

## Introduction to Supersymmetry

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ABSTRACT: This is a very lightly edited version of a set of lecture notes for five introductory lectures on supersymmetry given in June at the TASI summer school in Boulder CO. As such, it is often pretty telegraphic. My approach is basically quite old-fashioned, and mostly follows *Superspace* by Gates *et al.*

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## LECTURE I.

### I. Introduction to the introduction.

In the first lecture, I discuss superalgebras schematically and show how to construct representations by introducing superspace. This is a coset of the Super Poincaré group ( $SISO(D-1, 1)$ ) mod the Lorentz group ( $SO(D-1, 1)$ ), and is parametrized by spacetime coordinates  $x$  and anticommuting spinor coordinates  $\theta$ . Then I construct a representation of the  $SISO(D-1, 1)$  algebra as superdifferential operators acting on superfields  $\Phi(x, \theta)$ . I also find a supercovariant derivative that (anti)commutes with (super)translations and transforms as a spinor under the Lorentz group. I use this to define a superspace measure that allows one to construct supersymmetric actions, as well as to define components of superfields. Next, I describe how supersymmetry varies as one changes dimension. We will see that starting from one supercharge ( $N = 1$ ) in a higher dimension, we can get several supercharges ( $N > 1$ ) in lower dimensions. Finally I'll give the explicit supersymmetry algebras for  $N = 1, 2$  in  $D = 4, 3, 2$ .

### II. Introduction to supersymmetry.

What is supersymmetry? It is a symmetry between *bosons* and *fermions* with fermionic charges and fermionic parameters. Consider a schematic action

$$S \sim \int (B \square B + F \not{D} F); \quad (1)$$

Let's look for an invariance of the form

$$\delta B = i\varepsilon F$$

where  $F$  is commuting and  $\varepsilon$  is anticommuting. Dimensional analysis, or the form of the

action  $S$  implies

$$\delta F = -i(\not{D}B)\varepsilon$$

What is the symmetry algebra? We compute and find

$$[\delta_\varepsilon, \delta_\eta]B = +2\varepsilon\gamma^\mu\eta D_\mu B \Rightarrow \{Q, Q\} \sim \not{D}, \quad (2)$$

where  $Q$  is the generator of supersymmetry transformations:

$$\delta B = [i\varepsilon Q, B], \quad D_\mu B = [iP_\mu, B], \text{ etc.}$$

Eq. (2) is what I mean by supersymmetry: A graded algebra with fermionic charges that anticommute to give a spacetime translation. Thus, for example, the phenomenologically generated graded symmetry that nuclear physicists call supersymmetry is not supersymmetry in my sense: it is an internal graded symmetry, and doesn't have the form of (2).

Why is supersymmetry interesting? There are several reasons, some mathematical, and some physical. Historically, supersymmetry arose in two contexts: as a symmetry on the world sheet of the NSR formulation of the superstring, which is the only known consistent quantum theory of gravity, and as a symmetry of certain four dimensional theories; in such supersymmetric theories, quadratic ultraviolet divergences are absent, and thus these theories are “natural” in a certain technical sense.

### III. Introduction to superspace.

Still at a schematic level, let's construct superspace. This will give us a more or less systematic way of studying representations of supersymmetry.

Recall how one finds representations of the Poincaré group  $ISO(D-1, 1)$ . One defines spacetime as the coset  $ISO(D-1, 1)/SO(D-1, 1)$ . That is, for the  $ISO(D-1, 1)$  algebra

$$[J, J] \sim iJ, \quad [J, P] \sim iP, \quad [P, P] = 0,$$

we parametrize the quotient as

$$h(x) = e^{ix \cdot P} \text{mod } SO(D-1, 1).$$

We can act by left multiplication:

$$\begin{aligned} h(x') &= e^{iw \cdot J} e^{ix \cdot P} \text{mod } SO(D-1, 1) = e^{iw \cdot J} e^{ix \cdot P} e^{-iw \cdot J} \text{mod } SO(D-1, 1) \\ &= e^{ix \cdot (e^{iw \cdot J} P e^{-iw \cdot J})} \text{mod } SO(D-1, 1) = e^{i(e^w x) \cdot P} \text{mod } SO(D-1, 1) \end{aligned}$$

$\Rightarrow x' = e^w x$  for  $e^w$  the vector representation of  $SO(D-1, 1)$  induced by the commutator  $[J, P]$ . Similarly,

$$h(x') = e^{i\xi \cdot P} e^{ix \cdot P} \text{mod } SO(D-1, 1) = e^{i(\xi+x) \cdot P} \text{mod } SO(D-1, 1) \Rightarrow x' = x + \xi$$

(Linear) representations of  $ISO(D-1, 1)$  are found by introducing fields  $\phi(x)$  that transform as some matrix representation of  $SO(D-1, 1)$ :

$$[M, \phi] \sim \phi.$$

We define the transformation by:

$$\phi'(x') = e^{w \cdot M} \phi(x) e^{-w \cdot M},$$

which implies

$$\delta\phi \equiv \phi'(x) - \phi(x) = -\delta x \cdot \partial_x \phi + w \phi.$$

Identifying this as  $\delta\phi = i[w \cdot J + \xi \cdot P, \phi]$ , we find

$$P = i\partial_x, \quad J = ix \wedge \partial_x - iM$$

For superspace, we just make  $ISO(D-1, 1) \rightarrow SISO(D-1, 1)$  above; then

$$P \rightarrow (P, Q), \quad x \rightarrow (x, \theta), \quad \xi \rightarrow (\xi, \varepsilon).$$

This leads to:

$$P = i\partial_x, \quad Q = i(\partial_\theta - \frac{1}{2}\theta\partial_x), \quad J = i(x \wedge \partial_x + \theta\sigma\partial_\theta) - iM,$$

where  $\sigma$  is the spin  $\frac{1}{2}$  representation of the Lorentz group.

We find an important operator that (anti)commutes with  $Q, P$  by noting that left multiplication always commutes with right multiplication:

$$e^{-\varepsilon D} h(x, \theta) \equiv h(x, \theta) e^{i\varepsilon Q} \Rightarrow \{D, Q\} = 0, \quad [D, P] = 0.$$

Using the Baker-Campbell-Hausdorff theorem  $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots}$  we find

$$D = -iQ + \theta \not{P} = \partial_\theta + \frac{1}{2} \theta \not{\partial}_x$$

$D$  is the covariant generalization of  $\partial_\theta$ . If  $\Phi$  is a superfield, so is  $D\Phi$ ; neither  $Q\Phi$  nor  $\partial_\theta\Phi$  are (they do not transform simply under  $Q$ ).  $D$  is a  $SO(D-1, 1)$  spinor and obeys

$$\{D, D\} \sim i \not{\partial}_x. \quad (3)$$

It has several important uses:

- 1) Components of a superfield are best defined by covariant expansion:

$$\phi = \Phi|, \quad \Psi = D\Phi|, \quad \dots etc.$$

where  $|$  means the  $\theta$ -independent projection.

- 2) The basic superspace invariant measure is

$$\int dx(D)^{\mathcal{N}} \mathcal{L}(\Phi, D\Phi, \dots)$$

where  $\mathcal{N}$  is the number of (real) components of  $D$ . This is invariant because  $(D)^{\mathcal{N}+1} \sim \partial_x$  (because of (3)), and  $\delta\mathcal{L} = i\varepsilon Q(\mathcal{L}) = -\varepsilon D(\mathcal{L}) + (\partial_x \mathcal{L}\text{-term})$ .

#### IV. Supersymmetry in different dimensions.

Suppose we start with a basic supersymmetry algebra in some dimension  $D$ :

$$\{Q, Q\} \sim \not{P}, \quad [J, Q] \sim iQ, \quad [J, P] \sim iP, \quad [J, J] \sim iJ.$$

We can reduce to a lower  $D$ ; that is we keep the following subalgebra:

$$P \rightarrow (P, 0), \quad J_D \rightarrow \begin{pmatrix} J_{D-n} & 0 \\ 0 & T_n \end{pmatrix}, \quad \text{and all the } Q's,$$

where the  $T$ 's are generators of some internal group  $G$  and the  $Q$ 's are now in (the spinor representation of  $SO(D-n, 1)$ )  $\times$  (a representation of  $G$ ). The remaining subalgebra is

$$\{Q, Q\} \sim \not{P}, \quad [J, J] \sim iJ, \quad [J, Q] \sim iQ, \quad [J, P] \sim iP, \quad [T, Q] \sim iQ, \quad [T, T] \sim iT.$$

Thus  $N = 1$  in higher dimensions becomes, in general, higher  $N$  in lower dimensions; just how depends on the properties of the spinor representations of  $SO(D-1, 1)$ . The following table summarizes the relations:

$D$	11	10	9	8	7	6	5	4	3	2
Spinor type	M	MW	M	W	D	W	D	W	M	MW
Real spinor dim	32	16	16	16	16	8	8	4	2	1
Real/Complex	R	R	R	C	C	C	C	C	R	R
$N$	1	2	2	2	2	4	4	8	16	(16,16)
		1	1	1	1	2	2	4	8	(8,8)
						1	1	2	4	(4,4)
								1	2	(2,2)
									1	(1,1)

Table I

Key: M–Majorana, W–Weyl, D–Dirac.

Comments:  $D = 4$  spinors can be either Majorana *or* Weyl but not both; thus in  $D = 4$ , one can have either a real 4-dimensional spinor, or a complex 2-dimensional one.  $D = 2$  is special, and allows  $(p, q)$  supersymmetry because it has real Weyl spinors;  $(p, q)$  counts the number of left and right handed spinors. (Actually, you can have this in any dimension where there is a real or pseudo real Weyl representation, *e.g.*,  $D = 6, 10$ .)

