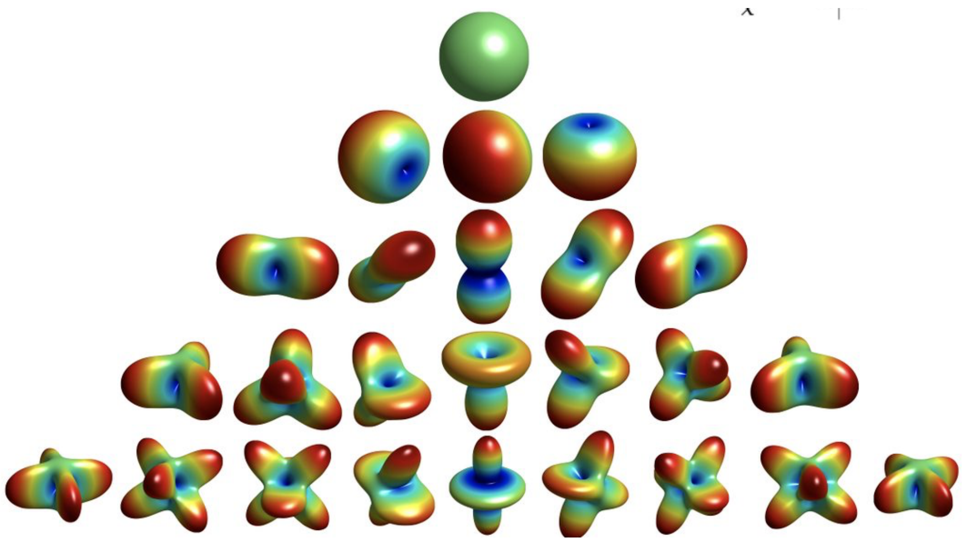
A review of PDE



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David Gontier

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INTRODUCTION

The present manuscript contains the notes for a one-week course (15h) given at Université Paris-Dauphine, entitled *A review in PDE*.

The presentation of the notes, the results, and most of the remarks are taken from the following books:

- *Analysis* by E. Lieb and M. Loss (in English) [LL01];
- *Analyse Fonctionelle* by H. Brezis (in French) [Bre99];
- *Éléments d'analyse fonctionnelle* by F. Hirsh and G. Lacombe (in French, with many exercices) [HL09].

Some other references are

- *Théorie des Distributions* by L. Schwartz (in French, for the chapter on distributions) [Sch66];
- *Partial Differential Equations* by L.C. Evans (in English, very complete, hence quite long) [Eva10].

Most of the proofs of the theorems are simplified, and there will be links to the corresponding theorems in these books.

CONTENTS

CHAPTER 1

THE LEBESGUE L^P SPACES

In this Chapter, we define the Lebesgue L^p spaces. We focus on the special case where the measure is the Lebesgue one on $\mathbb{R}^d$.

1.1 Notation and first facts

First, we recall basic facts about measure theory and integration. The reader who is not familiar with the theory can refer to [LL01, Chapter 1].

1.1.1 Basics in measure theory

The open ball of $\mathbb{R}^d$ of center $x \in \mathbb{R}^d$ and radius $r > 0$ is denoted by

$$\mathcal{B}(x,r) = \left\{ y \in \mathbb{R}^d, \quad |x-y| < r \right\}.$$

The **Borel sigma-algebra** of $\mathbb{R}^d$ is the one generated by the family of all open balls of $\mathbb{R}^d$. There is a *natural* **measure** on this sigma-algebra, called the **Lebesgue measure**, denoted $\mathcal{L}^d(A)$ or $|A|$ or $\mathrm{d}x(A)$, which is the one for which

$$|\mathcal{B}(x,r)| := \mathcal{L}^d(\mathcal{B}(x,r)) := \frac{|\mathbb{S}^{d-1}|}{d} r^d, \quad \text{with} \quad |\mathbb{S}^{d-1}| := \frac{2\pi^{d/2}}{\Gamma(d/2)},$$

where Γ is the usual Euler's Gamma function. Recall that $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$, and that $\Gamma(1/2) = \sqrt{\pi}$ while $\Gamma(1) = 1$. This gives the usual well-known formulae

$$\mathcal{L}^1(\mathcal{B}(x,r)) = 2r, \quad \mathcal{L}^2(\mathcal{B}(x,r)) = \pi r^2, \quad \mathcal{L}^3(\mathcal{B}(x,r)) = \frac{4}{3}\pi r^3, \quad \text{etc.}$$

By construction, the Lebesgue measure in translation invariant: $\mathcal{L}^d(A) = \mathcal{L}^d(A+y)$ for all $y \in \mathbb{R}^d$.

We say that a property $P : \mathbb{R}^d \to \{\text{True}, \text{False}\}$ holds **almost everywhere** if $P^{-1}\{\text{False}\}$ is (contained in a Borel set) of measure 0.

Example 1.1 (Countable sets have 0 measure)**.** *For all $x \in \mathbb{R}^d$, we have $\mathcal{L}^d(\{x\}) = 0$. By countable additivity, we deduce that if $C \in \mathbb{R}^d$ is countable, then C is Borel-measurable, and $\mathcal{L}^d(C) = 0$.*
For instance, the assertion x is irrational *holds almost everywhere.*

In the sequel, Ω always denotes an **non empty open set** of $\mathbb{R}^d$.

We say that a function $f : \Omega \to \mathbb{R}$ is (Borel or Lebesgue)-**measurable** if, for all $\lambda \in \mathbb{R}$, the set

$$\{f > \lambda\} := \{x \in \Omega, \quad f(x) > \lambda\}$$

is Borel-measurable.

Exercice 1.2

Prove that if f is measurable, then for all $\lambda \in \mathbb{R}$, the set $\{f < \lambda\}$ is measurable.
Hint: prove that $\{f \le \lambda + 1/n\}$ *is measurable.*

1.1.2 Integrable functions

We now focus on **Integrable** functions.

Definition 1.3 (Lebesgue integration)**.** *A* **positive** *measurable function* $f : \Omega \to \mathbb{R}_+$ *is (Lebesgue)-integrable if the function* $F_f(\lambda) := \mathcal{L}^d(\{f > \lambda\})$ *is Riemann integrable. In this case, its integral is*

$$\boxed{\int_\Omega f(x)\mathrm{d}x := \int_0^\infty F_f(\lambda)\mathrm{d}\lambda.} \tag{1.1}$$

Remark 1.4. *The function* $F_f : \mathbb{R}_+ \to \mathbb{R}_+$ *is positive and decreasing, so it is always Riemann integrable (why?). Being integrable only means* $\int_\Omega f = \int_0^\infty F_f < \infty$*.*

This formula is formally easy to understand from the Fubini's theorem (see Theorem 1.11 below). Indeed, if $\mathbb{1}(x > 0)$ denotes the Heaviside function, we have

$$\int_0^\infty F_f(\lambda)\mathrm{d}\lambda = \int_0^\infty \left(\int_\Omega \mathbb{1}(f(x) > \lambda)\mathrm{d}x \right) \mathrm{d}\lambda = \int_\Omega \left(\int_0^\infty \mathbb{1}(f(x) > \lambda)\mathrm{d}\lambda \right) \mathrm{d}x = \int_\Omega f(x)\mathrm{d}x.$$

If $f : \Omega \to \mathbb{R}$ is not positive valued, we introduce

$$f_+ := \max\{f, 0\} \quad \text{and} \quad f_- := \max\{-f, 0\}.$$

These two functions are positive valued, and we have $f = f_+ - f_-$ and $|f| = f_+ + f_-$. In this case, we say that f is integrable if f_+ and f_- are integrable.

Exercice 1.5

Prove that a measurable function f is integrable iff $|f|$ is integrable.

We admit the following.

Lemma 1.6. *If f is Riemann integrable, then it is Lebesgue integrable, and the two integrals coincide.*

Remark 1.7. *There is a small change of notation between the Riemann and Lebesgue integral in the case the one-dimensional case $d = 1$. If f is positive, we can write*

$$\int_{[a,b]} f(x)\mathrm{d}x \quad \textit{(Lebesgue notation)} \qquad \textit{versus} \qquad \int_a^b f(x)\mathrm{d}x \quad \textit{(Riemann notation)}$$

The first integral is always positive, while the second one is positive if $a < b$, and negative if $b < a$.

It is unclear from Definition 1.3 that the integral is linear: $\int(f+g) = \int f + \int g$. It is however the case (this is a consequence of Theorem 1.13 below). We can use Lebesgue integration *as usual*.

1.1.3 The «powerful» theorems

There are three *powerful* theorems that describe how limits of functions behave with the integral. These theorems are enunciated in [Bre99, Thm IV.1 and IV.2] and proved in [LL01, Thm 1.6, 1.7 and 1.8]

Theorem 1.8: Monotone Convergence Theorem

If (f_j) is a sequence of measurable functions, **increasing** in the sense $f_j(x) \le f_{j+1}(x)$ a.e., then $f(x) := \lim f_j(x)$ is measurable, and

$$\int_\Omega f(x)\mathrm{d}x = \lim_{j\to\infty} \int_\Omega f_j(x)\mathrm{d}x.$$

The last value can be infinite. In other words, increasing sequence implies $\lim \int = \int \lim$.

Proof. By replacing f_j by $f_j - f_1$, we may always assume that $f_j \ge 0$ is positive valued.
We see that the sequence $F_j := F_{f_j}$ is increasing: $F_{j+1}(\lambda) > F_j(\lambda)$. So (F_j) is a monotone family of decreasing functions, which converges point-wise to $F_f(\lambda)$. The rest of the proof is left is a classical exercise in the theory of Riemann integration. □

Theorem 1.9: Fatou's theorem

Let (f_j) be a sequence of **positive** measurable functions. Then $f := \liminf_{j\to\infty} f_j$ is positive, measurable, and

$$0 \le \int_\Omega f(x)\mathrm{d}x \le \liminf_{j\to\infty} \int_\Omega f_j(x)\mathrm{d}x.$$

In other words, positivity implies $\int \operatorname{argmin} \le \operatorname{argmin} \int$. A mnemotechnic trick is that *the sum of minimum is always lower than the minimum of the sum.*

Proof. Define $g_k(x) := \inf_{j\ge k} f_j(x)$. The sequence g_k is measurable, increasing, with $\lim_{k\to\infty} g_k = \liminf_{j\to\infty} f_j = f$. By the Monotone Convergence Theorem 1.8, we have

$$\int_\Omega f(x)\mathrm{d}x = \lim_{k\to\infty} \int_\Omega g_k(x)\mathrm{d}x.$$

Now, we see that $g_k \le f_j$ for all $k \le j$, so $\int g_k \le \inf_{j\ge k} \int f_j$, and the result follows. □

Finally, we have the *master* theorem

Theorem 1.10: Dominated Convergence Theorem

Let (f_j) be a sequence of measurable functions which converges point-wise to f a.e. Assume there is an integrable function G so that $|f_j|(x) \le G(x)$ (**domination**). Then $|f| \le G(x)$ and

$$\int_\Omega f(x)\mathrm{d}x \le \lim_{j\to\infty} \int_\Omega f_j(x)\mathrm{d}x.$$

In other words, domination implies $\lim \int = \int \lim$.

Proof. We only do the proof for positive functions f_j. By Fatou's theorem 1.9, we have

$$\liminf \int f_j \ge \int f.$$

On the other hand, since $G - f_j$ is also a family of positive functions, we have, by Fatou's lemma again

$$\liminf \int G - f_j \geq \int G - f, \quad \text{so} \quad \limsup \int f_j \leq \int f.$$

This proves $\limsup \int f_j \leq \int f \leq \liminf \int f_j$, and the result follows. □

To see why the domination is important, the reader should keep in mind the following **counterexamples**.

- **The mass goes to infinity**. Let $\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^+)$ and $e \in \mathbb{R}^d \setminus \{0\}$. Then $f_j(x) := \psi(x - je)$ converges point-wise to $f = 0$. However, $\int f_j = \int \psi > 0$, while $\int f = 0$.
- **The mass spreads over**. Consider now $f_j(x) = j^{-d}\psi(x/j)$. Again, f_j converges point-wise to $f = 0$. However, $\int f_j = \int \psi > 0$, while $\int f = 0$.
- **The mass concentrates**. Take $f_j(x) = j^d \psi(jx)$. Then f_j converge point-wise to f for all $x \neq 0$, so a.e. However, $\int f_j = \int \psi > 0$, while $\int f = 0$.

The following theorem is of different nature, but we include it here, as it is also *powerful*. We skip the proof, which can be found in [LL01, Thm 1.12]

Theorem 1.11: Fubini's theorem

If $f : \mathbb{R}^d \times \mathbb{R}^s \to \mathbb{R}^+$ is measurable, then

$$\int_{\mathbb{R}^{d+s}} f(x,y) \mathrm{d}^{d+s}((x,y)) = \int_{\mathbb{R}^s} \left(\int_{\mathbb{R}^d} f(x,y) \mathrm{d}^d x \right) \mathrm{d}^s y = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^s} f(x,y) \mathrm{d}^s y \right) \mathrm{d}^d x.$$

1.2 The L^p spaces

We now introduce the Lebesgue $L^p(\Omega)$ space. We focus on the case $1 \leq p \leq \infty$. The case $p = \infty$ is always a bit *special*.

1.2.1 Definitions

For $1 \leq p < \infty$, we define

$$\mathcal{L}^p(\Omega) := \{ f \text{ measurable from } \Omega \text{ to } \mathbb{C}, \quad \|f\|_{L^p} < \infty \}, \quad \text{where} \quad \boxed{\|f\|_{L^p} := \left(\int_\Omega |f|^p(x) \mathrm{d}x \right)^{1/p}.}$$

We have

- $\|f\|_{L^p} = 0$ iff $f = 0$ almost-everywhere;
- $\|\lambda f\|_{L^p} = |\lambda| \cdot \|f\|_{L^p}$;
- $\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$ (Minkowski inequality, see Theorem 1.18 below).

This proves that the map $\| \cdot \|$ is a **semi**-norm. We have $\|f\|_{L^p} = 0$ for any f which is 0 **almost everywhere**. To cure this problem, we introduce the equivalent relation $f \sim g$ iff $f = g$ almost everywhere, and define $L^p(E) := \mathcal{L}^p(E)/\sim$. In practice, this means that elements of $L^p(E)$ are not functions, but classes of functions. However, we usually say *a function f in $L^p(E)$*, with the implicit convention that f is only defined a.e. For instance, we say $f \in L^p(E)$ is continuous to state that there is a continuous representation of f.

For $p = \infty$, we set

$$L^\infty(\Omega) := \{ f \text{ measurable from } \Omega \text{ to } \mathbb{C}, \text{ bounded a.e.} \}, \quad \text{and} \quad \boxed{\|f\|_\infty := \inf\{\lambda \geq 0, \quad \mathcal{L}^d(\{f > \lambda\}) = 0\}.}$$

Really simple functions

A special case of L^p functions is given by the so-called *really simple functions*. These are somehow the most simple functions one can think of. They form a dense subset of L^p for $p < \infty$, which allows to simplify some of the proofs.

A set $A \subset \mathbb{R}^d$ is called **half-open rectangular** if A is of the form

$$A = (a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_d, b_d].$$

We say that $f \in L^p(\mathbb{R})$ is a **really simple function** if there is $N \in \mathbb{N}$ and half-open rectangular sets $A_1, \cdots, A_N$ and numbers $f_1, \cdots, f_N \in \mathbb{C}$ so that

$$f(x) = \sum_{j=1}^{N} f_j \mathbb{1}(x \in A_j).$$

In particular, $f(x)$ only takes finitely many values.

Exercice 1.12

Prove that we may always assume that the sets A_j are disjoint.

We admit the following result (see [LL01, Theorem 1.18]). It only holds for $p < \infty$.

Theorem 1.13: Density of very simple functions in L^p

For all $1 \leq p < \infty$, the set of really simple functions is dense in $L^p(\mathbb{R}^d)$. In other words, for all $f \in L^p(\mathbb{R}^d)$ and all $\varepsilon > 0$, there is a really simple function f_ε so that

$$\| f - f_\varepsilon \|_{L^p} < \varepsilon.$$

In the case $p = 1$ for instance, this allows to prove that integration is a linear operator (why?). The result cannot hold in $L^\infty(\mathbb{R})$. One reason is that really simple functions have compact support. However, the function $f : \mathbb{R} \to \mathbb{R}^+$ defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad \lfloor x \rfloor \text{ is even,} \\ 0 & \text{else,} \end{cases}$$

is in $L^\infty(\mathbb{R})$, but cannot be approximated by functions with compact support. On $L^\infty((-1,1))$, one counterexample is given by the function $g(x) := f(x/(1-x^2))$ with the previous f.

1.2.2 The useful inequalities

After the powerful theorems comes the powerful inequalities.

Theorem 1.14: Jensen's inequality

If $J : \mathbb{R}^+ \to \mathbb{R}^+$ is convex, and $f, \mu : \Omega \to \mathbb{R}^+$ are measurable with $\int \mu = 1$, then

$$\int_\Omega J(f)\mu \geq J\left(\int_\Omega f\mu \right).$$

This is somehow the generalisation of the definition of convexity.

Proof. Assume J differentiable for simplicity. Since J is convex, we have for all $a, b \in \mathbb{R}$,

$$J(a) \geq J(b) + J'(b)(a - b).$$

Taking $b = \int f\mu$, and $a = f(x)$ gives

$$J(f(x)) \geq \left(\int f\mu\right) + J'\left(\int_\Omega f\mu\right) \times \left[f(x) - \left(\int_\Omega f\mu\right)\right].$$

Multiplying by $\mu(x)$ and integrating gives the result (the term in bracket vanishes since $\int \mu = 1$.) □

Theorem 1.15: Hölder's inequality

Let $1 \leq p, q \leq \infty$ be such that

$$\frac{1}{p} + \frac{1}{q} = 1, \quad \text{or, equivalently,} \quad q = \frac{p}{p-1}.$$

Let $f \in L^p(\Omega)$ and $q \in L^q(\Omega)$. Then $fg \in L^1(\Omega)$, and

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

In the case $p = q = 2$, we recover Schwarz inequality $\int fg \leq \|f\|_{L^2} \|g\|_{L^2}$. In the sequel, we denote by

$$\boxed{p' := \frac{p}{p-1},} \quad \text{(dual exponent of } p). \tag{1.2}$$

Proof. Without loss of generality, we may assume that f and g are positive. By homogeneity, we may also assume that $\int g^q = 1$. We set $\mu(x) = g^q(x)$, $F(x) := f(x)/g^{q/p}(x)$ and $J(t) := t^p$, which is convex. We apply Jensen's inequality to this triplet, and notice that

$$\int_\Omega J(F)\mu = \int_\Omega \left(\frac{f}{g^{q/p}}\right)^p g^q = \int_\Omega f^p,$$

while, since $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_\Omega F\mu = \int_\Omega \left(\frac{f}{g^{q/p}}\right) g^q = \int_\Omega f g^{q(1-\frac{1}{p})} = \int_\Omega fg.$$

The result follows. □

The following form of the Hölder's inequality is often used.

