Géométrie différentielle et théorie de jauge, 10/03/2020
Spinors and Dirac operator

1. If $a, b \in C\ell(V)$ are of degrees $p, q = 0$ or 1, define their supercommutator by $[\![a, b]\!] = ab - (-1)^{pq}ba$. Denote $\sigma : C\ell(V) \to \Lambda V$ the usual identification. Prove that if $v \in V$ and $a \in C\ell(V)$ one has $\sigma([\![v, a]\!]) = -2i(v)\sigma(a)$.

2. Let (V, q) be an Euclidean vector space of even dimension n. One considers the chirality operator, defined in a direct orthonormal basis by

$$\Gamma = i^{\frac{n}{2}} e_1 \cdots e_n \in C\ell(V).$$

Prove that $\Gamma^2 = 1$ and $\Gamma v = -v\Gamma$ for all $v \in V$. Therefore any Clifford module E decomposes into $E = E_+ \oplus E_-$ for the eigenvalues ± 1 of Γ. Prove that for any $v \in V$ one has $vE_\pm = E_\mp$. Prove that in the case $E = S$, the decomposition coincides with the decomposition $S = S_+ \oplus S_-$.

3. Show that the spinor representation gives an injection $Spin_3 \hookrightarrow SU_2$, and deduce that $Spin_3$ is isomorphic to SU_2.

Similarly, show that $Spin_4$ is isomorphic to $SU_2 \times SU_2$.

4. The spinor representation of $Spin_8$ turns out to be real, that is

$$S = S_+ \oplus S_- = (\Delta_+ \oplus \Delta_-) \otimes \mathbb{C},$$

where $\Delta_\pm$ is a real 8-dimensional representation of $Spin_8$. Therefore we have three real 8-dimensional representations of $Spin_8$, the two previous $\Delta_\pm$ and $\mathbb{R}^8$ coming from the morphism $Ad : Spin_8 \to SO_8$.

Calculate the action of the central elements $\{1, -1, \Gamma, -\Gamma\} \subset Spin_8$ on the three representations. Deduce that they are not equivalent[1].

5. If E is a Clifford module with Clifford connection ∇, recall that $D^2 = \nabla^*\nabla + \mathscr{R}$ with $\mathscr{R}s = \sum_{i<j} e_i e_j R^{\nabla}_{e_i, e_j} s$. Consider $E = \Lambda T^*M$ with the Levi-Civita connection, so that $D = d + d^*$. Prove that for a 1-form α, one has

$$\mathscr{R}\alpha = \mathrm{Ric}(\alpha).$$

Deduce the Weitzenböck formula: for any 1-form α, denote $\Delta\alpha$ the Hodge Laplacian, then

$$\Delta\alpha = \nabla^*\nabla\alpha + \mathrm{Ric}(\alpha).$$

Deduce that, for a compact Riemannian manifold (M, g):

- if $\mathrm{Ric} \geq 0$ and $\mathrm{Ric} > 0$ at one point, then $H^1(M) = 0$;
- if $\mathrm{Ric} \geq 0$, then $b_1(M) \leq \dim M$. One can show that equality occurs if and only if M is a flat torus.

[1] They are actually exchanged by an order 3 outer automorphism of $Spin_8$, this is the phenomenon of triality.

6. Let (Σ, g) be a Riemann surface. Then in each tangent space there is an endomorphism J, the rotation of angle $\frac{\pi}{2}$. Then $J^2 = -1$, so there is a decomposition into eigenspaces for the eigenvalues $\pm i$ of J:

$$T\Sigma \otimes \mathbb{C} = T^{1,0}\Sigma \oplus T^{0,1}\Sigma,$$

which gives on the dual a decomposition $\Omega^1\Sigma(\mathbb{C}) = \Omega^{1,0}\Sigma \oplus \Omega^{0,1}\Sigma$.

A basic result in Riemann surface theory is the local existence of coordinates (x, y) such that $J\partial_x = \partial_y$ and $J\partial_y = -\partial_x$. Note $z = x + iy$. Then $T^{1,0}\Sigma = \langle \partial_z := \frac{1}{2}(\partial_x - i\partial_y)\rangle$, $T^{0,1}\Sigma = \langle \partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)\rangle$, and $\Omega^{1,0} = \langle dx + idy = dz\rangle$, $\Omega^{0,1} = \langle dx - idy = d\bar{z}\rangle$. The coordinate z is a holomorphic coordinate, and the transitions between holomorphic coordinates are given by holomorphic functions. The complex line bundle $\Omega^{1,0}$ is also called the canonical bundle and denoted K_Σ.

The notation is arranged so that if f is a complex function, then $df = (\partial_z f)dz + (\partial_{\bar{z}} f)d\bar{z}$. This gives a decomposition $df = \partial f + \bar{\partial} f$ of df in $\Omega^{1,0} \oplus \Omega^{0,1}$.

Let ω be the volume form of (Σ, g). Prove that $\omega(X, Y) = g(JX, Y)$ and deduce that $\nabla J = 0$. Deduce that ∇ preserves the decomposition $T\Sigma \otimes \mathbb{C} = T^{1,0}\Sigma \oplus T^{0,1}\Sigma$ and therefore induces a connection on both complex line bundles $T^{1,0}\Sigma$ and $T^{0,1}\Sigma$ (by duality this is true also for $\Omega^1(\mathbb{C}) = \Omega^{1,0} \oplus \Omega^{0,1}$).

Show that $E = \mathbb{C} + \Omega^{0,1}$ is a Clifford module, and that $S = E \otimes L$ is a spinor bundle for Σ if and only if $L^2 \simeq K_\Sigma$ (such L always exist).

Prove that the Dirac operator on E is

$$D = \sqrt{2}(\bar{\partial} + \bar{\partial}^*).$$

Prove that this formula extends to any Clifford module $E \otimes W$, where W is a complex line bundle with unitary connection, so that we can define $\bar{\partial} : \Gamma(W) \to \Omega^{0,1}(W)$ by taking $\bar{\partial}s$ be the (0,1)-part of ∇s.

(This formula extends to higher dimensional complex *Kähler* manifolds, it relates the Dirac operator and the complex geometry, and has important consequences for the topology of algebraic varieties).