M2 ICFP 2019/2020

Introduction to AdS/CFT TD n°1

Conformal Field Theory

1. Conformal Algebra

The generators of infinitesimal conformal transformations on functions of x^μ are :

$$P_\mu = -i\partial_\mu; \quad L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu);$$
$$D = -ix^\mu\partial_\mu; \quad K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^\nu x_\nu\partial_\mu)$$

Verify the commutation relations :

$$[D, P_\mu] = iP_\mu, \qquad [D, K_\mu] = -iK_\mu.$$

2. Special Conformal Transformations

We want to construct finite special conformal transformation, starting from the infinitesimal ones :

$$x^{\mu'} = 2(b^\nu x_\nu)x^\mu - b^\mu x^2 \qquad (x^2 \equiv x^\mu x_\mu).$$

1. Show that the *inversion*
$$I : x^\mu \to x^{\mu'} = \frac{x^\mu}{x^2}$$
is a conformal transformation. Can we obtain I from an infinitesimal transformation ?
2. Show that the infinitesimal special conformal generators K are related to the translation generators P by
$$K_\mu = -IP_\mu I$$
[use the action of the transformations on the coordinates]
3. From the previous result and the fact that $I^2 = 1$, show that
$$e^{ib^\mu K_\mu} = Ie^{-ib^\mu P_\mu}I.$$
Use this identity to find the form of the finite special conformal transformation with parameter b^μ.

3. Noether currents

Consider a classical field theory with $\{\phi_a\}$ and an infinitesimal transformation, with parameter ϵ, which we assume linear in ϵ (but not necessarily in ϕ) :

$$\delta\phi \equiv \phi'(x) - \phi(x) = X(\phi)\epsilon$$

If such a transformation leaves the action invariant, or equivalently if the functional variation of the Lagrangian is a total divergence,

$$\delta\mathcal{L} = \epsilon(\partial^\mu F_\mu),$$

then Noether's theorem ensures the existence of a current J^μ, conserved on the equations of motion, and given by

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}X(\phi) - F^\mu.$$

Below we consider the two kinds of transformations corresponding to translation and dilatation of the coordinates, and the corresponding action on a scalar field of conformal weight Δ,

$$x'_\mu = x_\mu - \epsilon_\mu, \quad \phi'(x') = \phi(x)$$

and

$$x'_\mu = x_\mu - \epsilon x_\mu, \quad \phi'(x') = (1 + \Delta\epsilon)\phi(x)$$

1. Write the functional variation $\delta\phi(x)$ of the field under the two transformations
2. Write functional variation of the Lagrangian, and verify that it is a total divergence in both cases (assuming the classical theory is scale invariant). This will lead you to identify the quantity F^μ. *[Recall that for a CFT, the Lagrangian is a scalar field of weight d]*
3. Construct the Noether currents for translation and dilatation using the general formula, and check they are given by the following expressions :

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)}\partial_\nu \phi - \eta_{\mu\nu}\mathcal{L}, \qquad J^D_\mu = x^\nu T_{\mu\nu} + \Delta\phi \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)}.$$

4. Improved stress tensor and conformal coupling

Consider the theory of a real scalar field in d dimensions with action

$$S = -\int d^d x \left[\frac{1}{2}\partial^\mu \phi \partial_\mu \phi + \frac{\lambda}{n}\phi^n\right]$$

3.1. Canonical stress tensor

i. What must be the conformal weight Δ of the field ϕ such that the kinetic term has the appropriate scaling dimension ?

ii. Write down the explicit form of the canonical (i.e. Noether) stress tensor $T_{\mu\nu}$, of its trace, and of the dilatation Noether current J^μ_D. For which value(s) of n is the dilatation current conserved (on a solution of the equations of motion) ? Is the stress tensor traceless in this case ?

3.2. Improved stress tensor

We want to construct an *improved* stress tensor $\Theta_{\mu\nu}$ such that 1) it is conserved 2) its tracelessness ensures conservation of the dilation current, or in other words such that

$$J^\mu_D = x^\mu \Theta_{\mu\nu}, \qquad EOM \Rightarrow \partial^\mu \Theta_{\mu\nu} = 0$$

i. Show that both can be achieved by defining

$$\Theta_{\mu\nu} = T_{\mu\nu} + \xi\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^\rho\partial_\rho\right)\phi^2$$

for an appropriate value of ξ (which depends on d).

ii. Show that the improved stress tensor $\Theta_{\mu\nu}$ is traceless (on solutions of the EOM) for the appropriate value of n (which you have found in part 1) for which the theory is conformally invariant.

3.3. Covariant version

Now consider the same scalar field theory minimally coupled to a background metric $g_{\mu\nu}$,

$$S = -\int d^d x \sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi + \frac{\lambda}{n}\phi^n\right]$$

i. Show that the definition

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}$$

gives back the canonical stress tensor (after setting $g_{\mu\nu} = \eta_{\mu\nu}$)

ii. Can you find a (covariant) term such that, when added to the action, gives the improved stress tensor of part 2 as

$$\Theta_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \ ?$$

[Hint : recall how the Ricci scalar changes under a variation of the metric. Remember to set $g_{\mu\nu} = \eta_{\mu\nu}$ at the end of the calculation, since we want to reproduce the flat space result of part 2.].

The resulting action is the action of a *conformally coupled* scalar field. It is conformal in any background, i.e. it is invariant under Weyl transformations (which are *not* coordinate transformations)

$$g_{\mu\nu} \to e^{2\omega(x)} g_{\mu\nu}, \quad \phi \to e^{-\Delta\omega(x)}\phi$$

for an arbitrary function $\omega(x)$.