AdS/CFT - TD 5

1 Planar AdS-Schwarzschild (Euclidean)

$$ds_5^2 = \frac{\ell^2}{z^2}\left[\frac{dt^2}{f(z)} + f(z)dz^2 + \delta_{ij}\,dx^i dx^j\right]$$

$$f(z) = 1 - \frac{z^4}{z_h^4} \qquad i,j = 1,2,3$$

1.1) Eistein's equation:

$$R_{ab} - \frac{1}{2} g_{ab} R = - g_{ab}\Lambda \qquad \Lambda = -\frac{6}{\ell^2}$$

- $\Gamma^a{}_{bc}$: composantes non-nulles:

$$\Gamma^{\tau}{}_{\tau z} = -\frac{1}{z} + \frac{1}{2}\frac{f'}{f} = \Gamma^{\tau}{}_{z\tau}$$

$$\Gamma^{z}{}_{\tau\tau} = \frac{f^2}{z} - \frac{1}{2} f' f \qquad \Gamma^{z}{}_{zz} = -\frac{1}{z} - \frac{f'}{2f}$$

$$\Gamma^{z}{}_{ij} = \frac{f}{z}\delta_{ij} \qquad \Gamma^{i}{}_{zj} = \Gamma^{i}{}_{jz} = -\frac{1}{z}\delta^{i}_{j}$$

For $\Gamma^a{}_{bc}$ and $R^a{}_{bcd}$ I am using the convention of David Tong's GR course:

$$\Gamma_{\mu\nu}{}^{\lambda} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right)$$

$$R^{\sigma}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\sigma}_{\nu\rho} - \partial_\nu \Gamma^{\sigma}_{\mu\rho} + \Gamma^{\lambda}_{\nu\rho}\Gamma^{\sigma}_{\mu\lambda} - \Gamma^{\lambda}_{\mu\rho}\Gamma^{\sigma}_{\nu\lambda}$$

If you are using the GRTensor mathematica package, these gives "wrong" results for the Euclidean because it is adopted to the $+---$ signature - You can use a Euclidean metric, but then you get different Riemann tensor, which is related to the above by:

$$\left(R^{\sigma}{}_{\rho\mu\nu}\right)_{\text{GRTensors}} = \left(-R^{\sigma}{}_{\nu\mu\rho}\right)_{\text{D. Tong}}$$

- $R^{a}{}_{bcd}$ - non-zero components

$$R^{z}{}_{zzz} = -\frac{f}{2z^2}\left(2f - 2\cancel{z}f' + z^2 f''\right)$$

$$R^{z}{}_{izj} = \left(-\frac{f}{z^2} + \frac{1}{2}\frac{f'}{z}\right)\delta_{ij} = -R^{z}{}_{ijz}$$

$$R^{z}{}_{zzz} = -\frac{1}{z^2} + \frac{f'}{zf} - \frac{f''}{2f}$$

$$R^{z}{}_{izj} = \left(-\frac{f}{z^2} + \frac{1}{2}\frac{f'}{z}\right)\delta_{ij}$$

$$R^{i}{}_{zjz} = \left(-\frac{1}{z^2} + \frac{f'}{2zf}\right)\delta^{i}_{j}$$

$$R^{i}{}_{zjz} = \left(-\frac{f^2}{z^2} + \frac{ff'}{2z}\right)\delta^{i}_{j}$$

$$R^{i}{}_{jkl} = -\left(\delta^{i}_{k}\delta_{jl} - \delta^{i}_{l}\delta_{jk}\right)\frac{f}{z^2}$$

- R_{bd} :

$$R_{zz} = -4f^2/z^2 + \frac{5ff'}{2z} - \frac{1}{2}ff''$$

$$R_{ij} = \left(-\frac{4f}{z^2} + \frac{f'}{z}\right)\delta_{ij}$$

$$R_{zz} = -\frac{4}{z^2} + \frac{5f}{2zf'} - \frac{1}{2}\frac{f''}{f}$$

Ricci Scalar:

$$R = \left(-20f + 8zf' - z^2 f''\right)/\ell^2$$

now specialize to $f = 1 - \frac{z^4}{z_h^4}$

$$f' = -\frac{4z^3}{z_h^4} \qquad f'' = -\frac{12z^2}{z_h^4}$$

$$\Rightarrow \quad R = -\frac{20}{\ell^2}$$

Einstein tensor:

$$G_{zz} = \frac{3}{2}\frac{(4f - zf')}{z^2 f} \; ; \; G_{\tau\tau} = \frac{3}{2}\frac{(4f - zf')f}{z^2}$$

$$G_{ij} = \left(\frac{6f}{z^2} - \frac{3f'}{z} + \frac{f''}{2}\right)\delta_{ij}$$

Specialize to $f = 1 - z^4/z_h^4$:

$$\underline{G_{zz}} = \frac{6}{z^2}\frac{1}{1 - z^4/z_h^4} = \underline{-g_{zz}\Lambda}$$

$$\underline{G_{\tau\tau}} = \frac{6}{z^2}\left(1 - \frac{z^4}{z_h^4}\right) = \underline{-g_{\tau\tau}\Lambda}$$

$$\underline{G_{ij}} = \frac{6}{z^2}\delta_{ij} = \underline{-g_{ij}\Lambda}$$

1.2)

1. ~~Bulk term~~; take $f = 1$.