Theorem 1.16: Holder's inequality in the general case

Let $1 \leq p, q, r \leq \infty$ be such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then $fg \in L^r(\Omega)$, and

$$\|fg\|_{L^r}(\Omega) \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

Proof. We use Hölder's inequality with $F = |f|^r \in L^{p/r}(E)$ and $G = |g|^r \in L^{q/r}(E)$. We have $1/(p/r) + 1/(q/r) = 1$, so $FG \in L^1$, and

$$\|fg\|_{L^r}^r = \int_\Omega |f|^r |g|^r = \|FG\|_{L^1} \leq \|F\|_{L^{p/r}} \|G\|_{L^{q/r}} = \left(\int |f|^p\right)^{r/p} \left(\int |g|^q\right)^{r/q} = \left(\|f\|_{L^p} \cdot \|g\|_{L^q}\right)^r.$$

□

Another corollary which is very useful is the following.

Theorem 1.17: Interpolation

If $1 \le p_1 \le p_2 \le \infty$, and $f \in L^{p_1}(\Omega) \cap L^{p_2}(\Omega)$, then for all $p \in [p_1, p_2]$, we have $f \in L^p(\Omega)$ with

$$\|f\|_{L^p} \le \|f\|_{L^{p_1}}^{\alpha} \|f\|_{L^{p_2}}^{1-\alpha}, \qquad \text{where } 0 \le \alpha \le 1 \text{ is chosen so that } \frac{1}{p} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}.$$

Proof. Write $f = f^{\alpha} f^{(1-\alpha)}$, with $f^{\alpha} \in L^{p_1/\alpha}$ and $f^{(1-\alpha)} \in L^{p_2/(1-\alpha)}$, and apply the previous result. □

Finally, we prove Minkowski's inequality.

Theorem 1.18: Minkowski's inequality

For all $f, g \in L^p(\Omega)$ with $1 \le p \le \infty$, we have

$$\|f + g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}.$$

In particular, the map $f \to \|f\|$ is (strictly) convex.

Proof. First, we have the easy bound $|f+g|^p \le (2\max\{f,g\})^p \le |2f|^p + |2g|^p$, so $f+g$ is indeed in L^p. Then, we have

$$\int |f+g|^p = \int |f+g| \cdot |f+g|^{p-1} \le \int |f| \cdot |f+g|^{p-1} + \int |g| \cdot |f+g|^{p-1}.$$

We then use Hölder's inequality with $f \in L^p$ and $|f+g|^{p-1} \in L^q$ with $q = \frac{p}{p-1}$, and get

$$\int |f| \cdot |f+g|^{p-1} \le \left(\int |f|^p \right)^{1/p} \left(\int |f+g|^p \right)^{\frac{p-1}{p}}.$$

This gives $\|f+g\|^p \le (\|f\|_{L^p} + \|g\|_{L^p}) \|f+g\|^{p-1}$, and the result follows. □

1.2.3 Convolution in Lebesgue spaces

In this section, we take $\Omega = \mathbb{R}^d$ (convolution is only defined on the full space). Let f, g be two complex-valued functions. We define the **convolution** $f * g$ by

$$(f * g)(x) := \int_{\mathbb{R}^d} f(y) g(x-y) \mathrm{d}y.$$

It is classical that $f * g = g * f$, and that $(f * g) * h = f * (h * g)$, that is the convolution is **commutative** and **associative**.

Convolution from $L^p \times L^q \to L^r$

Thanks to Hölder's inequality 1.15, we see that this definition indeed makes sense whenever $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ with $1/p + 1/q = 1$. In this case, we have $f * g \in L^{\infty}(\mathbb{R}^d)$ and

$$\|f * g\|_{L^{\infty}} \le \|f\|_{L^p} \|g\|_{L^q}.$$

On the other hand, if $f \in L^1(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d)$, we have by Fubini theorem that $f * g \in L^1(\mathbb{R}^d)$ with

$$\|f * g\|_{L^1} = \|f\|_{L^1} \|g\|_{L^1}.$$

The generalisation of this is known as Young's inequality (see [LL01, Theorem 4.2])

Theorem 1.19: Young's inequality, first form

Let $1 \le p, q, r \le \infty$ be such that
$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$
If $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, then $f * g \in L^r(\mathbb{R}^d)$, and
$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}.$$

One other way to state this theorem is as follows. The fact that these two inequalities are equivalent comes from the duality result presented below (see Theorem 1.25 below). In this second version, the variable r plays the role of $r/(r-1)$ of the first version). The proof is taken from [LL01, Theorem 4.2].

Theorem 1.20: Young's inequality, second form

Let $1 \le p, q, r \le \infty$ with
$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2.$$
Let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ and $h \in L^r(\mathbb{R}^d)$. Then the function $(f * g)h$ is in $L^1(\mathbb{R}^d)$, and
$$\left| \iint_{(\mathbb{R}^d)^2} f(x) g(y-x) h(y) \mathrm{d}x \mathrm{d}y \right| \le \|(f*g)h\|_{L^1} \le \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}.$$

Proof. Without loss of generality, we may assume that f, g, h are positive. Let p', q', r' the dual power of p, q, r, see Eqt (1.2), and set
$$\begin{cases} \alpha(x,y) & := f(x)^{p/r'} g(y-x)^{q/r'} \\ \beta(x,y) & := g(y-x)^{q/p'} h(y)^{r/p'} \\ \gamma(x,y) & := h(y)^{r/q'} f(x)^{p/q'}. \end{cases}$$
We have
$$\frac{1}{r'} + \frac{1}{p'} + \frac{1}{q'} = \left(1 - \frac{1}{r}\right) + \left(1 - \frac{1}{p}\right) + \left(1 - \frac{1}{q}\right) = 3 - 2 = 1,$$
so we can use Hölder's inequality (on $(\mathbb{R}^d)^2$) and get
$$\int_{(\mathbb{R}^d)^2} \alpha(x,y)\beta(x,y)\gamma(x,y) \mathrm{d}x \mathrm{d}y \le \|\alpha\|_{L^{r'}((\mathbb{R}^d)^2)} \|\beta\|_{L^{p'}((\mathbb{R}^d)^2)} \|\gamma\|_{L^{q'}((\mathbb{R}^d)^2)}.$$
The integrand in the left is also (we focus on the f terms for the computation)
$$\alpha(x,y)\beta(x,y)\gamma(x,y) = f(x)^{\frac{p}{r'} + \frac{p}{q'}} \cdots = f(x)^{p(1 - \frac{1}{p'})} \cdots = f(x) g(y-x) h(y).$$
On the other hand, we have, by Fubini's theorem, that
$$\|\alpha\|_{L^{r'}((\mathbb{R}^d)^2)}^{r'} = \iint_{(\mathbb{R}^d)^2} f(x)^p g(y-x)^q \mathrm{d}x \mathrm{d}y = \|f\|_{L^p}^p \|g\|_{L^q}^q,$$
so indeed $\alpha \in L^{r'}((\mathbb{R}^d)^2)$. Writing similar inequalities for β and γ gives the result. □

1.2.4 Convolution as a smoothing operator

We now prove that, in general, $f * g$ is more regular than f.

Smoothing sequences

Let $j \in C_c^\infty(\mathbb{R}^d)$ be such that j is radial decreasing, with $j(x) = 0$ for all $|x| > 1$, and $\int_{\mathbb{R}^d} j = 1$. The fact that such functions exist is classical. For $\varepsilon > 0$, we set

$$j_\varepsilon(x) := \frac{1}{\varepsilon^d} j\left(\frac{x}{\varepsilon}\right).$$

Since j is compactly supported in $\mathcal{B}(0,1)$, the function j_ε is compactly supported in $\mathcal{B}(0,\varepsilon)$. The scaling is chosen so that $\int_{\mathbb{R}^d} j_\varepsilon = \int_{\mathbb{R}^d} j$. The family (j_ε) is called a **smoothing sequence**, a **mollifier**, or an **approximation of the Dirac** (see Example 2.4 below).

Approximation by smooth functions

In the sequel, we say that a function f is **smooth** if $f \in C^\infty(\mathbb{R}^d)$. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$, we set

$$D^\alpha f := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} f.$$

Recall that if f is smooth, the order of the derivatives is irrelevant, thanks to Schwartz' Lemma. The following Lemma shows that convolution smooth functions (see [LL01, Theorem 2.16]).

Theorem 1.21: Convolution smooths functions

Let $1 \le p \le \infty$, and let (j_ε) be a smoothing sequence. For all $f \in L^p(\mathbb{R}^d)$, we set $f_\varepsilon := f * j_\varepsilon$. Then

- f_ε is smooth, and $D^\alpha(f * j_\varepsilon) = f * (D^\alpha j_\varepsilon)$;
- $f_\varepsilon \in L^p(\mathbb{R}^d)$ with $\|f_\varepsilon\|_{L^p} \le \|f\|_{L^p}$;
- if in addition $p < \infty$, then $\|f - f_\varepsilon\|_{L^p} \to 0$ as $\varepsilon \to 0^+$.

Before we give the proof, we emphasise that if $p = \infty$, the last result **is false**. Indeed, assume that $\|f - f_\varepsilon\|_\infty \to 0$. Then, f would the limit of the continuous functions f_ε for the uniform convergence. This would imply that f is also continuous. So the result fails whenever f is discontinuous.

Proof. The inequality $\|f_\varepsilon\|_{L^p} \le \|f\|_{L^p}$ is simply Young's inequality, together with the fact that $\|j\|_{L^1} = 1$. Let us prove the last result in the case $p = 1$ (see [LL01, Theorem 2.16] for the general case).

By density, it is enough to prove the result for f a really simple function, and by linearity, it is enough to prove it in the case $\chi(x) = \mathbb{1}(x \in A)$, where A is half-open rectangular, of the form

$$A = (a_1, b_1] \times \cdots \times (a_d, b_d].$$

Since j_ε is compactly supported in $\mathcal{B}(0,\varepsilon)$ with $\int j = 1$, the function $\chi_\varepsilon := \chi * j_\varepsilon$ satisfies

$$\mathbb{1}(x \in A_\varepsilon^-) \le \chi_\varepsilon(x) \le \mathbb{1}(x \in A_\varepsilon^+), \quad \text{with} \quad A_\varepsilon^\pm := (a_1 \mp \varepsilon, b_1 \pm \varepsilon] \times \cdots \times (a_d \mp \varepsilon, b_d \pm \varepsilon].$$

In particular, we have

$$\|\chi_\varepsilon - \chi\|_{L^1} \le \max\{\|\mathbb{1}(x \in A_\varepsilon^+) - \mathbb{1}(x \in A)\|, \|\mathbb{1}(x \in A) - \mathbb{1}(x \in A_\varepsilon^-)\|\} \approx C\varepsilon \xrightarrow[\varepsilon \to 0]{} 0.$$

This proves the last point.

It remains to prove the first point. We focus again on the case $p = 1$. We have

$$\frac{1}{t}\left(f_\varepsilon(x+t) - f_\varepsilon(x)\right) = \iint_{(\mathbb{R}^d)^2} f(y)\left[\frac{1}{t}\left(\psi_\varepsilon(x+t-y) - \psi_\varepsilon(x-y)\right)\right] \mathrm{d}y\mathrm{d}x.$$

The term in braket converges pointwise to $\psi_\varepsilon'(x-y)$ as $t \to 0$, and the integrand in pointwise dominated by $\|f\|_{L^1}\|\psi_\varepsilon'\|_\infty$. So, by the dominated convergence theorem, we deduce that

$$D\left(f * \psi_\varepsilon\right) = f * \left(D\psi_\varepsilon\right).$$

The result follows by induction. □

A direct corollary is the following density result, valid for $p < \infty$.

Theorem 1.22: Smooth functions are dense in L^p

For all $1 \le p < \infty$, and all $\Omega \subset \mathbb{R}^d$, the set $C^\infty(\Omega)$ is dense in $L^p(\Omega)$.
In addition, for $\Omega = \mathbb{R}^d$, $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$.

Proof. For the first point, we first notice that if $f \in L^p(\Omega)$, then the function

$$\widetilde{f}(x) := \begin{cases} f(x) & \text{if} \quad x \in \Omega, \\ 0 & \text{else,} \end{cases}$$

is in $L^p(\mathbb{R}^d)$. This function is called the **extension of** f. By the previous result, $(\widetilde{f_\varepsilon})$ is a family of smooth functions which converge to $\widetilde{f}$ in $L^p(\mathbb{R}^d)$. Restricting to Ω proves the first point.
For the second point, in the case $\Omega = \mathbb{R}^d$, it is enough to prove the result for really simple functions by density. But for such function f, the function f_ε is compactly supported. The result follows. □

1.3 L^p spaces as Banach spaces

We now focus on the completion properties of $L^p(\Omega)$ spaces. We first recall some basic notions of Banach spaces. We then focus on the special case of $L^p(\Omega)$ spaces.

1.3.1 Basics in Banach spaces

Let $(E, \|\cdot\|_E)$ be a **Banach space**, that is a vectorial space in which all Cauchy sequences have limits. This is equivalent to

$$\sum_{n\in\mathbb{N}} \|x_n\|_E < \infty \implies \sum_{n\in\mathbb{N}} x_n \text{ converges in } E.$$

A linear form $L : E \to \mathbb{C}$ is **continuous** (or **bounded**) if there is $C \in \mathbb{R}^+$ so that

$$\forall x \in E, \quad |L(x)| \le C\|x\|_E.$$

The set of all continuous form is called the **dual** of E, and is denoted by E^*. It is a Banach space when equipped with the norm

$$\|L\|_{\mathrm{op}} := \|L\|_{E^*} := \sup\left\{|L(x)|,\ x \in E,\ \|x\|_E \le 1\right\} = \sup\left\{\frac{|L(x)|}{\|x\|_E},\ x \in E \setminus \{0\}\right\}.$$

We sometime write

$$\langle L, x\rangle_{E',E} := L(x).$$

The **bidual** of E is the dual of the dual, that is $E^{**} := (E^*)^*$. We always have $E \subset E^{**}$ with the identification $E \ni x \mapsto M_x \in E^{**}$, where

$$M_x : E^* \to \mathbb{C}, \quad M_x : L \mapsto L(x).$$

If $E = E^{**}$, we say that E is **reflexive**.

Finally, we say that E is **separable** if there is a countable dense set in E.

1.3.2 Completion of L^p spaces

We now focus on $L^p(\Omega)$ space. We start with the following (see [LL01, Theorem 2.7]).

Theorem 1.23: L^p is complete

Let $1 \le p \le \infty$, and let (f_j) be a Cauchy sequence in $L^p(\Omega)$. Then there is $f \in L^p(\Omega)$ and a subsequence $\phi : \mathbb{N} \to \mathbb{N}$ so that

- $\|f_{\phi(n)} - f\|_{L^p} \to 0$;
- $f_{\phi(n)}(x) \to f(x)$ almost everywhere.

In particular, the space $L^p(\Omega)$ is complete.

Proof. We prove the result for $p < \infty$. Up to a subsequence, we may assume that $\|f_j - f_{j+1}\|_{L^p} \le 1/2^j$ (why?). We set

$$F_l(x) := |f_1(x)| + \sum_{k=1}^{l-1} |f_{l+1}(x) - f_l(x)|,$$

which is in $L^p(\Omega)$ by Minkowski's inequality with

$$\|F_l\|_{L^p} \le \|f_1\|_{L^p} + \sum_{k=1}^{l-1} \|f_{l+1} - f_l\|_{L^p} \le \|f_1\|_{L^p} + 1.$$

The sequence (F_l) is positive and increasing, hence, by the Monotone Convergence Theorem 1.8, the function $F(x) := \lim_{l\to\infty} F_l(x)$ is in $L^p(\Omega)$, and $\|F\|_{L^p} \le \|f_1\|_{L^p} + 1$. Then, we notice that

$$f_l(x) = f_1(x) + \sum_{k=1}^{l-1} \left(f_{l+1}(x) - f_l(x)\right).$$

For all fixed $x \in E$, the sequence converges absolutely (in $\mathbb{C}$), so it converges to some $f(x) := \lim f_l(x)$ point-wise. In addition, we have the domination $|f_l| \le F$, so, by the Dominated Convergence Theorem 1.10, we have $\|f\|_{L^p} = \lim_{l\to\infty} \|f_l\|_{L^p}$. Performing the same reasoning to the sequence $g_j := f_j - f$ gives the result. □

1.3.3 Separability of L^p spaces

Theorem 1.24: Separability of L^p

For all $1 \le p < \infty$, the space $L^p(\Omega)$ is separable. The space $L^\infty(\Omega)$ is not separable.

Proof. Consider first $1 \le p < \infty$, and consider the set $\mathcal{A}$ of really simple functions of the form $f = \sum_{j=1}^N f_j \mathbb{1}(x \in A_j)$, with $f_j \in (\mathbb{Q} + \mathrm{i}\mathbb{Q})$ and A_j the rectangles with rational boundaries. The set $\mathcal{A}$ is countable, and dense in $L^p(\Omega)$ for all $1 \le p < \infty$. So $L^p(\Omega)$ is separable.

In the case $p = \infty$, let us prove that $L^\infty(\mathbb{R})$ is not countable. We consider the following set of functions. For a subset $Q \in \mathbb{Z}$, we define

$$f_Q(x) = \begin{cases} 1 & \text{if} \quad \lfloor x \rfloor \in Q \\ 0 & \text{else} \end{cases}.$$

Then $(f_Q)_{Q\subset\mathbb{Z}}$ is an uncountable family, and $Q \neq Q'$ implies $\|f_Q - f_{Q'}\|_\infty = 1$. So $L^\infty(\mathbb{R})$ cannot be separable (why?). We can prove similarly that $L^\infty((-1,1))$ is not separable by considering the functions $g_Q(x) := f_Q(x/(1-x^2))$. □

1.3.4 Duality in L^p spaces

We enunciate the main result, which we will prove later in Section 3.1.1 in the case $p = 2$. We refer to [LL01, Theorem 2.14] for the proof in the general case. Recall that the dual exponent of p is $p' = p/(p-1)$ (see Eqt 1.2).

