Topological Quantum Field Theory

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Combined bachelor's thesis mathematics and physics

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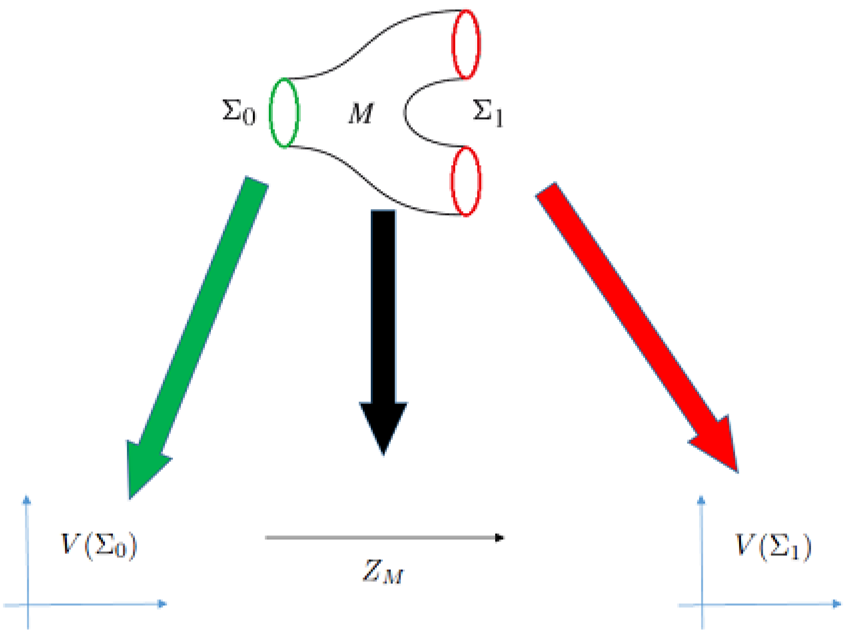

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Abstract

This thesis aims to provide an overview of the concepts needed to comprehend the simplest examples of Topological Quantum Field Theories (TQFT's). After briefly covering the essential mathematical preliminaries, the category **nCob** of n-dimensional cobordisms is described in detail, leading to the definition of a TQFT as a functor $\mathbf{nCob} \to \mathbf{Vect}_K$ that satisfies some additional properties. Next, the focus is turned to the definition of G-coverings and a description will be given of two ways to classify G-coverings over a given base space X. The first one uses the concept of Čech cocycles to classify G-coverings that are trivial over a specified open covering of X. The classifying space of a group, denoted BG, is introduced in order to look at a second way of classifying G-coverings, by considering homotopy classes of maps $X \to BG$. This classification uses Čech cocycles to construct a map $X \to BG$ from a G-covering. After covering all of these mathematical structures, Dijkgraaf-Witten theory is introduced. First, the focus is layed on the untwisted version, which heuristically counts the number of G-covering over a manifold with a weight corresponding to the number of automorphisms of the covering. The twisted version of Dijkgraaf-Witten theory generalizes this concept by including a second weight factor coming from a fixed cohomology class $\alpha \in H^n(BG, \mathbb{T})$. The untwisted version then corresponds to the case where α is trivial. Finally, Dijkgraaf-Witten theory is discussed within the framework of Quantum Field Theory (QFT). Both the untwisted and twisted version are covered from this point of view, checking their compatibility with the axioms of QFT.

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Contents

1 Introduction

"That is easy, in one sentence, at long distance Topological Quantum Field Theory is the relevant approximation, and why it's so important for, for instance, condensed matter physics." - Prof. Greg Moore on the importance of Topological Quantum Field Theory

Geometry plays a large role in theoretical physics. A particular area where geometry is heavily used is, for example, General Relativity. When describing global properties of physical systems, the topology of the systems involved becomes important. One remarkable interplay between physics and topology is described by Topological Quantum Field Theories (TQFT's) [A88]. The concept of TQFT was mathematically formalized in [A88] with the formulation of a set of axioms. Intuitively, a TQFT assigns a vector space $\mathcal{A}\Sigma$ to each $(n-1)$-dimensional space Σ and a linear map $\mathcal{A}\Sigma_0 \to \mathcal{A}\Sigma_1$ to each n-dimensional space-time that interpolates between the spaces Σ_0 and Σ_1. TQFT's are of interest to both mathematicians and physicists. In mathematics, they are studied because they assign topological invariants to manifolds of certain dimensions [K03]. An example of a mathematical theorem regarding TQFT's is given in [K03], where 2-dimensional TQFT's are related to Frobenius algebras. The physical interst in TQFT comes mainly from the fact that it is simple enough to do calculations and gives insight into more complicated quantum field theories (QFT's). At low energies, QFT's can be approximated by TQFT's, thus making computations easier (prof. Greg Moore, personal communication, June 7th 2017). This is the main application of TQFT to, amongst others, condensed matter physics (see for example [KTu15]). Another area of physics where TQFT plays a role is quantum computation. The importance of TQFT to quantum computation is described in [Na08].

1.1 Objective

The goal of this thesis is to provide an introduction to the concepts that are needed to understand TQFT and to formulate simple examples. Also, the analogy with operators and probabilities from quantum physics will be discussed. The example of a TQFT that will be covered is Dijkgraaf-Witten theory. This "toy-model" of a TQFT was first introduced in [DW90] and is widely accepted as one of the simplest TQFT's that still gives insight into the way the theory works (see for example [FQ93]). Dijkgraaf-Witten theory is a simplification of the more involved Chern-Simons theory. Whereas Chern-Simons theory uses bundles over a Lie Group, Dijkgraaf-Witten theory considers only finite groups. This reduces the complicated path integrals to finite sums, thus eliminating most of the analytical difficulty, while preserving the algebraic properties.

While writing this thesis it became clear that there is a lot of literature available on the concept of TQFT as well as on different approaches to Dijkgraaf-Witten theory (see for example [Mü07], [Fr92], [FQ93] and [Q91]). However, most of the literature either assumes extensive preknowledge or skips important details, leaving the reader in the dark about some of the necessary steps. Therefore, it is my intention that this thesis will provide a detailed exposition of the theory, that is accessible for mathematics students in the final phase of their undergraduate degree.

1.2 Physical relevance

As stated in the previous section, Dijkgraaf-Witten (DW) theory is useful as a TQFT because of its analytical simplicity. However, DW theory also has actual physical significance, coming from condensed matter physics. The physics behind phases of matter and phase transitions all comes from the underlying symmetries of the material [W16]. The so-called Landau-Ginzburg symmetry breaking formalism describes the physics of these symmetries and stands at the basis of condensed matter physics [HZK17]. Symmetries are described by a group G and phases of matter can be described at low-energy by a TQFT with symmetry group G. Thus, to understand these phases of matter, an understanding of such TQFT's is needed. In the case where G is finite, DW theory describes all possible TQFT's with symmetry group G. As will be seen in chapter 5, DW theories are classified by cocycles in $H^n(BG;\mathbb{T})$.[1] This implies that these special kind of phases are in bijection with the cohomology of the space BG [HZK17]. Unfortunately though, it turns out that this classification does not give the full picture [KTh15]. However, DW theory still provides a lot of insight into the physics governing the phases of matter of materials with finite symmetry group.

