps. Consider a manifold equipped with a metric $ds^2 = g_{\mu\nu}(r)dr^\mu dr^\nu$, where $g_{\mu\nu}(r)$ is the metric tensor. The quantity ds gives the length of the infinitesimal vector (dr^1, \dots, dr^d) , and may be thought of as the continuum limit of the distance on the lattice. The usual Euclidean metric is simply $g_{\mu\nu}(r) = \delta_{\mu\nu}$ (it corresponds to the continuum limit of a regular lattice). Consider a differentiable map $r \mapsto r'$. It naturally defines the transformation of the metric tensor:

$$g'_{\mu\nu} = g_{\sigma\rho} \frac{\partial r^\sigma}{\partial r'^\mu} \frac{\partial r^\rho}{\partial r'^\nu} , \quad (2.6)$$

so that $g'_{\mu\nu}(r')dr'^\mu dr'^\nu = g_{\sigma\rho}(r)dr^\sigma dr^\rho = ds^2$. The mapping is conformal iff the new tensor $g'_{\mu\nu}$ is proportional to $g_{\mu\nu}$:

$$g'_{\mu\nu}(r') = \lambda(r)^2 g_{\mu\nu} , \quad (2.7)$$

where $\lambda(r)$ is the local scaling factor.

It is easier to begin with infinitesimal conformal mappings:

$$r^\mu \mapsto r'^\mu = r^\mu + \alpha^\mu(r) .$$

At first order in α , the metric tensor transforms as

$$g'_{\mu\nu} = g_{\mu\nu} - \left(g_{\sigma\nu} \frac{\partial \alpha^\sigma}{\partial r^\mu} + g_{\mu\rho} \frac{\partial \alpha^\rho}{\partial r^\nu} \right) . \quad (2.8)$$

Hence, the mapping defined by $\alpha(r)$ is conformal if and only if there exists a scalar function $\lambda(r) = 1 + \beta(r)$ such that

$$g_{\sigma\nu}(r) \frac{\partial \alpha^\sigma}{\partial r^\mu} + g_{\mu\rho}(r) \frac{\partial \alpha^\rho}{\partial r^\nu} = -2\beta(r) g_{\mu\nu}(r) \quad (2.9)$$

If we specialise to the space \mathbb{R}^d with Euclidean metric $g_{\mu\nu}(r) = \delta_{\mu\nu}$, we obtain the condition:

$$\frac{\partial \alpha_\nu}{\partial r^\mu} + \frac{\partial \alpha_\mu}{\partial r^\nu} = -2\beta(r) \delta_{\mu\nu}. \quad (2.10)$$

Examples of infinitesimal conformal mappings are

- Translations: $\alpha^\mu(r) = b^\mu$, where b^μ is a constant vector,
- Dilatations: $\alpha^\mu(r) = \beta r^\mu$, where β is a constant scalar,
- Rotations: $\alpha^\mu(r) = \theta^{\mu\nu} r_\nu$, where $\theta^{\mu\nu}$ is a constant antisymmetric tensor,
- Special conformal transformations: $\alpha^\mu(r) = 2(b \cdot r)r^\mu - |r|^2 b^\mu$, where b^μ is a constant vector.

The special conformal transformations are obtained by conjugating an infinitesimal translation with the inversion map:

$$I^\mu(r) = \frac{r^\mu}{|r|^2}. \quad (2.11)$$

By explicit computation, one finds that the Euclidean metric maps to $g'_{\mu\nu} = |r|^4 \delta_{\mu\nu}$ under this inversion, and hence it is indeed conformal, with a local scaling factor $\lambda(r) = |r|^2$. For dimension $d > 2$, the space of infinitesimal conformal maps, *i.e.* the solution space of (2.10), is *finite-dimensional*.

In dimension $d = 2$, the situation is pretty different. With the complex variable $z = r^0 + ir^1$, the conditions (2.10) are equivalent to the Cauchy-Riemann equations for the function $f = \alpha_0 + i\alpha_1$. One can show that any holomorphic function $f(z)$ defines an infinitesimal conformal map $z \mapsto z + \epsilon f(z)$. Hence, in dimension $d = 2$, *the space of infinitesimal conformal maps is infinite-dimensional*. It generates a very rich symmetry for correlation functions, which allows a very deep analysis and classification of second-order phase transitions.

To describe *finite* conformal mappings, it is convenient to work with the compactified plane $\mathbb{C} \cup \{\infty\}$, which has the topology of a sphere. The conformal mappings from the compactified plane to itself are *called global conformal mappings*. One can show that they are all of the form:

$$z \mapsto w = \frac{az + b}{cz + d}, \quad (2.12)$$

where a, b, c, d are complex numbers satisfying the condition $ad - bc = 1$. These are called Moebius transformations (or homographic maps). More generally, a non-global conformal transformation will map the plane to some other surface. For instance, the function $[z \mapsto w = \frac{L}{2\pi} \log(z)]$ maps the plane to the infinite cylinder of circumference L .

2.1.3 Conformal covariance

We are now in the position to present the main assumption of conformal field theory: we expect that, at the critical point, there exists a collection of scaling fields, the *primary fields*, whose correlation functions obey a covariance equation generalising (1.54) to any conformal mapping. If the domains \mathcal{D} and \mathcal{D}' are related by a conformal mapping:

$$\begin{cases} \mathcal{D} & \rightarrow \mathcal{D}' \\ r & \mapsto r', \end{cases} \quad (2.13)$$

and ϕ_1, \dots, ϕ_n are scalar primary operators, then we have the identity:

$$\langle \phi_1(r'_1) \dots \phi_n(r'_n) \rangle_{\mathcal{D}'} = \left(\prod_{j=1}^n \left| \frac{\partial r'_j}{\partial r_j} \right|^{-x_j/d} \right) \langle \phi_1(r_1) \dots \phi_n(r_n) \rangle_{\mathcal{D}}, \quad (2.14)$$

where $|\partial r'/\partial r|$ denotes the determinant of the Jacobian matrix of the map (2.13). The quantity

$$\lambda(r) = |\partial r'/\partial r|^{-1/d} \quad (2.15)$$

is nothing but the scaling factor of the metric tensor associated to the conformal map, as in (2.7). It is important to stress that, on both sides of (2.14), the average values correspond to the continuum limit of the domains \mathcal{D} and \mathcal{D}' *with the same lattice discretisation*. If the RHS is the continuum limit of, say, the square lattice Ising model, then the lattice model corresponding to the LHS also lives on the square lattice, *not* its image by the conformal mapping.

The relation (2.14) assumes that the ϕ_j 's are scalar under rotations. Non-scalar operators will be described in the two-dimensional case.

A larger class of operators is given by the *quasi-primary operators*, for which one imposes the covariance relation (2.14) only for global conformal mappings, *i.e.* in the case $\mathcal{D} = \mathcal{D}' = \mathbb{R}^d$, where the conformal map is a composition of translations, dilatations, rotations and special conformal transformations. Of course, any primary operator is also quasi-primary.

2.1.4 Two- and three-point functions

As a simple consequence of the covariance of correlation functions under conformal maps, we have

$$\langle \phi_j(r_1) \phi_k(r_2) \rangle_{\mathbb{R}^d} = \begin{cases} \frac{\mathcal{A}_j}{|r_1 - r_2|^{2x_j}} & \text{if } x_j = x_k, \\ 0 & \text{otherwise.} \end{cases} \quad (2.16)$$

$$\langle \phi_j(r_1) \phi_k(r_2) \phi_\ell(r_3) \rangle_{\mathbb{R}^d} = \frac{\mathcal{C}_{jkl}}{|r_1 - r_2|^{x_j+x_k-x_\ell} |r_2 - r_3|^{x_k+x_\ell-x_j} |r_1 - r_3|^{x_j+x_\ell-x_k}}, \quad (2.17)$$

where $\phi_j, \phi_k, \phi_\ell$ are any quasi-primary operators with scaling dimensions x_j, x_k, x_ℓ , respectively. The \mathcal{A}_j are arbitrary normalisation factors for the operators. The constants \mathcal{C}_{jkl} cannot be obtained directly from (2.14), they are derived from a deeper analysis of each CFT model.

2.1.5 The two-dimensional case

Throughout this course, we shall concentrate on the case $d = 2$. We introduce the complex coordinates:

$$z = r^0 + ir^1, \quad \bar{z} = r^0 - ir^1. \quad (2.18)$$

The associated derivatives read:

$$\partial := \frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial r^0} - i \frac{\partial}{\partial r^1} \right), \quad \bar{\partial} := \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial r^0} + i \frac{\partial}{\partial r^1} \right). \quad (2.19)$$

To emphasize the fact that (correlation functions of) a scaling operator ϕ typically have nonzero derivatives $\langle \partial\phi \dots \rangle \neq 0$ and $\langle \bar{\partial}\phi \dots \rangle \neq 0$, we shall denote it by $\phi(z, \bar{z})$.

A scaling operator is called scalar if it transforms trivially under a rotation θ , *i.e.* $\phi(z, \bar{z}) \rightarrow \phi(e^{i\theta}z, e^{-i\theta}\bar{z})$ in a correlation function. Non-scalar operators are those which transform as

$$\phi(z, \bar{z}) \rightarrow e^{-i\theta s} \phi(e^{i\theta}z, e^{-i\theta}\bar{z}), \quad (2.20)$$

where the real number s is called the *conformal spin*. Hence, a general scaling operator ϕ_j is characterised by its scaling dimension x_j and its conformal spin s_j . Alternatively, we shall use the *conformal dimensions*, defined as

$$x_j = h_j + \bar{h}_j, \quad s_j = h_j - \bar{h}_j. \quad (2.21)$$

The conformal covariance relation for general (possibly non-scalar) primary operators now reads

$$\langle \phi_1(w_1, \bar{w}_1) \dots \phi_n(w_n, \bar{w}_n) \rangle_{\mathcal{D}'} = \left[\prod_{j=1}^n \left(\frac{\partial w_j}{\partial z_j} \right)^{-h_j} \left(\frac{\partial \bar{w}_j}{\partial \bar{z}_j} \right)^{-\bar{h}_j} \right] \langle \phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) \rangle_{\mathcal{D}},$$

(2.22)

where we have considered the conformal map:

$$\begin{cases} \mathcal{D} & \rightarrow \mathcal{D}' \\ z & \mapsto w = w(z). \end{cases}$$

Equation (2.22) is the fundamental relation for primary operators. As we shall see, it encodes a very rich symmetry, and it will be the basis for the construction of CFT.

By convention, we shall choose a basis of primary operators $\{\phi_j\}$ and a dual basis $\{\phi_j^*\}$ so that the two-point functions read:

$$\langle \phi_j(z_1, \bar{z}_1) \phi_k^*(z_2, \bar{z}_2) \rangle_{\mathbb{C}} = \frac{\delta_{jk}}{(z_1 - z_2)^{2h_j} (\bar{z}_1 - \bar{z}_2)^{2\bar{h}_j}}. \quad (2.23)$$

The three-point functions are then of the form:

$$\langle \phi_j(z_1, \bar{z}_1) \phi_k(z_2, \bar{z}_2) \phi_\ell(z_3, \bar{z}_3) \rangle_{\mathbb{C}} = \frac{C_{jkl}}{z_{12}^{h_{jk}^\ell} \bar{z}_{12}^{\bar{h}_{jk}^\ell} z_{13}^{h_{j\ell}^k} \bar{z}_{13}^{\bar{h}_{j\ell}^k} z_{23}^{h_{k\ell}^j} \bar{z}_{23}^{\bar{h}_{k\ell}^j}}, \quad (2.24)$$

where $z_{ab} = z_a - z_b$, and $h_{ab}^c = h_a + h_b - h_c$.

2.2 Space transformations

Unless otherwise specified we are working in the Euclidean setting : all directions are spacelike, and there is no time. This is mostly a matter of convention, as the following could also be formulated in the Minkowski setting. We will use indiscriminately the terms *space* and *spacetime*.

We will be mostly interested in the d -dimensional Euclidean space, that is \mathbb{R}^d equipped with the usual notion of distance. More generally we can (and we will) consider arbitrary Riemannian manifolds (M, g) , that is smooth manifolds M endowed with a (positive-definite) metric g .

While we will use some terminology of differential and Riemannian geometry, we will keep it to a minimum and try to give expressions in local coordinates as much as possible in order not to block the reader unfamiliar with these notions.

An extremely brief appendix on Riemannian geometry containing a few definitions and formulae is given at the end of these lecture notes, mostly to fix notations/conventions.

By a space transformation we mean a diffeomorphism $f : M \rightarrow M$, that is to say a smooth bijection whose inverse f^{-1} is also smooth. Note that space transformations as we just defined are *not* change of coordinates, but actual transformations of the manifold which may be visualized as smooth deformations of a continuous medium.

2.2.1 Flow generated by a vector field

The set of diffeomorphisms $\text{Diff}(M)$ of a given manifold M is a huge group, and describing all possible diffeomorphisms is a rather daunting task. There is however a very convenient way to construct many diffeomorphisms using vector fields.

Consider the stationary flow of a fluid : any point of the fluid has a particular velocity, and in this way there is a vector field associated to any flow. The converse is also true: it is possible to associate a flow to a vector field having that vector field as its velocity.

The crucial point is that the time evolution of such a flow is a diffeomorphism. Let's see how this works in more detail.

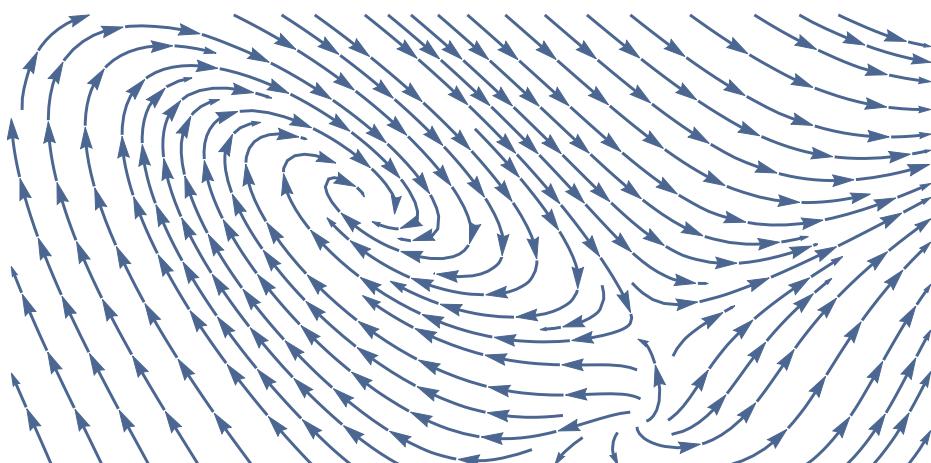


Figure 2.1: A two-dimensional flow

Let's start from an arbitrary vector field ξ . An integral curve $t \rightarrow c(t)$ is a solution of

$$c'(t) = \xi(c(t)). \quad (2.25)$$

In local coordinates $\xi = \xi^\mu(x)\partial_\mu$ the curve $c(t) = (c^1(t), \dots, c^d(t))$ obeys the following system of differential equations

$$\frac{dc^\mu}{dt} = \xi^\mu(c^1, \dots, c^d). \quad (2.26)$$

These equations simply say that the vector tangent to the curve at any point $c(t)$ along the curve is precisely the vector $\xi(c(t))$, and so the curve $t \rightarrow c(t)$ is tangent at each point to the vector field ξ .

For a given point x , there is a unique integral curve¹ $t \rightarrow c_x(t)$ with initial condition $c_x(0) = x$, and furthermore $c_x(t)$ depends smoothly on x . This means that the map

$$f_t : x \rightarrow c_x(t) \quad (2.27)$$

is a smooth map. Moreover it is clear that if $t \rightarrow c(t)$ is an integral curve, then so is $t \rightarrow c(t+s)$. From the abovementioned unicity this means that $c_x(t+s) = c_{c_x(s)}(t)$, or equivalently

$$f_{t+s} = f_t \circ f_s = f_s \circ f_t. \quad (2.28)$$

Because of the above relation, the functions f_t are sometimes called the one-parameter group associated to the vector field ξ . In particular f_t is bijective, with smooth inverse f_{-t} . All this means that $x \rightarrow f_t(x)$ is a diffeomorphism. It is the time evolution at time t of the flow induced by the vector field ξ . One should think of the vector field ξ as the generator of the infinitesimal transformation

$$f_t(x) = x + t\xi + O(t^2). \quad (2.29)$$

► **Example 1 :** The flow induced on \mathbb{R} by the constant vector field $\xi = a\partial_x$ is the translation by at

$$f_t(x) = x + at \quad (2.30)$$

Clearly $f_t \circ f_s = f_{t+s}$.

► **Example 2 :** The flow induced on \mathbb{R} by the vector field $\xi = x\partial_x$ is the dilation by a factor e^t :

$$f_t(x) = e^t x \quad (2.31)$$

It is easy to check that f_{-t} is indeed the inverse of f_t .

¹This curve may or may not be defined at all times : it can happen that the flow blows up to infinity in finite time (see example 2 and special conformal transformations for instance). For now we simply ignore this fact and suppose that the flow is well defined at all times. Vector fields with this property are called *complete*. For instance this is the case of any vector field with compact support, so *a fortiori* any vector field on a compact manifold.

► **Example 3 :** The flow induced on \mathbb{R}^2 by the vector field $\xi = -x_2\partial_1 + x_1\partial_2$ is the rotation

$$f_t \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \quad (2.32)$$

► **Example 4 :** The flow induced on \mathbb{R} by the vector field $\xi = x^2\partial_x$ is

$$f_t(x) = \frac{x}{1 - tx}. \quad (2.33)$$

It is only defined for $xt < 1$, since the flow blows up to infinity as $t \rightarrow 1/x$. An elegant way to cure this is to add a point at infinity. This amounts to work on the one-point compactification of the real line $\mathbb{R} \cup \{\infty\}$, *i.e.* the circle S^1 , as given by the inverse stereographic projection. The stereographic projection is the following diffeomorphism from the unit circle minus the north pole $N = (0, 1)$ to the real line (see Fig (2.2)) :

$$(x_1, x_2) \mapsto \frac{x_1}{1 - x_2} \quad (2.34)$$

If we parametrize the circle by θ as in (see Fig (2.2)), this means

$$x = \tan \frac{\theta}{2} \quad (2.35)$$

The (pullback of the) vector field ξ now reads on the circle $\xi = 2 \sin^2 \frac{\theta}{2} \partial_\theta$, and it is perfectly well behaved at the North pole $\theta = \pi$. Since it has compact support, the corresponding flow on the circle is defined at all times [exercise : compute this flow].

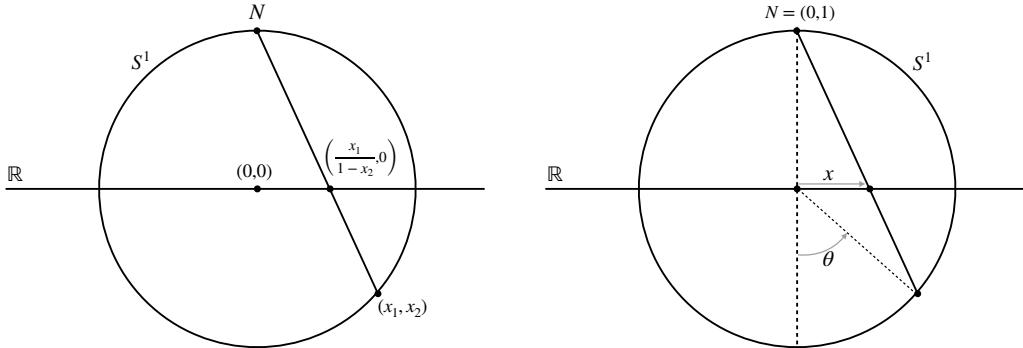


Figure 2.2: The one-point compactification of the real line is diffeomorphic to the unit circle $S^1 = \{(x_1, x_2) \in \mathbb{R}^2, x_1^2 + x_2^2 = 1\}$. The North pole $N = (0, 1)$ corresponds to the point added at infinity.

► **Exercise :** compute the flows of the following vector fields in \mathbb{R}^d

- $\xi = a^\mu \partial_\mu$ for some constant vector a^μ .
- $\xi = x^\mu \partial_\mu$
- $\xi = A^\mu_\nu x^\nu \partial_\mu$
- $\xi = (2a \cdot x x^\mu - x^2 a^\mu) \partial_\mu$.

For the last one, use the one-point compactification of \mathbb{R}^d (as given by the inverse stereographic projection) to construct a flow on the d -dimensional sphere S^d .

2.2.2 Isometries in Euclidean space

Perhaps the most familiar transformations of Euclidean space \mathbb{R}^d are isometries *i.e.* distance-preserving diffeomorphisms. In Euclidean space the distance between two points x and y is of course $d(x, y) = \|x - y\|$, and a map f is distance-preserving when

$$d(f(x), f(y)) = d(x, y) \quad (2.36)$$

Isometries of the Euclidean space form a $\frac{d(d+1)}{2}$ -dimensional Lie group called the Euclidean group² $\text{ISO}(d)$. Isometries are compositions of a translation and an orthogonal transformation

$$f(x) = a + R \cdot x, \quad R \in O(d) \quad (2.37)$$

where R is an orthogonal matrix, *i.e.* $RR^t = 1$. The proof is left as an exercise to the reader. The corresponding Lie algebra is spanned by the vector fields

$$\xi^\mu = a^\mu + \omega^\mu_\nu x^\nu \quad (2.38)$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$. From a previous exercise the reader will recognize the d generators of translations and the $\frac{d(d-1)}{2}$ generators of rotations (there are no reflections of the infinitesimal persuasion). All this is certainly very standard material, but we will use isometries as an excuse to prepare the way towards conformal transformations, and to introduce a few notions of differential geometry such as the pushforward and the pullback.

In order to make contact with isometries in the more general setting of Riemannian manifolds, we are going to give a slightly different but equivalent formulation. An isometry is a diffeomorphism that preserves the length of curves, in the sense that if $c : [t_0, t_1] \rightarrow \mathbb{R}^d$ is a smooth curve, then $\text{length}[c] = \text{length}[f \circ c]$:

$$\int_{t_0}^{t_1} \|c'(t)\| dt = \int_{t_0}^{t_1} \|(f \circ c)'(t)\| dt. \quad (2.39)$$

where $c' = \frac{dc}{dt}$ is the velocity of the curve. Differentiating with respect to t_1 , we find that this is equivalent to

$$\|c'(t)\| = \|(f \circ c)'(t)\| \quad (2.40)$$

In the above expression $(f \circ c)'(t)$ is the image of the tangent vector $c'(t)$ by the map f

$$(f \circ c)'(t) = df_{c(t)} \cdot c'(t) \quad (2.41)$$

where $df_{c(t)}$ is the Jacobian of f at $x = c(t)$

$$df_x = \begin{pmatrix} \partial_1 f^1(x) & \dots & \partial_d f^1(x) \\ \vdots & & \vdots \\ \partial_1 f^d(x) & \dots & \partial_d f^d(x) \end{pmatrix}. \quad (2.42)$$

²this is the analogue of the Poincaré group for Minkowski space, which is the group of transformation preserving spacetime intervals.

So we have found that f is an isometry if and only if its Jacobian matrix is everywhere orthogonal $df_x df_x^t = 1$ or in coordinates

$$\partial_\mu f^\rho(x) \delta_{\rho\sigma} \partial_\nu f^\sigma(x) = \delta_{\mu\nu} \quad (2.43)$$

Let us introduce some terminology that will be useful later. The differential of f at x is the linear map $v \rightarrow df_x \cdot v$. Given a vector field $\xi(x)$, its image by a diffeomorphism f is called the pushforward $f_* \xi$

$$(f_* \xi)(x) = df_x \cdot \xi(f(x)) \quad (2.44)$$

Given a differential form α (say of degree p), the pullback by f is the p -form $f^* \alpha$ defined by

$$(f^* \alpha)_x(v_1, \dots, v_p) = \alpha_{f(x)}(df_x \cdot v_1, \dots, df_x \cdot v_p) \quad (2.45)$$

The relation (2.43) can be rewritten as

$$f^* \eta = \eta \quad (2.46)$$

where $\eta = \delta_{\mu\nu} dx^\mu \otimes dx^\nu$ is the Euclidean metric.

To sum up the above discussion about $\text{Iso}(d)$, we have three equivalent characterizations of isometries. A diffeomorphism f is an isometry of the Euclidean space \mathbb{R}^d iff

- f is distance preserving : $d(f(x), f(y)) = d(x, y)$
- f preserves the length of curves : $\text{length}[c] = \text{length}[f \circ c]$
- f leaves the Euclidean metric invariant : $f^* \eta = \eta$

2.2.3 Isometries in Riemannian geometry, Killing vector fields

This discussion can be extended straightforwardly to Riemannian geometry, as soon as one has defined a notion of distance. The Riemannian metric allows one to define the length $\|v\| = \sqrt{g_p(v, v)}$ of a vector $v \in T_p M$ in the tangent space at p . Then one can define the length of a curve $c : [t_0, t_1] \rightarrow M$ as

$$L(c) = \int_{t_0}^{t_1} \|c'(t)\| dt \quad (2.47)$$

Finally the distance $d(x, y)$ between two points x and y is defined as the length of the shortest curve joining x and y . With this setting in place, we have again several equivalent characterizations of isometries.

Let $f : (M, g) \rightarrow (\tilde{M}, \tilde{g})$ be a diffeomorphism between two Riemannian manifolds, then the following are equivalent

- f is an isometry

- f is distance preserving : $d(f(x), f(y)) = d(x, y)$
- f preserves the length of curves : $\text{length}[c] = \text{length}[f \circ c]$
- $f^* \tilde{g} = g$ i.e. in local coordinates $g_{\mu\nu}(x) = \tilde{g}_{\rho\sigma}(f(x)) \partial_\mu f^\rho(x) \partial_\nu f^\sigma(x)$

The Riemannian manifolds (M, g) and (\tilde{M}, \tilde{g}) are then completely equivalent as far as geometry is concerned. Indeed isometries leave everything metric-related (lengths, angles, curvature, parallel transport) invariant, the image of a geodesic is a geodesic, etc...

Before moving on to conformal transformations, let us consider infinitesimal isometries. The diffeomorphisms f_t induced by the flow of a vector field ξ on a Riemannian manifold (M, g) are isometries iff

$$\mathcal{L}_\xi g = 0 \quad (2.48)$$

where $\mathcal{L}_\xi g$ is the Lie derivative of g with respect to ξ , which reads in local coordinates

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\nu\rho} \partial_\mu \xi^\rho + g_{\mu\rho} \partial_\nu \xi^\rho = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu. \quad (2.49)$$

So the condition for a vector field to be an infinitesimal isometry is that

$$\boxed{\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = 0.} \quad (2.50)$$

This is called the Killing equation, and infinitesimal generators of isometries are called Killing vector fields or Killing vectors.

Exercise : recover the flat space solutions (2.38) from the Killing equation.

The isometries of a Riemannian manifold (M, g) form a Lie group of dimension at most $\frac{d(d+1)}{2}$. In that sense the Euclidean space is maximally symmetric, while a generic Riemannian manifold (M, g) has no isometries (besides the trivial one $x \rightarrow x$). This should not come as a surprise : isometries leave curvature invariant. For instance a space of constant sectional curvature saturates the number $\frac{d(d+1)}{2}$ of (linearly independent) Killing vector fields (although this does not imply the existence of global isometries, as the flow might not be well defined). But it so happens that the infinitesimal transformations will prove sufficient for the purpose of deriving Ward identities. This will turn out to be a crucial point for conformal invariance in two dimensions. While the number of globally defined conformal transformations is always finite dimensional, the infinitesimal generators form an infinite dimensional Lie algebra in Euclidean space !

2.3 Conformal transformations

On a Riemannian manifold the metric allows one to define the angle θ between two vectors v_1 and v_2 tangent to the manifold (at the same point) through

$$\cos \theta = \frac{g(v_1, v_2)}{\|v_1\| \|v_2\|} \quad (2.51)$$

If two curves $c_1(t)$ and $c_2(t)$ intersect each other - say at $t = 0$ - with a non-zero velocity, then the angle between the two curves is defined as the angle between $v_1 = c'_1(0)$ and $v_2 = c'_2(0)$ (see Fig (2.3)). It is clear that isometries are angle-preserving, since we can

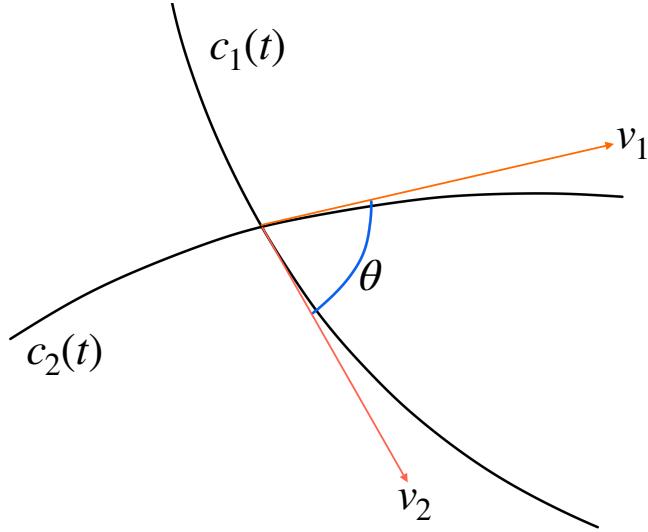


Figure 2.3: Two curves c_1 and c_2 intersect each other.

rewrite (2.51) as

$$\cos \theta = \frac{\|v_1 + v_2\|^2 - \|v_1\|^2 \|v_2\|^2}{2\|v_1\| \|v_2\|} \quad (2.52)$$

but the reverse is not true : there exists angle-preserving maps that do not preserve length. The canonical example are scale transformations (or dilations) $x \rightarrow \lambda x$ in Euclidean space. More generally a diffeomorphism that preserves angles is called a *conformal transformation*. We can rephrase the above definition in terms of curve by saying that the angles as measured by the metrics g and $f^*\tilde{g}$ must agree. This means

$$f^*\tilde{g} = \Omega^2 g$$

(2.53)

for some function Ω called the conformal factor. In coordinates this reads

$$g_{\rho\sigma}(f(x))\partial_\mu f^\rho \partial_\nu f^\sigma = \Omega^2(x)g_{\mu\nu}(x). \quad (2.54)$$

This follows from the following fact : two inner products on a vector space induce the same notion of angle if and only if they are proportional. This is a simple exercise in linear algebra which is left to the reader.

Let $f : (M, g) \rightarrow (\tilde{M}, \tilde{g})$ be a diffeomorphism between two Riemannian manifolds, then the following are equivalent

- f is a conformal map
- f preserves angles between curves
- $f^*\tilde{g} = \Omega^2 g$ for some smooth function Ω

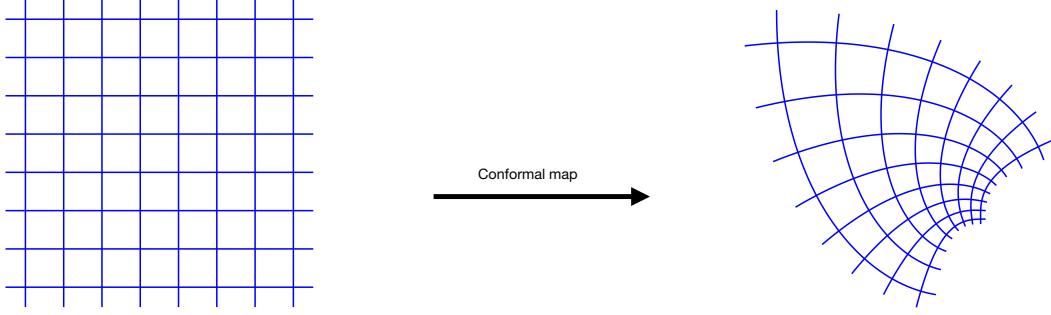


Figure 2.4: Image of a grid by a conformal map : lengths are not preserved, but angles between curves are.

Some remarks.

Two Riemannian metrics g_1 and g_2 are called conformally equivalent if $g_1 = \Omega^2 g_2$ for some function Ω , *i.e.* when they define the same angles. An equivalence class of such metrics is called a *conformal structure* or *conformal class*. A change of metric within a given conformal class is called a Weyl transformation :

$$g_1 \rightarrow g_2 = \Omega^2 g_1 . \quad (2.55)$$

One can think of the Weyl transformation as a deformation of the Riemannian manifold, or as the following conformal map

$$\begin{aligned} f : (M, g_1) &\rightarrow (M, g_2) \\ x &\mapsto x . \end{aligned} \quad (2.56)$$

Since conformal transformations only care about angles, not about lengths, Weyl transformations are a very natural deformation to consider. If $f : (M, g) \rightarrow (\tilde{M}, \tilde{g})$ is a conformal map, then it remains a conformal map after a Weyl rescaling of g and/or \tilde{g} . In that sense one does not need to specify a metric to talk about a conformal map, but only a conformal structure. It follows that if ξ is a conformal Killing vector field *w.r.t.* some metric, then it is a conformal Killing vector field *w.r.t.* all metrics in the same conformal class.

Equivalently, a conformal map $f : (M, g) \rightarrow (\tilde{M}, \tilde{g})$ such that $f^* \tilde{g} = \Omega^2 g$ can be decomposed into a Weyl rescaling $g \rightarrow \Omega^2 g$ followed by an isometry. For instance the inversion $x \rightarrow x / \|x\|^2$ is an isometry from $\mathbb{R}^d \setminus \{0\}$ with the metric $g_{\mu\nu} = \frac{1}{\|x\|^4} \delta_{\mu\nu}$ to $\mathbb{R}^d \setminus \{0\}$ with the Euclidean metric.

Three important examples.

- The inversion

$$\begin{aligned} f : \mathbb{R}^d \setminus \{0\} &\rightarrow \mathbb{R}^d \setminus \{0\} \\ x &\mapsto \frac{x}{\|x\|^2} \end{aligned} \quad (2.57)$$

- The Euclidean space with the origin removed $\mathbb{R}^d \setminus \{0\}$ is conformally equivalent to the d -dimensional “cylinder” $\mathbb{R} \times S^{d-1}$ with the canonical metric, through the conformal map

$$f(x) = \left(\log \|x\|, \frac{x}{\|x\|} \right) \quad (2.58)$$

In particular in two dimensions this yield the following map

$$\begin{aligned} f : \mathbb{C}^* &\rightarrow \mathbb{C}/2i\pi\mathbb{Z} \\ z &\mapsto \log z \end{aligned} \quad (2.59)$$

- **Stereographic projection** The d -dimensional sphere S^d (with a point removed) is conformally equivalent to the Euclidean space, through the stereographic projection. To define this map, we first consider the canonical embedding $S^d = \{x \in \mathbb{R}^{d+1}, \|x\|^2 = 1\}$, and we define

$$f : S^d \setminus \{N\} \rightarrow \mathbb{R}^d \quad (2.60)$$

$$(x_1, \dots, x_{d+1}) \mapsto \frac{1}{1 - x_{d+1}}(x_1, \dots, x_d) \quad (2.61)$$

where N is the North pole $(0, \dots, 0, 1)$. This means that the sphere S^d (with the round metric) is *conformally flat* : each point has a neighborhood that can be mapped to flat space by a conformal transformation.

$$f^* \tilde{g} = \sum_{i=1}^d d\left(\frac{x_i}{1-x_{d+1}}\right) \otimes d\left(\frac{x_i}{1-x_{d+1}}\right) = \frac{1}{(1-x_{d+1})^2} \sum_{i=1}^{d+1} dx_i \otimes dx_i = \frac{1}{(1-x_{d+1})^2} g \text{ using } \sum_{i=1}^{d+1} x_i dx_i = 0$$

2.3.1 Conformal Killing vector fields

Let us analyse the condition $f^* g = \Omega^2 g$ for an infinitesimal transformation $f^\mu(x) = x^\mu + \epsilon \xi^\mu(x)$. Expanding the conformal factor $\Omega(x) = 1 + \epsilon \omega(x)$, we get

$$\mathcal{L}_\xi g = 2\omega g. \quad (2.62)$$

This is a constraint on the vector field ξ , which in local coordinates reads

$$\xi^\rho \partial_\rho g_{\mu\nu} + g_{\nu\rho} \partial_\mu \epsilon^\rho + g_{\mu\rho} \partial_\nu \epsilon^\rho = 2\omega(x) g_{\mu\nu}(x), \quad (2.63)$$

or equivalently

$$\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = 2\omega(x) g_{\mu\nu}(x). \quad (2.64)$$

for some *a priori* arbitrary function $\omega(x)$. However tracing the above equation (i.e. contracting with $g^{\mu\nu}$) yields $\omega(x) = \frac{1}{d} \nabla_\mu \xi^\mu = \frac{1}{d} \text{div}(\xi)$, and we get

$$\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = \frac{2}{d} \text{div}(\xi) g_{\mu\nu}(x) \quad (2.65)$$

or equivalently

$$\mathcal{L}_\xi g = \frac{2}{d} \text{div}(\xi) g. \quad (2.66)$$

Such a vector field is called a conformal Killing vector field. We will see that for $d \geq 3$ there are finitely many solutions, but in two dimensions there are infinitely many conformal Killing vector fields !

In Euclidean space \mathbb{R}^d with $d \geq 3$, conformal Killing vector fields form a $\frac{(d+1)(d+2)}{2}$ -dimensional Lie algebra. In addition to translations and rotations, it contains

- dilations $\xi^\mu(x) = x^\mu$
- special conformal transformations $\xi^\mu(x) = 2(a \cdot x)x^\mu - x^2 a^\mu$

The derivation is a standard exercise (left to the reader) and can be found in any textbook about conformal field theory.

The only new thing here are special conformal transformations, whose flow is

$$f_t(x) = \frac{x - bx^2}{1 - 2b \cdot x + b^2 x^2}, \quad b = ta \quad (2.67)$$

which is an inversion $x \rightarrow x/x^2$ followed by a translation $x \rightarrow x - b$, followed by another inversion. This map is not well-defined at $x = \frac{b}{b^2}$: the flow blows up. There are several ways out of this conundrum. The first one is to work on a domain of \mathbb{R}^d that does not contain b/b^2 . The other one is to consider the one-point compactification of \mathbb{R}^d using the inverse stereographic projection. In this case one is really discussing the conformal transformations of the d -dimensional sphere S^d (endowed with the round metric). This is the conformal group $\text{SO}(d+1, 1)$.

Similarly to the case of isometries, on a Riemannian manifold there are at most $\frac{(d+1)(d+2)}{2}$ (linearly independent) conformal Killing vectors, and generically there is the only the trivial one $\xi = 0$.

2.3.2 Conformal maps on the complex plane

In two dimensions the story is quite different, and conformal geometry is very closely related to complex analysis. This is true on any (oriented) two-dimensional Riemannian manifold, but let us first see how this works on the Euclidean plane.

Let's denote by η the flat Euclidean metric ($\eta_{\mu\nu} = \delta_{\mu\nu}$), and recall that a diffeomorphism $f : (\mathbb{R}^2, \eta) \rightarrow (\mathbb{R}^2, \eta)$ from the Euclidean plane to itself is conformal iff $f^*\eta = \Omega^2\eta$ for some smooth function Ω , i.e.

$$\partial_\mu f^\rho \delta_{\rho\sigma} \partial_\nu f^\sigma = \Omega^2(x) \delta_{\mu\nu}. \quad (2.68)$$

This can be rephrased as $df_x^T df_x = \Omega^2(x) \mathbb{I}$, and furthermore taking the determinant of this equation we learn that $\Omega^2(x) = |\det df_x|$. Thus in flat two-dimensional space we have the following alternative characterization

$$df_x^T df_x = |\det df_x| \mathbb{I}. \quad (2.69)$$

or equivalently

$$\text{com}(df_x) = \pm df_x \quad (2.70)$$

where $\text{com}(df_x)$ stands for the comatrix of f , and the \pm sign is the sign of $\det df_x$. Note that this sign does not depend on x ($\det df_x$ cannot vanish by virtue of f being a diffeomorphism), and it simply tells us whether f is orientation preserving ($\det df_x > 0$) or orientation reversing ($\det df_x < 0$). Writing (2.70) explicitly yields

$$\begin{pmatrix} \partial_2 f^2 & -\partial_1 f^2 \\ -\partial_2 f^1 & \partial_1 f^1 \end{pmatrix} = \pm \begin{pmatrix} \partial_1 f^1 & \partial_2 f^1 \\ \partial_1 f^2 & \partial_2 f^2 \end{pmatrix}. \quad (2.71)$$

Depending on the sign, we recognise the Cauchy-Riemann equations or the anti-analytic version of Cauchy-Riemann

$$\begin{cases} \partial_1 f^1 = \pm \partial_2 f^2 \\ \partial_1 f^2 = \mp \partial_2 f^1 \end{cases} \quad (2.72)$$

Thus conformal maps from the Euclidean plane to itself are

- **holomorphic if they are orientation preserving,**
- **anti-holomorphic when they reverse orientation.**

Note that an orientation reversing conformal map is nothing but an orientation-preserving one followed by a reflection (complex conjugation). Thus it is sufficient to discuss orientation-preserving conformal maps.

At this stage it may look like there are many conformal maps in two-dimensions. In some sense this is true, but there is a caveat. So far we have only looked at the local constraint of being angle-conserving, namely $f^* \eta = \Omega^2 \eta$. But let's not forget that the function f must be a diffeomorphism³. This global constraint is very sensitive to the domain of f .

Let's start with conformal maps $f : \mathbb{C} \rightarrow \mathbb{C}$, that is automorphisms of the complex plane \mathbb{C} . It is a fact that the (orientation-preserving) conformal maps $f : \mathbb{C} \rightarrow \mathbb{C}$ are of the form

$$z \rightarrow az + b, \quad a \neq 0. \quad (2.73)$$

that is translations, rotations and dilations. If one adds a point at infinity $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, one gets the Möbius group

$$z \rightarrow \frac{az + b}{cz + d}, \quad ad - bc \neq 0. \quad (2.74)$$

and this now includes special conformal transformations. Proving the above is a standard exercise in complex analysis, and is left to the reader. We recover the full conformal group of the previous section, and indeed at the level of globally defined conformal maps there is no difference between the two-dimensional case and the case $d \geq 3$. So in what sense are there more conformal maps in two-dimensions ?

³A holomorphic diffeomorphism is called an automorphism

To understand this, let us recall the inversion function theorem : a smooth map $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is invertible in a neighborhood of some point x_0 as soon as $\det df_{x_0} \neq 0$. This means that - given a holomorphic map f , and a point x_0 such that $f'(x_0) \neq 0$ - there exist an open neighborhood U of x_0 such that $f : U \rightarrow f(U)$ is a diffeomorphism, and therefore a conformal map. So in this sense any holomorphic map yields a conformal map, albeit not on the whole complex plane. This is to be contrasted with the paucity of conformal maps in higher dimensions : any conformal map on *any domain* of \mathbb{R}^d is an element of the conformal group $\text{SO}(d+1, 1)$. Let us mention a striking and important result that illustrates the wealth of conformal maps in two-dimensions :

Riemann mapping theorem : any simply-connected open subset $U \subset \mathbb{C}$ (with $U \neq \emptyset, \mathbb{C}$) is conformally equivalent to the unit disk

$$\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\} \quad (2.75)$$

This result is remarkable and somehow counter-intuitive, as the domain U can be highly complicated, with a non-differentiable or even fractal boundary. Moreover the result is extremely simple. For non simply-connected domains, the story is more complicated. For instance let's consider the annulus $A_{r,R} = \{z \in \mathbb{C}, r < |z| < R\}$. Schottky theorem states that two annuli A_{r_1,R_1} and A_{r_2,R_2} are conformally equivalent if and only if $R_1/r_1 = R_2/r_2$.