## V. Explicit supersymmetry algebras in $D = 4, 3, 2$ .

We now examine some explicit examples of superalgebras. We begin in  $D = 4$ , where

we have complex Weyl spinor supercharges  $Q_\alpha$ ,  $\bar{Q}_{\dot{\alpha}} = -(Q_\alpha)^\dagger$  for  $\alpha = (+, -)$ . We can raise and lower spinor indices by  $C_{\alpha\beta} = -C_{\beta\alpha}$ ,  $C_{\alpha\beta}C^{\gamma\beta} = \delta_\alpha^\gamma$ ; for example,  $Q^\alpha = C^{\alpha\beta}Q_\beta$ ,  $Q_\alpha = Q^\beta C_{\beta\alpha}$ .\* The spinors  $Q$  are in the  $(\frac{1}{2}, 0)$  representation of  $SL(2\mathbf{C})$ , their conjugates  $\bar{Q}$  are in the  $(0, \frac{1}{2})$  representation, and vectors are in the  $(\frac{1}{2}, \frac{1}{2})$  representation. Thus vectors are naturally described as  $2 \times 2$  matrices. Then the  $N = 1$  superalgebra has charges  $Q_\alpha, \bar{Q}_{\dot{\alpha}}, P_{\alpha\dot{\alpha}}, J_{\alpha\beta}, \bar{J}_{\dot{\alpha}\dot{\beta}}$  which obey the following algebra:

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} &= P_{\alpha\dot{\alpha}} \quad , \\ [J_{\alpha\beta}, Q_\gamma] &= \frac{1}{2}i(C_{\gamma\alpha}Q_\beta + C_{\gamma\beta}Q_\alpha) \quad , \\ [J_{\alpha\beta}, P_{\gamma\dot{\gamma}}] &= \frac{1}{2}i(C_{\gamma\alpha}P_{\beta\dot{\gamma}} + C_{\gamma\beta}P_{\alpha\dot{\gamma}}) \quad , \\ [J_{\alpha\beta}, J_{\gamma\delta}] &= \frac{1}{2}i(C_{\gamma\alpha}J_{\beta\delta} + C_{\gamma\beta}J_{\alpha\delta} + C_{\delta\alpha}J_{\beta\gamma} + C_{\delta\beta}J_{\alpha\gamma}) \quad , \end{aligned}$$

and all other commutation relations either vanish or follow by hermitian conjugation. The spinor derivatives are:

$$\begin{aligned} D_\alpha &= \partial_\alpha + \frac{i}{2}\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \quad \bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} + \frac{i}{2}\theta^\alpha\partial_{\alpha\dot{\alpha}} \\ \{D_\alpha, D_\beta\} &= \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0, \quad \{D_\alpha, \bar{D}_{\dot{\alpha}}\} = i\partial_{\alpha\dot{\alpha}} \end{aligned}$$

The algebra for  $N = 2$  in  $D = 4$  is essentially the same, except that  $Q$  and  $D$  get doubled:

$$\{D_{a\alpha}, \bar{D}_{\dot{\beta}}^b\} = i\delta_a^b\partial_{\alpha\dot{\beta}}, \quad (a = 1, 2).$$

One may also include an  $SU(2)$  internal symmetry under which the  $Q$ 's transform as isospinors.

In  $D = 3$ , for  $N = 2$ , we just drop the “ $\cdot$ ” from  $N = 1$  above, impose  $P_{\alpha\beta} = P_{\beta\alpha}$ , and let  $J_{\alpha\beta}$  be real.

For  $D = 3$ ,  $N = 1$ , there is only one real  $Q_\alpha$ , and of course  $P_{\alpha\beta}$  and  $J_{\alpha\beta}$ ; the algebra

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\* Two frequently used identities for two component spinors are  $X_{[\alpha\beta]} \equiv X_{\alpha\beta} - X_{\beta\alpha} = -C_{\alpha\beta}X^\gamma{}_\gamma$  and  $X_{[\alpha\beta\gamma]} = 0$ .



is:

$$\begin{aligned}
\{Q_\alpha, Q_\beta\} &= 2P_{\alpha\beta} \quad , \\
[J_{\alpha\beta}, Q_\gamma] &= \frac{1}{2}i(C_{\gamma\alpha}Q_\beta + C_{\gamma\beta}Q_\alpha) \quad , \\
[J_{\alpha\beta}, P_{\gamma\delta}] &= \frac{1}{2}i(C_{\gamma\alpha}P_{\beta\delta} + C_{\gamma\beta}P_{\alpha\delta} + C_{\delta\alpha}P_{\beta\gamma} + C_{\delta\beta}P_{\alpha\gamma}) \quad , \\
[J_{\alpha\beta}, J_{\gamma\delta}] &= \frac{1}{2}i(C_{\gamma\alpha}J_{\beta\delta} + C_{\gamma\beta}J_{\alpha\delta} + C_{\delta\alpha}J_{\beta\gamma} + C_{\delta\beta}J_{\alpha\gamma}) \quad ,
\end{aligned}$$

and the spinor derivatives are

$$D_\alpha = \partial_\alpha + i\theta^\alpha \partial_{\alpha\beta}, \quad \{D_\alpha, D_\beta\} = 2i\partial_{\alpha\beta}.$$

Finally, we consider  $D = 2$ . For  $N = (1, 1)$ , we have the algebra of  $D = 3$ ,  $N = 1$  with  $P_{+-} = J_{++} = J_{--} = 0$ . This leaves  $Q_\pm, P_{\pm\pm}, J \equiv J_{+-}$ ; these obey

$$\begin{aligned}
Q_\pm^2 &= P_{\pm\pm} \quad , \\
[J, Q_\pm] &= \pm \frac{1}{2}Q_\pm \quad , \\
[J, P_{\pm\pm}] &= \pm P_{\pm\pm} \quad .
\end{aligned}$$

The corresponding spinor derivatives are

$$D_\pm = \partial_\pm + i\theta^\pm \partial_{\pm\pm}, \quad D_\pm^2 = i\partial_{\pm\pm}.$$

Often, I'll Wick rotate; then

$$D_\pm = \partial_\pm + \theta^\pm \partial_{z,\bar{z}}, \quad D_\pm^2 = \partial_{z,\bar{z}}.$$

For  $N = (2, 2)$ , we have

$$D_\pm = \partial_\pm + \frac{i}{2}\bar{\theta}^\pm \partial_{\pm\pm}, \quad \bar{D}_\pm = \bar{\partial}_\pm + \frac{i}{2}\theta^\pm \partial_{\pm\pm},$$

with

$$D_\pm^2 = \bar{D}_\pm^2 = 0, \quad \{D_\pm, \bar{D}_\pm\} = i\partial_{++}, \quad \{D_+, \bar{D}_-\} = \{D_-, \bar{D}_+\} = 0.$$

The Wick rotated form is

$$D_\pm = \partial_\pm + \frac{1}{2}\bar{\theta}^\pm \partial_{z,\bar{z}}, \quad \bar{D}_\pm = \bar{\partial}_\pm + \frac{1}{2}\theta^\pm \partial_{z,\bar{z}}, \quad \{D_\pm, \bar{D}_\pm\} = \partial_{z,\bar{z}}.$$

Finally, for  $N = (4, 4)$ , we have four complex spinor derivatives  $D_{\pm a}, \bar{D}_{\pm}^a$  obeying

$$\{D_{\pm a}, \bar{D}_{\pm}^b\} = i\delta_a^b \partial_{\pm\pm}, \quad (a = 1, 2).$$

## LECTURE II.

### I. Introduction and summary.

In this lecture, I construct several  $N = 1$  models in  $D = 3, 2$ . I write down actions in superspace, define components, and work out component actions. We will find actions that have the usual form for bosons and fermions, but with particular couplings dictated by supersymmetry. For example, a  $D = 3$  or  $2$  scalar superfield has components:

$$\Phi^i \rightarrow \begin{cases} \phi^i & \text{scalar} \\ \psi_{\alpha}^i & \text{spinor} \\ F^i & \text{auxiliary} \end{cases}$$

which together form a scalar supermultiplet. The superspace action for a generalized WZW-model, when expanded in components, will reveal an arbitrary bosonic WZW-model coupled to fermions in a way that is completely specified by supersymmetry. We also will explore the role of the auxiliary field  $F^i$ , and will find that it is needed for two things: (1) Dynamics-independent transformations, and (2) off-shell closure of the algebra of transformations.

The next model that we'll consider is  $D = 3, N = 1$  super Yang-Mills theory. The basic superfield here is a spinor potential

$$\Gamma_{\alpha} \rightarrow \begin{cases} A_{\alpha\beta} & \text{- gauge potential} \\ \lambda_{\alpha} & \text{- a spinor.} \end{cases}$$

$\Gamma_\alpha$  has more components, but they can be gauged away by the gauge transformations  $\delta\Gamma_\alpha = \nabla_\alpha K$ . In superspace, we'll find that all other connections and all field strengths can be constructed in terms of the basic object  $\Gamma_\alpha$ . We'll write down the superspace action for the usual Yang-Mills kinetic term as well as for the Chern-Simons term. The resulting component action has just the usual bosonic terms with minimally coupled adjoint representation fermions. The Chern-Simons term give an ordinary mass to the fermions.

Next, we'll couple  $D = 3$ ,  $N = 1$  super Yang-Mills theory to matter, and find the usual type of couplings with Yukawa couplings that have the *same* coupling constant as the gauge coupling. This is again typical of supersymmetry: supersymmetric theories are usual field theories with particular restrictions on the types of fields and on their couplings. Finally, I'll briefly discuss coupling gauge fields to a nonlinear  $\sigma$ -model with some nonlinear field dependent isometry generated by killing vectors.

## II. Models: Scalar multiplets

Let's start, for example, with a supersymmetric nonlinear  $\sigma$ -model for  $D = 3$ ,  $N = 1$ :  $S = \frac{1}{8} \int d^3x D^2 (D^\alpha \Phi^i D_\alpha \Phi^j g_{ij}(\Phi))$ . This describes maps from  $D = 3$  superspace into a target space manifold with metric  $g$ . We can easily reduce to  $D = 2$ :\* recall that  $D_\alpha$  is best written in terms of its real chiral components  $D_\pm, D_\pm^2 = \partial_{z, \bar{z}}$ . Then

$$\begin{aligned} S &= -\frac{1}{4} \int d^2x D_+ D_- (D_+ \Phi^i D_- \Phi^j - D_- \Phi^i D_+ \Phi^j) g_{ij}(\Phi) \\ &= -\frac{1}{4} \int d^2x D_+ D_- (D_+ \Phi^i D_- \Phi^j + D_+ \Phi^j D_- \Phi^i) g_{ij}(\Phi) \\ &= -\frac{1}{2} \int d^2x D_+ D_- (D_+ \Phi^i D_- \Phi^j) g_{ij}(\Phi) \end{aligned}$$

where the last equality follows because  $g$  is symmetric. A natural generalization of this

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\* We reduce the  $D = 3$  action in the standard way: we assume that all fields are independent of one coordinate, and we drop the integration over that coordinate.

