Holography at Finite Temperature

Finite temperature in QM/Field theory

We want to consider a field theory at finite temperature in a thermal equilibrium state. This can be done by

1 - Going Euclidean
2 - Make time periodic

Consider a transition amplitude in QM:

$$\langle q'; t_1 | q; t_0 \rangle = \langle q' | e^{-i\hat{H}(t_1 - t_0)} | q \rangle$$

Heisenberg states

$$= \sum_{n,m}' \langle q' | n \rangle \langle n | e^{-i\hat{H}(t_1 - t_0)} | m \rangle \langle m | q \rangle$$

$|n\rangle, |m\rangle \equiv$ basis of energy eigenstates

$$= \sum_{n}' \psi_n(q') \psi_m^*(q) e^{-iE_n(t_1 - t_0)}$$

Now do a Wick rotation:

$$t \longrightarrow -it_E \quad \text{with } t_1 - t_0 = -i\beta$$

so $-i(t_0 - t_1) = -\beta$

and

$$\langle q' ; \beta | q ; 0 \rangle = \sum_n{}' \psi_n(q') \psi_n^*(q) e^{-\beta E_n}$$

Now take $q'(\beta) = q(0) = q$ and integrate over q!

$$\int dq \, \langle q ; \beta | q , 0 \rangle = \int dq \sum_n{}' |\psi_n(q)|^2 e^{-\beta E_n}$$

$$= \sum_n{}' \langle n | e^{-\beta \hat{H}} | n \rangle = \mathrm{Tr}\, e^{-\beta \hat{H}}$$

So this object is the <u>partition function</u> in an equilibrium state at temperature

$$T = \frac{1}{\beta}$$

$Z[\beta]$ = integral of transition amplitude in <u>euclidean</u> <u>periodic</u> time $\boxed{\Delta t_E = \beta}$

On the other hand we have the path-integral representation

$$\langle q'_{t_1} | q_{t_0} \rangle = \int_{q(t_0)=q,\ q(t_1)=q'} \mathcal{D}[q(t)]\, e^{iS[q(t)]}$$

Now do a Wick rotation:

$$t = -i t_E$$

$$iS[q(t)] = i \int_{t_0}^{t_1} dt \left[\frac{1}{2}\left(\frac{dq}{dt}\right)^2 - V(q) \right]$$

$$= i \int -i\, dt_E \left[-\frac{1}{2}\left(\frac{dq}{dt_E}\right)^2 - V(q) \right]$$

$$= -S_E[q(t_E)]$$

$$S_E = \int dt_E \left(\frac{1}{2} \dot{q}^2 + V(q) \right)$$

Energy!

therefore:

$$Z[\beta] \equiv \mathrm{Tr}\, e^{-\beta \hat{H}} = \int_{q(t_E+\beta)=q(t_E)} \mathcal{D}[q]\, e^{-S_E}$$

To put the system in thermal equilibrium at the temperature β, make time Euclidean and compactify it on a circle or leght β. Same in QFT:

<u>Vacuum state</u> : theory on $\mathbb{R}^{1,3}$ with S

<u>thermal state</u> (equilibrium) : theory on $S_1 \times \mathbb{R}^3$ with action S_E

In a field theory with a gravity dual:

$$Z_{QFT}[\beta] = \int_{\substack{\text{Gravity}\\ \text{side}}} \mathcal{D}[\varphi]\, e^{-S_E^{grav}}$$

Gravitational path integral over Euclidean geometries where time is on a circle.

Crash review of equilibrium thermodynamics

$Z \equiv e^{-\beta F}$ $\quad F$ = free energy $= -\frac{1}{\beta}\log Z$

$\{ \beta = \frac{1}{T}$ fixed; V fixed $\}$

$\Rightarrow$ canonical ensemble

S, P, E = functions of β, T, V.

entropy, pressure

• $\boxed{dF = -P\,dV - S\,dT}$ First law

(change in free energy caused by a change in temperature and volume)

$$\bullet\ S = -\frac{\partial F}{\partial T} = -\frac{\partial}{\partial 1/\beta}\left(-\frac{1}{\beta}\log Z\right)$$

$$= -\beta^2 \frac{\partial}{\partial \beta}\left(\frac{1}{\beta}\log Z\right) =$$

$$= \log Z - \frac{1}{Z}\beta\frac{\partial}{\partial \beta}Z$$

$\log Z \to -\beta F$

$-\frac{1}{Z}\beta\frac{\partial}{\partial \beta}Z \to \frac{\beta}{Z}\mathrm{Tr}\left(\hat{H}e^{-\beta H}\right) = \beta E$

$\to \quad S = \beta(E - F) \Rightarrow \boxed{F = E - TS}$

- $p = -\dfrac{\partial F}{\partial V} = -\dfrac{F}{V}$ (for a homogeneous state)

$\to$ $-$pressure = free energy density.

Now consider a <u>conformal field theory</u> at finite T and volume V:

$$Z_{CFT}[\lambda\beta, \lambda^3 V] = Z_{CFT}[\beta, V]$$

by conformal invariance

$$\Rightarrow \quad \lambda^4 \beta V p(\lambda\beta) = \beta V p(\beta)$$

(p is independent of V)

$$\to \quad \boxed{\lambda^4 p(\lambda\beta) = p(\beta)}$$

$$\to \quad p = \text{cost}\, \beta^{-4} \Rightarrow \boxed{p = c\, T^4}$$

(cfr Stefan-Boltzmann law!)

C is a constant that depends on the specific CFT.

- In a free CFT: (e.g. photon gas)

$$p = +\sigma T^4 \times \#\text{ of degrees of freedom}$$

($\sim N^2$ in a Matrix theory)

σ $\equiv$ Stefan-Boltzmann constant
(>0) (up to a numerical coefficient)

- $$s = \frac{S}{V} = +\frac{1}{V}\frac{\partial(pV)}{\partial T} = \underline{4\sigma T^3}$$

- $$\rho = \frac{E}{V} = \frac{1}{V}(F + TS) = -p + Ts = \underline{3\sigma T^4}$$

- Specific heat:

$$C_V = \frac{1}{V}\frac{\partial E}{\partial T} \sim \sigma T^3$$

$\sigma > 0 \Rightarrow$ thermal stability

(increasing the temperature increases the energy)