Exercise : show that the unit disk \mathbb{D} and the complex plane \mathbb{C} are not conformally equivalent (hint : use Liouville's theorem : every bounded entire function is constant).

The abundance of conformal maps in two-dimensions is also manifest at the level of conformal-Killing vectors. In the Euclidean plane the conformal Killing condition reads for a vector field $\xi = \xi_1 \partial_1 + \xi_2 \partial_2$

$$\partial_1 \xi_2 + \partial_2 \xi_1 = 0, \quad \partial_1 \xi_1 = \partial_2 \xi_2 \quad (2.76)$$

which is equivalent to

$$(\partial_1 + i\partial_2)(\xi_1 + i\xi_2) = 0 \quad (2.77)$$

and we find

$$\boxed{\xi = \xi(z)\partial + \overline{\xi(z)}\bar{\partial}} \quad (2.78)$$

where $\xi(z)$ is **any** holomorphic function, and $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$. Of course the flow of most of these vector fields will not be globally defined, but this will not be an issue for quantum field theories.

2.3.3 Conformal maps between Riemann surfaces

The above discussion can be extended to arbitrary two-dimensional Riemannian manifolds, and in fact the main result remains.

Conformal maps from a surface (M, g) to another (M, \tilde{g}) are exactly the (anti-)holomorphic maps.

But for such a claim to make sense one first need to define what is meant by a holomorphic map on a generic surface. To do so, the relevant mathematical notion is that of a *complex structure*. Let us first briefly recall how one defines coordinates on a surface M . This is done through an *atlas*, which is a collection of *charts*. A chart is a homeomorphism φ from an open subset $U \subset M$ to an open subset of Euclidean space $\mathbb{R}^2 \simeq \mathbb{C}$. An atlas is a collection $\{(U_\alpha, \varphi_\alpha)\}$ of charts which covers M . In order to define the manifold through an atlas, one needs to know how to piece together the different charts. This is encoded in the *transition functions*. Whenever U_α and U_β overlaps, the transition function is simply

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta) \quad (2.79)$$

On a generic (topological) manifold transition functions are homeomorphisms⁴. This

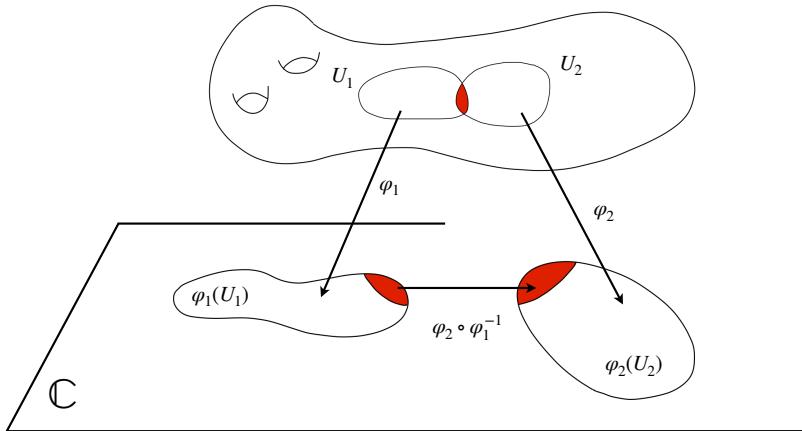


Figure 2.5: Transition function for a two-dimensional manifold.

fixes the topology of the surface, thus the notion of a continuous function $f : M \rightarrow \mathbb{R}$ makes sense. But what does it mean for f to be smooth ? Since the surface is locally like \mathbb{R}^2 , it would seem reasonable to declare f to be smooth on a chart U_1 as long as $f \circ \varphi_1^{-1}$ is smooth (notice that $f \circ \varphi_1^{-1}$ is map defined on a subset of \mathbb{R}^d , for which the notion of smoothness makes sense). However if U_2 is another chart overlapping U_1 , it is not necessarily true that $f \circ \varphi_2^{-1}$ is also smooth, unless the transition function $\varphi_1 \circ \varphi_2^{-1}$ is smooth. Thus in order to talk about smooth functions on a manifold, a *smooth structure* is required, that is an atlas whose transition functions are smooth.

⁴A homeomorphism is a continuous bijection whose inverse is also continuous.

The very same line of reasoning applies to holomorphic functions. A *complex structure* is an atlas whose transition functions are holomorphic. Note that a manifold endowed with such an atlas is necessarily orientable (why?).

But so far we are working on a smooth surface endowed with a metric. So where is the complex structure? It turns out that - as long as the surface is oriented - the metric induces a complex structure. This is based on the fact that two-dimensional Riemannian manifolds are conformally flat, *i.e.* there exists around any point a coordinate system (*i.e.* a chart) such that the metric is of the form $g_{\mu\nu}(x) = e^{2\sigma(x)}\delta_{\mu\nu}$. These are called isothermal coordinates, since each coordinate x^μ is harmonic (the reader is invited to check that $\Delta(x^\mu) = 0$ using Appendix 3.4.3), thus a steady-state solution of the heat equation. For a proof of the existence of these isothermal coordinates (also known as "conformal gauge" in Lorentzian signature), the reader is invited to have a look at Donaldson's *Riemann surfaces*, or Schottenloher's *A Mathematical Introduction to Conformal Field Theory* for the Lorentzian case. So on a Riemannian surface there exists an atlas in which all charts are made of isothermal coordinates. But notice that transition functions between isothermal coordinate systems are conformal by construction. Indeed an isothermal chart (U, φ) is nothing but a conformal map φ from U to Euclidean space. Since the composition of conformal maps is conformal, clearly transition functions are also conformal. The reader not satisfied with the above explanation can also check directly that the pullback of the metric $e^{2\sigma_2}\delta_{\mu\nu}$ by the transition function $\varphi_2 \circ \varphi_1^{-1}$ is $e^{2\sigma_1}\delta_{\mu\nu}$:

$$(\varphi_2 \circ \varphi_1^{-1})^* e^{2\sigma_2} \delta_{\mu\nu} = e^{2\sigma_1} \delta_{\mu\nu} \quad (2.80)$$

According to the analysis done in the previous section, this implies that transition functions between isothermal coordinates are either holomorphic or anti-holomorphic. Provided the surface is oriented, we can always choose our charts to be compatible with the orientation, in which case all transition functions are also orientation preserving, thus holomorphic functions.

A simple and explicit illustration of this construction is given by the two-sphere S^2 endowed with the round metric. Consider the canonical embedding $S^2 = \{x \in \mathbb{R}^3, \|x\|^2 = 1\}$, and denote by $N = (0, 0, 1)$ and $S = (0, 0, -1)$ the North and South pole, respectively. The standard atlas is made of two charts $U_1 = S^2 \setminus N$, and $U_2 = S^2 \setminus S$, with coordinates $\varphi_i : U_i \rightarrow \mathbb{C}$ given by the stereographic projections, namely:

$$\varphi_1(x) = \frac{1}{1-x_3}(x_1 + ix_2) \quad \varphi_2(x) = \frac{1}{1+x_3}(x_1 + ix_2) \quad (2.81)$$

We have already seen in a previous exercise that these are isothermal coordinates (to be precise we have seen that stereographic projections are conformal maps from the round sphere to Euclidean space, which is the same thing). Is the transition function holomorphic? A direct calculation⁵ show that it is fact anti-holomorphic

$$\begin{aligned} \varphi_2 \circ \varphi_1^{-1} : \mathbb{C} \setminus \{0\} &\rightarrow \mathbb{C} \setminus \{0\} \\ z &\rightarrow \frac{1}{\bar{z}} \end{aligned} \quad (2.82)$$

⁵One can first check that $\varphi_1^{-1}(x+iy) = \frac{1}{1+x^2+y^2}(2x, 2y, x^2+y^2-1)$

This stems from the fact that the atlas we chose was not oriented : the two charts have opposite orientations. This is easily fixed, we simply change the orientation of say φ_2 , while not spoiling the isothermal condition. To do so one simply composes φ_2 with any orientation-reversing conformal map of the plane. For instance $(x, y) \rightarrow (x, -y)$, *i.e.* complex conjugation. Thus we replace the chart (U_2, φ_2) by $(U_2, \bar{\varphi}_2)$, and the transition function is now holomorphic :

$$\begin{aligned} \bar{\varphi}_2 \circ \varphi_1^{-1} : \mathbb{C} \setminus \{0\} &\rightarrow \mathbb{C} \setminus \{0\} \\ z &\rightarrow \frac{1}{z} \end{aligned} \tag{2.83}$$

We have ourselves a complex structure on the sphere.

So we have seen how a Riemannian metric g on an oriented surface induces a natural complex structure, via isothermal coordinates. But of course two metrics in the same conformal class induce the same complex structure since they have the same isothermal coordinates. So what we have is that a *conformal structure* induces a unique *complex structure* on a two-dimensional oriented manifold. In fact this also goes the other way. Given a complex structure, *i.e.* an atlas whose transition functions are holomorphic, there is a unique compatible conformal structure. So we have the following

On a two-dimensional oriented manifold a choice of *conformal structure* is equivalent to a choice of *complex structure*.

A two-dimensional manifold equipped with a complex structure is called a *Riemann surface*. Now remember that conformal maps are insensitive to Weyl rescaling, we can work locally with the flat metric $dz d\bar{z}$, and the analysis done in the Euclidean case applies :

A map between oriented Riemannian surfaces is conformal if and only if it is holomorphic or anti-holomorphic.

Lecture 3

Conformal invariance in field theories

3.1 Conformal invariance in classical field theories

After this rather lengthy discussion about space transformations, let us get back to the subject of field theories and conformal invariance. We first describe the response of a classical field theory to infinitesimal space transformations, for which an important protagonist is the *stress-energy tensor* $T^{\mu\nu}$.

3.1.1 The stress-energy tensor

For pedagogical purposes we will initially restrict ourselves to field theories defined on flat space, where traditionally the stress-energy tensor is presented as the conserved Noether current associated with spacetime translations, see Appendix 3.4.1. However this leads to ambiguities in defining the stress-energy tensor, and a more modern and powerful definition of the stress-energy tensor involves working in curved space, see Appendix 3.4.2.

We consider a field theory in flat Euclidean space \mathbb{R}^d , as characterized by a Lagrangian density \mathcal{L}

$$S[\Phi] = \int \mathcal{L}(\Phi(x), \partial_\mu \Phi(x)) d^n x \quad (3.1)$$

which we assume depends only on the fields at x and their first derivatives. Here Φ stands for a collection of fields ϕ_1, ϕ_2, \dots which can be of different type (scalar, vector, spinor, etc). In these lecture notes we will only consider translation invariant theories.

The stress-energy tensor $T^{\mu\nu}$ encodes the response of the action to an arbitrary infinitesimal spacetime transformation (*i.e.* a diffeomorphism) regarded as an active transformation, that is, a displacement of the fields/configurations. Consider an arbitrary (smooth) vector field $\xi^\mu(x)$ and its corresponding flow $x \rightarrow \varphi_t(x)$ as defined by

$$\frac{d\varphi_t(x)}{dt} = \xi(\varphi_t(x)). \quad (3.2)$$

Imagine now that this flow carries the local degrees of freedom for some time t . If the initial configuration is Φ , we will denote by $\Phi' = \varphi_t^* \Phi$ the new configuration. Infinitesimally

(at time $t = \epsilon \ll 1$) the flow is given by

$$x^\mu \rightarrow x^\mu + \epsilon^\mu(x) + O(\epsilon^2) \quad (3.3)$$

where $\epsilon^\mu(x)$ stands for the infinitesimal vector field $\epsilon\xi^\mu(x)$. For a scalar field ϕ , the above transformation means $\phi'(x) = \phi(x + \epsilon(x)) \simeq \phi(x) + \epsilon^\mu \partial_\mu \phi(x)$:

$$\delta_\epsilon \phi(x) = \epsilon^\mu \partial_\mu \phi(x) + O(\epsilon^2),$$

but more generally fields transform covariantly. A gauge field for instance (such as the electromagnetic field) is a field which is locally represented by a differential 1-form $A_\mu dx^\mu$, and therefore¹

$$\delta_\epsilon A_\mu = \epsilon^\rho \partial_\rho A_\mu + (\partial_\mu \epsilon^\rho) A_\rho + O(\epsilon^2), \quad (3.5)$$

while the associated field strength $F = dA$ is a two-form, so

$$\delta_\epsilon F_{\mu\nu} = \epsilon^\rho \partial_\rho F_{\mu\nu} + (\partial_\mu \epsilon^\rho) F_{\rho\nu} + (\partial_\nu \epsilon^\rho) F_{\mu\rho} + O(\epsilon^2). \quad (3.6)$$

More generally the change of a vector field along the flow of a vector field is called the Lie derivative. The Lie derivative is sometimes called fisherman's derivative: the flow carries all fields past the fisherman, and the fisherman sits there and differentiates them.

In a more geometric language, fields are sections of some fiber bundles, and prescribing how they transform under a given diffeomorphism amounts to lifting this diffeomorphism into the corresponding bundle. For tensor fields there is a natural way to do this, called the pullback, and the infinitesimal variation is given by the Lie derivative. For spinors the story is a bit more subtle since diffeomorphisms cannot be lifted globally (for instance spinors transform with a minus sign under a full rotation), but infinitesimal diffeomorphisms can be lifted by introducing a connection on the spinor bundle, called a *spin connection*.

In practice however we will not have to worry about how the fields appearing in the action transform under diffeomorphisms² since we will not even have an action to begin with! All we need to know is that the transformation is local, in the sense that

$$\begin{aligned} \delta_\epsilon \phi(x) &= \epsilon^\rho(x) \partial_\rho \phi(x) + \text{finite number of terms involving} \\ &\quad \text{derivatives of } \epsilon^\mu(x) \text{ and } \phi(x) \end{aligned} \quad (3.7)$$

and that there exists an object called the *stress-energy tensor* $T^{\mu\nu}$ that describes the infinitesimal variation of the action $\delta S = S[\Phi'] - S[\Phi]$ at first order in the vector field ϵ :

$$\delta S = -\frac{1}{2\pi} \int T^{\mu}_{\nu} \partial_\mu \epsilon^\nu d^n x. \quad (3.8)$$

¹Here we ignore gauge invariance, which in principle adds some freedom in defining the action of infinitesimal diffeomorphisms. Indeed there is no reason not to couple space transformations to a gauge transformations, such as

$$\delta_\epsilon A_\mu = \epsilon^\rho \partial_\rho A_\mu + (\partial_\mu \epsilon^\rho) A_\rho - \partial_\mu (\epsilon^\rho A_\rho) + O(\epsilon^2). \quad (3.4)$$

²All we will need is how they transform under conformal transformations.

The prefactor is conventional, and has been chosen for future convenience.

The stress-energy tensor is conserved on-shell by construction³

$$\partial_\mu T^{\mu\nu} = 0 \quad \text{on-shell.} \quad (3.9)$$

Physically the components of the stress-energy tensor describe the density and flux of energy and momentum in spacetime (strictly speaking this interpretation requires a Lorentzian metric), and the associated conserved charges are energy and momentum

$$E = \int T^{00}(t, x) d^{n-1}x, \quad P^i = \int T^{0i}(t, x) d^{n-1}x \quad (3.10)$$

At the classical level this interpretation follows from Noether's theorem (see Appendix 3.4.1). For a quantum field theory these charges will become the generators of spacetime translations. We will come back to this interpretation once we have derived the Ward identity (3.60)).

It must be stressed that the stress-energy tensor is a particularly subtle notion of field theory, with several more or less equivalent definitions available (see Appendices 3.4.1 and 3.4.2). Consider for instance the field theory of electromagnetism. The setup is 3+1 dimensional Minkowski space with metric $\eta_{\mu\nu}$, and we choose the signature $(-+++)$. The Lagrangian density is

$$\mathcal{L}[A_\mu] = -\frac{1}{8\pi} F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (3.11)$$

where the components of the field strength are related to the electric and magnetic fields through $F_{0i} = E_i$ and $F_{ij} = \epsilon^{ijk} B_k$.

The correct stress-energy tensor is well known from Maxwell's equations. The energy density of the electromagnetic field is $T^{00} = \frac{1}{2}(E^2 + B^2)$, while the density of momentum is given by the Poynting vector $T^{0i} = (E \times B)_i$. Finally $T^{ij} = \frac{1}{2}(E^2 + B^2)\delta_{ij} - E_i E_j - B_i B_j$ is (minus the) Maxwell stress-tensor. More compactly

$$T^{\mu\nu} = F^{\mu\sigma} F^\nu_\sigma - \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \quad (3.12)$$

Exercise : Compare the above stress-energy tensor with the various definitions of the stress-energy tensor, namely

- the canonical SET as defined through Noether's theorem (see eq. (3.103) in appendix 3.4.1)
- the definition used in this section, as in Eq.(3.8)
- the Hilbert SET (see eq. (3.125) in appendix 3.4.2)

$$T_{\mu\nu} = -4\pi \frac{\delta S}{\delta g^{\mu\nu}} = -4\pi \left(\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\mathcal{L} \right) \quad (3.13)$$

Exercise : Same question for the abelian Chern-Simons theory in 2+1 dimensions as defined by $\mathcal{L}[A_\mu] = \frac{1}{2\pi}\epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho}$.

³Indeed as long as the action is extremal, $\delta S = 0$ for all $\delta\Phi$, so in particular for $\delta\Phi = \delta_\epsilon\Phi$.

3.1.2 Ambiguities, symmetries and improvements

With equation (3.8) we have postulated the existence of the stress-energy tensor, but we have not commented on its unicity. In fact (3.8) does not define a unique stress-energy tensor but rather a class of stress-energy tensors that differ from each other by a divergence-free piece. Indeed one can always add to T^μ_ν any tensor Θ^μ_ν such that

$$\int \Theta^\mu_\nu \partial_\mu \epsilon^\nu d^n x = 0 \quad (3.14)$$

for all vector fields $\epsilon^\mu(x)$. Integrating by parts, the above condition boils down to Θ^μ_ν being identically conserved (*i.e.* conserved off-shell)

$$\partial_\mu \Theta^\mu_\nu = 0 \quad (3.15)$$

For instance,

$$\Theta^\mu_\nu = \partial_\rho \Sigma^{\rho\mu}_\nu, \quad \Sigma^{\rho\mu}_\nu = -\Sigma^{\mu\rho}_\nu \quad (3.16)$$

for an arbitrary $\Sigma^{\rho\mu}_\nu$. Adding such terms goes under the general name of *improving* the stress-energy tensor.

Such an ambiguity is not surprising. Think for instance of a one-dimensional such as the *XXX* chain, with Hamiltonian

$$H = \sum_n h_n, \quad h_n = \vec{\sigma}_n \cdot \vec{\sigma}_{n+1} \quad (3.17)$$

While the notion of total energy is clear (it is nothing but the Hamiltonian H), the notion of energy density is not. One could declare it to be $h_n = \vec{\sigma}_n \cdot \vec{\sigma}_{n+1}$. It is indeed local, hermitian, and their sum yield the total energy. But one could just as well add local terms of the form $X_{n+1} - X_n$ to h_n without spoiling any of these properties.

In fact defining a notion of energy density amounts to define how this theory couples to gravity ! More on this later.

Rotational invariance

It is possible to exploit this ambiguity to choose a “nice” form of the stress-energy tensor. We are now going to show that rotational invariance⁴ is equivalent to the existence of a symmetric stress-energy tensor

$$T^{\mu\nu} = T^{\nu\mu} \quad (3.18)$$

A space transformation $\Phi \rightarrow \Phi + \delta_\epsilon \Phi$ is a symmetry when the variation of the action $\delta_\epsilon S$ vanishes, *i.e.* when the vector field ϵ^μ is such that

$$\int T^\mu_\nu \partial_\mu \epsilon^\nu d^n x = 0 \quad (3.19)$$

Since we want the above identity to be valid for all field configurations, this boils down to

$$T^\mu_\nu \partial_\mu \epsilon^\nu = \partial_\rho J^\rho \quad (3.20)$$

⁴We have already implicitly assumed and exploited translation invariance in (3.8).

for some J^ρ . For the generators of rotations

$$\epsilon^\mu = \omega_\nu^\mu x^\nu \quad (3.21)$$

this means

$$T^{\mu\nu} - T^{\nu\mu} = \partial_\rho Y^{\rho\mu\nu} \quad (3.22)$$

for some tensor $Y^{\rho\mu\nu} = -Y^{\rho\nu\mu}$. If we now add an improvement term *a la* (3.16) where

$$\Sigma^{\rho\mu\nu} = \frac{1}{2} (Y^{\rho\mu\nu} - Y^{\mu\rho\nu} - Y^{\nu\rho\mu}) \quad (3.23)$$

then the new stress-energy tensor is symmetric.

Conformal invariance

Let us now consider a rotational invariant field theory, and choose a stress-energy tensor representative that is symmetric. Infinitesimal conformal invariance in flat space is tantamount to

$$\omega(x) T^\mu_\mu = \partial_\mu K^\mu \quad (3.24)$$

for all functions $\omega(x)$ susceptible to appear in the *r.h.s.* of the conformal Killing constraint

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 2\omega(x) \delta_{\mu\nu}. \quad (3.25)$$

In dimension $d \geq 3$, the function $\omega(x)$ can be any linear function $\omega(x) = a + b_\mu x^\mu$, in which the a term comes from pure dilations, and the $b_\mu x^\mu$ term from special conformal transformations. So we must have

$$T^\mu_\mu = \partial_\mu A^\mu \quad \text{and} \quad x^\nu T^\mu_\mu = \partial_\mu B^{\mu\nu} \quad (3.26)$$

And therefore $x^\nu \partial_\mu A^\mu = \partial_\mu B^{\mu\nu}$, yielding

$$A^\mu = \partial_\rho C^{\rho\mu}, \quad C^{\mu\nu} = B^{\mu\nu} - x^\nu A^\mu \quad (3.27)$$

So infinitesimal conformal invariance in flat space with $d \geq 3$ is tantamount to

$T^\mu_\mu = \partial_\mu \partial_\nu C^{\mu\nu}$

(3.28)

In two-dimensions the above relation is necessary but by no means sufficient since $\omega(x)$ can now be any harmonic⁵ function. Therefore for any harmonic function $h(x)$ the quantity

$$h(x) T^\mu_\mu \quad (3.29)$$

must be a total derivative. Following a similar line of reasoning as in the case $d \geq 3$, we reach the conclusion that T^μ_μ must be of the form ΔC for some C and Δ is the Laplacian.

⁵A real function f of two real variables is harmonic (*i.e* its laplacian $\Delta\omega = \partial_1^2\omega + \partial_2^2\omega = 4\partial\bar{\partial}\omega$ vanishes) iff it is the real part of a holomorphic function : $\omega = g + \bar{g}$ for some g such that $\bar{\partial}g = 0$. This equivalence holds on any simply connected domain of \mathbb{C} .

A two-dimensional, rotational invariant field theory in flat Euclidean space (for which a symmetric stress-energy tensor has been chosen) is invariant under conformal transformations if and only iff there exist a functional $C = C(\Phi, \partial_\mu \Phi, \dots)$ of the fields such that

$$T^\nu_\nu = \Delta C \quad (3.30)$$

In both cases it is possible to improve the stress-energy tensor in order to make it traceless. A way to improve the stress-energy tensor without spoiling $T^{\mu\nu} = T^{\nu\mu}$ is to add terms of the form

$$\Theta^{\mu\nu} = \partial_\rho \partial_\sigma \Omega^{\rho\mu\sigma\nu} \quad (3.31)$$

for some tensor $\Omega^{\rho\mu\sigma\nu}$ such that

$$\Omega^{\rho\mu\sigma\nu} = -\Omega^{\mu\rho\sigma\nu}, \quad \Omega^{\rho\mu\sigma\nu} = -\Omega^{\rho\mu\nu\sigma}, \quad \Omega^{\rho\mu\sigma\nu} = \Omega^{\sigma\nu\rho\mu}. \quad (3.32)$$

Note that these are the same symmetries as the Riemann curvature tensor (see (3.144) in the Appendix). This is not an accident.

In $d \geq 3$

$$\Omega^{\rho\mu\sigma\nu} = \frac{1}{d-2} (-\eta^{\mu\nu} C^{\rho\sigma} - \eta^{\rho\sigma} C^{\mu\nu} + \eta^{\rho\nu} C^{\mu\sigma} + \eta^{\mu\sigma} C^{\rho\nu}) \quad (3.33)$$

$$+ \frac{1}{(d-1)(d-2)} (\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\rho\nu} \eta^{\mu\sigma}) C^\lambda_\lambda \quad (3.34)$$

does the trick, while for $d = 2$ we can use

$$\Omega^{\rho\mu\sigma\nu} = (\eta^{\rho\nu} \eta^{\mu\sigma} - \eta^{\mu\nu} \eta^{\rho\sigma}) C \quad (3.35)$$

A field theory in flat Euclidean space is invariant under conformal transformations if and only if it admits a symmetric and traceless stress-energy tensor.

3.2 Ward identities

After having considered conformal invariance in classical field theories in the previous sections, we can now delve into quantum field theories. We are going to restrict our attention to two-dimensional theories, although large parts of the discussion in this section can be readily generalized to any dimension.

For the sake of argument let us assume that the theory is described by a Lagrangian density \mathcal{L} . Even though in practice we will rarely rely on an explicit action, and in fact it might not even exist, it can be useful at a formal and heuristic level to consider the path

integral formulation of quantum field theories. Generically the partition function is given in terms of an action $S[\Phi]$ by

$$Z = \int e^{-S[\Phi]} D[\Phi]. \quad (3.36)$$

whereas in the previous section Φ stands for a collection of fields ϕ_1, ϕ_2, \dots . From a statistical physics perspective Φ describes (a coarse-grained version of) the lattice degrees of freedom, and $e^{-S[\Phi]}$ corresponds to the Boltzmann weight of a given configuration Φ . Correlation functions are given by

$$\langle O_1(x_1) \cdots O_p(x_p) \rangle = \frac{1}{Z} \int O_1(x_1) \cdots O_p(x_p) e^{-S[\Phi]} [D\Phi]. \quad (3.37)$$

The truly new ingredient as compared to a classical field theory is the functional measure $D[\Phi]$, which typically is not mathematically sound and physically requires subtle regularisation schemes (see the tutorial on zeta regularization for instance). We will work on a formal level and ignore this issue, all the while assuming a few reasonable properties of the functional measure. Indeed we will merely use the path-integral approach as an intuitive and heuristic way to derive the Ward identities. These Ward identities are relations between correlation functions that follow from the continuous symmetries of the QFT. This is the quantum version of the classical current conservation of Noether's theorem. As such Ward identities are very generic, and they can be formulated independently from any Lagrangian or functional measure. Later on we will bypass the path-integral formalism and simply postulate that the Ward identities hold (this amounts to assume that conformal invariance is not broken at the quantum level), and in this way we will construct two-dimensional CFTs without resorting to path-integrals or assuming the existence of a Lagrangian.

3.2.1 Schwinger-Dyson equation, redundant operators

In the classical field theory the equations of motion are obtained by demanding that the action $S[\Phi]$ is extremal, i.e. $\frac{\delta S}{\delta \Phi} = 0$. Under a small variation $\Phi + \delta\Phi$ (recall that Φ stands for a collection of fields ϕ_1, ϕ_2, \dots) the variation of the action is

$$\delta S = \int \left(\frac{\delta \mathcal{L}}{\delta \phi_a} \delta \phi_a(x) + \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_a)} \partial_\mu \delta \phi_a(x) \right) d^2x \quad (3.38)$$

$$= \int \left(\frac{\delta \mathcal{L}}{\delta \phi_a} - \partial_\mu \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_a)} \right) \delta \phi_a(x) d^2x \quad (3.39)$$

The equations of motion are precisely

$$\frac{\delta \mathcal{L}}{\delta \phi_a} - \partial_\mu \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_a)} = 0 \quad (3.40)$$

Let's denote by $R_a(x)$ the *l.h.s.* of the above equation. In the quantum theory these equations of motion also have a meaning. Consider the path-integral formulation of a correlation function

$$\langle O_1(x_1) \cdots O_p(x_p) \rangle = \frac{1}{Z} \int O_1(x_1) \cdots O_p(x_p) e^{-S[\Phi]} D[\Phi]. \quad (3.41)$$

and let us perform a change of variable $\Phi \rightarrow \tilde{\Phi} = \Phi + \delta\Phi$. The function $\delta\Phi(x)$ is arbitrary but independent of $\Phi(x)$ (the change of variable we consider is the analogue of a rigid translation in functional space : all configurations Φ are shifted by the *same* function $\delta\Phi$). In the quantum theory it is quite reasonable and natural to assume that the measure $D[\Phi]$ is invariant under such rigid translations :

$$D[\Phi + \delta\Phi] = D[\Phi] \quad (3.42)$$

while the variation of the action is

$$\delta S = \int R_a(x) \delta\phi_a(x) d^2x. \quad (3.43)$$

This leads to the Schwinger-Dyson equation

$$\sum_{i=1}^p \langle O_1(x_1) \cdots \delta O_i(x_i) \cdots O_p(x_p) \rangle = \sum_a \int \delta\phi_a(x) \langle R_a(x) O \rangle d^2x \quad (3.44)$$

where O stands for the product of fields $O_1(x_1) \cdots O_p(x_p)$. $\delta O_i(x)$ is the variation of the composite field $O_i(x)$ under the infinitesimal shift $\phi_a \rightarrow \phi_a + \delta\phi_a$, which generically is of the form

$$\delta O_i(x) = \int \frac{\delta O_i(x)}{\delta\phi_a(y)} \delta\phi_a(y) d^d y \quad (3.45)$$

Since we consider only local fields, the above expression may only involve the functions $\delta\phi_a$ and their derivatives taken at the point x . Therefore we find that

$$\langle R_a(x) O_1(x_1) \cdots O_p(x_p) \rangle = 0 \quad \text{as long as } x \neq x_1, \dots, x_p \quad (3.46)$$

At coincident points - when $x = x_i$ - there are *contact terms* (typically δ function and possibly their derivatives), but apart from these points the above expression vanishes. A field that vanishes in any correlation function away from coincident points is called a *redundant field*, and we write

$$R_a(x) \simeq 0 \quad (3.47)$$

3.2.2 Quantum stress-energy tensor and Ward identities

Let us now repeat the above discussion for an infinitesimal spacetime transformation

$$\tilde{\Phi} = f_\epsilon^* \Phi = \Phi + \delta_\epsilon \Phi + O(\epsilon^2) \quad (3.48)$$

as generated by an arbitrary infinitesimal vector field (with compact support) $\epsilon^\mu = \epsilon^\nu \xi^\mu$. In this case the measure $D[\Phi]$ has no reason to be invariant. Assuming translation invariance the variation must vanish if ϵ^ν is constant, so we expect that the measure changes according to the following local form

$$D[\tilde{\Phi}] = \left(1 + \int \partial_\mu \epsilon^\nu(y) \tau^\mu_\nu(y) d^2y + O(\epsilon^2) \right) D[\Phi] \quad (3.49)$$

We can absorb such a term in a redefinition of the stress-energy tensor, and in a QFT the stress-energy tensor T^μ_ν is defined as

$$e^{-S[\tilde{\Phi}]} D[\tilde{\Phi}] = \left(1 + \frac{1}{2\pi} \int \partial_\mu \epsilon^\nu(y) T^\mu_\nu(y) d^2y + O(\epsilon^2) \right) e^{-S[\Phi]} D[\Phi] \quad (3.50)$$

It contains, in addition to the classical part, quantum corrections coming from the variation of the measure. From now on, unless otherwise specified, whenever we mention the stress-energy tensor we will mean the quantum one.

It is not always possible to preserve all classical symmetries at the quantum level. When a classical symmetry is violated by quantum corrections the symmetry is said to be *anomalous*. Implicitly we have already assumed that the quantum theory remains translation invariant when writing (3.49). We will further assume that rotational invariance is also preserved, thus $T^{\mu\nu} = T^{\nu\mu}$ (up to possible improvements).

Upon performing the change of variable $\Phi \rightarrow \tilde{\Phi} = \Phi + \delta_\epsilon \Phi$, we get

$$\langle O \rangle = \frac{1}{Z} \int O e^{-S[\Phi]} D[\Phi] = \frac{1}{Z} \int \tilde{O} e^{-S[\tilde{\Phi}]} D[\tilde{\Phi}] \quad (3.51)$$

$$= \frac{1}{Z} \int (O + \delta_\epsilon O) \left(1 + \frac{1}{2\pi} \int \partial_\mu \epsilon^\nu(y) T^\mu_\nu(y) d^2y \right) e^{-S[\Phi]} D[\Phi], \quad (3.52)$$

yielding

$$\sum_{i=1}^p \langle O_1(x_1) \cdots \delta_\epsilon O_i(x_i) \cdots O_p(x_p) \rangle = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \partial_\mu \epsilon^\nu(y) \langle T^\mu_\nu(y) O \rangle d^2y \quad (3.53)$$

or equivalently

$$\boxed{\sum_{i=1}^p \langle O_1(x_1) \cdots \delta_\epsilon O_i(x_i) \cdots O_p(x_p) \rangle = \frac{1}{2\pi} \int_{\mathbb{R}^2} \epsilon^\nu(y) \langle (\partial_\mu T^\mu_\nu)(y) O \rangle d^2y} \quad (3.54)$$

This equation contains a lot of extremely interesting informations. First remember that fields transform in a local way, thus $\delta_\epsilon O_i(x_i)$ may only involve the vector field $\epsilon^\mu(x)$ and finitely many of its derivatives taken at the point x_i . This implies

$$\boxed{\partial_\mu T^{\mu\nu} \simeq 0}, \quad (3.55)$$

which means that $T^{\mu\nu}$ is conserved in a QFT up to contact terms. Thus we can rewrite eq. (3.54) as

$$\sum_{j=1}^p \langle O_1(x_1) \cdots \delta_\epsilon O_j(x_j) \cdots O_p(x_p) \rangle = \sum_{j=1}^p \frac{1}{2\pi} \int_{B_j} \epsilon^\nu(y) \langle (\partial_\mu T^\mu_\nu)(y) O \rangle d^2y \quad (3.56)$$

where B_1, \dots, B_p are non-overlapping neighborhoods of x_1, \dots, x_p as in Fig. 3.1. We will write

$$\delta_\epsilon O_j(x) \simeq \frac{1}{2\pi} \int_{B_j} \epsilon^\nu(y) (\partial_\mu T^\mu_\nu)(y) O_j(x_j) d^2y \quad (3.57)$$

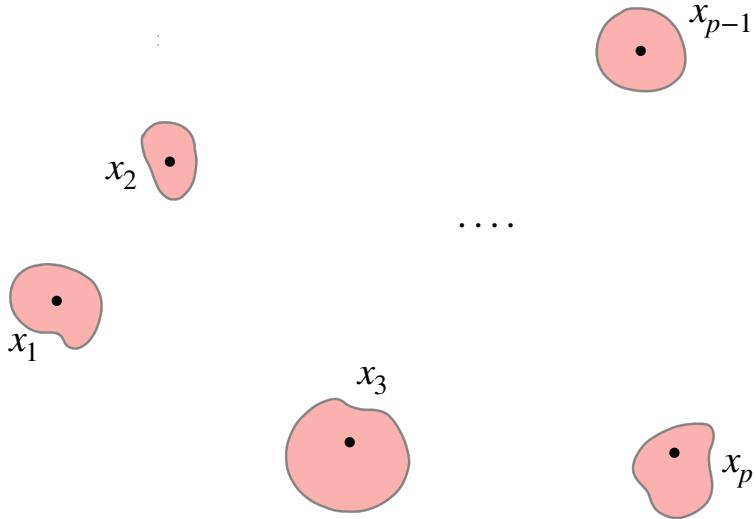


Figure 3.1: Non-overlapping neighborhoods of x_1, \dots, x_p .

as a short-hand notation for (3.56). The symbol \simeq is here as a reminder that the above identity is valid in correlation functions away from coincident points, and with the domain B_j containing no other insertion than $O_j(x_j)$. Note that the above identity is insensitive to improvements of the stress-energy tensor.

Integrating by parts, we can also rewrite (3.57) as

$$\begin{aligned} \delta_\epsilon O_j(x) &\simeq -\frac{1}{2\pi} \int_{B_j} \partial_\mu \epsilon^\nu(y) T^\mu_\nu(y) O_j(x_j) d^2y \\ &\quad + \frac{1}{2\pi} \oint_{\partial B_j} \epsilon_{\mu\rho} \epsilon^\nu(y) T^\mu_\nu(y) O_j(x_j) dy^\rho \end{aligned} \quad (3.58)$$

in which the boundaries ∂B_j are oriented in the counterclockwise direction and $\epsilon_{\mu\rho}$ is the completely antisymmetric tensor ($\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$).

While the above relation is valid for an arbitrary infinitesimal diffeomorphism, a particular role is played by symmetries and in particular by isometries. Consider the case in which the vector field ϵ^μ satisfies the Killing constraint inside all the B_j (and is arbitrary outside, while being smooth and compactly supported to avoid unpleasant issues). Then the integral over B_j in (3.58) vanishes (remember we have assumed that isometries are not anomalous, and that the stress-tensor is symmetric) and we get the Ward identity

$$\delta_\epsilon O_j(x) \simeq \frac{1}{2\pi} \oint_{\partial B_j} \epsilon_{\mu\rho} \epsilon^\nu(y) T^\mu_\nu(y) O_j(x_j) dy^\rho$$

(3.59)

for translations and rotations. In particular for translations $\delta_\epsilon O_j = \epsilon^\nu \partial_\nu O_j$, and we find

$$\partial_\nu O_j(x_j) \simeq \frac{1}{2\pi} \oint_{\partial B_j} \epsilon_{\mu\rho} T^\mu_\nu(y) O_j(x_j) dy^\rho \quad (3.60)$$

3.3 Conformal Ward identities and anomalous behavior

Following the discussion in section 2.3.1, an infinitesimal conformal transformation corresponds to a conformal Killing vector field ϵ^μ , *i.e.* such that

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \delta_{\mu\nu} \partial_\rho \epsilon^\rho. \quad (3.61)$$

For such a vector field Eq.(3.58) yields

$$\begin{aligned} \delta_\epsilon O_j(x) &\simeq -\frac{1}{4\pi} \int_{B_j} \partial_\rho \epsilon^\rho(y) T^\mu_\mu(y) O_j(x_j) d^2y \\ &\quad + \frac{1}{2\pi} \oint_{\partial B_j} \epsilon_{\mu\rho} \epsilon^\nu(y) T^\mu_\nu(y) O_j(x_j) dy^\rho \end{aligned} \quad (3.62)$$

Naively we would expect the first integral to vanish, as was the case for isometries. Indeed for a conformal invariant field theory, the classical stress-energy tensor can be chosen traceless, and we could expect the quantum stress-energy tensor to also be traceless. In flat space this turns out to be almost true, in the sense that T^μ_μ is redundant :

$$T^\mu_\mu \simeq 0. \quad (3.63)$$

However there are contact terms. These are extremely important, as they encode the quantum corrections to the behavior of fields under conformal transformations, and in particular their anomalous dimension. We interpret the term

$$-\frac{1}{4\pi} \int_{B_j} \partial_\rho \epsilon^\rho(y) T^\mu_\mu(y) O_j(x_j) d^2y \quad (3.64)$$

as a quantum correction to $\delta_\epsilon O_j(x)$. What we are saying is that fields in a QFT might not transform as their classical counterpart. Technically these anomalous terms come from the need to introduce a length scale when regulating a QFT (see for instance the regularization of vertex operators in the tutorials). Following the same philosophy as for the stress-energy tensor (3.50), we absorb these quantum corrections in a redefinition of $\delta_\epsilon O_j(x)$:

$$\tilde{\delta}_\epsilon O_j(x) = \delta_\epsilon O_j(x) + \frac{1}{4\pi} \int_B \partial_\rho \epsilon^\rho(y) T^\mu_\mu(y) O_j(x) d^2y \quad (3.65)$$

where B is a small neighborhood of x that does not contain any other field insertion. Notice that these quantum corrections do not spoil the local nature of the transformation law (3.7).

Consider for instance the behavior of fields under an infinitesimal dilatation $\epsilon^\mu = \epsilon x^\mu$. At the classical level the behavior of a field is completely fixed (for instance for scalar/vector/tensor field it is given by the Lie derivative) and is of the form

$$\delta_\epsilon O = \epsilon (x^\rho \partial_\rho + \Delta_0) O \quad (3.66)$$

where Δ_0 is the naive/canonical scaling dimension. Suppose the contact terms of T^μ_μ with the field O are of the form

$$T^\mu_\mu(y)O(x) = 2\pi\gamma\delta(y-x)O(x) + \text{higher derivatives of } \delta(y-x) \quad (3.67)$$

we find at the quantum level

$$\tilde{\delta}_\epsilon O = \epsilon(x^\rho \partial_\rho + \Delta) O, \quad \Delta = \Delta_0 + \gamma \quad (3.68)$$

This violation of the naive scaling dimension is our first encounter with quantum anomalies, and the extra term γ is called the *anomalous dimension*. These scaling dimensions are of tremendous importance in physics and are often directly accessible in experiments, but they are notoriously difficult to compute exactly. In statistical physics they correspond to critical exponents. One of the main achievements of two-dimensional CFT is to provide exact results for these scaling dimensions.

We can now rewrite the conformal Ward identity (3.58) in the following compact form

$$\tilde{\delta}_\epsilon O_j(x) \simeq \frac{1}{2\pi} \oint_{\partial B} \epsilon_{\mu\rho} \epsilon^\nu(y) T^\mu_\nu(y) O_j(x_j) dy^\rho. \quad (3.69)$$

Of course the vector field ϵ^μ must satisfy the conformal Killing equation inside B , and the sign \simeq is there to recall that the above equation is only valid when inserted in a correlation function with no other insertion inside B (nor inside its closure : we do not want any contact terms with the stress-energy tensor). Equivalently

$$\begin{aligned} & \sum_{i=1}^p \langle O_1(x_1) \cdots \tilde{\delta}_\epsilon O_i(x_i) \cdots O_p(x_p) \rangle \\ &= \sum_{i=1}^p \frac{1}{2\pi} \oint_{\partial B_i} \epsilon_{\mu\rho} \epsilon_\nu(y) \langle T^{\mu\nu}(y) O_1(x_1) \cdots O_p(x_p) \rangle dy^\rho \end{aligned} \quad (3.70)$$

From now on we will drop the tilde on $\tilde{\delta}_\epsilon O_j(x)$.