(special to  $D = 2$ ) is

$$S_{gen} = -\frac{1}{2} \int d^2x D_+ D_- (D_+ \Phi^i D_- \Phi^j) (g_{ij}(\Phi) + b_{ij}(\Phi)) \quad (4)$$

where  $b_{ij} = -b_{ji}$  is an antisymmetric tensor on the target space. Let's find the component expansion; we define components

$$\phi^i = \Phi^i|, \quad \psi^i = D_+ \Phi^i|, \quad \bar{\psi}^i = D_- \Phi^i|, \quad F^i = iD_+ D_- \Phi^i|.$$

Then

$$\begin{aligned} S_{gen} &= -\frac{1}{2} \int d^2x D_+ \left[ (-D_+ D_- \Phi^i D_- \Phi^j - D_+ \Phi^i \bar{\partial} \Phi^j) (g_{ij} + b_{ij}) \right. \\ &\quad \left. + D_+ \Phi^i D_- \Phi^j (g_{ij,k} + b_{ij,k}) D_- \Phi^k \right] \\ &= -\frac{1}{2} \int d^2x \left[ \left( -(\partial \bar{\psi}^i) \bar{\psi}^j + F^i F^j - \partial \phi^i \bar{\partial} \phi^j + \psi^i \bar{\partial} \psi^j \right) (g_{ij} + b_{ij}) \right. \\ &\quad \left. + \left[ (-iF^i \bar{\psi}^j + \psi^i \bar{\partial} \phi^j) \psi^k + (\partial \phi^i \bar{\psi}^j + i\psi^i F^j) \bar{\psi}^k - i\psi^i \bar{\psi}^j F^k \right] (g_{ij,k} + b_{ij,k}) \right. \\ &\quad \left. + \psi^i \bar{\psi}^j \psi^l \bar{\psi}^k (g_{ij,kl} + b_{ij,kl}) \right], \end{aligned}$$

where  $g_{ij,k} = \frac{\partial}{\partial \phi^k} g_{ij}$ , etc. We collect terms and integrate by parts to get:

$$\begin{aligned} S &= \frac{1}{2} \int d^2x \left[ \partial \phi^i \bar{\partial} \phi^j (g_{ij} + b_{ij}) - (\psi^i \bar{\nabla} \psi^j + \bar{\psi}^i \nabla \bar{\psi}^j) g_{ij} \right. \\ &\quad \left. - \frac{1}{2} R_{ij,kl}^- \psi^i \psi^j \bar{\psi}^k \bar{\psi}^l - g_{ij} (F^i + i\Gamma_{lk}^{-i} \psi^l \bar{\psi}^k) (F^j + i\Gamma_{mn}^{-j} \psi^m \bar{\psi}^n) \right] \end{aligned}$$

where

$$\begin{aligned} \bar{\nabla} \psi^i &= \bar{\partial} \psi^i + \Gamma_{jk}^{+i} \bar{\partial} \phi^j \psi^k \\ \nabla \bar{\psi}^i &= \partial \bar{\psi}^i + \Gamma_{jk}^{-i} \partial \phi^j \bar{\psi}^k. \end{aligned}$$

$R_{ij,kl}^-$  is the Riemann curvature of  $\Gamma^-$ , and

$$\Gamma_{jk}^{\pm l} = \left\{ \begin{matrix} l \\ jk \end{matrix} \right\} \pm g^{li} \underbrace{\frac{1}{2} (b_{ij,k} + b_{jk,i} + b_{ki,j})}_{T_{ijk} \text{ the torsion}} \quad (5)$$

This is an  $N = 1$  supersymmetric generalized WZW model. We will discuss this model later at great length, but let me make one comment here: if  $R^- = 0$ , then this is precisely a Wess-Zumino-Witten (WZW) model.

We can also add a superpotential:

$$i \int d^2x D_+ D_- V(\Phi) = \int d^2x V_{,i} F^i + i V_{,ij} \psi^i \bar{\psi}^j$$

Finally, let's find the supersymmetry transformations of the components:

$$\delta\phi^i = -(\varepsilon_+ Q_- - \varepsilon_- Q_+) \Phi^i = i(\varepsilon_+ D_- - \varepsilon_- D_+) \Phi^i = i(\varepsilon_+ \bar{\psi}^i - \varepsilon_- \psi^i)$$

(recall  $D = -iQ + 2\theta P$ )

$$\begin{aligned} \delta\psi^i &= -(\varepsilon_+ Q_- - \varepsilon_- Q_+) D_+ \Phi^i = i(\varepsilon_+ D_- - \varepsilon_- D_+) D_+ \Phi^i \\ &= -(\varepsilon_+ F^i + i\varepsilon_- \partial_z \phi^i) \end{aligned}$$

What is  $F^i$ ? Note that it enters the action only *algebraically*. It is called an auxiliary field.

It plays a very interesting role; its transformation is:

$$\begin{aligned} \delta F^i &= -(\varepsilon_+ Q_- - \varepsilon_- Q_+) (i D_+ D_- \Phi^i) = (\varepsilon_+ D_- - \varepsilon_- D_+) D_+ D_- \Phi^i \\ &= -(\varepsilon_+ \bar{\partial} \psi^i + \varepsilon_- \partial \bar{\psi}^i). \end{aligned}$$

Note that  $\delta\phi^i$ ,  $\delta\psi^i$ , and  $\delta F^i$  are *independent* of the form of  $S$ ; the transformations are valid without the use of field equations, and the algebra closes without any field equations. If we eliminate  $F^i$  by its equation of motion,  $F^i = -i\Gamma_{lk}^{-i} \psi^l \bar{\psi}^k$ , then  $\delta\psi^i$  depends on  $\Gamma^-$ , which in turn depends on the form of  $S$ . Consider the simplest example:  $g + b = 1, \Gamma = R = 0$ . Then on-shell  $F^i = 0$ , and  $\delta\psi^i = -i\varepsilon_- \partial_z \phi^i$ . Let's check the algebra; on  $\phi$ , we find

$$\begin{aligned} [\delta_\varepsilon, \delta_\eta] \phi^i &= 2i(\eta_+(i\varepsilon_+ \bar{\partial} \phi^i) - \eta_-(-i\varepsilon_- \partial \phi^i)) \\ &= +2(\varepsilon_+ \eta_+ \bar{\partial} \phi^i + \varepsilon_- \eta_- \partial \phi^i), \end{aligned}$$

which is fine. However, on  $\psi^i$  we have:

$$\begin{aligned} [\delta_\varepsilon, \delta_\eta] \psi^i &= -i\eta_- \partial (i\varepsilon_+ \bar{\psi}^i - i\varepsilon_- \psi^i) - \eta \leftrightarrow \varepsilon \\ &= +2\varepsilon_- \eta_- \partial \psi^i + (\eta_- \varepsilon_+ - \varepsilon_- \eta_+) \partial \bar{\psi}^i \\ &= \underbrace{2(\varepsilon_+ \eta_+ \bar{\partial} + \varepsilon_- \eta_- \partial) \psi^i}_{\text{good}} \\ &\quad - \underbrace{2\varepsilon_+ \eta_+ \bar{\partial} \psi^i + (\eta_- \varepsilon_+ - \varepsilon_- \eta_+) \partial \bar{\psi}^i}_{\text{Field equations}} \end{aligned}$$

Without auxiliary fields, in general, the algebra only close *on-shell*, that is, modulo equations of motion. Note that the action is still invariant without any equations of motions; since these are defined by extremizing the action, they can *never* be used in verifying a symmetry. Sadly, there are supersymmetric systems that we don't really understand in superspace, *i.e.*, we only have on-shell representations of the algebra.

We will eventually run into this, but let me say now that this is typically the situation for high  $D$  and/or  $N$ .

Another comment, referring back to off-shell supersymmetry transformations: how do you break supersymmetry ? Well, you need  $\langle \delta(\text{something}) \rangle \neq 0$ . Note that  $\langle \phi^i \rangle \neq 0$  does *not* break supersymmetry ( $\phi$  enters the transformation laws only through its derivatives), but  $\langle F^i \rangle \neq 0$  does. This is a very generic feature.

### III. Super Yang-Mills theory.

Let's look at some more models. Back in  $D = 3, N = 1$ , we can consider Super Yang-Mills theory. We do this by covariantizing the *spinor* derivative  $D_\alpha$ :

$$D_\alpha \rightarrow \nabla_\alpha \equiv D_\alpha - i\Gamma_\alpha.$$

We can construct everything in terms of the basic object  $\Gamma_\alpha$ . For example, we can define the vector potential  $\Gamma_{\alpha\beta}$  by

$$\{\nabla_\alpha, \nabla_\beta\} = 2i\nabla_{\alpha\beta} \equiv 2i(\partial_{\alpha\beta} - i\Gamma_{\alpha\beta}) \Rightarrow \Gamma_{\alpha\beta} = -\frac{1}{2}(iD_{(\alpha}\Gamma_{\beta)} + \{\Gamma_\alpha, \Gamma_\beta\})$$

(which means we absorb  $F_{\alpha\beta}$  into  $\Gamma_{\alpha\beta}$ ). The components follow from the pieces of  $\Gamma_\alpha$  that remain after we use the gauge transformation

$$\delta\Gamma_\alpha = \nabla_\alpha K$$

as follows:

$K|$  is the component gauge parameter

$\nabla_\alpha K|$  gauges away  $\Gamma_\alpha|$

$\nabla^2 K|$  gauges away  $D^\alpha \Gamma_\alpha|$

This leaves the component gauge field  $A_{\alpha\beta}$  and spinor  $\lambda_\beta$ :

$$D_{(\alpha} \Gamma_{\beta)}| \sim A_{\alpha\beta} \quad D^\alpha D_\beta \Gamma_\alpha| \sim \lambda_\beta.$$

A more careful definition of the components is given below.

The Bianchi identities

$$[\nabla_{(\alpha} \{\nabla_\beta, \nabla_\gamma\}] = 0$$

imply

$$[\nabla_\alpha, \nabla_\beta] \equiv -iF_{\alpha\beta} = C_{\alpha(\beta} F_{\gamma)}$$

(that is  $F_{(\alpha, \beta\gamma)} = 0$ ), which allows us to compute the superfield strength

$$F_\alpha = \frac{1}{3} \left( i(D^\beta \Gamma_{\alpha\beta} - \partial_{\alpha\beta} \Gamma^\beta) + [\Gamma^\beta, \Gamma_{\alpha\beta}] \right).$$

We also can find the usual field strength as follows:

$$[\nabla_{\alpha\beta}, \nabla_{\gamma\delta}] = -\frac{i}{2} [\nabla_{\alpha\beta}, \{\nabla_\gamma, \nabla_\delta\}] = \frac{i}{2} (\{\nabla_\gamma, [\nabla_\delta, \nabla_{\alpha\beta}]\} + \{\nabla_\delta, [\nabla_\gamma, \nabla_{\alpha\beta}]\}),$$

and hence

$$\begin{aligned} [\nabla_{\alpha\beta}, \nabla_{\gamma\delta}] &= \frac{i}{2} \left( C_{\delta\alpha} \nabla_\gamma F_\beta + C_{\delta\beta} \nabla_\gamma F_\alpha + C_{\gamma\alpha} \nabla_\delta F_\beta + C_{\gamma\beta} \nabla_\delta F_\alpha \right) \\ &= \frac{-i}{4} \left[ \left( C_{\alpha\gamma} (\nabla_\beta F_\delta + \nabla_\delta F_\beta) + \alpha \leftrightarrow \beta \right) + \gamma \leftrightarrow \delta \right]. \end{aligned}$$

Thus we have found the spacetime Yang-Mills field strength in terms of the spinor field strength  $F_\alpha$ .\* (Note that we also have:  $\{\nabla^\alpha, [\nabla^\beta, \nabla_{\alpha\beta}]\} = 0$  and hence  $\nabla^\alpha F_\alpha = 0$ .)