1.3 Overview

This thesis is structured as follows. Chapter 2 starts by briefly covering the concepts required to understand this thesis. The relevant constructions and theorems from differential and algebraic topology are stated and references are given to readers that wish to read a more detailed discussion of these concepts. This chapter is primarily intended for readers that have a basic knowledge of differential and algebraic topology, but who might not have an active knowledge of all the concepts needed. Readers with extensive knowledge of these subjects can choose to skip this chapter, while readers with very little understanding of the topics are adviced to turn to the references given at the start of the chapter. In chapter 3, first the concept of cobordisms is introduced. A detailed description is given of the category **nCob** that has closed oriented $(n-1)$-manifolds as its objects and equivalence classes of cobordisms as its morphisms. The main challenge in the definition of this category lies in the description of the composition of two cobordisms and the proof that this is well-defined. The chapter continues with the definition

[1] The notation BG for the classifying space of G will be introduced in chapter 4.

of a TQFT as a functor $\mathbf{nCob} \to \mathbf{Vect}_K$ satisfying two aditional properties, where $\mathbf{Vect}_K$ is the category of vector spaces over some ground field K. Finally, chapter 3 makes a connection between the language of quantum mechanics and that of TQFT. In order to comprehend Dijkraaf-Witten theory, one must be familiar with the concept of G-coverings. Therefore, chapter 4 starts by introducing the concept of G-coverings for a discrete group G. Afterwards, this chapter will cover two ways of classifying all G-coverings over a given base space up to isomorphisms. Using the concept of Čech cocycles a bijection is constructed from the set of isomorphism classes of G-coverings that are trivial over an open covering of the base space to the set of cohomology classes of Čech cocycles over the open covering. Next, the classifying space of G, denoted BG, is constructed using the geometric realization of (semi)simplicial sets. Using this space and the Čech cocycle classification, the homotopy classification of G-coverings is described. This construction associates a G-covering to every homotopy class of maps from the base space to BG using the so-called pull back. After having covered all of these concepts, chapter 5 turns to Dijkgraaf-Witten theory. This theory has a simple version, called untwisted Dijkgraaf-Witten theory, and a general version, called twisted Dijkgraaf-Witten theory. First, the untwisted version is covered, which assigns the space $\mathbb{C}[\Sigma, BG]$ to every closed $(n-1)$-manifold Σ. Then linear maps between two such vector spaces induced by n-manifolds M are constructed by counting the number of G-coverings over M times the inverse of the size of their automorphism group. A detailed discussion is given of the proof that this defines a TQFT. Following this paragraph is the explanation of twisted Dijkgraaf-Witten theory. This version associates an extra weight factor to every G-covering corresponding to a fixed cohomology class $\alpha \in H^n(BG; \mathbb{T})$, where $\mathbb{T} \subseteq \mathbb{C}^*$ is the group consisitng of all elements of $\mathbb{C}^*$ with unit norm. This cohomology class also alters the vector spaces that are assigned to closed $(n-1)$-manifolds. The final chapter of this thesis again covers Dijkgraaf-Witten theory, but this time as a QFT. First, this chapter covers the axioms of QFT. Then untwisted and twisted Dijkgraaf-Witten theory are treated within this framework, checking their compatibility with these axioms.

1.4 Acknowledgement

First of all, I want to thank dr. Hessel Posthuma and dr. Miranda Cheng for their supervision during the process of writing this thesis. A thank you also goes to dr. Chris Zaal for pushing me to arrange everything in time before going on exchange. Finally, I would like to thank my family for supporting me and reviewing my presentations, even though they probably did not understand a word of what I said.

2 Preliminaries

This chapter will briefly cover some of the concepts that will be used in the rest of this thesis. For a more detailed introduction to differential topology, see [Le03], and for algebraic topology, see [Br93].

2.1 Differential Topology

Recall that an n-dimensional topological manifold is a second countable Hausdorff space that is locally homeomorphic to $\mathbb{R}^n$. In this thesis the coordinate maps are assumed to be homeomorphisms onto $\mathbb{R}^n$, in contrast to the usual convention of using homeomorphisms onto open subsets of $\mathbb{R}^n$. However, these definitions are equivalent and our approach will avoid some of the technical details in the next chapter. A (smooth) manifold is a topological manifold where all coordinate charts are smoothly compatible. If M is an orientable manifold, then an orientation for M is a smooth choice of positive bases for the tangent spaces. When a map between oriented manifolds takes positive bases to positive bases, it is called *orientation preserving*.

For an oriented manifold with boundary, two types of boundary components can be distinguished: in-boundaries and out-boundaries [K03]. They are defined as follows.

Definition 2.1. *Let M be a manifold and $\iota : \Sigma \to M$ an orientation preserving diffeomorphism from an oriented manifold Σ onto a disjoint union of components of the boundary of M. For $x \in \iota(\Sigma)$ a* positive normal *is a vector $w \in T_xM$, such that $[v_1, \ldots, v_{n-1}, w]$ is a positive basis for T_xM, where $[v_1, \ldots, v_{n-1}]$ is a positive basis for $T_x\iota(\Sigma)$. The vector w can now locally be seen as a vector in $\mathbb{R}^n$ that is either pointing in or out of the halfspace $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n \geq 0\}$. If it points inwards for all $x \in \iota(\Sigma)$ then $\iota(\Sigma)$ is called an* in-boundary *of M and if it points outwards for all such x then $\iota(\Sigma)$ is called an* out-boundary*.*

With this definition, the boundary of a manifold can be divided into a disjoint union of in-boundaries and out-boundaries.

2.1.1 Morse functions

In the next chapter, a few results from Morse theory will be used. This subsection, which is based on [K03], will introduce the necessary vocabulary.

Definition 2.2. *For a smooth map $f : M \to I$ from a manifold to the unit interval, a point $x \in M$ is called a* critical point *if $df_x = 0$. If x is not a critical point, it is called a* regular point. *The images of critical and regular points under f are called critical and regular values respectively.*

For a critical point $x \in M$ one could look at the Hessian matrix $(\frac{\partial^2 f}{\partial x^i \partial x^j})_{i,j}$ in some chosen coordinate system. If this matrix is invertible, then x is called *nondegenerate.*

Definition 2.3. *A smooth map $f : M \to I$ is called a* Morse function *if all of its critical points are nondegenerate. Also it is assumed that $f^{-1}(\partial I) = \partial M$ and that $0, 1 \in I$ are regular values.*

When M is compact, it can be shown that a Morse function $M \to I$ has finitely many critical points. In particular, we can therefore find an $\epsilon > 0$ such that the intervals $[0, \epsilon]$ and $[1 - \epsilon, 1]$ contain only regular values. The most important result about Morse functions is that they exist for every manifold. This will be used in the construction of the composition of two cobordisms in the next chapter.