3.3.1 Conformal Ward identities in complex coordinates

Needless to say the whole thing looks simpler in complex coordinates, see Table (3.1). For starters the vector field $\epsilon^z \partial_z + \epsilon^{\bar{z}} \partial_{\bar{z}}$ is conformal Killing iff $\epsilon^z = \epsilon(z)$ is holomorphic (it follows that $\epsilon^{\bar{z}} = \bar{\epsilon}$ is anti-holomorphic). Then consider the stress-energy tensor. In complex coordinates it has components

$$T_{zz} = \overline{T_{\bar{z}\bar{z}}} = \frac{1}{4} (T_{11} - T_{22} - iT_{12} - iT_{21}) \quad (3.71)$$

$$T_{z\bar{z}} = \overline{T_{z\bar{z}}} = \frac{1}{4} (T_{11} + T_{22} + i(T_{12} - T_{21})) \quad (3.72)$$

but since T is symmetric

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4} T^\mu_\mu \simeq 0. \quad (3.73)$$

Cartesian coordinates	Complex coordinates
$x = x^1 = \frac{1}{2}(z + \bar{z})$, $y = x^2 = \frac{1}{2i}(z - \bar{z})$	$z = x + iy$, $\bar{z} = x - iy$
$\partial_x = \partial_z + \partial_{\bar{z}}$, $\partial_y = i(\partial_z - \partial_{\bar{z}})$	$\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$
$g_{\mu\nu} = g^{\mu\nu} = \delta_{\mu\nu}$	$g_{z\bar{z}} = \frac{1}{2}$, $g^{z\bar{z}} = 2$
$\epsilon_{12} = \epsilon^{12} = 1$	$\epsilon_{z\bar{z}} = \frac{i}{2}$, $\epsilon^{z\bar{z}} = -2i$
$T_{11} = T_{zz} + T_{z\bar{z}} + T_{\bar{z}z} + T_{\bar{z}\bar{z}}$	$T_{zz} = \overline{T_{\bar{z}\bar{z}}} = \frac{1}{4}(T_{11} - T_{22} - iT_{12} - iT_{21})$
$T_{22} = -T_{zz} + T_{z\bar{z}} + T_{\bar{z}z} - T_{\bar{z}\bar{z}}$	$T_{z\bar{z}} = \overline{T_{\bar{z}z}} = \frac{1}{4}(T_{11} + T_{22} + i(T_{12} - T_{21}))$
$T_{12} = i(T_{zz} - T_{z\bar{z}} + T_{\bar{z}z} - T_{\bar{z}\bar{z}})$	
$T_{21} = i(T_{zz} + T_{z\bar{z}} - T_{\bar{z}z} - T_{\bar{z}\bar{z}})$	

Table 3.1: From Cartesian to complex coordinates

Furthermore $\partial_\mu T^{\mu\nu} \simeq 0$, therefore

$$\partial_{\bar{z}} T_{zz} \simeq 0, \quad \partial_z T_{\bar{z}\bar{z}} \simeq 0 \quad (3.74)$$

Note that in the *r.h.s.* of eq. (3.70) the position y of the SET never coincides with a field insertion, thus we can effectively forget about contact terms (they have already played their part). So as far as the conformal Ward identities are concerned the SET has only two components $T(z)$ and $\overline{T}(\bar{z})$:

$$T(z) = T_{zz}, \quad \overline{T}(\bar{z}) = T_{\bar{z}\bar{z}}, \quad T_{z\bar{z}} = T_{\bar{z}z} = 0 \quad (3.75)$$

The conformal Ward identity (3.70) becomes

$$\begin{aligned} & \sum_{i=1}^p \langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \delta_\epsilon \mathcal{O}_i(z_i, \bar{z}_i) \dots \mathcal{O}_p(z_p, \bar{z}_p) \rangle \\ &= \frac{1}{2\pi i} \sum_{i=1}^p \oint_{\partial B_i} [\langle \dots T(z) \mathcal{O}_i(z_i, \bar{z}_i) \dots \rangle \epsilon(z) dz - \langle \dots \overline{T}(\bar{z}) \mathcal{O}_i(z_i, \bar{z}_i) \rangle \bar{\epsilon}(\bar{z}) d\bar{z}], \end{aligned} \quad (3.76)$$

where the integration over ∂B_i is performed in the counter-clockwise direction. We can rewrite the above Ward identity as

$$\delta_\epsilon \mathcal{O}(z, \bar{z}) \simeq \frac{1}{2\pi i} \oint_{\mathbb{C}} \epsilon(\xi) T(\xi) \mathcal{O}(z, \bar{z}) d\xi + \frac{1}{2\pi i} \oint_{\mathbb{C}} \bar{\epsilon}(\bar{\xi}) \overline{T}(\bar{\xi}) \mathcal{O}(z, \bar{z}) d\bar{\xi}, \quad (3.77)$$

in which the integration contour circles around z . As usual the above relation is valid in correlation functions away from coincident points, and with the domain B containing no other insertion than $\mathcal{O}(z, \bar{z})$.

3.3.2 From Ward identities to OPEs

Recall that under a global scaling $z \rightarrow \lambda z$ a scaling field ϕ with scaling dimension Δ behaves as

$$\phi'(z, \bar{z}) = \lambda^\Delta \phi(\lambda z, \lambda \bar{z}). \quad (3.78)$$

Under a rotation $z \rightarrow e^{i\theta}z$, we have

$$\phi'(z, \bar{z}) = e^{i\theta s} \phi(e^{i\theta}z, e^{-i\theta}\bar{z}) \quad (3.79)$$

where s is called the *spin* of the field ϕ . For local fields, we must have $s \in \mathbb{Z}$ (why ?), but we will also encounter quasi-local fields (such as the fermion field in the Ising model with $s = 1/2$) for which s can be a rational number. Dilations and rotations can be conveniently unified as transformations $z \rightarrow \lambda z$, with $\lambda \in \mathbb{C}$. Accordingly scaling fields transform as

$$\phi'(z, \bar{z}) = \lambda^h \bar{\lambda}^{\bar{h}} \phi(\lambda z, \bar{\lambda} \bar{z}), \quad \lambda \in \mathbb{C} \quad (3.80)$$

where we have introduced the left and right conformal dimensions (h, \bar{h}) . Evidently

$$\begin{cases} \Delta = h + \bar{h} \\ s = h - \bar{h} \end{cases} \quad (3.81)$$

Let's introduce some terminology. A field ϕ is called a *primary field* if it transforms as follows under a conformal transformation $z \rightarrow f(z)$

$$\phi'(z, \bar{z}) = (\partial f)^h (\bar{\partial} f)^{\bar{h}} \phi(f(z), \overline{f(z)}) \quad (3.82)$$

The numbers (h, \bar{h}) are the conformal dimensions of ϕ . Primary fields play a central role in CFTs. The reason will become clear when studying the representation theory of the conformal algebra (*a.k.a.* the Virasoro algebra). Note that not all fields are primary fields. For instance if ϕ is a primary field, then $\partial_z \phi$ is not⁶, since

$$\begin{aligned} \partial_z \phi'(z, \bar{z}) &= (\partial f)^{h+1} (\bar{\partial} f)^{\bar{h}} \partial_z \phi(f(z), \overline{f(z)}) \\ &\quad + h \partial^2 f (\partial f)^{h-1} (\bar{\partial} f)^{\bar{h}} \phi(f(z), \overline{f(z)}) \end{aligned} \quad (3.83)$$

Infinitesimally, for $f(z) = z + \epsilon(z)$, a primary field behaves as

$$\delta_\epsilon \phi(z, \bar{z}) = (\epsilon \partial_z + h \partial_z \epsilon) \phi(z, \bar{z}) + (\bar{\epsilon} \partial_{\bar{z}} + \bar{h} \partial_{\bar{z}} \bar{\epsilon}) \phi(z, \bar{z}) \quad (3.84)$$

Comparison with the Ward identity (3.77) yields the following Operator Product Expansion

$$\begin{aligned} T(z)\phi(w, \bar{w}) &= \frac{h}{(z-w)^2} \phi(w, \bar{w}) + \frac{1}{z-w} \partial_w \phi(w, \bar{w}) + \dots \\ \overline{T}(\bar{z})\phi(w, \bar{w}) &= \frac{\bar{h}}{(\bar{z}-\bar{w})^2} \phi(w, \bar{w}) + \frac{1}{\bar{z}-\bar{w}} \partial_{\bar{w}} \phi(w, \bar{w}) + \dots \end{aligned} \quad (3.85)$$

Note that in deriving the above we heedlessly treated the functions $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ as if they were independent, which they are not. This point, namely the factorization of CFT correlation functions into left and right degrees of freedom, which amounts to treat z and \bar{z} as being independent, will be commented more and to some extent justified in the next chapter.

⁶unless $h = 0$

For a generic scaling field, the OPEs $T\phi$ and $\bar{T}\phi$ are quite similar to (3.85) : the very same terms appear, but in general there are poles of higher degrees. For instance

$$T(z)\partial_w\phi(w, \bar{w}) = \frac{2h}{(z-w)^3}\phi(w, \bar{w}) + \frac{(h+1)}{(z-w)^2}\partial_w\phi(w, \bar{w}) + \frac{1}{z-w}\partial_w^2\phi(w, \bar{w}) + \dots \quad (3.86)$$

While primary fields are quite special, a larger class of field is given by *quasi-primary fields*, for which one imposes the covariance relation (3.82) only for global conformal mappings of the sphere, namely

$$f(z) = \frac{az+b}{cz+d} \quad (3.87)$$

Exercise : show that a field ϕ is quasi-primary if and only if the OPEs $T(z)\phi(w, \bar{w})$ (and $\bar{T}(\bar{z})\phi(w, \bar{w})$) has no pole of order 3.

Let's now consider the behavior of the SET itself under conformal maps. At the classical level

$$\delta_\epsilon T_{\mu\nu} = \epsilon^\rho \partial_\rho T_{\mu\nu} + \partial_\mu \epsilon^\rho T_{\rho\nu} + \partial_\nu \epsilon^\rho T_{\mu\rho} \quad (3.88)$$

which in complex coordinates (recall that T is traceless) reads

$$\delta_\epsilon T = \epsilon \partial_z T + 2\partial_z \epsilon T, \quad (3.89)$$

$$\delta_\epsilon \bar{T} = \bar{\epsilon} \partial_{\bar{z}} \bar{T} + 2\partial_{\bar{z}} \bar{\epsilon} \bar{T} \quad (3.90)$$

The naive dimension of the SET is $\Delta = 2$. Because it remains conserved in the quantum theory, its dimension cannot be modified (this is in fact a generic properties of conserved current : they cannot acquire anomalous dimensions). However in the quantum theory the above transformation law under infinitesimal conformal transformations generically gets a quantum correction of the form

$$\delta_\epsilon T = \epsilon \partial_z T + 2\partial_z \epsilon T + \frac{c}{12} \partial_z^3 \epsilon, \quad (3.91)$$

$$\delta_\epsilon \bar{T} = \bar{\epsilon} \partial_{\bar{z}} \bar{T} + 2\partial_{\bar{z}} \bar{\epsilon} \bar{T} + \frac{c}{12} \partial_{\bar{z}}^3 \bar{\epsilon}. \quad (3.92)$$

where the number c is called the central charge. Note that the extra term does not spoil the scaling dimension of the SET. The origin of this anomalous term will be made clear in the next section. Comparison with (3.77) yields the following Operator Product Expansion

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots \quad (3.93)$$

and likewise for $\bar{T}(\bar{z})\bar{T}(\bar{w})$. Notice that T and \bar{T} are not primary fields (unless $c=0$), but they are quasi-primary. In fact the transformation (3.91) is the infinitesimal version of

$$T'(z) = (\partial f)^2 T(f(z)) + \frac{c}{12} S(f)(z), \quad (3.94)$$

where $S(f)$ (also denoted $\{f, z\}$) is the Schwarzian derivative of f , namely

$$S(f) = \partial_z \left(\frac{\partial_z^2 f}{\partial_z f} \right) - \frac{1}{2} \left(\frac{\partial_z^2 f}{\partial_z f} \right)^2 = \frac{\partial_z^3 f}{\partial_z f} - \frac{3}{2} \left(\frac{\partial_z^2 f}{\partial_z f} \right)^2 \quad (3.95)$$

Exercise : Check that (3.94) is indeed the integrated version of the infinitesimal transformation (3.91). Show that $S(f \circ g) = g'^2 S(f) \circ g + S(g)$. Check that $S(f)$ vanishes iff f is a Möbius transformation $z \rightarrow \frac{az+b}{cz+d}$.

3.4 Appendix

3.4.1 Canonical stress-energy tensor

Noether's theorem

Consider a classical field theory on flat Euclidean d -dimensional spacetime with metric $\eta_{\mu\nu} = \delta_{\mu\nu}$. The dynamics of the various fields (ϕ_1, ϕ_2, \dots) which we will denote collectively by Φ is described by a Lagrangian density \mathcal{L} . For simplicity we will assume that $\mathcal{L} = \mathcal{L}(\Phi, \partial_\mu \Phi)$ depends only on the fields at x and their first derivatives. The equations of motion (*a.k.a.* the Euler-Lagrange equations) are obtained as usual from extremizing the action

$$S[\Phi] = \int \mathcal{L}(\Phi(x), \partial_\mu \Phi(x)) d^n x \quad (3.96)$$

Fields are called "on-shell" when they obey the equation of motions

$$\frac{\partial \mathcal{L}}{\partial \phi_a} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \right) \quad (3.97)$$

An infinitesimal transformation $\Phi \rightarrow \Phi + \delta\Phi$ is said to be a symmetry if the infinitesimal variation of the Lagrangian is a pure divergence

$$\delta\mathcal{L} = \mathcal{L}[\Phi + \delta\Phi] - \mathcal{L}[\Phi] = \partial_\mu K^\mu \quad (3.98)$$

for **all** configurations $\phi(x)$ (not just on-shell⁷)

Exercise Compute the equation of motion for

- the complex scalar field $\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - V(|\phi|^2)$
- the Proca action $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + m^2 A_\mu A^\mu$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
- the Abelian 2 + 1 Chern-Simons theory $\mathcal{L} = \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$.

and list their symmetries.

The corresponding Noether current is derived as follows. Assume the fields are on-shell, and consider an arbitrary variation $\delta\phi$. At first order the variation of the action is

$$\delta S = \partial_\mu \left(\delta\phi_a \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \right) + \delta\phi_a \left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \right) = \partial_\mu \left(\delta\phi_a \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \right) \quad (3.99)$$

where we have used the fact that the second term vanishes when ϕ satisfies Euler-Lagrange. If $\Phi \rightarrow \Phi + \delta\Phi$ is also a symmetry, comparing with (3.98), we find that

$$\partial_\mu j^\mu = 0, \quad j^\mu = \delta\phi_a \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} - K^\mu \quad (3.100)$$

⁷While this definition of symmetry is perfectly sensible for a quantum field theory, it is not very natural for a classical one. Indeed a classical theory is fully characterized by its equations of motion, and it does not require a Lagrangian. It could happen that the equations of motion enjoy a symmetry that the Lagrangian does not. More importantly off-shell quantities are simply undefined/meaningless for a classical system : many Lagrangians can share the same equations of motion. This caveat will not be an issue in this lecture since we are just preparing the grounds for the quantum case.

provided the fields are on shell. The corresponding conserved charge is

$$Q = \int j^0(t, x) d^{n-1}x$$

Indeed

$$\dot{Q} = \int \partial_0 j^0(t, x) d^{n-1}x = - \int \partial_i j^i(t, x) d^{n-1}x = 0$$

by Stokes theorem, provided the current j^i vanishes sufficiently rapidly at infinity (or if space is compact, e.g. a torus).

Noether theorem associates a conservation law to each continuous symmetry of the system. Let us however note a potential pitfall. While the identification of the global charge Q is quite clear, there is an intrinsic ambiguity in defining the corresponding local current j^μ by simply demanding $\partial_\mu j^\mu = 0$. Indeed if j^μ is conserved, then so is $j^\mu + \partial_\nu b^{\mu\nu}$ for any antisymmetric tensor $b^{\mu\nu}$. Indeed the only characterization of the canonical Noether current j^μ is that it satisfies the continuity equation $\partial_\mu j^\mu = 0$. But *without changing the total charge Q* , we are free to add to j^μ any term of the form $\partial_\nu b^{\mu\nu}$, as long as $b^{\mu\nu}$ is antisymmetric !

Spacetime translations

Under an infinitesimal translation of spacetime (most) fields transform as $\Phi(x) \rightarrow \Phi(x^\mu + \epsilon^\mu) = \Phi(x) + \epsilon^\mu \partial_\mu \Phi(x)$. It is a symmetry as soon as \mathcal{L} does not depend explicitly on x^μ . Indeed the variation of the Lagrangian density \mathcal{L} is a total derivative

$$\delta\mathcal{L} = \left(\epsilon^\mu \partial_\mu \phi_a \frac{\partial \mathcal{L}}{\partial \phi_a} + \epsilon^\mu \partial_\mu \partial_\nu \phi_a \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi_a)} \right) = \partial_\mu (\epsilon^\mu \mathcal{L}) \quad (3.101)$$

where we used the chain rule $\partial_\mu \mathcal{L}(\Phi, \partial_\nu \Phi) = \partial_\mu \phi_a \frac{\partial \mathcal{L}}{\partial \phi_a} + \partial_\mu (\partial_\nu \phi_a) \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi_a)}$.

According to Noether's theorem, the associated current is

$$j^\mu = \epsilon^\nu \left(\partial_\nu \phi_a \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} - \delta_\nu^\mu \mathcal{L} \right) = -\frac{1}{2\pi} \epsilon^\nu T^\mu_\nu \quad (3.102)$$

so the canonical energy-momentum tensor is

$$T^{\mu\nu} = 2\pi \left(-\partial^\nu \phi_a \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} + \eta^{\mu\nu} \mathcal{L} \right) \quad (3.103)$$

The prefactor 2π is conventional, but the overall sign is not. It has been chosen so that the energy density T^{00} is positive in Minkowski space with signature $(-, +, +, +)$. In signature $(+, -, -, -)$ one must change the sign of $T^{\mu\nu}$.

The conserved charges associated to spacetime translations are energy and momentum

$$E = \int T^{00}(t, x) d^{n-1}x, \quad P^i = \int T^{0i}(t, x) d^{n-1}x \quad (3.104)$$

In the quantum setting these will become the generators of spacetime translations. For instance it is straightforward to check that E is (proportional to) the Hamiltonian

$$H = \int (\pi^a \dot{\phi}_a - \mathcal{L}) d^{n-1}x, \quad \pi^a = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} \quad (3.105)$$

However the naive canonical stress-energy tensor as defined above suffers from a number of severe issues and is in general unphysical :

- For gauge theories, it is in general not gauge invariant, as can be checked with electrodynamics
- It is not symmetric in general¹⁸
- It can fail to be traceless for scale invariant theories (more on this later)

All these issues are illustrated with free electrodynamics

$$\mathcal{L}[A_\mu] = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (3.106)$$

for which the canonical stress-energy tensor is

$$T^{\mu\nu} = F^{\mu\rho} \partial^\nu A_\rho - \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \quad (3.107)$$

It is manifestly not gauge invariant, nor symmetric, nor is it traceless in four-dimensions (in $4d$ this theory is scale invariant). An even more serious matter in $3+1$ dimensions is that T^{00} and T^{0i} fail to reproduce the well-known, experimentally tested expressions for the electromagnetic energy density $\frac{1}{2}(E^2 + B^2)$ and energy flux density (the Poynting vector) $E \times B$. In this particular case these issues can be traced back to the gauge symmetry of the action.

Improved stress-energy tensor

The canonical SET suffers from the same ambiguity as one described in section 3.1.2. This can be exploited to improve the canonical SET and yield a partial cure to the above-mentioned issues. Indeed for electrodynamics choosing $\Sigma^{\rho\mu\nu} = F^{\rho\mu} A^\nu$ leads to the correct stress-energy tensor

$$\begin{aligned} T^{\mu\nu} &= F^{\mu\rho} F^\nu_\rho + A^\nu \partial_\rho F^{\rho\mu} - \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \\ &\simeq F^{\mu\rho} F^\nu_\rho - \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \quad \text{on-shell} \end{aligned} \quad (3.108)$$

where we have used the on-shell relation $\partial_\mu F^{\mu\nu} = 0$. This tensor is now symmetric, gauge-invariant, and traceless in four dimensions, **as long as we are on-shell**. In fact it is the correct one (up to a possible global sign depending on the choice of signature of $\eta^{\mu\nu}$),

¹⁸This is required in a Lorentz invariant theory for which the angular momentum current $T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu$ should be conserved and it is also required to couple to curvature in general relativity (more on this to come).

since $T^{00} = \frac{1}{2}(E^2 + B^2)$ and $T^{0i} = (E \times B)^i$.

There it however a better way to obtain the correct SET from Noether's theorem. Instead of $\delta A_\mu = \epsilon^\alpha \partial_\alpha A_\mu$, consider a transformation that combines a naive translation to a specific gauge transformation as follows :

$$\delta A_\mu = \epsilon^\alpha \partial_\alpha A_\mu - \partial_\mu(\epsilon^\alpha A_\alpha) = \epsilon^\alpha F_{\alpha\mu} \quad (3.109)$$

The variation of the Lagrangian density \mathcal{L} is the same as for $\delta A_\mu = \epsilon^\alpha \partial_\alpha A_\mu$ since \mathcal{L} is gauge invariant. So $K^\mu = \epsilon^\mu \mathcal{L}$. Noether's theorem now yields a different conserved SET, which turns out to be the correct one (and not just on-shell).

In general curing the canonical SE tensor by adding such an improvement term has to be done on a case by case basis, and is a rather ad hoc procedure. It should be stressed that improving the stress-energy tensor is not merely a formal manipulation. Improvements amount to a relocalization of the energy and momentum densities, which are observable quantities, and as such they must agree with experiments. Generically the canonical SE tensor is an unphysical object coming from a formal and rather ambiguous construction.

Exercise : Compute the canonical SET for the 2+1 dimensional Chern-Simons theory $\mathcal{L} = \frac{1}{2\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$. Show that it can be improved to achieve $T^{\mu\nu} = 0$ on-shell (which is what we expect since for a topological field theory). Compare with the SET obtained using the "improved translation" (3.109).

3.4.2 Hilbert stress-energy tensor

Given a global symmetry, Noether theorem yields a well defined total charge, but the local distribution remains ambiguous. What is missing is a physically sound prescription of the notion of local charge and local currents. An elegant way to address this is to define how this charge couples to external fields.

Electric current as the source of the electromagnetic field

This is best illustrated for the electric charge. Consider for instance a complex scalar field with Lagrangian density

$$\mathcal{L} = \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi^* - V(|\phi|^2)$$

The global U(1) symmetry $\phi(x) \rightarrow e^{i\theta} \phi(x)$ yields the conservation of the total (electric) charge through the canonical Noether current

$$j^\mu = i(\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi) \quad (3.110)$$

but this current is only defined up to possible improvements. A way to circumvent this ambiguity is to couple this theory to an arbitrary external (electromagnetic) gauge field A_μ , for instance

$$\mathcal{L} = \eta^{\mu\nu} D_\mu \phi (D_\nu \phi)^* - V(|\phi|^2), \quad D_\mu = \partial_\mu - iA_\mu \quad (3.111)$$

By prescribing how the complex scalar field couples to the background U(1) gauge field, we are promoting the global U(1) symmetry to a local one $\phi(x) \rightarrow e^{i\theta(x)} \phi(x)$, $A_\mu \rightarrow A_\mu + \partial_\mu \theta$.

From the Lagrangian density (3.111) we can define unambiguously the local current as the response of the system to a variation of the external gauge field⁹ :

$$\delta S = - \int j^\mu \delta A_\mu d^n x, \quad \text{or equivalently} \quad j^\mu = - \frac{\delta S}{\delta A_\mu} \quad (3.114)$$

and we recover the expression (3.110) when $A_\mu = 0$. That this current is conserved on-shell is a consequence of gauge invariance. Indeed the action is invariant under a simultaneous change $\delta\phi(x) = i\theta(x)\phi(x)$ and $\delta A_\mu(x) = \partial_\mu\theta(x)$, therefore

$$0 = \delta S = \int \left(\frac{\delta S}{\delta \phi} \delta\phi + \frac{\delta S}{\delta \phi^*} \delta\phi^* \right) d^n x - \int j^\mu \partial_\mu \theta d^n x \quad (3.115)$$

Since $\frac{\delta S}{\delta \phi} = \frac{\delta S}{\delta \phi^*} = 0$ on-shell, we get for an arbitrary function $\theta(x)$

$$\int j^\mu \partial_\mu \theta d^n x = 0 \quad \text{on-shell} \quad (3.116)$$

which is equivalent to $\partial_\mu j^\mu = 0$ on-shell. Furthermore from

$$\int \left(\frac{\delta S}{\delta \phi} \delta\phi + \frac{\delta S}{\delta \phi^*} \delta\phi^* \right) d^n x = \int j^\mu \partial_\mu \theta d^n x \quad (3.117)$$

we find

$$\delta S = \int j^\mu(x) \partial_\mu \theta(x) d^n x, \quad \text{under} \quad \delta\phi(x) = i\theta(x)\phi(x) \quad (3.118)$$

which is also valid for the original problem of the theory that is **not** coupled to the gauge field (*i.e.* the case $A_\mu = 0$).

Linear response to spacetime deformations

The same approach can be used to define the stress-energy tensor. What plays the role of the external field is now the background metric : we are coupling the theory to gravity. Indeed in general relativity, the stress-energy tensor acts as the source of spacetime curvature.

One must prescribe how to extend the theory from flat space to an arbitrary curved space. Once our theory is defined on arbitrary curved background, we can define the stress

⁹Equivalently, upon considering a dynamical gauge field, the current j^μ can be seen as the source of the electromagnetic field. Indeed from

$$\mathcal{L} = \mathcal{L}_m + \mathcal{L}_A, \quad \mathcal{L}_m = \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi^* - V(|\phi|^2), \quad \mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (3.112)$$

the equation of motion of the electromagnetic field reads

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad j^\mu = - \frac{\delta S}{\delta A_\mu} \quad (3.113)$$

tensor as the susceptibility of the system with respect to the variations of the background metric :

$$\delta S = \frac{1}{4\pi} \int T^{\mu\nu} \delta g_{\mu\nu} \sqrt{|g|} d^n x = -\frac{1}{4\pi} \int T_{\mu\nu} \delta g^{\mu\nu} \sqrt{|g|} d^n x \quad (3.119)$$

where the minus sign arises because $\delta g^{\mu\nu} = -g^{\mu\sigma} \delta g_{\sigma\tau} g^{\tau\nu}$. This is equivalent to¹⁰

$$T_{\mu\nu} = -4\pi \frac{\delta S}{\delta g^{\mu\nu}} \quad (3.120)$$

This defines the so-called Hilbert stress-energy tensor. It suffers none of the unpleasant properties of the canonical one, since it is by construction gauge-invariant and symmetric.

In order to define the electric charge density, we promoted the U(1) symmetry to local gauge invariance. We now have defined the stress-energy tensor by promoting translation invariance to general (or diffeomorphism) covariance. What is left is to derive the analogue of (3.118), which will turn out to be nothing but the characterisation of the stress-energy tensor used in the main text, namely (3.8). To see this, consider once again an infinitesimal diffeomorphism $x^\mu \rightarrow x^\mu + \epsilon^\mu(x)$, acting this time both on the fields and the metric :

$$\delta g_{\mu\nu}(x) = \epsilon^\rho \partial_\rho g_{\mu\nu}(x) + g_{\mu\rho}(x) \partial_\nu \epsilon^\rho(x) + g_{\sigma\nu}(x) \partial_\mu \epsilon^\sigma(x) = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \quad (3.121)$$

where ∇ is the Levi-Civita connection and $\nabla_\mu \epsilon_\nu$ stands for $(\nabla_\mu \epsilon)_\nu$, namely

$$(\nabla_\mu \epsilon)_\nu = g_{\nu\lambda} (\nabla_\mu \epsilon)^\lambda = g_{\nu\lambda} (\partial_\mu \epsilon^\lambda + \Gamma^\lambda_{\rho\mu} \epsilon^\rho) = g_{\nu\lambda} \partial_\mu \epsilon^\lambda + \epsilon^\rho \Gamma_{\nu\rho\mu} \quad (3.122)$$

where $\Gamma_{\nu\rho\mu}$ is the Christoffel symbol

$$\Gamma_{\nu\rho\mu} = \frac{1}{2} (\partial_\mu g_{\nu\rho} + \partial_\rho g_{\nu\mu} - \partial_\nu g_{\mu\rho}) \quad (3.123)$$

In particular in flat space (and in flat coordinates, *i.e.* such that $g_{\mu\nu} = \eta_{\mu\nu}$)

$$\delta g_{\mu\nu}(x) = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \quad (3.124)$$

But general covariance tells us that the theory remains unchanged under arbitrary diffeomorphisms (note that such diffeomorphisms correspond to symmetries only when they leave the metric unchanged, which infinitesimally means Killing vector fields). Therefore we have

$$\begin{aligned} 0 &= \delta S = S[\Phi + L_\epsilon \Phi, g + L_\epsilon g] - S[\Phi, g] \\ &= S[\Phi + L_\epsilon \Phi, g] - S[\Phi, g] + \int \left. \frac{\delta S}{\delta g_{\mu\nu}} \right|_{g=\eta} \delta g_{\mu\nu} \sqrt{|g|} d^n x \end{aligned}$$

and we get

$$\begin{aligned} S[\Phi + L_\epsilon \Phi, g] - S[\Phi, g] &= - \int \frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu} \sqrt{|g|} d^n x \\ &= -\frac{1}{2\pi} \int T^{\mu\nu} \nabla_\mu \epsilon_\nu \sqrt{|g|} d^n x \end{aligned}$$

¹⁰We recall our convention for functional derivatives : $\delta S = \int \frac{\delta S}{\delta g^{\mu\nu}(x)} \delta g^{\mu\nu}(x) \sqrt{|g|} d^n x$ and not $\frac{\delta S}{\delta g^{\mu\nu}(x)} \delta g^{\mu\nu}(x) d^n x$.

In flat space and in flat coordinates we recover (3.8)

$$S[\Phi + L_\epsilon \Phi] - S[\Phi] = \int T^{\mu\nu} \partial_\mu \epsilon_\nu d^n x, \quad T^{\mu\nu} = -4\pi \frac{\delta S}{\delta g_{\mu\nu}} \Big|_{g=\eta} \quad (3.125)$$

Matter tells spacetime how to curve

Alternatively one can couple the theory (referred to as *matter* in this context, with action S_m) to a *dynamical* metric with the Einstein-Hilbert action

$$S_g = \frac{1}{2} \int R \sqrt{|g|} d^n x$$

where R is the Ricci scalar (see Appendix (3.4.3)). The total action is now

$$S = S_m + S_g$$

and the equation of motion of the metric is

$$0 = \frac{\delta S_g}{\delta g_{\mu\nu}} + \frac{\delta S_m}{\delta g_{\mu\nu}} \quad (3.126)$$

In (3.4.3) we compute

$$\frac{\delta S_g}{\delta g_{\mu\nu}} = -\frac{1}{2} \left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \right) \quad (3.127)$$

while by definition

$$\frac{\delta S_m}{\delta g_{\mu\nu}} = -\frac{1}{2} T^{\mu\nu} \quad (3.128)$$

and we obtain the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu} \quad (3.129)$$

which is the analogue of $\partial_\mu F^{\mu\nu} = j^\nu$ for gravitation : the stress-energy tensor acts as the source of the gravitational field.

3.4.3 Some basic notions in Riemannian geometry

While Riemannian geometry does not play an essential role throughout this lecture, notations such as metric, Levi-Civita connection and curvature are mentioned at times. This section - aimed at the reader already familiar with these concepts - is simply a reminder of basic definitions and an excuse to fix our notations.

Covariant derivative, Levi-Civita connection, Christoffel symbols

We consider a Riemannian manifold (M, g) where g is the metric. Given a local chart (coordinate system) x^μ , the components of the metric are

$$g_{\mu\nu} = g(\partial_\mu, \partial_\nu), \quad g^{\mu\nu} = g(dx^\mu, dx^\nu), \quad (3.130)$$

and these two matrices are inverses of each other

$$g_{\mu\nu}g^{\nu\rho} = \delta_\mu^\rho. \quad (3.131)$$

The Levi-Civita connection (or covariant derivative) ∇ is the only torsion-free, metric-compatible connection on the tangent bundle of Riemannian manifold. Torsion-free means

$$\nabla_X Y - \nabla_Y X = [X, Y] \quad (3.132)$$

while metric-compatibility is $\nabla g = 0$, *i.e.*

$$\nabla_Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \quad (3.133)$$

In local coordinates the Christoffel symbols $\Gamma^\rho_{\mu\nu}$ are defined by

$$\nabla_{\partial_\mu}\partial_\nu = \Gamma^\rho_{\mu\nu}\partial_\rho \quad (3.134)$$

and they can be expressed in terms of the metric as

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}) \quad (3.135)$$

The Levi-Civita connection has a natural extension to the dual bundle (*i.e.* the cotangent bundle) and more generally to arbitrary tensor bundles :

$$(\nabla_\rho T)^{\nu_1\dots\nu_s}_{\mu_1\dots\mu_r} = \frac{\partial}{\partial\rho}T^{\nu_1\dots\nu_s}_{\mu_1\dots\mu_r} \quad (3.136)$$

$$+ \Gamma^{\nu_1}_{\sigma\rho}T^{\sigma\nu_2\dots\nu_s}_{\mu_1\dots\mu_r} + \dots + \Gamma^{\nu_s}_{\sigma\rho}T^{\nu_1\dots\nu_{s-1}\sigma}_{\mu_1\dots\mu_r} \quad (3.137)$$

$$- \Gamma^\sigma_{\mu_1\rho}T^{\nu_1\dots\nu_s}_{\sigma\mu_2\dots\mu_r} - \dots - \Gamma^\sigma_{\mu_r\rho}T^{\nu_1\dots\nu_s}_{\mu_1\dots\mu_{r-1}\sigma}. \quad (3.138)$$

Following standard notations we will often drop the parenthesis and write $\nabla_\rho T^{\nu_1\dots\nu_s}_{\mu_1\dots\mu_r}$ in place of $(\nabla_\rho T)^{\nu_1\dots\nu_s}_{\mu_1\dots\mu_r}$. One should be careful about this abuse of notation. For instance the metric-compatibility reads

$$\nabla_\rho g_{\mu\nu} = 0 \quad (3.139)$$

being understood that $\nabla_\rho g_{\mu\nu}$ stands for $(\nabla_\rho g)_{\mu\nu}$ and not for $\nabla_\rho(g_{\mu\nu})$. Indeed the latter is in general non vanishing since

$$\nabla_\rho(g_{\mu\nu}) = \partial_\rho g_{\mu\nu}. \quad (3.140)$$

A last remark is that this extension to arbitrary tensors is natural in the sense that it is compatible with Leibniz rule and with trace operations (tensor contraction). For instance

$$g^{\mu\nu}\nabla_\rho T_{\mu\nu} = \nabla_\rho(g^{\mu\nu}T_{\mu\nu}) \quad (= \partial_\rho T^\mu_\mu). \quad (3.141)$$

Curvature

The Riemann curvature tensor is defined by

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u, v]} w \quad (3.142)$$

In local coordinates, we work with the convention

$$R_{\mu\nu\rho\sigma} = g(R(\partial_\mu, \partial_\nu)\partial_\sigma, \partial_\rho) \quad (3.143)$$

This tensor enjoys the following properties

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}, \quad R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}, \quad R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}. \quad (3.144)$$

In terms of the Christoffel symbols we have

$$R^\rho_{\mu\sigma\nu} = \partial_\sigma \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\sigma\lambda} \Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\sigma\mu} \quad (3.145)$$

The Ricci tensor is given by

$$Ric(u, v) = \text{Tr}_g(x \rightarrow R(x, u)v) \quad (3.146)$$

which means in local coordinates

$$R_{\mu\nu} = Ric(\partial_\mu, \partial_\nu) = g^{\rho\sigma} R_{\rho\mu\sigma\nu} = R^\rho_{\mu\rho\nu} \quad (3.147)$$

This is a symmetric tensor, whose local expression in terms of the Christoffel symbols is

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu} \quad (3.148)$$

Finally the Ricci scalar is simply $R = R^\mu_{\mu}$.

Variation of the Einstein-Hilbert action

We want to compute the variation δS_g of the Einstein-Hilbert action

$$S_g = \frac{1}{2} \int R \sqrt{|g|} d^n x \quad (3.149)$$

under an infinitesimal change of metric $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$. The answer is

$$\delta S_g = \frac{1}{2} \int G_{\mu\nu} \delta g^{\mu\nu} \sqrt{|g|} d^n x \quad (3.150)$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor.

A first useful formula is

$$\delta \sqrt{|g|} = -\frac{1}{2} \sqrt{|g|} g_{\mu\nu} \delta g^{\mu\nu} = \frac{1}{2} \sqrt{|g|} g^{\mu\nu} \delta g_{\mu\nu} \quad (3.151)$$

Indeed for any invertible matrix A one has

$$\det(A + H) = \det A(1 + A^{-1}H) = \det A + \det A \text{Tr}(A^{-1}H) + O(H^2) \quad (3.152)$$

which can be written as $\frac{\partial \det A}{\partial A_{\mu\nu}} = \det AA^{\nu\mu}$, where $A^{\mu\nu}$ stands for $(A^{-1})_{\mu\nu}$. So under $A \rightarrow A + \delta A$ we get

$$\delta \det A = \det AA^{\nu\mu}\delta A_{\mu\nu} \quad (3.153)$$

and (3.151) follows.

The variation of the Ricci curvature R is in itself of interest and will prove useful :

$$\delta R = (R_{\mu\nu} - \nabla_\mu \nabla_\nu + g_{\mu\nu} \nabla_\rho \nabla^\rho) \delta g^{\mu\nu} \quad (3.154)$$

where $\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}$. Deriving this identity is a rather lengthy but standard calculation. We start with

$$\delta R = (\delta g^{\mu\nu})R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \quad (3.155)$$

and

$$\delta R^\rho_{\sigma\mu\nu} = \partial_\mu \delta \Gamma^\rho_{\nu\sigma} - \partial_\nu \delta \Gamma^\rho_{\mu\sigma} + \delta \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} \quad (3.156)$$

$$+ \Gamma^\rho_{\mu\lambda} \delta \Gamma^\lambda_{\nu\sigma} - \delta \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\nu\lambda} \delta \Gamma^\lambda_{\mu\sigma} \quad (3.157)$$

$$= \nabla_\mu \delta \Gamma^\rho_{\nu\sigma} - \nabla_\nu \delta \Gamma^\rho_{\mu\sigma} \quad (3.158)$$

Indeed $\delta \Gamma^\rho_{\nu\sigma}$ being the difference between two connections, it is a tensor, and it makes sense to take its covariant derivative. This yields the *Palatini identity*

$$\delta R_{\mu\nu} = \nabla_\rho \delta \Gamma^\rho_{\nu\mu} - \nabla_\nu \delta \Gamma^\rho_{\rho\mu}. \quad (3.159)$$

Since covariant derivative commutes with contraction we get

$$\delta R = (\delta g^{\mu\nu})R_{\mu\nu} + \nabla_\rho (g^{\mu\nu}\delta \Gamma^\rho_{\nu\mu}) - \nabla_\nu (g^{\mu\nu}\delta \Gamma^\rho_{\rho\mu}) \quad (3.160)$$

$$= (\delta g^{\mu\nu})R_{\mu\nu} + \nabla_\rho (g^{\mu\nu}\delta \Gamma^\rho_{\nu\mu} - g^{\mu\rho}\delta \Gamma^\sigma_{\sigma\mu}) \quad (3.161)$$

This is enough to get (3.150), since the term in ∇_ρ above yields a pure divergence in δS_g and therefore does not contribute. But to get (3.154) we must further massage this divergence term.

$$g^{\mu\nu}\delta \Gamma^\rho_{\nu\mu} - g^{\mu\rho}\delta \Gamma^\sigma_{\sigma\mu} = \delta(g^{\mu\nu}\Gamma^\rho_{\mu\nu}) - (\delta g^{\mu\nu})\Gamma^\rho_{\mu\nu} - g^{\mu\rho}\delta \partial_\mu \log \sqrt{|g|} \quad (3.162)$$

where we have used

$$\Gamma^\sigma_{\mu\sigma} = \frac{1}{2}g^{\rho\sigma}\partial_\mu g_{\rho\sigma} = \frac{1}{2g}\partial_\mu g = \partial_\mu \log \sqrt{|g|} \quad (3.163)$$

We now use

$$g^{\mu\nu}\Gamma^\rho_{\nu\mu} = -\frac{1}{\sqrt{|g|}}\partial_\mu (\sqrt{|g|}g^{\mu\rho}) = -\partial_\mu g^{\mu\rho} - g^{\mu\rho}\partial_\mu \log \sqrt{|g|} \quad (3.164)$$

and we get

$$g^{\mu\nu}\delta\Gamma_{\nu\mu}^\rho - g^{\mu\rho}\delta\Gamma_{\sigma\mu}^\sigma = -\partial_\mu\delta g^{\mu\rho} - \Gamma_{\mu\sigma}^\sigma\delta g^{\mu\rho} \quad (3.165)$$

$$-\Gamma_{\mu\nu}^\rho\delta g^{\mu\nu} - 2g^{\mu\rho}\partial_\mu\delta\log\sqrt{|g|} \quad (3.166)$$

$$= -\nabla_\mu\delta g^{\mu\rho} - g^{\mu\rho}\partial_\mu\delta\log|g| \quad (3.167)$$

$$= -\nabla_\mu\delta g^{\mu\rho} - g^{\mu\rho}\partial_\mu\frac{\delta g}{g} \quad (3.168)$$

$$= -\nabla_\mu\delta g^{\mu\rho} + g^{\mu\rho}\partial_\mu(g_{\nu\sigma}\delta g^{\nu\sigma}) \quad (3.169)$$

$$= -\nabla_\mu\delta g^{\mu\rho} + g^{\mu\rho}\nabla_\mu(g_{\nu\sigma}\delta g^{\nu\sigma}) \quad (3.170)$$

$$= -\nabla_\mu\delta g^{\mu\rho} + g^{\mu\rho}g_{\nu\sigma}\nabla_\mu\delta g^{\nu\sigma} \quad (3.171)$$

and this yields (3.154).

Peculiarities in two dimensions

In two-dimensions, due to the symmetries (3.144) of the Riemann curvature tensor, there is only one independent component :

$$R_{\mu\nu\rho\sigma} = \frac{R}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \quad (3.172)$$

and the Einstein tensor vanishes¹¹

$$R_{\mu\nu} = \frac{R}{2}g_{\mu\nu} \quad (3.174)$$

In fact one can say much more : any two-dimensional Riemannian manifold is (locally) conformally flat, in the sense that there exist local coordinates in which the metric is conformal to the Euclidean metric :

$$g_{\mu\nu}(x) = e^{\sigma(x)}\delta_{\mu\nu} \quad (3.175)$$

Such coordinates are called *isothermal coordinates*. To say things differently, in two dimensions each point has a neighborhood that can be mapped to flat space by a conformal map.

Furthermore if the surface under consideration is oriented, then it is naturally a Riemann surface, *i.e.* a one-dimensional complex manifold. This is a simple consequence of the existence of isothermal coordinates. The main observation is that transition functions between isothermal coordinates are conformal maps (and therefore holomorphic or anti-holomorphic). If the surface under consideration is oriented, then we can choose our atlas of isothermal coordinates to be oriented, in which case the transition functions - being orientation preserving - are holomorphic. What we have on our hands is an atlas whose

¹¹As a cultural remark, this makes two-dimensional gravity quite peculiar. In fact the Einstein-Hilbert action becomes a topological invariant since on a compact surface of genus g the Gauss-Bonnet theorem states that

$$\int R\sqrt{|g|}d^2x = 8\pi(1-g). \quad (3.173)$$

Deforming the surface (*i.e.* changing the metric) leaves the Einstein-Hilbert action unchanged, which is another way to see that the Einstein tensor must vanish (from (3.150)).

transition functions are holomorphic, *i.e.* a Riemann surface. To say things differently (oriented) isothermal coordinates are complex coordinates.