Theorem 1.25: The dual of $L^p(\Omega)$

For all $1 < p < \infty$, the dual space of $L^p(\Omega)$ is $(L^p(\Omega))^* = L^{p'}(\Omega)$.
(**Case** $p = 1$). The dual space of $L^1(\Omega)$ is $L^\infty(\Omega)$.
(**Case** $p = \infty$). We have the **strict** inclusion $L^1(\Omega) \subsetneq L^\infty(\Omega)$.

For $1 < p < \infty$, the space $L^p(\Omega)$ is reflexive, while $L^1(\Omega)$ and $L^\infty(\Omega)$ **are not** reflexive.

The dual for $L^\infty(\Omega)$ is a set of measures. We do not elaborate on this point.

We postpone the proof of this Theorem, and just remark that the inclusion $L^{p'}(\Omega) \subset (L^p(\Omega))*$ comes from Hölder's inequality. Indeed, for all $g \in L^{p'}(\Omega)$, one can consider the linear form $L_g : L^p(\Omega) \to \mathbb{C}$ defined by $L_g(f) := \int_\Omega gf$. Thanks to Hölder's inequality, we have

$$|L_g(f)| \le \int_\Omega |gf| \le \|g\|_{L^{p'}} \|f\|_{L^p}.$$

This proves that $L_g \in (L^p(\Omega))'$, and that $\|L_g\|_{\mathrm{op}} \le \|g\|_{L^{p'}}$. In addition, consider $f_0 := |g|^{p'-1}\overline{\theta(g)}$, where $\theta(g) \in \mathbb{S}^1$ is the phase of g. Since $g \in L^{p'}(\Omega)$ and since $p = p'/(p'-1)$, we have $f_0 \in L^p(\Omega)$ with

$$\|f_0\|_{L^p}^p = \int_\Omega |f_0|^p = \int_\Omega |f_0|^{\frac{p'}{p'-1}} = \int_\Omega |g|^{p'} = \|g\|_{L^{p'}}^{p'}.$$

In addition,

$$L_g(f_0) = \int_\Omega g f_0 = \int_\Omega |g|^{p'} = \|g\|_{L^{p'}}^{p'} = \|g\|_{L^{p'}} \|f_0\|_{L^p}.$$

We deduce that $\|L_g\|_{\mathrm{op}} = \|g\|_{L^{p'}}$. In practice, we identify L_g and g.

1.4 Topologies of L^p spaces

We now focus on the different topologies of L^p spaces.

1.4.1 Basics in topologies in Banach spaces

Let E be a Banach space. We can consider several topologies on E. The more natural one in the **strong topology**, defined by the following notion of convergence:

$$x_n \xrightarrow[n\to\infty]{} x, \quad \text{iff} \quad \|x_n - x\|_E \to 0. \qquad (\textbf{strong convergence}).$$

We can also define the **weak topology** of E. This one is defined by the following notion:

$$x_n \xhookrightarrow[n\to\infty]{\text{weak}} x, \quad \text{iff} \quad \forall L \in E^*, \quad L(x_n) \to L(x). \qquad (\textbf{weak convergence}).$$

Finally, we sometime use the **weak-$*$ topology**. This only applies if $E = F^*$ is already the dual space of another Banach space F. Then

$$x_n \xhookrightarrow[n\to\infty]{\text{weak-}*} x, \quad \text{iff} \quad \forall f \in F^*, \quad x_n(f) \to x(f). \qquad (\textbf{weak-* convergence if } E = F^*).$$

The weak-* topology will be used in the space $E = L^\infty(\Omega)$, which is the dual of $F = L^1(\Omega)$.

If (x_n) strongly converges to x, then (x_n) weakly converges to the same x. Also, since $F \subset F^{**}$, if $E = F^*$ and (x_n) weakly converges to x, then (x_n) weakly-* converges to the same x. If F is reflexive, we have $F = F^{**}$, and weak-* convergence is equivalent to weak convergence. Finally, if (x_n) converges to x for any of these topologies, then (x_n) is bounded in E (not so trivial, see [Bre99, Proposition III.12]).

Theorem 1.26

If $x_n \to x$ strongly in E, and $L_n \to L$ weakly-(*) in E^*, then $L_n(x_n) \to L(x)$ in $\mathbb{C}$.

Proof. We write

$$|L_n(x_n) - L(x)| \le |L_n(x_n - x)| + |(L_n - L)(x)| \le \left(\max_n \|L_n\|\right) \|x_n - x\|_E + |(L_n - L)(x)|\,.$$

Let $n \to \infty$, and the result follows. $\square$

1.4.2 Weak convergence which are not strong in L^p

Let $1 \le p < \infty$, and consider the special case $E = L^p(\mathbb{R})$. There are several ways for a subsequence (f_n) to weakly converge to f, and fails to strongly converge to f.

- The mass goes to infinity;
- The mass vanishes;
- The mass concentrate;
- Oscillations.

We already saw the first three ones (see the counter examples after Theorem 1.10). For oscillations, consider $f_0 \in C_0^\infty(\mathbb{R})$ and set

$$f_n(x) := \mathrm{e}^{inx} f_0(x),$$

which is in $L^p(\mathbb{R})$. For all $\psi \in C_c^\infty(\mathbb{R}) \subset L^{p'}$, we have

$$\langle f_n, \psi \rangle_{L^p, L^{p'}} = \int f_n \psi = \int_{\mathbb{R}} \mathrm{e}^{inx} \psi(x) f_0(x) \mathrm{d}x.$$

We recognise the Fourier transform of the smooth function $f_0\psi$. By the Riemann-Lebesgue theorem, this integral goes to 0 as $n \to \infty$. By density of $C_c^\infty(\mathbb{R})$ in $L^{p'}(\mathbb{R})$, we deduce that $f_n \to 0$ weakly in $L^p(\mathbb{R})$. However, we always have $\|f_n\|_{L^p} = \|f_0\|_{L^p}$, so the convergence is not strong.

1.4.3 Banach-Alaoglu theorem in L^p

The importance of the weak (or weak-*) topology comes from the following Theorem (see [LL01, Theorem 2.18]). We enunciate in the case of the L^p space, but it can easily be generalised for any reflexive separable Banach space E.

Theorem 1.27: Banach-Alaoglu theorem

Let $1 < p < \infty$. If (f_n) is a bounded sequence in $L^p(\Omega)$, then there is a subsequence $\phi(n)$ and an element $f \in L^p(\Omega)$ so that $(f_{\phi(n)})$ weakly converges to f.
Case $p = \infty$. If (f_n) is a bounded sequence in $L^\infty(\Omega)$, then there is a subsequence $\phi(n)$ and an element $f \in L^\infty(\Omega)$ so that $(f_{\phi(n)})$ weakly-* converges to f.

In other words, bounded sequences have weak-limits up to subsequences. This theorem **fails** for $p = 1$. Indeed, consider for instance the smoothing sequence (j_ε). The sequence point-wise converges to 0 a.e., so the weak-limit, if exists, can only be 0. However, taking the constant function $1 \in L^\infty(\mathbb{R}) = (L^1(\mathbb{R}))^*$, we have

$$\langle j_\varepsilon, 1 \rangle_{L^1, L^\infty} = \int_{\mathbb{R}} j_\varepsilon = \|j_\varepsilon\|_{L^1} = 1,$$

and the sequence does not go to 0 as $\varepsilon \to 0$.

Proof. The dual space of $L^p(\Omega)$ is $L^{p'}(\Omega)$ with $1 < p' < \infty$, and $L^{p'}(\Omega)$ is separable. Let (g_j) be a dense countable family in $L^{p'}(\Omega)$. We apply a Cantor diagonal argument to the family (g_j).

• The sequence $\int g_1 f_n$ is bounded in $\mathbb{C}$, so there is a subsequence ϕ_1 so that $g_1(f_{\phi_1(n)})$ converges to some $C_1 \in \mathbb{C}$.

• The sequence $\int g_2 f_{\phi_1(n)}$ is bounded in $\mathbb{C}$, so there is a subsequence ϕ_2 so that $g_2(f_{\phi_1(\phi_2(n))})$ converges to some $C_2 \in \mathbb{C}$.

We go on, and construct a family of subsequences ϕ_n. Finally, we set

$$\phi(n) := \phi_1(\phi_2 \cdots (\phi_n(n))).$$

By construction, for all $j \in \mathbb{N}$, the sequence $\int g_j f_{\phi(n)}$ converges to some $C_j \in \mathbb{C}$ as $n \to \infty$. We now introduce the functional $L : L^{p'} \to \mathbb{C}$ by

$$\forall g \in L^{p'}(\Omega), \quad L(g) := \lim_{n \to \infty} \int g f_{\phi(n)}.$$

This functional is clearly linear, and bounded on the dense subset (g_j). We can extend it by continuity to the whole set $L^{p'}(\mathbb{R})$. So $L \in (L^{p'}(\Omega))^* = L^p(\Omega)$. In other words, there is $f \in L^p(\Omega)$ so that $L(g) = \int fg$. This proves that $f_{\phi(n)}$ weakly converges to f in $L^p(\Omega)$.

The proof in the case $p = \infty$ is similar. This time, we use that $L^1(\Omega)$ is separable and that $L^\infty(\Omega) = L^1(\Omega)^*$. □

1.5 Lower-semi continuity and convexity

Let E be a Banach space. We say that a map $J : E \to \mathbb{R}$ is **lower-semi-continuous** (lsc) if

$$x_n \to x \quad \text{implies that} \quad J(x) \le \liminf J(x_n).$$

The convergence $x_n \to x$ can be interpreted with several topologies: a function J can be **strongly lsc**, or **weakly lsc** (or even **weakly-* lsc**). If a map J is weakly lsc, then it is strongly lsc.

Theorem 1.28: Lower semi-continuity for convex functions

Let E be a Banach space. If $J : E \to \mathbb{R}$ is convex, then J is strongly lsc iff J is weakly lsc. In particular, the map $\| \cdot \|_E$ is lsc: If $x_n \to x$ weakly in E, then $\|x\| \le \liminf \|x_n\|$.

Proof. We admit that if J is convex and strongly lsc, then, there is a continuous linear operator L_x (called **support plane**) so that

$$\forall y \in E, \quad J(y) \ge J(x) + L_x(y - x).$$

(Think of L as the differential $DJ(x)$). In particular, we have, for a sequence (x_n) that weakly converges to x, that

$$J(x_n) \ge J(x) + L_x(x_n - x).$$

Passing to the limit $n \to \infty$ gives $J(x) \leq \liminf J(x_n)$.

It is clear that the norm map $\|\cdot\|$ is convex (triangle inequality), and, *by definition*, it is strongly continuous, hence strongly lsc. So $\|\cdot\|_E$ is weakly lsc. $\square$

In the case $E = L^p(\Omega)$, we have a stronger result (see [LL01, Theorem 2.11])

Theorem 1.29

In the case $1 < p < \infty$, if $f_n \to f$ weakly in $L^p(\Omega)$, then $\|f\|_{L^p} \leq \liminf \|f_n\|_{L^p}$. If in addition, $\|f\|_{L^p} = \liminf \|f_n\|_{L^p}$, then the convergence is strong.

Proof. The first point is the previous Theorem in the case $E = L^p(\Omega)$. We prove the second point only in the case $p = 2$. We have

$$\|f - f_n\|_{L^2}^2 = \|f\|_{L^2}^2 + \|f_n\|_{L^2}^2 - 2\mathrm{Re} \int_\Omega \overline{f} f_n \xrightarrow[n\to\infty]{} 0.$$

$\square$

1.6 Additional exercices

Exercice 1.30

Let $1 \leq p < \infty$, and let $f \in L^p(\mathbb{R}^d)$. Prove that for all $\varepsilon > 0$, there is $R > 0$ so that

$$\int_{B(0,R)^c} |f(x)|^p \mathrm{d}x < \varepsilon.$$

Exercice 1.31

Let $1 \leq p < \infty$, and let $f \in L^p(\mathbb{R}^d)$. Prove that for all $\varepsilon > 0$, there is $h^* > 0$ so that

$$\forall h \in B(0, h^*), \quad \int_{\mathbb{R}^d} |f(x-h) - f(x)|^p \, \mathrm{d}x < \varepsilon.$$

Exercice 1.32

Let $f \in C_0^\infty(\Omega)$. For $1 \leq p < \infty$, we denote by $\alpha := 1/p \in (0,1)$. Prove that the following map in convex on $(0,1]$:

$$\alpha \mapsto \log\left(\|f\|_{L^{\frac{1}{\alpha}}}\right)$$

CHAPTER 2

DISTRIBUTIONS

In this chapter, we introduce the set of distributions $\mathcal{D}'(\Omega)$ and the Sobolev spaces $W^{1,m}(\Omega)$. Some references are [LL01, Chapter 6], and the original book by Schwartz [Sch66]. In the references [Bre99; Eva10], only the Sobolev spaces $W^{m,p}(\Omega)$ are defined (not all distributions). The main idea of distributions is to create an extremely weak notion of "functions", which we can differentiate any number of times. This allows to prove that some equations have solutions "*in the distributional sense*". We usually prove in a next step that these solutions are actually "*real functions*".

2.1 Distributions

2.1.1 Definition and examples

We denote by $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$ the set of smooth functions with compact support, with the following notion of convergence/topology: A sequence $(\phi_n) \in \mathcal{D}(\Omega)$ converges to $\phi \in \mathcal{D}(\Omega)$ if there is a *fixed* compact set $K \subset \Omega$ so that the support of $\phi_n - \phi$ is contained in K, and such that, for all $\alpha \in \mathbb{N}^d$, we have

$$\sup_{x \in K} |D^\alpha \phi_n(x) - D^\alpha \phi(x)| \xrightarrow[m \to \infty]{} 0.$$

Definition 2.1 (Distribution)**.** *A* **distribution** *T is a linear map from $\mathcal{D}(\Omega) \to \mathbb{C}$, which is continuous in the following sense: for all $\phi \in \mathcal{D}(\Omega)$ and all sequences (ϕ_n) that converges to ϕ in $\mathcal{D}(\Omega)$, we have*

$$T(\phi_n) \to T(\phi).$$

The set of distributions is denoted by $\mathcal{D}'(\Omega)$. We sometime write

$$\langle T, \phi \rangle_{\mathcal{D}', \mathcal{D}} := T(\phi).$$

Example 2.2 (Dirac mass)**.** *The map $\delta_0 : \phi \mapsto \phi(0)$ is a distribution, called the* Dirac mass.

Locally integrable functions

For $1 \le p \le \infty$, we set

$$L^p_{\text{loc}}(\Omega) := \{ f \quad \text{measurable on } \Omega \text{ such that, for all compact } K \in \Omega, \quad f \in L^p(K) \}. \tag{2.1}$$

The sets $L^p_{\text{loc}}(\Omega)$ **are not normed spaces**. Of course, $L^p(\Omega) \subset L^p_{\text{loc}}(\Omega)$. In addition, by Hölder's inequality, and since K is a bounded set, we always have

$$\|f\|_{L^1(K)} \le \int_K |f| = \int_K 1|f| \le \left(\int_K 1^{p'} \right)^{1/p'} \left(\int_K |f|^p \right)^{\frac{1}{p}} = |K|^{1/p'} \|f\|_{L^p(K)}.$$

This proves that $L^1_{\text{loc}}(\Omega) \subset L^p_{\text{loc}}(\Omega)$. More generally, by Theorem 1.17, we have

$$1 \leq p \leq q \leq \infty \implies L^p_{\text{loc}} \subset L^q_{\text{loc}}.$$

A mnemotechnic way to remember this is that a bounded function ($q = \infty$) is always locally integrable. The space $L^1_{\text{loc}}(\Omega)$ is somehow the *weaker* space, as it is included in all other ones.

The following theorem shows that one can distinguish these functions among distributions.

Theorem 2.3: L^1_{loc} functions are determined by distributions

If $f \in L^1_{\text{loc}}(\Omega)$, then $T_f : \phi \mapsto \int_\Omega f\phi$ is a distribution.
If $f, g \in L^1_{\text{loc}}$, then $T_f = T_g$ in $\mathcal{D}'(\Omega)$ iff $f = g$ a.e.

Proof. For the first part, we write that

$$|T_f(\phi_n) - T_f(\phi)| = \left| \int_K f\,[\phi_n - \phi] \right| \leq \|f\|_{L^1(K)} \sup_K |\phi_n - \phi| \to 0.$$

We now prove the second part. Consider the inner neighbourhood of Ω,

$$\Omega_\delta := \{x \in \Omega,\ \forall y \in \mathcal{B}(0,\delta),\ x + y \in \Omega\}.$$

Let (j_ε) be a smoothing sequence. Fix $x \in \Omega_\delta$, and set, for $\varepsilon < \delta$, $\phi(y) := j_\varepsilon(x - y) \in \mathcal{D}(\Omega)$. Then

$$\int_\Omega f(y)\phi(y)\mathrm{d}y = \int_\Omega f(y) j_\varepsilon(x-y)\mathrm{d}y = f * j_\varepsilon(x).$$

So we have $f * j_\varepsilon = g * j_\varepsilon$ on Ω_δ. Taking $\varepsilon \to 0$ gives $f = g$ a.e. in Ω_δ. Finally, taking $\delta \to 0$ gives $f = g$ a.e. on Ω. □

Convergence of distributions

We say that a sequence (T_n) of distributions converge to T **in the distributional sense**, or in $\mathcal{D}'(\Omega)$, if, for all $\phi \in \mathcal{D}(\Omega)$, we have $T_n(\phi) \to T(\phi)$.

Example 2.4. *If (j_ε) is a smoothing sequence, then $j_\varepsilon \to \delta_0$ in $\mathcal{D}'(\Omega)$.*

2.1.2 Operations on distributions

Mimicking what happens for smooth functions, we can define many operations for distributions.

Derivatives

If $f \in C^1(\Omega)$ and $\phi \in C_c^\infty(\Omega)$, we have the integration by part formula (there are no boundary terms since ϕ is compactly supported)

$$\int_\Omega (\partial_{x_i} f)\,\phi = -\int_\Omega f\,(\partial_{x_i}\phi)\,.$$

We therefore extend this property and *define* the derivative of a distribution as follows.

Definition 2.5 (Derivative of a distribution)**.** *If $T \in \mathcal{D}'(\Omega)$, the D^α derivative of T is the distribution noted $D^\alpha T$ and defined by*

$$\langle D^\alpha T, \phi\rangle_{\mathcal{D}',\mathcal{D}} := (-1)^{|\alpha|} \langle T, D^\alpha \phi\rangle_{\mathcal{D}',\mathcal{D}}\,.$$

Remark 2.6. *Since ϕ is smooth, we may use Schwartz Lemma on ϕ, and deduce that*[1] $\partial^2_{xy} T = \partial^2_{yx} T$.