$$R_{ab} - \frac{1}{2} g_{ab} R = - g_{ab} \Lambda$$

$$\Rightarrow \quad R\left(1 - \frac{5}{2}\right) = -5\Lambda$$

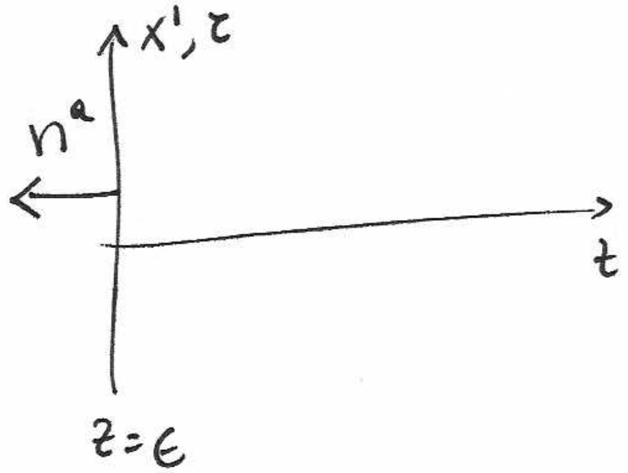
$$\Rightarrow \quad R = 10/3\,\Lambda \quad \Rightarrow \quad R - 2\Lambda = \frac{4}{3}\Lambda = -\frac{8}{\ell^2}$$

$$\Rightarrow \quad \cdot \; S^{\epsilon}_{bulk} = -M_p^3 \int_{\epsilon}^{\infty} dz \int d^4x \, \frac{\ell^5}{z^5}\left(-8/\ell^2\right)$$

$$= + \underline{\underline{2 M_p^3 \ell^3 \frac{V_3 \beta}{\epsilon^4}}} \qquad V_3 = \text{Volume of space}$$

- $$S^{\epsilon}_{ct} = 6 \frac{M_p^3}{\ell} \int d^4x \, \frac{\ell^4}{\epsilon^4} = \underline{\underline{6 M_p^3 \ell^3 \frac{V_3 \beta}{\epsilon^4}}}$$

- To compute S_{GH} we need K, for which we need n^a, which should be a unit vector, normal to the boundary, and directed outward:



$$n^a = \left(-z/\ell\, , 0\, , \vec{0}\right)$$

$$g_{ab}\, n^a n^b = 1$$

$$K = \nabla_a n^a = \frac{1}{\sqrt{g}}\, \partial_a \sqrt{g}\, g^{ab} n_b$$

$$= \left(\ell/z\right)^{-5} \partial_z \left(\ell/z\right)^{5} \cdot \left(-z/\ell\right) = 4/\ell$$

$$\rightarrow\quad S^{\epsilon}_{GH} = -2 M_p^3 \int d^4x\, \frac{\ell^4}{\epsilon^4} \cdot \frac{4}{\ell} = \underline{\underline{-\frac{8 M_p^3 \ell^3 V_3 \beta}{\epsilon^4}}}$$

$$\Rightarrow\quad \underline{S^{\epsilon}_{Bulk} + S^{\epsilon}_{GH} + S^{\epsilon}_{ct} = 0}$$

2. Now we need the same calculation, but in the BH background.

- It is still true that $R_{ab} - \frac{1}{2} g_{ab} R = - g_{ab} \Lambda$

$\Rightarrow \; R - 2\Lambda = -8/\ell^2$ again.

- $$S^{\epsilon}_{bulk} = - M_p^3 \int \underbrace{\sqrt{g}}_{\ell^5/z^5} \underbrace{(R - 2\Lambda)}_{-8/\ell^2} =$$

$$= 8 M_p^3 \ell^3 V_3 \beta \int_{\epsilon}^{z_h} \frac{dz}{z^5}$$

$$= 8 M_p^3 \ell^3 V_3 \beta \left(\frac{1}{4\epsilon^4} - \frac{1}{4 z_h^4} \right)$$

$$= 2 M_p^3 \ell^3 \frac{V_3 \beta}{\epsilon^4} \underbrace{\left(1 - \frac{\epsilon^4}{z_h^4} \right)}_{\to f(\epsilon)}$$

- $$S^{\epsilon}_{ct} = \frac{6 M_p^3}{\ell} V_3 \beta \sqrt{\gamma}$$

$$ds^2_{ind} = \frac{\ell^2}{\epsilon^2} \left[f(\epsilon) d\tau^2 + dx_i^2 \right]$$

$$= 6 M_p^3 \ell^3 \frac{V_3 \beta}{\epsilon^4} \sqrt{f(\epsilon)}$$

- the unit normal is now: $n^a = \left(-\frac{z\sqrt{f}}{\ell}, 0\right)$

and

$$K = \nabla_a n^a = -\frac{\ell^{-5}}{z^{-5}} \partial_z\left(\frac{\ell^5}{z^5} \frac{z}{\ell} f^{1/2}\right)$$

$$= -\frac{z}{\ell}\frac{1}{2}\frac{f'}{f^{1/2}} + \frac{4}{\ell} f^{1/2}$$

$$\Rightarrow \quad S_{GH}^{\epsilon} = -2M_p^3 V_3 \beta \underbrace{\frac{\ell^4}{\epsilon^4}\sqrt{f(\epsilon)}}_{\sqrt{\gamma}} \underbrace{\left(\frac{4}{\ell}\sqrt{f} - \frac{\epsilon}{2\ell}\frac{f'(\epsilon)}{\sqrt{f}}\right)}_{K}$$

$$= -8M_p^3 \ell^3 \frac{V_3 \beta}{\epsilon^4}\left(f(\epsilon) - \frac{\epsilon}{8} f'(\epsilon)\right)$$

- Putting together the 3 terms:

$$S_{bulk}^{\epsilon} + S_{ct}^{\epsilon} + S_{GH}^{\epsilon} = \frac{M_p^3 \ell^3}{\epsilon^4} V_3 \beta \times$$

$$\left[2 - \frac{2\epsilon^4}{z_h^4} + 6\sqrt{1 - \frac{\epsilon^4}{z_h^4}} - 8\left(1 - \frac{\epsilon^4}{z_h^4} + \frac{\epsilon^4}{2z_h^4}\right)\right]$$

$$\underset{\epsilon\to 0}{\sim} M_p^3 \ell^3 \frac{V_3 \beta}{\epsilon^4}\left[\cancel{(2+6-8)} + \frac{\epsilon^4}{z_h^4}(-2-3+4)\right]$$

$$\xrightarrow[\epsilon\to 0]{} -\frac{M_p^3 \ell^3 V_3}{z_h^4}\beta \qquad \underline{\text{Finite}}$$

3. $S_{\text{Euclidean}} = \beta F_{BH} \Rightarrow \boxed{F_{BH} = -\dfrac{M_p^3 \ell^3 V_3}{z_h^4}}$