2.2 Algebraic topology

This section is mostly based on [Br93]. A map $Y \to X$ between path connected Hausdorff spaces is called a *covering map* if each point in X has an open neigbourhood that is evenly covered. The fiber over $x \in X$ of this covering is denoted Y_x. An important property of coverings is that homotopies can be lifted over them, as the following theorem states.

Theorem 2.4. *Let $p : Y \to X$ be a covering and $H : W \times I \to X$ a homotopy, where W is a locally connected space. Suppose there is a lift $h : W \to Y$ of $H_0 := H|_{W\times\{0\}}$. Then we can find a unique $\tilde{H} : W \times I \to Y$ such that $\tilde{H}_0 = h$ and $p \circ \tilde{H} = H$.*

An important topological invariant of a space is its fundamental group. This concept can be generalized to arbitrary homotopy groups. The n-th homotopy group of a pointed space (X, x_0) is defined as

$$\pi_n(X, x_0) = [(S^n, *), (X, x_0)],$$

where $* \in S^n$ is any point and the brackets denote equivalence classes of maps $S^n \to X$ under homotopy relative the basepoint. For $n = 1$, this is just the fundamental group and for $n = 0$, it consists of the path components of X.

2.2.1 (Co)homology groups

Other topological invariants are the homology and cohomology of a space. Recall that a singular n-simplex of a space X is a map $\Delta_n \to X$, where $\Delta_n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n x_i = 1, x_i \geq 0 \text{ for all } i\}$ is the standard n-simplex. The free abelian group on the singular n-simplices of X is called the singular n-chain group $S_n(X)$. These groups define a chain complex with a boundary map ∂ that maps a singular simplex to the alternating sum of its faces. The homology of this complex is called the singular homology of X and is denoted as $H_*(X)$. In other words:

$$H_n(X) = \frac{\ker(\partial : S_n(X) \to S_{n-1}(X))}{\mathrm{im}(\partial : S_{n+1}(X) \to S_n(X))}.$$

To define cohomology, first fix an abelian group F and define $S^n(X; F) := \mathrm{Hom}(S_n(X), F)$, where Hom denotes the group of group homomorphisms. Now the boundary map $\partial : S_n(X) \to S_{n-1}(X)$ induces a map $\delta : S^{n-1}(X; F) \to S^n(X; F)$ by sending a group homomorphism f to $f \circ \partial$. This makes $S^*(X; F)$ into a cochain complex, whose cohomology $H^*(X; F)$ is called the singular cohomology of X with coefficients in F. So

$$H^n(X; F) = \frac{\ker\left(\delta : S^n(X; F) \to S^{n+1}(X; F)\right)}{\mathrm{im}(\delta : S^{n-1}(X; F) \to S^n(X; F))}.$$

2.2.2 Homology of manifolds

Now we turn to the calculation of the homology of an n-dimensional manifold. See [Ma99] as a reference. Let M be an orientable n-manifold without boundary and take a point $x \in M$. By definition there is an open subset $U \subseteq M$ with $x \in U$ and $U \cong \mathbb{R}^n$. Since $\overline{M \backslash U} \subseteq \mathrm{int}(M \backslash \{x\})$ excision gives an isomorphism $H_i(M, M\backslash\{x\}) \cong H_i(U, U\backslash\{x\})$ for all i. Then the long exact sequence for the pair $(U, U\backslash\{x\})$ and the fact that the reduced homology of a point is trivial induces an isomorphism $H_i(U, U\backslash\{x\}) \cong \tilde{H}_{i-1}(U\backslash\{x\})$. Finally by homotopy invariance this is again isomorphic to $\tilde{H}_{i-1}(S^{n-1})$, which is $\mathbb{Z}$ for $i = n$ and trivial otherwise. So $H_n(M, M\backslash\{x\})$ is generated by a single element.[1]

Definition 2.5. *If $[M] \in H_n(M)$ is an element such that $(\iota_x)_*([M])$ generates $H_n(M, M\backslash\{x\})$ for all $x \in M$, then $[M]$ is called a fundamental class of M. Here, ι_x is the inclusion $M \to (M, M\backslash\{x\})$.*

Note that since $\mathbb{Z}$ has two generators (plus and minus one), there are two choices for a fundamental class of M (provided that M is orientable). These correpond to the two orientations on M.

[1]For an explanation of the axioms of homology, including exactness, excision and homotopy invariance, see [Br93].

3 Axioms for TQFT

This chapter will introduce the basic concepts needed to describe a toplogical quantum field theory (TQFT). First, the notion of cobordisms is introduced and the category of cobordisms **nCob** is defined. The main part of this definition lies in the composition of two cobordisms. Afterwards, we define a TQFT as a functor from **nCob** to $\mathbf{Vect}_K$. Finally, the relationship between TQFT and quantum mechanics is discussed. Most of this chapter is based on chapter 1 from [K03].

3.1 The category nCob

Recall that to describe a category, we need to define its objects and morphisms, along with the composition of morphisms and the identity morphisms. As the objects of **nCob** we take the closed oriented $(n-1)$-manifolds. Note that ´closed´ in this context means compact without boundary. To define the morphisms, we first need the following definition.

Definition 3.1. *Let Σ_0 and Σ_1 be two objects in* **nCob** *and M a compact oriented n-manifold. If $\iota_j : \Sigma_j \to M$ $(j = 0, 1)$ are two maps as in definition 2.1, the triple (M, ι_0, ι_1) is called an* (oriented) cobordism *from Σ_0 to Σ_1, provided that $\iota_0(\Sigma_0)$ is an in-boundary of M, $\iota_1(\Sigma_1)$ is an out-boundary of M and $\partial M = \iota_0(\Sigma_0) \coprod \iota_1(\Sigma_1)$.*

Intuitively a cobordism can be seen as a "smooth" way to transform system Σ_0 into Σ_1. On the class of cobordims the following relation is defined.

Definition 3.2. *For Σ_0, Σ_1 objects in* **nCob** *let $M, M' : \Sigma_0 \Rightarrow \Sigma_1$ be two cobordims. Denote the embeddings of Σ_j into M and M' by ι_j respectively ι'_j for $j = 0, 1$. Define $M \sim M'$ if there is an orientation preserving diffeomorphism $\psi : M \to M'$ such that the following diagram commutes.*

$$\begin{tikzcd} & M \arrow[dd, "\psi"] & \\ \Sigma_0 \arrow[ur, "\iota_0"] \arrow[dr, "\iota'_0"'] & & \Sigma_1 \arrow[ul, "\iota_1"'] \arrow[dl, "\iota'_1"] \\ & M' & \end{tikzcd} \tag{3.1}$$

Note that this condition says that ψ maps the in-boundary of M to the in-boundary of M' and similarly for the out-boundary.