[1]To quote a teacher of mine: "*students like distributions, since they are infinitely differentiable, and the order of derivatives do not matter*".

The next Lemma shows that if $(T_m) \to T$ in $\mathcal{D}'(\Omega)$ and $(D^\alpha T_m) \to T_\alpha$ in $\mathcal{D}'(\Omega)$, then necessarily $T_\alpha = D^\alpha T$. In other words, all derivatives to T_m converge to the associate derivative of T.

Theorem 2.7

If (T_m) converges to T in $\mathcal{D}'(\Omega)$, then for all $\alpha \in \mathbb{N}^d$, $(D^\alpha T_m)$ converges to $D^\alpha T$ in $\mathcal{D}'(\Omega)$.

Proof. For all $\phi \in \mathcal{D}(\Omega)$, we have $D^\alpha \phi \in \mathcal{D}(\Omega)$, so

$$\langle D^\alpha T_m, \phi \rangle_{\mathcal{D}',\mathcal{D}} = (-1)^{|\alpha|} \langle T_m, D^\alpha \phi \rangle_{\mathcal{D}',\mathcal{D}} \xrightarrow[m\to\infty]{} (-1)^{|\alpha|} \langle T, D^\alpha \phi \rangle_{\mathcal{D}',\mathcal{D}} = \langle D^\alpha T, \phi \rangle .$$

□

Multiplication by a smooth function

Let g be a $C^\infty(\Omega)$ function (not necessarily compactly supported). If $\phi \in \mathcal{D}(\Omega)$, then $g\phi \in \mathcal{D}(\Omega)$ as well, so we can define the distribution $gT = Tg$ by

$$\langle gT, \phi \rangle_{\mathcal{D}',\mathcal{D}} = \langle Tg, \phi \rangle_{\mathcal{D}',\mathcal{D}} := \langle T, g\phi \rangle_{\mathcal{D}',\mathcal{D}}.$$

Again, one can check that gT is indeed a distribution.

Exercice 2.8

Prove that $\partial_{x_i}(gT) = (\partial_{x_i} g)T + g(\partial_{x_i} T)$ in $\mathcal{D}'(\Omega)$.

Convolution

We focus here in the case $\Omega = \mathbb{R}^d$, although one can generalise to any $\Omega \subset \mathbb{R}^d$ by first checking the support of functions.

For all f, g, ϕ, we have, by a change of variable

$$\int_{\mathbb{R}^d} (f * g)\phi := \iint_{(\mathbb{R}^d)^2} f(x)g(x-y)\phi(y)\mathrm{d}y\mathrm{d}x = \iint_{(\mathbb{R}^d)^2} f(x)g(z)h(x-z)\mathrm{d}x\mathrm{d}z = \int_{\mathbb{R}^d} f(g * \phi).$$

In addition, if $\phi \in \mathcal{D}(\Omega)$ and g is an $L^1(\mathbb{R}^d)$ function with compact support, then $g * \phi \in \mathcal{D}(\Omega)$ (see Theorem 1.21). This suggests to define, for all $g \in L^1(\mathbb{R}^d)$ with compact support, the distribution $g * T = T * g$ by

$$\langle T * g, \phi \rangle_{\mathcal{D}',\mathcal{D}} = \langle g * T, \phi \rangle_{\mathcal{D}',\mathcal{D}} := \langle T, \phi * g \rangle_{\mathcal{D}',\mathcal{D}} .$$

Again, one can check that $g * T$ is indeed a distribution.

Exercice 2.9

Prove that $\partial_{x_i}(g * T) = (\partial_{x_i} g) * T = g * (\partial_{x_i} T)$ in $\mathcal{D}'(\mathbb{R}^d)$.

Theorem 2.10: Convolution smooths distributions

Let (j_ε) be a smoothing sequence. Then the distribution $T_\varepsilon := T * j_\varepsilon$ converges to T in $\mathcal{D}'(\mathbb{R}^d)$. In addition, T_ε can be identified with a $C^\infty(\mathbb{R}^d)$ function, and $D^\alpha T_\varepsilon = T * (D^\alpha j_\varepsilon)$.

Proof. For $\phi \in \mathcal{D}(\mathbb{R}^d)$, we have

$$T_\varepsilon(\phi) - T(\phi) = \langle T, \phi * j_\varepsilon - \phi \rangle_{\mathcal{D}',\mathcal{D}}.$$

For all $0 < \varepsilon < 1$, the smooth functions $\phi * j_\varepsilon$ all have support in $K + \mathcal{B}(0,1) =: K'$. In addition, since ϕ is smooth, $\phi * j_\varepsilon$ converges uniformly to ϕ in $\mathcal{D}(\mathbb{R}^d)$ (why). So $\phi * j_\varepsilon \to \phi$ in $\mathcal{D}'(\mathbb{R}^d)$, and

$T_\varepsilon(\phi) \to T(\phi)$, as wanted.

We now prove that T_ε can be seen as a smooth function. Assume first that T is a smooth function. Then

$$t_\varepsilon(x) := T * j_\varepsilon(x) = \int_\Omega T(y) j_\varepsilon(x-y)\mathrm{d}y = \langle T, j_\varepsilon(x - \cdot)\rangle_{\mathcal{D}',\mathcal{D}}.$$

and, similarly,

$$D^\alpha t_\varepsilon(x) = \langle T, (D^\alpha j_\varepsilon)(x - \cdot)\rangle_{\mathcal{D}',\mathcal{D}}.$$

The reader can check that these manipulations are still valid when T is any distribution, in the sense that indeed $T * j_\varepsilon = t_\varepsilon$ in $\mathcal{D}'(\mathbb{R}^d)$. In addition, t_ε is infinitely differentiable. □

Exercice 2.11

Prove that $(D^\alpha T) * j_\varepsilon = T * (D^\alpha j_\varepsilon) = D^\alpha (T * j_\varepsilon)$.

A similar result holds in $\Omega \subset \mathbb{R}^d$ by using techniques similar to the proof of Theorem 2.3. We deduce the following.

Theorem 2.12: Density of smooth functions

The set $C^\infty(\Omega)$ is dense in $\mathcal{D}'(\Omega)$: for all $T \in \mathcal{D}'(\Omega)$, there is a sequence $(T_n) \in C^\infty(\Omega)$ so that $T_n \to T$ in $\mathcal{D}'(\Omega)$.

As an example of how to use the last theorem, we state the following.

Theorem 2.13: Constant distributions

Let $T \in \mathcal{D}'(\Omega)$ be such that $\nabla T = 0$. Then T is a constant: there is $C \in \mathbb{C}$ so that $T(\phi) = C \int \phi$.

Proof. We have

$$0 = j_\varepsilon * (\nabla T) = \nabla (j_\varepsilon * T) =: \nabla T_\varepsilon.$$

Since T_ε is a smooth function with null derivatives, we have $T_\varepsilon = cst(\varepsilon)$. Differentiating in ε shows that the constant is independent of ε, so there is $C \in \mathbb{C}$ so that $T_\varepsilon = C$. Letting $\varepsilon \to 0$ proves the result. □

2.2 Example: the Poisson's equation

2.2.1 Integration by parts on domains

We recall here the so-called divergence formula. The notion of **domain with boundary of class C^1** will be detailed later in Section 4.2.2. We start with the divergence theorem.

Theorem 2.14: Divergence theorem

Let Ω be an open set of $\mathbb{R}^d$ with boundary $\partial\Omega$ of class C^1. For all F of class $C^1(\overline{\Omega}, \mathbb{R}^d)$, we have

$$\int_\Omega \mathrm{div}(F) = \int_{\partial\Omega} F \cdot \nu \mathrm{d}\omega,$$

where ν is the outward normal of Ω, and where $\mathrm{d}\omega$ is the surface measure on $\partial\Omega$.

We admit the result, since its proof need the definitions of surface measure, of smooth domains, and so on. In the sequel however, we use the theorem mainly in the case $\Omega = \mathcal{B}(0, x)$, in which case all these notions are easily understood.

One important application of this formula is the second Green's identity.

Theorem 2.15: Second Green's identity

Let Ω be an open set of $\mathbb{R}^d$ with boundary $\partial\Omega$ of class C^1. For all $A, B \in C^2(\overline{\Omega}, \mathbb{R})$, we have

$$\int_\Omega (\Delta A)B - A(\Delta B) = \int_{\partial\Omega} (B\nabla A - A\nabla B) \cdot \nu \mathrm{d}\omega.$$

Proof. Take $F = (\nabla A)B - A(\nabla B)$ and apply the Divergence Theorem 2.14. □

2.2.2 Green's functions and the Poisson's equation

We define the following Green's functions. Recall that $d \in \mathbb{N}$ is the dimension.

$$\begin{cases} G_0(x) := -\frac{1}{2}|x|, & \text{if} \quad d = 1, \\ G_0(x) := -\frac{1}{2\pi}\ln(|x|), & \text{if} \quad d = 2, \\ G_0(x) := \frac{1}{4\pi}\frac{1}{|x|}, & \text{if} \quad d = 3, \end{cases} \quad \text{and} \quad G_0(x) = \frac{1}{(d-2)|\mathbb{S}^{d-1}|}\frac{1}{|x|^{d-2}} \quad \text{if} \quad d > 3.$$

(Actually, the last formula is also valid for $d = 1$ and $d = 3$).

Theorem 2.16: Distributional Laplacian of Green's functions

We have $-\Delta G_0 = \delta_0$ in $\mathcal{D}'(\mathbb{R}^d)$, where δ_0 is the Dirac mass at 0.

Proof. We only do the proof in the case $d = 3$ for clarity. The function G_0 is smooth away from $x = 0$, and satisfies

$$\forall x \neq 0, \quad \nabla G_0(x) = -\frac{1}{4\pi}\frac{x}{|x|^3}, \quad \Delta G_0(x) = 0.$$

It remains to check what happens at $x = 0$. First, G_0 is locally integrable since, for all $a > 0$, we have

$$\int_{\mathcal{B}(0,a)} |G_0| = \frac{1}{4\pi}\int_{\mathcal{B}(0,a)} \frac{1}{|x|}\mathrm{d}x = \frac{1}{4\pi}|\mathbb{S}^2| \int_0^a \frac{1}{r} r^2 \mathrm{d}r = \frac{a^2}{2}.$$

Take $\phi \in \mathcal{D}(\Omega)$ with support contained in $\mathcal{B}(0, R)$ with $R > a$. The last equality shows that $\int G_0\phi$ on $\mathcal{B}(0, a)$ goes to 0 as $a \to 0$. We evaluate the integral on the annulus $\{x \in \mathbb{R}^d, a \le |x| \le R\}$. On this section, G_0 is smooth with $\Delta G_0 = 0$. The second Green's identity with $A = \phi$ and $B = G_0$ on the annulus gives

$$\int_{a \le |x| \le R} (\Delta\phi) G_0 = \int_{|x|=a} [(\nabla\phi)G_0 - \phi(\nabla G_0)] \cdot \nu \mathrm{d}\omega.$$

As before, the part with $(\nabla\phi)G_0$ goes to 0 as $a \to 0$. We focus on the term $\phi(\nabla G_0)$. Using the explicit formula for ∇G_0 and that $\nu(x) = -\frac{x}{|x|}$, we get (note that $|a\mathbb{S}^2| = 4\pi a^2$)

$$-\int_{|x|=a} \phi(\nabla G_0) \cdot \nu \mathrm{d}\omega = \frac{1}{4\pi a^2}\int_{a\mathbb{S}^2} \phi(\omega)\mathrm{d}\omega \xrightarrow[a \to 0]{} \phi(0),$$

where we used that ϕ is continuous in the last line, and by recognising the average of ϕ on $a\mathbb{S}^2$. This concludes the proof. □

We can now prove the general case.

Theorem 2.17: The Poisson equation in the whole space

Let $f \in L^1_{\rm loc}(\mathbb{R}^d)$ be such that $|f| * G_0$ is well-defined a.e.. Then $u := f * G_0$ is in $L^1_{\rm loc}(\mathbb{R}^d)$, and satisfies

$$-\Delta u = f, \quad \text{in} \quad L^1_{\rm loc}(\mathbb{R}^d).$$

We skip the proof (see [LL01, Theorem 6.21]), and just remark that, formally, this follows from

$$-\Delta (f * G_0) = f * (-\Delta G_0) = f * \delta_0 = f.$$

However, this line does not make sense if f is not smooth. Actually, it is unclear at this point that $f * G_0$ is well-defined, since neither f nor G_0 are smooth. However, we admit that everything indeed makes sense whenever $|f| * G_0$ is well-defined (see [LL01, Theorem 6.21] for details).

As we can see, the weakening of the notion of functions provides a very simple answer for the existence of u solution to $-\Delta u = f$ on the whole space $\mathbb{R}^d$. In the sequel, we will prove that, if f is regular, then so is u. For now, let us just mention the following.

Lemma 2.18. *Assume $d \geq 3$. Let $q \geq \frac{d}{2}$, and assume that*

$$f \in L^{q-\varepsilon}(\mathbb{R}^d) \cap L^{q+\varepsilon}(\mathbb{R}^d) \quad \text{for some} \quad \varepsilon > 0.$$

Then $u = f * G_0$ is in $L^r(\mathbb{R}^d)$, where $r \geq 1$ is defined by

$$\frac{1}{q} - \frac{2}{d} := \frac{1}{r}.$$

This lemma is a prototype! It can be generalised in many different ways.

Proof. Write

$$G_0(x) = G_0(x)\mathbb{1}(x < R) + G_0(x)\mathbb{1}(x \geq R).$$

Since $G_0(x) \approx |x|^{d-2}$, the first part is in $L^p(\mathbb{R}^d)$ for all $p < p_0 := \frac{d}{d-2}$, while the second part is in $L^p(\mathbb{R}^d)$ for all $p > p_0$. The dual exponent of p_0 is $q_0 = \frac{d}{2}$. The result follows from Young's inequality 1.19. □

2.3 Sobolev spaces $W^{m,p}(\Omega)$ and $H^m(\Omega)$

2.3.1 Definition

We proved that $L^p(\Omega) \subset L^p_{\rm loc}(\Omega) \subset L^1_{\rm loc}(\Omega) \subset \mathcal{D}'(\Omega)$. In particular, we can consider the distributional derivatives of L^p functions. For $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, we define the **Sobolev spaces**

$$W^{m,p}(\Omega) := \left\{ f \in L^p(\Omega), \quad \forall \alpha \in \mathbb{N}^d, \ |\alpha| \leq m, \quad D^\alpha f \in L^p(\Omega) \right\},$$

and the corresponding norm

$$\| f \|_{W^{m,p}(\Omega)} := \left(\| f \|^p_{L^p(\Omega)} + \sum_{|\alpha| \leq m} \| D^\alpha f \|^p_{L^p(\Omega)} \right)^{1/p}.$$

Note that $W^{0,p}(\Omega) := L^p(\Omega)$. When $p = 2$, we set

$$H^m(\Omega) := W^{m,2}(\Omega).$$

The H^m spaces will play a special role, since they will be Hilbert spaces when equipped with the natural inner product

$$\langle f, g \rangle_{H^m(\Omega)} := \int_\Omega \overline{f} g + \sum_{|\alpha| \leq m} \int_\Omega \overline{D^\alpha f}\, D^\alpha g.$$

We can also define the spaces $W^{m,p}_{\rm loc}(\Omega)$ as in (2.1), but we will not use them in this course.

We start with the following density result (see [LL01, Theorem 6.15]). We skip its proof since it is similar to Theorem 2.12.

Theorem 2.19

For all $m \in \mathbb{N}$ and all $1 \le p < \infty$, the set $C^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$.

We warn the reader at this point that functions of $W^{m,p}(\Omega)$ may *explode* near the boundary of Ω. So $C^\infty(\overline{\Omega})$ is usually **not dense** in $W^{m,p}(\Omega)$. This will be the topic of Section 4.4.

2.3.2 Completion of Sobolev spaces

As one can expect, Sobolev spaces are complete (see [Bre99, Proposition VIII.1]).

Theorem 2.20: The Sobolev spaces $W^{m,p}(\Omega)$ are complete.

For all $m \in \mathbb{N}$ and all $1 \le p \le \infty$, the set $W^{m,p}(\Omega)$ is a Banach space. It is separable if $p < \infty$, and reflexive if $1 < p < \infty$.
In particular, $H^m(\Omega)$ is a (separable) Hilbert space.

Proof. Let (f_j) be a Cauchy sequence in $W^{m,p}(\Omega)$. Then (f_j) is a Cauchy sequence in $L^p(\Omega)$, and, for all $|\alpha| \le$, $(D^\alpha f_j)$ is also Cauchy a sequence in $L^p(\Omega)$. The space $L^p(\Omega)$ is complete, so there are $f \in L^p(\Omega)$ and $f_\alpha \in L^p(\Omega)$, so that

$$f_j \xrightarrow{L^p} f, \quad D^\alpha f_j \xrightarrow{L^p} f_\alpha.$$

It remains to prove that $f_\alpha = D^\alpha f$. Since we have convergence in L^p, we also have convergence in the distributional sense, that is $f_j \to f$ in $\mathcal{D}'(\Omega)$. In particular, by Theorem 2.7, we must have $D^\alpha f_j \to D^\alpha f$ in $\mathcal{D}'(\Omega)$. By uniqueness of the limit in $\mathcal{D}'(\Omega)$, we indeed have $f_\alpha = D^\alpha f$. □

The reader might ask what is the dual space of $W^{1,p}(\Omega)$. We refer to [LL01, Theorem 6.24] and to the discussion in [Bre99, p.174] for this difficult (and not so interesting) question.

2.4 Sobolev spaces $W_0^{m,p}(\Omega)$ and $H_0^m(\Omega)$

2.4.1 Definition

We now study whether $\mathcal{D}(\Omega)$ is dense in $W^{m,p}(\Omega)$. In (2.19), we proved the density of $C^\infty(\Omega)$. Now we ask for the density of $C_c^\infty(\Omega)$. In general, this set **is not** dense. We therefore define

$$W_0^{m,p}(\Omega) := \overline{C_c^\infty(\Omega)} \quad \text{the closure of } C_c^\infty(\Omega) \text{ for the norm of } W^{m,p}(\Omega).$$

Similarly, we set $H_0^m(\Omega) := W_0^{m,2}(\Omega)$. Since $W_0^{m,p}(\Omega)$ is a closed subset of the Banach space $W^{m,p}(\Omega)$, it is a Banach space for the same norm.