For an action, we have:

$$S = \frac{1}{4} \int d^3x D^\alpha D_\alpha \frac{1}{g^2} \text{Tr}(F^\beta F_\beta).$$

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\* We use the two component spinor identities from the footnote of Section V, Lecture I.

We can also add a super Chern-Simons term:

$$S_{CS} = \frac{1}{2} \int d^3x D^\alpha D_\alpha \frac{m}{g^2} \text{Tr}(\Gamma^\beta (F_\beta - \frac{1}{6}[\Gamma^\gamma, \Gamma_{\beta\gamma}])).$$

This is, of course, a special feature of  $D = 3$ . We can define components by projection as follows: The gauge field is

$$\Gamma_{\alpha\beta}| = A_{\alpha\beta},$$

which leads to a component field strength

$$\sim \nabla_{(\alpha} F_{\beta)}|.$$

The physical spinor is

$$F_\alpha| = \lambda_\alpha|.$$

As described above, the rest of  $\Gamma_\alpha$  can be gauge away.

In components, the sum of the two actions is simply a Yang-Mills field coupled to a massive adjoint representation spinor with mass equal to the topological mass of the Yang-Mills field given by the Chern-Simons term.

Finally, let us consider matter couplings; that is, for some scalar fields transforming as  $\delta\Phi^i = i(K\Phi)^i \equiv iK^A(T_A\Phi)^i$  we couple to Yang-Mills fields by the usual minimal coupling prescription:  $D_\alpha\Phi^i \rightarrow (\nabla_\alpha\Phi)^i = D_\alpha\Phi^i - i(\Gamma_\alpha\Phi)^i$ . This gives rise to a matter action

$$\begin{aligned} S &= -\frac{1}{8} \int d^3x D^\alpha D_\alpha \left[ (\nabla^\beta\Phi)_i (\nabla_\beta\Phi)^i \right] = \frac{1}{8} \int d^3x D^\alpha D_\alpha \left[ \Phi_i (\nabla^\beta\nabla_\beta\Phi)^i \right] \\ &= \frac{1}{8} \int d^3x \left[ \nabla^\alpha\nabla_\alpha \left( \Phi_i (\nabla^\beta\nabla_\beta\Phi)^i \right) \right] \\ &= \frac{1}{8} \int d^3x \left[ (\nabla^\alpha\nabla_\alpha\Phi)_i (\nabla^\beta\nabla_\beta\Phi)^i + 2(\nabla^\alpha\Phi)_i (\nabla_\alpha\nabla^\beta\nabla_\beta\Phi)^i + \Phi_i (\nabla^\alpha\nabla_\alpha\nabla^\beta\nabla_\beta\Phi)^i \right] \\ &= \frac{1}{2} \int d^3x \left[ \frac{1}{4} F_i F^i - i\psi_i^\alpha (\nabla_{\alpha\beta}\psi^\beta)^i - \frac{1}{2} (\nabla^{\alpha\beta}\phi)_i (\nabla_{\alpha\beta}\phi)^i + i(\psi_i^\alpha (\lambda_\alpha\phi)^i - \phi_i (\lambda^\alpha\psi_\alpha)^i) \right] \end{aligned}$$

with  $F^i$  auxiliary again. More generally, we can consider an isometry that acts on some scalar fields by  $\delta\Phi^i = K^A X_A^i(\Phi)$ , where  $X_A^i$  are killing vectors satisfying

$$X_{i;j} + X_{j;i} = 0.$$



This leaves the action  $S = \int d^3x D^\alpha D_\alpha g_{ij} D^\beta \Phi^i D_\beta \Phi^j$  invariant, as can be seen as follows:

$$\begin{aligned}
\delta S &= \int d^3x D^\alpha D_\alpha \left( 2g_{ij} D^\beta \Phi^i D_\beta (K^A X_A^j) + g_{ij,k} D^\beta \Phi^i D_\beta \Phi^j K^A X_A^k \right) \\
&= \int d^3x D^\alpha D_\alpha \left( 2K^A D^\beta \Phi^i D_\beta \Phi^j X_{Ai;j} - D^\beta \Phi^i D_\beta \Phi^j K^A X_A^k (g_{ij,k} + g_{ik,j} - g_{jk,i}) \right. \\
&\quad \left. + g_{ij,k} D^\beta \Phi^i D_\beta \Phi^j K^A X_A^k \right) \\
&= 0 .
\end{aligned}$$

Now minimal coupling takes the form  $(\nabla_\alpha \Phi)^i = D_\alpha \Phi^i - i\Gamma_\alpha^A X_A^i(\Phi)$ .

### LECTURE III.

#### I. Preview and summary.

I describe  $D = 4$ ,  $N = 1$  supersymmetric models, in particular, I write down the most general renormalizable  $D = 4$  supersymmetric model. This will involve scalar multiplets and Yang-Mills theories.\* However, the superspace formulation of these theories is quite different from the  $D = 3$ ,  $N = 1$  theories described in the previous lecture. We will see that we will have to introduce superfields that are independent of some  $\theta$ 's — constrained superfields. These exist for  $D = 4$ ,  $N = 1$  because the superderivatives have an abelian subalgebra (that is *not* real). These “chiral” superfields resemble complex  $D = 3$ ,  $N = 1$  scalar superfields, and describe matter supermultiplets with a complex scalar, a Weyl spinor, and a complex auxiliary field. The Yang-Mills multiplets are described by an unrestricted superfield. The structure of the theory is somewhat novel and is determined by the structure of the matter multiplets: that is, by the principle that the restricted (chiral) superfields that we use to describe matter should be allowed to carry Yang-Mills charge.

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\* Supersymmetric standard models are of this form; so is the only (proven) finite  $D = 4$  theory,  $N = 4$  Super Yang-Mills theory.

## II. $D = 4, N = 1$ models.

By way of motivation, let's consider some dimensional analysis. In  $D = 4$ , we have 4 spinor derivatives (see table I):  $D_\alpha, \bar{D}_{\dot{\alpha}}$ . Thus the measure is  $\int d^4x D^\alpha D_\alpha \bar{D}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}}$ . Now  $\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = i\partial_{\alpha\dot{\alpha}}$ , which implies the mass dimension of  $[D] = \frac{1}{2}$ . Thus the measure has dimension 2. We expect, from our experience in  $D = 3$ , that  $\phi^i = \Phi^i|$ , and hence the physical scalar has  $[\phi] = 1$ . Then a dimensionless action should have the form

$$\int d^4x D^2 \bar{D}^2 |\Phi|^2.$$

But for an unconstrained superfield, this gives no dynamics; we want a  $\partial^{\alpha\beta} \phi \partial_{\alpha\beta} \phi$  term. We need a new idea. Let's have a look at the  $D = 4$  algebra of spinor derivatives:

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = i\partial_{\alpha\dot{\alpha}}, \quad \{D_\alpha, D_\beta\} = 0, \quad \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0.$$

Notice that by themselves, the  $D$ 's or the  $\bar{D}$ 's are abelian. In particular, there is no inconsistency with constraining a superfield to be *chiral*:

$$\bar{D}_{\dot{\alpha}} \Phi = 0.$$

Loosely, this means  $\Phi$  depends on  $\theta$  but not  $\bar{\theta}$ ; I say loosely, because the constraint is  $\bar{D}_{\dot{\alpha}} \Phi = 0$ , not  $\bar{\partial}_{\dot{\alpha}} \Phi = 0$ .  $\Phi$  had better be a complex superfield, because we have:

$$D_\alpha \bar{\Phi} = 0$$

and if  $\Phi = \bar{\Phi}$ , then  $\{D_\alpha, \bar{D}_{\dot{\alpha}}\} \Phi = 0$ , which implies  $\partial_{\alpha\dot{\alpha}} \Phi = 0$ , *i.e.*,  $\Phi$  is a constant and that is not good for field theory. What are the components of  $\Phi$ ?

$$\phi = \Phi|, \quad \psi_\alpha = D_\alpha \Phi|, \quad F = \frac{1}{2} D^\alpha D_\alpha \Phi|.$$

Does this solve our problem? Let's see:

$$\begin{aligned} S_0 &= \frac{1}{4} \int d^4x D^2 \bar{D}^2 (\Phi \bar{\Phi}) = \frac{1}{4} \int d^4x D^2 (\Phi \bar{D}^2 \bar{\Phi}) \\ &= \frac{1}{4} \int d^4x (D^2 \Phi \bar{D}^2 \bar{\Phi} + 2 D^\alpha \Phi D_\alpha \bar{D}^2 \bar{\Phi} + \Phi D^2 \bar{D}^2 \bar{\Phi}). \end{aligned}$$

We need a few identities:

$$D_\alpha \bar{D}^2 = \bar{D}^2 D_\alpha - 2i \partial_{\alpha\dot{\alpha}} \bar{D}^{\dot{\alpha}}$$

and

$$D^2 \bar{D}^2 \bar{\Phi} = 2 \partial^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \bar{\Phi}$$

which implies

$$S_0 = \int d^4x (F\bar{F} - i\psi^\alpha \partial_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} + \frac{1}{2} \phi \partial^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \bar{\phi}),$$

which is just what we want! Other constraints are possible; there is a systematic way to study all linear constraints, but it is far too technical to describe here (see Gates *et al.*, section 3.11.a).

How about masses and interactions? Well, now we're hoisted by our own petard:  $\int d^4x D^2 \bar{D}^2 |\Phi|^2$  is a kinetic term, not a mass! How do we get no derivatives? We need a new superspace measure. We can get a clue as follows: The kinetic action can be written as  $\int d^4x D^2 (\Phi \bar{D}^2 \bar{\Phi})$ ; this almost looks like  $D = 3$ . Can we use  $D^2$  as a measure? Not in general, but for Lagrangians that are chiral, we can!

This leads to the following theorem:

For  $\mathcal{L}$  such that  $\bar{D}\mathcal{L} = 0$ ,  $S_{\text{chiral}} = \int d^4x D^2 \mathcal{L}$  is a superinvariant.