Notice also that two oriented Riemannian manifolds that are conformally equivalent yield the same complex structure, so in this sense Weyl transformations are compatible with complex structure. In layman terms : if a function is holomorphic for the metric g , it is also holomorphic for $e^\sigma g$.

Finally a map between two oriented Riemannian manifolds is conformal if and only if it is holomorphic or anti-holomorphic (in the sense that $(\partial_1 \mp i\partial_2)f = 0$ in isothermal coordinates).

It should be clear from the discussion above that using (local) complex coordinates is going to prove rather beneficial. We list below some related basic formulas.

Let (x^1, x^2) be isothermal coordinates, and define $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$. A local frame of the (complexified) cotangent space is given by

$$dz = dx^1 + idx^2, \quad d\bar{z} = dx^1 - idx^2 \quad (3.176)$$

and the dual frame in the tangent space reads

$$\partial = \frac{1}{2}(\partial_1 - i\partial_2), \quad \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2) \quad (3.177)$$

In particular the metric

$$g = e^{\sigma(x)}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2) \quad (3.178)$$

reads in complex coordinates

$$g = \frac{1}{2}e^{\sigma(x)}(dz \otimes d\bar{z} + d\bar{z} \otimes dz) \quad (3.179)$$

which is to say in components

$$g_{zz} = g_{\bar{z}\bar{z}} = 0, \quad g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}e^\sigma \quad (3.180)$$

The inverse metric is given by

$$g^{zz} = g^{\bar{z}\bar{z}} = 0, \quad g^{z\bar{z}} = g^{\bar{z}z} = 2e^{-\sigma} \quad (3.181)$$

A real vector field $\xi = \xi^1\partial_1 + \xi^2\partial_2$ can be decomposed in the basis $(\partial, \bar{\partial})$ as

$$\xi = \xi^z\partial + \xi^{\bar{z}}\bar{\partial}, \quad \xi^z = \bar{\xi^{\bar{z}}} = \xi^1 + i\xi^2 \quad (3.182)$$

Complex indices are raised and lowered according to

$$\xi_z = g_{z\bar{z}}\xi^{\bar{z}} = \frac{1}{2}e^\sigma \xi^{\bar{z}}, \quad \xi^z = 2e^{-\sigma}\xi_{\bar{z}} \quad (3.183)$$

The stress-energy tensor $T_{\mu\nu}$ has components

$$T_{zz} = \overline{T_{\bar{z}\bar{z}}} = \frac{1}{4}(T_{11} - T_{22} - iT_{12} - iT_{21}) \quad (3.184)$$

$$T_{z\bar{z}} = \overline{T_{\bar{z}z}} = \frac{1}{4}(T_{11} + T_{22} + i(T_{12} - T_{21})) \quad (3.185)$$

The Riemannian volume form $\omega = \sqrt{|g|}dx^1 \wedge dx^2$ is

$$\omega = e^\sigma \frac{d\bar{z} \wedge dz}{2i} \quad (3.186)$$

The covariant derivative becomes particularly simple

$$\nabla_\partial = (\partial\sigma)\partial, \quad \nabla_{\bar{\partial}} = (\bar{\partial}\sigma)\bar{\partial} \quad (3.187)$$

which is to say the only non zero Christoffel symbols are

$$\Gamma^z_{zz} = \partial\sigma, \quad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \bar{\partial}\sigma \quad (3.188)$$

as well as the scalar curvature

$$R = -4e^{-\sigma}\partial\bar{\partial}\sigma = -\Delta\sigma \quad (3.189)$$

where Δ is the Laplacian.

Lecture 4

Conformal field theories in curved space

So far we have discussed conformal invariance for a quantum field theory on the complex plane, and we now consider a field theory living on an arbitrary two-dimensional curved space (M, g) . This essentially brings two new features in the game

- the *topology* of space-time (*i.e.* the smooth surface M itself, as characterized by its genus¹),
- the *geometry* of space-time, as described by the Riemannian metric g .

One must prescribe how to extend the theory from flat Euclidean space to an arbitrary curved space, through an action that now depends explicitly on the metric g (and on the manifold M)

$$S[\Phi, g] = \int_M \mathcal{L}(\Phi(x), \partial_\mu \Phi(x), g_{\mu\nu}(x)) dV(x) \quad (4.1)$$

where $dV(x) = \sqrt{|g(x)|} d^2x$ is the Riemannian volume element, and $|g(x)|$ stands for the determinant of the matrix $g_{\mu\nu}(x) = g(\partial_\mu, \partial_\nu)$. Besides allowing us to work on curved space, introducing an arbitrary metric g has an important benefit : the action now takes the same mathematical form in all coordinate systems (even in flat space, one could decide to work in polar coordinates for instance). One says that such a theory is *generally covariant*, and it means that the theory does not require the choice of a coordinate system in order to be well-defined. For instance the free scalar field can be coupled to the metric as follow

$$S[\phi, g] = \int g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi dV(x) \quad (4.2)$$

where $g^{\mu\nu} = g(dx^\mu, dx^\nu)$ is the inverse matrix of $g_{\mu\nu}$. It is not very difficult to check that the above expression remains the same in all coordinate systems.

On a generic curved manifold there is no longer a preferred coordinate system (nor a globally defined one), so having a coordinate-independent formulation of physics is a very natural requirement indeed. One can push this idea even further and use a coordinate-free

¹For now we will restrict ourselves to compact, oriented manifolds without boundary. Conformal field theories can also be defined on unorientable manifolds and in the presence of boundaries, but this is a more advanced topic which we will not address at this stage.

formalism in which fields become purely geometric objects. For instance the action (4.2) is just the expression in local coordinates of the intrinsic quantity

$$S = \int d\phi \wedge \star d\phi = \int g(d\phi, d\phi) dV = \|d\phi\|_g^2 \quad (4.3)$$

At the classical level general covariance means that the action is invariant under isometries, in the sense that for any diffeomorphism $f : M \rightarrow M$

$$S[f^*\Phi, f^*g] = S[\Phi, g] \quad (4.4)$$

which essentially means that the Lagrangian density \mathcal{L} is a scalar : $f^*\mathcal{L} = \mathcal{L} \circ f$.

We assume that general covariance still holds at the quantum level. In the path integral this amounts to

$$e^{-S[f^*\Phi, f^*g]} D_{f^*g}[f^*\Phi] = e^{-S[\Phi, g]} D_g[\Phi] \quad (4.5)$$

where we write $D_g[\Phi]$ to emphasise that the functional measure does in general depend on the metric. Why is that ? Naively it does not, but the fact that it has to be regularized (for instance through a short-distance cut-off) makes the story more subtle. For instance a short-distance cut-off - such as a lattice constant - introduces a UV length, de facto making the integration measure scale-dependent. In practice assuming general covariance means finding a way to regularize the QFT in such a way as to preserve this invariance. We will see in the next tutorial how this is done for the free boson. If we denote by Z_g the partition function of the QFT on (M, g) , we have for any diffeomorphism $f : M \rightarrow M$

$$\boxed{Z_g = Z_{f^*g}} \quad (4.6)$$

where we dropped the subscript M . Furthermore for a correlation function

$$\langle O \rangle_{f^*g} = \frac{1}{Z_{f^*g}} \int O e^{-S[\Phi, f^*g]} D_{f^*g}[\Phi] = \frac{1}{Z_g} \int (f^*O) e^{-S[f^*\Phi, f^*g]} D_{f^*g}[f^*\Phi] \quad (4.7)$$

$$= \frac{1}{Z_g} \int (f^*O) e^{-S[\Phi, g]} D_g[\Phi] = \langle f^*O \rangle_g \quad (4.8)$$

or more explicitly

$$\boxed{\langle (f^*O_1)(x_1) \cdots (f^*O_p)(x_p) \rangle_g = \langle O_1(x_1) \cdots O_p(x_p) \rangle_{f^*g}} \quad (4.9)$$

Another way to think about (4.6) and (4.9) is in term of active versus passive transformation. An elementary but useful example is that of a scale transformation $f(x) = \lambda x$ on the Euclidean plane (with metric $\eta_{\mu\nu} = \delta_{\mu\nu}$). The active transformation amounts to send the point x to λx , hence multiplying the length of curves by a factor λ . In the passive transformation points do not move, but the definition of distance is rescaled by changing the metric to $f^*\eta = \lambda^2\eta$. Again the length of all curves is multiplied by λ . Clearly these two transformations are equivalent. For instance in a CFT the two point function of a scalar primary field ϕ_Δ with scaling dimension Δ obeys

$$\langle \phi_\Delta(x_1) \phi_\Delta(x_2) \rangle_{\lambda^2\eta} = \lambda^{-2\Delta} \frac{1}{\|x_1 - x_2\|^{2\Delta}} = \langle \phi_\Delta(\lambda x_1) \phi_\Delta(\lambda x_2) \rangle_\eta. \quad (4.10)$$

In the first equality we used the behavior of a correlation function under a Weyl rescaling (4.35) which we'll derive shortly.

4.1 Stress-energy tensor revisited

In section 3.1 we have defined the classical stress-energy tensor through (3.8), that is as the response of the theory to an infinitesimal diffeomorphism *while leaving the (flat) metric unchanged*. The generalization to curved space is immediate :

$$S[\Phi + \delta_\epsilon \Phi, g] - S[\Phi, g] = -\frac{1}{2\pi} \int_M T^{\mu\nu} \nabla_\mu \epsilon_\nu dV(x) \quad (4.11)$$

but just as in flat space this leaves the freedom to add to $T^{\mu\nu}$ any covariantly conserved tensor. General covariance allows to bypass this ambiguity by defining the stress-energy tensor as the response of the theory to an infinitesimal change of the metric (see appendix (3.4.2))

$$S[\Phi, g + \delta g] - S[\Phi, g] = \frac{1}{4\pi} \int_M T^{\mu\nu} \delta g_{\mu\nu} dV(x) = -\frac{1}{4\pi} \int_M T_{\mu\nu} \delta g^{\mu\nu} dV(x) \quad (4.12)$$

or equivalently

$$T_{\mu\nu}(x) = -4\pi \frac{\delta S}{\delta g^{\mu\nu}(x)}. \quad (4.13)$$

Note that we work with the following convention for functional derivatives :

$$\delta F = \int_M \frac{\delta F}{\delta f(x)} \delta f(x) dV(x)$$

for a generic functional F of a function f defined on M . This means

$$\frac{\delta f(x)}{\delta f(y)} = \delta_y(x) \quad (4.14)$$

where $\delta_y(x)$ is the delta-function normalized with respect to the volume form, in the sense that

$$\int_M \delta_y(x) h(x) \sqrt{|g(x)|} d^2x = h(y). \quad (4.15)$$

for any test function h .

Equation (4.17) defines the so-called Hilbert stress-energy tensor, and the fact that it coincides with our previous definition (4.11) in flat space is a simple consequence of general covariance. This stress-energy tensor is symmetric by construction. The reader might wonder what happened to the intrinsic ambiguity underlying the definition of the stress-energy tensor in flat space. It is not really gone, and it hides in the fact that the extension of the theory from flat space to curved space may not be unique. For instance we could decide that the action of a free scalar field in curved space is

$$S[\phi, g] = \int (g^{\mu\nu} \partial^\mu \phi \partial_\nu \phi + \alpha R \phi) dV(x). \quad (4.16)$$

Both theories (4.2) and (4.16) are indistinguishable in the flat space limit since the scalar curvature R vanishes, but even in flat space the corresponding Hilbert stress-energy tensors as defined by (4.17) do not coincide : they differ by an improvement term.

At the quantum level there may be contributions coming from the integration measure, thus we define

$$\begin{aligned} & \langle T_{\mu_1\nu_1}(y_1) \cdots T_{\mu_r\nu_r}(y_r) O_1(x_1) \cdots O_p(x_p) \rangle_g \\ &= \frac{(4\pi)^r}{Z_g} \frac{\delta^r}{\delta g^{\mu_1\nu_1}(y_1) \cdots \delta g^{\mu_r\nu_r}(y_r)} Z_g \langle O_1(x_1) \cdots O_p(x_p) \rangle_g \end{aligned} \quad (4.17)$$

This allows to write the infinitesimal change of the partition function and correlation functions under a change of metric $g \rightarrow g + \delta g$ as

$$\frac{Z_{g+\delta g}}{Z_g} = \frac{1}{4\pi} \int \langle T_{\mu\nu}(x) \rangle \delta g^{\mu\nu}(x) \sqrt{|g(x)|} d^2x \quad (4.18)$$

and

$$\langle O \rangle_{g+\delta g} = \frac{1}{4\pi} \int (\langle T_{\mu\nu}(x) O \rangle_g - \langle T_{\mu\nu}(x) \rangle_g \langle O \rangle_g) \delta g^{\mu\nu}(x) \sqrt{|g(x)|} d^2x \quad (4.19)$$

where O stands for $O_1(x_1) \cdots O_p(x_p)$.

The terminology *tensor* is now fully justified, since general covariance implies that the Hilbert stress-energy tensor indeed transforms as a tensor under isometries :

$$\langle (f^*T)_{\mu\nu}(x) (f^*O_1)(x_1) \cdots (f^*O_p)(x_p) \rangle_g = \langle T_{\mu\nu}(x) O_1(x_1) \cdots O_p(x_p) \rangle_{f^*g} \quad (4.20)$$

where

$$(f^*T)_{\mu\nu}(x) = T_{\rho\sigma}(f(x)) \frac{\partial f^\rho}{\partial x^\mu} \frac{\partial f^\sigma}{\partial x^\nu}. \quad (4.21)$$

Another benefit of defining $T_{\mu\nu}$ through (4.17) is that the Ward identity becomes an elementary consequence of general covariance. Indeed for an infinitesimal diffeomorphism $f^\mu(x) = x^\mu + \epsilon^\mu(x)$ the relation (4.9) together with (4.19) yields

$$\sum_j \langle O_1(x_1) \cdots \delta_\epsilon O_j(x_j) \cdots O_p(x_p) \rangle_g = -\frac{1}{4\pi} \int \delta_\epsilon g_{\mu\nu} \langle T^{\mu\nu}(x) O \rangle_g dV(x) \quad (4.22)$$

$$= -\frac{1}{2\pi} \int \nabla_\mu \epsilon_\nu(x) \langle T^{\mu\nu}(x) O \rangle_g dV(x) \quad (4.23)$$

and we recover the Ward identity (3.53) in flat space.

4.2 Conformal/Weyl covariance

As explained in section (2.3), a conformal transformation can be split into an isometry and a Weyl rescaling. Since we have assumed invariance under isometries (this is the content of general covariance) conformal invariance is then equivalent to Weyl invariance. Under an infinitesimal Weyl rescaling $\delta g_{\mu\nu}(x) = 2\sigma(x)g_{\mu\nu}(x)$, one has from (4.17)

$$\delta Z = - \left(\frac{1}{2\pi} \int_M \langle T^\mu_\mu \rangle \sigma dV \right) Z \quad (4.24)$$

Naïvely conformal symmetry requires $T^\mu_\mu \simeq 0$, but for a QFT in curved space there is an anomaly. From general covariance, the trace of the stress-energy tensor must be invariant

under isometries, and it must vanish in the flat space limit. Furthermore, it must be of scaling dimension 2. In two-dimensions this only leaves one possibility : T^μ_μ can only be proportional to the scalar curvature R . The usual convention is to write the numerical prefactor as $-c/12$, where c is called the *central charge* :

$$T^\mu_\mu \simeq -\frac{c}{12}R \quad (4.25)$$

This is called the Weyl anomaly, and under an infinitesimal Weyl rescaling we have

$$Z_{e^{2\delta\sigma}g} = \left(1 + \frac{c}{24\pi} \int_M \delta\sigma(x) R_g(x) dV_g(x) + O(\delta\sigma^2)\right) Z_g. \quad (4.26)$$

where R_g is the scalar curvature associated to the metric g , and $dV = dV_g = \sqrt{|g|}d^2x$. This can be rephrased as

$$\frac{\delta}{\delta\sigma(x)} \log Z_{e^{2\sigma}g} = \frac{c}{24\pi} R_g(x) \quad (4.27)$$

Starting from some partition function Z_{g_0} , (4.27) determines the partition function for all metrics in the same conformal class, *i.e.* of the form $g = e^{2\sigma}g_0$. Indeed from (4.26) we have

$$Z_{e^{2\delta\sigma}g} = \left(1 + \frac{c}{24\pi} \int_M \delta\sigma(x) [R_0 - 2\Delta_0\sigma(x)] dV_0(x) + O(\delta\sigma^2)\right) Z_g. \quad (4.28)$$

where we used $R_{e^{2\sigma}g_0} = e^{-2\sigma}(R_0 - 2\Delta_0\sigma)$, with R_0 and Δ_0 begin the curvature and Laplacian of the reference metric g_0 . Thus

$$\frac{\delta}{\delta\sigma(x)} \log Z_{e^{2\sigma}g_0} = \frac{c}{24\pi} (R_0(x) - 2\Delta_0\sigma) \quad (4.29)$$

Upon integrating (4.29) we find the following behavior of the partition function under a finite Weyl rescaling (as long as the function σ has compact support)

$$Z_{e^{2\sigma}g} = \exp\left(\frac{c}{24\pi} \int_M (g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + R_g\sigma) dV_g\right) Z_g \quad (4.30)$$

Exercise : Wess-Zumino consistency condition. Check that (4.30) is consistent under the composition of Weyl transformations, in the sense that $g \rightarrow e^{2\sigma_1+2\sigma_2}g$ yields the same answer as $g \rightarrow e^{2\sigma_1}g$ followed by $g \rightarrow e^{2\sigma_2}g$.

Exercise : For a given background metric g_0 , consider the functional differential equation

$$\frac{\delta}{\delta\sigma(x)} F[\sigma] = R_0(x) - 2\Delta_0\sigma \quad (4.31)$$

with boundary condition $F[0] = 0$. It is clear that the solution, if it exists, is unique. Check that

$$F[\sigma] = \int_M (g_0^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + R_0\sigma) dV_0 \quad (4.32)$$

is a solution, as long as the function σ has compact support.

We now know how the partition function of a conformal field theory behaves under Weyl rescaling. What about correlation functions ? The definition (4.17) of the stress-energy tensor *a la* Hilbert tells us that under an infinitesimal Weyl rescaling $\delta g_{\mu\nu}(x) = 2\sigma(x)g_{\mu\nu}(x)$ a correlation function $\langle O \rangle = \langle O_1(x_1)\cdots O_p(x_p) \rangle$ behaves as

$$\delta(Z\langle O \rangle) = -\left(\frac{1}{2\pi} \int_M \langle T^\mu_\mu O \rangle \sigma dV\right) Z \quad (4.33)$$

i.e.

$$\delta\langle O \rangle = -\frac{1}{2\pi} \int_M (\langle T^\mu_\mu O \rangle - \langle T^\mu_\mu \rangle \langle O \rangle) \sigma dV. \quad (4.34)$$

The Weyl anomaly cancels out in $\langle T^\mu_\mu O \rangle - \langle T^\mu_\mu \rangle \langle O \rangle$, and only the contact terms of T^μ_μ can contribute : as in flat space, they may modify the naive classical transformation of fields. While classical fields are invariant under Weyl transformation, quantum fields have to be regularized, and hence become sensitive to change of scales. Vertex operators in the free scalar theory provide a good and explicit example. For instance for scalar primary fields

$$\langle O_1(x_1)\cdots O_p(x_p) \rangle_{e^{2\sigma}g} = \prod_{i=1}^p e^{-\gamma_i \sigma(x_i)} \langle O_1(x_1)\cdots O_p(x_p) \rangle_g. \quad (4.35)$$

which we will write as

$$(O_i)_{e^{2\sigma}g} = e^{-\gamma_i \sigma} (O_i)_g \quad (4.36)$$

where γ_i is the anomalous dimension of the field O_i .

Exercise : By computing $\frac{4\pi}{Z} \frac{\delta}{\delta g^{\mu\nu}} Z \langle T^\rho_\rho O \rangle$ using (4.25), show that the contact term between T^ρ_ρ and $T_{\mu\nu}$ is

$$T^\rho_\rho(x) T_{\mu\nu}(y) = 4\pi \frac{c}{12} \frac{\delta R(x)}{\delta g^{\mu\nu}(y)} \quad (4.37)$$

and evaluate the functional derivative using eq. (3.154). Deduce that in a locally flat metric

$$T^\rho_\rho(x) T_{zz}(y) = -4\pi \frac{c}{12} \partial_z^2 \delta(x-y) \quad (4.38)$$

and recover (3.91).

4.3 Behavior of the stress-energy tensor under Weyl rescaling

In a classical field theory enjoying Weyl invariance, the stress tensor behaves as

$$(T_{\mu\nu})_{e^{2\sigma}g} = (T_{\mu\nu})_g \quad (4.39)$$

Indeed Weyl invariance of the action implies $S[\Phi, g + \delta g] - S[\Phi, g] = S[\Phi, e^{2\sigma}(g + \delta g)] - S[\Phi, e^{2\sigma}g]$, yielding

$$\int_M (T_{\mu\nu})_g \delta g^{\mu\nu} dV = \int_M (T_{\mu\nu})_{e^{2\sigma}g} \delta g^{\mu\nu} dV \quad (4.40)$$

since $\delta \tilde{g}^{\mu\nu} = e^{-2\sigma} \delta g^{\mu\nu}$ and $d\tilde{V} = e^{2\sigma} dV$ for $\tilde{g} = e^{2\sigma} g$.

At the quantum level however the anomalous behavior (4.30) means that the quantum stress tensor transforms as

$$(T_{\mu\nu})_{e^{2\sigma}g} = (T_{\mu\nu})_g + \frac{c}{6} \left(\partial_\mu \sigma \partial_\nu \sigma - \frac{1}{2} g_{\mu\nu} \partial^\rho \sigma \partial_\rho \sigma + g_{\mu\nu} \Delta \sigma - \nabla_\mu \nabla_\nu \sigma \right) \quad (4.41)$$

Indeed under $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$ we have

$$\frac{1}{Z_{e^{2\sigma}g}} \delta \left(Z_{e^{2\sigma}g} \langle O \rangle_{e^{2\sigma}g} \right) = \frac{1}{4\pi} \int \langle T_{\mu\nu} O \rangle_{e^{2\sigma}g} \delta \tilde{g}^{\mu\nu} d\tilde{V} \quad (4.42)$$

$$= \frac{1}{4\pi} \int \langle T_{\mu\nu} O \rangle_{e^{2\sigma}g} \delta g^{\mu\nu} dV \quad (4.43)$$

Thus

$$\langle T_{\mu\nu} O \rangle_{e^{2\sigma}g} = \frac{4\pi}{Z_{e^{2\sigma}g}} \frac{\delta}{\delta g^{\mu\nu}} \left(Z_{e^{2\sigma}g} \langle O \rangle_{e^{2\sigma}g} \right) \quad (4.44)$$

When the functional derivative $\frac{\delta}{\delta g^{\mu\nu}}$ acts on O , it only produces contact terms. So away from coinciding points we can write

$$(T_{\mu\nu})_{e^{2\sigma}g} = \frac{4\pi}{Z_{e^{2\sigma}g}} \frac{\delta}{\delta g^{\mu\nu}} Z_{e^{2\sigma}g} \quad (4.45)$$

Plugging in (4.30) we get

$$(T_{\mu\nu})_{e^{2\sigma}g} = \frac{4\pi}{Z_{e^{2\sigma}g}} \frac{\delta}{\delta g^{\mu\nu}} Z_{e^{2\sigma}g} \quad (4.46)$$

$$= (T_{\mu\nu})_g + \frac{c}{6} \frac{\delta}{\delta g^{\mu\nu}} \int_M (g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma + R_g \sigma) dV \quad (4.47)$$

Computing the functional derivatives (see (3.151) and (3.154) in the Appendix) yields (4.41).

Exercise : using equation (4.41), check that T^μ_μ transforms as

$$(T^\mu_\mu)_{e^{2\sigma}g} = e^{-2\sigma} \left((T^\mu_\mu)_g + \frac{c}{6} \Delta \sigma \right) \quad (4.48)$$

and in particular we recover $(T^\mu_\mu)_{e^{2\sigma}dzd\bar{z}} = \frac{c}{6} e^{-2\sigma} \Delta \sigma = -\frac{c}{12} R$.

We are now going to work in isothermal coordinates according to which we can define complex coordinates. We now have

$$g_{\mu\nu}(x) = e^{2\sigma(x)} \delta_{\mu\nu}, \quad R_g = -2\Delta_g \sigma = -2e^{-2\omega} (\partial_1^2 + \partial_2^2) \sigma = -8e^{-2\sigma} \partial \bar{\partial} \sigma \quad (4.49)$$

We get

$$(T_{zz})_{e^{2\sigma}dzd\bar{z}} = (T_{zz})_{dzd\bar{z}} + \frac{c}{6} ((\partial \sigma)^2 - \partial^2 \sigma) \quad (4.50)$$

4.4 Behavior of the stress-energy tensor under conformal maps

Since we know how the stress-tensor transforms under isometries and Weyl transformation, we know how it behaves under conformal transformations. Consider a conformal map

$$f : (M, g) \rightarrow (M, \tilde{g}) \quad (4.51)$$

$$z \rightarrow f(z) \quad (4.52)$$

In practice we can always exploit Weyl transformations to make the metric locally flat around all insertions. So we consider a point z and its image $w = f(z)$ with metric (locally, *i.e.* in some neighborhood) of the form $g = dz d\bar{z}$ and $\tilde{g} = dw d\bar{w}$, so that $f^* \tilde{g} = e^{2\sigma} g$ with $2\sigma = \log \frac{\partial f}{\partial z} + \log \frac{\partial \bar{f}}{\partial \bar{z}}$, at least in some neighborhood of z . From general covariance

$$\langle (f^* T)(z) \cdots \rangle_{\tilde{g}} = \langle T(z) \cdots \rangle_{f^* \tilde{g}} = \langle T(z) \cdots \rangle_{e^{2\sigma} g} \quad (4.53)$$

Now remember that T stands for T_{zz} , and therefore $(f^* T)(z) = (\partial_z f)^2 T(f(z)) = (\partial_z w)^2 T(w)$. Now (4.50) gives the behavior under a Weyl rescaling :

$$\langle T(z) \cdots \rangle_{e^{2\sigma} g} = \left\langle \left(T(z) - \frac{c}{12} S(f)(z) \right) \cdots \right\rangle_g \quad (4.54)$$

where $S(f)$ is the Schwarzian derivative as defined in eq. (3.95)

$$S(f) = \partial_z \left(\frac{\partial_z^2 f}{\partial_z f} \right) - \frac{1}{2} \left(\frac{\partial_z^2 f}{\partial_z f} \right)^2 = \frac{\partial_z^3 f}{\partial_z f} - \frac{3}{2} \left(\frac{\partial_z^2 f}{\partial_z f} \right)^2 \quad (4.55)$$

Thus we recover the anomalous transformation eq. (3.94)

$$\left\langle \left((\partial_z f)^2 T(f(z)) + \frac{c}{12} S(f)(z) \right) \cdots \right\rangle_{\tilde{g}} = \langle T(z) \cdots \rangle_g \quad (4.56)$$

We are going to write the above slightly abusively as

$$T(w) = \left(\frac{\partial w}{\partial z} \right)^{-2} \left[T(z) - \frac{c}{12} \{w, z\} \right]$$

(4.57)

keeping in mind however that we are working in locally flat metrics (hence the metric need *not* be the same in the l.h.s. and the r.h.s.).

Lecture 5

The Virasoro algebra

In the previous lectures, we have seen that the conformal invariance of the action simply translates into conformal Ward identities, which in turn allow us to express *all* the singular terms in the OPE of the stress-energy tensor $T(z)$ with primary operators, and in the OPE of $T(z)$ with itself. We shall use this as a basis to construct families of “descendant” scaling operators, by acting with the stress-energy tensor on the primary operators. This linear action can be described in terms of an infinite-dimensional Lie algebra: *the Virasoro algebra*.

5.1 Mode expansion of the stress-energy tensor

Any correlation function of the form $\langle T(z) \dots \rangle$ is holomorphic in z , with possible singularities at the insertion points of other local operators denoted by “ \dots ”. Let us look at the behaviour of $T(z)$ around the origin: it may be singular, but since the correlation function is still single-valued, this singularity should be of the form of a (possibly multiple) pole at $z = 0$. Hence we have the Laurent series expansion:

$$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n, \quad L_n = \frac{1}{2i\pi} \oint_{C_0} dz z^{n+1} T(z), \quad (5.1)$$

where the integration contour is a closed curve around the origin, anti-clockwise oriented. Similarly, for the anti-holomorphic component:

$$\bar{T}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n, \quad \bar{L}_n = \frac{1}{2i\pi} \oint_{C_0} d\bar{z} \bar{z}^{n+1} \bar{T}(\bar{z}). \quad (5.2)$$

In the following, we shall consider the action of the L_n ’s and the \bar{L}_n ’s *on the space \mathcal{A} of operators inserted at the origin*. This makes sense because of the OPE principle:

$$T(z)\mathcal{O}(0,0) = \sum_{n \in \mathbb{Z}} z^{-n-2} (L_n \cdot \mathcal{O})(0,0), \quad (5.3)$$

where \mathcal{O} can be any local scaling operator inserted at $z = 0$. From this point of view, the L_n ’s now act linearly on such operators. If no other operator is inserted close to the

origin, one can shrink the integration contour for L_n as much as we like:

$$(L_n \cdot \mathcal{O})(0,0) = \frac{1}{2i\pi} \oint_{|z|=\epsilon} dz z^{n+1} T(z) \mathcal{O}(0,0), \quad (5.4)$$

with ϵ arbitrarily small. For this reason, $(L_n \cdot \mathcal{O})$ is indeed a local operator inserted at the origin.

Actually, because of the OPE principle:

$$\mathcal{O}_1(z, \bar{z}) \mathcal{O}_2(0,0) = \sum_j C_{12}^j z^{-h_1-h_2+h_j} \bar{z}^{-\bar{h}_1-\bar{h}_2+\bar{h}_j} \mathcal{O}_j(0,0), \quad (5.5)$$

the local operators themselves may be considered as acting linearly on \mathcal{A} .

Let us compute explicitly the action of L_0 and \bar{L}_0 on a scaling operator. Recall the conformal Ward identity associated to an infinitesimal conformal mapping $z \mapsto z + \epsilon(z)$:

$$\delta_\epsilon \mathcal{O}(z, \bar{z}) \simeq \frac{1}{2\pi i} \oint_C \epsilon(\xi) T(\xi) \mathcal{O}(z, \bar{z}) d\xi + \frac{1}{2\pi i} \oint_C \bar{\epsilon}(\bar{\xi}) \bar{T}(\bar{\xi}) \mathcal{O}(z, \bar{z}) d\bar{\xi}. \quad (5.6)$$

In the case of a dilatation $z \mapsto \lambda z$ (with $\lambda = 1 + \beta$, and β infinitesimal), from the RG argument, the scaling operator transforms as

$$\mathcal{O}(z, \bar{z}) \mapsto \lambda^x \mathcal{O}(\lambda z, \lambda \bar{z}) = \mathcal{O}(z, \bar{z}) + \delta \mathcal{O}(z, \bar{z}), \quad \delta \mathcal{O}(z, \bar{z}) = \beta(x + z\partial + \bar{z}\bar{\partial}) \mathcal{O}(z, \bar{z}). \quad (5.7)$$

Setting $z = 0$, we get $x \mathcal{O}(0,0) = ((L_0 + \bar{L}_0) \cdot \mathcal{O})(0,0)$. For rotations, we get: $s \mathcal{O}(0,0) = ((L_0 - \bar{L}_0) \cdot \mathcal{O})(0,0)$, where s is the conformal spin of \mathcal{O} . As a result, we get, for any scaling operator \mathcal{O} with conformal dimensions h, \bar{h} :

$$(L_0 \cdot \mathcal{O})(0,0) = h \mathcal{O}(0,0), \quad (\bar{L}_0 \cdot \mathcal{O})(0,0) = \bar{h} \mathcal{O}(0,0). \quad (5.8)$$

Exercise: using an infinitesimal rotation, show that

$$(L_{-1} \cdot \mathcal{O})(0,0) = \partial \mathcal{O}(0,0), \quad (\bar{L}_{-1} \cdot \mathcal{O})(0,0) = \bar{\partial} \mathcal{O}(0,0). \quad (5.9)$$

5.2 Commutation relations

When composing the L_n 's, one should choose the integration contours in such a way that the OPEs are performed in the correct order. For instance:

$$(L_m \cdot L_n \cdot \mathcal{O})(0,0) = \frac{1}{(2i\pi)^2} \oint_{|z_1|=\epsilon_1} dz_1 \oint_{|z_2|=\epsilon_2} dz_2 z_1^{m+1} z_2^{n+1} T(z_1) T(z_2) \mathcal{O}(0,0), \quad (5.10)$$

with $\epsilon_1 > \epsilon_2$. Similarly, for the composition of L_n with a primary operator $\phi(z, \bar{z})$:

$$(L_n \cdot \phi(z, \bar{z}) \cdot \mathcal{O})(0,0) = \frac{1}{2i\pi} \oint_{|w|=R} dw w^{n+1} T(w) \phi(z, \bar{z}) \mathcal{O}(0,0), \quad (5.11)$$

with $R > |z|$.

We recall the OPE of the stress-energy tensor with a primary operator:

$$T(\xi)\phi(z, \bar{z}) = \frac{h\phi(z, \bar{z})}{(\xi - z)^2} + \frac{\partial\phi(z, \bar{z})}{\xi - z} + \text{reg.} \quad (5.12)$$

Viewing L_n and $\phi(z, \bar{z})$ as acting on \mathcal{A} , we now consider the commutator:

$$[L_n, \phi(z, \bar{z})] = L_n\phi(z, \bar{z}) - \phi(z, \bar{z})L_n. \quad (5.13)$$

In the first term, the integration contour for L_n should enclose the origin *and* the point z , whereas in the second term, it should only enclose the origin. As a result, we obtain:

$$[L_n, \phi(z, \bar{z})] = \frac{1}{2i\pi} \oint_{C_z} d\xi \xi^{n+1} T(\xi)\phi(z, \bar{z}), \quad (5.14)$$

where C_z is a small contour enclosing only the point z , anti-clockwise oriented. Inserting the OPE into this integral, we get:

$$[L_n, \phi(z, \bar{z})] = h(n+1)z^n \phi(z, \bar{z}) + z^{n+1}\partial\phi(z, \bar{z}). \quad (5.15)$$

Similarly,

$$[\bar{L}_n, \phi(z, \bar{z})] = \bar{h}(n+1)\bar{z}^n \phi(z, \bar{z}) + \bar{z}^{n+1}\bar{\partial}\phi(z, \bar{z}). \quad (5.16)$$

Through a similar computation, from the OPE of T with itself, we can derive the commutation relations of the L_n 's (and the \bar{L}_n 's):

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \quad (5.17)$$

$$[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \quad (5.18)$$

$$[L_m, \bar{L}_n] = 0. \quad (5.19)$$

The commutation rules (5.17) define the Virasoro algebra of central charge c . Note that the modes L_n and \bar{L}_n generate two independent Virasoro algebras.

5.3 Descendant operators

The action of L_n on a scaling operator \mathcal{O} located at the origin is given by:

$$(L_n\mathcal{O})(0, 0) = \frac{1}{2i\pi} \oint_{C_0} d\zeta \zeta^{n+1} T(\zeta)\mathcal{O}(0, 0). \quad (5.20)$$

If we perform a translation by z , we may define

$$(L_n\mathcal{O})(z, \bar{z}) := \frac{1}{2i\pi} \oint_{C_z} d\zeta (\zeta - z)^{n+1} T(\zeta)\mathcal{O}(z, \bar{z}). \quad (5.21)$$

This defines a local scaling operator at any position. The operator $L_n\mathcal{O}$ (and, after more iterations, any operator of the form $L_{n_1} \dots L_{n_k}\mathcal{O}$) is called a *descendant* of \mathcal{O} .

Suppose \mathcal{O} has conformal dimensions h, \bar{h} . If we perform a dilatation $z \mapsto z/\lambda$, we see that this operator transforms as

$$\langle (L_n\mathcal{O})(z, \bar{z}) \dots \rangle = \frac{1}{2i\pi} \oint_{C_z} d\zeta (\zeta - z)^{n+1} \lambda^{-2-h-\bar{h}} \langle T(\zeta/\lambda)\mathcal{O}(z/\lambda, \bar{z}/\lambda) \dots \rangle \quad (5.22)$$

$$= \lambda^{-h-\bar{h}+n} \frac{1}{2i\pi} \oint_{C_z} \frac{d\zeta}{\lambda} \left(\frac{\zeta - z}{\lambda} \right)^{n+1} \langle T(\zeta/\lambda)\mathcal{O}(z/\lambda, \bar{z}/\lambda) \dots \rangle \quad (5.23)$$

$$= \lambda^{-h-\bar{h}+n} \langle (L_n\mathcal{O})(z/\lambda, \bar{z}/\lambda) \dots \rangle, \quad (5.24)$$

and hence it has scaling dimension $h + \bar{h} - n$. Reasoning similarly with rotations, we find the spin $h - \bar{h} - n$. Hence the descendant $L_n \mathcal{O}$ has conformal dimensions:

$$h - n, \quad \bar{h}. \quad (5.25)$$

Let $\phi(z, \bar{z})$ be a primary operator with conformal dimensions h, \bar{h} . Recall the OPE:

$$T(\xi)\phi(z, \bar{z}) = \frac{h\phi(z, \bar{z})}{(\xi - z)^2} + \frac{\partial\phi(z, \bar{z})}{\xi - z} + \text{reg.} \quad (5.26)$$

Comparing with

$$T(\xi)\phi(z, \bar{z}) = \sum_{m \in \mathbb{Z}} (\xi - z)^{-m-2} (L_m \cdot \phi)(z, \bar{z}), \quad (5.27)$$

we get:

$$(L_{m>0}\phi)(z, \bar{z}) = 0, \quad (L_0\phi)(z, \bar{z}) = h\phi(z, \bar{z}), \quad (L_{-1}\phi)(z, \bar{z}) = \partial\phi(z, \bar{z}), \quad (5.28)$$

and similarly:

$$(\bar{L}_{m>0}\phi)(z, \bar{z}) = 0, \quad (\bar{L}_0\phi)(z, \bar{z}) = \bar{h}\phi(z, \bar{z}), \quad (\bar{L}_{-1}\phi)(z, \bar{z}) = \bar{\partial}\phi(z, \bar{z}). \quad (5.29)$$

Hence, the space of descendants of a primary operator ϕ is generated by

$$\{L_{-m_1} L_{-m_2} \dots L_{-m_k} \bar{L}_{-\bar{m}_1} \bar{L}_{-\bar{m}_2} \dots \bar{L}_{-\bar{m}_{\bar{k}}} \phi\}_{\substack{m_1 \geq m_2 \geq \dots \geq m_k \geq 1 \\ \bar{m}_1 \geq \bar{m}_2 \geq \dots \geq \bar{m}_{\bar{k}} \geq 1}}. \quad (5.30)$$

Clearly, this space is itself a representation of the conformal algebra $\text{Vir} \otimes \overline{\text{Vir}}$. In the following, we shall “forget” the physical setting of conformal invariance for a while, and present some basics of the representation theory of the Virasoro algebra. Since the actions of the L_n 's and \bar{L}_n 's are totally decoupled, it is convenient to consider the representations of a single Virasoro algebra Vir , rather than the full conformal algebra $\text{Vir} \otimes \overline{\text{Vir}}$.

5.4 Representations of the Virasoro algebra

5.4.1 Verma modules

Consider a *highest weight* state $|h\rangle$ such that

$$L_0|h\rangle = h|h\rangle, \quad L_{n>0}|h\rangle = 0. \quad (5.31)$$

The representation generated by all the descendants of $|h\rangle$ is called the Verma module \mathcal{V}_h . It looks like:

$$\begin{aligned} (L_0 = h) & |h\rangle \\ (L_0 = h+1) & L_{-1}|h\rangle \\ (L_0 = h+2) & L_{-2}|h\rangle \quad L_{-1}^2|h\rangle \\ (L_0 = h+3) & L_{-3}|h\rangle \quad L_{-2}L_{-1}|h\rangle \quad L_{-1}^3|h\rangle \\ (L_0 = h+4) & L_{-4}|h\rangle \quad L_{-3}L_{-1}|h\rangle \quad L_{-2}^2|h\rangle \quad L_{-2}L_{-1}^2|h\rangle \quad L_{-1}^4|h\rangle \\ & \dots \end{aligned} \quad (5.32)$$

5.4.2 Virasoro scalar product

In the Verma module, the “scalar product” is defined by

$$\langle h|h \rangle := 1, \quad L_n^\dagger := L_{-n}. \quad (5.33)$$

The scalar product only couples the states at the same level: this is because we are dealing with the eigenstates of L_0 , which is self-adjoint. Using the Virasoro commutation relations, we get at level 1:

$$\langle h|L_1L_{-1}|h \rangle = 2h, \quad (5.34)$$

and at level 2:

$$\langle h|L_{-1}^2L_1^2|h \rangle = 4h(2h+1), \quad \langle h|L_2L_{-1}^2|h \rangle = 6h, \quad (5.35)$$

$$\langle h|L_{-1}^2L_2|h \rangle = 6h, \quad \langle h|L_2L_{-2}|h \rangle = 4h + \frac{c}{2}. \quad (5.36)$$

Clearly, we see that the basis of level 2 $\{|h^{(1,1)}\rangle = L_{-1}^2|h\rangle, |h^{(2)}\rangle = L_{-2}|h\rangle\}$ is not orthogonal. Let us compute explicitly its dual basis $\{\langle \widehat{h}^{(2)}|, \langle \widehat{h}^{(1,1)}|\}$. We write the dual vector of $L_{-1}^2|h\rangle$ as

$$\langle \widehat{h}^{(1,1)}| = u \langle h|L_2 + v \langle h|L_1^2.$$

The conditions $\langle \widehat{h}^{(1,1)}|h^{(1,1)}\rangle = 1, \langle \widehat{h}^{(1,1)}|h^{(2)}\rangle = 0$ yield the linear system

$$\begin{aligned} 6h u + 4h(2h+1) v &= 1 \\ (4h+c/2) u + 6h v &= 0, \end{aligned} \quad (5.37)$$

and the solution is

$$\langle \widehat{h}^{(1,1)}| = \frac{-12h \langle h|L_2 + (8h+c) \langle h|L_1^2}{4h[16h^2 + (2c-10)h + c]}. \quad (5.38)$$

Similarly, we get

$$\langle \widehat{h}^{(2)}| = \frac{(4h+2) \langle h|L_2 - 3\langle h|L_1^2}{16h^2 + (2c-10)h + c}. \quad (5.39)$$

More generally, if we denote

$$|h^{(m_1, \dots, m_k)}\rangle := L_{-m_1} \dots L_{-m_k} |h\rangle, \quad (5.40)$$

with $m_1 \geq m_2 \geq \dots \geq m_k \geq 1$, and $m_1 + \dots + m_k = M$ then the dual basis elements are of the form

$$\langle \widehat{h}^{(m_1, \dots, m_k)}| = \sum_{\substack{m'_1, \dots, m'_p=1 \\ m'_1 + \dots + m'_p=M}}^M a_{m'_1, \dots, m'_p}(h) \langle h|L_{m'_1} \dots L_{m'_p}, \quad (5.41)$$

where $a_{m'_1, \dots, m'_p}(h)$ is a rational function of h .