Gravity side

$$Z = e^{-\beta F} \underset{\text{large-}N}{\simeq} e^{-S_E^{\text{on-shell}}}$$

$$S_E = -S_{\text{Lorentzian}} \quad \left(\begin{array}{c}\text{for time-} \\ \text{independent} \\ \text{configurations}\end{array}\right)$$

$$\to \boxed{\beta F = S_E}$$

Consider the dual of $N=4$ SYM (forget the S^5):

$$S = \frac{M_p^3}{2}\int (R - 2\Lambda)\sqrt{g}$$

to evaluate on an AdS_5 solution with periodic time: $\quad t_E \sim t_E + \beta$

$$ds^2 = \frac{\ell^2}{z^2}\left[dz^2 + \underbrace{dt_E^2 + dx_i dx^i}_{S^1_\beta \times \mathbb{R}^3}\right] \qquad i = 1\ldots 3$$

we can compute the on-shell action easily: the solution satisfies $(\forall \beta)$:

$$R_{ab} - \frac{1}{2} g_{ab} R = - g_{ab} \Lambda$$

$$\Rightarrow R = \frac{10}{3}\Lambda = -20/\ell^2$$

$$\rightarrow S_{\text{on-shell}} = \frac{M_p^3}{2} \int_\epsilon^\infty \frac{\ell^5}{z^5} \left(\frac{10}{3}\Lambda - 2\Lambda\right) d^3x \, dt_E \, dz$$

$$= \frac{2}{3} M_p^3 \ell^3 \, \ell^2 \underbrace{\left(-\frac{6}{\ell^2}\right)}_{\Lambda} \times V \times \beta \, \frac{1}{4\epsilon^4}$$

$$\rightarrow F \sim (M_p \ell)^3 \frac{V \beta}{\epsilon^4}$$

Problems:

1) it's divergent (this can be fixed by renormalization)

2) it does <u>not</u> scale as expected
$\sim \beta$ instead of β^{-4}.

Both problems ~~are~~ "solved" by renormalization:

define

$$S' = S'_{bare} + S_{ct}$$

$$S_{ct} = \frac{cst}{\epsilon^4} \int_{z=\epsilon} \sqrt{\gamma}\, d^3x\, dt_E$$

$\mathcal{F}$ is now finite, but it is also $= 0$ at leading order in N^2.

On the other hand we expect

$$\mathcal{F}_{\substack{N=4 \\ SYM}} \sim c N^2 T^4 + O(1)$$

Where does the leading order answer come from? there are other solutions with the same asymptotic geometry $S^1 \times \mathbb{R}^3$: AdS-Black-holes!

Interlude: Black holes as thermal states

Consider ordinary Schwarzschild black holes (in 4d):

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega_2^2$$

this is valid outside the horizon, $r > r_H$

$(r_H \equiv 2GM)$

Now go to Euclidean: $-dt^2 \longrightarrow +d\tau^2$

Near $r \simeq r_H$ the metric becomes: $(r = r_H + \epsilon)$

$$ds_E^2 \simeq +\left(1 - \frac{r_H}{r_H+\epsilon}\right)d\tau^2 + \frac{d\epsilon^2}{\left(1 - \frac{r_H}{r_H+\epsilon}\right)}$$

$$+ r_H^2 d\Omega_2^2$$

$$\simeq \frac{\epsilon}{r_H}d\tau^2 + \frac{r_H}{\epsilon}d\epsilon^2 + r_H^2 d\Omega_2^2$$

Now change variables to $\tilde{\rho} = \sqrt{\epsilon}$

$$ds_E^2 \simeq \tilde{\rho}^2 \frac{d\tau^2}{r_H} + 4 r_H \, d\tilde{\rho}^2 + r_H^2 d\Omega_2^2$$

$$= 4 r_H \left(d\tilde{\rho}^2 + \frac{\tilde{\rho}^2}{(2r_H)^2} d\tau^2 \right)$$

$$+ \underbrace{r_H^2 \, d\Omega_2^2}_{\text{finite } S^2}$$

the horizon must be <u>regular</u>

→ the sphere S^2 is finite

→ the other piece has a vanishing $d\tau^2$ at $\tilde{\rho} = 0$. this is a singularity <u>unless</u> $\frac{\tau}{2r_H}$ is periodic with period 2π: define $\frac{\tau}{2r_H} = \theta$

$$\rightarrow ds_E^2 \simeq 4 r_H \underbrace{\left(d\tilde{\rho}^2 + \tilde{\rho}^2 d\theta^2 \right)}_{\mathbb{R}^2} + \underbrace{r_H^2 d\Omega_2^2}_{S^2}$$

$\mathbb{R}^2 \times S^2$

(in polar coordinates)

$\theta \sim \theta + 2\pi$ identification

$\Rightarrow \quad \tau \sim \tau + 4\pi r_H$

$\Rightarrow$ Euclidean time periodic with period $\beta = 4\pi r_H$

$\rightarrow$ state with Temperature

$$\boxed{T_H = \frac{1}{4\pi r_H}}$$

Hawking temperature!

Not only BH's have temperature, but,

- they have entropy

$$S = \frac{A_H}{4G} \quad \left(= \frac{4\pi r_H^2}{4G} = 4\pi G M^2 \right)$$

horizon area

- Defining the Free energy by

$\beta_H F = S_E = \int \sqrt{g}\, R$ we have

the relation $\beta F = \beta E - S$

Black Hole mass

Are Schwarzschild BH's stable?

$$T_H = \frac{1}{8\pi G M} \Rightarrow M = \frac{1}{8\pi G}\frac{1}{T}$$

$$C_V = \frac{\partial M}{\partial T} = -\frac{1}{8\pi G}\frac{1}{T^2} < 0$$

$\Rightarrow$ BH's cool down by absorbing energy (unstable).

AdS - Schwarzschild Black holes

the gravitational path integral is

$$Z_{grav} = \int_{g_{ab} \to \frac{1}{z^2} \times \left(\substack{\text{metric of} \\ S^1 \times \mathbb{R}^3}\right)} \mathcal{D}[g_{ab}]\, e^{-S_E[g]}$$

where $S_E = \frac{M_p^3}{2} \int \sqrt{g}\,(R - 2\Lambda)$

In the semiclassical limit the path integral concentrates on classical solutions, but if there is more than one solution with the same boundary condition (i.e. conformal boundary $\simeq S^1 \times \mathbb{R}^3$) we have to sum over geometries:

$$Z \underset{\text{semiclassical}}{\simeq} e^{-S_E(g_1)} + e^{-S_E(g_2)} + \cdots$$

$g_1, g_2 \cdots$ solutions of $R_{ab} - \frac{1}{2} g_{ab} R = -g_{ab}\Lambda$

All solutions must behave asymptotically as :

$$ds^2 = \frac{\ell^2}{z^2}\Big[dz^2 + \underbrace{\Big(d\tau^2 + \sum_{i=1}^{3} dx_i^2\Big)}_{S^1_\beta \times \mathbb{R}^3} + O(z^2) \Big] \quad \text{as } z \to 0$$