Loosely speaking $W_0^{m,p}(\Omega)$ is the set of functions of $W^{m,p}(\Omega)$ that *vanish* at the boundary $\partial\Omega$. However, since the measure of $\partial\Omega$ is usually null, this only has an *affective* meaning.

Theorem 2.21: $W_0^{m,p}(\mathbb{R}^d) = W^{m,p}(\mathbb{R}^d)$

In the case $\Omega = \mathbb{R}^d$, we have equality $W_0^{1,p}(\mathbb{R}^d) = W^{1,p}(\mathbb{R}^d)$.

Proof. We know that $C^\infty(\mathbb{R}^d)$ is dense in $W^{m,p}(\mathbb{R}^d)$ (see Theorem 2.19). In addition, for $f \in C^\infty(\mathbb{R}^d) \cap W^{m,p}(\mathbb{R}^d)$, we set $f_n := \chi(x/n)f$, where χ is a smooth cut-off function. Then, $f_n \in C_c^\infty(\mathbb{R}^d)$, and, by the Dominated Convergence Theorem 1.10, we have $\|f_n - f\|_{W^{m,p}} \to 0$, which proves the result. □

For $1 < p < \infty$, the dual space of $W_0^{m,p}(\Omega)$ is noted $W^{-m,p'}(\Omega)$, with $\frac{1}{p} + \frac{1}{p'} = 1$, as in (1.2). By density, $f \in W^{-m,p'}(\Omega) \subset \mathcal{D}'(\Omega)$ iff there is a constant $C > 0$ so that

$$\forall \phi \in \mathcal{D}(\Omega), \quad \left| \int_\Omega f\phi \right| := |\langle f, \phi \rangle_{\mathcal{D}',\mathcal{D}}| \le C\|\phi\|_{W^{m,p}}.$$

2.4.2 Poincaré's inequalities

In the case where Ω is **bounded**, there are several important inequalities related to $W_0^{1,p}(\Omega)$.

Theorem 2.22: Poincaré's inequality

Let Ω be a bounded open set in $\mathbb{R}^d$ and let $1 \le p \le \infty$. There is a constant $C = C(\Omega, p)$ so that, for all $u \in W_0^{1,p}(\Omega)$, we have

$$\|u\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)}.$$

Proof. We do the proof for $p < \infty$. We denote by L the diameter of Ω. By density of $C_c^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$, it is enough to prove the result for $u \in C_c^\infty(\Omega)$. Let $x \in \Omega$, and consider a point $a \in \partial\Omega$, so that $a_1 = x_1$ (same first coordinate). On the segment $[a, x]$, we have the point-wise bound

$$\begin{aligned} |u(x)| = |u(x) - u(a)| &\le \int_a^x |\partial_{x_1} u|(s, x_2, \cdots, x_N)\mathrm{d}s \le L \int_a^{a+L} |\nabla u|(s, x_2, \cdots, x_N)\mathrm{d}s \\ &\le L \left(\int_a^{a+L} 1^{p'}\mathrm{d}s \right)^{1/p'} \left(\int_a^{a+L} |\nabla u|^p(s, x_2, \cdots x_N)\mathrm{d}s \right)^{1/p}, \end{aligned}$$

where we used the Hölder's inequality on the last line. We take the p power and integrate to obtain the result. □

A consequence of Poincaré's inequality is that the map $u \mapsto \|\nabla u\|_{L^p}$ is a norm on $W_0^{1,p}(\Omega)$, which is equivalent to the usual $W^{1,p}(\Omega)$ norm. Indeed, we have

$$\|\nabla u\|_{L^p} \le \left(\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p\right)^{1/p} = \|u\|_{W^{1,p}} \le \left(C\|\nabla u\|_{L^p}^p + \|\nabla u\|_{L^p}^p\right)^{1/p} = (C+1)^{1/p}\|\nabla u\|_{L^p}.$$

If $u \notin W_0^{m,p}(\Omega)$, we have a similar result, that we state for completeness. The proof uses the Rellich's Theorem 4.10 (see below).

Theorem 2.23: Poincaré-Wirtinger's inequality

Let Ω be a bounded **connected** open set of $\mathbb{R}^d$, with boundary $\partial\Omega$ of class C^1, and let $1 \le p < \infty$. There is a constant $C = C(\Omega, p)$ so that, for all $u \in W^{1,p}(\Omega)$, we have

$$\left\| u - \fint_\Omega u \right\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)},$$

where we set $\displaystyle\fint_\Omega u := \frac{1}{|\Omega|}\int_\Omega u$ the average of u.

Proof. Assume otherwise, and let $u_n \in W^{1,p}(\Omega)$ be such that

$$\left\| u_n - \fint_\Omega u_n \right\|_{L^p} \geq n \|\nabla u_n\|_{L^p}.$$

We set $w_n := \frac{u_n - \fint u_n}{\|u_n - \fint u_n\|_{L^p}}$, so that $\fint w_n = 0$ and

$$1 = \|w_n\|_{L^p} \geq n \|\nabla w_n\|_{L^p}.$$

The sequence (w_n) is bounded in $W^{1,p}$, hence converges (up to a subsequence) weakly to $w_* \in W^{1,p}(\Omega)$. In addition, we have $\|\nabla w_n\|_{L^p} \to 0$, so $\nabla w_* = 0$. By Theorem 2.13 and the fact that Ω is connected, we deduce that w_* is constant. In addition, by weak-convergence, we have

$$0 = \int_\Omega w_n = \langle w_n, 1 \rangle_{L^p, L^{p'}} \xrightarrow[n \to \infty]{} \langle w_*, 1 \rangle_{L^p, L^{p'}} = \int_\Omega w_*.$$

This proves that $w_* = 0$. However, by the Rellich's Theorem 4.10, the sequence (w_n) strongly converges to w_* in $L^p(\Omega)$, so we also have $\|w_*\|_{L^p} = \lim \|w_n\|_{L^p} = 1$, a contradiction. $\square$

CHAPTER 3

HILBERT SPACES, AND LAX-MILGRAM THEOREM

In this chapter, we recall the basic theory of (separable) Hilbert spaces. We prove the Lax-Milgram theorem, and provide some examples of applications.

3.1 Hilbert spaces

An Hilbert space is a Banach space with an inner (or sesquilinear) product. Our convention is that the inner product is linear on the right, and antilinear on the left. The natural Hilbert space we will be working with is the $L^2(\Omega)$ space, with the inner product

$$\langle f, g \rangle_{L^2(\Omega)} := \int_\Omega \overline{f} g$$

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ be an Hilbert space. We recall the Cauchy-Schwarz inequality $|\langle f, g \rangle| \le \|f\| \cdot \|g\|$, and the parallelogram equality

$$\|f+g\|^2 + \|f-g\|^2 = 2\left(\|f\|^2 + \|g\|^2\right).$$

Theorem 3.1: Projection on convex set

Let K be a closed convex set of $\mathcal{H}$. Then, for all $f \in \mathcal{H}$, the minimisation problem

$$P_K(f) := \operatorname{argmin}\left\{\|f-k\|_{\mathcal{H}}, \quad k \in K\right\}$$

is well-posed. It is the unique element $k_* \in K$ so that

$$\langle f - k_*, k - k_* \rangle \le 0, \quad \text{for all } k \in K.$$

Proof. Consider a minimising sequence for the problem, that is a sequence $k_n \in K$ so that $\|f - k_n\| \to d := \inf\{\|f - K\|\}$. We claim that k_n is a Cauchy sequence. Indeed, we have, using the parallelogram equality

$$\|k_n - k_{n+p}\|^2 = 2\left(\|k_n - f\|^2 + \|k_{n+p} - f\|^2\right) - \|k_n + k_{n+p} - 2f\|^2.$$

The last term is also $4\|\frac{k_n + k_{n+p}}{2} - f\|^2$, and since K is convex, it is greater than d, so

$$\|k_n - k_{n+p}\|^2 \le 2(d_n + d_{n+p}) - 4d \xrightarrow[n\to\infty,\forall p]{} 0.$$

So (k_n) is a Cauchy sequence in $\mathcal{H}$, hence converges to some $k_* \in \mathcal{H}$. Since K is closed, $k_* \in K$. By continuity of the norm, we have $d = \|f - k_*\|$.

Uniqueness comes from the convexity of K, together with the parallelogram identity.

For the second part of the Theorem, we consider $k \in K$. Since K is convex, we have $tk+(1-t)k_* \in K$. So

$$\|f - [tk + (1-t)k_*]\|^2 \geq \|f - k_*\|^2.$$

Taking the derivative at $t = 0$ gives the inequality. $\square$

There is a similar result in $L^p(\mathbb{R}^d)$ spaces, see for instance [LL01, Lemma 2.8].

3.1.1 The dual of an Hilbert space

The previous Theorem allows to prove that $\mathcal{H}$ is isomorphic to its dual $\mathcal{H}^*$. For $f \in \mathcal{H}$, we denote by $L_f \in \mathcal{H}^*$ the linear functional defined by

$$\forall v \in \mathcal{H}, \quad L_f(v) := \langle f, v \rangle_{\mathcal{H}}.$$

By Cauchy-Schwarz, we have $\|L_f\|_{\rm op} \leq \|f\|_{\mathcal{H}}$, and taking $v = f$ shows that $\|L_f\|_{\rm op} = \|f\|_{\mathcal{H}}$. In other words, we have $\mathcal{H} \subset \mathcal{H}^*$. The next result proves that there is equality.

Theorem 3.2: Riesz's theorem

For all $L \in \mathcal{H}^*$, there is a unique $f \in \mathcal{H}$, so that $L = L_f$.

In particular, this proves that $(L^2(\Omega))^* = L^2(\Omega)$ (this is the case $p = 2$ in Theorem 1.25). The proof that $(L^p)^* = L^{p'}$ is similar (see [LL01, Theorem 2.14]).

Proof. Let $L \in \mathcal{H}^*$ and let $K := \ker L$. Since L is continuous, K is a closed vectorial space (hence convex) of $\mathcal{H}$. If $K = \mathcal{H}$, then $L = 0$, and we can choose $f = 0$. Otherwise, there is $v_0 \in \mathcal{H}$ so that $L(v_0) = 1$.

Consider $v_1 := P_K(v_0) \in K$, and take $v_* := v_0 - v_1$. We have $L(v_*) = L(v_0) - L(v_1) = 1$. In addition, by the second part of Theorem 3.1, we have $\langle v_*, k \rangle = 0$ for all $k \in K$ (why?).

For $v \in \mathcal{H}$, we define

$$w := v - L(v)v_*, \quad \text{so that} \quad v = L(v)v_* + w.$$

By linearity, and since $L(v_*) = 1$, we have $L(w) = L(v) - L(v)L(v_*) = 0$, so $w \in K$. This gives

$$\langle v_*, v \rangle = L(v)\|v_*\|^2 + \langle v_*, w \rangle = L(v)\|v_*\|^2, \quad \text{or, equivalently,} \quad L(v) = \left\langle \frac{v_*}{\|v_*\|^2}, v \right\rangle.$$

This proves that $L = L_f$ with the vector $f := v_*/\|v_*\|^2$. $\square$

Example 3.3. *We take $\mathcal{H} = (H^1(\mathbb{R}), \|\cdot\|_{H^1})$, with the inner product*

$$\langle f, g \rangle_{\mathcal{H}} := \int_{\mathbb{R}} \overline{f} g + \overline{f'} g'.$$

We will see later in Theorem 4.7 that the linear map $\delta_0 : f \mapsto f(0)$ is continuous on $H^1(\mathbb{R})$. By the Riesz' theorem, there is $f_0 \in \mathcal{H}$ so that $\langle f, f_0 \rangle_{\mathcal{H}} = f(0)$. A computation reveals that

$$f_0(x) := \frac{1}{2} \mathrm{e}^{-|x|}.$$

Indeed, we have

$$\langle f_0, f \rangle_{H^1} = \frac{1}{2} \int_{\mathbb{R}} f(x) \mathrm{e}^{-|x|} \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}} f'(x)(-\mathrm{sgn} x) \mathrm{e}^{-|x|} \mathrm{d}x.$$

With an integration by part, the can compute the last integral in $\mathbb{R}^+$ with

$$-\frac{1}{2} \int_{\mathbb{R}^+} f'(x) \mathrm{e}^{-x} \mathrm{d}x = -\frac{1}{2} \int_{\mathbb{R}^+} f(x) \mathrm{e}^{-x} \mathrm{d}x - \frac{1}{2} \left[f(x) \mathrm{e}^{-x} \right]_0^\infty = -\frac{1}{2} \int_{\mathbb{R}^+} f(x) \mathrm{e}^{-x} \mathrm{d}x + \frac{1}{2} f(0).$$

and the result follows.

3.1.2 Basis of an Hilbert space

A **countable orthonormal basis** of $\mathcal{H}$ is an orthonormal family of vectors $(e_1, e_2, \cdots)$ which is dense in $\mathcal{H}$. This means that any $x \in \mathcal{H}$ is of the form

$$x = \sum_{i=1}^{\infty} x_i e_i, \quad \text{with} \quad x_i \in \mathbb{C}, \quad \text{in the sense that} \quad \left\| x - \sum_{i=1}^{N} x_i e_i \right\|_{\mathcal{H}} \xrightarrow[N\to\infty]{} 0.$$

Taking the inner product with e_i gives $x_i = \langle x, e_i \rangle_{\mathcal{H}}$. Similarly, taking the inner product of x with itself shows that $\|x\|^2 = \sum |x_i|^2$. So

$$\boxed{x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i. \quad \text{and} \quad \|x\|_{\mathcal{H}}^2 = \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2.}$$

Theorem 3.4. *If $\mathcal{H}$ is a separable Hilbert space, then $\mathcal{H}$ admits a countable basis.*

Proof. Let (x_n) be a dense sequence in $\mathcal{H}$, and consider $F_k := \mathrm{Ran}\{x_1, \cdots, x_k\}$, of dimension at most k. First erase all k so that $\dim F_k = \dim F_{k-1}$ to obtain a new sequence F_k so that $\dim F_k = k$. Then perform a Gram-Schmidt algorithm on the sequence F_k. □

In practice, all interesting Hilbert spaces are separable.

3.2 Lax-Milgram theorem

Let $a : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a bilinear (or sesquilinear) map. We say that a is **continuous** if there is $\beta > 0$ so that

$$\forall u, v \in \mathcal{H}, \quad |a(u, v)| \le \beta \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}.$$

We say that a is **coercive** if there is $\alpha > 0$ so that

$$\forall u \in \mathcal{H}, \quad a(u, u) \ge \alpha \|u\|_{\mathcal{H}}^2.$$

Finally, we say that a is **symmetric** (or **hermitian**) if

$$\forall u, v \in \mathcal{H}, \quad a(u, v) = \overline{a(v, u)}.$$

Theorem 3.5: Lax-Milgram

Let $a : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a bilinear form which is continuous and coercive, and let $L : \mathcal{H} \to \mathbb{C}$ be a continuous linear map. Then there is unique $u \in \mathcal{H}$ so that

$$\forall f \in \mathcal{H}, \quad a(u, f) = L(f).$$

In addition, if a is symmetric, then u is the unique minimiser of

$$\min \left\{ \frac{1}{2} a(f, f) - L(f), \quad f \in \mathcal{H} \right\}.$$

Remark 3.6. *If we take $a(u, v) := \langle u, v \rangle_{\mathcal{H}}$ the usual inner product, we recover Riesz' Theorem 3.2.*

Before we give the proof, let us re-state this theorem in the case $\mathcal{H} = \mathbb{R}^n$. In this case, $a(\cdot, \cdot)$ and $L(\cdot)$ are of the form

$$a(u, v) := \langle Au, v \rangle_{\mathbb{R}^n}, \quad \text{and} \quad L(f) = \langle b, f \rangle_{\mathbb{R}^n}$$

for some $A \in \mathcal{M}_n(\mathbb{R})$ and $b \in \mathbb{R}^n$. In this case, $a(\cdot, \cdot)$ is always continuous (why?), and coercivity implies that A is injective. The equation $a(u, f) = L(f)$ is then equivalent to $Au = b$. Since A is injective, it is invertible, and we find $u = A^{-1}b$. Assume in addition that A is symmetric. Then coercivity implies $A > 0$. In particular, the map $J : \mathbb{R}^n \to \mathbb{R}$ defined by

$$J(f) := \frac{1}{2}\langle Af, f\rangle_{\mathbb{R}^d} - \langle b, f\rangle_{\mathbb{R}^d}$$

is strictly convex and coercive, hence admits a unique minimiser u_*. Solving $\nabla J(u) = 0$ proves that $u_* = A^{-1}b$.

The Lax-Milgram Theorem is somehow a generalisation for the invertibility of an operator

Proof. Let $(e_n)_{n\in\mathbb{N}}$ be a basis of $\mathcal{H}$, and consider the finite vectorial space

$$E_n := \mathrm{Vect}\{e_1, \cdots, e_n\}.$$

We consider the approximate problem with the restriction $a : E_n \times E_n \to \mathbb{R}$ and $L : E_n \to \mathbb{R}$. By the previous paragraph (finite dimension), there is unique $u_n \in E_n$ so that

$$\forall f \in E_n, \quad a(u_n, f) = L(f).$$

In addition, we have the *a priori* estimate

$$\alpha\|u_n\|^2_{\mathcal{H}} \le a(u_n, u_n) = L(u_n) \le \|L\|_{\mathrm{op}}\|u_n\|_{\mathcal{H}}, \quad \text{so} \quad \|u_n\|_{\mathcal{H}} \le \alpha^{-1}\|L\|_{\mathrm{op}}.$$

The sequence (u_n) is bounded in the (separable) Hilbert space $\mathcal{H}$. By the Banach-Alaoglu Theorem 1.27, there is a subsequence $\phi(n)$, and an element $u \in \mathcal{H}$ so that $u_{\phi(n)} \hookrightarrow u$ weakly in $\mathcal{H}$. The weak-convergence already proves that

$$\forall f \in \mathcal{H}, \quad a(u, f) = L(f).$$

Let us prove uniqueness. If u_1, u_2 solves the equation, then

$$\begin{aligned}\alpha\|u_1 - u_2\|^2 &\le a(u_1 - u_2, u_1 - u_2) = a(u_1, u_1 - u_2) - a(u_2, u_1 - u_2)\\ &= L(u_1 - u_2) - L(u_1 - u_2) = 0.\end{aligned}$$

So there is a unique solution. In particular, the whole sequence (u_n) converges (weakly) to u.