In terms of temperature: $z_h^4 = \dfrac{1}{\pi^4 T^4}$

$$\Rightarrow \underline{F_{BH} = -\pi^4 M_p^3 \ell^3 V_3 T^4}$$

$F(T) \leq 0 \quad \forall T \Rightarrow$ BH always dominates over Euclidean AdS ($F_{AdS} = 0$)

4. Entropy:

$$\underset{\substack{\uparrow \\ \text{(entropy)}}}{S} = -\frac{\partial F}{\partial T} = 4 M_p^3 \ell^3 \pi^4 V_3 T^3$$

(As $M_p^3 = \dfrac{1}{16\pi G_N} \Rightarrow S = \dfrac{\ell^3 \pi^3 T^3 V_3}{4 G_N}$

(Area $= V_3 \ell^3 / z_h^3$) $= A/4G_N$

= BH entropy

$$\Rightarrow s = \frac{S}{V} = 4 M_p^3 \ell^3 \pi^4 T^3$$

Energy density :

$$E = \mathcal{F} + TS \Rightarrow \rho = \frac{\mathcal{F}}{V} + Ts$$

$$\rho = (-1 + 4)\, \pi^4 M_p^3 \ell^3 T^4$$

$$\Rightarrow \quad \underline{\rho = 3\pi^4 M_p^3 \ell^3 T^4}$$

Equation of state :

$$p = -\frac{\mathcal{F}}{V} = \pi^4 M_p^3 \ell^3 T^4$$

$$\Rightarrow \quad \boxed{p = \frac{1}{3}\rho}$$

Conformal equation of state $(T_\mu{}^\mu = 0)$

5. $$c_v = \frac{\partial \rho}{\partial T} = 12\pi^4 M_p^3 \ell^3 T^3$$

$$c_s^2 = \frac{dp}{d\rho} = \frac{1}{3}$$

(sound speed $= \frac{1}{\sqrt{3}} \times$ light speed.
this is a property of a conformal fluid)

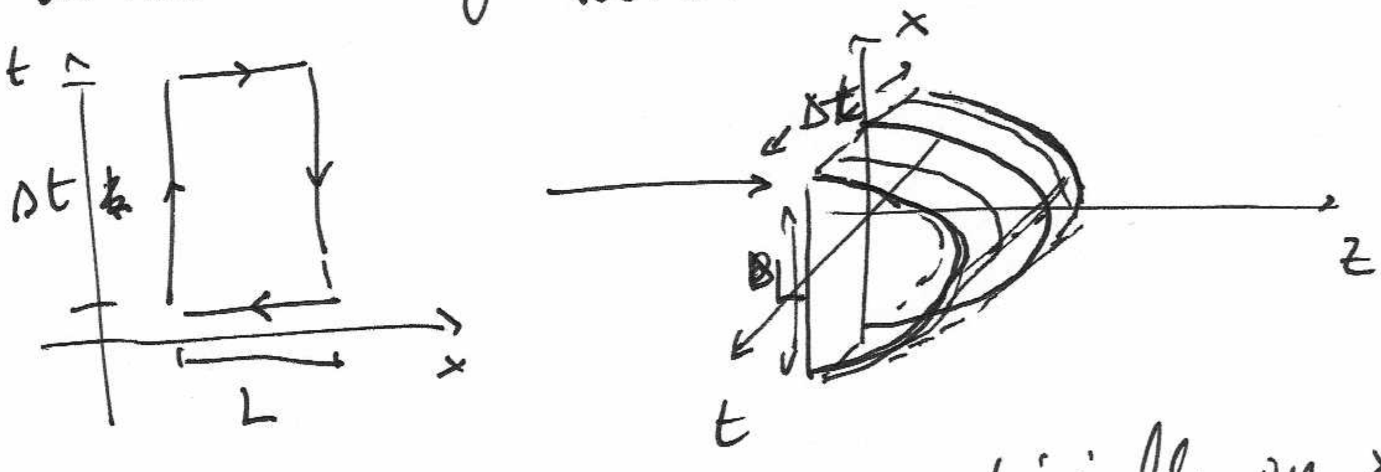
2 - The AdS Soliton

$$ds_5^2 = \frac{\ell^2}{z^2}\left[\frac{dz^2}{f(z)} + f(z)\,d\theta^2 \underbrace{- dt^2 + \delta_{ij}dx^i dx^j}_{\text{2+1 dim Minkowski}}\right]$$

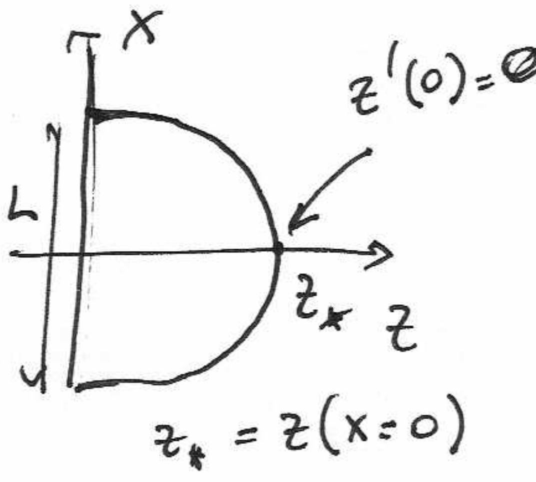
$$f(z) = 1 - \frac{z^4}{z_0^4}$$

2.1) Wilson loop [see hep-th/9811192 for a nice general discussion.]