It turns out that $\sim$ is an equivalence relation on the cobordisms from Σ_0 to Σ_1. Reflexivity and transitivity are not hard to prove. To see that it is symmetric, let M

and M' be two cobordisms from Σ_0 to Σ_1, such that $M \sim M'$. This means that there is an orientation preserving diffeomorphism $\psi : M \to M'$ that makes diagram 3.1 commute. Now $\psi^{-1} : M' \to M$ is a diffeomorphism as well. We prove that it is also orientation preserving. Take $p \in M'$ and $q = \psi^{-1}(p)$. For a positive basis $\mathcal{B}_p$ for T_pM' we want to prove that $(\psi^{-1})_*(\mathcal{B}_p)$ is a positive basis for T_qM. Assume that this is not the case. Then $\psi_*((\psi^{-1})_*(\mathcal{B}_p))$ is a negative basis for T_pM', since ψ preserves orientation. But $\psi_* \circ (\psi^{-1})_* = (\psi \circ \psi^{-1})_* = \mathrm{Id}_{T_pM'}$, so $\mathcal{B}_p$ is a negative basis for T_pM', which contradicts our assumption. So ψ^{-1} is orientation preserving. Also, for $j = 0, 1$

$$\psi^{-1} \circ \iota'_j = \psi^{-1} \circ \psi \circ \iota_j = \iota_j,$$

so ψ^{-1} maps the in-boundary of M' to the in-boundary of M and similarly for the out-boundary. So $M' \sim M$.

This equivalence relation induces equivalence classes of cobordisms from Σ_0 to Σ_1. The equivalence class of a cobordism M is denoted $[M]$. We define the class of morphisms $\mathrm{Hom}_{\mathbf{nCob}}(\Sigma_0, \Sigma_1,)$ as the class of these equivalence classes.

3.1.1 Composition

Given a morphism in $\mathrm{Hom}_{\mathbf{nCob}}(\Sigma', \Sigma,)$ and one in $\mathrm{Hom}_{\mathbf{nCob}}(\Sigma, \Sigma'',)$, the goal is to define a morphism in $\mathrm{Hom}_{\mathbf{nCob}}(\Sigma', \Sigma'',)$. That is, from two cobordisms we want to define a new manifold with a smooth structure that is unique up to diffeomorphisms. We start by defining the manifold.

The topological composition

Let $M_0 : \Sigma' \Rightarrow \Sigma$ and $M_1 : \Sigma \Rightarrow \Sigma''$ be two cobordisms with embeddings $\iota_0 : \Sigma \to M_0$ and $\iota_1 : \Sigma \to M_1$. On the disjoint union $M_0 \coprod M_1$, the following equivalence relation is defined: let every point be equivalent to itself and let $m_0 \in M_0$ be equivalent to $m_1 \in M_1$ if there is an $x \in \Sigma$ such that $\iota_0(x) = m_0$ and $\iota_1(x) = m_1$. This is clearly an equivalence relation and we denote the quotient space as $M_0 \coprod_\Sigma M_1$ (or as $M_0 \coprod_\Sigma^{\iota_0,\iota_1} M_1$ if we want to stress the embeddings used in the equivalence relation). On this space, we take the quotient topology induced by the projection map $\pi : M_0 \coprod M_1 \to M_0 \coprod_\Sigma M_1$. From now on, M_0 and M_1 will be seen as subspaces of $M_0 \coprod_\Sigma M_1$ where we identify these spaces with their image under π. We want to make $M_0 \coprod_\Sigma M_1$ into a topological manifold. For points that are not on the gluing edge $\iota_0(\Sigma) = \iota_1(\Sigma)$ (seen as subspaces of $M_0 \coprod_\Sigma M_1$) we already have charts. For a point p on this "edge", we have to find an open neighbourhood U and a homeomorphism with $\mathbb{R}^n$. Pick an open neighbourhood U of this point. By definition, the sets $U_0 := U \cap M_0$ and $U_1 := U \cap M_1$ are open in M_0 and M_1 respectively. Note that $U = U_0 \coprod_\Sigma U_1$. By shrinking U if necessary, we may assume that U_0 and U_1 are the domains of two coordinate maps $\psi_0 : U_0 \to \mathbb{R}^n_- = \{x \in \mathbb{R}^n : x_n \leq 0\}$ and $\psi_1 : U_1 \to \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n \geq 0\}$. Define $\Xi := U \cap \iota_0(\Sigma) = U_0 \cap U_1$. The maps $p_0 := \psi_0|_\Xi$ and $p_1 := \psi_1|_\Xi$ give embeddings of Ξ in $\mathbb{R}^n_\pm$, whose images are the boundary of these spaces. Therefore, we can define the space $\mathbb{R}^n_- \coprod_\Xi^{p_0,p_1} \mathbb{R}^n_+ \cong \mathbb{R}^n$. When $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$ are seen as subspaces of $\mathbb{R}^n$ under this identification, ψ_0 and ψ_1 can be regarded as maps to $\mathbb{R}^n$ that give homeomorphisms to their image. Because these images agree on Ξ by defintion, they induce a continuous map $\psi : U \to \mathbb{R}^n$. By the same reasoning, also ψ^{-1} is continuous and therefore ψ is a homeomorphism. Left to prove is that the structure found this way is unique. Suppose we had chosen different coordinate maps from U_0 and U_1, say χ_0 and χ_1. These can be glued together again to form a homeomorphism $\chi : U \to \mathbb{R}^n$. The transitionfunctions between the ψ_j's and the χ_j's are homeomorphisms $\alpha_0 : \mathbb{R}^n_- \to \mathbb{R}^n_-$ and $\alpha_1 : \mathbb{R}^n_+ \to \mathbb{R}^n_+$. Define for $j = 0, 1$ the embeddings $q_j : \alpha_j \circ p_j$. Then we can glue the images of the α_j's again to form $\mathbb{R}^n_- \coprod_{\Xi}^{q_0,q_1} \mathbb{R}^n_+ \cong \mathbb{R}^n$. Now we get a continuous map $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ that is precisely the transitionfunction between ψ and χ. So both coordinate maps belong to the same maximal atlas, which is therefore uniquely defined.

Smooth structure

Next we want to assign a smooth structure to the topological manifold formed above. The next theorem shows that when we find a smooth structure, it is unique up to diffeomorphisms. This theorem will not be proven here. See the references given by [K03] for more details.