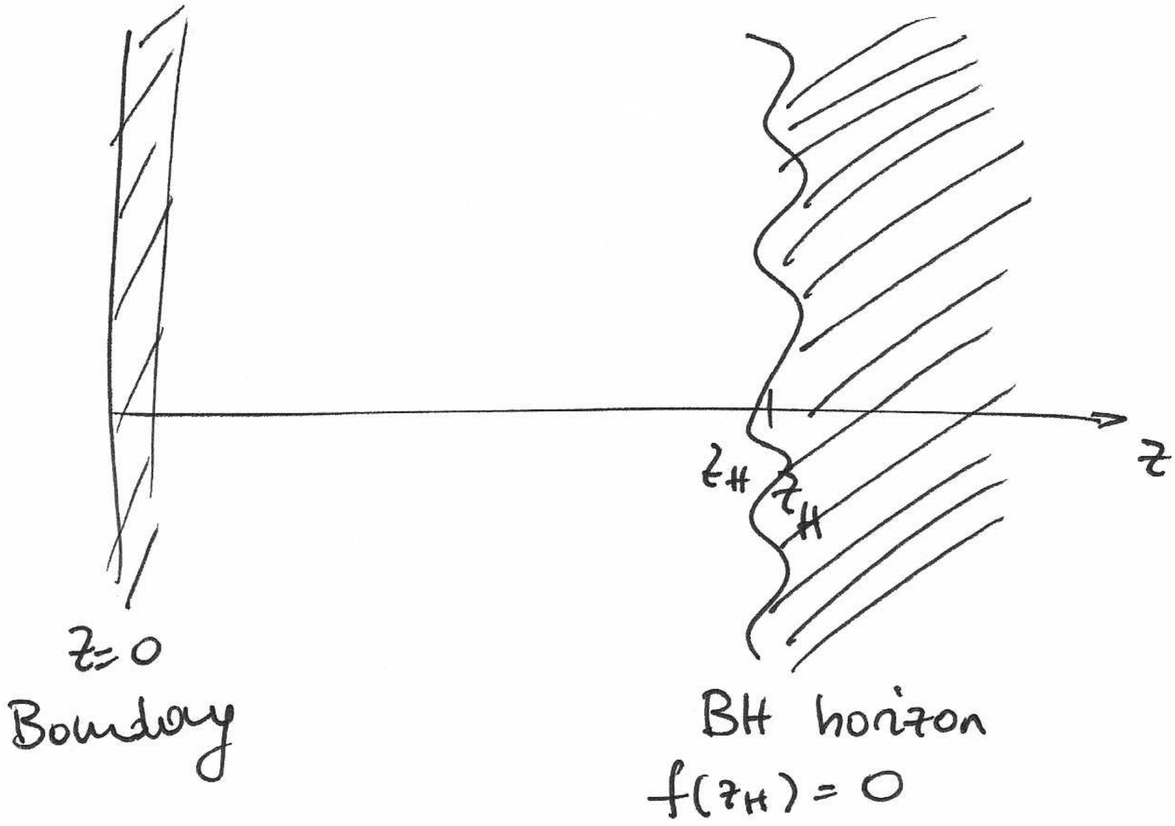
For fixed β there are <u>Two</u> such solutions

- $EAdS_5$ with periodic time $\tau \sim \tau + \beta$
- <u>AdS_5 black holes</u> :

$$ds^2 = \frac{\ell^2}{z^2}\left[\frac{dz^2}{f(z)} + f(z)d\tau^2 + \sum_i dx_i^2 \right]$$

$$f(z) = 1 - \frac{z^4}{z_H^4} \qquad \begin{array}{l} z_H = \text{horizon} \\ 0 < z_H < \infty \end{array}$$

(in general $d+1$: $f(z) = 1 - \left(\frac{z}{z_H}\right)^d$)



- As $z \to 0$ the metric asymptotes AdS_5 compactified on a circle :

$$f \longrightarrow 1 + O(z^4)$$

- Subleading contribution $\sim z^4$

$\Rightarrow \quad \langle T_\mu{}^\nu \rangle \neq 0$ in the QFT

- At fixed z, τ, the black hole is <u>planar</u> : geometry of horizon is flat $\mathbb{R}^3$ (we can put it in finite volume V by making x_1, x_2, x_3 compact)

Fixing β is equivalent to fixing z_H :

go to near-horizon limit $z = z_H + \epsilon$, $\epsilon \longrightarrow 0$:

$$f(z) \simeq \frac{4\epsilon}{z_H}, \quad ds^2 \simeq \frac{\ell^2}{z_H^2}\left[z_H \frac{d\epsilon^2}{4\epsilon} + \frac{4\epsilon}{z_H} d\tau^2 + dx_i^2 \right]$$

set $\rho = \sqrt{\epsilon}$, get (as before)

$$ds^2 \simeq \frac{\ell^2}{z_H}\left[\underbrace{d\rho^2 + \rho^2 \left(\frac{2}{z_H}\right)^2 d\tau^2}_{\mathbb{R}^2 \text{ in polar coordinates if}} + \underbrace{dx_i^2}_{\mathbb{R}^3} \right]$$

$$\frac{2\tau}{z_H} = \theta$$

$\Rightarrow$ <u>Regularity</u> fixes the periodicity of

$$\theta \sim \theta + 2\pi \quad \Rightarrow \quad \tau \sim \tau + \pi z_H$$

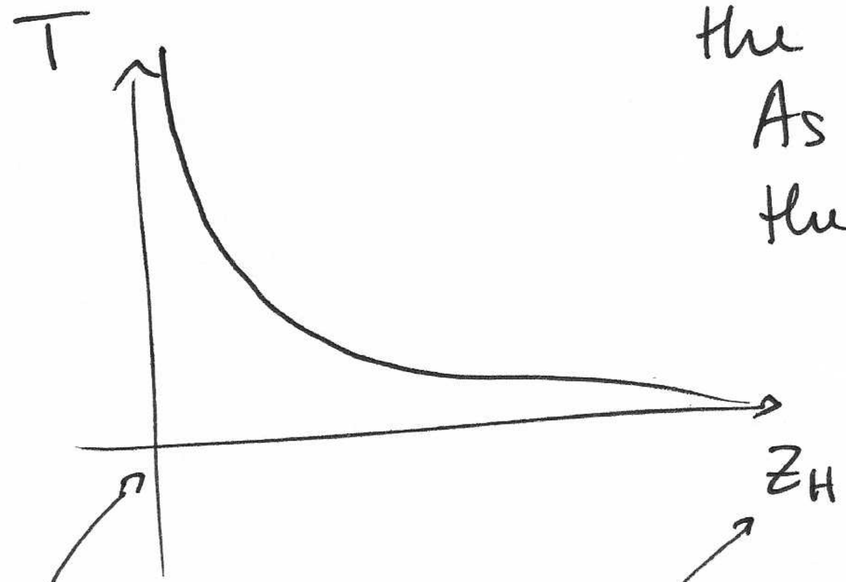
$$\boxed{\beta = \pi z_H}$$

[As for the scalar field calculation, regularity in the <u>interior</u> fixes the solution]

$$T = \frac{1}{\pi z_H}$$ $\Rightarrow$ the further from the boundary is the horizon, the <u>colder</u> the BH.