We consider a rectangular Wilson loop on the boundary. We want to find a corresponding worldsheet in the bulk:



- only z depends nontrivially on x (or x on z):
- θ is constant (e.g. $\theta = 0$)
- t does not depend on x



Surface embedded in terms of worldsheet
coordinates $\{\eta, \sigma\}$ as:

w.s. time ↗ ↖ w.s. distance

$$\begin{cases} x = \sigma \\ t = \eta \\ z = z(\sigma) \\ \theta = 0 \\ y = 0 \end{cases} \Rightarrow ds_2^2 = \frac{\ell^2}{z^2}\left[\left(\frac{z'^2}{f(z)} + 1\right)d\sigma^2 - d\eta^2\right]$$

where $z = z(\sigma)$

1) We can change variable to $u = 1/z$

$$\Rightarrow ds_2^2 = \ell^2 u^2(\sigma)\left[\left(1 + \frac{u'^2}{u^4 f(u)}\right)d\sigma^2 - d\eta^2\right]$$

where $f(u) = 1 - \frac{u_0^4}{u^4}$

2)
$$S'_{NG} = -\frac{1}{2\pi\alpha'}\int d\eta d\sigma \sqrt{\det G}$$

$$= -\frac{\ell^2}{2\pi\alpha'}\int_0^{\Delta t} d\eta \int_{-\frac{L}{2}}^{\frac{L}{2}} d\sigma \sqrt{u^4(\sigma) + \left(\frac{du}{d\sigma}\right)^2 \frac{1}{f(u(\sigma))}}$$

2 Dynamics is 1-dimensional problem with Lagrangian

$$\mathcal{L} = -\left(u^4 + \frac{(u')^2}{f} \right)^{1/2} \quad \text{and } \sigma = \text{"time"}$$

3) since $\mathcal{L}$ does not depend explicitly on σ, the "Hamiltonian" is conserved:

$$H = u' \frac{\partial L}{\partial u'} - \mathcal{L} =$$

$$= -\frac{u' \cdot u'/f}{\left(u^4 + (u')^2/f\right)^{1/2}} + \left(u^4 + \frac{(u')^2}{f}\right)^{1/2}$$

$$= -\frac{(u')^2/f - \left(u^4 + (u')^2/f\right)}{\left(u^4 + (u')^2/f\right)^{1/2}}$$

$$= \quad + \frac{u^4}{\left[u^4 + (u'^2)/f\right]^{1/2}} = -\frac{u^4}{\mathcal{L}}$$

the value of H determines one of the integration constants and it is related to the turning point u^* where $u'(\sigma) = 0$ by:

$$H = u_*^2 \qquad (u_* = u(\sigma = 0))$$

4. For fixed u_* we can find a ~~one~~-first-order differential equation for $u(\sigma)$:

$$u_*^2 = \frac{u^4(\sigma)}{\left(u^4(\sigma) + \frac{u'(\sigma)^2}{f} \right)^{1/2}}$$

$$\Rightarrow \left(\frac{u_*}{u}\right)^4 = \frac{1}{1 + \frac{u'^2}{u^4 f}}$$

$$\Rightarrow u'(\sigma) = \pm \sqrt{f}\, u^2 \sqrt{\frac{u^4}{u_*^4} - 1}$$

(the $\pm$ corresponds to the two branches for $\sigma > 0$ $(+)$ and $\sigma < 0$ $(-)$)

we can solve this by separation of variables:

$$\int_0^{L/2} dx = \int_{u_*}^{\infty} \frac{du}{u^2 \sqrt{f(u)\left(\frac{u^4}{u_*^4}-1\right)}}$$

when we have used the "boundary condition"

$\sigma(u=\infty) = L/2$ (the lenght of the string on the boundary) $\frac{1}{2}$

and the "initial condition" $\Rightarrow u(\sigma=0)=u_*$

this is the same expression as the one found in AdS, except for the function $f(u)$ in the denominator.

$$\Rightarrow \frac{L}{2} = \int_{u_*}^{\infty} \frac{du}{u^2} \frac{1}{\left(1-\frac{u_0^4}{u^4}\right)^{1/2}\left(\frac{u^4}{u_*^4}-1\right)^{1/2}}$$

- Recall that the variable z runs between $0 < z < z_0 \Rightarrow u_0 < u < +\infty$

$\Rightarrow u_* > u_0$

- For any $u_* \neq u_0$, the integral is finite, since

$$\text{Integrand} \sim \frac{1}{u^4} \qquad u \to \infty$$

$$\sim \frac{1}{\sqrt{u-u_0}} \qquad u \to u_*$$

- However, for $u_* = u_0$, the integral diverges:

$$\text{Integrand} \sim \frac{1}{(u-u_0)^{1/2}(u-u_0)^{1/2}}$$

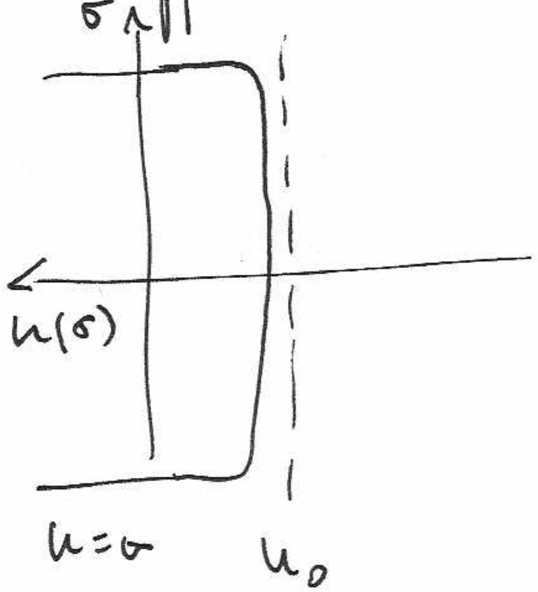
$$\sim \frac{1}{u-u_0}$$

$\Rightarrow$

$$\underline{L \longrightarrow \infty} \quad \text{as} \quad \underline{u_* \longrightarrow u_0} \;!$$

[Recall in AdS: $L \to \infty$ as $u_* \to 0$ and $L \sim \frac{1}{u_*}$!]

therefore, as we stretch the string on the boundary more and more, the endpoint approaches u_0 without ever reaching it



(one can check that

$$L \propto \log \frac{1}{\sqrt{u_* - u_0}}$$)

as $u_* \longrightarrow u_0$. see #6 below.