Theorem 3.3 ([K03]). *Let Σ be an out-boundary of M_0 and an in-boundary of M_1. If α and β are two smooth structures on $M_0 \coprod_\Sigma M_1$, such that for every coordinate chart (U, φ) in α or β the coordinate charts $((\pi|_{M_j})^{-1}(U), \varphi \circ \pi|_{M_j})$ are in the smooth structure on M_j for $j = 0, 1$, then there is a diffeomorphism $\phi : (M_0 \coprod_\Sigma M_1, \alpha) \to (M_0 \coprod_\Sigma M_1, \beta)$ such that $\phi|_\Sigma = \mathrm{Id}_\Sigma$. Here, $\pi : M_0 \coprod M_1 \to M_0 \coprod_\Sigma M_1$ is again the quotient map.*

First, we will discuss how to glue together cylinders. Take two cobordisms $M_0 : \Sigma_0 \Rightarrow \Sigma_1$ and $M_1 : \Sigma_1 \Rightarrow \Sigma_2$, such that $[M_0] = [\Sigma_1 \times [0, 1]]$ and $[M_1] = [\Sigma_1 \times [1, 2]]$. Define $S := \Sigma_1 \times [0, 2]$. The standard smooth structure on S matches the structures on $\Sigma_1 \times [0, 1]$ and $\Sigma_1 \times [1, 2]$. Let

$$\begin{aligned} i_0 &: \Sigma_1 \times [0, 1] \to \Sigma_1 \times [0, 2] \\ i_1 &: \Sigma_1 \times [1, 2] \to \Sigma_1 \times [0, 2] \end{aligned}$$

be the inclusions and

$$\begin{aligned} \alpha_0 &: M_0 \to \Sigma_1 \times [0, 1] \\ \alpha_1 &: M_1 \to \Sigma_1 \times [1, 2] \end{aligned}$$

be orientation preserving diffeomorphisms. Define

$$\alpha : M_0 \coprod\nolimits_{\Sigma_1} M_1 \to S$$
$$\alpha(x) = \begin{cases} i_0 \circ \alpha_0(x) & x \in M_0 \\ i_1 \circ \alpha_1(x) & x \in M_1. \end{cases}$$

This clearly defines a homeomorphism. The restrictions $\alpha|_{M_0}$ and α_{M_1} are orientation preserving diffeomorphisms. Define a smooth structure on $M_0 \coprod_{\Sigma_1} M_1$ by pulling back the smooth structure on S through α. This gives the composition of the two cylinders.

The next theorem, coming from Morse theory, gives a criterium for when a cobordism is diffeomorphic to a cylinder.

Theorem 3.4 ([K03]). *Let $M : \Sigma_0 \Rightarrow \Sigma_1$ be a cobordism and $f : M \to [0,1]$ a smooth map without critical points. Suppose that $f^{-1}(\{0\}) = \Sigma_0$ and $f^{-1}(\{1\}) = \Sigma_1$. Then there is a diffeomorphism $\phi : \Sigma_0 \times [0,1] \to M$ such that $f \circ \phi = \pi_2$, where $\pi_2 : \Sigma_0 \times [0,1] \to [0,1]$ is the projection on the second coordinate.*

With the tools developed so far, we are now able to glue together arbitrary manifolds. Take two cobordisms $M_0 : \Sigma_0 \Rightarrow \Sigma_1$ and $M_1 : \Sigma_1 \Rightarrow \Sigma_2$. Then there exist two Morse functions $f_0 : M_0 \to [0,1]$ and $f_1 : M_1 \to [1,2]$ that can be glued together to a function $f : M_0 \coprod_{\Sigma_1} M_1 \to [0,2]$. Choose $\epsilon > 0$ small enough such that the intervals $[1-\epsilon, 1]$ and $[1, 1+\epsilon]$ are regular for f_0 and f_1 respectively. Then using theorem 3.4, it can be seen that $N_0 := f_0^{-1}([1-\epsilon, 1])$ and $N_1 := f_1^{-1}([1, 1+\epsilon])$ are diffeomorphic to cylinders. Let

$$\begin{aligned} \alpha_0 &: N_0 \to \Sigma_1 \times [0,1] \\ \alpha_1 &: N_1 \to \Sigma_1 \times [1,2] \end{aligned}$$

be the corresponding diffeomorphisms. Because we know already how to glue together cylinders, a homeomorphism $\alpha : N_0 \coprod_{\Sigma_1} N_1 \to S := \Sigma_1 \times [0,2]$ is found, such that $\alpha|_{N_0}$ and α_{N_1} are diffeomorphisms. Define an atlas on $M_0 \coprod_{\Sigma_1} M_1$ as follows:

- For every chart (U, ψ) in M_j let (U, ψ) be a chart in $M_0 \coprod_{\Sigma_1} M_1$ (where we regard $U \subseteq M_j$ as a subset of $M_0 \coprod_{\Sigma_1} M_1$);
- For every chart (V, Ψ) in S let $(\alpha^{-1}(V), \Psi \circ \alpha)$ be a chart in $M_0 \coprod_{\Sigma_1} M_1$.

We now have to prove that the transition maps between these charts are diffeomorphisms. Suppose that (U, ψ) is a chart in M_j and (V, Ψ) in S, such that $W := \alpha^{-1}(V) \cap U \neq \varnothing$. Then on W

$$\begin{aligned} \psi \circ (\Psi \circ \alpha)^{-1} &= \psi \circ \alpha^{-1} \circ \Psi^{-1} \\ &= \psi \circ (\alpha|_{N_j})^{-1} \circ \Psi^{-1} \end{aligned}$$

is a diffeomorphism, since ψ, Ψ and $\alpha|_{N_j}$ are diffeomorphisms. So this atlas is well defined and gives a smooth structure on $M_0 \coprod_{\Sigma_1} M_1$. The resulting cobordism is denoted $M_0 M_1$.[1] On this cobordism we choose the unique orientation that makes this into a cobordism from Σ_0 to Σ_2 and makes the embeddings $\Sigma_j \to M_0 M_1$ (for $j = 0, 2$) into orientation preserving maps. By theorem 3.3, the structure found this way is unique up

[1] Note that the smooth structure depends on the maps f_0 and f_1. Therefore, a better notation would be $M_0 \cdot^{f_0 f_1} M_1$. However, we will only be interested in the equivalence class of this cobordism, which does not depend on these maps, according to theorem 3.3.

to diffeomorphisms. The corresponding diffeomorphisms clearly have to preserve orientation and map in-boundaries to in-boundaries and out-boundaries to out-boundaries. Therefore the equivalence class $[M_0M_1]$ is uniquely defined in $\mathrm{Hom}_{\mathbf{nCob}}(\Sigma_0, \Sigma_2,)$. Starting with two morphisms $[M_0]$ and $[M_1]$ the composition that we write as $[M_0][M_1] = [M_0M_1]$ is hereby well defined, since it does not depend on the chosen representative.

Associativity

The last thing left to check is that the defined composition is associative. Let $M_0 : \Sigma_0 \Rightarrow \Sigma_1$, $M_1 : \Sigma_1 \Rightarrow \Sigma_2$ and $M_2 : \Sigma_2 \Rightarrow \Sigma_3$ be three cobordisms. First note that as topological spaces, the equality $(M_0 \coprod_{\Sigma_1} M_1) \coprod_{\Sigma_2} M_2 = M_0 \coprod_{\Sigma_1}(M_1 \coprod_{\Sigma_2} M_2)$ applies, when regarding the original spaces as subsets of the formed quotientspace like discussed earlier. Also, observe that we can choose the ϵ's in the definition of the composition small enough such that the corresponding "cylinders" are disjoint. Therefore, the two compositions do not affect each other. These comments can be combined to see that $([M_0][M_1])[M_2] = [M_0]([M_1][M_2])$.