As $z_H \to \infty$ the BH disappears ($T = 0$)

z decompactifi



High temperature $\Rightarrow$ $z_H \sim 0$

low temperature $\Rightarrow$ $z_H \sim \infty$

- <u>Entropy</u>: the BH (Black hole, or Beckenstein-Hawking) entropy is

$$\boxed{S_{BH} = \frac{A_H}{4G_N} = 2\pi V \frac{(M_p \ell)^3}{z_H^3}}$$

$$A_H = \frac{\ell^3}{z_H^3} \underbrace{\int d^3x}_{\text{3-d Volume}} = \frac{V \ell^3}{z_H^3}$$

$$G_N = \frac{1}{8\pi M_P^3}$$

now, $M_P^3 \ell^3 \sim N^2$: in fact, in the <u>10</u> <u>D</u> solution, $AdS_5 \times S^5$, we had

$$M_P^8 \ell^8 \sim N^2$$ where M_P is the <u>10D</u> Planck scale

the AdS_5 we are using in a <u>5D</u> theory obtained reducing the <u>10D</u> theory on the S^5 :

$$S_{10} = \frac{1}{2} M_{10}^8 \int d^{10}x \sqrt{g_{10}} \left(R^{10} + \ldots F_5^2 + \ldots \right)$$

now reduce on S^5 : take all fields independent on the S^5 coordinates and write an ansatz where

$$ds_{10}^2 = \underbrace{g_{ab}\, dx^a dx^b}_{\substack{\text{5d metric,} \\ \text{independent of the} \\ S^5 \text{ coordinates}}} + \underbrace{\ell^2 d\Omega_5^2}_{S^5}$$

$$F_5 = (1 + *) \, Vol(S^5)$$

we can do the integral over S^5 and get a 5D action for g_{ab} alone. (F_5 is non-dynamical in 5D and gives a negative cosmological constant):

$$S_{10} = \frac{1}{2} M_{10}^8 \underbrace{\int d^5y \sqrt{g_{S^5}}}_{\sim \ell^5} \int d^5x \sqrt{g(x)} \left(R^{(g)} - \ell^{-2} \right)$$

From S^5 curvature + F_5^2

$$= \frac{1}{2} M_5^3 \int d^5x \sqrt{g} \left(R^{(g)} - 2\Lambda \right) = S_5$$

where $M_5^3 = M_{10}^8 \ell^5$

$$\rightarrow \quad M_5^3 \ell^3 = M_{10}^8 \ell^8 \sim N^2.$$

So we have:

$$S = \frac{2\pi \underbrace{(M_p \ell)^3}_{\sim N^2}}{z_H^3} \quad , \quad z_H = \frac{1}{\pi T}$$

$$\Rightarrow \quad \boxed{S \simeq (2\pi)^4 N^2 T^3}$$

- Entropy scales like N^2 and T^3 as it should.
- From S we can get the pressure, knowing that

$$S = -\frac{1}{V}\frac{\partial F}{\partial T}$$

$$\Rightarrow \quad F_{BH} = -\frac{(2\pi)^4}{4}(M_p \ell)^3 V T^4 + \text{cst}$$

- ~~the cst is fixed by conformal invariance, cst=0, or by continuity with the T=0 solution,~~

the additive constant can be calculated explicitly by computing $S_E = \beta \mathcal{F}$ on the BH solution, but we can use a shortcut: as $T \to 0$ $(z_H \to \infty)$ the geometry reduces to $EAdS_5$, with $\beta\mathcal{F}$ that we computed before:

$$\beta \mathcal{F}\Big|_{EAdS} = \beta \frac{(M_p \ell)^3}{\epsilon^4} V \Rightarrow \text{cst} = \frac{(M_p \ell)^3 V}{\epsilon^4}$$

(independent of T)

Altenatively, we can use the renormalized Free energy where we subtract the divergent term, then $\mathcal{F}^{(ren)}_{EAdS} = 0$ to order N^2

In any event:

$$\boxed{\mathcal{F}_{BH} = -\frac{(2\pi)^4}{4}(M_p \ell)^3 V T^4 + \mathcal{F}_{EAdS}}$$

- <u>BH energy density</u> : we can use BH thermodynamics,

$$E = \mathcal{F} + TS = \frac{3}{4}(2\pi)^4 (M_{pl})^3 V T^4$$

(+cst)

<u>or</u> compute directly using the ADM mass (= generalisation of the BH mass that works in any stationary space-time with suitable asymptotics)

- the quantities $T, V, \mathcal{F}$ etc are measured in field-theory-side units : recall that in general :

$$ds^2_{grav} \xrightarrow[z\to 0]{} \frac{\ell^2}{z^2}\Big[dz^2 + \big[g_{FT}(x)_{\mu\nu} + O(z^2)\big] dx^\mu dx^\nu\Big]$$

here, $g_{FT}(x)_{\mu\nu}$ here is the $\mathbb{R}^3_x \times S^1_\beta$ metric, so β is the (inverse) temperature of the dual QFT. same for E, $\mathcal{F}$, etc.

Let us go back to the <u>partition function</u>: we found 2 solutions with same temperature $T = 1/\beta$:

- $EAdS_5$ compactified on S^1_β
- AdS-Schwarzschild BH with

$$z_H = \frac{1}{\pi T}$$

$$Z[\beta] \simeq A_1 e^{-\beta \mathcal{F}_1^\beta} + A_2 e^{-\beta \mathcal{F}_2^\beta}$$

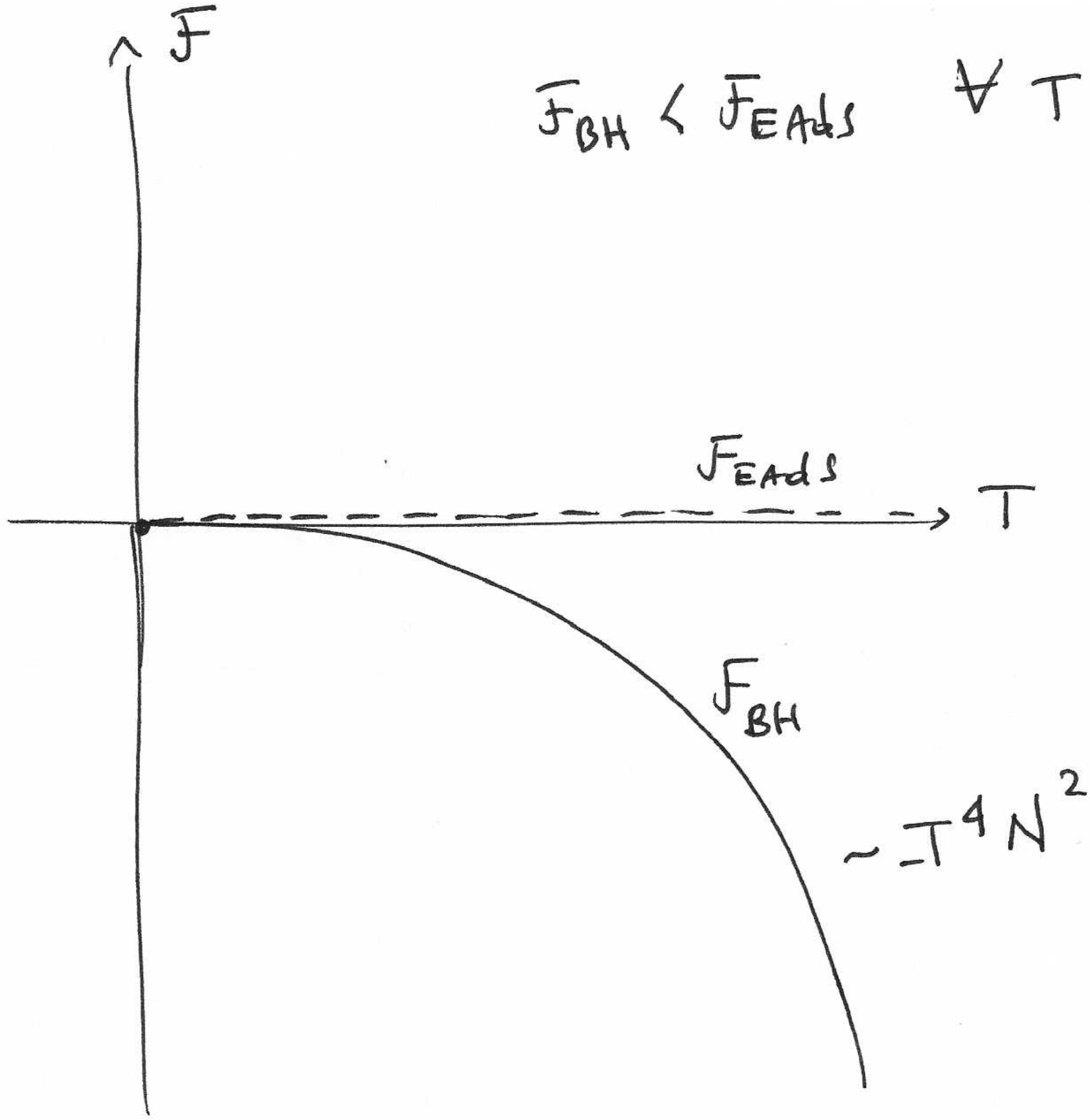
where A_1 and A_2 are 1-loop determinants and $O(N^0)$.