5. L becomes small when $u_* \longrightarrow \infty$. ($u_* \gg u_0$)

In this limit, the integral is over a range where $u \gg u_0$, we can approximate it by: $1 - \frac{u_0^4}{u^4} \simeq 1$

$$\frac{L}{2} \underset{u_* \gg u_0}{\sim} \int_{u_*}^{\infty} \frac{du}{u^2} \frac{1}{\sqrt{u^4/u_*^4 - 1}} = \quad y = u/u_*$$

As in AdS

$$= \frac{1}{u_*} \underbrace{\int_1^{\infty} \frac{dy}{y^2} \frac{1}{\sqrt{y^4 - 1}}}_{\text{constant}} \Rightarrow L \doteq u_*^{-1}$$

$$\ll u_0^{-1}$$

6. As $u_* \to u_0$ we have $L \to \infty$
so in particular $L \gg u_0^{-1}$
In this limit the integral can be written as:

$$\frac{L}{2} \simeq \int_{u_*}^{\infty} \frac{du}{u^2} \frac{u^4 u_*^4}{\left(\left(u^4-u_0^4\right)\left(u^4-u_*^4\right)\right)^{1/2}}$$

$$\simeq \frac{u_0^{-2}}{4} \int_{u_*}^{\infty} \frac{du}{\sqrt{(u-u_0)(u-u_*)}}$$

Because the largest contribution comes from the region $u \simeq u_* \simeq u_0$, so we can set $u = u_* = u_0$ anywhere except where there would be a singularity.
The integral can be done and gives

$$\int \frac{1}{(u-u_0)(u-u_*)} \sim \operatorname{ArcSinh} \sqrt{\frac{u-u_*}{u_*-u_0}}$$

so if we set $u_* = u_0 + \epsilon$ and evaluate

$$\Rightarrow \quad \frac{L}{2} \sim \text{Arc Sinh}\left(\frac{Cst}{\sqrt{\epsilon}}\right)$$

$$\simeq \log \frac{1}{\sqrt{\epsilon}} \quad \text{as } \epsilon \to 0$$

7. Now we wat to compute the NG action as a function of L:

$$S_{NG} = \frac{\ell^2}{2\pi\alpha'}\int_0^{\Delta t} d\eta \int d\sigma\, \mathcal{L} = \frac{2\Delta t\,\ell^2}{2\pi\alpha'}\int_{u_*}^{\infty} du\, \frac{1}{u'}\,\mathcal{L}$$

$$= \frac{2\Delta t\,\ell^2}{2\pi\alpha'}\int_{u_*}^{\infty} du\, \frac{\mathcal{L}}{u^2\sqrt{f\left(\frac{u}{u_*^4}-1\right)}}$$

use $\mathcal{L} = -u^4/H = -u^4/u_*^2$

$$= \frac{-2\Delta t\,\ell^2}{2\pi\alpha'}\int_{u_*}^{\infty} \frac{du}{u^2}\, \frac{u^4}{u_*^2\sqrt{f(u)\left(\frac{u^4}{u_*^4}-1\right)}}$$

As $L \ll u_0^{-1}$ we are in the limit $u_* \longrightarrow \infty$ (see point 5)

$\Rightarrow$ we can set $f \simeq 1$ and we have the same as in AdS:

$$S \sim -\frac{2\Delta t\, \ell^2}{2\pi\alpha'} \int_{u_*}^{\infty} \frac{du}{u^2} \frac{u^4}{u_*^2\left(\frac{u^4}{u_*^4}-1\right)^{1/2}}$$

we have to make the integral finite by subtracting the "bare" mass of two isolated quarks

$$S - 2S'_{quark} = -\frac{2\Delta t}{2\pi\alpha'}\ell^2 \int_{u_*}^{\infty}\left(\frac{du\, u^2}{u_*^2\left(\frac{u^4}{u_*^4}-1\right)^{1/2}} - 1\right)$$

$\simeq \quad u_* \times$ some finite integral

$$\sim \frac{1}{L}\Delta t \, \frac{\ell^2}{2\pi\alpha'}$$

8. We want to now how this behaves as $u_* \longrightarrow u_0$ and $L \to \infty$

$\Rightarrow$ the integral is dominated by the region $u \simeq u_* \simeq u_0$

so we can replace u^4 in the numerator by u_0^4 and u_*^2 in the denominator by u_0^2:

$$= -\frac{2\Delta t \ell^2}{2\pi\alpha'}\int \frac{du}{u^2}\frac{1}{\sqrt{f(u)\left(\frac{u^4}{u_*^4}-1\right)}}\frac{u^4 \rightarrow u_0^4}{u_*^2 \rightarrow u_0^2}$$

$$\simeq -\frac{2\Delta t\, u_0^2 \ell^2}{2\pi\alpha'}\underbrace{\int_{u_*}^{\infty}\frac{du}{u^2}\frac{1}{\sqrt{f(u)\left(\frac{u^4}{u_*^4}-1\right)}}}_{\frac{L}{2}}$$

$$\simeq -\underbrace{\Delta t \cdot L}_{\text{Area low}} \cdot \underbrace{\frac{u_0^2 \ell^2}{2\pi\alpha'}}_{\text{string tension}}$$

$$\tau_c = \frac{u_0^2\ell^2}{2\pi\alpha'}$$

2.2 Deconfinement transition

1. the calculation is the same as in 2.1
but now the $f(u)$ is in front of $d\eta^2$:
the induced metric is:

$$ds^2_{ind} = \ell^2 u^2 \left[\left(1 + \frac{u'^2}{u^4 f(u)}\right) d\sigma^2 + f(u) d\eta^2 \right]$$

where $f(u) = 1 - u_h^4/u^4$ $\qquad u_h = \frac{1}{z_h}$

therefore: $\mathcal{L} = -\sqrt{u^4 f + u'^2}$

$$H = -\frac{u^4 f}{\mathcal{L}} = \sqrt{u_*^4 f(u_*)}$$

~~Done:~~ and we get:

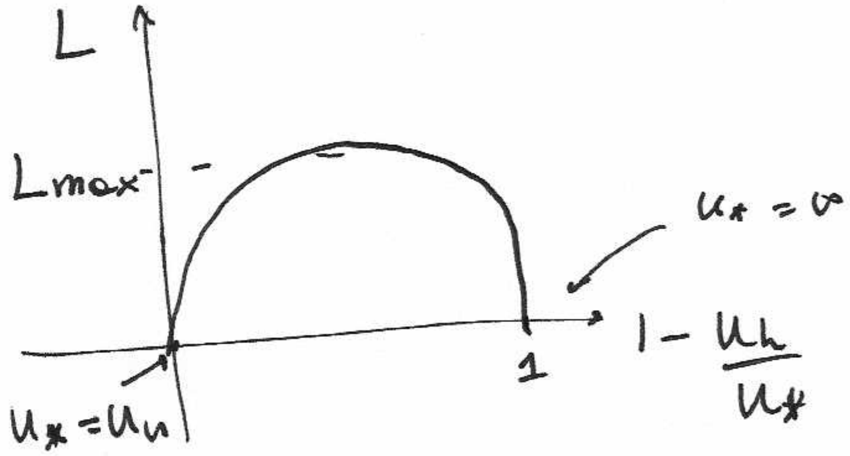
$$u' = \pm\sqrt{u^4 - u_h^4} \left(\frac{u^4 - u_h^4}{u_*^4 - u_h^4} - 1 \right)^{1/2}.$$

$$= \pm \left(\frac{(u^4 - u_h^4)(u^4 - u_*^4)}{(u_*^4 - u_h^4)} \right)^{1/2}$$

integrating:

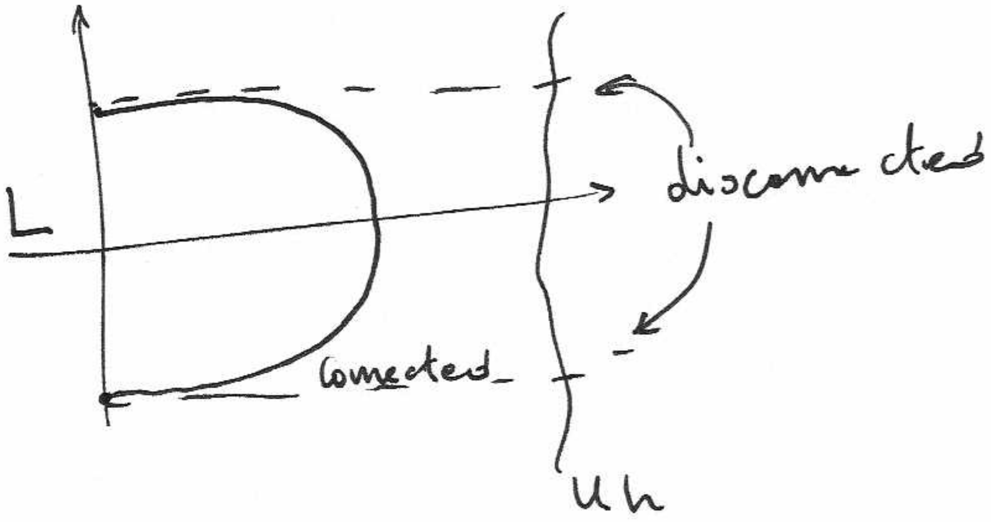
$$\frac{L}{2} = \sqrt{u_*^4 - u_h^4} \int_{u_*}^{\infty} du \, \frac{1}{\sqrt{(u^4-u_h^4)(u^4-u_*^4)}}$$

this has the usual behavior as $u_* \to \infty$ $(L \sim 1/u_*)$ but it goes back to 0 as $u_* \to u_h$



$\Rightarrow$ ~~Po~~ For $L > L_{max}$ there is no solution.

Infact, before we get to L_{max}, there is a value $L_c < L_{max}$ for which the energy of the connected worldsheet is larger than that of 2 disconnected straight strings extending to u_h

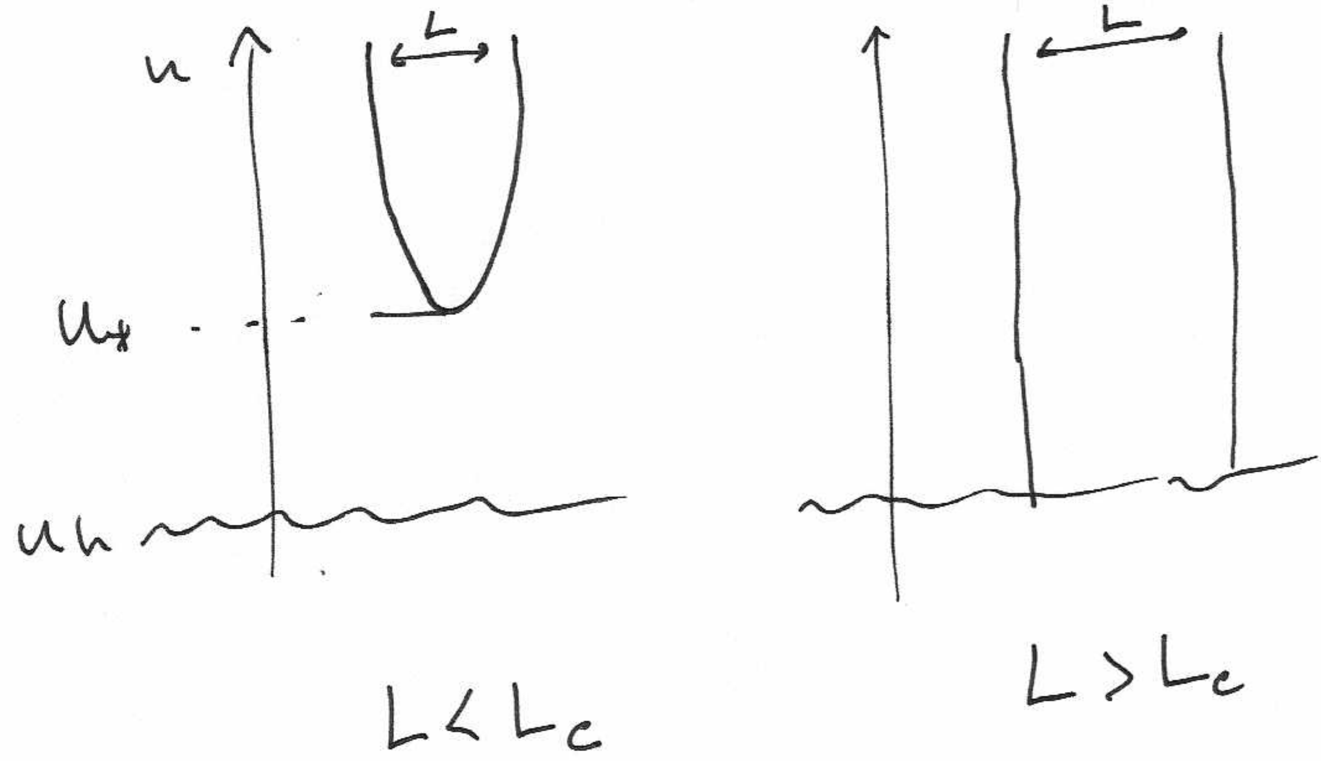


$\Rightarrow$ for $L > L_c$ the action is that of 2 disconnected strings $\Rightarrow$ independent of L.

this means that in the black hole phase the temperature has the effect of <u>screening</u> the $Q\bar{Q}$ pairs, which can be separated as much as they want

$\Rightarrow$ color is deconfined.