Proof:

$$\begin{aligned} \delta S_{\text{chiral}} &= i \int d^4x D^2 (\epsilon^\alpha Q_\alpha + \bar{\epsilon}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}) \mathcal{L} \\ &= - \int d^4x D^2 (\epsilon^\alpha D_\alpha + \bar{\epsilon}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}}) \mathcal{L} + \text{total derivatives} \\ &= \int d^4x D^2 D_\alpha (\epsilon^\alpha \mathcal{L}) = 0, \end{aligned}$$

where the last equality follows from  $D^3 = 0$ .  $\int d^4x D^2$  is a *chiral* measure good for *chiral* Lagrangians. Now we're in great shape:

$$\begin{aligned} S_V &= \frac{1}{2} \int d^4x D^2 V(\Phi^i) + c.c. \\ &= \int d^4x (V_{,i} F^i + \frac{1}{2} V_{,ij} \psi^{i\alpha} \psi_\alpha^j) + c.c. \end{aligned}$$

For  $V = \frac{1}{2}m\Phi^2 + \frac{1}{3!}\lambda\Phi^3$ , we get

$$S_{m,\lambda} = \int d^4x \left[ m\left(\frac{1}{2}\psi^\alpha\psi_\alpha + \phi F\right) + \frac{1}{2}\lambda(\phi\psi^\alpha\psi_\alpha + F\phi^2) + c.c. \right]$$

(There is an old nomenclature that is still used: terms constructed with  $\int d^4x D^2 \bar{D}^2$  are called “ $D$ ” terms, whereas terms constructed with  $\int d^4x D^2 + c.c.$  are called “ $F$ ” terms.)

Eliminating  $F$ , we get

$$\begin{aligned} \int d^4x \left[ \bar{\phi} \left( \frac{1}{2} \partial^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} - m^2 \right) \phi - i \bar{\psi}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \psi^\alpha + \frac{1}{2} m (\psi^\alpha \psi_\alpha + \bar{\psi}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}) \right. \\ \left. - \frac{1}{2} m \lambda (\phi \bar{\phi}^2 + \bar{\phi} \phi^2) - \frac{1}{4} \lambda^2 \phi^2 \bar{\phi}^2 + \frac{1}{2} \lambda (\phi \psi^\alpha \psi_\alpha + \bar{\phi} \bar{\psi}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}) \right] \end{aligned}$$

This is the most general renormalizable supersymmetric model with one complex scalar  $\phi$  and one Weyl spinor  $\psi$ .

We can easily generalize to many  $\Phi$ ’s:

$$S = \int d^4x D^2 \bar{D}^2 \Phi^i \bar{\Phi}_i + \int d^4x D^2 V(\Phi^i) + \int d^4x \bar{D}^2 \bar{V}(\bar{\Phi}^i)$$

for  $V(\Phi^i)$  a cubic polynomial.

### III. D=4 Super Yang-Mills theory.

As for the scalar multiplet, we immediately have a problem: where can we start? We can’t use minimal coupling, as there are no derivatives in the Lagrangian! Let’s examine the constraints on the matter superfields. Suppose we start with a chiral superfield; we may ask that it remain chiral after a gauge transformation. This implies that the gauge parameter must be chiral:

$$\bar{D}_{\dot{\alpha}} \Phi^i = 0; \text{ we ask } \bar{D}_{\dot{\alpha}} (\delta \Phi^i) = 0, \quad \text{but } \delta \Phi^i = \underbrace{i \Lambda^A (T_A)^i_j}_{\Lambda_j^i} \Phi^j,$$

which implies that  $\Lambda$  better be chiral. This implies, however, that  $\bar{D}_{\dot{\alpha}}$  is gauge-covariant as it stands:

$$\delta(\bar{D}_{\dot{\alpha}} \Phi^i) = \bar{D}_{\dot{\alpha}} (i \Lambda_j^i \Phi^j) = i \Lambda_j^i \bar{D}_{\dot{\alpha}} \Phi^j.$$

This implies:  $F_{\dot{\alpha}\dot{\beta}} = i\{\nabla_{\dot{\alpha}}, \nabla_{\dot{\beta}}\} = i\{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0$ . But what about  $D_{\alpha}$ ? We certainly need  $F_{\alpha\beta} = i\{\nabla_{\alpha}, \nabla_{\beta}\} = 0$ , but  $D_{\alpha}$  is not covariant:  $\delta(D_{\alpha}\Phi^i) = i(D_{\alpha}\Lambda_j^i)\Phi^j + i\Lambda_j^i D_{\alpha}\Phi^j$ . To satisfy  $F_{\alpha\beta} = 0$ , we must have  $\nabla_{\alpha} = e^{-V}D_{\alpha}e^V$ ; how should  $V$  transform to fix things? Consider

$$\delta(\nabla_{\alpha}\Phi) = (\delta\nabla_{\alpha})\Phi + i[\nabla_{\alpha}, \Lambda]\Phi + i\Lambda\nabla_{\alpha}\Phi$$

which implies

$$\delta\nabla_{\alpha} = -i[\nabla_{\alpha}, \Lambda].$$

this is accomplished by:

$$(e^V)' = e^{i\bar{\Lambda}}e^Ve^{-i\Lambda}, \quad V = V^{\dagger} \Rightarrow \delta V = i(\bar{\Lambda} - \Lambda) + \dots$$

Then the gauge covariant action is

$$\int d^4x D^2 \bar{D}^2 (\bar{\Phi} e^V \Phi)$$

where  $\Phi' = e^{i\Lambda}\Phi$ ,  $\bar{\Phi}' = \bar{\Phi}e^{-i\bar{\Lambda}}$ . This may seem like magic, but it is as deductive as I can make it. We see that  $V$  acts as a “bridge” or “converter” between chiral and antichiral representations of the gauge group. What does  $V$  contain?

$$V|, D_{\alpha}V|, \bar{D}_{\dot{\alpha}}V|, D^2V|, \bar{D}^2V|$$

can be gauged away by

$$Im\Lambda|, D_{\alpha}\Lambda|, \bar{D}_{\dot{\alpha}}\bar{\Lambda}|, D^2\Lambda|, \bar{D}^2\bar{\Lambda},$$

respectively. What remains is the physical vector

$$A_{\alpha\dot{\alpha}} \sim [D_{\alpha}, \bar{D}_{\dot{\alpha}}]V|,$$

the physical spinor

$$\lambda_{\alpha} \sim \bar{D}^2 D_{\alpha}V|, \quad \bar{\lambda}_{\dot{\alpha}} \sim D^2 \bar{D}_{\dot{\alpha}}V|,$$

and a hermitian auxiliary field:

$$\mathcal{D} \sim \bar{D}^{\dot{\alpha}} D^2 \bar{D}_{\dot{\alpha}} V|.$$

The full nonlinear expressions for these components can be found from the Bianchi identities and the constraints:

$$\{\nabla_{\alpha}, \nabla_{\beta}\} = \{\bar{\nabla}_{\dot{\alpha}}, \bar{\nabla}_{\dot{\beta}}\} = 0, \quad \{\nabla_{\alpha}, \bar{\nabla}_{\dot{\alpha}}\} = i\nabla_{\alpha\dot{\alpha}}.$$

The first two constraints can be interpreted as integrability conditions for the existence of chiral and anti-chiral superfields, and the third is merely a matter of convenience: it defines the vector connection in terms of the spinor connections, just as in  $D = 3$ . The Bianchi identities now imply:

$$[\nabla_{\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] = C_{\dot{\alpha}\dot{\beta}} W_{\beta}, \quad W_{\beta} = \frac{i}{2} \bar{D}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} (e^{-V} D_{\beta} e^V)$$

The covariant expressions for the components of the Yang-Mills multiplet are:

$$A_{\alpha\dot{\alpha}} = \Gamma_{\alpha\dot{\alpha}}|, \quad \lambda_{\alpha} = W_{\alpha}|, \quad \bar{\lambda}_{\dot{\alpha}} = \bar{W}_{\dot{\alpha}}|,$$

$$\mathcal{D} = \nabla^{\alpha} W_{\alpha}|, \quad f_{\alpha\beta} \equiv \partial_{\alpha}^{\dot{\alpha}} A_{\beta\dot{\alpha}} + \partial_{\beta}^{\dot{\alpha}} A_{\alpha\dot{\alpha}} = \nabla_{(\alpha} W_{\beta)}|.$$

We note that  $\bar{D}_{\dot{\alpha}} W_{\beta} = 0$ . This allows us to write a consistent gauge invariant action for the Yang-Mills field:

$$S_{YM} = \frac{1}{4g^2} \text{tr} \int d^4x D^2 (W^{\alpha} W_{\alpha}).$$

This is real up to a topological  $F\tilde{F}$  term. We can also add a term

$$\int d^4x D^2 \bar{D}^2 \text{tr}(\nu V)$$

for an abelian subgroup; I'll leave it as an exercise to prove that this is gauge invariant when  $\nu$  commutes with all the  $T_A$ 's. Such a term is called the Fayet-Illiopoulos term, and leads to spontaneous supersymmetry breaking.

So finally, as promised, we have the most general  $D = 4$  renormalizable supersymmetric action:

$$\begin{aligned}
S = \int d^4x \Big[ & -\frac{1}{2}(\nabla^{\alpha\dot{\alpha}}\phi)^i(\nabla_{\alpha\dot{\alpha}}\bar{\phi})_i - i\psi^{\alpha i}(\nabla_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}})_i + i(\bar{\phi}_i(\lambda^\alpha\psi_\alpha)^i - \bar{\psi}_i^{\dot{\alpha}}(\bar{\lambda}_{\dot{\alpha}}\phi)^i) \\
& + \bar{\phi}_i(\mathcal{D}\phi)^i + F^i\bar{F}_i + tr(-i\lambda^\alpha[\nabla_{\alpha\dot{\alpha}}, \bar{\lambda}^{\dot{\alpha}}] - \frac{1}{2}f^{\alpha\beta}f_{\alpha\beta} + \mathcal{D}^2) \\
& + tr(\nu\mathcal{D}) + (V_{,i}F^i + \frac{1}{2}V_{,ij}\psi^{\alpha i}\psi_\alpha^j + \bar{V}^i\bar{F}_i + \frac{1}{2}\bar{V}^{ij}\bar{\psi}_i^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}j}) \Big].
\end{aligned}$$

Eliminating the auxiliary fields  $F, \bar{F}$ , and  $\mathcal{D}$  gives a scalar potential:

$$-[V_{,i}\bar{V}^i + \frac{1}{4}(\bar{\phi}(T_A\phi) + Tr(\nu T_A))^2].$$

## LECTURE IV.

### I. Preview and summary.

In this lecture I discuss the relation between supersymmetry and the target space geometry of supersymmetric sigma-models. The main motivations for this are

- 1) String theory — many conformal field theories (CFT's) can be described as  $\sigma$ -models. This is an important way to give a spacetime interpretation to conformal field theories. (The other known way is Landau-Ginzburg models.)
- 2) Mathematics — supersymmetric sigma-models have led to important results in complex manifold theory, and in particular, to the hyperkähler quotient.