Clearly, the partition function is dominated by the solution with the <u>smallest $\mathcal{F}$</u>. We have:

$$\mathcal{F}_{BH}^\beta - \mathcal{F}_{EAdS}^\beta = -\frac{(2\pi)^4}{4}(M_p l)^3 V T^4 < 0$$

$$\Rightarrow \mathcal{F}_{BH} < \mathcal{F}_{EAdS} \quad \forall\, T$$

$\mathcal{F}_{BH} < \mathcal{F}_{EAdS} \quad \forall\, T$



Notice that the specific heat of the BHs is positive:

$$E \sim c T^4 \implies C_V = c T^3 > 0$$

$\implies$ these are thermodynamically stable (unlike Schwarzschild in asymptotically flat space)

$\Rightarrow$ At any temperature it is the BH solution that dominates the ensemble, so

$$Z[\beta] \simeq A\, e^{-\beta \mathcal{F}_{BH}(\beta)} \Big[1 + \underbrace{B\, e^{-\beta(\mathcal{F}_{EAdS} - \mathcal{F}_{BH})}}_{\sim\, e^{-\cancel{N^2} V T^3}} \Big]$$

subleading non-perturbative correction.

$$\rightarrow \mathcal{F}(\beta) = -\frac{1}{\beta} \log Z \simeq \underbrace{\mathcal{F}_{BH}(\beta)}_{\to\, O(N^2)} + \underbrace{O\left(e^{-\beta \cancel{N^2} V T^4}\right)}_{\to\, O(N^0)}$$

- Notice that we are taking contributions from different macroscopic geometries ("Quantum gravity!")

- As expected, $\mathcal{F} = c N^2 V T^4$, but from the D3-brane construction in 10D we know the exact relation between $(M_p \ell)^3$ and N^2 $\Rightarrow$ we can get the actual number and compare with the result from the field theory calculation (at weak coupling)

- $N=4$ SYM at small λ (actually $\lambda = 0$)

$$\mathcal{F}_{SYM} = -\frac{\pi^2}{6} N^2 V T^4$$

- Gravity calculation (with identification $M_p^3 \ell^3 = c N^2$)

$$\mathcal{F}_{BH} = -\frac{\pi^2}{8} N^2 V T^4$$

$$\boxed{\mathcal{F}_{Grav} = \frac{3}{4} \mathcal{F}_{SYM}}$$

In a fluid in thermal equilibrium, the stress-tensor has the form:

$$T_\mu{}^\nu = \begin{pmatrix} \rho & & & 0 \\ & -p & & \\ & & -p & \\ 0 & & & -p \end{pmatrix}$$

where ρ = energy density
p = pressure.

Here: $$\rho = \frac{3}{4}(2\pi)^4 (M_p\ell)^3 T^4$$
$$p = \frac{1}{4}(2\pi)^4 (M_p\ell)^3 T^4$$

$$\Rightarrow \rho = 3p \Rightarrow \boxed{T_\mu{}^\mu = 0}$$

(like photon gas, but interacting)

Conformal invariance is broken softly (by the state, not by a source)

$\Rightarrow$ trace identity still holds like in unbroken case

Problem? not really: $\mathcal{F}$ depends on λ, and the SYM and Gravity calculations capture the limit $\lambda \to 0$ and $\lambda \to \infty$, so the actual result should rather be

$$\mathcal{F} = \mathcal{F}(\beta, \underset{\uparrow}{\lambda}) \longrightarrow \begin{cases} c N^2 T^4 V & (\lambda \to 0) \\ \frac{3}{4} c N^2 T^4 V & (\lambda \to \infty) \end{cases}$$

[it is striking that, going to ∞-ly strong coupling only changes the results by a factor 3/4...]

- this is being used extensively as a model for strongly coupled QCD at finite temperature, in the <u>deconfined phase</u> (Quark-gluon Plasma).

Thermal phase transitions

Planar AdS_5 black holes capture the thermodynamics of $N=4$ SYM on $\mathbb{R}^3$ and at infinite coupling.

this thermodynamics is rather boring ($F = cT^4$, no characteristic temperature, no phase transitions)

To get something more interesting we can introduce a different, independent source of conformal breaking. For example, put $N=4$ SYM on a curved spacetime (e.g. on S^3)

now we have to study the theory on

$$S^1_\beta \times S^3_R$$

two independent radii $\rightarrow$ can form dimensionless parameter βR.

Now we have to look at geometries which asymptote to:

$$ds^2 \sim \frac{\ell^2}{z^2}\left[dz^2 + d\tau^2 + R^2 d\Omega_3^2\right]$$

+ subleading

as $z \to 0$

By rescaling z we can always make ~~R=l~~ $R = \ell$ (we can always eliminate one scale).

then this metric has the same asymptotics as global AdS_5:

$$ds^2 = \ell^2\left[\cosh^2\rho\, d\varphi^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2\right]$$

$$\xrightarrow[\rho\to\infty]{} \ell^2\left[e^{2\rho} d\varphi^2 + e^{2\rho} d\Omega_3^2 + d\rho^2\right]$$