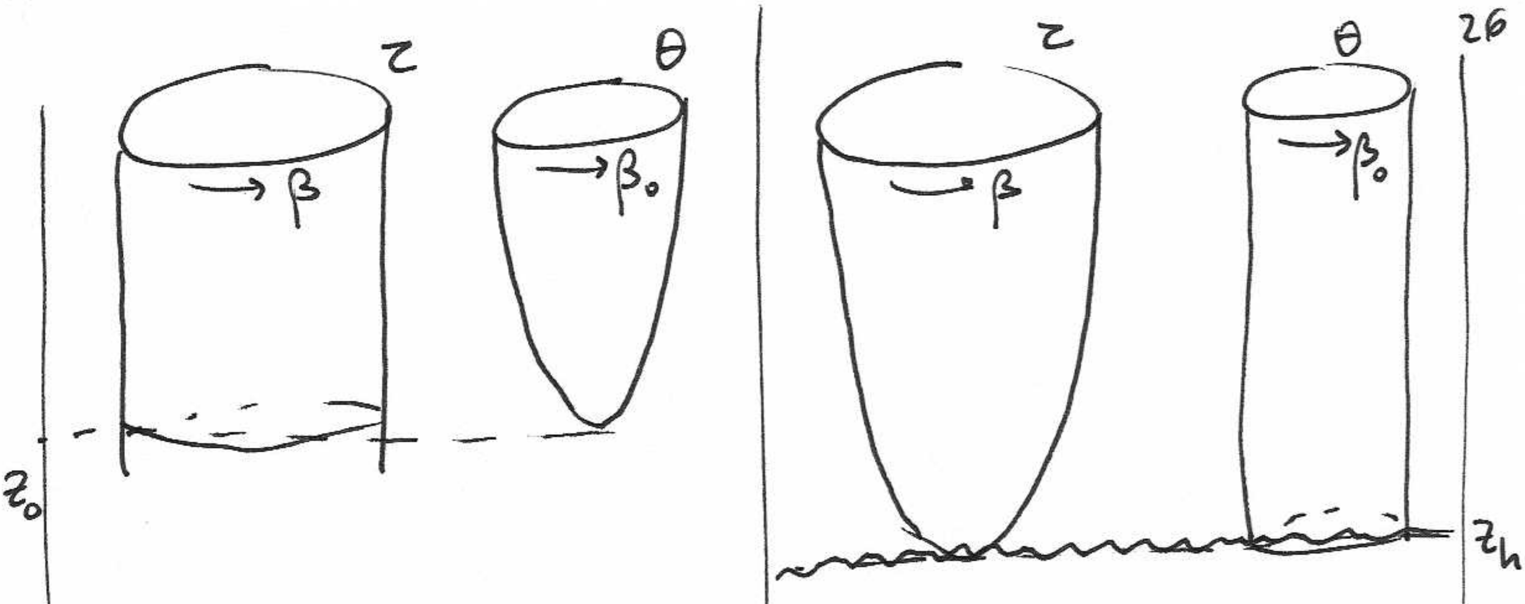
[For details, see hep-th/9803137]



2. So $\forall\, T$ we have two solutions:

- One is confining, with $f_0(z)$ in front of $d\theta^2$, and τ periodic of size β
- One is deconfined, with $f_h(z)$ in front of $d\tau^2$.

Both solution have θ with period β_0 and τ with period β at infinity (in u) (or at $z=0$) but in the deconfined solution the size of the τ-circle varies, that of the θ-circle not.



Confining solution

S^1_τ not-contractible
S^1_θ contractible

Space-time ends at z_0

Deconfined solution

S^1_τ contractible
S^1_θ non-contractible

space-time ends at z_h

which solution dominates at a given T ?

$$\beta F = S_E$$ for each of the two solution.

We already computed S_E in 1.2 :

- Confining solution:

$$S_c = -M_p^3 \ell^3 \underbrace{V_2}_{\text{this used to be } V_3} \beta \, \frac{\beta_0}{z_0^4}$$

this used to be β

this used to be z_h

- Deconfined solution:

$$S_d = -M_p^3 \ell^3 \underbrace{V_2}_{\text{this used to be } V_3} \beta_0 \, \frac{\beta}{z_h^4}$$

So:

$$F_d - F_c = -M_p^3 \ell^3 V_2 \beta_0 \left(\frac{1}{z_h^4} - \frac{1}{z_0^4} \right)$$

$$= -M_p^3 \ell^3 V_2 \beta_0 \pi^4 \left(T^4 - T_c^4 \right)$$

where $\boxed{T_c \equiv \frac{1}{\pi z_0}}$

- For $T > T_c$ $\mathcal{F}_d < \mathcal{F}_c$

$\Rightarrow$ the black hole (deconfined phase) dominates

- For $T < T_c$ $\mathcal{F}_d > \mathcal{F}_c$

$\Rightarrow$ the confined phase dominate

$\Rightarrow$ phase transition at $T = T_c = \frac{1}{\pi z_0}$.
(deconfinement at high-T)

