M2 ICFP 2019/2020

Introduction to AdS/CFT TD n°4

More fields in Anti-de Sitter spacetime

Throughout, the AdS radius is denoted by ℓ. The metric in Poincaré coordinates is :

$$ds^2 = \frac{\ell^2}{z^2}\eta_{ab}dx^a dx^b = \frac{\ell^2}{z^2}\left[dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right] \qquad 0 < z < +\infty$$

Throught the text, we denote $(d+1)$-dimensional (bulk) indices by $a, b, \ldots$ and d-dimensional (boundary) indices by $\mu, \nu \ldots$.

1. Bulk-to-boundary propagator

The bulk-to-boundary propagator for a massive scalar in $(E)AdS_{d+1}$ is the solution of

$$\left(-\Box + m^2\right) K(z, x^\mu) = 0, \qquad K(z, x^\mu) \to z^{\Delta_-}\delta^{(d)}(x^\mu), \;\; z \to 0$$

with regular behavior in the interior (in Eucliedan signature). Here $\Box = g^{ab}\nabla_a\nabla_b$ is the $(E)AdS_{d+1}$ Laplacian/D'Alambertian. We define Δ_- and Δ_+ as the two solutions of the equation

$$\Delta(\Delta - d) = m^2\ell^2,$$

with $\Delta_- < \Delta_+$. They are related by $\Delta_+ = d - \Delta_-$.

1. Show that the expression

$$\phi(z, x) = \int d^d x' K(z, x - x')\phi_-(x')$$

solves the scalar wave equation together with boundary conditions specified by $\phi_-(x)$,

$$\left(-\Box + m^2\right)\phi(z, x) = 0; \qquad \phi(z, x) \to z^{\Delta_-}\phi_-(x'), \; z \to 0.$$

2. Show that the expression

$$K(z; x) = C\frac{z^{\Delta_+}}{(z^2 + x^2)^{\Delta_+}}, \qquad x^2 \equiv \eta^{\mu\nu}x_\mu x_\nu$$

is the bulk-to-boundary propagator for an approproate choice of the constant C.

3. The bulk Green's function $G(z, x; z', x')$ is defined as the solution of

$$\left(-\Box_{z,x} + m^2\right) G(z, z'; x - x') = \frac{1}{\sqrt{g}}\delta^d(x - x')\delta(z - z'),$$

such that it is regular in the interior and normalizable at the boundary.
Show that the bulk-to-boundary propagator is realted to the bulk Green's function $G_B(z, z'; x)$ by

$$K(z, x) = \lim_{z' \to 0}\frac{\Delta_+ - \Delta_-}{z'^{\Delta_+}}G(z, z'; x)$$

(here we have set set $x' = 0$)
[It is not necessary to use the explicit formula for K or G_B to show this.]

2. Vector fields in AdS

2.1 Gauge field

We consider an abelian gauge field minimally coupled to gravity, on the spacetime AdS_{d+1}. The action is

$$S = -\frac{1}{\ell^{d-1}} \int dz \int d^d x \frac{1}{4} F_{ab} F^{ab}, \qquad F_{ab} = \partial_a A_b - \partial_b A_a$$

The action has the usual gauge-invariance under

$$A_a \to \tilde{A}_a = A_a + \partial_a \alpha$$

We want to understand the field-operator correspondence in this case. In particular, this exercice will show that the operator dual to a gauge field in AdS is a conserved current. Conversely, every global symmetry on the field theory side must correspond to a gauge symmetry on the gravity side.

1. Show that the following is a consistent gauge choice :

$$A_z = \partial^\mu A_\mu = 0 \tag{1}$$

   i.e. given any field configurationm A_a it is always possible to find a gauge transformation such that the new field $\tilde{A}_a$ satisfies the condition above.

2. Write the field eqution in a general gauge, then with the choice (1).
3. Find the solution near the boundary, and determine the scaling of the two independent power-like solutions, with two appropriate exponents $\Delta_\pm$ (they are *not* the same as those for a scalar),

$$A^\pm_\mu(x, z) \simeq z^{\Delta_\pm} v^\pm_\mu(x),$$

4. Find the weight of $v^\pm_\mu(x)$ under scale transformations. What are the dimensions of the operator dual to A_μ and of the corresponding coupling, on the field theory side ?
5. Show that gauge-invariance in the AdS side implies, on the field theory side, the conservation of the operator J_μ dual to A_μ.

2.2 Massive vector field

Consider now a massive vector field in AdS_{d+1}. The action is

$$S = -\frac{1}{\ell^{d-1}} \int dz \int d^d x \left[\frac{1}{4} F_{ab} F^{ab} + \frac{m^2}{2} A_a A^a \right], \qquad F_{ab} = \partial_a A_b - \partial_b A_a$$

1. Show that the *transverse polarizations* of A_μ (i.e. those such that $\partial_\mu A^\mu = 0$) are decoupled from the A_z and longitudinal polarizations, and satisfy the equation of motion :

$$A''_\mu - \frac{(d-3)}{z} A'_\mu + \eta^{\rho\sigma} \partial_\rho \partial_\sigma A_\mu - \frac{m^2 \ell^2}{z^2} A_\mu = 0.$$

   where $' \equiv \partial_z$.

2. Find the the near-boundary scaling behavior the solution in terms of $\Delta_\pm$ and determine the dimension of the corresponding operator and coupling on the field theory side.