For the string theory applications, we want to study  $D = 2, N > 1$  models, but for the mathematical applications, as well as for pedagogical reasons, I'll start at  $D = 4$  and work my way down.  $D = 4, N = 1$  sigma models expressed in terms of chiral superfields have a unique and natural superspace Lagrangian:  $\mathcal{L} \sim K(\Phi^i, \bar{\Phi}^i)$ . Working out the relevant part of the component action, we'll find that these describe manifolds that are

*Kähler*:  $ds^2 = K_{i\bar{j}} d\Phi^i d\bar{\Phi}^j$ . I'll explain what Kähler geometry is. It is possible to show that this is a general result for  $D = 4$   $\sigma$ -models, and doesn't depend on using chiral superfields. However, it is easier to do this after descending to  $D = 3$ . There we can formulate the general question in  $N = 1$  superspace; we'll find that extra supersymmetries correspond to covariantly constant hermitian complex structures on the target space. This proves that we can have  $N = 2$  supersymmetry if and only if the target space manifold is Kähler, and  $N = 4$  supersymmetry if and only if the target space manifold is hyperkähler. Descending further to  $D = 2$ , we have the option of adding torsion. One way to do this is to take advantage of the existence of new kinds of constrained superfields: twisted chiral superfields, which are left chiral and right antichiral, and can exist only in  $D = 2$ . However, this does not give the most general model, and the general case is still not fully understood. The final classification theorem is summarized in the table below:

$D$	6/5	4	3	2 (T=0)	Target Space	
					Dimension	Geometry
$N$	1	2	4	(4,4)	4k	Hyperkähler
		1	2	(2,2)	2k	Kähler
			1	(1,1)	k	Riemannian

Table 2

Here  $k$  is an arbitrary integer.

For non-zero torsion, new geometries arise; these don't have nice names, so I haven't put them in the table. I will try to derive as much of this table as possible.

## II. $D = 4$ , $N = 1$ sigma Models.

The most general bosonic  $D = 4$   $\sigma$ -model can be described by an action:

$$S_0 = -\frac{1}{2} \int d^4x (\partial^{\alpha\dot{\alpha}} \phi^i \partial_{\alpha\dot{\alpha}} \phi^j) g_{ij}(\phi)$$



and describes maps from 4 dimensions into a target space with a metric  $g_{ij}$ . On dimensional grounds (counting  $D$ 's), the superspace analog for chiral superfields must be:

$$S = \frac{1}{4} \int d^4x D^2 \bar{D}^2 K(\Phi^i, \bar{\Phi}^i), \quad \bar{D}_{\dot{\alpha}} \Phi^i = D_{\alpha} \bar{\Phi}^i = 0.$$

Let's work out the terms in this corresponding to  $S_0$ :

$$\begin{aligned} S &= \frac{1}{4} \int d^4x \left( K_{\bar{i}} D^2 \bar{D}^2 \bar{\Phi}^i - 2K_{i\bar{j}} (D^{\alpha} \bar{D}^{\dot{\alpha}} \bar{\Phi}^i) (D_{\alpha} \bar{D}_{\dot{\alpha}} \bar{\Phi}^j) \right. \\ &\quad \left. + F \text{ terms} + \psi \text{ terms} \right) \\ &= \frac{1}{2} \int d^4x (K_{\bar{i}} \partial^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \bar{\Phi}^i + K_{i\bar{j}} \partial^{\alpha\dot{\alpha}} \bar{\Phi}^i \partial_{\alpha\dot{\alpha}} \bar{\Phi}^j) + \dots \\ &= -\frac{1}{2} \int d^4x K_{i\bar{j}} \partial^{\alpha\dot{\alpha}} \bar{\Phi}^i \partial_{\alpha\dot{\alpha}} \bar{\Phi}^j + \dots \end{aligned}$$

where we have integrated by parts in the last step. Thus the metric  $g_{ij}$  is written as the matrix of second derivatives of the function  $K$ :

$$g_{ij} \sim \frac{\partial^2 K}{\partial \phi^i \partial \bar{\phi}^j}, \quad ds^2 = K_{i\bar{j}} d\phi^i d\bar{\phi}^j$$

This is the form of an arbitrary Kähler metric. Notice that the geometry depends only on the Hessian of  $K$ ; in particular, shifting  $K$  by the real part of a holomorphic function  $f(\phi) + \bar{f}(\bar{\phi})$  does not change the metric. This is also true for the superspace action, because the full superspace measure  $\int d^4x D^2 \bar{D}^2$  annihilates a chiral or antichiral integrand.

We have shown that for chiral superfields, the  $N = 1$ ,  $D = 4$  target space must be Kähler; conversely, for any Kähler manifold, we can write an  $N = 1$ ,  $D = 4$   $\sigma$ -model in terms of chiral superfields. This result is originally due to Zumino. To proceed further, we need to know more about Kähler geometry. Notice that the manifold is complex: the coordinates can be chosen to be complex ( $\phi$  and  $\bar{\phi}$ ). This gives an intuitive explanation of why supersymmetry knows about target space geometry: chiral superfields correspond to complex coordinates.

We can define a natural tensor:

$$J_i^j = i\delta_i^j, J_{\bar{i}}^{\bar{j}} = J_{\bar{i}}^j = 0, J_i^{\bar{j}} = -i\delta_i^{\bar{j}}.$$

Then  $J$  has the following important properties:

0. The manifold on which it is defined is even (real) dimensional.
1.  $J^2 = -1$  ( $J$  is an almost complex structure).
2. Let  $\partial_{\pm I} = \partial_I \pm iJ_I^J \partial_J$ ,  $\{I\} = \{i, \bar{j}\}$ . Then  $[\partial_{\pm I}, \partial_{\pm J}] = C_{IJ}^K \partial_{\pm K}$  ( $J$  is an *integrable* complex structure.)
3.  $g_{IJ} J_K^J = -g_{KJ} J_I^J$  (The metric is *hermitian*.)
4.  $\nabla_I J_K^J = 0$  ( $J$  is covariantly constant).
5. This implies  $[\nabla_I, \nabla_J] J_L^K = 0$ , which further implies
6.  $R_{IJ}^{KM} J_L^M + R_{IJL}^M J_M^K = 0$  (The holonomy of the Levi-Civita connection is not  $O(2K)$  but  $U(K)$ .)

These properties characterize Kähler manifolds.

### III. $D = 3$ , here we come!

As described in the first lecture, we can reduce to  $D = 3$ . We let:

$$D_\alpha \rightarrow D_\alpha, \quad \bar{D}_{\dot{\alpha}} \rightarrow \bar{D}_{\dot{\alpha}}, \quad \Rightarrow \{D_\alpha, \bar{D}_{\dot{\beta}}\} = i\partial_{\alpha\dot{\beta}} = i\partial_{\dot{\beta}\alpha}.$$

The action remains

$$S = \frac{1}{4} \int d^3x D^2 \bar{D}^2 K(\Phi^i, \bar{\Phi}^i), \quad \bar{D}_\alpha \Phi^i = D_\alpha \bar{\Phi}^i = 0$$

But, because we're in  $D = 3$ , we can write  $N = 1$   $\sigma$ -models:

$$S = \frac{1}{8} \int d^3x D^2 \left( D^\alpha \Phi^i D_\alpha \Phi^j g_{ij}(\Phi) \right)$$

(Recall lecture 2). Here,  $i, j$  are real indices. For future use we write down the superfield equation of motion:  $D^\alpha D_\alpha \Phi^i + \Gamma_{jk}^i D^\alpha \Phi^j D_\alpha \Phi^k = 0$ .) Let's try and see what happens

if we look for a second supersymmetry using  $N = 1$  components. We want it to anti-commute with the first supersymmetry, so we should write it with  $D_\alpha$ ; the most general transformation that we can write down is:

$$\delta\Phi^i = J_j^i \varepsilon^\alpha D_\alpha \Phi^j; \quad (6)$$

let's study the consequences of

(A) imposing the supersymmetry algebra on Eq. (6); this will lead to conditions 0. + 1. + 2. above, and hence to the conclusion that the target space manifold is complex.

(B) imposing invariance of  $S$  on Eq. (6); this will lead to 3. + 4., and to the conclusion that the manifold is Kähler.

(A) Starting only from the supersymmetry algebra, we have

$$\begin{aligned} [\delta_\varepsilon, \delta_\eta]\Phi^i &= J_j^i \eta^\alpha D_\alpha (J_k^j \varepsilon^\beta D_\beta \Phi^k) + J_{j,k}^i J_l^k (\varepsilon^\beta D_\beta \Phi^l) (\eta^\alpha D_\alpha \Phi^j) - \eta \leftrightarrow \varepsilon \\ &= -J_j^i J_k^j \eta^\alpha \varepsilon^\beta \{D_\alpha, D_\beta\} \Phi^k - J_j^i J_{[k,l]}^j \eta^\alpha \varepsilon^\beta D_\alpha \Phi^l D_\beta \Phi^k \\ &\quad + J_{[j|,k}^i J_{|l]}^k \eta^\alpha \varepsilon^\beta D_\beta \Phi^l D_\alpha \Phi^j \end{aligned}$$

which leads to

$$-J_j^i J_k^j = \delta_k^i \quad \Rightarrow \text{(conditions 0. + 1.)},$$

and

$$J_j^i J_{[k,l]}^j + J_{[k}^j J_{l],j}^i = 0 \quad \Rightarrow \text{(condition 2.)}.$$

(B) Invariance of the action implies:

$$0 = \delta S = \frac{1}{8} \int d^3x D^2 \left[ -2\varepsilon^\beta (D_\beta \Phi^l) \underbrace{J_l^i g_{ij}}_{J_{jl}} (D^\alpha D_\alpha \Phi^i + \Gamma_{jk}^i D^\alpha \Phi^j D_\alpha \Phi^k) \right].$$

A nontrivial calculation reveals that this vanishes if and only if  $J_{il} = -J_{li}$ , which implies condition 3. (the metric is hermitian), and if  $J_{il,k} - J_{ij}\Gamma_{lk}^j - J_{jl}\Gamma_{ik}^j = 0$ , which implies condition 4. (the metric is Kähler).

So any  $N = 2$ ,  $D = 3$  model is Kähler, and hence can be written with chiral superfields, and hence the same is true for any  $N = 1$ ,  $D = 4$  model.