5.4.3 Reducible Verma modules

In the generic case, a Verma module \mathcal{V}_h is irreducible. However, for some special values of (h, c) , \mathcal{V}_h may become *reducible*, *i.e.* it may admit a non-trivial subspace \mathcal{W} which is stable under Vir . Let us quickly examine the consequences of reducibility, and then find for which values of (h, c) this may happen.

Suppose \mathcal{V}_h is reducible: in this case, the non-trivial stable subspace \mathcal{W} must contain a vector $|\chi\rangle$ with the minimal eigenvalue $h + N$ for L_0 , where the integer $N > 0$:

$$|\chi\rangle = (\#L_{-N} + \dots + \#L_{-1}^N)|h\rangle, \quad L_0|\chi\rangle = (h + N)|\chi\rangle. \quad (5.42)$$

For any $m > 0$, we have $L_0 \cdot L_m |\chi\rangle = (h + N - m)L_m |\chi\rangle$, and hence $L_m |\chi\rangle$ must vanish (otherwise $h + N$ would not be a minimal eigenvalue for L_0 in \mathcal{W}). We say that $|h\rangle$ is degenerate at level N . We then have

$$\forall m > 0, \quad L_m |\chi\rangle = 0. \quad (5.43)$$

This means that the descendant state $|\chi\rangle$ is itself a highest-weight state. The action of the Virasoro algebra on $|\chi\rangle$ thus generates a submodule \mathcal{W}_χ , isomorphic to \mathcal{V}_{h+N} . We may define the quotient space $\mathcal{V}'_h = \mathcal{V}_h / \mathcal{W}_\chi$. If \mathcal{V}'_h is still reducible, then we need to identify a new highest-weight vector $|\chi'\rangle$ at level $N' \geq N$, and repeat the quotient procedure, to obtain a space \mathcal{V}''_h , and so on. Once the subspaces generated by highest-weight vectors not proportional to $|h\rangle$ are quotiented out, one gets an irreducible module.

Consider the scalar product between $|\chi\rangle$ and any state in $|v\rangle \in \mathcal{V}_h$ of the form: $|v\rangle = L_{-m_k} \dots L_{-m_1} |h\rangle$ with $m_k \geq \dots \geq m_1 \geq 1$. If $m_1 + \dots + m_k \neq N$, then trivially $\langle v|\chi\rangle = 0$. If $m_1 + \dots + m_k = N$, then $\langle v|\chi\rangle = \langle h|L_{m_1} \dots L_{m_k} |\chi\rangle = 0$. Hence, we have:

$$\forall |v\rangle \in \mathcal{V}_h, \quad \langle v|\chi\rangle = 0. \quad (5.44)$$

We say that $|\chi\rangle$ is a *null vector*. In particular, $|\chi\rangle$ has norm zero: $\langle \chi|\chi\rangle = 0$.

Exercise : Prove that any descendant of $|\chi\rangle$ is also orthogonal to \mathcal{V}_h :

$$\forall |v\rangle \in \mathcal{V}_h, \quad \forall n_1, \dots, n_\ell \geq 1, \quad \langle v|L_{-n_1} \dots L_{-n_\ell} |\chi\rangle = 0. \quad (5.45)$$

In particular, $L_{-1}|\chi\rangle$ is also a null vector. However, by the simple computation:

$$L_1 \cdot L_{-1} |\chi\rangle = (L_{-1}L_1 + 2L_0) |\chi\rangle = 2(h + N) |\chi\rangle \neq 0 \quad \text{if } h \neq -N, \quad (5.46)$$

we see that, in general, a null vector is *not* a highest-weight vector.

Let us consider a state $|h\rangle$ degenerate at level $N = 1$. This means that $|\chi\rangle = L_{-1}|h\rangle$ is a primary: $L_{n>0}|\chi\rangle = 0$. For $n > 1$, we always have $L_n L_{-1}|h\rangle = L_{-1}L_n|h\rangle + (n+1)L_{n-1}|h\rangle = 0$. Hence, the only non-trivial condition is

$$L_1 L_{-1} |h\rangle = 0 \quad \Leftrightarrow \quad (L_{-1}L_1 + 2L_0) |h\rangle = 0 \quad \Leftrightarrow \quad 2h|h\rangle = 0. \quad (5.47)$$

The only operator which is degenerate at level $N = 1$ is the identity, with $h = 0$.

Let us do the same exercice for level $N = 2$. Writing $|\chi\rangle = (uL_{-2} + vL_{-1}^2)|h\rangle$, and imposing $L_1|\chi\rangle = L_2|\chi\rangle = 0$, we get the linear system

$$\begin{aligned} 3u + (4h + 2)v &= 0 \\ \left(4h + \frac{c}{2}\right)u + 6hv &= 0. \end{aligned} \quad (5.48)$$

The system admits a non-trivial solution if and only if its determinant is zero, which yields the relation:

$$16h^2 + (2c - 10)h + c = 0 \quad \Leftrightarrow \quad h = \frac{(5 - c) \pm \sqrt{(1 - c)(25 - c)}}{16}. \quad (5.49)$$

For generic values of the central charge, there are two possible values for the dimension of an operator degenerate at level $N = 2$.

For higher levels N , it becomes non-trivial to enumerate the degenerate dimensions. Let us state the result, which is known as the Kac parametrisation of degenerate states. We write the central charge

$$c = 1 - 6(b^{-1} - b)^2, \quad (5.50)$$

with some nonzero parameter b . Then the degenerate states at level N are of the form $|h_{r,s}\rangle$ with $r, s \in \{1, 2, 3, \dots\}$ and $N = rs$, and their dimensions are given by

$$h_{rs} = \frac{(rb^{-1} - sb)^2 - (b^{-1} - b)^2}{4}. \quad (5.51)$$

In particular, we have $h_{12} = (3b^2 - 2)/4$, and the corresponding null vector is

$$|\chi_{12}\rangle = (L_{-2} - b^{-2}L_{-1}^2)|h_{12}\rangle. \quad (5.52)$$

Lecture 6

Radial quantisation

In the previous lecture, we have worked in the Lagrangian formalism: implicitly or explicitly, we have assumed that the theory is governed by some action $\mathcal{A}[\Phi]$ defined on the local degrees of freedom $\{\Phi(r)\}$, so that the correlation functions are given by:

$$\langle \dots \rangle = \frac{1}{Z} \int [D\Phi] e^{-\mathcal{A}[\Phi]}(\dots), \quad Z = \int [D\Phi] e^{-\mathcal{A}[\Phi]}, \quad (6.1)$$

where the dots denote any functional of $\Phi(r)$. In the present lecture, we will show how to construct an equivalent viewpoint based on a Hilbert space. It is instructive to begin with the lattice analogs of these two viewpoints.

6.1 Critical model on the cylinder

6.1.1 Conserved charges from the Ward identities

We start from the conformal Ward identity for any local operator $\mathcal{O}(u, \bar{u})$, under an infinitesimal conformal transformation $\epsilon(z)$:

$$\delta_\epsilon \mathcal{O}(u, \bar{u}) = \frac{1}{2i\pi} \oint \epsilon(w) T_{\text{cyl}}(w) \mathcal{O}(u, \bar{u}) dw + \frac{1}{2i\pi} \oint \bar{\epsilon}(\bar{w}) \bar{T}_{\text{cyl}}(\bar{w}) \mathcal{O}(u, \bar{u}) d\bar{w}, \quad (6.2)$$

where $T_{\text{cyl}}(w)$ is the stress-energy tensor on the cylinder, and the contour integrals are closed circuits around the operator $\mathcal{O}(u, \bar{u})$. These integrals can be written as the difference of two integrals over the circumference of the cylinder:

$$\frac{1}{2i\pi} \oint \epsilon(w) T_{\text{cyl}}(w) \mathcal{O}(u, \bar{u}) dw = \frac{1}{2i\pi} \left(\oint_{C_+} - \oint_{C_-} \right) \epsilon(w) T_{\text{cyl}}(w) \mathcal{O}(u, \bar{u}) dw, \quad (6.3)$$

where C_{u+} (resp. C_{u-}) is a circumference placed on the right (resp. left) of $\mathcal{O}(u, \bar{u})$. Using the Heisenberg formalism, we get:

$$\delta_\epsilon \mathcal{O}(u, \bar{u}) = [Q_\epsilon, \mathcal{O}(u, \bar{u})] \quad (6.4)$$

where

$$Q_\epsilon = \frac{1}{2i\pi} \oint \epsilon(w) T_{\text{cyl}}(w) dw + \frac{1}{2i\pi} \oint \bar{\epsilon}(\bar{w}) \bar{T}_{\text{cyl}}(\bar{w}) d\bar{w}. \quad (6.5)$$

Through the mapping $z = \exp(2\pi w/L)$, the stress-energy tensor transforms as

$$T_{\text{cyl}}(w) = (2\pi/L)^2 [z^2 T_{\text{pl}}(z) - c/24]. \quad (6.6)$$

For a time translation along the axis of the cylinder (ϵ =real constant), we get the expression for the Hamiltonian:

$$\mathcal{H} = \frac{2\pi}{L} \left(L_0 + \bar{L}_0 - \frac{c}{12} \right). \quad (6.7)$$

Similarly, a space (cyclic) translation is generated by

$$\mathcal{P} = \frac{2\pi}{L} (L_0 - \bar{L}_0). \quad (6.8)$$

6.1.2 The transfer-matrix formalism

Let us consider a lattice model, with local interactions – say, a spin model with nearest-neighbour interactions on the square lattice. The analog of $\{\Phi(r)\}$ is then a spin configuration $\{S_i\}$ on the lattice, and the action $\mathcal{A}[\Phi]$ corresponds to the energy of the spin configuration:

$$E[S] = \sum_{\langle ij \rangle} \varepsilon(S_i, S_j), \quad (6.9)$$

where $\langle ij \rangle$ denotes a pair of adjacent sites. The analog of (6.1) is

$$\langle \dots \rangle_{\text{lattice}} = \frac{1}{Z} \sum_{\{S_i\}} e^{-E[S]}(\dots), \quad Z = \sum_{\{S_i\}} e^{-E[\sigma]}. \quad (6.10)$$

Let us impose periodic boundary conditions (PBC) in the vertical direction, and set some specific boundary conditions on the left and right boundaries. We thus consider a square lattice of $M \times N$ sites, on a cylinder. The simple idea of the transfer-matrix formalism is to decompose any spin configuration $\{S_i\}$ on the lattice into M column configurations $[S] = (\alpha^{(1)}, \dots, \alpha^{(M)})$, and to introduce a matrix which encodes the Boltzmann weights inside a single column. Thus, for any pair of column spin configurations $\alpha = (\alpha_1, \dots, \alpha_L)$ and $\alpha' = (\alpha'_1, \dots, \alpha'_L)$, we define the matrix element

$$(\tau_N)_{\alpha\alpha'} = \prod_{n=1}^N \exp \left[-\frac{1}{2} \varepsilon(\alpha_n, \alpha_{n+1}) - \frac{1}{2} \varepsilon(\alpha'_n, \alpha'_{n+1}) - \varepsilon(\alpha_n, \alpha'_n) \right], \quad (6.11)$$

where $\alpha_{N+1} := \alpha_1$ and $\alpha'_{N+1} := \alpha'_1$. As a linear operator, the matrix τ_N acts on *configuration vectors*. The linear space for these vectors is called the *quantum space* V_N . If each individual spin S_j can take q values, then $\dim V_N = q^N$. The pure configuration vectors $|\alpha\rangle$ form a basis of V_N .

The sum over spin configurations can be performed by iterating the matrix product, while the left and right boundary conditions are encoded into some vectors $|b_\ell\rangle$ and $|b_r\rangle$: fixed boundary conditions are given by $|b\rangle = |\alpha^{(0)}\rangle$ for some column configuration $\alpha^{(0)}$, free boundary conditions correspond to $|b\rangle = \sum_\alpha |\alpha\rangle$, etc. For instance, the partition function is given by

$$Z_{MN} = \langle b_r | (\tau_N)^M | b_\ell \rangle. \quad (6.12)$$

Similarly, correlation functions are obtained by inserting local operators S_n , with matrix elements of the form:

$$(S_n)_{\alpha\alpha'} = \alpha_n \prod_{k=1}^N \delta_{\alpha_k \alpha'_k}. \quad (6.13)$$

For example, if $m' \geq m$, the two-point function reads (we simply write τ for the transfer matrix):

$$\langle S(m', n') S(m, n) \rangle = \frac{1}{Z_{MN}} \langle b_r | \tau^{M/2-m'} S_{n'} \tau^{m'-m} S_n \tau^{m+M/2} | b_\ell \rangle. \quad (6.14)$$

As seen from this correspondence, the transfer matrix can be considered as the imaginary-time evolution operator of a 1d quantum model whose wavefunctions live in V_N . We thus consider the horizontal direction as imaginary time t , and the vertical direction as space x .

6.1.3 The Heisenberg picture

At this point it is natural to introduce the Heisenberg picture for lattice operators:

$$\widehat{S}(0, n) = S_n, \quad \widehat{S}(m, n) = \tau^{-m} S_n \tau^m, \quad (6.15)$$

so that, for $m' \geq m$, we have

$$\langle S(m', n') S(m, n) \rangle = \frac{1}{Z} \langle b_r | \tau^{M/2} \widehat{S}(m', n') \widehat{S}(m, n) \tau^{M/2} | b_\ell \rangle. \quad (6.16)$$

More generally, for a k -point function, if $M/2 \geq m_1 \geq m_2 \geq \dots \geq m_k \geq -M/2$, we have

$$\langle S(m_1, n_1) \dots S(m_k, n_k) \rangle = \frac{\langle b_r | \tau^{M/2} \widehat{S}(m_1, n_1) \dots \widehat{S}(m_k, n_k) \tau^{M/2} | b_\ell \rangle}{\langle b_r | \tau^M | b_\ell \rangle}. \quad (6.17)$$

The same construction can be applied to any correlation function of scaling operators Φ_j on the cylinder:

$$\langle \Phi_1(m_1, n_1) \dots \Phi_k(m_k, n_k) \rangle = \frac{\langle b_r | \tau^{M/2} \widehat{\Phi}_1(m_1, n_1) \dots \widehat{\Phi}_k(m_k, n_k) \tau^{M/2} | b_\ell \rangle}{\langle b_r | \tau^M | b_\ell \rangle}, \quad (6.18)$$

$M/2 \geq m_1 \geq m_2 \geq \dots \geq m_k \geq -M/2.$

6.1.4 Radial ordering

From the above discussion, we see that, when looking at a CFT on the complex plane, it is very natural to consider the radial coordinate as (imaginary) time, and the angular coordinate as space. In this picture, states in the Hilbert space represent (linear combinations) of configurations of a 1d quantum system on a circle centered around zero. Time evolution takes the system to a circle with higher radius. The origin of times is at $z = 0$, and the end of times is at $z = \infty$. In the Heisenberg formalism, one needs to impose *radial ordering* to ensure the correspondence of correlation functions:

$$\langle \phi_1(z_1, \bar{z}_1) \dots \phi_1(z_n, \bar{z}_n) \rangle_{\text{pl}} = \langle 0 | \mathcal{R} [\phi_{H,1}(z_1, \bar{z}_1) \dots \phi_{H,n}(z_n, \bar{z}_n)] | 0 \rangle, \quad (6.19)$$

where

$$\mathcal{R} [\phi_{H,1}(z_1, \bar{z}_1) \dots \phi_{H,n}(z_n, \bar{z}_n)] := \phi_{H,k_1}(z_{k_1}, \bar{z}_{k_1}) \dots \phi_{H,k_n}(z_{k_n}, \bar{z}_{k_n}), \quad (6.20)$$

with $|z_{k_1}| > |z_{k_2}| > \dots > |z_{k_n}|$.

6.1.5 Infinite cylinder limit

Consider the limit of an infinitely long cylinder: $M \rightarrow \infty$, while N is kept fixed. To prepare for the continuum limit, we consider the renormalised scaling operators

$$\phi_j(w = am + ian, \bar{w} = am - ian) = \text{const} \times a^{-x_j} \Phi_j(m, n), \quad (6.21)$$

where a is the lattice step, and $L = aN$ is the physical circumference of the cylinder.

Since the transfer matrix τ_N has positive entries, from the Perron-Frobenius theorem its dominant eigenvalue λ_0 is positive and non-degenerate. The corresponding eigenvector is called the *vacuum state*, and is denoted by $|0\rangle_N$. As $M \rightarrow \infty$, we have

$$\tau^{M/2}|b_\ell\rangle \sim \beta_0 \lambda_0^{M/2}|0\rangle, \quad \beta_\ell := \langle 0|b_\ell\rangle, \quad (6.22)$$

$$\langle b_r|\tau^{M/2} \sim \beta_r \langle 0|\lambda_0^{M/2}, \quad \beta_r := \langle b_r|0\rangle. \quad (6.23)$$

We assume that the boundary conditions are in the correct symmetry sector, so that the overlaps β_ℓ and β_r are non-zero. Hence the cylinder correlation functions read:

$$\boxed{\begin{aligned} \langle \phi_1(w_1, \bar{w}_1) \dots \phi_k(w_k, \bar{w}_k) \rangle_{\text{cyl}} &\xrightarrow[M \rightarrow \infty]{} \langle 0|\widehat{\phi}_1(w_1, \bar{w}_1) \dots \widehat{\phi}_k(w_k, \bar{w}_k)|0\rangle, \\ \text{Re}(w_1) \geq \text{Re}(w_2) \geq \dots \geq \text{Re}(w_k). \end{aligned}} \quad (6.24)$$

This relation provides a simple dictionary between the “Lagrangian” language (sums over 2d lattice configurations) and the “Hamiltonian” language (expectation values of operators in a 1d quantum system). The Heisenberg operators should be ordered in imaginary time, to ensure the correspondence with correlation functions in the statistical model.

6.2 Constructing the quantum states of a CFT

6.2.1 The vacuum state

If we use the conformal mapping:

$$w \mapsto z = \exp(2\pi w/L), \quad (6.25)$$

The $Ma \times L$ cylinder gets mapped to the annulus $\epsilon < |z| < R$, where $\epsilon = \exp(-\pi aM/L)$, and $R = \exp(+\pi aM/L)$. In this picture, the initial state of the system, on the inner boundary $|z| = \epsilon$, is given by $|b_\ell\rangle$, and every iteration of the transfer matrix inserts a layer of width $\delta|z| \simeq 2\pi a|z|/L$. Now when we let $\epsilon \rightarrow 0$, and no operator is inserted in the region $\epsilon < |z| < r$, then it takes $m = \frac{L}{2\pi} \log(r/\epsilon)$ iterations to get from ϵ to r . The state of the system on $|z| = r$ is given by $\tau^m|b_\ell\rangle \propto |0\rangle$. In this situation, any correlation of the form

$$\langle \dots T(z) \rangle \quad (6.26)$$

is regular at $z = 0$, since no operator is inserted at the origin: this means that the state $T(z)|0\rangle$ should be regular at $z = 0$. In terms of Virasoro modes, we get

$$\boxed{\begin{aligned} L_n|0\rangle &= \frac{1}{2i\pi} \oint dz z^{n+1} T(z)|0\rangle = 0 \\ \bar{L}_n|0\rangle &= \frac{1}{2i\pi} \oint d\bar{z} \bar{z}^{n+1} \bar{T}(\bar{z})|0\rangle = 0 \quad \text{for } n \geq -1. \end{aligned}} \quad (6.27)$$

6.2.2 From operators to states

The vacuum state will typically sit in the most symmetric sector (otherwise the system would be in a broken-symmetry phase). For instance, if the Boltzmann weights are symmetric under spin reversal R , then we will get

$$R|0\rangle = +|0\rangle, \quad R\widehat{S}(m, n)R^{-1} = -\widehat{S}(m, n). \quad (6.28)$$

In particular, we will have

$$\langle 0|\widehat{S}(m, n)|0\rangle = 0. \quad (6.29)$$

Let us denote the subdominant eigenvalues of τ_N by $\lambda_0 > |\lambda_1| \geq |\lambda_2| \geq \dots$ and the corresponding eigenvectors by $\{|u_0\rangle = |0\rangle, |u_1\rangle, |u_2\rangle, \dots\}$. Among these $|u_j\rangle$'s, we denote by $|\sigma\rangle$ the dominant eigenvector of τ in the “spin” sector, *i.e.* such that $R|\sigma\rangle = -|\sigma\rangle$. Then typically, the overlap $\langle \sigma|\widehat{S}(m, n)|0\rangle$ will be non-zero.

The eigenstate $|\sigma\rangle$ can simply be recovered on the infinite cylinder, by sending the lattice operator \widehat{S} to $-\infty$, while keeping the boundary conditions associated to the vacuum state:

$$\widehat{S}(m, n)|0\rangle = \tau^{-m}\widehat{S}(0, n)\tau^m|0\rangle \quad (6.30)$$

$$= \sum_j |u_j\rangle \langle u_j| \tau^{-m} \widehat{S}(0, n) \tau^m |0\rangle \quad (6.31)$$

$$= \sum_j (\lambda_j/\lambda_0)^{-m} \langle u_j| \widehat{S}(0, n) |0\rangle \times |u_j\rangle \quad (6.32)$$

$$\underset{m \rightarrow -\infty}{\sim} (\lambda_\sigma/\lambda_0)^{|m|} \langle \sigma| \widehat{S}(0, n) |0\rangle \times |\sigma\rangle. \quad (6.33)$$

Hence, in the limit $m \rightarrow -\infty$, we get $\widehat{S}(m, n)|0\rangle \propto |\sigma\rangle$. This is easily extended to obtain the eigenstate associated to a (renormalised) scaling operator \mathcal{O} :

$$|\mathcal{O}\rangle = \lim_{w \rightarrow -\infty} \frac{\widehat{\mathcal{O}}(w, \bar{w})|0\rangle}{\|\widehat{\mathcal{O}}(w, \bar{w})|0\rangle\|}, \quad \tau|\mathcal{O}\rangle = \lambda|\mathcal{O}\rangle. \quad (6.34)$$

This relation is called the state-operator correspondence.

6.2.3 From states to operators

Given a generic eigenstate $|u\rangle$ of the transfer matrix τ , with an eigenvalue of the form

$$\lambda = \lambda_0 \times \exp\left(-\frac{2\pi a}{L}x\right), \quad (6.35)$$

as expected from the the form of the Hamiltonian $\mathcal{H} = \frac{2\pi}{L}(L_0 + \bar{L}_0 - c/12)$.

The following procedure associates a scaling operator \mathcal{O}_u . We simply need to define the correlation function:

$$\langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_k(z_k, \bar{z}_k) \mathcal{O}_u(z, \bar{z}) \rangle_{\mathbb{C}} := \lim_{\epsilon \rightarrow 0} \left[e^{-x} \langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_k(z_k, \bar{z}_k) \rangle_{\mathcal{D}_u(\epsilon)} \right], \quad (6.36)$$

where $\mathcal{D}_u(\epsilon)$ is the complex plane with a disc of radius ϵ removed around z , and with boundary conditions given by the state $|u\rangle$. Let us consider the case when \mathcal{O}_u is inserted

at the origin. In the cylinder geometry, this corresponds to

$$(\lambda/\lambda_0)^{-M/2} \frac{\langle 0 | \tau^{M/2} \widehat{\mathcal{O}}_1(w_1, \bar{w}_1) \dots \widehat{\mathcal{O}}_k(w_k, \bar{w}_k) \tau^{M/2} | u \rangle}{\langle 0 | \tau^M | 0 \rangle} = \langle 0 | \widehat{\mathcal{O}}_1(w_1, \bar{w}_1) \dots \widehat{\mathcal{O}}_k(w_k, \bar{w}_k) | u \rangle, \quad (6.37)$$

with the identification $\epsilon = \exp(-\pi a M/L)$ and $M \gg L$.

On one hand, by definition we have

$$\langle \dots \mathcal{O}_u(0, 0) \rangle = \langle 0 | \dots \mathcal{O}_u(0, 0) | 0 \rangle. \quad (6.38)$$

On the other hand, we have just found that

$$\langle \dots \mathcal{O}_u(0, 0) \rangle = \langle 0 | \dots | u \rangle. \quad (6.39)$$

Hence, through the above construction, we have built an operator such that

$$\boxed{\mathcal{O}_u(0, 0)|0\rangle = |u\rangle.} \quad (6.40)$$

6.2.4 Primary states

Let us now specialise to the case of primary operators. If we set $w = am$ with integer $m \rightarrow -\infty$, the square norm $\|\widehat{\phi}(w, \bar{w})|0\rangle\|^2$ can be written:

$$\|\widehat{\phi}(w, \bar{w})|0\rangle\|^2 = \langle 0 | \tau^m \widehat{\phi}(0, 0) \tau^{-m} \cdot \tau^{-m} \widehat{\phi}(0, 0) \tau^m | 0 \rangle \quad (6.41)$$

$$= \langle 0 | \widehat{\phi}(-w, -\bar{w}) \widehat{\phi}(w, \bar{w}) | 0 \rangle \quad (6.42)$$

$$= \langle \phi(-w, -\bar{w}) \phi(w, \bar{w}) \rangle_{\text{cyl}}. \quad (6.43)$$

Using the mapping from the cylinder to the plane, we have, for a scalar primary operator:

$$\langle \phi(w_1, \bar{w}_1) \phi(w_2, \bar{w}_2) \rangle_{\text{cyl}} = \left| \frac{L}{2\pi} \sinh \frac{\pi(w_1 - w_2)}{L} \right|^{-2x}, \quad (6.44)$$

and hence we get:

$$\|\widehat{\phi}(w, \bar{w})|0\rangle\| \underset{w \rightarrow -\infty}{\sim} \left| \frac{2\pi}{L} e^{2\pi w/L} \right|^x. \quad (6.45)$$

As a consequence, the normalised eigenvector $|\phi\rangle$ associated to the primary operator ϕ is

$$\boxed{|\phi\rangle = \lim_{w \rightarrow -\infty} \left[\left| \frac{2\pi}{L} e^{2\pi w/L} \right|^x \widehat{\phi}_{\text{cyl}}(w, \bar{w}) |0\rangle \right] \equiv \lim_{z \rightarrow 0} [\widehat{\phi}_{\text{plane}}(z, \bar{z}) |0\rangle].} \quad (6.46)$$

In the following, we shall simply write $\phi(z, \bar{z})$ instead of $\widehat{\phi}_{\text{plane}}(z, \bar{z})$. From the relations:

$$L_0 \phi(0, 0) = h \phi(0, 0), \quad L_{n>0} \phi(0, 0) = 0, \quad (6.47)$$

$$\bar{L}_0 \phi(0, 0) = \bar{h} \phi(0, 0), \quad \bar{L}_{n>0} \phi(0, 0) = 0, \quad (6.48)$$

we get the corresponding action on primary states:

$$\boxed{L_0 |\phi\rangle = h |\phi\rangle, \quad L_{n>0} |\phi\rangle = 0, \\ \bar{L}_0 |\phi\rangle = \bar{h} |\phi\rangle, \quad \bar{L}_{n>0} |\phi\rangle = 0.} \quad (6.49)$$

This means that the primary states are *highest-weight states* for the two Virasoro algebras Vir and $\overline{\text{Vir}}$.

6.2.5 Descendant states

Recall the definition for a descendant of a primary operator ϕ :

$$(L_{-m} \cdot \phi)(z, \bar{z}) = \frac{1}{2i\pi} \oint_{C_z} d\xi (\xi - z)^{-m+1} T(\xi) \phi(z, \bar{z}) \quad (6.50)$$

Let us construct the corresponding state $|L_{-m} \cdot \phi\rangle$:

$$(L_{-m} \cdot \phi)(z, \bar{z})|0\rangle = \frac{1}{2i\pi} \oint_{C_z} d\xi (\xi - z)^{-m+1} \mathcal{R}[T(\xi) \phi(z, \bar{z})] |0\rangle \quad (6.51)$$

$$= \frac{1}{2i\pi} \oint_{C_{0,z}} d\xi (\xi - z)^{-m+1} T(\xi) \phi(z, \bar{z}) |0\rangle \quad (6.52)$$

$$- \frac{1}{2i\pi} \oint_{C_0} d\xi (\xi - z)^{-m+1} \phi(z, \bar{z}) T(\xi) |0\rangle. \quad (6.53)$$

In the first term, we can let $z \rightarrow 0$, which gives $L_{-m} \phi(0, 0)|0\rangle$. In the second term, keeping z fixed and letting $\xi \rightarrow 0$, we get $(-z)^{-m+1} \phi(z, \bar{z}) L_{-1}|0\rangle = 0$. Hence, we obtain the identity:

$$|L_{-m} \cdot \phi\rangle = \lim_{z \rightarrow 0} [(L_{-m} \cdot \phi)(z, \bar{z})|0\rangle] = L_{-m}|\phi\rangle. \quad (6.54)$$

By iterating this argument, we get for any sequences $m_1 \geq \dots \geq m_k$ and $\bar{m}_1 \geq \dots \geq \bar{m}_{\bar{k}}$:

$$|L_{-m_1} \cdots L_{-m_k} \cdot L_{-\bar{m}_1} \cdots L_{-\bar{m}_{\bar{k}}} \cdot \phi\rangle = \lim_{z \rightarrow 0} [(L_{-m_1} \cdots L_{-m_k} \cdot L_{-\bar{m}_1} \cdots L_{-\bar{m}_{\bar{k}}} \cdot \phi)(z, \bar{z})|0\rangle]$$

$$= L_{-m_1} \cdots L_{-m_k} L_{-\bar{m}_1} \cdots L_{-\bar{m}_{\bar{k}}} |\phi\rangle.$$

Using the Virasoro commutation relations, we get

$$L_0|L_{-m_1} \cdots L_{-m_k} \cdot L_{-\bar{m}_1} \cdots L_{-\bar{m}_{\bar{k}}} \cdot \phi\rangle = (h + M)|L_{-m_1} \cdots L_{-m_k} \cdot L_{-\bar{m}_1} \cdots L_{-\bar{m}_{\bar{k}}} \cdot \phi\rangle, \quad (6.55)$$

$$\bar{L}_0|L_{-m_1} \cdots L_{-m_k} \cdot L_{-\bar{m}_1} \cdots L_{-\bar{m}_{\bar{k}}} \cdot \phi\rangle = (\bar{h} + \bar{M})|L_{-m_1} \cdots L_{-m_k} \cdot L_{-\bar{m}_1} \cdots L_{-\bar{m}_{\bar{k}}} \cdot \phi\rangle.$$

The state $L_{-2}|0\rangle$ is of particular interest, because it corresponds to the operator:

$$(L_{-2} \cdot \mathbf{1})(z, \bar{z}) = \frac{1}{2i\pi} \oint_{C_z} dw (w - z)^{-1} T(w) \mathbf{1}(z, \bar{z}) = T(z). \quad (6.56)$$

Hence, we will write

$$L_{-2}|0\rangle = |T\rangle, \quad \bar{L}_{-2}|0\rangle = |\bar{T}\rangle. \quad (6.57)$$

6.2.6 Scalar product

From the interpretation of the L_n 's as Fourier coefficients on the cylinder, and by choosing the canonical scalar product on the basis of lattice configurations, we see that the Hermitian conjugates of the modes are given by:

$$L_n^\dagger = L_{-n}, \quad \bar{L}_n^\dagger = \bar{L}_{-n}. \quad (6.58)$$

Consider the scalar product between two generic basis states, *i.e.* two descendants of primary states:

$$\langle \phi_i | L_{m_1} \cdots L_{m_k} \cdot L_{-n_1} \cdots L_{-n_\ell} | \phi_j \rangle, \quad (6.59)$$

where m_1, \dots, m_k and n_1, \dots, n_ℓ are positive integers. For primary states, we have already established from general conformal invariance arguments that

$$\langle \phi_i | \phi_j \rangle = \langle \phi_i(\infty, \infty) \phi_j(0, 0) \rangle = \delta_{ij}. \quad (6.60)$$

Inserting a mode L_{-n} gives:

$$\langle \phi_i | L_{-n} | \phi_j \rangle = \frac{1}{2i\pi} \oint_{C_0} dz z^{-n+1} \langle \phi_i(\infty, \infty) T(z) \phi_j(0, 0) \rangle \quad (6.61)$$

$$= \frac{1}{2i\pi} \oint_{C_0} dz z^{-n+1} \langle \phi_i(\infty, \infty) \left[\frac{h_j \phi_j(0, 0)}{z^2} + \frac{\partial \phi_j(0, 0)}{z} + \text{reg.} \right] \rangle \quad (6.62)$$

$$= 0 \quad \text{if } i \neq j. \quad (6.63)$$

Through a similar argument, we find that

$$\boxed{\forall i \neq j, \quad \langle \phi_i | L_{m_1} \dots L_{m_k}, L_{-n_1} \dots L_{-n_\ell} | \phi_j \rangle = 0.} \quad (6.64)$$

6.3 The full Hilbert space

Let us consider the two-point function of a lattice operator ϕ , with $m_1 > m_2$:

$$\langle \Phi(m_1, n_1) \Phi(m_2, n_2) \rangle = \langle 0 | \widehat{\Phi}(m_1, n_1) \widehat{\Phi}(m_2, n_2) | 0 \rangle \quad (6.65)$$

$$= \lambda_0^{-m_1+m_2} \langle 0 | \widehat{\Phi}(0, n_1) \tau^{m_1-m_2} \widehat{\Phi}(0, n_2) | 0 \rangle \quad (6.66)$$

$$\underset{m_1-m_2 \gg N}{\sim} (\lambda_\phi / \lambda_0)^{m_1-m_2} \langle 0 | \widehat{\Phi}(0, n_1) | \phi \rangle \langle \phi | \widehat{\Phi}(0, n_2) | 0 \rangle \quad (6.67)$$

Hence, for any scaling operator \mathcal{O} , we have:

$$\langle \mathcal{O}(w_1, \bar{w}_1) \mathcal{O}(w_2, \bar{w}_2) \rangle_{\text{cyl}} \underset{\text{Re}(w_1-w_2) \gg L}{\propto} \exp \left[-\frac{\text{Re}(w_1-w_2)}{\xi_{\mathcal{O}}} \right], \quad \xi_{\mathcal{O}} = \frac{a}{\log(\lambda_0/\lambda)}. \quad (6.68)$$

Since the system is critical, the correlation length should behave as $\xi_{\mathcal{O}} \propto L$. Hence we get:

$$-\log(\lambda/\lambda_0) \propto \frac{a}{L}. \quad (6.69)$$

Equivalently, since the transfer matrix is related to the quantum Hamiltonian through

$$\tau = \text{const}^N \times e^{-a\mathcal{H}}, \quad (6.70)$$

the energy gap should behave as

$$E - E_0 \propto \frac{1}{L} \quad (6.71)$$

As we have shown above, since the Hamiltonian is given by

$$\mathcal{H} = \frac{2\pi}{L} \left(L_0 + \bar{L}_0 - \frac{c}{12} \right), \quad (6.72)$$

any primary state or descendant has an energy of the form

$$\mathcal{H}|\mathcal{O}\rangle = E_{\mathcal{O}}|\mathcal{O}\rangle, \quad E_{\mathcal{O}} = \frac{2\pi}{L} \left(x_{\mathcal{O}} - \frac{c}{12} \right), \quad (6.73)$$

where $x_{\mathcal{O}}$ is the scaling dimension of the operator. This confirms the expected behaviour of energy gaps. The basic assumption we shall make is that *all* the states in the Hilbert space of the CFT are (linear combinations of) primaries and their descendants:

Assumption: The Hilbert space of a CFT is generated by all the admissible primary states, and their descendants under the Virasoro algebra $\text{Vir} \otimes \overline{\text{Vir}}$.

Some descendant states may be null vectors, *i.e.* they are orthogonal to the whole Hilbert space.

Exercise: Let ϕ be a primary operator, and $\chi(z, \bar{z}) = ((\#L_{-N} + \dots + \#L_{-1}^N) \cdot \phi)(z, \bar{z})$ be the descendant operator associated to a null vector $|\chi\rangle$. Prove that any correlation function $\langle \dots \chi(z, \bar{z}) \dots \rangle$ vanishes.

The physical Hilbert space is obtained from the direct sum

$$\bigoplus_{\phi_j \text{ primary}} \mathcal{V}_{h_j} \otimes \bar{\mathcal{V}}_{\bar{h}_j}, \quad (6.74)$$

by quotienting out all the possible null vectors.

A CFT with a finite number of primary operators is called *rational*. A CFT in which $h_j = \bar{h}_j$ for every primary operator is called *diagonal*.

6.4 Appendix : Mapping from the plane to the infinite cylinder

We consider the infinite cylinder of circumference $L = Na$, where the coordinates are written $w = t + ix$. We use the mapping:

$$\begin{cases} \text{cylinder} & \rightarrow \text{plane} \\ w & \mapsto z = \exp(2\pi w/L). \end{cases} \quad (6.75)$$

Let us begin by the two-point function of a primary field ϕ on the infinite cylinder, with conformal dimensions h, \bar{h} :

$$\langle \phi(w_1, \bar{w}_1) \phi(w_2, \bar{w}_2) \rangle_{\text{cyl}} = \left[\frac{L}{\pi} \sinh \frac{\pi(w_1 - w_2)}{L} \right]^{-2h} \left[\frac{L}{\pi} \sinh \frac{\pi(\bar{w}_1 - \bar{w}_2)}{L} \right]^{-2\bar{h}}. \quad (6.76)$$

Take the limit $\text{Re}(t_1 - t_2) \rightarrow +\infty$:

$$\langle \phi(w_1, \bar{w}_1) \phi(w_2, \bar{w}_2) \rangle_{\text{cyl}} \sim \left(\frac{L}{2\pi} \right)^{-4h} \exp \left[-\frac{2\pi}{L} (h + \bar{h})(t_1 - t_2) + \frac{2i\pi}{L} (h - \bar{h})(x_1 - x_2) \right]. \quad (6.77)$$

This is consistent with the energy gap and momentum:

$$E_\phi - E_0 = \frac{2\pi}{L} (h + \bar{h}), \quad p_\phi = \frac{2\pi}{L} (h - \bar{h}). \quad (6.78)$$

Lecture 7

Structure constants of the OPE

7.1 The Operator Product Expansion in CFTs

Consider two scaling operators $\mathcal{O}_i(r'), \mathcal{O}_j(r)$, and let $r' \rightarrow r$ inside a correlation function, where all the other operators are located at a larger distance to $\mathcal{O}_j(r)$. Then, for $r' = r + \epsilon$ with $\epsilon \rightarrow 0$, the product $\mathcal{O}_i(r + \epsilon), \mathcal{O}_j(r)$ may be viewed, at large distance, as a single local operator located at r . The latter may be, in turn, decomposed on the basis of scaling operators, with ϵ -dependant coefficients:

$$\mathcal{O}_i(r + \epsilon) \mathcal{O}_j(r) \underset{\epsilon \rightarrow 0}{\sim} \sum_k c_{ij}^k(\epsilon) \mathcal{O}_k(r). \quad (7.1)$$

This relation is called the Operator Product Expansion (OPE). It defines an algebra on operators, simply called the operator algebra. The coefficients $c_{ij}^k(\epsilon)$ are the structure constants of this algebra.

In the case of a 2d CFT, the explicit space dependance of the c_{ij}^k 's reads, in complex coordinates

$$\mathcal{O}_i(z, \bar{z}) \mathcal{O}_j(0, 0) = \sum_{\substack{\mathcal{O}_k \text{ scaling op.}}} \mathcal{C}_{ij}^k z^{-h_i - h_j + h_k} \bar{z}^{-\bar{h}_i - \bar{h}_j + \bar{h}_k} \mathcal{O}_k(0, 0). \quad (7.2)$$

In principle, any n -point correlation function could be computed by repeatedly applying the OPE relations, until one gets to a linear combination of two-point functions.

Let us focus on the OPE of two primary operators:

$$\begin{aligned} \phi_i(z, \bar{z}) \phi_j(w, \bar{w}) &= \\ &\sum_{\substack{\phi_k \text{ primary} \\ [m], [\bar{m}]}} \mathcal{C}_{ij}^k([m], [\bar{m}]) (z-w)^{-h_{ij}^k + M} (\bar{z}-\bar{w})^{-\bar{h}_{ij}^k + \bar{M}} \phi_k^{[m], [\bar{m}]}(w, \bar{w}), \end{aligned} \quad (7.3)$$

where

$$h_{ij}^k := h_i + h_j - h_k, \quad \bar{h}_{ij}^k := \bar{h}_i + \bar{h}_j - \bar{h}_k, \quad (7.4)$$

and $[m] = (m_1, \dots, m_\ell)$ with $m_1 \geq m_2 \geq \dots \geq m_\ell \geq 1$, and similarly for $[\bar{m}]$, and we have use the short-hand notations

$$M = |m| := m_1 + \dots + m_\ell, \quad \bar{M} = |\bar{m}| := \bar{m}_1 + \dots + \bar{m}_{\bar{\ell}}, \quad (7.5)$$

$$\phi_k^{[m], [\bar{m}]} := L_{-m_1} \cdots L_{-m_\ell} \cdot \bar{L}_{-\bar{m}_1} \cdots \bar{L}_{-\bar{m}_{\bar{\ell}}} \cdot \phi_k. \quad (7.6)$$

For a given pair ϕ_i, ϕ_j , only a subset of primary operators ϕ_k , contribute to the r.h.s. of the OPE $\phi_i \cdot \phi_j$. We denote this as

$$\phi_i \times \phi_j \rightarrow \sum_k N_{ij}^k \phi_k, \quad (7.7)$$

where the fusion integer $N_{ij}^k = 1$ if the fusion $\phi_i \times \phi_j \rightarrow \phi_k$ is allowed, and $N_{ij}^k = 0$ otherwise. A descendant $\phi_k^{[m], [\bar{m}]}$ can only appear in the r.h.s. of the OPE if the primary operator ϕ_k is present.

Hence, the fundamental data defining a CFT are:

- The conformal algebra, generating the “spacetime” symmetries, and possibly the some additional internal symmetries – if one ignores the internal symmetries, one gets the conformal algebra $\text{Vir} \times \overline{\text{Vir}}$;
- The set of primary operators, *i.e.* the operators which transform covariantly under the conformal algebra;
- The fusion rules between primary operators, encoded by the N_{ij}^k ;
- The structure constants between the scaling operators.

The fusion integers and the structure constants cannot take any values: they are constrained by the conformal symmetry and other conditions. From the above data, in principle, any n -point correlation function of scaling operators can be reconstructed.