4. To check the order of the transition, compute the entropy density:

$$s = -\frac{1}{V_2}\frac{\partial F}{\partial T}$$

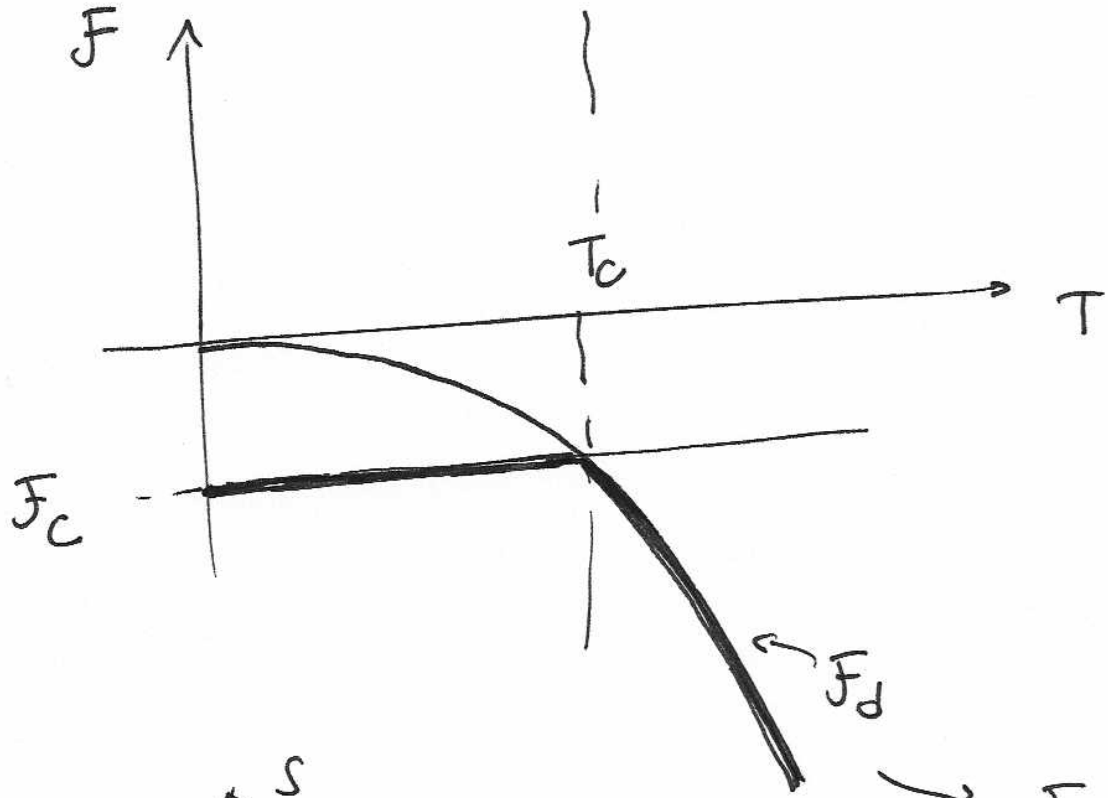
$$s_d = 4\pi^4 M_p^3 \ell^3 \beta_0 T^3 \quad , \quad s_c = 0$$

$\mathcal{F}_c$ does <u>not</u> depend on T

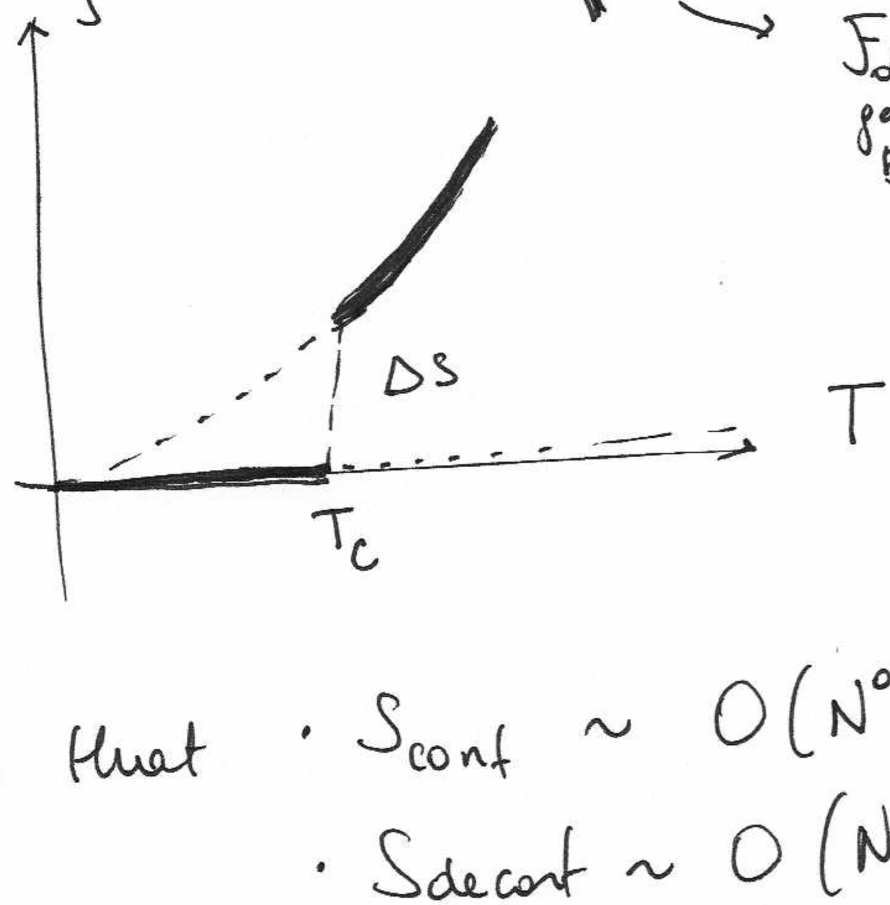
$$L_h = T_c \Delta s \Big|_{T=T_c} = \underline{4\pi^4 M_p^3 \ell^3 \beta_0 T^4}$$

$=$

(s jumps at $T_c \Rightarrow 1^{st}$ order transition)



$\mathcal{F}_d \sim T^4$ we get back the 5-d conformal behavior.



notice that
- $S_{conf} \sim O(N^0)$
- $S_{deconf} \sim O(N^2)$

entropy counts the # of degrees of freedom

$\sim O(N^0)$ for confined hadrons

$\sim O(N^2)$ for deconfined quarks + gluons