Assume now that a is symmetric. The function $J(f) := \frac{1}{2}a(f, f) - L(f)$ is (strongly) continuous by definition. It is strictly convex and coercive, so has a unique minimum $u_* \in \mathcal{H}$. Since J is differentiable, se must have $D_u J(u^*) = 0$, which gives

$$\forall f \in \mathcal{H}, \quad a(u_*, f) = L(f).$$

This proves that $u_* = u$. □

If a is symmetric, then the convergence of (u_n) to u is strong. Indeed, we have

$$\begin{aligned}\alpha\|u - u_n\|^2 &\le a(u - u_n, u - u_n) = a(u, u - u_n) - 2a(u, u_n) + a(u_n, u_n)\\ &= L(u) - 2L(u_n) + L(u_n) \to 0.\end{aligned}$$

In addition, the sequence $J(u_n)$ is decreasing, and we have

$$J(u) = \lim_{n\to\infty} J(u_n).$$

An operator vision

By the Riesz theorem, there is $f \in \mathcal{H}$ so that $L(v) = \langle f, v \rangle_{\mathcal{H}}$. Also, for all $u \in \mathcal{H}$, the map $v \mapsto a(u, v)$ is continuous linear, so by the Riesz' theorem again, there if an element denoted $Au \in \mathcal{H}$ so that $a(u, v) = \langle Au, v \rangle_{\mathcal{H}}$. The map $A : \mathcal{H} \to \mathcal{H}$ is linear. The Lax-Milgram theorem finds a solution to the equation

$$Au = f.$$

It states that if A is continuous and coercive, then it is invertible (or bijective).

3.3 Some examples of applications

3.3.1 The Laplace equation

Let $f \in L^2(\mathbb{R}^d)$. We would like to **solve** the equation

$$\boxed{-\Delta u + u = f, \quad \text{on} \quad \mathbb{R}^d.}$$

Assume first that we find a **strong solution**, that is a solution $u \in C^2(\mathbb{R}^d)$. Then multiplying the equation by $\phi \in \mathcal{D}(\mathbb{R}^d)$ and integrating by parts would give

$$\int_{\mathbb{R}^d} \nabla u \cdot \nabla \phi + \int_{\mathbb{R}^d} u\phi = \int_{\mathbb{R}^d} f\phi.$$

This suggests to recast the problem in the following Lax-Milgram form. Take $\mathcal{H} = H^1(\mathbb{R}^d)$ with its natural inner product, and consider

$$a(u, v) := \int_{\mathbb{R}^d} uv + \int_{\mathbb{R}^d} \nabla u \cdot \nabla v.$$

(We recognise the usual inner product of $H^1(\mathbb{R}^d)$). This map is clearly bilinear, continuous and coercive. In addition, it is symmetric. Now, consider the linear form $L : \mathcal{H} \to \mathbb{R}$ defined by

$$L(u) := \int_{\mathbb{R}^d} fu.$$

Since $f \in L^2(\mathbb{R}^d)$, we have by Cauchy-Schwarz

$$|L(u)| \le \|f\|_{L^2}^2 \|u\|_{L^2}^2 \le \|f\|_{L^2} \|u\|_{H^1}.$$

So L is continuous. We can apply the Lax-Milgram theorem, and deduce that there is a unique $u \in H^1(\mathbb{R}^d)$ so that

$$\forall v \in H^1(\mathbb{R}^d), \qquad \int uv + \int \nabla u \cdot \nabla v = \int fv.$$

Taking $v \in \mathcal{D}(\mathbb{R}^d) \subset H^1(\mathbb{R}^d)$ proves that u a **weak solution**, that is a solution in the distributional sense. Actually, we proved that it is the unique solution in $H^1(\mathbb{R}^d)$.

We have $-\Delta u = f - u$ in $\mathcal{D}'(\mathbb{R}^d)$. In addition, we have $u \in H^1(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$, so $f - u \in L^2(\mathbb{R}^d)$. We deduce that $-\Delta u \in L^2(\mathbb{R}^d)$ as well. Using the characterization of $H^2(\mathbb{R}^d)$ functions using the Fourier transform (not in this course), this eventually proves $u \in H^2(\mathbb{R}^d)$. We deduce that the equation $-\Delta u + u = f$ actually holds in $L^2(\mathbb{R}^d)$ (this is much better than $\mathcal{D}'(\mathbb{R}^d)$).

3.3.2 The Dirichlet equation in a bounded domain

Let $\Omega \subset \mathbb{R}^d$ be a **bounded**, and let $f \in L^2(\Omega)$. We would like to solve the equation

$$\boxed{\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}}$$

The second condition, called **Dirichlet boundary conditions**, is a shortcut notation! Indeed, u is a distribution, so $u(x)$ has no meaning *a priori*. The correct meaning is

$$(u = 0 \quad \text{on} \quad \partial\Omega) \quad \text{means} \quad u \in H_0^1(\Omega).$$

Without the second condition, there is trivially an infinity of solutions, since if u is solution, then so is $u + cst$.

As before, we recast the problem in a Lax-Milgram form. We set $\mathcal{H} = H_0^1(\Omega)$, with the $H^1(\Omega)$ inner product. We set

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v.$$

This is clearly a bounded bilinear form. Thanks to the Poincaré's inequality (see Theorem 2.22 and the remark afterwards), it is also coercive, since

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \le (C^2 + 1)\|\nabla u\|_{L^2}^2 = (C^2 + 1)a(u, u).$$

We set $L(v) := \int_\Omega fv$. Using again Cauchy-Schwarz and Poincaré inequality, we have

$$|L(v)| \le \int_\Omega |fv| \le \|f\|_{L^2}\|v\|_{L^2} \le \|f\|_{L^2} C \|v\|_{H_0^1},$$

so L is indeed continuous. Applying Lax-Milgram proves that there is unique solution $u \in H_0^1(\Omega)$ to the equation $-\Delta u = f$ in $\mathcal{D}'(\Omega)$.

Since $f \in L^2(\Omega)$, we have $-\Delta u \in L^2(\Omega)$. This eventually proves that $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

3.3.3 Neumann problem on bounded domain

We want to solve the Neumann problem

$$\boxed{\begin{cases} u - \Delta u = f & \text{in} \quad \Omega \\ \partial_\nu u = 0 & \text{on} \quad \partial\Omega, \end{cases}}$$

where $\partial_\nu u = \nabla u \cdot \nu$ is the **normal derivative** on $\partial\Omega$ (we assume that this one is well-defined). Assume first that u is a strong solution, in $C^2(\overline{\Omega})$. By the divergence Theorem 2.14, we have, for $\psi \in C^\infty(\overline{\Omega})$ that

$$\int_\Omega (-\Delta u)\psi = \int_\Omega \nabla u \cdot \nabla \psi - \int_{\partial\Omega} (\partial_\nu u)\psi = \int_\Omega \nabla u \cdot \nabla \psi.$$

This suggests to consider this time $\mathcal{H} = H^1(\Omega)$ and $a(u, v) = \langle u, v \rangle_{H^1}$. Again, applying Lax-Milgram gives a weak solution $u \in H^1(\Omega)$, in the sense that

$$\forall v \in H^1(\Omega), \qquad \int_\Omega uv + \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv.$$

Taking $v \in \mathcal{D}(\Omega)$ shows that $u - \Delta u = f$ in the distributional sense. In particular, since $u \in H^1(\Omega) \subset L^2(\Omega)$, we have $-\Delta u = f - u \in L^2(\Omega)$, so $u \in H^2(\Omega)$.

Now, taking $v \in H^1(\Omega)$ and using the divergence theorem again shows that $\partial_\nu u = 0$ on the boundary. This justifies the Neumann boundary conditions.

CHAPTER 4

COMPLEMENTS ON SOBOLEV SPACES

In this section, we discuss embedding of the form $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$. As we will see, the theory is not so difficult if $\Omega = \mathbb{R}^d$ is the whole space, and there might be difficulties if $\Omega \neq \mathbb{R}^d$, because of its boundary.

4.1 Basics in operator theory

We recall here some basic notions for operator theory. We state all results for operator $A : \mathcal{H} \to \mathcal{H}$, but the definitions can be extended for operator $A : E \to F$ where E and F are general Banach spaces.

Let $\mathcal{H}$ be a separable Hilbert space. A **bounded operator** on $\mathcal{H}$ is a linear map $A : \mathcal{H} \to \mathcal{H}$ so that there is $C \in \mathbb{R}$ with

$$\forall x \in E, \quad \|Ax\|_{\mathcal{H}} \le C\|x\|_{\mathcal{H}}.$$

The smallest C satisfying this property is the **operator norm** of A, so

$$\|A\|_{\rm op} = \sup \left\{ \|Ax\|_{\mathcal{H}}, \quad x \in E, \ \|x\| = 1 \right\}.$$

The **dual** of A is the application $A^* : \mathcal{H} \to \mathcal{H}$ so that

$$\forall x, y \in \mathcal{H}, \quad \langle x, Ay \rangle = \langle A^* x, y \rangle.$$

An bounded operator $A : \mathcal{H} \to \mathcal{H}$ is **compact** if

$$A\left(\mathcal{B}_{\mathcal{H}}(0,1)\right) \quad \text{is (relatively) compact in } \mathcal{H}.$$

Equivalently, if (x_n) is a bounded sequence in $\mathcal{H}$, there is a subsequence $\phi(n)$ and an element $y \in \mathcal{H}$ so that $Ax_{\phi(n)}$ converges to y in $\mathcal{H}$.

We are interested in the operator $\lambda - A := \lambda \mathbb{I}_{\mathcal{H}} - A$ for $\lambda \in \mathbb{C}$. The **resolvent set** of A is the set $\rho(A) \subset \mathbb{C}$ defined by

$$\rho(A) := \left\{ \lambda \in \mathbb{C}, \quad \lambda - A \quad \text{if bijective from } \mathcal{H} \text{ to } \mathcal{H} \right\}.$$

If $\lambda \in \rho(A)$, then automatically $(\lambda - A)^{-1}$ is a bounded operator (this result, known as the Banach-Steinhaus theorem is non trivial, see [Bre99, Corollaire II.6]).

The **spectrum** of A is the complement of $\rho(A)$, that is $\sigma(A) := \mathbb{C} \setminus \rho(A)$. The set $\rho(A)$ is open in $\mathbb{C}$, and the set $\sigma(A)$ is closed in $\mathbb{C}$.

A number $\lambda \in \mathbb{C}$ is an **eigenvalue** of A if $\ker\{\lambda - A\} \neq \{0\}$. In this case any $u \in \ker\{\lambda - A\}$ is a **corresponding eigenvector**. The **multiplicity** of λ is $\dim \ker\{\lambda - A\}$.

4.2 Extension operators

Let $\Omega \subset \mathbb{R}$. An **extension** of $u \in W^{m,p}(\Omega)$ is a function $\widetilde{u} \in W^{m,p}(\mathbb{R}^d)$ so that $\widetilde{u}(x) = u(x)$ a.e. in Ω. An **extension operator** is a bounded linear operator $E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^d)$ so that Eu is an extension of u for all $u \in W^{m,p}(\Omega)$. Here, bounded means that there is $C > 0$ so that

$$\forall u \in W^{m,p}(\Omega), \quad \|Eu\|_{W^{m,p}(\mathbb{R}^d)} \leq \|u\|_{W^{m,p}(\Omega)}.$$

For instance, in the case $L^p(\Omega)$ corresponding to $m = 0$, we can define

$$\widetilde{u}(x) := \begin{cases} u(x) & \text{if} \quad x \in \Omega \\ 0 & \text{else} \end{cases} \quad \text{which satisfies} \quad \|\widetilde{u}\|_{L^p(\mathbb{R}^d)} = \|u\|_{L^p(\Omega)}.$$

So the map $E : u \mapsto \widetilde{u}$ is an extension operator of $L^p(\Omega)$, with norm $\|E\|_{\mathrm{op}} = 1$. However, the function $\widetilde{u}$ has discontinuities at the boundary $\partial\Omega$, hence it is not smooth, and such construction will not work for general $W^{m,p}(\Omega)$.

4.2.1 Extension operator on half-space

We start with the case of the half-space $\Omega = \mathbb{R}^d_+ := \mathbb{R}^{d-1} \times \mathbb{R}^+$, with boundary $\partial\Omega = \mathbb{R}^d_0 := \mathbb{R}^{d-1} \times \{0\}$. First, we prove the following (see [Eva10, Chapter 5.3.3]).

Theorem 4.1: Density of smooth functions, up to the boundary

In the case $\Omega = \mathbb{R}^d_+$, $C^\infty(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$ for all $m \in \mathbb{N}$ and all $1 \leq p < \infty$.

Proof. We introduce the translation operator

$$\tau_h(u)(x_1, \cdots, x_{d-1}, x_d) := u(x_1, \cdots, x_{d-1}, x_d + h),$$

and we set $u_\varepsilon := \tau_\varepsilon(u) * j_\varepsilon = \tau_\varepsilon(u * j_\varepsilon)$. We have moved u upwards so that the convolution is well-defined everywhere in $\mathbb{R}^d_+$. In particular, we directly have $u_\varepsilon \in C^\infty(\overline{\Omega})$. In addition, we have

$$\|D^\alpha u_\varepsilon - D^\alpha u\|_{L^p} \leq \|D^\alpha u_\varepsilon - D^\alpha(\tau_\varepsilon u)\|_{L^p} + \|D^\alpha(\tau_\varepsilon u) - D^\alpha u\|_{L^p}.$$

The first term is goes to zero by the properties of smoothing sequences, and the second goes to zero since translations are continuous in L^p. □

Remark 4.2. *The same proof shows that $C^\infty(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$, if Ω is bounded with $\partial\Omega$ of class C^1.*

Theorem 4.3: Extension on half space

For all $m \in \mathbb{N}$ and all $1 \leq p \leq \infty$, there is an extension operator $P : W^{m,p}(\mathbb{R}^d_+) \to W^{m,p}(\mathbb{R}^d)$.

Proof. We only do the proof in the case $d = 1$, $m = 1$ and $p \neq \infty$. We refer to [Eva10, Chapter 5.4] for the general case.
For $u \in W^{1,p}(\mathbb{R}^+)$, we define the **first order reflection**

$$\bar{u}(x) = \begin{cases} u(x) & \text{if} \quad x > 0, \\ -3u(-x) + 4u(-\frac{x}{2}) & \text{else.} \end{cases}$$

By density, it is enough to prove the result for $u \in C^1([0,\infty))$. We claim that $\bar{u}$ is of class C^1 as well. Indeed, first, we check that $\bar{u}(0^-) = -3u(0) + 4u(0) = u(0) = \bar{u}(0^+)$, so $\bar{u}$ is continuous at $x = 0$. Next, we have

$$\forall x < 0, \quad \bar{u}'(x) = 3u'(-x) - 2u'(-\tfrac{x}{2})$$

Taking $x \to 0$ proves that $\bar{u}'$ is also continuous at $x = 0$, so $\bar{u}$ is indeed C^1. In addition, by standard inequalities, there is $C > 0$ so that

$$\|\bar{u}\|^p_{W^{1,p}(\mathbb{R})} = \int_{\mathbb{R}} |\bar{u}|^p + \int_{\mathbb{R}} |\bar{u}'|^p \le C \left(\int_{\mathbb{R}^+} |u|^p + \int_{\mathbb{R}_+} |u'|^p \right) = C\|u\|^p_{W^{1,p}(\mathbb{R}^+)}.$$

This proves that the map $E : u \mapsto \bar{u}$ is a bounded extension operator.

The reader can check that the same construction works on $\mathbb{R}^d$. The proof for higher derivatives $m \ge 1$ necessitates high order reflections. □

4.2.2 Extension operators on domains with smooth boundary

We say that a domain $\Omega \in \mathbb{R}^d$ has a **boundary $\partial\Omega$ of class C^k** if, for all $x \in \partial\Omega$, there is $\varepsilon > 0$, $\delta > 0$, and a C^k diffeomorphism

$$\Psi : \mathcal{B}(x,\varepsilon) \to \mathcal{B}(0,\delta),$$

so that $\Psi(x) = 0$ and $\Psi(\mathcal{B}(x,\varepsilon) \cap \Omega) = \Psi(\mathcal{B}(0,\delta) \cap \mathbb{R}^d_+)$. In particular, this implies that $\Psi(\partial\Omega) = \mathbb{R}^d_0$.

Thanks to the diffeomorphism Ψ, we can *transport* the previous result to general domains. If Ω is a **bounded** domain of class C^1, then its boundary $\partial\Omega$ is closed and bounded, hence compact.

Theorem 4.4: Extension operators on smooth domains

If Ω is bounded with $\partial\Omega$ of class C^k, then for all $m \le k$ and all $1 \le p \le \infty$, there is an extension operator $E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^d)$.

We require the boundary to be at least as regular as the functions inside: $m \le k$.

Proof. For all $x \in \partial\Omega$, construct $\varepsilon_x > 0$ and a C^k diffeomorphism Ψ_x as above. The collection of open sets $\bigcup_{x\in\partial\Omega} \mathcal{B}(x,\varepsilon_x)$ covers the compact boundary $\partial\Omega$, so there is a finite collection of points $\{x_1, \cdots, x_N\} \in (\partial\Omega)^N$ so that $\bigcup_{i=1}^N \mathcal{B}(x_i,\varepsilon_i)$ covers $\partial\Omega$. We set $\varepsilon_i = \varepsilon_{x_i}$, $\Psi_i := \Psi_{x_i}$ and $U_i := \mathcal{B}(x_i,\varepsilon_i)$.

We also consider $U_0 \subset \Omega$ an open set so that $K := \overline{U_0} \subset \Omega$ as well, and so that $\Omega = U_0 \cup U_1 \cup \cdots \cup U_N$.

Next we consider a **partition** of unity subordinated to $(U_0, U_1, \cdots U_N)$, that is a family of smooth functions $\theta_0, \theta_1, \cdots, \theta_N$ so that

- For all $x \in \Omega$, we have $\sum_{i=0}^N \theta_i(x) = 1$;
- The function θ_i is compactly supported with support (strictly) included in U_i.

For $u \in W^{m,p}(\Omega)$ we write $u = \sum_{i=0}^N u_i$ with $u_i := \theta_i u$. Since the functions θ_i are smooth, we see that there is $C > 0$ independent of u so that

$$\|u\|_{W^{m,p}(\Omega)} \le C \sum_{i=0}^N \|u_i\|_{W^{1,p}(\Omega)}, \quad \text{and} \quad \|u_i\|_{W^{m,p}(\Omega)} \le C\|u\|_{W^{m,p}(\Omega)}.$$

The function u_0 is compactly supported in Ω, so it has a trivial extension $\widetilde{u_0}$ to $W^{m,p}(\mathbb{R}^d)$.