7.2 Structure constants for primary operators

Start with the OPE

$$\phi_i(z, \bar{z})\phi_j(0, 0) = \sum_{p, [m], [\bar{m}]} z^{-h_{ij}^p + M} \bar{z}^{-\bar{h}_{ij}^p + \bar{M}} C_{ij}^p([m], [\bar{m}]) \phi_p^{[m], [\bar{m}]}(0, 0), \quad (7.8)$$

In the following, the structure constant for three primary operators will simply be denoted:

$$C_{ij}^k := C_{ij}^k(\cdot, \cdot). \quad (7.9)$$

We insert the above OPE into the average value $\langle \phi_k | \dots | 0 \rangle$, where ϕ_k is a primary operator:

$$\langle \phi_k | \phi_i(z, \bar{z})\phi_j(0, 0) | 0 \rangle = \sum_{p, [m], [\bar{m}]} z^{-h_{ij}^p + M} \bar{z}^{-\bar{h}_{ij}^p + \bar{M}} C_{ij}^p([m], [\bar{m}]) \langle \phi_k | \phi_p^{[m], [\bar{m}]} \rangle \quad (7.10)$$

$$= z^{-h_{ij}^k} \bar{z}^{-\bar{h}_{ij}^k} C_{ij}^k. \quad (7.11)$$

On the other hand, the LHS is exactly equal to the three-point function

$$\langle \phi_k(\infty, \infty)\phi_i(z, \bar{z})\phi_j(0, 0) \rangle = \lim_{R \rightarrow \infty} R^{2h_k} \bar{R}^{2\bar{h}_k} \langle \phi_k(R, \bar{R})\phi_i(z, \bar{z})\phi_j(0, 0) \rangle.$$

Recall the general expression of the three-point function of primary operators:

$$\langle \phi_k(z_1, \bar{z}_1)\phi_i(z_2, \bar{z}_2)\phi_j(z_3, \bar{z}_3) \rangle = \frac{A_{ijk}}{z_{12}^{h_{ki}^j} z_{23}^{h_{ij}^k} z_{13}^{h_{kj}^i} \times \bar{z}_{12}^{\bar{h}_{ki}^j} \bar{z}_{23}^{\bar{h}_{ij}^k} \bar{z}_{13}^{\bar{h}_{kj}^i}}, \quad (7.12)$$

where $z_{ij} := z_i - z_j$. As we let $z_1 = R \rightarrow \infty$, with $z_2 = z$ and $z_3 = 0$, we get

$$\langle \phi_p(R, \bar{R}) \phi_i(z, \bar{z}) \phi_j(0, 0) \rangle \sim R^{-2h_k} \bar{R}^{-2h_k} z^{-h_{ij}^k} \bar{z}^{-\bar{h}_{ij}^k} A_{ijk}. \quad (7.13)$$

By identifying the two expressions, we find that the structure constant for primary fields is nothing but the multiplicative factor of the three-point function:

$$C_{ij}^k = A_{ijk}. \quad (7.14)$$

Because of the normalisation of two-point functions, when $\phi_k = \mathbf{1}$ we have $C_{ij}^{\mathbf{1}} = \delta_{ij}$. The determination of the non-trivial structure constants of primary operators requires some specific approach, which will be exposed in subsequent lectures.

7.3 Structure constants for descendants

The dual basis of descendants $\langle \widehat{\phi}_k^{[m], [\bar{m}]} |$ is of the form:

$$\langle \widehat{\phi}_k^{[m], [\bar{m}]} | = \sum_{\substack{[n], |n|=|m| \\ [\bar{n}], |\bar{n}|=|\bar{m}|}} a_k([m][n]) \bar{a}_k([\bar{m}][\bar{n}]) \langle \phi_k | L_{n_1} \dots L_{n_p} \bar{L}_{\bar{n}_1} \dots \bar{L}_{\bar{n}_{\bar{p}}}, \quad (7.15)$$

with

$$\langle \widehat{\phi}_k^{[m], [\bar{m}]} | \phi_k^{[n], [\bar{n}]} \rangle = \delta_{[m], [n]} \delta_{[\bar{m}], [\bar{n}]} \quad (7.16)$$

From the OPE $\phi_i \cdot \phi_j$ we can write:

$$C_{ij}^k([m], [\bar{m}]) = z^{h_{ij}^k - M} \bar{z}^{\bar{h}_{ij}^k - \bar{M}} \langle \widehat{\phi}_k^{[m], [\bar{m}]} | \phi_i(z, \bar{z}) \phi_j(z, \bar{z}) \rangle. \quad (7.17)$$

Hence this $C_{ij}^k([m], [\bar{m}])$ is a linear combination of terms of the form

$$\langle \phi_k | L_{n_1} \dots L_{n_p} \bar{L}_{\bar{n}_1} \dots \bar{L}_{\bar{n}_{\bar{p}}} \phi_i(z, \bar{z}) \phi_j(z, \bar{z}) \rangle. \quad (7.18)$$

If only one L_n is inserted (with $n > 0$), we get:

$$\langle \phi_k | L_n \phi_i(z, \bar{z}) \phi_j(z, \bar{z}) \rangle = \frac{1}{2i\pi} \left(\oint_{C_z} + \oint_{C_0} \right) dw w^{n+1} \langle \phi_k(\infty) T(w) \phi_i(z, \bar{z}) \phi_j(0) \rangle \quad (7.19)$$

$$= \frac{1}{2i\pi} \oint_{C_z} dw w^{n+1} \langle \phi_k(\infty) \left[\frac{h_i \phi_i(z, \bar{z})}{(w-z)^2} + \frac{\partial_z \phi_i(z, \bar{z})}{w-z} \right] \phi_j(0) \rangle$$

$$+ \langle \phi_k(\infty) \phi_i(z, \bar{z}) (L_n \cdot \phi_j)(0) \rangle \quad (7.20)$$

$$= z^m [(n+1)h_i + z\partial_z] \langle \phi_k(\infty) \phi_i(z, \bar{z}) \phi_j(0) \rangle \quad (7.21)$$

$$= z^{-h_{ij}^k + n} \bar{z}^{\bar{h}_{ij}^k} (nh_i - h_j + h_k) C_{ij}^k. \quad (7.22)$$

By recursion, after multiple insertions of L_n 's and $\bar{L}_{\bar{n}}$, we get the same form for the overlaps

$$\langle \phi_k | L_{n_1} \dots L_{n_p} \bar{L}_{\bar{n}_1} \dots \bar{L}_{\bar{n}_{\bar{p}}} \phi_i(z, \bar{z}) \phi_j(z, \bar{z}) \rangle = z^{-h_{ij}^k + |n|} \bar{z}^{\bar{h}_{ij}^k + |\bar{n}|} b_{ij}^k([n]) \bar{b}_{ij}^k([\bar{n}]) C_{ij}^k, \quad (7.23)$$

where the prefactor $b_{ij}^k([n])$ (resp. $\bar{b}_{ij}^k([\bar{n}])$) depends on the triplet (h_i, h_j, h_k) (resp. $(\bar{h}_i, \bar{h}_j, \bar{h}_k)$). Hence the structure constants can be factorised as

$$C_{ij}^k([m], [\bar{m}]) = C_{ij}^k \times \beta_{ij}^k([m]) \times \bar{\beta}_{ij}^k([\bar{m}]), \quad (7.24)$$

where the coefficients $\beta_{ij}^k([m])$ and $\bar{\beta}_{ij}^k([\bar{m}])$ are given by

$$\beta_{ij}^k([m]) := \sum_{[n], |n|=|m|} a_k([m][n]) b_{ij}^k([n]), \quad (7.25)$$

$$\bar{\beta}_{ij}^k([\bar{m}]) := \sum_{[\bar{n}], |\bar{n}|=|\bar{m}|} a_k([\bar{m}][\bar{n}]) \bar{b}_{ij}^k([\bar{n}]). \quad (7.26)$$

Note that the factors β_{ij}^k and $\bar{\beta}_{ij}^k$ are “purely kinematic”, *i.e.* they are completely determined by the Virasoro commutation rules and the OPEs $T.\phi_i$. In contrast, as we shall see in the following lectures, the structure constants for primary operators are “dynamic”, *i.e.* they are determined non-trivially from the full structure of the operator algebra.

7.4 Explicit computation of descendant structure constants

Let us illustrate how the coefficients $\beta_{ij}^k([m])$ and $\bar{\beta}_{ij}^k([\bar{m}])$ may be computed simply from the commutation rules of the Virasoro algebra. Start with the OPE of primary operators:

$$\phi_i(z, \bar{z})\phi_j(0, 0) = \sum_{k, [m], [\bar{m}]} C_{ij}^k \beta_{ij}^k([m]) \bar{\beta}_{ij}^k([\bar{m}]) z^{-h_{ij}^k + |m|} \bar{z}^{-\bar{h}_{ij}^k + |\bar{m}|} \phi_k^{[m], [\bar{m}]}(0, 0). \quad (7.27)$$

In terms of states, we get

$$\phi_i(z, \bar{z})|\phi_j\rangle = \sum_{k, M, \bar{M}} C_{ij}^k z^{-h_{ij}^k + M} \bar{z}^{-\bar{h}_{ij}^k + \bar{M}} |\psi_{ij}^k(M, \bar{M})\rangle, \quad (7.28)$$

where we have grouped the terms from each level (M, \bar{M}) :

$$|\psi_{ij}^k(M, \bar{M})\rangle = \sum_{\substack{[m], |m|=M \\ [\bar{m}], |\bar{m}|=\bar{M}}} \beta_{ij}^k([m]) \bar{\beta}_{ij}^k([\bar{m}]) L_{-m_1} \dots L_{-m_\ell} L_{-\bar{m}_1} \dots L_{-\bar{m}_{\bar{\ell}}} |\phi_k\rangle. \quad (7.29)$$

We now apply the operator L_n with $n > 0$. On one hand, we have immediately:

$$L_n \phi_i(z, \bar{z}) |\phi_j\rangle = \sum_{k, M, \bar{M}} C_{ij}^k z^{-h_{ij}^k + M} \bar{z}^{-\bar{h}_{ij}^k + \bar{M}} L_n |\psi_{ij}^k(M, \bar{M})\rangle. \quad (7.30)$$

On the other hand, using the commutation relation

$$[L_n, \phi_i(z, \bar{z})] = z^n ((n+1)h_i + z\partial_z) \phi_i(z, \bar{z}), \quad (7.31)$$

we obtain:

$$L_n \phi_i(z, \bar{z}) |\phi_j\rangle = \sum_{k, M, \bar{M}} C_{ij}^k z^{-h_{ij}^k + M + n} \bar{z}^{-\bar{h}_{ij}^k + \bar{M}} (nh_i - h_j + h_k + M) |\psi_{ij}^k(M, \bar{M})\rangle. \quad (7.32)$$

By identifying the two expansions for $L_n \phi_i(z, \bar{z}) |\phi_j\rangle$, we get

$$L_n |\psi_{ij}^k(M + n, \bar{M})\rangle = (nh_i - h_j + h_k + M) |\psi_{ij}^k(M, \bar{M})\rangle.$$

(7.33)

This equation directly relates linearly the coefficients β_{ij}^k at level $(M+n)$ to the coefficients at level M .

For example, let us compute $\beta_{ij}^k([1])$. We have

$$|\psi_{ij}^k(0,0)\rangle = |\phi_k\rangle, \quad |\psi_{ij}^k(1,0)\rangle = \beta_{ij}^k([1]) L_{-1} |\phi_k\rangle. \quad (7.34)$$

Applying L_1 , we get

$$L_1 |\psi_{ij}^k(1,0)\rangle = (h_i - h_j + h_k) |\psi_{ij}^k(0,0)\rangle. \quad (7.35)$$

Using the Virasoro commutation rules $[L_1, L_{-1}] = 2L_0$, we get

$$2h_k \beta_{ij}^k([1]) |\phi_k\rangle = (h_i - h_j + h_k) |\phi_k\rangle. \quad (7.36)$$

So we have, when $h_k \neq 0$:

$$\beta_{ij}^k([1]) = \frac{h_i - h_j + h_k}{2h_k}. \quad (7.37)$$

Lecture 8

Fusion rules and minimal models

8.1 Action of the L_n 's in correlation functions

To study the action of negative Virasoro modes, we can start with any scaling operator \mathcal{O} , and consider the operator $(L_{-m}\mathcal{O})(z, \bar{z})$ with $m \geq 1$, when inserted into a correlation function with primary operators:

$$\langle (L_{-m}\mathcal{O})(z, \bar{z})\phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) \rangle \quad (8.1)$$

We can deform the integration contour for L_{-m} :

$$\langle (L_{-m}\mathcal{O})(z, \bar{z})\phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) \rangle \quad (8.2)$$

$$= \frac{1}{2i\pi} \oint_{C_z} dw (w-z)^{1-m} \langle T(w)\mathcal{O}(z, \bar{z})\phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) \rangle \quad (8.3)$$

$$= \frac{1}{2i\pi} \left(\oint_{C_\infty} - \sum_{j=1}^n \oint_{C_j} \right) dw (w-z)^{1-m} \langle T(w)\mathcal{O}(z, \bar{z})\phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) \rangle, \quad (8.4)$$

where C_∞ is an anti-clockwise oriented contour enclosing $\{z, z_1, \dots, z_n\}$, and C_j is an anti-clockwise oriented contour enclosing only z_j .

Since $\langle T(w) \dots \rangle \propto w^{-4}$ as $w \rightarrow \infty$, the integral on C_∞ vanishes. In the integral over C_j , we can use the OPE $T(w)\phi_j(z_j, \bar{z}_j)$:

$$\frac{1}{2i\pi} \oint_{C_j} dw (w-z)^{1-m} \langle \dots T(w)\phi_j(z_j, \bar{z}_j) \dots \rangle \quad (8.5)$$

$$= \frac{1}{2i\pi} \oint_{C_j} dw (w-z)^{1-m} \langle \dots \left[\frac{h_j\phi_j(z_j, \bar{z}_j)}{(w-z_j)^2} + \frac{\partial\phi_j(z_j, \bar{z}_j)}{w-z_j} + \text{reg}_{w \rightarrow z_j} \right] \dots \rangle \quad (8.6)$$

$$= \langle \dots \left[\frac{(1-m)h_j\phi_j(z_j, \bar{z}_j)}{(z_j-z)^m} + \frac{\partial\phi_j(z_j, \bar{z}_j)}{(z_j-z)^{m-1}} \right] \dots \rangle \quad (8.7)$$

Finally, we obtain

$$\langle (L_{-m}\mathcal{O})(z, \bar{z})\phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) \rangle = \mathcal{L}_{-m} \langle \mathcal{O}(z, \bar{z})\phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) \rangle,$$

$$\mathcal{L}_{-m} := \sum_{j=1}^n \left[\frac{(m-1)h_j}{(z_j-z)^m} - \frac{1}{(z_j-z)^{m-1}} \partial_{z_j} \right].$$

(8.8)

By recursion, for a primary operator $\phi(z, \bar{z})$ we obtain:

$$\begin{aligned} & \langle (L_{-m_1} \dots L_{-m_k} \phi)(z, \bar{z}) \phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) \rangle \\ &= \mathcal{L}_{-m_1} \dots \mathcal{L}_{-m_k} \langle \phi(z, \bar{z}) \phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) \rangle. \end{aligned} \quad (8.9)$$

8.2 Fusion rules of degenerate primary operators

Recall that a primary operator ϕ is degenerate at level N , if one of its descendants $\chi = (L_{-N} + \dots + \#L_{-1}^N)\phi$ at this level is itself a primary. This operator χ is then called a null-vector, and decouples from the theory, *i.e.* it behaves in correlation functions as

$$\chi(z, \bar{z}) = [(L_{-N} + \dots + \#L_{-1}^N)\phi](z, \bar{z}) \equiv 0. \quad (8.10)$$

The above relation is called the null-vector condition for the degenerate operator ϕ . Let us illustrate the consequences of a null-vector condition on the OPEs of ϕ with other primary fields, with the example of ϕ_{12} . To simplify the discussion, we consider the case of a diagonal CFT *i.e.* all primary fields are scalars: $h_k = \bar{h}_k$. As we have seen in the previous lecture, the null-vector condition reads

$$\left(L_{-2} - \frac{3}{4h_{12} + 2} L_{-1}^2 \right) \phi_{12} \equiv 0. \quad (8.11)$$

When inserted into a correlation function with primary operators ϕ_1, \dots, ϕ_n , this gives the second-order partial differential equation (PDE):

$$\left(\mathcal{L}_{-2} - \frac{3}{4h_{12} + 2} \mathcal{L}_{-1}^2 \right) \langle \phi_{12}(z, \bar{z}) \phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) \rangle = 0, \quad (8.12)$$

with

$$\mathcal{L}_{-2} = \sum_{k=1}^n \left[\frac{h_k}{(z - z_k)^2} + \frac{1}{z - z_k} \partial_{z_k} \right], \quad \mathcal{L}_{-1} = - \sum_{k=1}^n \partial_{z_k} \equiv \partial_z. \quad (8.13)$$

Let ϕ_i, ϕ_j be two primary fields, and suppose that the fusion rule $\phi_{12} \times \phi_i \rightarrow \phi_j$ is allowed: this means that the three-point function $\langle \phi_{12}(z, \bar{z}) \phi_i(z_1, \bar{z}_1) \phi_j(z_2, \bar{z}_2) \rangle$ is nonzero, and satisfies the above differential equation. From general conformal invariance arguments, we know that this function is of the form:

$$\begin{aligned} & \langle \phi_{12}(z, \bar{z}) \phi_i(z_1, \bar{z}_1) \phi_j(z_2, \bar{z}_2) \rangle \\ &= \frac{C}{(z - z_1)^{h_{12} + h_i - h_j} (z - z_2)^{h_{12} + h_j - h_i} (z_1 - z_2)^{h_i + h_j - h_{12}} \times (\text{anti-holo})}. \end{aligned} \quad (8.14)$$

Inserting this function into the PDE (8.12) yields the relation:

$$3(h_i - h_j)^2 - (2h_{12} + 1)(h_i + h_j) + h_{12}(1 - h_{12}) = 0. \quad (8.15)$$

This means that, for a given dimension h_i , only two values of h_j are allowed in the fusion $\phi_{12} \times \phi_i \rightarrow \phi_j$. It is easier to express the results in terms of the Kac parameterisation:

$$c = 1 - 6(b^{-1} - b)^2, \quad (8.16)$$

$$h_{rs} = \frac{(rb^{-1} - sb)^2 - (b^{-1} - b)^2}{4}. \quad (8.17)$$

With these notations, a generic primary operator ϕ_h can be identified with $\phi_{1,\mu}$, with

$$h_{1,\mu} = \frac{(\mu - 1)b}{2} \left[\frac{(\mu - 1)b}{2} - (b^{-1} - b) \right]. \quad (8.18)$$

For generic (real) values of μ , the explicit fusion rules of ϕ_{12} with $\phi_{1,\mu}$ are:

$$\phi_{12} \times \phi_{1,\mu} \rightarrow \phi_{1,\mu-1} + \phi_{1,\mu+1}. \quad (8.19)$$

Alternatively, we can use the parameter $\alpha = (\mu - 1)b/2$, and write

$$h_\alpha = \alpha \left[\alpha - (b^{-1} - b) \right]. \quad (8.20)$$

We then denote $V_\alpha := \phi_{h_\alpha}$. In terms of α , the fusion rules are:

$$\phi_{12} \times V_\alpha \rightarrow V_{\alpha-b/2} + V_{\alpha+b/2}. \quad (8.21)$$

From a similar argument, one gets the fusion rules for ϕ_{21} :

$$\phi_{21} \times V_\alpha \rightarrow V_{\alpha-b^{-1}/2} + V_{\alpha+b^{-1}/2}. \quad (8.22)$$

More generally, for any pair of positive integers r, s , using the null-state condition at level $N = rs$ for ϕ_{rs} , we get a PDE of order N for the three-point function $\langle \phi_{rs} \phi_\alpha \phi_{\alpha'} \rangle$, which translates into the fusion rule with $N = rs$ terms in the r.h.s. :

$$\phi_{rs} \times V_\alpha \rightarrow \sum_{\substack{k=1-r \\ k=r-1 \bmod 2}}^{r-1} \sum_{\substack{\ell=1-s \\ \ell=s-1 \bmod 2}}^{s-1} V_{\alpha+kb^{-1}/2+\ell b/2}. \quad (8.23)$$

Note that, if $\alpha = [(1-\mu)b^{-1} - (1-\nu)b]/2$ with μ, ν positive integers, then $V_\alpha = \phi_{\mu\nu}$ is also degenerate, and some additional constraints, deriving from its null-vector condition, may occur. In this case, some of the terms in the fusion $\phi_{rs} \times V_\alpha$ may not be allowed. For instance, we have:

$$\phi_{12} \times \phi_{11} = \phi_{12} \times \mathbf{1} = \phi_{12}, \quad \phi_{12} \times \phi_{21} = \phi_{22}. \quad (8.24)$$

8.3 Minimal models

8.3.1 Kac table and fusion rules

Consider a CFT, and assume that it includes the primary operators of dimensions ϕ_{12} and ϕ_{21} . By the fusion process:

$$\phi_{12} \times \phi_{12} \rightarrow \phi_{11} + \phi_{13}, \quad (8.25)$$

we see that a primary operator with dimension h_{13} should also be included. If we repeat this process, we produce the operators:

$$\phi_{11}, \phi_{12}, \phi_{13}, \phi_{14}, \dots \quad (8.26)$$

Fusing repeatedly with ϕ_{21} , we produce all the possible degenerate operators ϕ_{rs} , and the CFT has an infinite number of primary operators. A possible mechanism which leads to

a finite operator algebra, is if one of the operators produced by fusion, say ϕ_{rs} , coincides with an existing operator $\phi_{r',s'}$. By identifying the dimensions $h_{rs} = h_{r',s'}$, we find:

$$(r - r')b^{-1} = (s - s')b \quad \text{or} \quad (r + r')b^{-1} = (s + s')b. \quad (8.27)$$

In both cases, the parameter b^2 is *rational*.

Let us examine in more detail this case of a central charge $c = 1 - 6(b^{-1} - b)^2$, with $b^2 = p/p'$, where $p < p'$ are two coprime natural integers. The expressions for the central charge and the dimensions of degenerate operators take the form:

$$c = 1 - \frac{6(p' - p)^2}{pp'}, \quad h_{rs} = \frac{(rp' - sp)^2 - (p' - p)^2}{4pp'}. \quad (8.28)$$

The operator algebra of this model closes on a *finite* number of primary operators, which are all degenerate:

$$\{\phi_{rs}, \quad r = 1, \dots, p-1, \quad s = 1, \dots, p'-1\}. \quad (8.29)$$

This set of operators is called the Kac table, and it defines the minimal model $\mathcal{M}_{pp'}$. Each primary operator actually appears twice in this table, due to the relation:

$$h_{p-r,p'-s} = h_{rs} \quad \Rightarrow \quad \phi_{p-r,p'-s} \equiv \phi_{rs}. \quad (8.30)$$

As a consequence, in the minimal model $\mathcal{M}_{p,p'}$, the primary operator ϕ_{rs} is degenerate at level rs and at level $(p-r)(p'-s)$.

The total number of primary operators is $(p-1)(p'-1)/2$, and their fusion rules are given by

$$\phi_{r_1 s_1} \times \phi_{r_2 s_2} \rightarrow \sum_{\substack{k=1+|r_1-r_2| \\ k+r_1+r_2=1 \bmod 2}}^{p-1-|p-r_1-r_2|} \sum_{\substack{\ell=1+|s_1-s_2| \\ \ell+s_1+s_2=1 \bmod 2}}^{p'-1-|p'-s_1-s_2|} \phi_{k\ell}. \quad (8.31)$$

These fusion rules are of course compatible with the identification $\phi_{rs} \equiv \phi_{p-r,p'-s}$. Another important property of these fusion rules is that they obey a \mathbb{Z}_2 symmetry. We define the \mathbb{Z}_2 charge of a primary operator ϕ_{rs} as $(-1)^{m_{rs}}$, where

$$m_{rs} := \begin{cases} r - 1 \bmod 2 & \text{if } p \text{ is even,} \\ s - 1 \bmod 2 & \text{if } p' \text{ is even,} \\ r + s \bmod 2 & \text{otherwise.} \end{cases} \quad (8.32)$$

With this definition, we have $m_{rs} = m_{p-r,p'-s}$, and the fusion rules conserve this \mathbb{Z}_2 charge:

$$\text{If } \phi_{r_1 s_1} \times \phi_{r_2 s_2} \rightarrow \phi_{k\ell}, \quad \text{then } m_{r_1 s_1} + m_{r_2 s_2} = m_{k\ell} \bmod 2. \quad (8.33)$$

Hence, all the minimal models encode an internal \mathbb{Z}_2 symmetry, on top of conformal invariance.

8.3.2 Unitary and non-unitary models

For generic values of the integers p, q , the model $\mathcal{M}_{p,p'}$ is “non-unitary”, *i.e.* the “scalar product” defined by $\langle \phi_i | \phi_j \rangle = \delta_{ij}$ for primaries and $L_n^\dagger = L_{-n}$ is not positive (this scalar product is definite thanks to the quotienting with respect to null states, but some states

can have negative norm). Alternatively, if one changes to a positive definite scalar product, then the Hamiltonian $\mathcal{H} = \frac{2\pi}{L}(L_0 + \bar{L}_0 - c/12)$ is no longer Hermitian. This is clearly seen on the dimensions of the Kac table for the minimal model $\mathcal{M}_{p,p'}$. In a unitary CFT, all primary dimensions should be positive, because, for any primary state $|\phi\rangle$:

$$\|L_{-1}|\phi\rangle\|^2 = \langle\phi|L_1L_{-1}|\phi\rangle = \langle\phi|(L_{-1}L_1 + 2L_0)|\phi\rangle = 2h.$$

This is consistent with the relation with eigenvalues of the transfer matrix.

$$\log(\lambda_j/\lambda_0) = -\frac{2\pi}{N}(h_j + \bar{h}_j). \quad (8.34)$$

Now let us find the minimal dimension in the Kac table (8.29). From the Bézout theorem, one can find two positive integers u, v such that $uq - vp = 1$. Let $k = \lfloor u/p \rfloor = \lfloor v/p' \rfloor$, and $r_0 := u - kp, s_0 := v - kp'$. These two indices satisfy $1 \leq r_0 \leq p - 1$ and $1 \leq s_0 \leq p' - 1$, and the corresponding dimension is

$$h_0 = h_{r_0, s_0} = \frac{1 - (p' - p)^2}{4pp'}. \quad (8.35)$$

If $p' = p + 1$ then $h_0 = h_{11} = 0$, and all the other dimensions h_{rs} are positive. One can show that the models $\mathcal{M}_{p,p+1}$ are in fact unitary. In contrast, if $p' > p + 1$ then $h_0 < 0$, and the model cannot be unitary. This is in contradiction with the interpretation of the vacuum state $|0\rangle$ as the (scaling limit of the) Perron-Frobenius vector of the transfer matrix. Actually, non-unitary CFTs are associated to local lattice models with non-positive Boltzmann weights, where the transfer matrix cannot be set real and symmetric in the canonical basis of row configurations.

8.3.3 Example: the Ising model

The CFT \mathcal{M}_{34} has central charge $c = 1/2$, and consists of three primary fields:

$$\mathbf{1} = \phi_{11} = \phi_{23}, \quad \epsilon = \phi_{13} = \phi_{21}, \quad \sigma = \phi_{12} = \phi_{22}, \quad (8.36)$$

with dimensions

$$h_{\mathbf{1}} = 1, \quad h_{\epsilon} = \frac{1}{2}, \quad h_{\sigma} = \frac{1}{16}, \quad (8.37)$$

\mathbb{Z}_2 charges

$$(-1)^{m_{\mathbf{1}}} = +1, \quad (-1)^{m_{\epsilon}} = +1, \quad (-1)^{m_{\sigma}} = -1, \quad (8.38)$$

and fusion rules

$$\sigma \times \sigma \rightarrow \mathbf{1} + \epsilon, \quad \epsilon \times \epsilon \rightarrow \mathbf{1}, \quad \sigma \times \epsilon \rightarrow \sigma. \quad (8.39)$$

This corresponds to the smallest unitary model with \mathbb{Z}_2 symmetry, and a single odd operator σ . Hence, it can be identified with the continuum limit of the critical Ising model on a regular lattice.

8.3.4 Virasoro characters

The character of a representation \mathcal{W} of the Virasoro algebra is defined as

$$\mathrm{Tr}_{\mathcal{W}}(q^{L_0-c/24}), \quad (8.40)$$

where q is a formal parameter.

Let us first compute that the character of the Verma module \mathcal{V}_h . It reads

$$\begin{aligned} \mathrm{Tr}_{\mathcal{V}_h}(q^{L_0-c/24}) &= q^{h-c/24} \sum_{N=0}^{\infty} \dim_N(\mathcal{V}_h) q^N = q^{h-c/24} \sum_{N=0}^{\infty} p(N) q^N \\ &= \frac{q^{h+(1-c)/24}}{\eta(q)}, \end{aligned} \quad (8.41)$$

where $\dim_N(\mathcal{V}_h)$ is the dimension of level N in the representation \mathcal{V}_h . In a Verma module, this dimension is simply equal to $p(N)$, the number of partitions of the integer N . In the final expression, we have used $\eta(q)$, the Dedekind eta function, defined for $|q| < 1$ by

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad \frac{1}{\eta(q)} = q^{-1/24} \prod_{n=1}^{\infty} \sum_{k=0}^{\infty} q^{nk} = \sum_{N=0}^{\infty} p(N) q^N. \quad (8.42)$$

For $h = h_{rs}$, we shall use the short-hand notation for Verma modules $\mathcal{V}_{rs} := \mathcal{V}_{h_{rs}}$.

The primary operator ϕ_{rs} has null vectors of dimensions

$$h_{rs} + rs = h_{p+r,p'-s} = h_{p'-r,q+s}, \quad \text{and} \quad h_{rs} + (p-r)(p'-s) = h_{r,2p'-s} = h_{2p-r,s}.$$

The Kac module \mathcal{K}_{rs} is obtained by quotienting the Verma module \mathcal{V}_{rs} by the two corresponding submodules of null vectors, which are themselves isomorphic to two Verma modules:

$$\mathcal{K}_{rs} = \mathcal{V}_{rs}/(\mathcal{V}_{p+r,p'-s} + \mathcal{V}_{r,2p'-s}). \quad (8.43)$$

It turns out that the submodules $\mathcal{V}_{p+r,p'-s}$ and $\mathcal{V}_{r,2p'-s}$ of \mathcal{V}_{rs} are *not* in direct sum. Let us first identify the submodules of $\mathcal{V}_{p+r,p'-s}$. Since $\phi_{p+r,p'-s}$ is degenerate at level $(p+r, p'-s)$, the representation $\mathcal{V}_{p+r,p'-s}$ admits a submodule isomorphic to $\mathcal{V}_{2p+r,s}$. This same operator $\phi_{p+r,p'-s} \equiv \phi_{p-r,p'+s}$ is also degenerate at level $(p-r, p'+s)$, and thus it has a submodule $\mathcal{V}_{r,2p'+s}$. With the same reasoning, we find that $\mathcal{V}_{r,2p'-s}$ also has $\mathcal{V}_{2p+r,s}$ and $\mathcal{V}_{r,2p'+s}$ as submodules. Hence, the intersection is given by

$$\mathcal{V}_{p+r,q-s} \cap \mathcal{V}_{r,2p'-s} = \mathcal{V}_{2p+r,s} + \mathcal{V}_{r,2p'+s}, \quad (8.44)$$

and the character of the Kac module is

$$\chi_{rs}(q) := \mathrm{Tr}_{\mathcal{K}_{rs}}(q^{L_0-c/24}) = (\mathrm{Tr}_{\mathcal{V}_{rs}} - \mathrm{Tr}_{\mathcal{V}_{p+r,p'-s} + \mathcal{V}_{r,2p'-s}})(q^{L_0-c/24}) \quad (8.45)$$

$$= [\mathrm{Tr}_{\mathcal{V}_{rs}} - (\mathrm{Tr}_{\mathcal{V}_{p+r,p'-s}} + \mathrm{Tr}_{\mathcal{V}_{r,2p'-s}}) + \mathrm{Tr}_{\mathcal{V}_{2p+r,s} + \mathcal{V}_{r,2p'+s}}](q^{L_0-c/24}) \quad (8.46)$$

This process repeats recursively:

$$\mathcal{V}_{2p+r,s} \cap \mathcal{V}_{r,2p'+s} = \mathcal{V}_{3p+r,p'-s} + \mathcal{V}_{r,4p'-s}, \quad (8.47)$$

and so on. This yields the expression for the Kac character:

$$\chi_{rs}(q) = \frac{q^{(1-c)/24}}{\eta(q)} \left[q^{h_{rs}} + \sum_{k=1}^{\infty} (-1)^k (q^{h_{kp+r,s_k}} + q^{h_{r,kp'+s_k}}) \right], \quad (8.48)$$

where

$$s_k := \begin{cases} s & \text{for } k \text{ even,} \\ p' - s & \text{for } k \text{ odd.} \end{cases}$$

Using the notation

$$\Delta_{rs} := h_{rs} + \frac{1-c}{24} = \frac{(p'r - ps)^2}{4pp'}, \quad (8.49)$$

we can write

$$\chi_{rs}(q) = \frac{1}{\eta(q)} \left[q^{\Delta_{rs}} + \sum_{n=1}^{\infty} (q^{\Delta_{2np+r,s}} + q^{\Delta_{r,2np'+s}}) \right] - \frac{1}{\eta(q)} \sum_{n=1}^{\infty} (q^{\Delta_{(2n-1)p+r,p'-s}} + q^{\Delta_{r,2np'-s}}), \quad (8.50)$$

which can be re-arranged as the difference:

$$\chi_{rs}(q) = K_{rs}(q) - K_{r,-s}(q), \quad \text{where} \quad K_{rs}(q) := \frac{1}{\eta(q)} \sum_{n \in \mathbb{Z}} q^{(p'r - ps + 2npp')^2 / 4pp'}. \quad (8.51)$$

Lecture 9

Conformal blocks

9.1 Conformal block decompositions

In this lecture, we shall study the structure of four-point functions of primary operators. Recall that, for any quadruplet (z_1, z_2, z_3, z_4) , the Moebius transformation

$$z \mapsto \frac{z_2 - z_1}{z_2 - z_4} \times \frac{z - z_4}{z - z_1} \quad (9.1)$$

maps the points z_i as

$$(z_1, z_2, z_3, z_4) \mapsto (\infty, 1, z, 0), \quad \text{where } z = \frac{(z_2 - z_1)(z_3 - z_4)}{(z_2 - z_4)(z_3 - z_1)}. \quad (9.2)$$

Hence, we shall consider the following correlation function, with $|z| < 1$:

$$G(z, \bar{z}) = \langle \phi_1(\infty, \infty) \phi_2(1, 1) \phi_3(z, \bar{z}) \phi_4(0, 0) \rangle, \quad (9.3)$$

where ϕ_1, \dots, ϕ_4 are primary operators.

We can write, in Heisenberg formalism:

$$G(z, \bar{z}) = \sum_{k, [m], [\bar{m}]} \langle \phi_1 | \phi_2(1, 1) | \phi_k^{[m], [\bar{m}]} \rangle \langle \widehat{\phi}_k^{[m], [\bar{m}]} | \phi_3(z, \bar{z}) | \phi_4 \rangle. \quad (9.4)$$

Recall from the previous lectures that we have:

$$\langle \phi_k^{[m], [\bar{m}]} | \phi_2(1, 1) | \phi_1 \rangle = C_{21}^k b_{21}^k([m]) \bar{b}_{21}^k([\bar{m}]), \quad (9.5)$$

$$\langle \widehat{\phi}_k^{[m], [\bar{m}]} | \phi_3(z, \bar{z}) | \phi_4 \rangle = C_{34}^k \beta_{34}^k([m]) \bar{\beta}_{34}^k([\bar{m}]) z^{-h_{34}^k + |m|} \bar{z}^{-\bar{h}_{34}^k + |\bar{m}|}. \quad (9.6)$$

This gives:

$$\begin{aligned} G(z, \bar{z}) &= \sum_{k, [m], [\bar{m}]} (C_{21}^k)^* C_{34}^k (b_{21}^k([m]))^* \beta_{34}^k([m]) z^{-h_{34}^k + |m|} \\ &\quad \times (\bar{b}_{21}^k([\bar{m}]))^* \bar{\beta}_{34}^k([\bar{m}]) \bar{z}^{-\bar{h}_{34}^k + |\bar{m}|}, \end{aligned} \quad (9.7)$$

and hence the decomposition:

$$G(z, \bar{z}) = \sum_k (C_{21}^k)^* C_{34}^k \mathcal{F}_{12}^{34}(k|z) \bar{\mathcal{F}}_{12}^{34}(k|\bar{z}), \quad (9.8)$$

where

$$\mathcal{F}_{12}^{34}(k|z) := \sum_{[m]} (b_{21}^k([m]))^* \beta_{34}^k([m]) z^{-h_{34}^k + |m|}, \quad (9.9)$$

$$\bar{\mathcal{F}}_{12}^{34}(k|\bar{z}) := \sum_{[\bar{m}]} (\bar{b}_{21}^k([\bar{m}]))^* \bar{\beta}_{34}^k([\bar{m}]) \bar{z}^{-\bar{h}_{34}^k + |\bar{m}|}. \quad (9.10)$$

These functions, called the conformal blocks, are given by the power sums, where the coefficients are determined by the factors from descendant structure constants:

$$\begin{aligned} \mathcal{F}_{12}^{34}(k|z) &:= z^{-h_{34}^k} \sum_{n=0}^{\infty} a_n z^n, & a_n &:= \sum_{[m], |m|=n} (b_{21}^k([m]))^* \beta_{34}^k([m]), \\ \bar{\mathcal{F}}_{12}^{34}(k|\bar{z}) &:= \bar{z}^{-\bar{h}_{34}^k} \sum_{n=0}^{\infty} \bar{a}_n \bar{z}^n, & \bar{a}_n &:= \sum_{[m], |m|=n} (\bar{b}_{21}^k([m]))^* \bar{\beta}_{34}^k([m]). \end{aligned} \quad (9.11)$$

Using the invariance under the conformal map $u \mapsto 1 - u$, we have, for $|1 - z| < 1$:

$$G(z, \bar{z}) = \langle \phi_1(\infty, \infty) \phi_2(0, 0) \phi_3(1 - z, 1 - \bar{z}) \phi_4(1, 1) \rangle \quad (9.12)$$

$$= \langle \phi_1 | \phi_4(1, 1) \phi_3(1 - z, 1 - \bar{z}) | \phi_2 \rangle, \quad (9.13)$$

which yields the decomposition:

$$G(z, \bar{z}) = \sum_k (C_{41}^k)^* C_{32}^k \mathcal{F}_{14}^{32}(k, 1 - z) \bar{\mathcal{F}}_{14}^{32}(k, 1 - \bar{z}). \quad (9.14)$$

Similarly, if we use the map $u \mapsto 1/u$, we get for $|z| > 1$:

$$G(z, \bar{z}) = z^{-2h_3} \bar{z}^{-2\bar{h}_3} \langle \phi_1(0, 0) \phi_2(1, 1) \phi_3(1/z, 1/\bar{z}) \phi_4(\infty, \infty) \rangle \quad (9.15)$$

$$= z^{-2h_3} \bar{z}^{-2\bar{h}_3} \langle \phi_4 | \phi_2(1, 1) \phi_3(1/z, 1/\bar{z}) | \phi_1 \rangle, \quad (9.16)$$

which yields the decomposition:

$$G(z, \bar{z}) = z^{-2h_3} \bar{z}^{-2\bar{h}_3} \sum_k (C_{24}^k)^* C_{31}^k \mathcal{F}_{42}^{31}(k, 1/z) \bar{\mathcal{F}}_{42}^{31}(k, 1/\bar{z}). \quad (9.17)$$

At this point, it is convenient to introduce the *chiral* primary operators $\phi_i(z)$, acting on representations of a single Virasoro algebra, and defined by:

$$\phi_i(z)|h_j\rangle := \sum_{k,[m]} N_{ij}^k \beta_{ij}^k([m]) z^{-h_{ij} + |m|} |h_k^{[m]}\rangle, \quad (9.18)$$

$$[L_n, \phi_i(z)] := (n+1) h_i z^n \phi_i(z) + z^{n+1} \partial \phi_i(z). \quad (9.19)$$

From the above commutation relation, we get easily the chiral OPE:

$$T(w)\phi_i(z) = \frac{h_i \phi_i(z)}{(w-z)^2} + \frac{\partial \phi_i(z)}{w-z} + \text{reg}_{w \rightarrow z}. \quad (9.20)$$

A similar definition for $\bar{\phi}_i(\bar{z})$ can be introduced. Moreover, the orthogonal projector onto the module \mathcal{W}_{h_k} is given by

$$\mathcal{P}_k := \sum_{[m]} |h_k^{[m]}\rangle\langle\widehat{h}_k^{[m]}|, \quad [L_n, \mathcal{P}_k] = 0. \quad (9.21)$$

The conformal blocks may then be written as

$$\mathcal{F}_{12}^{34}(k|z) = \sum_{[m]} \langle h_1 | \phi_2(1) h_k^{[m]} \rangle \langle \widehat{h}_k^{[m]} | \phi_3(z) | h_4 \rangle = \langle h_1 | \phi_2(1) \mathcal{P}_k \phi_3(z) | h_4 \rangle,$$

$$\bar{\mathcal{F}}_{12}^{34}(k|\bar{z}) = \sum_{[\bar{m}]} \langle \bar{h}_1 | \bar{\phi}_2(1) \bar{h}_k^{[\bar{m}]} \rangle \langle \widehat{\bar{h}}_k^{[\bar{m}]} | \bar{\phi}_3(\bar{z}) | \bar{h}_4 \rangle = \langle \bar{h}_1 | \bar{\phi}_2(1) \bar{\mathcal{P}}_k \bar{\phi}_3(\bar{z}) | \bar{h}_4 \rangle.$$

9.2 Conformal blocks of degenerate operators

Reminder. Recall that a primary operator ϕ is degenerate at level N if it admits a level- N descendant χ which is itself primary. The associated state $|\chi\rangle$ is orthogonal to the full Hilbert space, and hence, in any correlation function we have:

$$\chi(z, \bar{z}) = [(\#L_{-N} + \dots + \#L_{-1}^N) \cdot \phi](z, \bar{z}) \equiv 0, \quad (9.23)$$

which is called the null-vector equation for ϕ . As we have seen in the previous lectures, the possible dimensions of degenerate operators at level N are given, in terms of the central charge, by the Kac parameterisation:

$$c = 1 - 6(b^{-1} - b)^2, \quad h_{rs} = \frac{(rb^{-1} - sb)^2 - (b^{-1} - b)^2}{4}, \quad (9.24)$$

with $r, s \geq 1$ positive integers and $N = rs$. The corresponding scalar primary operators are denoted $\phi_{rs}(z, \bar{z})$. The degenerate operators ϕ_{1n} obey the fusion rules¹:

$$\phi_{1n} \times \phi_h \rightarrow \phi_{h'_1} + \dots + \phi_{h'_n}, \quad (9.25)$$

$$\phi_{1m} \times \phi_{1n} \rightarrow \phi_{1,|m-n|+1} + \phi_{1,|m-n|+3} + \dots + \phi_{1,m+n-1}, \quad (9.26)$$

where, in the parameterisation $h_\alpha = \alpha[\alpha - (b^{-1} - b)]$, one has

$$h = h_\alpha, \quad h'_k = h_{\alpha+(2k-n-1)b/2}. \quad (9.27)$$

In particular, when n is odd, $n = 2\ell - 1$, we get $h'_\ell = h$, and hence, we have $\langle \phi_{1,2\ell-1} \phi_h \phi_h \rangle \neq 0$. Thus we expect, for the fusion of a generic primary operator with itself:

$$\phi_h \times \phi_h \rightarrow \mathbf{1} + \phi_{13} + \phi_{15} + \dots \quad (9.28)$$

Let us consider the specific example of the four-point function:

$$G(z, \bar{z}) = \langle \phi_h(\infty, \infty) \phi_{1N}(1, 1) \phi_{1N}(z, \bar{z}) \phi_h(0, 0) \rangle, \quad (9.29)$$

¹These fusion rules are valid for a generic central charge, *i.e.* when b^2 is *not* a rational. When b^2 is rational, the fusion rules may be truncated.

where ϕ_h is a scalar primary operator with conformal dimensions (h, h) , and h is some generic (non-degenerate) conformal dimension. In this correlation function, the limit $z \rightarrow 0$ corresponds to the fusion $\phi_{1N} \times \phi_h$. Hence, using the chiral operators, we can define N independent conformal blocks

$$\mathcal{I}_k(z) = \sum_{[m]} \langle h | \phi_{1N}(1) | h'_{k'}^{[m]} \rangle \langle \hat{h}_{k'}^{[m]} | \phi_{1N}(z) | h \rangle = z^{-h_{1N}-h+h'_k} \sum_{n=0} a_n z^n. \quad (9.30)$$

By construction, the leading term of $\mathcal{I}_k(z)$ is $a_0 = 1$. We have the decomposition:

$$G(z, \bar{z}) = \sum_{k=1}^N (C_{\phi_{1N}, \phi_h}^{\phi_{h'_k}})^2 |\mathcal{I}_k(z)|^2. \quad (9.31)$$

The limit $z \rightarrow 1$ corresponds to the fusion $\phi_{1N} \times \phi_{1N}$, and hence we can define the conformal blocks

$$\mathcal{J}_k(z) = \sum_{[m]} \langle h | \phi_h(1) | h_{1,2k-1}^{[m]} \rangle \langle \hat{h}_{1,2k-1}^{[m]} | \phi_{1N}(1-z) | h_{1N} \rangle \quad (9.32)$$

$$= (1-z)^{-2h_{1N}+h_{1,2k-1}} \sum_{n=0}^{\infty} b_n (1-z)^n. \quad (9.33)$$

We have the decomposition:

$$G(z, \bar{z}) = \sum_{k=1}^N C_{\phi_{1N}, \phi_{1N}}^{\phi_{1,2k-1}} C_{\phi_h, \phi_h}^{\phi_{1,2k-1}} |\mathcal{J}_k(z)|^2. \quad (9.34)$$

In the limit $z \rightarrow \infty$, we can write

$$G(z, \bar{z}) = \sum_{k=1}^N (C_{\phi_{1N}, \phi_h}^{\phi_{h'_k}})^2 |\mathcal{K}_k(z)|^2, \quad \mathcal{K}_k(z) := z^{-2h_{1N}} \mathcal{I}_k(1/z). \quad (9.35)$$

The identification of the three above conformal block decompositions of the physical correlation function $G(z, \bar{z})$ is called the *crossing symmetry* condition. To exploit this condition in a meaningful way, we shall characterise the conformal blocks families $\{\mathcal{I}_k(z)\}$, $\{\mathcal{J}_k(z)\}$ and $\{\mathcal{K}_k(z)\}$ as three bases of holomorphic solutions of the same differential equation.