$e^{\rho} = \ell/z$, $\tau = \ell\varphi$

$$= \frac{\ell^2}{z^2}\left[dz^2 + d\tau^2 + \ell^2 d\Omega_3^2\right]$$

now compactify on a circle of lenght β

$\to$ $S^1_\beta \times S^3_\ell$

there are other solutions with the same asymptotics: global AdS-Schwarzschild

$$ds^2 = f(r)\,d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2$$

$$f(r) = 1 + \frac{r^2}{\ell^2} - \frac{\mu}{r^2}$$

- now this has spherical symmetry (not planar).
- For $\mu = 0$ we get back global AdS_5: change coordinates to

$$\tau = \ell\psi\,, \quad r/\ell = \sinh\rho$$

and we find:

$$ds^2 = \left(1 + \frac{r^2}{\ell^2}\right) d\tau^2 + \frac{dr^2}{\left(1 + r^2/\ell^2\right)} + r^2 d\Omega_3^2$$

(yet another way to write global AdS_5 metric)

the boundary is at $r \longrightarrow \infty$

- Horizon : $f(r_H) = 0$

$$1 + \frac{r_H^2}{\ell^2} - \frac{\mu}{r_H^2} = 0 \Rightarrow r_H^2 = -\frac{\ell^2}{2} + \sqrt{\frac{\ell^4}{4} + \mu\ell^2}$$

$$r_H \to \begin{cases} \infty & \mu \to \infty \\ 0 & \mu \to 0 \quad \text{(no black hol} \end{cases}$$

- Temperature : by requiring regularity at r_H ~~k~~ we get:

$$\beta = \frac{4\pi}{f'(r_H)}$$

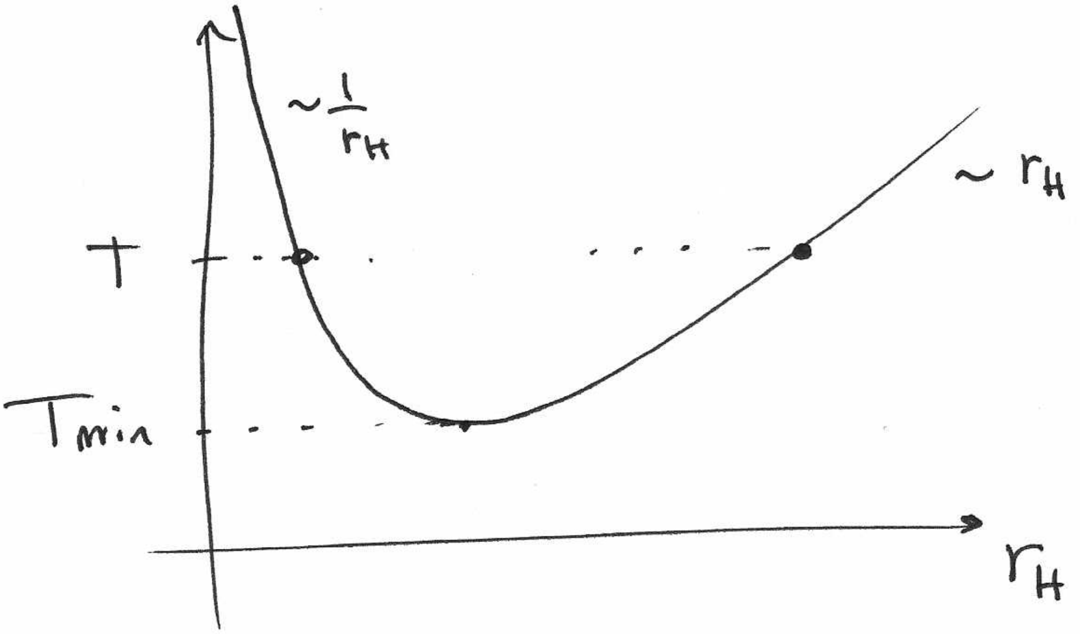
now : $$f'(r_H) = \frac{2r_H}{\ell^2} + \frac{2\mu}{r_H^3} = 2\frac{(2r_H^2 + \ell^2)}{\ell^2 r_H}$$

use equation that defines r_H

$$\Rightarrow \quad \beta = \frac{2\pi\ell^2 r_H}{\ell^2 + 2r_H^2}$$

so now T is a nontrivial function of r_H :

$$T = \frac{2r_H^2 + \ell^2}{(2\pi)\ell^2 r_H} \longrightarrow \begin{cases} +\infty & r_H \to \infty \\ +\infty & r_H \to 0 \end{cases}$$



- $T(r_H)$ is non-monotonic and it has a minimum, $T_{min} > 0$ (strictly positive)

$\Rightarrow$ there is a temperature gap:

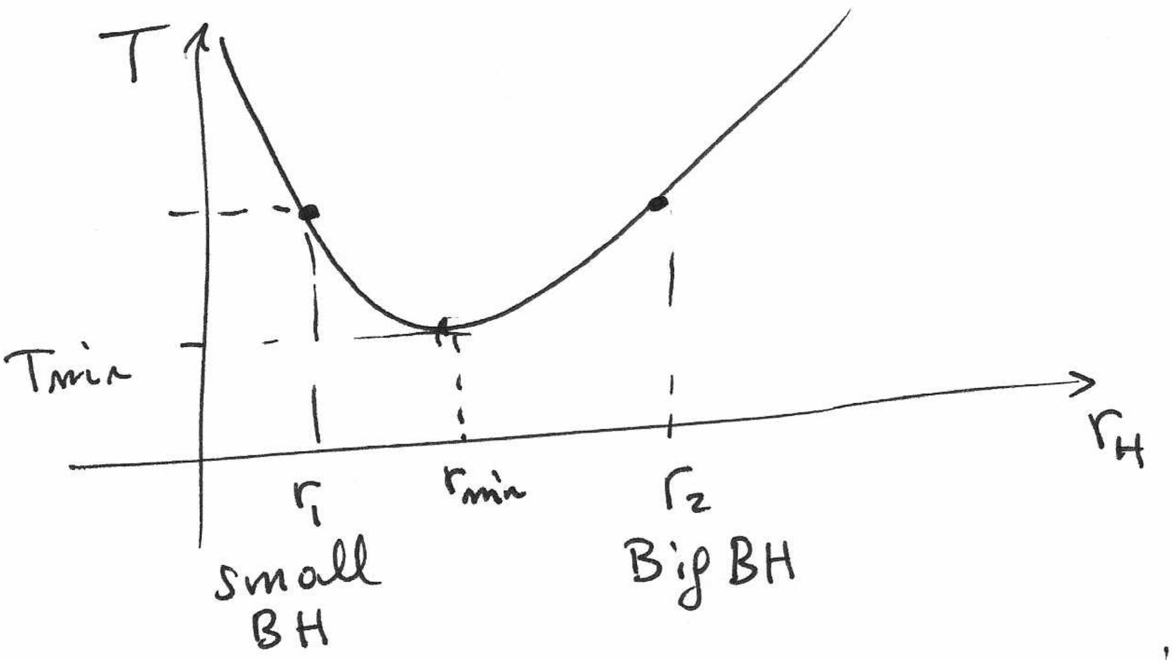
- $T < T_{min}$ only the Global AdS_5 solutions exist