For $1 \le i \le N$, the function u_i is compactly supported in $U_i = \mathcal{B}(x_i,\varepsilon_i)$. We define on $V_i := \mathcal{B}(0,\delta_i) \cap \mathbb{R}^d_+$ the function $v_i(x) := u_i(\Psi_i^{-1}(x))$. Since $\Psi^{-1} \in C^k$ with $k \le m$, we have $v_i \in W^{m,p}(\mathbb{R}^d_+)$. By Theorem 4.3, we can extend v_i on $W^{m,p}(\mathbb{R}^d_+)$ by a function $\bar{v}_i$. In addition, by construction, $\bar{v}_i$ is compactly supported in $\mathcal{B}(0, \delta_i)$. We finally set, for all $x \in U_i$, $\bar{u}_i(x) := \bar{v}_i(\Psi(x))$. Again, since $\Psi \in C^k$, we can check that $\bar{u}_i$ is compactly supported in U_i, and that $u_i \in W^{m,p}(\mathbb{R}^d)$.

Finally, we set $\bar{u} := \sum_{i=0}^N \bar{u}_i \in W^{m,p}(\mathbb{R}^d)$. For all $x \in \Omega$, we have $\bar{u}_i(x) = u_i(x)$, so indeed $\bar{u}(x) = u(x)$. Finally, by construction, the map $u \mapsto \bar{u}$ is linear, and bounded from $W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^d)$. □

Exercice 4.5

Let $U_1, \cdots U_N$ be a collection of open sets so that $\bigcup_{i=1}^N U_i = \mathbb{R}^d$. Define

$$\theta_i^\varepsilon(x) := \int_{U_i} j_\varepsilon(x-y) \mathrm{d}y.$$

Prove that $\sum_{i=1}^N \theta_i^\varepsilon = 1$ on $\mathbb{R}^d$, and that θ_i^ε is compactly supported in an ε-neighborhood of U_i.

4.3 Sobolev embeddings

In this section, we focus on the so-called Sobolev embeddings. This set of inequalities states that if $u \in W^{m,p}(\mathbb{R}^d)$, then $u \in L^q(\mathbb{R}^d)$ as well, for some q. In other words, *regularity* implies *integrability.*

4.3.1 Sobolev embeddings on the whole space

We begin with the case $\Omega = \mathbb{R}^d$, that is the whole space [Bre99, Theorem IX.9].

Theorem 4.6: Gagliardo, Nirengerg, Sobolev's inequality

For all $1 \le p < \frac{d}{m}$, there is a constant $C = C(m,p,d)$ so that,

$$\forall u \in W^{m,p}(\mathbb{R}^d), \quad \text{we have} \quad u \in L^q(\mathbb{R}^d) \quad \text{with} \quad \|u\|_{L^q} \le C \sum_{|\alpha|=m} \|D^\alpha u\|_{L^p}, \quad \text{where} \quad \frac{1}{q} = \frac{1}{p} - \frac{m}{d}.$$

In particular, we have $W^{m,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ with continuous embedding. In the case $m = 1$, the corresponding exponent q is written p^*, so

$$p^* \quad \text{satisfies} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}, \quad \text{that is} \quad \boxed{p^* = \frac{dp}{d-p}.}$$

Before we prove this theorem, let us note that the exponent q is *natural.* Indeed, consider the scaling $u_\lambda(x) = u(\lambda x)$. We compute

$$\|u_\lambda\|_{L^q} = \left(\int_{\mathbb{R}^d} |u(\lambda x)|^q \frac{\mathrm{d}(\lambda x)}{\lambda^d} \right)^{1/q} = \lambda^{-\frac{d}{q}} \|u\|_{L^q}.$$

On the other hand, we have, for $|\alpha| = m$,

$$\|D^\alpha u_\lambda\|_{L^p} = \left(\int_{\mathbb{R}^d} |\lambda^m (D^\alpha u)(\lambda x)|^p \frac{\mathrm{d}(\lambda x)}{\lambda^d} \right)^{1/p} = \lambda^{m - \frac{d}{p}} \|D^\alpha u\|_{L^p}.$$

Since the Sobolev inequality must be valid for all values of $\lambda > 0$, the homogeneity in λ must be similar, so $-\frac{d}{q} = m - \frac{d}{p}$.

Proof. By induction, it is enough to prove the result for $m = 1$ only. We prove the result in the case $d = 2$, and refer to [Bre99, Theorem IX.9] for a full proof.
We start with $p = 1$. By density, it is enough to prove the result for $u \in \mathcal{D}(\mathbb{R}^2) \subset W^{1,1}(\mathbb{R}^2)$. We have

$$|u(x_1, x_2)| \le \int_{-\infty}^{x_1} |\partial_{x_1} u(s, x_2)| \mathrm{d}s \le \int_{-\infty}^{\infty} |\partial_{x_1} u(s, x_2)| \mathrm{d}s =: v_1(x_2).$$

Similarly, we have with similar notation that $|u(x_1, x_2)| \le v_2(x_1)$. So

$$\begin{aligned} \|u\|^2_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} |u(x_1, x_2)|^2 \mathrm{d}x_1 \mathrm{d}x_2 \le \int_{\mathbb{R}^2} |v_1(x_2)| \cdot |v_2(x_1)| \mathrm{d}x_1 \mathrm{d}x_2 \\ &= \|v_1\|_{L^1(\mathbb{R})} \|v_2\|_{L^1(\mathbb{R})} = \|\partial_{x_1} u\|_{L^1(\mathbb{R}^2)} \|\partial_{x_2} u\|_{L^1(\mathbb{R}^2)} \le \|\nabla u\|^2_{L^1(\mathbb{R}^2)}. \end{aligned}$$

This proves the result in the case $p = 1$ and $d = 2$. For the case $1 \le p < 2$, we apply the result to the function $u_t := |u|^{t-1} u$. This function satisfies $\nabla u_t = t|u|^{t-1} \nabla u$. This gives

$$\|u\|^t_{L^{2t}} = \|u_t\|_{L^2} \le \|\nabla u_t\|_{L^1} = t \| |u|^{t-1} \nabla u\|_{L^1} \le t \| |u|^{t-1} \|_{L^{p'}} \|\nabla u\|_{L^p} = t \|u\|^{t-1}_{L^{(t-1)p'}} \|\nabla u\|_{L^p}.$$

Choosing t so that $2t = (t-1)p' = (t-1)\frac{p}{p-1}$, that is $t = \frac{p}{2-p}$, gives the result. □

Together with Theorem 1.17, we deduce that $u \in L^r(\mathbb{R}^d)$ for all $r \in [p, q]$.

4.3.2 Morrey's embedding in the whole space

The Sobolev inequality proves non trivial embedding for $p < \frac{d}{m}$. On the other side $p > \frac{d}{m}$, we have more information, see [Bre99, Theorem IX.12, and Corollary IX.13].

Theorem 4.7: Morrey's embedding

If $m \ge 1$ and $p > \frac{d}{m}$, then $W^{m,p}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$. In addition, setting

$$k := \left\lfloor m - \frac{d}{p} \right\rfloor, \quad \text{and} \quad \theta := m - \frac{d}{p} - k \in [0, 1),$$

and if $\theta \ne 0$, there is a constant $C = C(m, p, d)$ so that, for all α with $|\alpha| = k$, we have

$$\forall u \in W^{m,p}(\mathbb{R}^d), \quad |D^\alpha u(x) - D^\alpha u(y)| \le C|x - y|^\theta \|\nabla D^\alpha u\|_{L^p}, , \quad a.e.$$

The most important case is the case $m = 1$. Then we always have $k = 0$ and $0 < \theta < 1$. We deduce that u is θ-Hölder's continuous, that is

$$\forall x, y \in \mathbb{R}^d, \quad |u(x) - u(y)| \le C_u |x - y|^\theta.$$

In particular, u is continuous (in the sense "*there exists a continuous representation of u in $L^\infty(\mathbb{R})$*").

Proof. The full proof of Theorem 4.7 is quite complex, so we admit it. The case $d = 1$ however is quite simple to prove. By induction, we only need to consider the case $m = 1$. We write that

$$|u(x) - u(y)| \le \int_{[x,y]} |u'(s)| \mathrm{d}s \le \left(\int_{[x,y]} |u'|^p(s) \mathrm{d}s \right)^{1/p} \left(\int_{[x,y]} 1^{p'} \mathrm{d}s \right)^{1/p'} \le \|u'\|_{L^p} |x - y|^{1/p'}.$$

This proves the result with $\theta = \frac{1}{p'} = 1 - \frac{1}{p}$, as wanted. □

The reader may ask what happens at the critical point $p = \frac{d}{m}$. The answer is unfortunately not so easy. If $u \in W^{m, \frac{d}{m}}(\mathbb{R}^d)$, then

- u always belong to all $L^r(\mathbb{R}^d)$ space, for all $p \le r < \infty$;
- Sometime, u also belongs to $L^\infty(\mathbb{R}^d)$;
- Sometime, u also belongs to $C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

The last case happens for instance in the case $m = 1$ and $d = 1$ [Bre99, Theorem VIII.7].

Theorem 4.8: Critical case in dimension 1.

For all $1 \le p \le \infty$, we have $W^{1,p}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \cap C^0(\mathbb{R}^d)$.

Proof. The proof for $p > 1$ comes from Morrey's inequality. Let us prove the result for $p = 1$. We have

$$|u(x) - u(y)| \le \int_{[x,y]} |u'(s)|\mathrm{d}s \le |x - y| \cdot \|u'\|_{L^\infty}.$$

This proves that u is Lipschitz, hence continuous. Similarly, we have, for $u \in C_c^\infty(\mathbb{R})$,

$$|u(x)| \le \int_{(-\infty,x]} |u'(s)|\mathrm{d}s \le \|u'\|_{L^1}.$$

So $\|u\|_\infty \le \|u'\|_{L^1}$. This proves the result for $u \in C_c^\infty(\mathbb{R})$. We conclude by density of $C_c^\infty(\mathbb{R})$ into $W^{1,1}(\mathbb{R})$. $\square$

4.3.3 Compact embedding in bounded domains

Let Ω be a bounded domain. If Ω has a smooth enough boundary $\partial\Omega$, we may use an extension operator to have similar theorems than Sobolev and Morrey. We only state a simple version of these theorems [Bre99, Corollaire IX.14].

Theorem 4.9: Embedding theorems in bounded domains

Let Ω is a bounded domain with boundary $\partial\Omega$ of class C^k. Then, for all $m \le k$ and all $1 \le p \le \infty$, the conclusions of Theorems **??** and 4.7 hold.

We emphasize that the boundary must be at least as smooth as the functions inside.

Now, since Ω is bounded, we actually gain *compactness*. This is the Rellich-Kontrachov result [Bre99, p. IX.16]. Recall that an operator $T : E \to F$ is **compact** if, $T(\mathcal{B}_E(0,1))$ is relatively compact in F. It is equivalent to the following: for all bounded sequence $(x_n) \in E$, there is a subsequence $\phi(n)$ and a limit $f \in F$ so that $T(x_{\phi(n)})$ converges to f in F.

Theorem 4.10: Rellich Kontrachov Theorem

Let Ω is a bounded domain with boundary $\partial\Omega$ of class C^k, and let $m \le k$.
- If $1 \le p < \frac{m}{d}$, then for all $r \in [p, q)$, where $\frac{1}{q} = \frac{1}{p} - \frac{m}{d}$, the embedding $W^{m,p}(\Omega) \to L^r(\Omega)$ is **compact**.
- If $p \ge \frac{m}{d}$, then for all $r \in [p, \infty)$, the embedding $W^{m,p}(\Omega) \to L^r(\Omega)$ is **compact**.
- If $p > \frac{m}{d}$, the embedding $W^{m,p}(\Omega) \to C^0(\overline{\Omega})$ is **compact**.

Again, we skip the proof. This one is not difficult, but uses tools (Ascoli theorem) that goes beyond the scope of this course.

This theorem combines nicely with the Banach-Alaoglu Lemma: weak-limits in $W^{m,p}(\Omega)$ becomes strong limits in $L^r(\Omega)$. We will see some applications in the next Chapter.

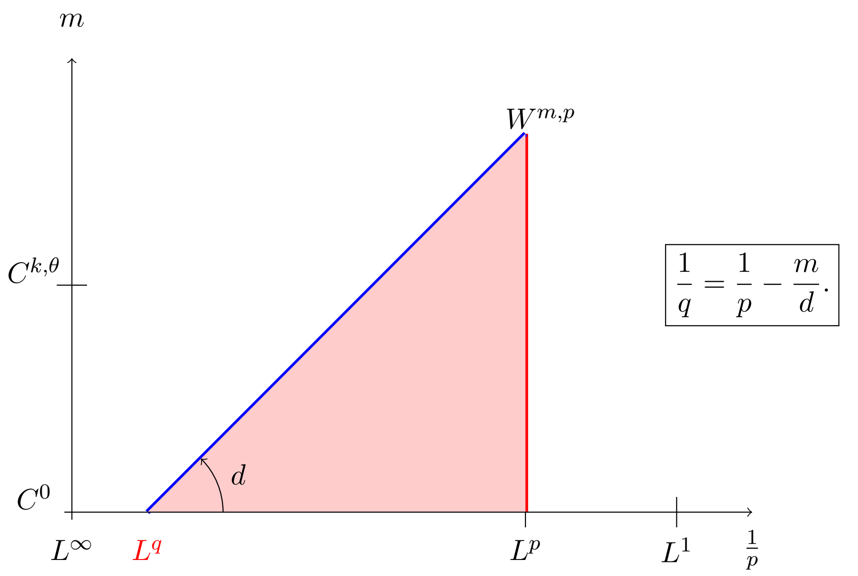

Figure 4.1: Sobolev's and Morrey's embbedings in a single graph. The Lebesgue spaces L^p are at $m=0$, the Hölder's space $C^{k,\theta}$ are at $\frac{1}{p}=0$, and the Sobolev's spaces $W^{m,p}$ are in the $(m,\frac{1}{p})$ quarter space. The point $(0,0)$ represent either L^p for all $p\geq 1$, or L^∞, or C^0, depending on the dimension. If $u\in W^{m,p}$, then u belongs to all Sobolev/Hölder's spaces in the red area (including the critical blue line). If Ω is bounded with smooth boundary, the embeddings are compact in the red area (excluding the critical blue line).

4.4 Trace operators

In this section, we study what are the properties of u restricted to the boundary $\partial\Omega$. The next Theorem can be found in [Eva10, Chapter 5.5].

4.4.1 Trace operators on the half-space

We begin with the half-space $\Omega := \mathbb{R}^d_+$. In this case, the boundary $\partial\Omega = \mathbb{R}^d_0 = \mathbb{R}^{d-1}\times\{0\}$ can be identified with $\mathbb{R}^{d-1}$. In particular, it has a well-defined $(d-1)$-dimensional Lebesgue measure.

Theorem 4.11: Traces in half-space

For all $1\leq p<\infty$ and all $u\in W^{1,p}(\mathbb{R}^d_+)$, the function $u|_{\partial\Omega}:\mathbb{R}^{d-1}\to\mathbb{C}$ belongs to $L^p(\mathbb{R}^{d-1})$.

In other words, the intersection of a $W^{1,p}(\mathbb{R}^d)$ function with a plane is a $L^p(\mathbb{R}^{d-1})$ function.

Proof. Using an extension operator, we can consider $u\in W^{1,p}(\mathbb{R}^d)$ and study its intersection with the plane $\{x_d=0\}$. Assume first that $u\in C_c^\infty(\mathbb{R}^d)$ is smooth and compactly supported, and positive valued. For all $\underline{x}\in\mathbb{R}^{d-1}=\partial\Omega$, we have

$$\left|u|_{\partial\Omega}\right|^p(\underline{x}) = |u|^p(\underline{x},0) \leq \int_{[0,\infty)} \partial_{x_d}\left(|u|^p\right)(\underline{x},s)\mathrm{d}s = p\int_{[0,\infty)} |u|^{p-1}|\partial_{x_d}u|(\underline{x},s)\mathrm{d}s$$

Using the inequality $|a|^{p-1}|b|\leq |a|^p+|b|^p$ (consider the case $|a|\leq|b|$ and $|b|\leq|a|$), this gives the point-wise estimate

$$\left|u|_{\partial\Omega}\right|^p(\underline{x}) \leq p\int_{[0,\infty)} (|u|^p+|\nabla u|^p)(\underline{x},s)\mathrm{d}s.$$

Integrating in $\underline{x}\in\partial\Omega$ proves that $\|u\|_{L^p(\partial\Omega)}\leq p\|u\|_{W^{1,p}(\mathbb{R}^d)}$. By density of $C_c^\infty(\mathbb{R}^d)$ in $W^{1,p}(\mathbb{R}^d)$, the result holds on the whole space $W^{1,p}(\mathbb{R}^d)$. □

The corresponding map $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is called the **trace operator**. The following Lemma identifies the kernel of T.

Theorem 4.12

Let $u \in W^{1,p}(\Omega)$, we have $Tu = 0$ iff $u \in W_0^{1,p}(\Omega)$.

This justifies *a posteriori* the notation "$u = 0$ on $\partial\Omega$" for $u \in W_0^{1,p}(\Omega)$.

Proof. If $u \in W_0^{1,p}(\Omega)$, it is the limit of functions in $C_c^\infty(\Omega)$. We easily deduce that $Tu = 0$. For the converse, we refer to [Eva10, Chapter 5.5]. □

We can wonder whether the map T is *surjective*. Unfortunately, it is not the case, and characterising precisely the image is a difficult task. In the case $p = 2$ however, the image of T is usually denoted by $H^{1/2}(\partial\Omega)$. Indeed, this notation has an interpretation as a Sobolev space (with partial derivatives) on $\partial\Omega \approx \mathbb{R}^{d-1}$.

Of course, as the notation suggests, we indeed have $L^2(\mathbb{R}^{d-1}) \subset H^{1/2}(\mathbb{R}^{d-1}) \subset H^1(\mathbb{R}^d)$, with continuous embeddings. We do not comment more on this space, and just recap the discussion with the following Theorem.

Theorem 4.13

The trace operator $T : H^1(\mathbb{R}^d_+) \to H^{1/2}(\mathbb{R}^{d-1})$ is **surjective**: for any $\gamma \in H^{1/2}(\mathbb{R}^{d-1})$, there is $u \in H^1(\mathbb{R}^d_+)$ so that $Tu = \gamma$.
In addition, if $v \in H^1(\mathbb{R}^d_+)$ is such that $Tv = \gamma$, then $v - u \in H^1_0(\mathbb{R}^d_+)$.