9.3 Ordinary differential equations

9.3.1 Action of the L_n 's inside a four-point function

Let us derive the action of Virasoro operators when inserted into a four-point conformal block. We start with the conformal block:

$$\mathcal{F}_k(t, z) := \langle h_1 | \phi_2(t) \mathcal{P}_k \phi_3(z) | h_4 \rangle, \quad (9.36)$$

where $|t| > |z|$, and ϕ_2, ϕ_3 are chiral primary operators. When acting on $\phi_3(z)$ with L_{-n} , where $n \geq 1$, we get:

$$\mathcal{F}_k^{(-n)}(t, z) = \langle h_1 | \phi_2(t) \mathcal{P}_k (L_{-n} \phi_3)(z) | h_4 \rangle \quad (9.37)$$

$$= \frac{1}{2i\pi} \oint_{C_z} dw (w-z)^{-n+1} \langle h_1 | \phi_2(t) \mathcal{P}_k \mathcal{R}[T(w) \phi_3(z)] | h_4 \rangle \quad (9.38)$$

$$= \frac{1}{2i\pi} \left(\oint_{C_\infty} - \oint_{C_t} - \oint_{C_0} \right) dw (w-z)^{-n+1} \langle h_1 | \mathcal{R}[T(w) \phi_2(t) (\mathcal{P}_k \phi_3)(z)] | h_4 \rangle. \quad (9.39)$$

- On the contour C_∞ , we can let $w \rightarrow \infty$, so that

$$\langle h_1 | T(w) = \sum_{n \geq 0} w^{-n-2} \langle h_1 | L_n \underset{w \rightarrow \infty}{\sim} h_1 w^{-2} \langle h_1 | , \quad (9.40)$$

and hence the integral over C_∞ vanishes for $n \geq 1$.

- On the contour C_t , one can use the OPE

$$T(w)\phi_h(t) = \frac{h\phi_h(t)}{(w-t)^2} + \frac{\partial\phi_h(z)}{w-t} + \text{reg}_{w \rightarrow t}. \quad (9.41)$$

The corresponding residue is

$$[(-n+1)(t-z)^{-n} + (t-z)^{-n+1}\partial_t] \mathcal{F}_k(t, z) \quad (9.42)$$

- On the contour C_0 , we let $w \rightarrow 0$, and we have

$$T(w)|h_4\rangle = \sum_{n \leq 0} w^{-n-2} L_n |h_4\rangle = (w^{-2} h_4 + w^{-1} L_{-1}) |h_4\rangle + \text{reg}_{w \rightarrow 0}. \quad (9.43)$$

The corresponding residue is

$$(-n+1)(-z)^{-n} h_4 \mathcal{F}_k(t, z) + (-z)^{-n+1} \langle h_1 | \phi_2(t) \mathcal{P}_k \phi_3(z) L_{-1} | h_4 \rangle. \quad (9.44)$$

As a result, we get

$$\begin{aligned} \mathcal{F}_k^{(-n)}(t, z) &= \left\{ (n-1)[h_2(t-z)^{-n} + h_4(-z)^{-n}] - (t-z)^{-n+1} \partial_t \right\} \mathcal{F}_k(t, z) \\ &\quad - (-z)^{-n+1} \langle h_1 | \phi_2(t) \mathcal{P}_k \phi_3(z) L_{-1} | h_4 \rangle. \end{aligned} \quad (9.45)$$

To obtain an explicit result, we need to express $\langle h_1 | \phi_2(t) \mathcal{P}_k \phi_3(z) L_{-1} | h_4 \rangle$ and $\partial_t \mathcal{F}_k(t, z)$ in terms of $\mathcal{F}_k(t, z)$ and $\partial_z \mathcal{F}_k(t, z)$. To do this, we write:

$$\begin{aligned} &\langle h_1 | L_m \phi_2(t) \mathcal{P}_k \phi_3(z) | h_4 \rangle - \langle h_1 | \phi_2(t) \mathcal{P}_k \phi_3(z) L_m | h_4 \rangle \\ &\quad = \langle h_1 | [L_m, \phi_2(t)] \mathcal{P}_k \phi_3(z) | h_4 \rangle + \langle h_1 | \phi_2(t) \mathcal{P}_k [L_m, \phi_3(z)] | h_4 \rangle, \end{aligned} \quad (9.46)$$

and using the commutation relations

$$[L_m, \phi_h(z)] = [(m+1)h z^m + z^{m+1} \partial_z] \phi_h(z), \quad (9.47)$$

we get, for $m = 0$ and $m = -1$:

$$(h_1 - h_4) \mathcal{F}_k(t, z) = (h_2 + h_3 + t \partial_t + z \partial_z) \mathcal{F}_k(t, z), \quad (9.48)$$

$$- \langle h_1 | \phi_2(t) \mathcal{P}_k \phi_3(z) L_{-1} | h_4 \rangle = (\partial_t + \partial_z) \mathcal{F}_k(t, z). \quad (9.49)$$

After combining these results and setting $t = 1$, we get

$$\begin{aligned} \langle h_1 | \phi_2(1) \mathcal{P}_k (L_{-n} \phi_3)(z) | h_1 \rangle &= \mathcal{L}_{-n}(h_1, \dots, h_4) \cdot \langle h_1 | \phi_2(1) \mathcal{P}_k \phi_3(z) | h_1 \rangle, \\ \mathcal{L}_{-n}(h_1, \dots, h_4) &:= \frac{\alpha_{n-1}(z)}{z^{n-1}(1-z)^{n-1}} \partial_z + \frac{\beta_n(z)}{z^n(1-z)^n}, \end{aligned} \quad (9.50)$$

where

$$\boxed{\begin{aligned}\alpha_0(z) &= 1, \\ \alpha_n(z) &= z^{n+1} - (z-1)^{n+1} = \sum_{k=0}^n z^k (z-1)^{n-k}, \\ \beta_n(z) &= (n-1)[h_2 z^n + h_4 (z-1)^n] + (h_1 - h_2 - h_3 - h_4)z(z-1)\alpha_{n-2}(z).\end{aligned}} \quad (9.51)$$

Note that α_n, β_n are polynomials of degree n [including $\alpha_{-1}(z) := 0$]. Here is an important property of the operator \mathcal{L}_{-n} under the change of parameters associated to the conformal mapping $z \mapsto 1-z$:

$$\boxed{\begin{aligned}(h_2, h_4, z, \partial_z) &\rightarrow (h_4, h_2, 1-z, -\partial_z) \\ \mathcal{L}_{-n} &\rightarrow (-1)^n \mathcal{L}_{-n}.\end{aligned}} \quad (9.52)$$

Finally, to compute the iterated action of the L_{-n} 's inside a conformal block, we need to replace the primary chiral operator $\phi_3(z)$ by a descendant $\phi_3^{[m]}(z)$. From the basic commutation relations $[L_m, L_n]$ and $[L_m, \phi_h(z)]$, we easily obtain

$$[L_0, \phi_3^{[n]}(z)] = (h_3 + |M| + z\partial_z)\phi_3^{[m]}(z), \quad [L_{-1}, \phi_3^{[n]}(z)] = \partial_z\phi_3^{[m]}(z). \quad (9.53)$$

Hence, when $\phi_3(z)$ is replaced by $\phi_3^{[m]}(z)$ in the above derivation, the only change is $h_3 \rightarrow h_3 + |m|$. As a result, we obtain:

$$\boxed{\begin{aligned}&\langle h_1 | \phi_2(1) \mathcal{P}_k(L_{-n_1} \cdots L_{-n_p} \phi_3)(z) | h_1 \rangle \\ &= \mathcal{L}_{-n_1}^{(N_1)} \cdot \mathcal{L}_{-n_2}^{(N_2)} \cdots \mathcal{L}_{-n_p}^{(N_p)} \cdot \langle h_1 | \phi_2(1) \mathcal{P}_k \phi_3(z) | h_1 \rangle, \\ &N_\ell := n_{\ell+1} + n_{\ell+2} + \cdots + n_p, \\ &\mathcal{L}_{-n}^{(N)} := \mathcal{L}_{-n}(h_1, h_2, h_3 + N, h_4).\end{aligned}}$$

For instance, for $k = 2$ we have

$$\mathcal{L}_{-n_1}^{(n_2)} \cdot \mathcal{L}_{-n_2} = \frac{1}{z^{n_1+n_2-2} (1-z)^{n_1+n_2-2}} \left[A_{n_1, n_2}(z) \partial_z^2 + \frac{B_{n_1, n_2}(z)}{z(1-z)} \partial_z + \frac{C_{n_1, n_2}(z)}{z^2(1-z)^2} \right],$$

where $A_{n_1, n_2}, B_{n_1, n_2}, C_{n_1, n_2}$ are polynomials of degree n in z :

$$A_{n_1, n_2}(z) = \alpha_{n_1-1}(z) \alpha_{n_2-1}(z), \quad (9.54)$$

$$\begin{aligned}B_{n_1, n_2}(z) &= [\beta_{n_1}(z) + n_2 z(1-z) \alpha_{n_1-2}(z)] \alpha_{n_2-1}(z) + \alpha_{n_1-1}(z) \beta_{n_2}(z) \\ &\quad + \alpha_{n_1-1}(z) [z(1-z) \alpha'_{n_2-1}(z) - (n_2-1)(1-2z) \alpha_{n_2-1}(z)],\end{aligned} \quad (9.55)$$

$$\begin{aligned}C_{n_1, n_2}(z) &= [\beta_{n_1}(z) + n_2 z(1-z) \alpha_{n_1-2}(z)] \beta_{n_2}(z) \\ &\quad + \alpha_{n_1-1}(z) [z(1-z) \beta'_{n_2}(z) - n_2(1-2z) \beta_{n_2}(z)].\end{aligned} \quad (9.56)$$

Of course, the above derivation is well adapted to the situation of conformal blocks, but it also holds for a physical four-point function:

$$\boxed{\begin{aligned}&\langle \phi_1(\infty, \infty) \phi_2(1, 1) (L_{-n_1} \cdots L_{-n_p} \cdot \phi_3)(z, \bar{z}) \phi_1(0, 0) \rangle \\ &= \mathcal{L}_{-n_1}^{(N_1)} \cdot \mathcal{L}_{-n_2}^{(N_2)} \cdots \mathcal{L}_{-n_p}^{(N_p)} \langle \phi_1(\infty, \infty) \phi_2(1, 1) \phi_3(z, \bar{z}) \phi_1(0, 0) \rangle,\end{aligned}}$$

with the same expression for the operator $\mathcal{L}_{-n}^{(N)}$.

9.3.2 Differential equation from the null-vector condition

We get back to our four-point function with degenerate operators:

$$G(z, \bar{z}) = \langle \phi_h(\infty, \infty) \phi_{1N}(1, 1) \phi_{1N}(z, \bar{z}) \phi_h(0, 0) \rangle. \quad (9.57)$$

From the holomorphic and anti-holomorphic null-vector equations for ϕ_{1N} , the correlation function G obeys the ordinary differential equations (ODEs) of order N :

$$\left[\# \mathcal{L}_{-N} + \dots + \# \mathcal{L}_{-1}^{(N-1)} \dots \mathcal{L}_{-1}^{(0)} \right] G(z, \bar{z}) = 0, \quad (9.58)$$

$$\left[\# \bar{\mathcal{L}}_{-N} + \dots + \# \bar{\mathcal{L}}_{-1}^{(N-1)} \dots \bar{\mathcal{L}}_{-1}^{(0)} \right] G(z, \bar{z}) = 0 \quad (9.59)$$

From the explicit form of the operators $\mathcal{L}_{-n}^{(N)}$, these equations are of the form:

$$\begin{aligned} & \left[\partial_z^N + \frac{p_1(z)}{z(1-z)} \partial_z^{N-1} + \dots + \frac{p_N(z)}{z^N(1-z)^N} \right] G(z, \bar{z}) = 0, \\ & \left[\partial_{\bar{z}}^N + \frac{p_1(\bar{z})}{\bar{z}(1-\bar{z})} \partial_{\bar{z}}^{N-1} + \dots + \frac{p_N(\bar{z})}{\bar{z}^N(1-\bar{z})^N} \right] G(z, \bar{z}) = 0, \end{aligned} \quad (9.60)$$

where $p_j(z)$ is a polynomial of degree j or smaller.

The conformal blocks $\{\mathcal{I}_k(z)\}$ are solutions of the same holomorphic ODE. Due to the symmetry of the \mathcal{L}_{-n} 's under $z \rightarrow 1-z$, the blocks $\{\mathcal{J}_\ell(z)\}$ are solutions of the same ODE. Hence, $\{\mathcal{I}_k(z)\}$ and $\{\mathcal{J}_\ell(z)\}$ are two bases of solutions of the holomorphic ODE, which are holomorphic on $\mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$. Hence, any $\mathcal{I}_k(z)$ can be expressed as a linear combination of the $\mathcal{J}_\ell(z)$'s. This change of basis is encoded in the *fusion matrix* P :

$$\mathcal{I}_k(z) = \sum_{\ell=1}^N P_{k\ell} \mathcal{J}_\ell(z). \quad (9.61)$$

As we shall see in the following, it is quite straightforward to obtain the power series for the conformal blocks \mathcal{I}_k and \mathcal{J}_ℓ from the ODE. In contrast, except for special cases, it is very non-trivial to obtain the fusion matrix between the two bases of conformal blocks.

Lecture 10

The conformal bootstrap

10.1 Introduction to Fuchsian differential equations

10.1.1 Regular singularities

At $z = 0$, the term $\partial_z^{N-j}G$ has a pole of order at most j : we say that $z = 0$ is a *regular singularity* of the ODE. Similarly, the point $z = 1$ is a regular singularity.

To study the solutions of (9.60) in the vicinity of a singular point w , it will be convenient to express the ODE in terms of the differential operator

$$\theta_w := (z - w)\partial_z,$$

with the property

$$(z - w)^k \partial_z^k = \theta_w(\theta_w - 1) \dots (\theta_w - k + 1). \quad (10.1)$$

For instance, at $w = 0$, after multiplying the ODE by z^N , we get an equation of the form:

$$[\theta^N + q_1(z)\theta^{N-1} + \dots + q_{N-1}(z)\theta + q_N(z)]\mathcal{G} = 0, \quad (10.2)$$

where we have denoted simply $\theta := \theta_0$, and the functions $q_j(z)$ are linear combinations of $p_\ell(z)/(1-z)^\ell$, and thus, they are regular as $z \rightarrow 0$:

$$q_j(z) := \sum_{\ell=1}^N \frac{\alpha_{j\ell} p_\ell(z)}{(1-z)^\ell}, \quad p_\ell(z) = \sum_{n=0}^j p_{\ell,n} z^n \quad (10.3)$$

For $w = \infty$, we use the variable $u = 1/z$, and define $\widehat{\theta} := u\partial_u = -z\partial_z$. Equation (10.2) takes the form:

$$[\widehat{\theta}^N + \widehat{q}_1(u)\widehat{\theta}^{N-1} + \dots + \widehat{q}_{N-1}(u)\widehat{\theta} + \widehat{q}_N(u)]\mathcal{G} = 0, \quad (10.4)$$

where

$$\widehat{q}_j(u) := \sum_{\ell=1}^N \frac{\alpha_{j\ell} \widehat{p}_\ell(u)}{(1-u)^\ell}, \quad \widehat{p}_\ell(u) = \sum_{n=0}^{\ell} p_{\ell,\ell-n} u^n. \quad (10.5)$$

Since the coefficients $\widehat{q}_j(u)$ are regular as $u \rightarrow 0$, the point $z = \infty$ is also a regular singularity.

Hence the three singular points $\{0, 1, \infty\}$ of the ODE (9.60) are *regular* singularities: this type of complex differential equation is called a *Fuchsian differential equation*.

10.1.2 Basis of solutions in the vicinity of a singularity

Let us look for solutions which behave as a power law in the z variable in the vicinity of the singularity $z = 0$. Inserting $\mathcal{G}(z, \bar{z}) \sim z^\lambda$ into the ODE (10.2), and using $\theta.z^\lambda = \lambda z^\lambda$, we find that the exponent λ should be a solution of the characteristic equation:

$$R(\lambda) := \lambda^N + \sum_{j=0}^{N-1} q_{N-j}(0) \lambda^j = 0. \quad (10.6)$$

The roots of R are called the local exponents of the ODE at $z = 0$. Now we consider a local exponent λ , and we shall construct a solution of the form

$$z^\lambda f(z) = z^\lambda \sum_{n=0}^{\infty} a_n z^n, \quad a_0 = 1, \quad (10.7)$$

for $0 < |z| < \rho$, and $-\pi < \text{Arg } z < \pi$, where ρ is the radius of convergence of the solution ($\rho \leq 1$, because $z = 1$ is a singular point). Let us write the series expansion of the coefficients q_j as

$$q_j(z) = \sum_{n=1}^{\infty} q_{j,n} z^n \quad \text{for } |z| < 1. \quad (10.8)$$

Then, inserting $\mathcal{G}(z) = z^\lambda f_\lambda(z)$ into (10.2), we get

$$\sum_{n=0}^{\infty} \left[(\lambda + n)^N a_n + \sum_{j=0}^{N-1} \sum_{k=0}^n (\lambda + k)^j q_{N-j,n-k} a_k \right] z^{\lambda+n} = 0. \quad (10.9)$$

Introducing the polynomials

$$Q_k(\lambda) := \sum_{j=0}^{N-1} q_{N-j,k} \lambda^j \quad \text{for } k \geq 1,$$

This can be rewritten as

$$z^\lambda \sum_{n=0}^{\infty} \left[R(\lambda + n) a_n + \sum_{k=0}^{n-1} Q_{n-k}(\lambda + k) a_k \right] z^n = 0. \quad (10.10)$$

For any n , the coefficients a_0, \dots, a_n of the function $f_\lambda(z)$ are determined by an $(n+1) \times (n+1)$ lower-triangular linear system:

$$\begin{aligned} R(\lambda) a_0 &= 0 \\ Q_1(\lambda) a_0 + R(\lambda + 1) a_1 &= 0 \\ &\vdots \\ Q_n(\lambda) a_0 + \cdots + Q_1(\lambda + n - 1) a_{n-1} + R(\lambda + n) a_n &= 0 \end{aligned} \quad (10.11)$$

Since $R(\lambda) = 0$, the leading coefficient can always be set to $a_0 = 1$. Let us consider the “generic” case, when the roots of R do not differ by an integer, so that, for any integer $n \geq 1$, $R(\lambda + n) \neq 0$. The entire function $f_\lambda(z)$ is then defined uniquely by the recursion relation:

$$a_0 = 0, \quad a_n = -\frac{1}{R(\lambda + n)} \sum_{k=0}^{n-1} Q_{n-k}(\lambda + n - k) a_k \quad \text{for } n \geq 1. \quad (10.12)$$

From this procedure, called the *Frobenius method*, we get a basis of solutions to the ODE:

$$\{z^{\lambda_1} f_1(z), z^{\lambda_2} f_2(z), \dots, z^{\lambda_N} f_N(z)\}, \quad (10.13)$$

where $\lambda_1, \dots, \lambda_N$ are the roots of the characteristic equation. It is possible to show that each entire function $f_j(z)$ admits an analytic continuation on $\mathbb{C} \setminus \{0, 1, \infty\}$.

10.1.3 The case of degenerate exponents

If R admits two roots of the form $(\lambda_1, \lambda_2) = (\lambda, \lambda + r)$ with $r \in \mathbb{N}$, we say that the local exponent λ is degenerate. Let us first treat the case $r \geq 1$. The recursion system (10.11) for λ_2 has a unique solution, which defines the function $z^{\lambda+r} f_2(z)$, with $f_2(0) = 1$. For the solution associated to λ_1 , after setting $a_0 = 1$, the coefficients a_1, \dots, a_{r-1} are fixed uniquely, but a_r can be set to any value $a_r = \alpha$, which defines the one-parameter family of solutions $z^\lambda f_1^{(\alpha)}(z)$, and we have $f_1^{(\alpha)}(z) = f_1^{(0)}(z) + \alpha z^r f_2(z)$. For any value of α , the functions $z^\lambda f_1^{(\alpha)}(z)$ and $z^{\lambda+r} f_2(z)$ are independent solutions of the form (10.7). In the case of a multiply-degenerate exponent λ , where the roots of R are of the form

$$(\lambda_1, \dots, \lambda_m) = (\lambda, \lambda + r_1, \dots, \lambda + r_{m-1}), \quad (10.14)$$

with $0 < r_1 < r_2 < \dots < r_{m-1}$, using the same method, one can construct a set of m independent solutions

$$\{z^\lambda f_1(z), z^{\lambda+r_1} f_2(z), \dots, z^{\lambda+r_{m-1}} f_m(z)\} \quad (10.15)$$

with $f_j(0) = 1$. The function f_m is unique, whereas, for the entire function f_j associated to $\lambda + r_j$ with $1 \leq j < m-1$, we can choose freely the values of the coefficients a_j, \dots, a_{m-1} .

In the case of a double root $\lambda_1 = \lambda_2 = \lambda$, if no other root is of the form $\lambda + \mathbb{Z}$, the recursion system (10.11) gives a unique solution $z^\lambda f_\lambda(z)$. To build intuitively a second, independent solution, it is useful to think of this situation as the limit $\epsilon \rightarrow 0$ of an ODE with exponents $\lambda, \lambda + \epsilon$. Under this hypothesis, the difference of the two independent solutions $z^\lambda f_\lambda(z), z^{\lambda+\epsilon} f_{\lambda+\epsilon}(z)$ would be of the form:

$$z^{\lambda+\epsilon} \sum_{n=0}^{\infty} a_n(\lambda + \epsilon) z^n - z^\lambda \sum_{n=0}^{\infty} a_n(\lambda) z^n \underset{\epsilon \rightarrow 0}{\sim} \epsilon z^\lambda \left[\ln z \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n \right], \quad (10.16)$$

where $a_n = a_n(\lambda)$ and $b_n = \partial_\lambda a_n(\lambda)$. Hence, we look for a solution of the form

$$z^\lambda [g_\lambda(z) + \ln z f_\lambda(z)], \quad \text{where } g_\lambda(z) = \sum_{n=0}^{\infty} b_n z^n, \quad (10.17)$$

and $z^\lambda f_\lambda(z)$ is the “regular” solution. Inserting the above form into the ODE, and using

$$\theta^k.(z^\lambda \ln z) = \lambda^k z^\lambda \ln z + k \lambda^{k-1} z^\lambda,$$

we get the relations for $n \geq 0$:

$$R(\lambda + n)a_n + \sum_{k=0}^{n-1} Q_{n-k}(\lambda + k)a_k = 0, \quad (10.18)$$

$$\left[R(\lambda + n)b_n + \sum_{k=0}^{n-1} Q_{n-k}(\lambda + k)b_k \right] + \left[R'(\lambda + n)a_n + \sum_{k=0}^{n-1} Q'_{n-k}(\lambda + k)a_k \right] = 0. \quad (10.19)$$

From the first relation, we recover the regular solution if we set $a_0 = 1$. In the second relation, since $R(\lambda) = R'(\lambda) = 0$, we can set b_0 to any value $b_0 = \alpha$, and this defines uniquely the function $g_\lambda^{(\alpha)}(z) = g_\lambda^{(0)}(z) + \alpha f_\lambda(z)$. For any value of α , the functions

$$\{z^\lambda f_\lambda(z), z^\lambda [g_\lambda^{(0)}(z) + (\alpha + \ln z) f_\lambda(z)]\} \quad (10.20)$$

are independent solutions.

Similarly, for a root λ of multiplicity m , we can construct linearly independent solutions $\{\varphi_1(z), \dots, \varphi_m(z)\}$ of the form

$$\varphi_1(z) = z^\lambda f_1(z), \quad (10.21)$$

$$\varphi_2(z) = z^\lambda [\ln z f_1(z) + f_2(z)], \quad (10.22)$$

$$\dots \quad (10.23)$$

$$\varphi_k(z) = z^\lambda \sum_{r=0}^{k-1} \binom{k-1}{r} (\ln z)^{k-1-r} f_{r+1}(z), \quad (10.24)$$

To describe the determination of the entire functions $f_1(z), \dots, f_m(z)$, let us introduce the polynomial of degree N in t :

$$\mathcal{D}(z, t) = t^N + q_1(z)t^{N-1} + \dots + q_N(z), \quad (10.25)$$

so that the ODE reads $\mathcal{D}(z, \theta)\mathcal{G} = 0$. Using the identity

$$\theta^p.[(\ln z)^r \varphi(z)] = \sum_{\ell=0}^{\min(r,p)} \binom{r}{\ell} (\ln z)^{r-\ell} \frac{p!}{(p-\ell)!} \theta^{p-\ell}. \varphi(z), \quad (10.26)$$

we get

$$\mathcal{D}(z, \theta).[(\ln z)^r \varphi(z)] = \sum_{\ell=0}^r \binom{r}{\ell} (\ln z)^{r-\ell} \mathcal{D}^{(\ell)}(z, \theta).\varphi(z), \quad (10.27)$$

where $\mathcal{D}^{(\ell)}(z, t) := (\partial_t)^\ell \mathcal{D}(z, t)$ is also a polynomial in t . The function $f_1(z)$ is determined by the homogeneous ODE:

$$\mathcal{D}(z, \theta).[z^\lambda f_1(z)] = 0. \quad (10.28)$$

Then, the condition $\mathcal{D}(z, \theta).\varphi_2 = 0$ yields:

$$\mathcal{D}^{(1)}(z, \theta).[z^\lambda f_1(z)] + \mathcal{D}(z, \theta).[z^\lambda f_2(z)] = 0, \quad (10.29)$$

which is an ODE for $f_2(z)$, with a source term given by the knowledge of $f_1(z)$. The functions $f_k(z)$ are determined recursively by the coefficient of $(\ln z)^0$ in the ODE $\mathcal{D}(z, \theta).\varphi_k = 0$, which reads:

$$\sum_{r=0}^{k-1} \binom{k-1}{r} \mathcal{D}^{(r)}(z, \theta).[z^\lambda f_{k-r}(z)] = 0. \quad (10.30)$$

The value of the polynomials at $z = 0$:

$$\mathcal{D}^{(\ell)}(0, \lambda) = R^{(\ell)}(\lambda) = 0 \quad \text{for } \ell = 0, \dots, m-1, \quad (10.31)$$

allow us to set $f_k(0) = 1$. Then, the higher coefficients in the power series for $f_k(z)$ are uniquely determined by the above ODE.

10.1.4 Riemann scheme of the differential equation

For each singular point $w \in \{0, 1, \infty\}$, we can apply the Frobenius method, and construct a basis with local exponents given by the roots of the characteristic polynomial $R_w(\lambda)$.

We denote $\{\lambda_1, \dots, \lambda_N\}$ the local exponents at $z = 0$, $\{\mu_1, \dots, \mu_N\}$ those at $z = 1$, and $\{\nu_1, \dots, \nu_N\}$ those at $z = \infty$. These data are stored in the Riemann scheme:

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \vdots & \vdots & \vdots \\ \lambda_N & \mu_N & \nu_N \end{array}$$

Using the characteristic equations (10.6), one can prove Fuchs' relation:

$$\sum_{j=1}^N (\lambda_j + \mu_j + \nu_j) = \frac{N(N-1)}{2}. \quad (10.32)$$

If we change the unknown function to $g(z)$ with $\mathcal{G}(z) = z^\alpha g(z)$, then from the relation

$$\theta.[z^\alpha \varphi(z)] = z^\alpha (\theta + \alpha). \varphi(z), \quad (10.33)$$

we get

$$\left[(\theta + \alpha)^N + \sum_{j=0}^{N-1} q_{N-j}(z) (\theta + \alpha)^j \right] g(z) = 0, \quad (10.34)$$

and hence, the local exponents for g at $z = 0$ are $\{\lambda_1 - \alpha, \dots, \lambda_N - \alpha\}$, and those at $z = \infty$ are $\{\nu_1 + \alpha, \dots, \nu_N + \alpha\}$. We can similarly introduce a factor $(1-z)^\beta$, to shift the exponents μ_j and ν_j by $-\beta$ and $+\beta$, respectively. Overall, if we make the change of function

$$\mathcal{G}(z) = z^\alpha (1-z)^\beta g(z), \quad (10.35)$$

the new function $g(z)$ satisfies a Fuchsian equation with Riemann scheme

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline \lambda_1 - \alpha & \mu_1 - \beta & \nu_1 + \alpha + \beta \\ \lambda_2 - \alpha & \mu_2 - \beta & \nu_2 + \alpha + \beta \\ \vdots & \vdots & \vdots \\ \lambda_N - \alpha & \mu_N - \beta & \nu_N + \alpha + \beta \end{array}$$

Of course, the degeneracies of exponents, as well as the Fuchs relation are preserved by this operation.

10.1.5 Monodromy of solutions

Let $\{\varphi_1(z), \dots, \varphi_N(z)\}$ be a basis of solutions of the Fuchsian ODE (9.60). Then, for any singular point w , if we apply a rotation of 2π around w , we get the functions $\{\varphi_1(w + e^{2i\pi}(z-w)), \dots, \varphi_N(w + e^{2i\pi}(z-w))\}$, which are again solutions of the ODE. Hence, there exists an invertible matrix $M^{(w)}$ (the monodromy matrix around w) such that

$$\varphi_j(w + e^{2i\pi}(z-w)) = \sum_{k=1}^N M_{jk}^{(w)} \varphi_k(z). \quad (10.36)$$

In the case when the characteristic polynomial $R(\lambda)$ at $z = 0$ has no multiple roots, the monodromy matrix for the basis $\{z^{\lambda_1} f_1(z), \dots, z^{\lambda_N} f_N(z)\}$ constructed above is the diagonal matrix:

$$M^{(0)} = \text{diag}(e^{2i\pi\lambda_1}, \dots, e^{2i\pi\lambda_N}).$$

If $P(\lambda)$ has multiple roots, then $M^{(0)}$ has some additional off-diagonal terms, and is no longer diagonalisable.

Given a basis of solutions $\{\varphi_1(z), \dots, \varphi_N(z)\}$ expressed in terms of series at $z = 0$, it is a harder task to determine its monodromy around $z = 1$. A way to do this, is to use the *fusion matrix* from $\{\varphi_1(z), \dots, \varphi_N(z)\}$ to a basis $\{\psi_1(z), \dots, \psi_N(z)\}$ of solutions defined in the vicinity of $z = 1$:

$$\varphi_j(z) = \sum_{k=1}^N P_{jk} \psi_k(z). \quad (10.37)$$

Then, the monodromy of the basis $\{\varphi_1(z), \dots, \varphi_N(z)\}$ around $z = 1$ is given by:

$$M^{(1)} = P \cdot \text{diag}(e^{2i\pi\mu_1}, \dots, e^{2i\pi\mu_N}) \cdot P^{-1}. \quad (10.38)$$

10.1.6 Example: the hypergeometric equation

Consider a second-order Fuchsian equation with the Riemann scheme:

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & A \\ 1-C & C-A-B & B \end{array}$$

By identifying the coefficients of the characteristic polynomials at $z = 0, 1, \infty$, one can show that the corresponding ODE must be given by the *hypergeometric equation*:

$$\left[\partial_z^2 + \frac{C - (A + B + 1)z}{z(1-z)} \partial_z - \frac{AB}{z(1-z)} \right] g = 0. \quad (10.39)$$

As long as $C \notin \mathbb{Z}$, the Frobenius method at $z = 0$ and $z = 1$ respectively provides the bases of solutions:

$$I_1(z) = {}_2F_1(A, B; C|z), \quad I_2(z) = z^{1-C} {}_2F_1(1+B-C, 1+A-C; 2-C|z), \quad (10.40)$$

$$J_1(z) = {}_2F_1(A, B; 1-D|1-z), \quad J_2(z) = (1-z)^D {}_2F_1(C-A, C-B; 1+D|1-z), \quad (10.41)$$

where $D = C - A - B$. The function ${}_2F_1(A, B; C|z)$ is Gauss's hypergeometric series

$${}_2F_1(A, B; C|z) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{(C)_n n!} z^n, \quad (10.42)$$

where we have used the Pochhammer symbol:

$$(u)_n = u(u+1)\dots(u+n-1) = \frac{\Gamma(u+n)}{\Gamma(u)}. \quad (10.43)$$

Using the Euler Beta function

$$\mathbf{B}(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \quad \text{for } \operatorname{Re}(u), \operatorname{Re}(v) > 0, \quad (10.44)$$

One can write the hypergeometric function as

$${}_2F_1(A, B; C|z) = \frac{1}{\mathbf{B}(B, C-B)} \int_0^1 x^{B-1} (1-x)^{C-B-1} (1-zx)^{-A} dx, \quad (10.45)$$

for $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$ and $z \in \mathbb{C} \setminus [1, +\infty[$. Suppose $z \in]0, 1[$. Using the change of variables $x = 1/t$ and $x = t/z$, respectively, we get

$$I_1(z) = \frac{1}{N_1} \int_1^\infty t^{A-C} (t-z)^{-A} (t-1)^{C-B-1} dt, \quad (10.46)$$

$$I_2(z) = \frac{1}{N_2} \int_0^z t^{A-C} (z-t)^{-A} (1-t)^{C-B-1} dt, \quad (10.47)$$

where $N_1 = \mathbf{B}(B, C - B)$ and $N_2 = \mathbf{B}(1 - A, 1 + A - C)$. Similarly, with the changes $x = 1/(1-t)$ and $x = (1-t)/(1-z)$, we get

$$J_1(z) = \frac{1}{M_1} \int_{-\infty}^0 (-t)^{A-C} (z-t)^{-A} (1-t)^{C-B-1} dt, \quad (10.48)$$

$$J_2(z) = \frac{1}{M_2} \int_z^1 t^{A-C} (t-z)^{-A} (1-t)^{C-B-1} dt, \quad (10.49)$$

where $M_1 = \mathbf{B}(B, 1 + A - C)$ and $M_2 = \mathbf{B}(1 - A, C - B)$.

If we define the holomorphic function

$$f(t) = t^{A-C} (t-z)^{-A} (t-1)^{C-B-1} \quad (10.50)$$

for $t \in \mathbb{C} \setminus]-\infty, 0]$, the closed-contour relations

$$\int_{-\infty+i\epsilon}^{+\infty+i\epsilon} f(t) dt = 0, \quad (10.51)$$

for $\epsilon \rightarrow 0^\pm$ we get the two equations:

$$e^{-i\pi(B+1)} \widehat{J}_1(z) + e^{i\pi(D-1)} \widehat{I}_2(z) + e^{i\pi(C-B-1)} \widehat{J}_2(z) + \widehat{I}_1(z) = 0, \quad (10.52)$$

$$e^{+i\pi(B+1)} \widehat{J}_1(z) + e^{i\pi(1-D)} \widehat{I}_2(z) + e^{i\pi(1-C+B)} \widehat{J}_2(z) + \widehat{I}_1(z) = 0, \quad (10.53)$$

where $\widehat{I}_k = N_k I_k(z)$ and $\widehat{J}_k = M_k J_k(z)$ are the unnormalised integrals. We obtain the relations

$$\widehat{I}_1(z) = \frac{1}{\sin \pi D} [\sin \pi(C-A) \widehat{J}_1(z) - \sin \pi A \widehat{J}_2(z)], \quad (10.54)$$

$$\widehat{I}_2(z) = \frac{1}{\sin \pi D} [\sin \pi B \widehat{J}_1(z) - \sin \pi(C-B) \widehat{J}_2(z)]. \quad (10.55)$$

From the identity $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$, we get the change of basis:

$$\begin{bmatrix} I_1(z) \\ I_2(z) \end{bmatrix} = \begin{bmatrix} \frac{\Gamma(C)\Gamma(D)}{\Gamma(C-A)\Gamma(C-B)} & \frac{\Gamma(C)\Gamma(-D)}{\Gamma(A)\Gamma(B)} \\ \frac{\Gamma(2+C)\Gamma(D)}{\Gamma(1-A)\Gamma(1-B)} & \frac{\Gamma(2-C)\Gamma(-D)}{\Gamma(1-C+A)\Gamma(1-C+B)} \end{bmatrix} \cdot \begin{bmatrix} J_1(z) \\ J_2(z) \end{bmatrix}. \quad (10.56)$$

We have just proven these relations for $z \in]0, 1[$, but by analytic continuation, they extend to $z \in \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$.

10.2 Crossing symmetry of the four-point function

10.2.1 Conformal block decomposition

The physical correlation function can be decomposed on both bases of conformal blocks:

$$\mathcal{G}(z, \bar{z}) = \sum_{k=1}^N (C_{\phi_{1N}, \phi_h}^{\phi_{h'_k}})^2 |\mathcal{I}_k(z)|^2 = \sum_{\ell=1}^N C_{\phi_{1N}, \phi_{1N}}^{\phi_{1,2\ell-1}} C_{\phi_h, \phi_h}^{\phi_{2\ell-1}} |\mathcal{J}_\ell(z)|^2. \quad (10.57)$$

This identity between the two expansions is known as *crossing symmetry*. Given the (model-independent) expression of conformal blocks as power-series solutions of (9.60), and especially their fusion matrix P , (10.57) becomes a quadratic equation on the OPE structure constants $\{C_{ij}^k\}$.

If we generalise to an expansion of the form:

$$\mathcal{G}(z, \bar{z}) = \sum_{i,j=1}^N X_{ij} \bar{\mathcal{I}}_i(\bar{z}) \mathcal{I}_j(z), \quad (10.58)$$

after the change of basis, we get:

$$\mathcal{G}(z, \bar{z}) = \sum_{k,\ell=1}^N Y_{k\ell} \bar{\mathcal{J}}_k(\bar{z}) \mathcal{J}_\ell(z), \quad Y = P^\dagger X P. \quad (10.59)$$

The particular choice

$$X_{ij} = X_i \delta_{ij}, \quad Y_{k\ell} = Y_k \delta_{k\ell}, \quad (10.60)$$

with

$$X_i = (C_{\phi_{1N}, \phi_h}^{\phi_{h'_i}})^2, \quad Y_k = C_{\phi_{1N}, \phi_{1N}}^{\phi_{1,2k-1}} C_{\phi_h, \phi_h}^{\phi_{1,2k-1}}, \quad (10.61)$$

comes from the assumption that all primary operators in the CFT model under consideration (in particular, the $\phi_{h'_k}$'s and the $\phi_{1,2\ell-1}$'s) are scalar. Different forms for the matrices X and Y may arise in other models, *e.g.* if we allow \mathcal{G} to have non-trivial monodromies as $\phi_{1N}(z, \bar{z})$ winds around $\phi_h(0,0)$ or $\phi_{1N}(1,1)$. The conditions $Y_{k\ell} = 0$ (resp. $X_{k\ell} = 0$) for $k \neq \ell$ leads to the linear system for the X_i 's (resp. Y_i 's):

$$\forall k \neq \ell, \quad \sum_{i=1}^N \bar{P}_{ik} P_{i\ell} X_i = 0, \quad \sum_{i=1}^N (\bar{P}^{-1})_{ik} (P^{-1})_{i\ell} Y_i = 0. \quad (10.62)$$

The expansion coefficients $\{X_i\}$ and $\{Y_i\}$ determine the OPE structure constants of the model, as well as the expression of the physical four-point function $\mathcal{G}(z, \bar{z})$.