- $T > T_{min}$ 2 Black holes (+ the Global AdS_5) solutions

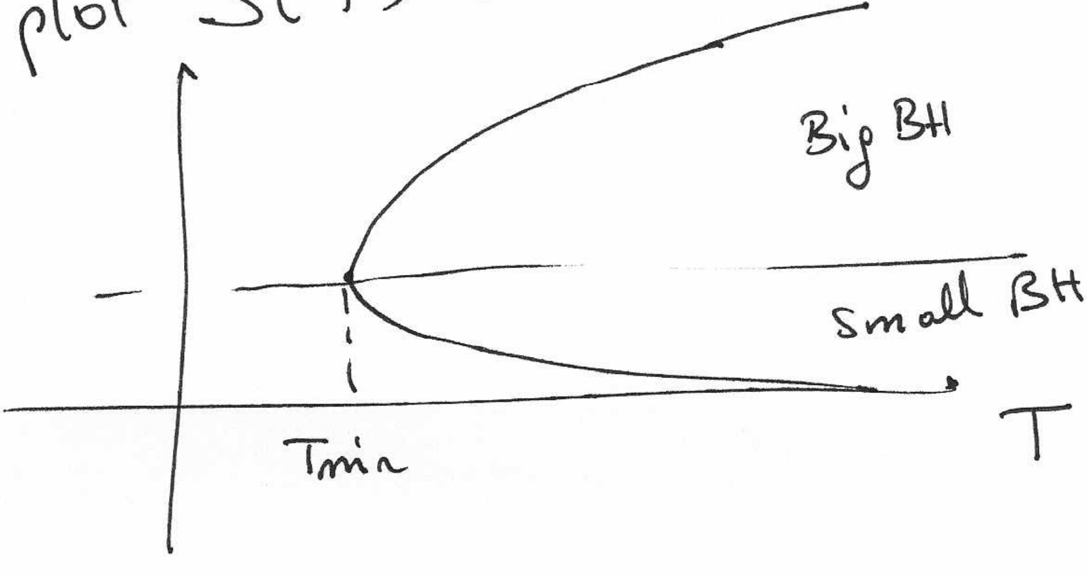
- Horizon area (aka entropy):

$$S_{BH} = \frac{A_H}{4G} = r_H^3 \, Vol(S^3) \times 2\pi M_P^3$$

so the larger r_H, the bigger the black hole



now S is monotonic in r_H so if we plot $S(T)$ it looks like this



Can we get the Free energy ?

We can get the free energy <u>difference</u> between each BH and the thermal AdS_5 solution by integrating

$$\mathcal{S} = -\frac{\partial \mathcal{F}}{\partial T}$$

But $\mathcal{S}(T)$ is not a function → we have to do it in 2 rounds!

On each branch :

$$\mathcal{F}_{SBH}(T) = -\int_{T_{min}}^{T} \mathcal{S}_{SBH}\, dT + C_1$$

$$\mathcal{F}_{BBH}(T) = -\int_{T_{min}}^{T} \mathcal{S}_{BBH}\, dT + C_2$$

now change integration variable to r_H : both $\mathcal{S}$ and T are proper functions of r_H :

$$\mathcal{F}_{SBH}(r_H) = -\int_{r_{min}}^{r_H} \mathcal{S}(u)\frac{dT}{du}\, du + C_1 \qquad (r_H < r_{min})$$

$$\mathcal{F}_{BBH}(r_H) = -\int_{r_{min}}^{r_H} \mathcal{S}(u)\frac{dT}{du}\, du + C_2 \qquad (r_H > r_{min})$$

At $r_H = r_{min}$ the solutions coincide $\Rightarrow$

$$\underset{\substack{\parallel \\ C_1}}{\mathcal{F}_{SBH}(r_{min})} = \underset{\substack{\parallel \\ C_2}}{\mathcal{F}_{BBH}(r_{min})}$$

$$\Rightarrow \quad C_1 = C_2 .$$

$\Rightarrow$ we can write a single expression:

$$\mathcal{F}_{BH}(r_H) = - \int_{r_{min}}^{r_H} S(u) \frac{dT}{du} \, du + C$$

(valid for any $r_H \in (0, +\infty)$)

to determine C one has to really compute the on-shell action on the solution (done by Hawking and Page in 1983). However, to understand the thermodynamics, we don't need to.

All we need in fact is (to see which is the dominant sol.):

$$\Delta F = \mathcal{F}_{BH}(r_H) - \mathcal{F}_{AdS_5}$$

and we also know that

$$\mathcal{F}_{AdS_5} = \mathcal{F}_{BH}(r_H = 0) = C + \int_0^{r_{min}} S \frac{dT}{du} du$$

$$\to \mathcal{F}_{BH}(r_H) - \mathcal{F}_{AdS_5} = -\int_{r_{min}}^{r_H} S \frac{dT}{du} du + \cancel{C}$$

$$\cancel{-C} - \int_0^{r_{min}} S \frac{dT}{du} du$$

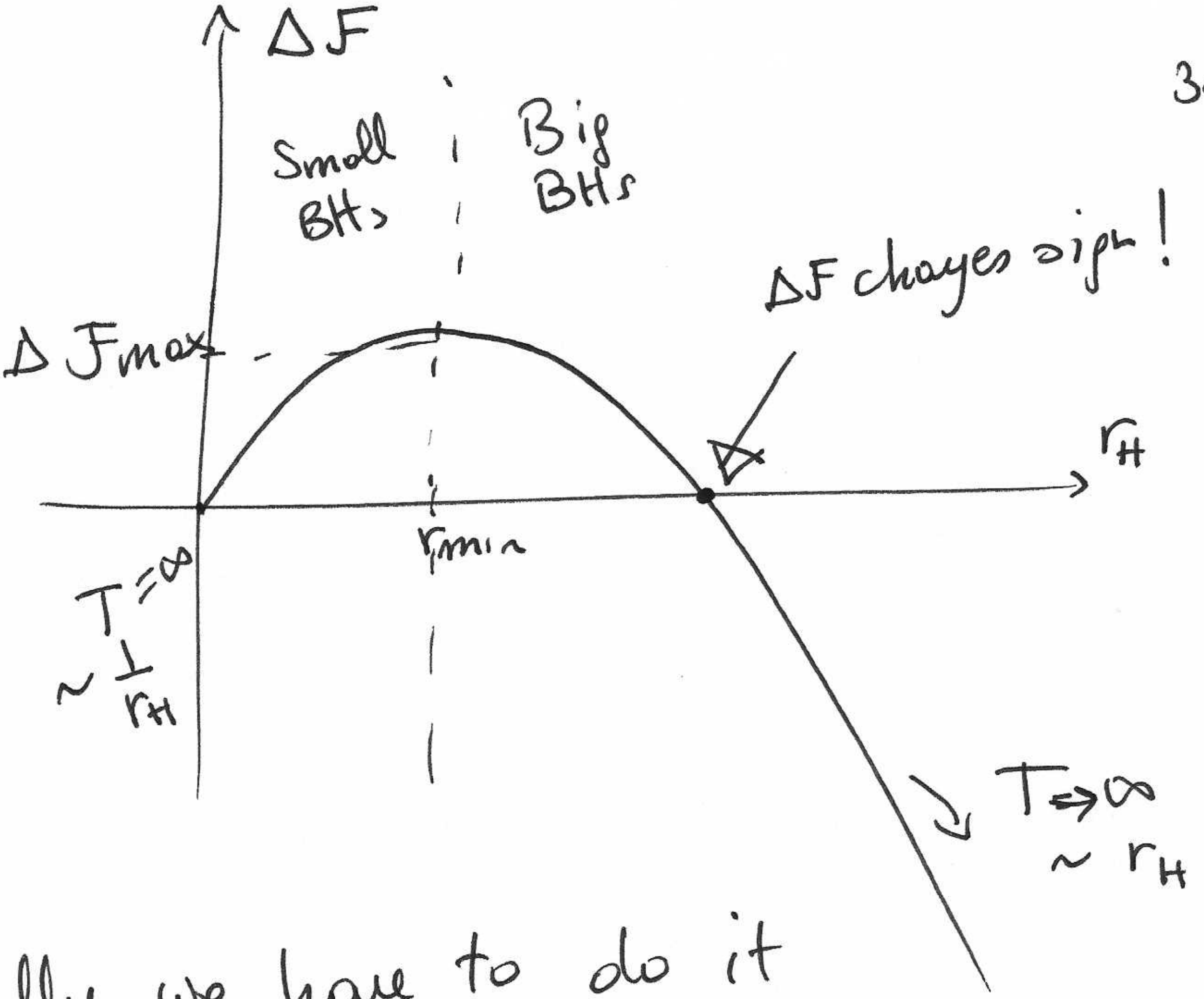
$$\to \Delta \mathcal{F}(r_H) = -\int_0^{r_H} S(u) \frac{dT}{du} du$$