3.1.2 Identity morphisms

The last thing to do in order to complete the description of the category of cobordisms, is to show that every object has an identity morphism. Let Σ be a closed oriented $(n-1)$-manifold. We claim that the morphism $[C]$ is the identity on Σ, where $C := \Sigma \times [0, 1]$. To show this, take a cobordism $M : \Sigma \Rightarrow \Sigma'$. As mentioned in the contruction of the composition of two cobordisms, M can be seen as the composition of $M_{[0,\epsilon]}$ and $M_{[\epsilon,1]}$, where $M_{[0,\epsilon]}$ is diffeomorphic to a cylinder over Σ. Two cylinders can clearly be glued together to form another cylinder, so there is an orientation preserving diffeomorphism $CM_{[0,\epsilon]} \to M_{[0,\epsilon]}$. So now

$$\begin{aligned} [C]\,[M] &= [C]\left(\left[M_{[0,\epsilon]}\right]\,\left[M_{[\epsilon,1]}\right]\right) \\ &= \left([C]\,\left[M_{[0,\epsilon]}\right]\right)\,\left[M_{[\epsilon,1]}\right] \\ &= \left[M_{[0,\epsilon]}\right]\,\left[M_{[\epsilon,1]}\right] \\ &= [M]\,. \end{aligned}$$

This proves that $[C]$ is a left unit for Σ. A similar proof shows that it is also a right unit. In conclusion, $[C]$ is the identity on Σ.

3.2 Definition of a TQFT

Now that all the relevant concepts are introduced, we can finally define what a TQFT is. Let K be some fixed ground field.

Definition 3.5. *An* n-dimensional TQFT *is a functor* $\mathcal{A}$ *from* **nCob** *to the category* $\mathbf{Vect}_K$ *of vector spaces over* K *that satisfies the following two conditions:*

1. *If Σ and Σ' are two objects in* **nCob**, *then* $\mathcal{A}(\Sigma \coprod \Sigma') \cong \mathcal{A}(\Sigma) \otimes_K \mathcal{A}(\Sigma')$.

2. $\mathcal{A}(\varnothing) = K$, *where $\varnothing$ is the empty $(n-1)$-manifold.*

Note that the second condition implies that the empty cobordism gets sent to Id_K, *since the empty cobordism is the cylinder over the empty manifold.*

Mathematically the two conditions reflect the monoidal structure of the two categories involved. To be precise, $\mathcal{A}$ is a symmetric monoidal functor (see chapter 3 from [K03]). To give some physical intuition, a TQFT can be seen as a rule that assigns to a physical system its state space and to a time evolution between two systems a linear map between the corresponding state spaces. The first condition in the defintion then corresponds to the fact that in quantum mechanics the state space of two independent systems is the tensor product between the state spaces of the seperate systems.

3.3 Relationship with quantum mechanics

This section will explain the relationship between TQFT and quantum mechanics (QM). In QM, a physical system is associated to its space of states. Operators on the system correspond to linear transformations of the state space. By computing inner products, probabilities can be calculated that measurements give certain outcomes. We will discuss how these operators and probabilities can be seen in the context of TQFT. This section is based on [Ba95].

The following notation for a TQFT will be used: a boundary Σ gets mapped to a vector space $V(\Sigma)$ and a cobordism M to a linear map Z_M. Note that a cobordism $M : \varnothing \Rightarrow \partial M$ corresponds to a state $Z_M \in V(\partial M)$, since the map $Z_M : \mathbb{C} \to V(\partial M)$ is completely determined by the image of 1. Denoting $\overline{\Sigma}$ for the manifold Σ with opposite orientation, there is a canonical isomorphism $V(\overline{\Sigma}) \cong V(\Sigma)^*$, where $V(\Sigma)^*$ is the dual space of $V(\Sigma)$ consisting of the linear functionals $V(\Sigma) \to \mathbb{C}$ [DW90]. This section will assume the existence of a linear map $a_\Sigma : V(\Sigma) \to V(\overline{\Sigma})$ for all boundaries Σ, corresponding to the inner product $< \sigma, \tau > := a_\Sigma(\sigma)(\tau)$, with $\sigma, \tau \in V(\Sigma)$. We will assume that $a_{\partial M}(Z_M) = Z_{\overline{M}}$. Also, for closed M, we will assume $a(Z_M) = Z_{\overline{M}} = \overline{Z_M} \in \mathbb{C}$, where the last bar denotes complex conjugation in $\mathbb{C}$. Throughout this chapter, the ¨bra-ket" notation from QM will be used:

$$
\begin{aligned}
|M\rangle &:= Z_M \in V(\partial M) \\
\langle M| &:= Z_{\overline{M}} \in V(\overline{\partial M}).
\end{aligned}
$$

Then if $\partial N = \partial M$ the inner producted between Z_M and Z_N can be denoted $\langle M|N\rangle = Z_{\overline{M}}(Z_N) = a(Z_M)(Z_N) =< Z_M, Z_N >$.

3.3.1 Operators and probabilities

As mentioned at the end of section 3.2, a boundary Σ can be identified with a physical system and the vector space $V(\Sigma)$ with the space of possible states of the system.

A manifold C with in-boundary Σ and out-boundary Σ corresponds to a linear map $V(\Sigma) \to V(\Sigma)$. If Z_C happens to be corresponding to an operator on the system Σ, we can look at the result of applying the operator C to the state $Z_M \in V(\Sigma)$, where $\partial M = \Sigma$. This is done by gluing M and Σ to get the manifold $M \coprod_\Sigma C$ corresponding to a state in $V(\Sigma)$, which can be denoted $C\,|M\rangle$. Now if there is another operator D on Σ, then the manifold $M \coprod_\Sigma C \coprod_\Sigma D$ can be constructed. Here, the order of gluing reflects the time order in which the operators are applied to the system. If operator C is applied first and then operator D is applied, we must glue the in-boundary of C to M and the in-boundary of D to the out-boundary of C.

If $\pi : \Sigma \Rightarrow \Sigma$ is an operator corresponding to an orthogonal projection in $V(\Sigma)$, then in QM π is associated to the proposition of a measurement resulting in a certain outcome. Here $\pi\,|M\rangle$ asserts the desired outcome and $(1-\pi)\,|M\rangle$ asserts the opposite. We can look at the probabilities of this happening:

$$\begin{aligned} p_{\text{yes}} &= \frac{\langle M|\pi|M\rangle}{\langle M|M\rangle} \\ p_{\text{no}} &= \frac{\langle M|1-\pi|M\rangle}{\langle M|M\rangle}, \end{aligned}$$

where of course $p_{\text{yes}} + p_{\text{no}} = 1$. If there are two such projections π_1, π_2, they can be combined to give:

$$\begin{aligned} p_{\text{yes,yes}} &= \frac{\langle M|\pi_1\pi_2\pi_1|M\rangle}{\langle M|M\rangle} \\ p_{\text{yes,no}} &= \frac{\langle M|\pi_1(1-\pi_2)\pi_1|M\rangle}{\langle M|M\rangle} \\ p_{\text{no,yes}} &= \frac{\langle M|(1-\pi_1)\pi_2(1-\pi_1)|M\rangle}{\langle M|M\rangle} \\ p_{\text{no,no}} &= \frac{\langle M|(1-\pi_1)(1-\pi_2)(1-\pi_1)|M\rangle}{\langle M|M\rangle}. \end{aligned}$$

In conclusion, we have shown that many of the fundamental concepts of QM can be translated to the language of TQFT.