4.4.2 Trace operators on bounded domains

Similar results holds in the case where Ω is a bounded domain of $\mathbb{R}^d$ with boundary $\partial\Omega$ of class C^1. In this case, the Lebesgue space $L^p(\partial\Omega)$ is defined in terms of the **surface measure** $\mathrm{d}s$:

$$\|f\|_{L^p(\partial\Omega)} := \int_{\partial\Omega} |f|^p \mathrm{d}s.$$

We do not elaborate more on this point, and just enunciate the main result.

Theorem 4.14

Let $1 \leq p < \infty$, and let Ω be a bounded domain of $\mathbb{R}^d$ with boundary $\partial\Omega$ of class C^1. Then the trace operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is bounded. In addition,

- for all $u \in W^{1,p}(\Omega)$, we have $Tu = 0$ iff $u \in W_0^{1,p}(\Omega)$;
- **(Case** $p = 2$**)** T is surjective from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$.

As in the previous section, the proof uses localisation and flattening of the boundary to recover the half-space case, see [Eva10, Chapter 5.5].

CHAPTER 5

OPTIMISATION

We this chapter, we introduce the Euler-Lagrange equations. We use it to prove the spectral decomposition of compact operators, and we provide some examples.

5.1 Euler-Lagrange equations

We recall the Implicit Function Theorem. In the sequel, X, Y and Z are Banach spaces.

Theorem 5.1: Implicit Function Theorem

Let $F : X \times Y \to Z$ be a function of class C^k with $k \geq 1$, and let $(x_0, y_0) \in X \times Y$ be such that $F(x_0, y_0) = 0$. Assume that $D_y F(x_0, y_0)$ is invertible, as a linear operator from Y to Z. Then,

- there is a neighbourhood U_x of x_0 in X, and a neighbourhood U_y of y_0 in Y,
- there is a (unique) map $\Psi : U_x \to U_y$, which is of class C^k,

so that

$$\forall x, y \in U_x \times U_y, \quad F(x, y) = 0 \quad \text{iff} \quad y = \Psi(x).$$

In particular, we have $F(x, \Psi(x)) = 0$ for all $x \in U_x$, and

$$D_x \Psi(x_0) = - \left[D_y F(x_0, y_0) \right]^{-1} D_x F(x_0, y_0).$$

This theorem states that, under the *mild* condition $D_y F$ invertible, a solution to $F(x_0, y_0) = 0$ belongs to a unique *branch of solutions*. We do not prove this Theorem, as it is classical.

Our main application of this theorem concerns the properties of optimisers for problems under constraints. We are interested in problems of the form

$$\inf \left\{ F(x), \quad x \in X, \ G(x) = 0 \right\},$$

where $F : X \to \mathbb{R}$ (always real-valued, otherwise *optimisation* is not possible), and $G : X \to Y$ is a set of constraints. For our purpose, we usually need one constraint, so we assume $G : X \to \mathbb{R}$. Recall that an optimisation problem is **well-posed** if the infimum is a minimum: an optimiser exists.

Theorem 5.2: Euler-Lagrange equations

Let F and G be functions of class C^1 from X to $\mathbb{R}$. Assume that $x_* \in X$ is an optimiser of the minimisation problem
$$\inf \{F(x), \quad x \in X,\ G(x) = 0\},$$
Assume in addition that $D_x G(x_*) \neq 0$. Then there is $\lambda \in \mathbb{R}$ so that

$$D_x F(x_*) = \lambda D_x G(x_*) \qquad \text{(Euler-Lagrange equations).}$$

The number $\lambda \in \mathbb{R}$ is called the **Lagrange multiplier**.

Proof. Since $D_x G(x_*) \neq 0$, there is $e_1 \in X$ so that $D_x G(x_*) \cdot e_1 = 1$. We set $E_1 := \mathrm{Vect}\{e_1\}$, and consider a complement E of E_1 in E, that is $E = V \oplus E_1$. We write $x = (\underline{x}, x_1) =: (\underline{x}, y)$ in this decomposition. We have $D_y G(x_*) = D_x G(x_*) \cdot e_1 = 1$ by construction, so we can apply the Implicit Function Theorem to G. We deduce that there is C^1 map $\Psi : V \to E_1$ so that, locally around x_*, we have
$$G(\underline{x}, x_1) = 0 \quad \text{iff} \quad x_1 = \Psi(\underline{x}).$$
In addition, since $D_y G = 1$, we have

$$\Psi'(\underline{x_*}) = -D_{\underline{x}} G(x_*), \quad \text{so that} \quad D_x G(x_*) = \begin{pmatrix} -\Psi'(\underline{x_*}) \\ 1 \end{pmatrix}$$

We found a parametrisation of the constraint $G(x) = 0$ around x_*. In particular, $\underline{x_*}$ is the minimum of the function $\underline{x} \mapsto F(\underline{x}, \Psi(\underline{x}))$. We deduce that

$$D_{\underline{x}} F(x_*) + D_y F(x_*) \Psi'(\underline{x_*}) = 0, \quad \text{so that} \quad D_x F(x_*) = \begin{pmatrix} D_{\underline{x}} F(x_*) \\ D_y F(x_*) \end{pmatrix} = D_y F(x_*) \begin{pmatrix} -\Psi'(\underline{x_*}) \\ 1 \end{pmatrix},$$

and the result follows. Actually, we proved that $\lambda = D_y F(x_*)$. □

5.1.1 Application, NLS in bounded domain

As an application, we would like to find a non trivial solution (λ, u) to the following NLS problem. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary $\partial\Omega$ of class C^1. We consider the problem

$$\boxed{\begin{cases} -\Delta u + u^p = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \quad \partial\Omega. \end{cases}}$$

Again, the second line means $u \in H_0^1(\Omega)$. We consider the functional $F : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$F(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4} \int_\Omega |u|^4,$$

and the problem
$$J := \inf \left\{ F(u), \quad u \in H_0^1(\Omega), \quad \|u\|_{L^2} = 1 \right\}.$$
The functional F is positive, hence bounded from below. Let $(u_n) \in H_0^1(\Omega)$ be a minimising sequence for this problem, that is $\|u_n\| = 1$ and $F(u_n) \to J$.

Existence of a minimum. The sequence $F(u_n)$ converges, hence is bounded. So the sequence $\|\nabla u_n\|_{L^2}$ is bounded as well. In particular, (u_n) is a bounded sequence in $H_0^1(\Omega)$, hence converge weakly to some $u_* \in H_0^1(\Omega)$.

Let us first prove that $\|u_*\|_{L^2} = 1$. By Rellich's theorem 4.10, and since Ω is bounded with C^1 boundary, the embedding $H_0^1(\Omega) \to L^2(\Omega)$ is **compact**, so (u_n) converges strongly to u_* in $L^2(\Omega)$. We deduce that $\|u_*\|_{L^2} = 1$.

We now prove that $F(u_*) = J$. The function $u \mapsto \|\nabla u\|_{L^2}^2$ is convex, and strongly continuous from $H_0^1(\Omega) \to \mathbb{R}$. So, by Theorem 1.28, it is weakly lower semi continuous. So

$$\|\nabla u_*\|_{L^2}^2 \le \liminf_{n\to\infty} \|\nabla u_n\|_{L^2}^2.$$

We would to use a similar reasoning for the second term $u \mapsto \|u\|_{L^4}^4$. This map is indeed convex. However, we do not know whether it is strongly continuous. According to Rellich's embedding 4.10, it would be the case if $d \le 2$. For $d > 3$, we need another argument. Here, positivity saves us.

Since the map (u_n) strongly converges to u_* in $L^2(\Omega)$, by Theorem 1.23, we may (up to a subsequence) assume that (u_n) also converges point-wise to u_*. Then $|u_n|^4$ converges point-wise to $|u_*|^4$. By Fatou Lemma, we deduce that

$$\int_\Omega |u_*|^4 \le \liminf \int_\Omega |u_n|^4.$$

This proves that $F(u_*) \le J$, and since u_* is a valid test function (that is $\|u_*\|_{L^2} = 1$), we have $F(u_*) = J$, so u_* is an optimiser. Replacing u_* by $|u_*|$, we may assume that $u_* \ge 0$ a.e..

Euler-Lagrange equations. We can therefore derive the Euler-Lagrange equations. First, we note that

$$F(u_* + h) = F(u_*) + \int_\Omega \nabla u_* \cdot \nabla h + \int_\Omega u_*^3 h + O\left(\|h\|_{H^1}^2\right),$$

so $D_u F(u_*)$ is the linear map from $H_0^1(\Omega)$ to $\mathbb{R}$ defined by

$$D_u F(u_*) : h \mapsto \int_\Omega \nabla u_* \cdot \nabla h + \int_\Omega u^3 h.$$

The Euler-Lagrange equations shows that there is $\lambda \in \mathbb{R}$ so that

$$\forall h \in H_0^1(\Omega), \quad \int_\Omega \nabla u_* \cdot \nabla h + \int_\Omega u^3 h = \lambda \int_\Omega uh.$$

Taking $h \in \mathcal{D}(\Omega)$ shows that $-\Delta u_* + u_*^3 = \lambda u_*$ in $\mathcal{D}'(\Omega)$. Since $\|u_*\|_{L^2} = 1$, we have $u_* \neq 0$. We conclude that u_* is a non trivial solution to the NLS equation.

5.2 Spectral decomposition of compact symmetric operators

5.2.1 Decomposition of compact symmetric operators

In this section, we prove the spectral decomposition. We refer to [Bre99, Chapitre VI] for a full presentation.

Let A be a symmetric compact operator on $\mathcal{H}$. For all $x \in \mathcal{H}$, we have

$$\langle x, Ax \rangle = \langle Ax, x \rangle = \overline{\langle x, Ax \rangle},$$

so $\langle x, Ax \rangle$ is always a real number. In particular, we must have $\sigma(A) \subset \mathbb{R}$.

We set

$$m := \inf\{\langle x, Ax \rangle_{\mathcal{H}},\ \|x\| = 1\}, \quad \text{and} \quad M := \sup\{\langle x, Ax \rangle_{\mathcal{H}},\ \|x\| = 1\}.$$

Theorem 5.3

If $\lambda \neq 0$ is a non-null eigenvalue of A, it is of finite multiplicity.
The numbers m and M are eigenvalues of A.

Proof. Let us prove that for all $\lambda \neq 0$, $E_\lambda := \ker\{\lambda - A\}$ of finite dimension. For all $x \in E_\lambda$, we have $\lambda x = Ax$, so the unit ball of E_λ satisfies $\mathcal{B}_{E_\lambda} \subset \lambda^{-1} A(\mathcal{B}_E)$, hence is relatively compact. By the Riesz' lemma (see [Bre99, Theorem VI.5]), E_λ is finite dimensional.

Let us prove that M is an eigenvalue (the proof for m is similar). The problem defining M is an optimisation problem under constraint. Let us prove that this problem is well-posed.

Let (x_n) be a maximising sequence for this problem, so $\|x_n\| = 1$ and $\langle x_n, Ax_n \rangle \to M$. Since x_n is bounded in the reflexive Banach space $\mathcal{H}$, there is a subsequence, still noted n and an element $x_* \in \mathcal{H}$ so that (x_n) weakly converges to x_* in $\mathcal{H}$. In addition, since A is compact, we have $Ax_n \to Ax_*$ strongly. In particular, by Theorem 1.26, we have $M = \lim \langle x_n, Ax_n \rangle = \langle x_*, Ax_* \rangle$.

We claim that $\|x_*\| = 1$. Assume otherwise. Then, by Theorem 1.29 (the norm is weakly lsc), we have $\|x_*\| < 1$. Then, we would have

$$\left\langle \frac{x_*}{\|x_*\|}, A \frac{x_*}{\|x_*\|} \right\rangle = \frac{1}{\|x_*\|^2} \langle x_*, Ax_* \rangle = \frac{M}{\|x_*\|} > M,$$

a contradiction. So $\|x_*\| = 1$. In particular, x_* is an optimiser of the problem defining M.

We can therefore write the Euler-Lagrange equations. There is $\lambda \in \mathbb{R}$ so that $Ax_* = \lambda x_*$. By identification, we must have $\lambda = M$, which proves the second part of the theorem.

□

This allows to prove the following important theorem.

Theorem 5.4: Spectral decomposition of symmetric compact operators

Let $\mathcal{H}$ be a separable Hilbert space, and let A be a symmetric compact operator on $\mathcal{H}$. Then there is a basis $(e_1, e_2, \cdots)$ of $\mathcal{H}$ where all elements are eigenvectors of A. We can order the basis so that

$$Ae_n = \lambda_n e_n, \quad |\lambda_1| \geq |\lambda_2| \geq \cdots \geq 0.$$

Finally, 0 is the only accumulation point of $(\lambda_n)_{n \in \mathbb{N}}$.

In the sequel, we often use the Dirac notations. An element $x \in \mathcal{H}$ is denoted $|x\rangle$ (ket) if x is seen as an element of $\mathcal{H}$, and $\langle x|$ (bra), if it is seen as an element of the dual $\mathcal{H}* = \mathcal{H}$. This allows to write

$$A = \sum_{n \in \mathbb{N}} \lambda_n |e_n\rangle\langle e_n|, \quad \text{in the sense} \quad Ax = \sum_{n \in \mathbb{N}} \left(\lambda_n \langle e_n, x \rangle_{\mathcal{H}}\right) e_n.$$

With this notation, we may have $\lambda_n = \lambda_{n+1}$: we repeat the eigenvalues as many times as their multiplicities.

Proof. We do not prove fully this Theorem, and refer to [Bre99, Theorem VI.11] for a complete proof. Let $\lambda_0 := 0$, let (λ_n) be the set of non-null eigenvalues of A (counting multiplicities), and set $E_n := \ker\{\lambda_n - A\}$. First, we notice that if $x \in E_n$ and $y \in E_m$ are normalised vectors, then, since A is symmetric,

$$(\lambda_n - \lambda_m)\langle x, y \rangle = \langle \lambda_n x, y \rangle - \langle x, \lambda_m y \rangle = \langle Ax, y \rangle - \langle x, Ay \rangle = 0,$$

so $\lambda_n \neq \lambda_m$ implies $\langle x, y \rangle = 0$. In other words, the spaces E_n are mutually orthogonal.

Let $F := E_0 \oplus E_1 \oplus E_2 \oplus \cdots$. We have $AF \subset F$, so $AF^\perp \subset F^\perp$ (why?). The operator $A_{F^\perp} : F^\perp \to F^\perp$, that is the operator A *restricted to* $F^\perp$, is a compact symmetric operator. Since $A_{F^\perp}$ cannot have eigenvalues (since otherwise, they would have been considered in F), we deduce by Theorem 5.3 that $F^\perp = 0$. In other words, F is dense in $\mathcal{H}$. It remains to choose a basis in all (finite dimensional) spaces E_n, and the result follows. □

5.2.2 Application: the spectrum of Dirichlet Laplacien in bounded domains

In this section, we study the operator $-\Delta$ as an operator on $\mathcal{H} := L^2(\Omega)$. Unfortunately, this operator is not compact, so we cannot directly apply the result of the previous section. One idea is to consider the operator $(-\Delta)^{-1}$. However, to define such operator, one needs to precise the *boundary conditions*.

Here, we study the **Dirichlet Laplacian** $-\Delta_0$ that we define now. Recall that, in Section 3.3.2, we proved that

$$\forall f \in L^2(\Omega), \quad \text{there is a unique } u \in H^2(\Omega) \cap H_0^1(\Omega) \text{ so that } \quad -\Delta u = f.$$

In addition, we have $\|u\|_{H^2}^2 \le \|f\|_{L^2}$. Let $\widetilde{A_0} : f \mapsto u$ be the corresponding bounded linear map. So $\widetilde{A_0}$ is a map from $L^2(\Omega)$ to $H^2(\Omega) \cap H_0^1(\Omega)$. It is invertible, and the Dirichlet Laplacian is the inverse

$$-\Delta_0 := \widetilde{A_0}^{-1}, \quad \text{from } H^2(\Omega) \cap H_0^1(\Omega) \text{ to } L^2(\Omega).$$

The operator $\widetilde{A_0}$ has different starting and ending spaces. We rather study the operator $A_0 = I\widetilde{A_0}$, where I is the trivial injection map $H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$. If Ω has a boundary $\partial\Omega$ of class C^1, then, by Sobolev's embedding, I is compact. In particular, $A_0 : L^2(\Omega) \to L^2(\Omega)$ is compact. Applying Theorem 5.4 gives the following.

Theorem 5.5: Spectrum of the Dirichlet Laplacian

Let Ω be a bounded domain in $\mathbb{R}^d$ with boundary $\partial\Omega$ of class C^1. Then, there is a basis $(e_1, e_2, \cdots)$ of $L^2(\Omega)$ so that

$$\forall n \in \mathbb{N}^*, \quad e_n \in H^2(\Omega) \cap H_0^1(\Omega), \quad \text{and} \quad -\Delta_0 e_n = \lambda_n e_n.$$

In addition, we have $0 < \lambda_1 \le \lambda_2 \le \cdots$, and the sequence (λ_n) goes to infinity.

Proof. For the first part, we apply the spectral decomposition to A_0, and deduce that

$$A_0 = \sum_{n=1}^{\infty} \mu_n |e_n\rangle\langle e_n|.$$

Note that since A_0 is injective, 0 is not an eigenvalue of A_0, so A_0 is indeed invertible. We then set $\lambda_n := 1/\mu_n$, and deduce that

$$(-\Delta_0) = \sum_{n=1}^{\infty} \lambda_n |e_n\rangle\langle e_n|.$$

It remains to prove that $\lambda_n \ge 0$. Since $e_n \in H^2(\Omega) \cap H_0^1(\Omega)$, we can integrate by part, and get

$$\lambda_n = \langle e_n, (-\Delta_0) e_n \rangle_{L^2} = \int_\Omega |\nabla e_n|^2 \ge 0.$$

This concludes the proof. □

One can perform the same analysis for the **Neumann Laplacian** $(-\Delta_N)$. Note however that the constant function 1_Ω is in $H^2(\Omega) \cap H^1(\Omega)$, and $(-\Delta_N)1_\Omega = 0$, so 0 is now an eigenvalue of $(-\Delta_N)$, and $(-\Delta_N)$ is not invertible. One needs to study the operator $A_N^{-1} := (1 - \Delta_N)$ to apply the theory.

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