- As $r_H \to 0$, $\Delta \mathcal{F} \to 0^+$
  ($dT/dr_H < 0$)

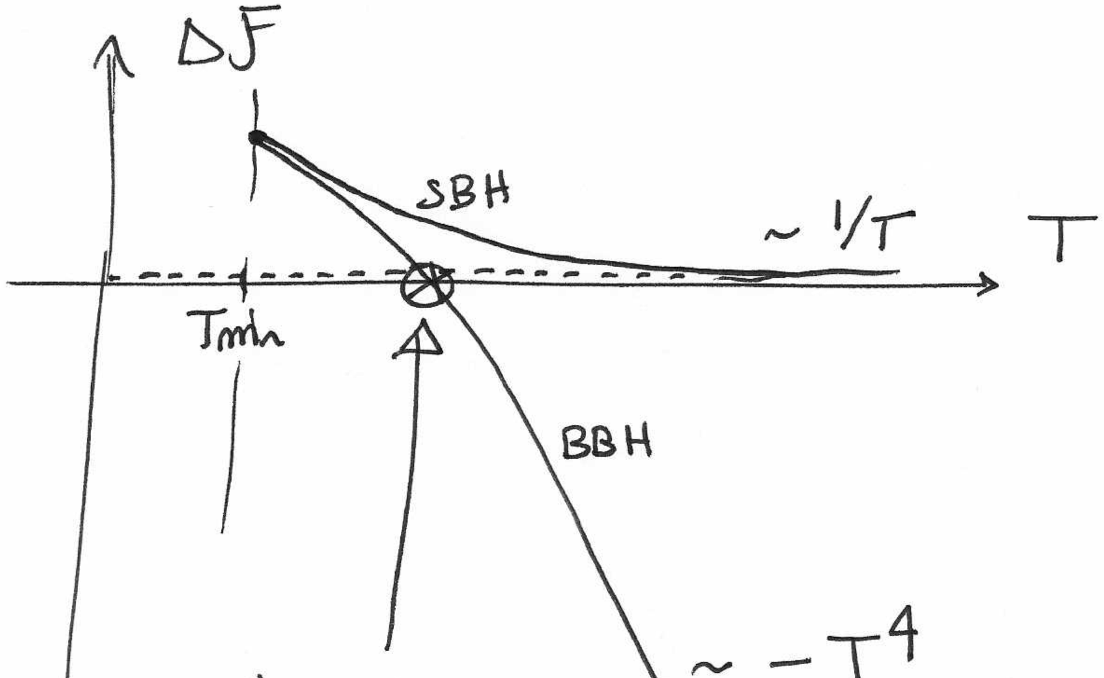
- As $r_H \to +\infty$ $\Delta \mathcal{F} \to -\infty$
  (as $-r_H^4$)

- $\Delta \mathcal{F}$ has an extremum at r_{min},
  where $dT/du = 0$ and $S \neq 0$

$\Rightarrow$



Finally we have to do it as a function of Temperature:



ΔF changes sign at this temperature T_c

High temperature : $F \sim -N^2 T^4$ (like in conformal case)

$\Rightarrow$ - For $T < T_c$ the global AdS_5 solution has lower free energy

- For $T > T_c$: the <u>Big Black Hole</u> has lower free energy

(- the small black hole is always subleading $\forall T$)

At $T = T_c$: <u>first order phase transition</u>

- Latent heat : $\Delta E \big|_{T_c} = + \cancel{\Delta F} T \Delta S$

$$\Delta E \sim T_c S_{BH} \div T_c N^2$$

- $T_c \sim 1/\ell$ (only scale around), but if we reinstate the radius of S^3, then we will find $T_c \sim 1/R$.

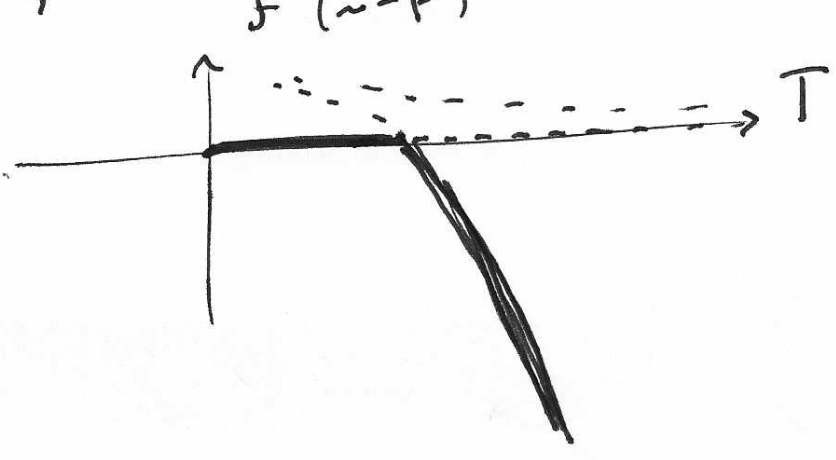
Direct calculation of the on-shell action gives

$$S_E(BH) - S_E(\text{Thermal AdS}) \left(\equiv \beta \Delta F\right)$$

$$= \frac{\pi^2 r_H^3 (\ell^2 - r_H^2)}{4 G_N (2 r_H^2 + \ell^2)}$$

this changes sign at $r_{H_c} = \ell$ and one can solve for T_c from

. Phase diagram is obtained by Maxwell contruction: the equilibrium state which dominates the ensemble at any temperature is the one with lowest F



P-T diagram