4 G-coverings and classification

In this chapter, the notion of G-coverings will be introduced. In sections 4.2 and 4.3, two ways to classify the G-coverings over a certain space will be described. Both classifications will be of importance in the next chapter to understand Dijkgraaf-Witten theory. In this chapter, the topological group G is always assumed to be equipped with the discrete topology.

4.1 G-coverings and isomorphisms

The following section is based on chapters 11 through 16 from [Fu95]. The informal definition of a G-covering is a covering space that arises from a group action on a topological space. Recall that a (left) group action from G on a topological space Y is a map $G \times Y \to Y$ denoted $(g, y) \mapsto g \cdot y$, that satisfies:

- Associativity: $g \cdot (h \cdot y) = (gh) \cdot y$ for all $g, h \in G$ and $y \in Y$;
- Neutral element: $e \cdot y = y$ for all $y \in Y$, where e denotes the identity element of G.

Furthermore, we always assume that for each $g \in G$ the map $y \mapsto g \cdot y$ is a homeomorphism of Y. This implies that if U is open in Y, then so is gU. When such a group action is defined, the set Y/G denotes the set of orbits with the quotient topology coming from the projection map $Y \to Y/G$. In order for this map to be a covering map, we have to put some extra condition on the group action. Therefore, we make the following definition.

Definition 4.1. *We call a group action* properly discontinuous *if every $y \in Y$ has a neighbourhood V such that $g \cdot V \cap h \cdot V = \varnothing$ for all $g \neq h$ in G.*

It turns out that this condition is sufficient to make the projection map a covering map as stated in the next lemma.

Lemma 4.2. *If the group action of G on Y is properly discontinuous, then the projection $p : Y \to Y/G$ is a covering map.*

Proof. By definition, p is continous. It is also open, since for an open $U \subseteq Y$ the set $p^{-1} \circ p(U) = \bigcup_{g \in G} gU$ is the union of open sets and is therefore itself open. By definition of the quotient topology, this means that $p(U)$ is open which then implies that the map p is in fact open. Left to prove is that each point $\bar{y} \in Y/G$ has an open neighbourhood that is evenly covered by p. If we take $y \in Y$ such that $p(y) = \bar{y}$, y has an open neighbourhood V that satisfies the condition in definition 4.1. Define $\bar{V} := p(V)$. This is an open neighbourhood of $\bar{y}$ since p is open. We want to show that every set in this disjoint union $p^{-1}(\bar{V}) = \coprod_{g \in G} gV$ is mapped homeomorphically to $\bar{V}$ by p. It is enough to show that $p|_{gV} : gV \to \bar{V}$ is a bijection for every $g \in G$, since p is open and continuous. Suppose that $p(gy_1) = p(gy_2)$ for some $y_1, y_2 \in V$. Then there is an $h \in G$ such that $hgy_1 = gy_2$, so this element lies in $hgV \cap gV$. Since the group action is properly discontinuous, this can only be true if $h = e$ and therefore $gy_1 = gy_2$. This proves the injectivity. The surjectivity follows from the fact that $\bar{V} = p(V) = p(gV)$. □

Now that we know that properly discontinuous group actions induce covering maps, it makes sense to consider coverings that arise in this way.

Definition 4.3. *i) A G-*covering *is a covering $p : Y \to X$ together with a properly discontinuous action of G on Y, such that there is a homeomorphism $\psi : X \to Y/G$ that makes the following diagram commute:*

$$\begin{tikzcd} Y \arrow[d, "p"'] \arrow[dr, "\pi"] & \\ X \arrow[r, "\psi"'] & Y/G. \end{tikzcd}$$

Here, π is just the projection of Y onto its orbits. A G-covering is often denoted as a triple (Y, p, X).

ii) Two G-coverings $p : Y \to X$ and $p' : Y' \to X$ over the same base space are called isomorphic *if there is a homeomorphism $\phi : Y \to Y'$ such that $p' \circ \phi = p$ and $\phi(gy) = g\phi(y)$ for all $y \in Y$ and $g \in G$. In this case ϕ is called an* isomorphism of G-coverings.

iii) The category $\mathcal{G}(X)$ is defined to have G-coverings over X as its objects and isomorphisms of G-coverings as morphisms.

For a given base space X, there is a natural way to create a G-covering by setting $Y = X \times G$ and defining $g(x, h) = (x, gh)$ for $g, h \in G$ and $x \in X$. The projection $X \times G \to X$ is called the trivial G covering over X. It turns out that every G-covering locally looks like a trivial G-covering. More formally, if $p : Y \to X$ is a G-covering then every point in X has a neigbourhood U such that $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is isomorphic as a G-covering to the trivial G-covering over U. This can easily be seen by setting U equal to $\bar{V}$ as in the proof of lemma 4.2 and defining $\phi : p^{-1}(U) \to U \times G$ as $\phi(gv) = (p(v), g)$. Since G is discrete and p is a homeomorphism when restricted to the appropriate domain and codomain, ϕ is a homeomorphism. Clearly ϕ also satisfies the other properties for being an ismorphism of G-coverings, which proves the claim.

Next, we want to describe the G-coverings over a given base space up to isomorphisms. There are two ways to do this. The first one uses the concept of so called Čech cocycles. The second method uses homotopy classes of maps to a fixed base space.

4.2 Čech cohomology

We know that every G-covering is locally trivial. In this section we will describe a way to classify the G-coverings that are trivial over a certain open cover of a space. First, we introduce the concept of Čech cocycles.

Definition 4.4. *Let X be a topological space with covering $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$. Then a* Čech cocycle *on $\mathcal{U}$ with coefficients in G is a collection $\{g_{\alpha\beta} : \alpha, \beta \in \mathcal{A}\}$ of locally constant functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ satisfying:*

1. *$g_{\alpha\alpha} = e$ for all $\alpha \in \mathcal{A}$,*
2. *$g_{\beta\alpha} = (g_{\alpha\beta})^{-1}$ for all $\alpha, \beta \in \mathcal{A}$,*
3. *$g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ for all $\alpha, \beta, \gamma \in \mathcal{A}$.*

Two Čech cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are called cohomologous *if for all $\alpha, \beta \in \mathcal{A}$ such that $U_\alpha \cap U_\beta \neq \varnothing$ there are locally constant functions $h_\alpha : U_\alpha \to G$ such that $g'_{\alpha\beta} = (h_\alpha)^{-1} g_{\alpha\beta} h_\beta$ on $U_\alpha \cap U_\beta$. It is not hard to see that being cohomologous defines an equivalence relation on the set of cocycles and we call the equivalence classes* Čech cohomology classes *on $\mathcal{U}$ with coefficients in G. The set $H^1(\mathcal{U}, G)$ is the set of these classes[1].*

We start with a technical lemma, describing isomorphsims of trivial G-coverings. The proof is based on [No15].