10.2.2 Explicit solution in the case of ϕ_{12}

The crucial ingredient in the determination of $\{X_i\}$ and $\{Y_k\}$ is the fusion matrix P relating the two bases of conformal blocks. A general method, known as the Coulomb-Gas approach, and based on the representation of \mathcal{I}_k and \mathcal{J}_k as contour integrals over $(N-1)$ auxiliary variables, may be used to calculate the matrix elements of A . However, in this lecture, we will only focus on the simplest case

$$\mathcal{G}(z, \bar{z}) = \langle \phi_h(\infty, \infty) \phi_{12}(1, 1) \phi_{12}(z, \bar{z}) \phi_h(0, 0) \rangle, \quad (10.63)$$

where the ODE is second order, and can be reduced to the *hypergeometric differential equation*. Using the Kac parameterisation $c = 1 - 6Q^2$, and $h_\alpha = \alpha(\alpha - 2Q)$, with $2Q = b^{-1} - b$, we have the fusion for $\phi_h = V_\alpha$ and $\phi_{12} = V_{b/2}$:

$$\phi_{12} \times V_\alpha = V_{\alpha+b/2} + V_{\alpha-b/2}. \quad (10.64)$$

From the simple identity

$$h_{\alpha+\beta} = h_\alpha + h_\beta + 2\alpha\beta,$$

we see that the conformal blocks at $z = 0$ with internal dimensions $h_{\alpha+b/2}$ and $h_{\alpha-b/2}$, respectively, are of the form:

$$\mathcal{I}_1(z) = z^{b\alpha}(1 + a_1z + \dots), \quad \mathcal{I}_2(z) = z^{b(2Q-\alpha)}(1 + b_1z + \dots). \quad (10.65)$$

In the channel $z \rightarrow 1$, we have the fusion

$$\phi_{12} \times \phi_{12} = \mathbf{1} + \phi_{13}, \quad (10.66)$$

and the conformal blocks at $z = 1$ are given by:

$$\mathcal{J}_1(z) = z^{-2h_{12}}(1 + c_1z + \dots), \quad \mathcal{J}_2(z) = z^{h_{13}-2h_{12}}(1 + d_1z + \dots). \quad (10.67)$$

To study the local exponents at $z \rightarrow \infty$, let us first use conformal invariance under $z \mapsto 1/z$, which yields the relation:

$$\mathcal{G}(z, \bar{z}) = |z|^{-4h_{12}} \mathcal{G}(1/z, 1/\bar{z}). \quad (10.68)$$

The local exponents for \mathcal{G} at $z \rightarrow \infty$ are thus $2h_{12} + b\alpha$ and $2h_{12} + b(2Q - \alpha)$. The Riemann scheme for the ODE satisfied by \mathcal{G} is:

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline b\alpha & -2h_{12} & 2h_{12} + b\alpha \\ b(2Q - \alpha) & h_{13} - 2h_{12} & 2h_{12} + b(2Q - \alpha) \end{array}$$

We define the function $g(z, \bar{z})$ as

$$\mathcal{G}(z, \bar{z}) := |z|^{2b\alpha} |1 - z|^{-4h_{12}} g(z, \bar{z}). \quad (10.69)$$

This function g satisfies a second-order Fuchsian equation, with Riemann scheme

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & 2b\alpha \\ 1 - b^2 - 2b\alpha & 2b^2 - 1 & 1 - b^2 \end{array}$$

which is of the form

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & A \\ 1 - C & C - A - B & B \end{array}$$

with

$$A = 2b\alpha, \quad B = 1 - b^2, \quad C = b^2 + 2b\alpha. \quad (10.70)$$

This means that the conformal blocks are given by

$$\mathcal{I}_k(z) = z^{2b\alpha} (1 - z)^{-4h_{12}} I_k(z), \quad \mathcal{J}_\ell(z) = z^{2b\alpha} (1 - z)^{-4h_{12}} J_\ell(z), \quad (10.71)$$

where $\{\mathcal{I}_1, \mathcal{I}_2\}$ and $\{\mathcal{J}_1, \mathcal{J}_2\}$ are the bases of holomorphic solutions of the hypergeometric equation. We recall the fusion matrix, and its inverse:

$$P = \begin{bmatrix} \frac{\Gamma(C)\Gamma(D)}{\Gamma(C-A)\Gamma(C-B)} & \frac{\Gamma(C)\Gamma(-D)}{\Gamma(A)\Gamma(B)} \\ \frac{\Gamma(2+C)\Gamma(D)}{\Gamma(1-A)\Gamma(1-B)} & \frac{\Gamma(2-C)\Gamma(-D)}{\Gamma(1-C+A)\Gamma(1-C+B)} \end{bmatrix},$$

$$P^{-1} = \begin{bmatrix} \frac{\Gamma(1-C)\Gamma(1-D)}{\Gamma(1-C+A)\Gamma(1-C+B)} & \frac{\Gamma(C-1)\Gamma(1-D)}{\Gamma(A)\Gamma(B)} \\ \frac{\Gamma(1-C)\Gamma(1+D)}{\Gamma(1-A)\Gamma(1-B)} & \frac{\Gamma(C-1)\Gamma(1+D)}{\Gamma(C-A)\Gamma(C-B)} \end{bmatrix}, \quad (10.72)$$

with $D = C - A - B = 2b^2 - 1$. Using $P = \bar{P}$, the linear system for Y_1, Y_2 yields the ratio:

$$\frac{Y_2}{Y_1} = \frac{\gamma(C-A)\gamma(C-B)\gamma(-D)}{\gamma(A)\gamma(B)\gamma(D)} = \frac{\gamma^2(b^2)\gamma(2-2b^2)\gamma(1-2b^2)}{\gamma(2b\alpha)\gamma[2b(2Q-\alpha)]}, \quad (10.73)$$

where $\gamma(x) := \Gamma(x)/\Gamma(1-x)$. The coefficients Y_1, Y_2 are related to the structure constants:

$$Y_1(\alpha) = C_{\phi_{12}, \phi_{12}}^1 \times C_{\phi_h, \phi_h}^1 = 1, \quad Y_2(\alpha) = C_{\phi_{12}, \phi_{12}}^{\phi_{13}} \times C_{\phi_h, \phi_h}^{\phi_{13}}, \quad (10.74)$$

with $h = \alpha(\alpha - 2Q)$. We obtain the non-trivial structure constant by combining the equations for $\alpha = b/2$ and generic α :

$$C_{\phi_h, \phi_h}^{\phi_{13}} = \frac{Y_2(\alpha)}{\sqrt{Y_2(b/2)}} = \frac{\sqrt{\gamma^3(b^2)\gamma(2-2b^2)\gamma(1-2b^2)\gamma(2-3b^2)}}{\gamma(2b\alpha)\gamma[2b(2Q-\alpha)]}. \quad (10.75)$$

We can then compute the coefficients X_1, X_2 :

$$\begin{aligned} X_1 &= (P^{-1})_{11}^2 Y_1 + (P^{-1})_{21}^2 Y_2 = \frac{\gamma(C-A)\gamma(C-B)}{\gamma(C)\gamma(D)}, \\ X_2 &= (P^{-1})_{12}^2 Y_1 + (P^{-1})_{22}^2 Y_2 = \frac{\gamma(C-1)\gamma(D)}{\gamma(A)\gamma(B)}. \end{aligned} \quad (10.76)$$

This yields the OPE structure constants:

$$\begin{aligned} C_{\phi_{12}, V_\alpha}^{V_{\alpha+b/2}} &= \sqrt{\frac{\gamma(b^2)\gamma(2-2b^2)}{\gamma[2b(\alpha+b/2)]\gamma[2b(2Q-\alpha)]}}, \\ C_{\phi_{12}, V_\alpha}^{V_{\alpha-b/2}} &= \sqrt{\frac{\gamma(b^2)\gamma(2-2b^2)}{\gamma[2b(2Q-\alpha+b/2)]\gamma(2b\alpha)}}. \end{aligned} \quad (10.77)$$

Lecture 11

Modular invariance

11.1 Complex tori

So far we have mostly considered conformal field theories on the Riemann sphere. Let's now focus on the torus. From a statistical physics point of view, this is quite a natural thing to do. Indeed as soon as one works with periodic boundary condition we are in fact working on the torus, see figure 11.1. Likewise in condensed matter physics, the torus appears naturally upon considering a periodic one-dimensional quantum system at finite temperature. At a more abstract level, having a consistent formulation of conformal field

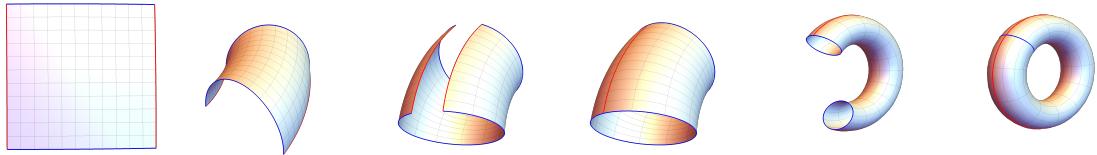


Figure 11.1: The square grid with periodic boundary condition is a torus

theories on higher genus surfaces, and in particular on the torus, yield deep insights into their structure.

11.1.1 Topological classification of closed surfaces

A closed¹ surface refers to a two-dimensional manifold that is compact and without boundary. As is probably well known to many readers, the topology of a connected closed surface is fully determined by two topological invariants : its genus g (or equivalently its Euler characteristic χ), and whether it is orientable or not. Given a closed surface, how can one figure out its genus g ? A very concrete and elementary way is to compute its Euler Characteristic χ . Given a triangulation of the surface, χ is defined as

$$\chi = V - E + F \tag{11.1}$$

where V, E and F are the total number of vertices, edges and faces of the triangulation, respectively. It is a fact that χ does not depend on the triangulation, and is indeed a topological invariant of the surface. The Euler characteristic is related to the genus as follows

¹not to be confused with *closed* in the sense of topology.

- if the surface is orientable, $\chi = 2 - 2g$
- if the surface is unorientable, $\chi = 2 - g$

For an informal discussion about the topological classification of closed surfaces including a sketch of proof, the reader is invited to read chapter 2 of Donaldson's *Riemann Surfaces*. This classification also holds for *smooth* surfaces : any two smooth manifolds with the same genus and that are both orientable (or both non-orientable) are equivalent, in the sense that there exists a diffeomorphism between them. This is because two-dimensional smooth manifolds are diffeomorphic if and only if they are homeomorphic (this is non trivial, and not valid in higher dimensions). From now on we will restrict our attention to oriented surfaces, for which the first few topological classes are depicted in fig. (11.2).

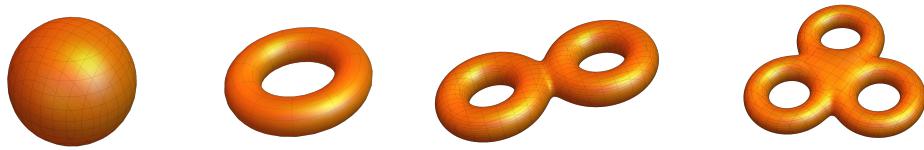


Figure 11.2: Oriented closed surfaces of genus $g = 0, 1, 2$ and 3 .

While the topological classification of surfaces is all well and good, in order to put a CFT on a surface one needs to endow the surface with a Riemannian metric. As soon as extra structure is added (beyond the smoothness structure), such as a Riemannian metric or a conformal structure, the topological classification is clearly no longer sufficient. As depicted in fig. (11.3), a two-dimensional sphere can have many different shapes ! Given



Figure 11.3: Two surfaces with the topology (and smooth structure) of the sphere S^2 . They are clearly not isometric : one displays constant curvature while the other does not. They are however conformally equivalent since the moduli space of the sphere is trivial.

the simple manner in which CFT correlation functions behave under Weyl rescaling, what we really care about is the complex structure induced by the metric². Thus within the

²Recall that two Riemannian metrics are called conformally equivalent when they differ by a Weyl transformation, *i.e.* when they define the same angles. An equivalence class of such metrics is called a *conformal structure* or *conformal class*. Furthermore if the surface is oriented a conformal structure induces a unique complex structure (and vice-versa) via isothermal coordinates.

framework of CFT one is concerned about the classification of (oriented) surfaces endowed with a complex structure, that is the classification or *Riemann surfaces*. The *moduli space* describes the various inequivalent complex structures (compatible with the given orientation) one can put on a surface of fixed genus. In the case of a genus 0 (topological sphere), the moduli space turns out to be trivial : all metrics are in the same conformal class (up to diffeomorphism). But for higher genus this is no longer the case. We will describe in detail the moduli space of the torus (genus $g = 1$) below. Let us simply mention that for genus $g \geq 2$ the moduli space is parametrized by $3g - 3$ complex parameters.

11.1.2 Curvature, genus and the uniformization theorem

One could wonder whether there is a notion of a canonical metric in a given conformal class. The existence of isothermal coordinates tells us that every metric is **locally** conformally flat : for each point x there exists a metric (in the same conformal class) which is flat in some neighborhood of x (*i.e.* there exist coordinates such that $g_{\mu\nu} = \delta_{\mu\nu}$). If we forget about the complex structure, this is rather intuitive. Imagine that the surface is made from dough. Then we can flatten it from the sides pushing all curvature away from x . While it is less clear that this can be done while preserving angles, it turns out to be true. However the Gauss-Bonnet theorem provides an obstruction to removing all the curvature and having a globally flat metric. Indeed the total curvature is related to the genus through

$$\int_M R dV = 8\pi(1-g). \quad (11.2)$$

So unless $g = 1$, it is not possible to make the metric globally flat. The next best thing is constant curvature, and this is the essence of the uniformization theorem : every smooth Riemannian metric on a two-dimensional surface is conformal to one with constant curvature. Whether this curvature is positive, null, or negative depends on the genus, as dictated by the Gauss-Bonnet theorem. In practice this means that the sphere ($g = 0$) admits a metric with constant positive curvature. This is the usual round metric as inherited from the canonical embedding in \mathbb{R}^3 . The torus ($g = 1$) can be endowed with a flat metric (zero curvature), while for higher genus there exist a metric of constant negative curvature.

Two oriented Riemannian surfaces (M_1, g_1) and (M_2, g_2) are conformally equivalent (*i.e.* related through a Weyl rescaling) if and only if they have the same genus and same moduli. Any Riemannian surface (M, g) is conformally equivalent to a surface of constant curvature.

11.1.3 Complex tori and moduli space

Topologically a (two-dimensional) torus \mathbb{T} is simply the cartesian product of two circles $S^1 \times S^1$, thus we may write

$$\mathbb{T} \simeq \mathbb{R}^2 / \mathbb{Z}^2 \quad (11.3)$$

where \simeq stands for diffeomorphic. This means that as a smooth surface, the torus is equivalent to the plane quotiented by the group \mathbb{Z}^2 of translations generated by $(x, y) \rightarrow (x+1, y)$

and $(x, y) \rightarrow (x, y + 1)$. This is nothing but a square with opposite side identified (*a.k.a.* periodic boundary conditions). If we consider instead a torus endowed with a complex structure (or equivalently a conformal class of Riemannian metrics), we are now looking at a *complex torus*, that is a genus 1 Riemann surface. The moduli space of the torus is not trivial, which amounts to say that two complex tori are in general not conformally equivalent³. This is strikingly different from the sphere !

A convenient and simple way to describe a complex torus is as a quotient of the plane by a lattice. Given two complex numbers ω_1 and ω_2 such that ω_1/ω_2 is not real, the lattice Λ generated by ω_1 and ω_2 is the subset

$$\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z} \quad (11.4)$$

as depicted in fig. (11.4). The complex numbers ω_1 and ω_2 are called *periods* of the lattice. Given a lattice Λ , the quotient space \mathbb{C}/Λ defines a complex torus. Concretely

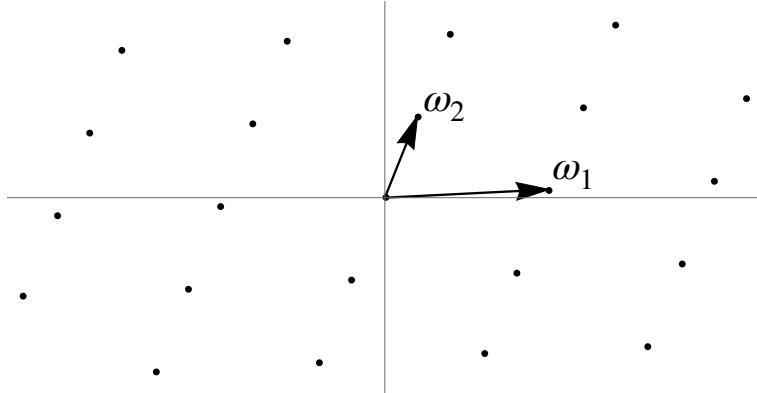


Figure 11.4: The lattice $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ generated by ω_1 and ω_2 .

this means identifying $z \equiv z + \omega_1 \equiv z + \omega_2$. As a Riemannian surface the torus inherits the flat metric $g = dx \otimes dx + dy \otimes dy$ of Euclidean space \mathbb{R}^2 . As a Riemann surface, the complex structure (or conformal class) of \mathbb{C} induces a complex structure on \mathbb{T} (the only one compatible with the above flat metric), and this complex structure depends on the lattice Λ .

It is a standard result of the theory of Riemann surfaces that the above construction exhausts all possible complex tori, in the sens that any complex torus is conformally equivalent to such a quotient (the reader is invited to read Donaldson chapter 6). But not all lattices yield inequivalent complex tori :

Fact⁴ : two complex tori \mathbb{C}/Λ and \mathbb{C}/Λ' are conformally equivalent if and only if the two lattices Λ and Λ' differ by a rotation/dilation, *i.e.* $\Lambda' = \alpha\Lambda$ for some $\alpha \in \mathbb{C}^*$.

³Recall that two Riemann surfaces are said to be (conformally) equivalent if there exist a biholomorphic isomorphism between them, *i.e.* a holomorphic map which is one-one and onto, and whose inverse is also holomorphic.

⁴The proof is rather elementary given a little knowledge about covering spaces and lifting of maps (for

In particular the lattice $\omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ is equivalent to

$$\Lambda = \mathbb{Z} + \tau\mathbb{Z}, \quad \tau = \frac{\omega_2}{\omega_1} \quad (11.5)$$

Note that we can assume without loss of generality that $\text{Im}(\tau) > 0$ (if it is negative, one can simply interchange ω_1 and ω_2 , or change $\omega_1 \rightarrow -\omega_1$). We will denote by \mathcal{H} the upper half-plane

$$\mathcal{H} = \{\tau \in \mathbb{C}, \text{Im } \tau > 0\}. \quad (11.6)$$

Furthermore given a lattice Λ , the choice of periods is not unique. Two bases (ω_1, ω_2) and (ω'_1, ω'_2) generate the same lattice if and only if they are related by a unimodular matrix A , *i.e.* a matrix with integer coefficients and determinant ± 1 . Namely

$$\begin{pmatrix} \omega'_2 \\ \omega'_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = \pm 1 \quad (11.7)$$

In terms of the parameter τ , this means

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \quad (11.8)$$

Note that $ad - bc = -1$ is no longer allowed once we restrict $\tau \in \mathcal{H}$. Indeed

$$\text{Im}\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{(ad - bc)}{|c\tau + d|^2} \text{Im}(\tau) \quad (11.9)$$

We have reached a full description of the moduli space of the torus as

$$\mathcal{H}/SL(2, \mathbb{Z}) \quad (11.10)$$

namely the quotient of the upper half-plane $\tau = \tau_1 + i\tau_2 \in \mathcal{H}$ by the group $SL(2, \mathbb{Z})$ acting as in eq. (11.8). The standard choice for the fundamental domain \mathcal{F} is depicted in fig. (11.5). In fact staring at (11.8) reveals that $SL(2, \mathbb{Z})$ does not have a faithful (or effective) action, in the sense that the matrices A and $-A$ have identical actions. Thus one introduces the *modular group* (*a.k.a.* the projective special linear group) $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2$ of matrices with integer coefficients and unit determinant, in which the matrices A and $-A$ are identified.

Complex tori are realized as quotient spaces

$$\mathbb{T}_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \quad (11.11)$$

where τ is in the upper half-plane. Furthermore \mathbb{T}_τ and $\mathbb{T}_{\tau'}$ are conformally equivalent if and only if τ and τ' are related by the modular group $PSL(2, \mathbb{Z})$, namely if there exists a, b, c, d integers with $ad - bc = 1$ such that

$$\tau' = \frac{a\tau + b}{c\tau + d} \quad (11.12)$$

which an excellent reference is *A. Hatcher, Algebraic topology* chapter 1) and rests on the two following facts

- (i) any bi-holomorphic map between \mathbb{C}/Λ and \mathbb{C}/Λ' lifts into a conformal automorphism of \mathbb{C}
- (ii) conformal automorphisms of \mathbb{C} are of the form $z \rightarrow \alpha z + \beta$

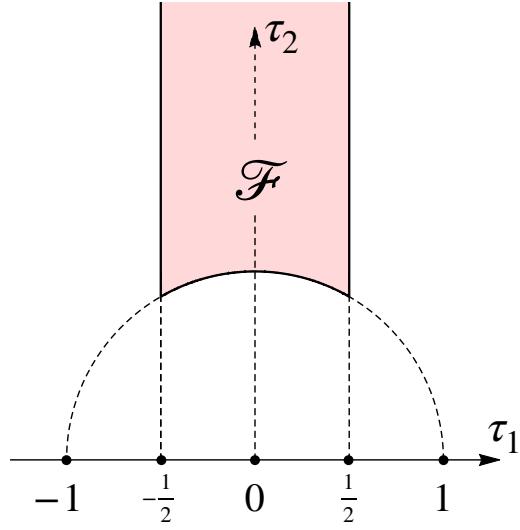


Figure 11.5: Fundamental domain \mathcal{F} of the torus moduli space.

In the following we will need the following fact about the modular group : it is generated by the so-called modular T - and S -transformations :

- $T : \tau \rightarrow \tau + 1$
- $S : \tau \rightarrow -1/\tau$

The proof of this statement is left as the following exercice.

Exercice : Let S, T and U be the following matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \quad (11.13)$$

Check that $U = TST$. Show that for two coprime integers a, c , Euclid's algorithm yields a sequence of integers n_1, \dots, n_{2p} such that

$$T^{n_1} U^{n_2} T^{n_3} \dots U^{n_{2p}} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (11.14)$$

Since $U = TST$ and $S \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the above implies that there exists m_1, \dots, m_k such that

$$T^{m_1} S T^{m_2} \dots S T^{m_k} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (11.15)$$

Deduce that S and T indeed generate $\mathrm{SL}(2, \mathbb{Z})$.

11.2 CFT partition function on the torus

Geometrically the complex torus \mathbb{T}_τ is a parallelogram spanned by 1 and $\tau = \tau_1 + i\tau_2$ with its opposite sides identified. Identifying the red pair of sides in figure (11.6) amounts to identify z and $z + 1$. This rolls the parallelogram into a cylinder of height τ_2 .

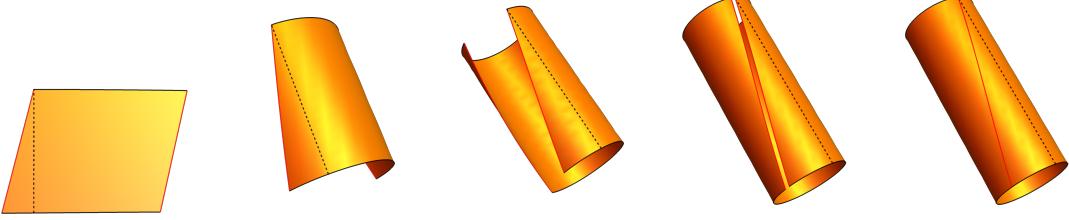


Figure 11.6: The cylinder obtained from the unit cell after the identification of the red edges.

We now need to close up our torus, by identifying z and $z + \tau$. Since we are on a cylinder, this is conveniently done in Hamiltonian formalism as follows. Decomposing $z = x + it$ we interpret x as the space coordinate (on a circle) and t as the (imaginary) time. If τ was pure imaginary, say $\tau = i\tau_2$, the partition function would coincide exactly with that of a one-dimensional system (living on a circle of length 1) at inverse temperature τ_2 , namely

$$Z = \text{Tr}(e^{-\tau_2 H})$$

However the real part τ_1 of τ requires an additional translation of space on top of the euclidean time evolution before sewing up (*i.e.* before taking the trace), therefore

$$Z(\tau) = \text{Tr}(e^{-\tau_2 H} e^{-\tau_1 P})$$

Notice the the slightly unusual form of the time and space translation operators, which stems from the fact that we working in imaginary time, *i.e.* in Euclidean signature. The cylinder Hamiltonian H and total momentum P can be obtained easily via conformal mapping to the plane through $w = e^{-i2\pi z}$, as in fig. (11.7). where we can use the formalism

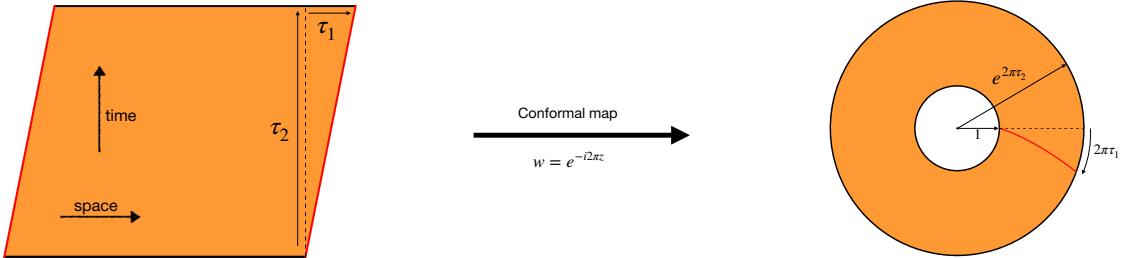


Figure 11.7: The flat torus is conformally equivalent to an annulus with edges identified. of radial quantization. Recall that P and H follow from the Ward identities :

$$P = \frac{1}{2\pi} \int_0^1 T_{21}(x, 0) dx, \quad H = \frac{1}{2\pi} \int_0^1 T_{22}(x, 0) dx. \quad (11.16)$$

where $T_{21} = i(T - \bar{T})$ and $T_{22} = -(T + \bar{T})$. Now since $T(z) = -(2\pi)^2 \left(w^2 T(w) - \frac{c}{24} \right)$ we find

$$\frac{1}{2\pi} \int_0^1 T dx = i \oint_{\mathbb{Q}} \left(w T(w) - \frac{c}{24} \frac{1}{w} \right) dw = -2\pi \left(L_0 - \frac{c}{24} \right) \quad (11.17)$$

leading to

$$P = 2\pi i (\bar{L}_0 - L_0), \quad H = 2\pi \left(L_0 + \bar{L}_0 - \frac{c}{12} \right) \quad (11.18)$$

Looking back at the annulus in fig. (11.7), this is not surprising. P implements (clockwise) rotations, while H is the generator of dilations. The only non-trivial part is the shift of energy proportional to $-\frac{c}{12}$, which comes from the Weyl transformation when going from the cylinder to the annulus.

We end up with the following expression for the partition function

$$Z(\tau) = \text{Tr} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right), \quad q = e^{2\pi i \tau}$$

in which the trace is taken over the whole Hilbert space of the CFT.

11.2.1 Rational CFT

Since we have an action of $\text{Vir} \otimes \overline{\text{Vir}}$, the Hilbert space can be decomposed into irreducible representations. Typically

$$\bigoplus_{i,j} N_{i,j} V_i \otimes \overline{V_j}$$

where $N_{i,j}$ is the multiplicity of the occurrence of the representation $V_i \otimes \overline{V_j}$, although in some cases the discrete sum can be an integral. It follows that the partition function can be written in terms of the Virasoro characters

$$Z(\tau) = \sum_{i,j} N_{i,j} \chi_i(q) \chi_j(\bar{q})$$

where $\chi_i(q)$ are the Virasoro characters as introduced in previous lectures. Note that $N_{i,j}$ are non-negative integers, and that the vacuum being non degenerate we must have $N_{0,0} = 1$ (in a unitary CFT the vacuum is the only state with a vanishing conformal dimension, therefore $V_0 \otimes \overline{V_0}$ can only appear once).

11.2.2 Modular bootstrap for rational CFTs

We have just obtained an expression of the partition function on the torus \mathbb{T}_τ that depends explicitly on the moduli τ . However we have seen that acting on τ with the modular group $\text{PSL}(2, \mathbb{Z})$ yield the same torus. Thus the partition function must be *modular invariant* :

$$Z(\tau) = Z \left(\frac{a\tau + b}{c\tau + d} \right), \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$$

From the above discussion about the modular group, modular invariance boils down to invariance under the modular T - and S -transformations :

$$Z(\tau) = Z(\tau + 1), \quad \text{and} \quad Z(\tau) = Z(-1/\tau)$$

Modular invariance of the partition function is analogous to crossing symmetry of 4-point functions on the sphere : it provides a constraint on how to piece together the holomorphic and anti-holomorphic parts of the CFT. One can think of the characters $\chi_i(q)$ as conformal block on the torus (notice that they are holomorphic in τ), and the modular group on the torus is the analogous of the braid group on the four-punctured sphere. In fact these are two examples of a more general problem, that of computing n -point correlation functions on a genus g surface. The braid group and modular group are particular instances of mapping class groups (see *Conformal field theory and mapping class groups* by T. Gannon, arXiv:0710.1329v1).

Very much like conformal blocs on the sphere, characters of a rational CFT form a finite dimensional representation of the modular group :

$$\chi_i(\tau + 1) = \sum_j T_{ij} \chi_j(\tau), \quad \chi_i(-1/\tau) = \sum_j S_{ij} \chi_j(\tau)$$

The T matrix is particularly simple as it is diagonal, namely

$$T_{ij} = \delta_{ij} e^{2i\pi(h_i - \frac{c}{24})}$$

This is straightforward since χ_i is of the form

$$\chi_i(\tau) = e^{2i\pi\tau(h_i - \frac{c}{24})} \sum_{n=0}^{\infty} N_i(n) e^{2i\pi n\tau}$$

although care has to be taken into defining the phase $e^{2i\pi(h_i - \frac{c}{24})}$. What we really mean by the modular T -transformation $\tau \rightarrow \tau + 1$ is the transformation of the characters χ_i under analytic continuation as we follow a path from τ to $\tau + 1$ in the fundamental domain of the moduli space.

11.2.3 $M(3,4)$ and the Ising model

Recall that the minimal model $M(3,4)$ has central charge $c = \frac{1}{2}$, and that there are 3 degenerate Verma modules.

The S matrix can be computed for the minimal models, see appendix 11.3.1. For the minimal model $M(3,4)$, the modular S matrix is given by

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} \quad (11.19)$$

in which the entries correspond to $\phi_{1,1}, \phi_{2,1}$ and $\phi_{2,2}$, respectively. The T -matrix reads

$$T = e^{-i\frac{\pi}{24}} \begin{pmatrix} 1 & -1 & \\ & e^{i\frac{\pi}{8}} & \end{pmatrix} \quad (11.20)$$

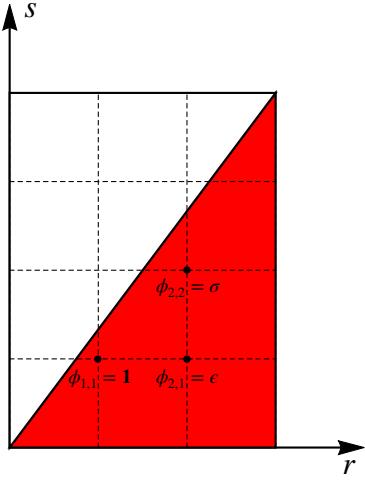
Figure 11.8: Kac Table for the minimal model $M(3,4)$.

The modular bootstrap amounts to find all 3 by matrices N of non negative integer coefficients (with $N_{1,1} = 1$) such that

$$S^\dagger NS = M, \quad T^\dagger NT = N \quad (11.21)$$

In the Ising model all the conformal dimensions being distinct modulo 1, the constraint $T^\dagger NT = N$ is tantamount to N being diagonal. Solving $S^\dagger NS = N$ yields $M = I$ as the unique solution. Thus the only modular invariant partition function that one can build is

$$Z = |\chi_{1,1}|^2 + |\chi_{2,1}|^2 + |\chi_{2,2}|^2 \quad (11.22)$$



11.2.4 The three-state Potts model and $M(5,6)$

The minimal model $M(5,6)$ has central charge $c = \frac{4}{5}$, and there are 10 degenerate representations of the Virasoro algebra. Let χ_i , $i = 1, \dots, 10$ be the corresponding characters, labelled in lexicographic order ($\chi_1 = \chi_{1,1}$, $\chi_2 = \chi_{1,2}$ etc...), where we are only taking the fields in the red region of the Kac table as in Fig.(11.9).

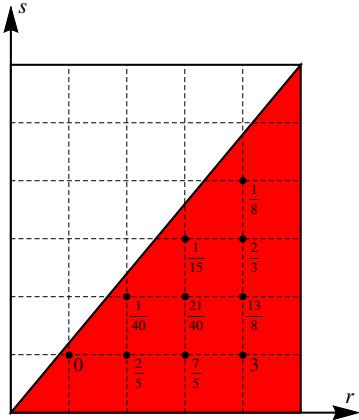


Figure 11.9: Kac Table for the minimal model $M(5,6)$.

Physical fields are product of holomorphic and anti-holomorphic fields, and they are naturally labelled by their left and right conformal dimensions (h, \bar{h}) . Thus will denote them by $\Phi_{h,\bar{h}}$.

. Example : non-compact boson (in zeta regularization and/or Hamiltonian formalism), followed by compact-boson (argue that summing over all windings has to be modular invariant ; then use using path-integral and decompose ϕ into harmonic ϕ_h with windings + quantum ϕ without windings ; basically do this without Hamiltonian formalism or characters). Exercise : cyclic orbifold, done with the same philosophy.

example : compact boson ; more interesting example : majorana fermion (recover diagonal minimal invariant from free fermion characters plus modular invariance, see Blumenhagen ch4 page 136)

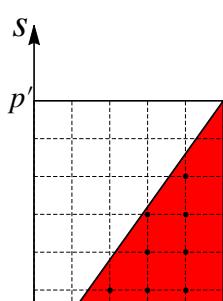
11.3 Appendix

11.3.1 Modular S matrix for minimal models

Recall that minimal models $M(p,p')$ are labelled by two coprime integers p, p' , with $2 \leq p < p'$, with central charge

$$c(p, p') = 1 - \frac{6(p - p')^2}{pp'}. \quad (11.23)$$

For this value of the central charge, the degenerate representations of the Virasoro algebra are labelled by a



two integers r, s , with highest weight

$$h_{r,s} = \frac{(p'r - ps)^2 - (p - p')^2}{4pp'} \quad (11.24)$$

The labels (r, s) are restricted to lie in the Kac table

$$\begin{cases} 1 \leq r \leq p - 1 \\ 1 \leq s \leq p' - 1 \end{cases} \quad (11.25)$$

and are subject to the identification $(r, s) = (p - r, p' - s)$. Thus we can restrict (r, s) to be in the range

$$\begin{cases} 1 \leq r \leq p - 1 \\ 1 \leq s \leq p' - 1 \\ ps < p'r \end{cases} \quad (11.26)$$

as in figure (11.10) for a total of $\frac{(p-1)(p'-1)}{2}$ inequivalent degenerate representations.

The corresponding characters are

$$\chi_{r,s}(\tau) = \frac{1}{\eta(\tau)} \left(\sum_{n \in \mathbb{Z}} q^{(p'r - ps + 2npp')^2 / 4pp'} - \sum_{n \in \mathbb{Z}} q^{(p'r + ps + 2npp')^2 / 4pp'} \right). \quad (11.27)$$

A point that will prove important is that the characters are linearly independent functions of τ . Indeed they are power series of the form $q^{h_{r,s}-c/24} \sum_n a_n q^n$, thus there can only be linear relations between characters with the same conformal dimensions. But it is easy to check⁵ that all the conformal dimensions $h_{r,s}$ are distincts for (r, s) in the range (11.26).

Computing the modular S -matrix

The characters are a representation of the modular group. One can prove that they are the solutions of a Fuchsian differential equation in τ (of order $\frac{(p-1)(p'-1)}{2}$) that follows from the null-vector at level $(p-1)(p'-1)$ in the module of the identity. In order to compute the S -matrix, we follow closely Di Francesco et al, chapter 10, section 10.6 (up to some change of notations). Let us first rewrite the characters as

$$\chi_{r,s}(\tau) = K_{\lambda_{r,s}}(\tau) - K_{\lambda_{r,-s}}(\tau), \quad (11.28)$$

where $\lambda_{r,s} = p'r - ps$ and $\lambda_{r,-s} = p'r + ps$ and

$$K_\lambda(q) := \frac{1}{\eta(\tau)} \sum_{k=\lambda \bmod N} q^{\frac{k^2}{2N}}, \quad N = 2pp'. \quad (11.29)$$

Since $K_\lambda = K_{\lambda+N}$, λ only matters mod N . The parameter $\lambda \in \mathbb{Z}/N\mathbb{Z}$ will soon turn out to be a simpler parametrization of the characters than (r, s) . But first we need to understand the reflexion $(r, s) \rightarrow (r, -s)$ at the level of $\mathbb{Z}/N\mathbb{Z}$.

⁵ $h_{r_1,s_1} = h_{r_2,s_2}$ iff $p'r_1 - ps_1 = \pm(p'r_2 - ps_2)$, which we can rewrite as $p'(r_1 \mp r_2) = p(s_1 \mp s_2)$. Thus $r_1 \mp r_2 = 0 \bmod p$ and $s_1 \mp s_2 = 0 \bmod p'$. In the range (11.25) the only solutions are $(r_1, s_1) = (r_2, s_2)$ and $(r_1, s_1) = (p - r_2, p' - s_2)$.

Fact : let (r_0, s_0) be two integers such that $p'r_0 - ps_0 = 1$, and let $\omega_0 = p'r_0 + ps_0 \bmod N$. Then the reflexion $(r, s) \rightarrow (r, -s)$ is implemented in $\mathbb{Z}/N\mathbb{Z}$ by multiplication by ω_0 :

$$\lambda_{r,-s} = \omega_0 \lambda_{r,s} \quad (11.30)$$

as follows easily from the identity

$$(p'r_0 - ps_0)(p'r + ps) = (p'r_0 + ps_0)(p'r - sps) \bmod N \quad (11.31)$$

Thus the characters can be conveniently written as

$$\chi_\lambda(\tau) = K_\lambda(\tau) - K_{\omega_0\lambda}(\tau) \quad (11.32)$$

At this stage a straightforward application of the Poisson resummation formula yields

$$K_\lambda(-1/\tau) = \frac{1}{\sqrt{N}} \sum_{\mu=0}^{N-1} e^{-i2\pi \frac{\lambda\mu}{N}} K_\mu(\tau) \quad (11.33)$$

and thus (since $\omega_0^2 = 1 \bmod N$)

$$K_{\omega_0\lambda}(-1/\tau) = \frac{1}{\sqrt{N}} \sum_{\mu=0}^{N-1} e^{-i2\pi \frac{\omega_0\lambda\mu}{N}} K_\mu(\tau) = \frac{1}{\sqrt{N}} \sum_{\mu=0}^{N-1} e^{-i2\pi \frac{\lambda\mu}{N}} K_{\omega_0\mu}(\tau) \quad (11.34)$$

It follows that

$$\chi_\lambda(-1/\tau) = \frac{1}{\sqrt{N}} \sum_{\mu=0}^{N-1} e^{-i2\pi \frac{\lambda\mu}{N}} \chi_\mu(\tau) \quad (11.35)$$

However this is not the end of the story, since

- we need to go back to the label (r, s) instead of λ
- the characters appearing in the *r.h.s.* are not linearly independent.

To resolve these two issues, we use the following facts.

Fact 1: The map

$$(r, s) \rightarrow \lambda_{r,s} = p'r - ps \bmod N \quad (11.36)$$

is a bijection from $\{(r, s), 0 \leq r \leq p-1, 1-p' \leq s \leq p'\}$ to \mathbb{Z}_N . Indeed since p and p' are coprime this map is injective⁶, and therefore a bijection since both sets are of size N . Thus we may write

$$\chi_\lambda(-1/\tau) = \frac{1}{\sqrt{N}} \sum_{r=0}^{p-1} \sum_{s=1-p'}^{p'} e^{-i2\pi \frac{\lambda\lambda_{r,s}}{N}} \chi_{r,s}(\tau) \quad (11.37)$$

⁶Suppose $\lambda_{r_1,s_1} = \lambda_{r_2,s_2} \bmod N$. This mean $p'(r_1 - r_2) = p(s_1 - s_2) \bmod 2pp'$, and in particular $p'(r_1 - r_2) = 0 \bmod p$. Given the range of r , this implies $r_1 = r_2$ and thus $s_1 - s_2 = 0 \bmod 2p'$, which in turns means $s_1 = s_2$.

Fact 2: For all values of λ such that $\lambda = \pm\omega_0\lambda \bmod N$, we have $\chi_\lambda = 0$, so we can remove them from the above sum. But $\lambda_{r,s} = \pm\omega_0\lambda_{r,s}$ iff $r = 0 \bmod p$ or $s = 0 \bmod p'$. Therefore we can exclude $r = 0$ and $s = 0$, yielding

$$\chi_\lambda(-1/\tau) = \frac{1}{\sqrt{N}} \sum_{r=1}^{p-1} \sum_{s=1}^{p'-1} \left(e^{-i2\pi \frac{\lambda\lambda_{r,s}}{N}} \chi_{r,s}(\tau) + e^{-i2\pi \frac{\lambda\lambda_{r,-s}}{N}} \chi_{r,-s}(\tau) \right) \quad (11.38)$$

$$= \frac{1}{\sqrt{N}} \sum_{r=1}^{p-1} \sum_{s=1}^{p'-1} \left(e^{-i2\pi \frac{\lambda\lambda_{r,s}}{N}} - e^{-i2\pi \frac{\lambda\lambda_{r,-s}}{N}} \right) \chi_{r,s}(\tau) \quad (11.39)$$

Finally we exploit $\chi_{p-r,p'-s} = \chi_{r,s}$, together with $\lambda_{p-r,p'-s} = -\lambda_{r,s}$ to restrict the sum to the fundamental domain of the Kac table (see Fig. (11.10))

$$\chi_\lambda(-1/\tau) = \frac{2}{\sqrt{N}} \sum_{r,s} \left(\cos 2\pi \frac{\lambda\lambda_{r,s}}{N} - \cos 2\pi \frac{\lambda\lambda_{r,-s}}{N} \right) \chi_{r,s}(\tau) \quad (11.40)$$

$$= \frac{4}{\sqrt{N}} \sum_{r,s} \sin 2\pi \frac{\lambda p' r}{N} \sin 2\pi \frac{\lambda p s}{N} \chi_{r,s}(\tau) \quad (11.41)$$

and we get the S -matrix

$$S_{(r_1,s_1),(r_2,s_2)} = \frac{4}{\sqrt{2pp'}} \sin \pi \frac{(p'r_1 - ps_1)p'r_2}{pp'} \sin \pi \frac{(p'r_1 - ps_1)ps_2}{pp'} \quad (11.42)$$

i.e.

$$S_{(r_1,s_1),(r_2,s_2)} = 2\sqrt{\frac{2}{pp'}} (-1)^{1+s_1r_2+s_2r_1} \sin \pi \frac{p'r_1r_2}{p} \sin \pi \frac{ps_1s_2}{p'} \quad (11.43)$$

In the above derivation we took (r_2, s_2) in the range (11.26), *i.e.* with $ps_2 < p'r_2$, but for obvious reasons the above S matrix has to be invariant under $(r_2, s_2) \rightarrow (p - r_2, p' - s_2)$. This elementary sanity check is left to the reader.

This S matrix enjoys

$$S^2 = 1, \quad S = \overline{S} = S^t \quad (11.44)$$

In particular S is unitary.

11.3.2 ADE classification

The proof of the ADE classification of modular invariants goes in two steps:

- find all invariants without demanding $N_{i,j}$ integer (well explained in Zuber et al, commun math phys 113, 1-26 (1987))
- impose $N_{i,j}$ non negative integer (read Ganon)