How about  $N = 3$ ,  $D = 3$ ? This means there are *two* complex structures  $J_1, J_2$ , and the metric is Kähler with respect to both of them. Imposing the supersymmetry algebra implies  $[\delta_1, \delta_2] = 0$  and hence  $J_1 J_2 + J_2 J_1 = 0$ . This implies that  $(J_1 J_2)^2 = -1$ , and hence

$$J_3 \equiv J_1 J_2$$

is another complex structure, and generates yet another symmetry (it is easy to show  $J_3$  is hermitian and covariantly constant). So  $N = 3 \Rightarrow N = 4$ , and the target space manifold is hyperkähler. Hyperkähler geometry is characterized by the existence of 3 covariantly constant complex structures  $J_x$ , obeying the algebra of the imaginary quaternions:

$$\{J_x, J_y\} = -2\delta_{xy}.$$

#### IV. $D = 2$ , nous sommes ici!

We now reduce once more to reach two dimensions. The resulting algebra of spinor derivatives is:

$$D_+^2 = D_-^2 = \{D_+, D_-\} = \{D_+, \bar{D}_-\} = 0, \quad \{D_+, \bar{D}_+\} = \partial, \quad \{D_-, \bar{D}_-\} = \bar{\partial}.$$

Again, we can define a chiral superfield that obeys:  $\bar{D}_\pm \Phi = 0$ . But in  $D = 2$ , and only in  $D = 2$ ,  $\{\bar{D}_+, D_-\} = 0$ , so we can also have *twisted chiral superfields*:

$$\begin{aligned} \bar{D}_+ \chi &= D_- \chi = 0, \\ D_+ \bar{\chi} &= \bar{D}_- \bar{\chi} = 0. \end{aligned}$$

Now let's consider

$$\int d^2x D^2 \bar{D}^2 K(\Phi^i, \chi^a).$$

by the usual manipulations, this gives a bosonic part with a metric and in addition an antisymmetric tensor  $b$ -field:

$$g_{i\bar{j}} = K_{i\bar{j}}, \quad g_{a\bar{b}} = -K_{a\bar{b}}, \quad b_{i\bar{a}} = -K_{i\bar{a}}, \quad b_{a\bar{i}} = K_{a\bar{i}}.$$

Two obvious questions arise: is this the most general (2,2) model? What kind of geometry describes its target space?

To get an answer, we go back to  $N = 1$  superspace. Because we are in  $D = 2$ , we can have separate left and right handed transformations with parameters  $\varepsilon^+$  and  $\varepsilon^-$ :

$$\delta\Phi^I = J_J^{(+I)} \varepsilon^+ D_+ \Phi^J + J_J^{(-I)} \varepsilon^- D_- \Phi^J. \quad (7)$$

Now we impose the supersymmetry algebra, and find

1.  $J^{2(\pm)} = -1$ .
2.  $J^{(\pm)}$  are integrable.
3. There is a term that involves the field equations and  $[J^{(+)}, J^{(-)}]$ :

$$\begin{aligned} & [\delta_{(+)}, \delta_{(-)}] \Phi^I \\ &= \delta_{(+)} (J_J^{(-I)} \varepsilon^- D_- \Phi^J) - (+ \leftrightarrow -) \\ &= J_J^{(-I)} \varepsilon^- D_- (J_K^{(+I)} \varepsilon^+ D_+ \Phi^K) + J_{J,K}^{(-I)} J_L^{(+I)K} \varepsilon^+ D_+ \Phi^L \varepsilon^- D_- \Phi^J - (+ \leftrightarrow -) \\ &= \left( J_J^{(+I)} J_K^{(-I)J} - J_J^{(-I)} J_K^{(+I)J} \right) \varepsilon^+ \varepsilon^- D_+ D_- \Phi^K \\ &\quad - \left( J_{[L}^{(+I)K} J_{|J|,K]}^{(-I)} + J_{[K}^{(-I)} J_{|L|,J]}^{(+I)K} \right) \varepsilon^+ \varepsilon^- D_+ \bar{\Phi}^L D_- \Phi^J \end{aligned}$$

If  $[J^{(+)}, J^{(-)}] = 0$  and if this is integrable, then the algebra closes off shell. Then we can conclude that the model can be described in terms of chiral and twisted chiral superfields, and has the geometry descibed above. Otherwise, we need some auxiliary superfields. For  $N = 2$  some cases are known, but the general case is *not solved*. Note that when  $J^{(+)} = J^{(-)}$ , we're fine: that's the Kähler case.

Let's now impose invariance of the action in Eq. (4) under the transformations in Eq. (7): we find

4.  $g$  is bihermitian, that is, hermitian with respect to both  $J^{(\pm)}$ .
5.  $\nabla_I^{(\pm)} J_k^{(\pm)J} = 0$  with respect to a connection  $\Gamma^\pm$  with torsion  $T = \frac{1}{2}db$  (see Eq. 5).

The holonomy of the curvature with torsion  $R_{IJKL}^\pm$  is  $U(k)$ . This is a lot of interesting structure that the mathematicians haven't analyzed yet, and that we don't fully understand.

For  $N = 4$ , we find left quaternionic structures and right quaternionic structures. This structure is also very rich and interesting. For these models, the  $\beta$ -functions vanish. Some things are known about the off-shell structure, but a lot remains obscure.

## LECTURE V. QUOTIENTS AND DUALITY

### I. Preview and summary.

This time, I want to describe two closely related notions: quotients and duality. I won't do fancy mathematics. The way physicists perform a quotient is very simple: gauge the symmetry that you want to quotient by, *without* a kinetic term for the gauge field, and then integrate out the gauge field. I'll describe this for  $D = 2$ ,  $N = 1, 2$ . A *duality* transformation is just as simple: gauge the symmetry that you want to transform with respect to, *without* a kinetic term, and add a Lagrange multiplier to constrain the gauge field to be pure gauge; then integrate out the gauge field. In the  $N = 2$  case, we'll see that duality swaps chiral superfields for twisted chiral ones, and vice-versa. Then I'll briefly discuss the geometry of  $N = 2$  quotients, which is another very beautiful example of how supersymmetry “knows” deep mathematics on the target space. Then, even more briefly,

I'll discuss CFT quotients, and explain how duality transformations can be understood as CFT quotients. I'll end with a translation of a bad Swedish pun.

## II. Quotients.

I will work with  $U(1)$  only; general groups are discussed in the literature (see Hull in particular) and  $U(1)$  is interesting enough.

Let's start with a bosonic  $\sigma$ -model

$$S = \int d^2x (g_{ij}(\phi) + b_{ij}(\phi)) \partial\phi^i \bar{\partial}\phi^j.$$

This is invariant under a symmetry  $\delta\phi^i = X^i(\phi)$  if

1.  $X^i$  is a killing vector  $X_{i;j} + X_{j;i} = 0$ , *i.e.*,  $(\mathcal{L}_X g = 0)$  and
2.  $X^i b_{jk,i} + X^i_{,j} b_{ik} + X^i_{,k} b_{ji} = w_{[k,j]}$  for some  $w_j$ , *i.e.*,  $(\mathcal{L}_X b = dw)$ .

We can always choose coordinates such that  $X^i$  is a constant, *i.e.*, the symmetry acts only on one coordinate  $\phi^0$  by  $\delta\phi^0 = \alpha$ . Then condition 1. implies that  $g$  is independent of  $\phi^0$ , and condition 2. implies that we can find a 1-form  $\Omega_i$  such that shifting  $b_{ij}$  by a curl  $\Omega_{[i,j]}$  makes  $b$  also independent of  $\phi^0$ . Then the action takes the form (where now  $i, j$  do *not* run over  $\phi^0$ ):

$$S = \int d^2x (g_{00} \partial\phi^0 \bar{\partial}\phi^0 + (g_{0i} + b_{0i}) \partial\phi^0 \bar{\partial}\phi^i + (g_{i0} + b_{i0}) \partial\phi^i \bar{\partial}\phi^0 + (g_{ij} + b_{ij}) \partial\phi^i \bar{\partial}\phi^j) \quad (i, j \neq 0).$$

We can gauge this by minimal coupling:  $\partial\phi^0 \rightarrow \partial\phi^0 + A$ ,  $\bar{\partial}\phi^0 \rightarrow \bar{\partial}\phi^0 + \bar{A}$ ; however, there is an ambiguity, peculiar to  $D = 2$ , which arises as follows: The action  $S$  is invariant under shifts  $b_{ij} \rightarrow b_{ij} + \Omega_{[i,j]}$ . However, under such a shift, the Lagrangian changes by a total derivative. After gauging, the resulting  $A$ -dependent term is *not* a total derivative, and contributes a gauge invariant term  $U(\phi^i) \underbrace{(\bar{\partial}A - \partial\bar{A})}_{F(A)}$ . In CFT, this ambiguity is fixed by

the requirement of conformal invariance. Eliminating  $A + \bar{A}$  by their field equations, and substituting back into the action  $S$ , one finds:

$$S_Q = \int d^2x (g_{ij} + b_{ij} - (g_{i0} + b_{i0})g_{00}^{-1}(g_{0j} + b_{0j})) \partial\phi^i \bar{\partial}\phi^j \\ \Rightarrow g_{ij}^Q = g_{ij} - g_{00}^{-1}(g_{i0}g_{0j} + b_{i0}b_{0j}), \quad b_{ij}^Q = b_{ij} + g_{00}^{-1}(g_{i0}b_{0j} + b_{i0}g_{0j}).$$

Notice that  $g_{ij}^Q$  and  $b_{ij}^Q$  change non-trivially under  $b_{0a} \rightarrow b_{0a} - U_{,a}$ ; this is precisely the ambiguity noted above.

This quotient can be used to construct  $\sigma$ -models, *e.g.*,  $CP^n$  models ( $S^{2n+1}/U(1)$ ). The generalization to  $N = 1$  superspace is straightforward: we just change the measure, and replace  $\partial, \bar{\partial}, A, \bar{A}$  by  $D_+, D_-, \Gamma_+, \Gamma_-$ .

$N = 2$  superspace is more interesting. Consider the case with chiral and twisted chiral multiplets; that is not the most general  $N = 2$  case, but it is interesting and includes the general Kähler case. Recall the action

$$\int d^2x D^2 \bar{D}^2 K(\Phi^i, \chi^a), \quad \bar{D}_+ \Phi = \bar{D}_- \Phi = 0, \bar{D}_+ \chi = D_- \chi = 0$$

Note that this action is invariant (up to total derivatives) under the following shifts:

$$K \rightarrow K + [f(\Phi, \chi) + g(\Phi, \bar{\chi}) + c.c.]$$

This again leads to some ambiguities in gauging. The only symmetries that we can gauge are those that respect the constraints, *i.e.*,  $\delta\Phi^i = X^i(\Phi)$  or  $\delta\Lambda^a = Y^a(\Lambda)$ . Without loss of generality, we can choose  $\delta\Phi^i = X^i(\Phi)$ . Furthermore, we can choose coordinates and shifts such that there is some  $\Phi^0$  for which the action becomes

$$K = K(\Phi^0 + \bar{\Phi}^0, \Phi^i, \chi^a)$$

Then gauging is simple:  $\Phi^0 + \bar{\Phi}^0 \rightarrow \Phi^0 + \bar{\Phi}^0 + V$ ; recall

$$\delta\Phi^0 = i\Lambda, \quad \delta\bar{\Phi}^0 = -i\bar{\Lambda}, \quad \delta V = i(\bar{\Lambda} - \Lambda).$$



We also include an explicit Fayet-Iliopoulos term  $-cV$ , although that is really just a gauging of a “shift term”  $-(\Phi^0 + \bar{\Phi}^0)$ .<sup>\*</sup> Thus we have:

$$\int d^2x D^2 \bar{D}^2 \left( K(\phi^0 + \bar{\phi}^0 + V, \Phi^i, \chi^a) - cV \right)$$

Eliminating  $V$ , we get:

$$K_Q = K(V(\Phi^i, \chi^a), \Phi^i, \chi^a) - cV(\Phi^i, \chi^a)$$

with  $V(\Phi^i, \chi^a)$  determined by  $\frac{\partial K}{\partial V}(V, \Phi^i, \chi^a) = c$ .

For example: consider  $K = \sum_{i=1}^n \Phi^i \bar{\Phi}^i = e^{\Phi^0 + \bar{\Phi}^0} (1 + \sum_{i=1}^n \varphi^i \bar{\varphi}^i)$ . Then  $K_Q = c \ln(1 + \sum_{i=1}^n \varphi^i \bar{\varphi}^i)$  is the Kähler potential for  $CP^n$ .

### III. Duality

Again, let's start with the bosonic case. We take the gauged action

$$S = \int d^2x \left[ g_{00}(\partial\phi^0 + A)(\bar{\partial}\Phi^0 + \bar{A}) + (g_{0i} + b_{0i})(\partial\phi^0 + A)\bar{\partial}\phi^i + (g_{i0} + b_{i0})\partial\phi^i(\bar{\partial}\phi^0 + \bar{A}) \right. \\ \left. + (g_{ij} + b_{ij})\partial\phi^i\bar{\partial}\phi^j \right]$$

and add a term  $\tilde{\phi}(\partial\bar{A} - \bar{\partial}A)$ . Integrating out  $\tilde{\phi}$  implies  $\partial\bar{A} - \bar{\partial}A = 0$  and hence,  $A = \partial\lambda$ ; after the shift  $\phi^0 + \lambda \rightarrow \phi^0$  we get back the original action. Actually there are subtleties:  $F = 0 \Rightarrow A = \partial\lambda$  only locally. If  $\tilde{\phi}$  is periodic, then we get  $A = \partial\lambda$  with  $\lambda$  periodic. If  $\tilde{\phi}$  and  $\phi$  have the right periods, we get exact equivalence; otherwise, we get orbifolds. (A discussion of orbifolds and why the last claims I made are true is beyond the scope of these lectures).

Let's consider what happens if we integrate out the gauge field to get the dual model;

we find:

$$\tilde{S} = \int d^2x \left[ \frac{1}{g_{00}} \partial\tilde{\phi}\bar{\partial}\tilde{\phi} + \frac{1}{g_{00}}(g_{0i} + b_{0i})\partial\tilde{\phi}\bar{\partial}\phi^i - \frac{1}{g_{00}}(g_{i0} + b_{i0})\partial\phi^i\bar{\partial}\tilde{\phi} + (g_{ij}^Q + b_{ij}^Q)\partial\phi^i\bar{\partial}\phi^j \right] \\ \Rightarrow \tilde{g}_{00} = \frac{1}{g_{00}}, \tilde{g}_{0i} = \frac{b_{0i}}{g_{00}}, \tilde{b}_{0i} = \frac{g_{0i}}{g_{00}}, \tilde{g}_{ij} = g_{ij}^Q, \tilde{b}_{ij} = b_{ij}^Q.$$

<sup>\*</sup> Actually,  $c$  can be  $f(\chi) + \bar{f}(\bar{\chi})$  for any  $f$  when there are twisted chiral fields  $\chi$  in the theory.

Notice that  $b_{0i}$  and  $g_{0i}$  exchange values, and that the quotient metric and  $b$ -field enter.

Once more, the  $N = 1$  generalization is tediously straightforward. The  $N = 2$  story is very interesting. We need the analog of  $\tilde{\phi}F(A)$ . In  $D = 4$ , we had a superfield strength  $W_\alpha = \bar{D}^2 D_\alpha V$  (in the abelian case). It turns out that in  $D = 2$  we can take a complex scalar superfield strength

$$W = \bar{D}_+ D_- V;$$

this is invariant under  $\delta V = i(\Lambda + \bar{\Lambda})$ . (The non-abelian version of  $W$  is  $\bar{D}_+(e^{-V} D_- e^V)$ , as follows from  $\{\bar{\nabla}_+, \nabla_-\} = W$ .) So we take

$$K(V + \Phi^0 + \bar{\Phi}^0, \Phi^i, \chi^a) + i\Psi \bar{D}_+ D_- V + i\bar{\Psi} D_+ \bar{D}_- V$$

where  $\Psi, \bar{\Psi}$  are unconstrained superfields. Integrating by parts and shifting  $V$  harmlessly, we find

$$K(V, \Phi^i, \chi^a) - (\tilde{\chi} + \tilde{\bar{\chi}})V$$

where  $\tilde{\chi} = +i\bar{D}_+ D_- \Psi$ ,  $\tilde{\bar{\chi}} = +iD_+ \bar{D}_- \bar{\Psi}$  are twisted chiral and antichiral superfields respectively. Now eliminating  $V$ , we find

$$\tilde{K} = K(V(\Phi^i, \chi^a, \tilde{\chi} + \tilde{\bar{\chi}}), \Phi^i, \chi^a) - (\tilde{\chi} + \tilde{\bar{\chi}})V$$

where  $V(\Phi^i, \chi^a, \tilde{\chi} + \tilde{\bar{\chi}})$  is found from

$$\frac{\partial K}{\partial V} = \tilde{\chi} + \tilde{\bar{\chi}};$$

$\tilde{K}$  is the Legendre transform of  $K$ . Notice that the chiral superfield  $\Phi^0$  has been replaced by the twisted chiral superfield  $\tilde{\chi}$ .

One final note here: For the reverse duality, that is if we want to replace a twisted chiral gauge field by a chiral one, we need a twisted gauge multiplet. This is found by switching  $D_-$  with  $\bar{D}_-$  while leaving  $D_+$  alone.

#### IV. Geometry of $N = 2$ Quotients.

$N = 2$  supersymmetry requires an even-dimensional target space. A component quotient by  $U(1)$  removes one dimension, so you must lose  $N = 2$  supersymmetry. How does the  $N = 2$  quotient manage to preserve  $N = 2$  supersymmetry? The answer is: *symplectic reduction*. In addition to a quotient, one imposes a constraint. The  $N = 2$  gauge multiplet has a physical vector, a spinor, and an auxiliary scalar. The vector performs the quotient, and the auxiliary field imposes the constraint. For simplicity, let's consider the Kähler case. Then we have  $X^i$  is a holomorphic killing vector. That implies  $\mathcal{L}_X J = 0$ ,  $\mathcal{L}_X g = 0$ , and hence  $\mathcal{L}_X(Jg) = 0$ . This gives:

$$X^i J_{jk,i} + X^i_{,j} J_{ik} + X^i_{,k} J_{ji} = 0.$$

We also have that  $\nabla J = 0$ , which in particular implies

$$J_{jk,i} + J_{ki,j} + J_{ij,k} = 0.$$

Then we find

$$-X^i J_{ki,j} - X^i J_{ij,k} + X^i_{,j} J_{ik} + X^i_{,k} J_{ji} = 0,$$

which finally implies

$$(X^i J_{ik})_{,j} - (X^i J_{ij})_{,k} = 0.$$

Thus the curl of  $X^i J_{ij}$  vanishes, and hence it is a gradient:

$$X^i J_{ij} = \mu_{,j}$$

$\mu$  is called the moment map.

If we now impose  $\mu = 0$  and take quotient on that submanifold, we preserve the Kähler structure. In our case,  $\mu = \frac{\partial K}{\partial V}|_{V=0} - c$ . The  $N = 2$  quotient described earlier gives precisely the symplectic reduction of the target space as described here.

## V. CFT Quotients.

The equations for invariance discussed so far only imply *global* invariance. The corresponding Noether current is conserved:  $\partial\bar{J} + \bar{\partial}J = 0$ , but for conformal field theory, we would like to have separate holomorphic currents:  $\bar{\partial}J = 0$ . This means that the symmetry should survive for an arbitrary parameter  $\alpha(\bar{z})$ . A straightforward calculation shows that this implies

$$\nabla_i^+ X^j = 0.$$

If  $T_{ijk}X^k = 0$ ,  $\phi^0$  is a distinct free field. If  $T_{ijk}X^k \neq 0$ , the action has the form (in special coordinates)

$$\int (\partial\phi^0\bar{\partial}\phi^0 + g_i(\phi^i)\partial\phi^i\bar{\partial}\phi^0) + S[\phi^i]$$

If we also want an independent  $\bar{J}$ , we are led to two coordinates

$$\begin{aligned} & \int \left[ \partial\phi_R\bar{\partial}\phi_R + \partial\phi_L\bar{\partial}\phi_L + 2B(\phi^i)\partial\phi_R\bar{\partial}\phi_L + G_i^R\partial\phi_R\bar{\partial}\phi^i + G_i^L\partial\phi^i\bar{\partial}\phi_L \right] \\ & + (\partial\phi_L\bar{\partial}\phi_R - \partial\phi_R\bar{\partial}\phi_L) + S[\phi^i] \end{aligned}$$

Then there are two natural quotients, vector:

$$\delta\phi_R = \alpha, \quad \delta\phi_L = \alpha,$$

or axial:

$$\delta\phi_R = \alpha, \quad \delta\phi_L = -\alpha.$$

Since these are quotients of by Kač-Moody symmetries, they make sense in CFT (for example, they could be implemented by the GKO construction). If we simply perform the quotients at the lagrangian level, we find that the two different quotients lead to theories that are a dual pair! So dual backgrounds are different CFT quotients of a higher dimensional theory.

## VI. The Surreal Pun:

A man from Helsinki goes into a shop, and asks, in Swedish, “Do you have any candles?” The salesperson answers, “Would you like apple juice or orange juice?” and the man replies “I would like Christmas candles.”

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