Lemma 4.5. *Consider the trivial G-covering $p : Y = X \times G \to X$ over X. Then for every isomorphism of G-coverings $\phi : Y \to Y$ there is a unique locally constant funcion $h : X \to G$ such that $\phi(x, g) = (x, g \cdot h(x))$.*

Proof. Let $\phi : Y \to Y$ be an isomorphism of G-coverings. Suppose that ϕ sends (x, e) to $(x', h(x))$, where h is a function from X to G. Then for $g \in G$

$$\phi(x, g) = g\phi(x, e) = g(x', h(x)) = (x', gh(x)).$$

Now $x = p(x, g) = p \circ \phi(x, g) = p(x', gh(x)) = x'$, so the first coordinate is preserved. Define $s : X \to Y$, $x \mapsto (x, e)$ and $\pi : Y \to G$, $(x, g) \mapsto g$. Clearly, both are continuous. Then

$$\begin{aligned} \pi \circ \phi \circ s(x) &= \pi \circ \phi(x, e) \\ &= \pi(x, h(x)) \\ &= h(x), \end{aligned}$$

and since π, ϕ and s are continuous, so is h. Now for a connected component $U \subseteq X$, we can write $U = \bigcup_{g \in G} (h|_U)^{-1}(\{g\})$, which is a disjoint union of open sets so only one can be non-empty. Therefore, h is locally constant. Clearly if h' also satisfies $\phi(x, g) = (x, g \cdot h'(x))$, then $h' = h$. □

[1] Note that G is not necessarily Abelian.

4.2.1 From G-covering to Čech cocycle

From a G-covering $p : Y \to X$ a Čech cocycle can be constructed by looking at a trivialisation of the covering and applying lemma 4.5. This is done in the following way. Since the G-covering is locally trivial, there is an open covering $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ of X, such that $p^{-1}(U_\alpha) \to U_\alpha$ is isomorphic to a trivial covering for every α. Therefore, we can find isomorphisms of G-coverings $\phi_\alpha : U_\alpha \times G \to p^{-1}(U_\alpha)$. By composing the restrictions of ϕ_α with ϕ_β^{-1} on $U_\alpha \cap U_\beta$ for $\alpha, \beta \in \mathcal{A}$ such that $U_\alpha \cap U_\beta \neq \varnothing$ we get isomorphisms of G-coverings $\phi_\beta^{-1} \circ \phi_\alpha : (U_\alpha \cap U_\beta) \times G \to (U_\alpha \cap U_\beta) \times G$. By lemma 4.5, these isomorphisms can be written as $(x, g) \mapsto (x, g \cdot g_{\alpha\beta}(x))$ for unique locally constant functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. These functions can easily be seen to satisfy the properties listed in definition 4.4. Therefore, we have constructed a Čech cocycle from the G-covering p. Note that the resulting cocycle depends on the trivialization $\{\phi_\alpha : \alpha \in \mathcal{A}\}$ chosen. We claim though, that if we had used a different trivialization $\{\phi'_\alpha : \alpha \in \mathcal{A}\}$ the resulting cocycles are cohomologous. To see this, first fix $\alpha \in \mathcal{A}$. Then ϕ_α and ϕ'_α define two isomorphism of G-coverings $U_\alpha \times G \to p^{-1}(U_\alpha)$. So the composition $(\phi'_\alpha)^{-1} \circ \phi_\alpha$ is an isomorphism of G-coverings from $U_\alpha \times G$ to itself. Therefore by using lemma 4.5 we find a unique locally constant function $h_\alpha : U_\alpha \to G$ such that this isomorphism can be written as $(x, g) \mapsto (x, g \cdot h_\alpha(x))$. Then we get the following relation:

$$\begin{aligned}
(x, g \cdot g_{\alpha\beta}(x) h_\beta(x)) &= ((\phi'_\beta)^{-1} \circ \phi_\beta) \circ (\phi_\beta^{-1} \circ \phi_\alpha)(x, g) \\
&= (\phi'_\beta)^{-1} \circ \phi_\alpha(x, g) \\
&= ((\phi'_\beta)^{-1} \circ \phi'_\alpha) \circ ((\phi'_\alpha)^{-1} \circ \phi_\alpha)(x, g) \\
&= (x, g \cdot h_\alpha(x) g'_{\alpha\beta}(x)),
\end{aligned}$$

where $\{g'_{\alpha\beta}\}$ is the cocycle formed from the trivialization $\{\phi'_\alpha\}$. So the two cocycles are related by $g'_{\alpha\beta} = (h_\alpha)^{-1} g_{\alpha\beta} h_\beta$, which means that they are cohomologous. In the same way we can prove that two isomorphic G-coverings induce cohomologous cocycles when we take the same sets U_α over which the coverings are trivial.

4.2.2 From Čech cocycle to G-covering

Now we want to reverse this process and use a Čech cocycle $\{g_{\alpha\beta} : \alpha, \beta \in \mathcal{A}\}$ on $\mathcal{U}$ to make a G-covering that is trivial over all the U_α's. To do this first define $\Sigma := \coprod_{\alpha \in \mathcal{A}} (U_\alpha \times G)$. Then we define for $x \in U_\alpha \cap U_\beta$ and $g \in G$ that $(x_\alpha, g) \sim (x_\beta, g \cdot g_{\alpha\beta}(x))$. Here, x_α denotes x regarded as element of U_α in the disjoint union. From the properties listed in definition 4.4, it can easily be seen that this defines an equivalence relation on Σ. We denote $q : \Sigma \to Y$ as the quotient map, where Y is the set of equivalence classes, and equip Y with the quotient topology. A group action on Y can be defined by $g' \cdot [(x, g)] = [(x, g' \cdot g)]$ where $g' \in G$ and $[(x, g)] = q((x, g))$. This action is well-defined, since the equivalence relation is compatible with left-multiplication by G. Also, it clearly satisfies the requirements of associativity and neutral element defined at the start of this chapter. We also need to check that for $g' \in G$ the map $\mu : Y \to Y, [(x, g)] \mapsto [(x, g' \cdot g)]$ is a homeomorphism. If $U \subseteq Y$ is open, then $V := q^{-1}(U) \subseteq \Sigma$ is open. $\mu(U) = \mu \circ q(V) = q(g'V)$, so

$q^{-1}(\mu(U)) = q^{-1} \circ q(g'V) = g'V$ is open. Therefore $\mu(U)$ is open. Since μ^{-1} is just μ with g' replaced by $(g')^{-1}$, μ^{-1} is also open. The map is clearly bijective and therefore μ is a homeomorphism. So this multiplication satisfies all the requirements for a group action. Left to check is that the action on Y is properly discontinuous. Take $[(x,g)] \in Y$ and let $V = q(U_\alpha \times \{g\})$. For elements in V we can find a unique representation $[(y,g)]$ with $(y,g) \in U_\alpha \times G \subseteq \Sigma$. Let $h_1, h_2 \in G$ and suppose that $h_1V \cap h_2V \neq \varnothing$. Then there exists an element in this intersection that we can represent in the representation described before. Since the element is in h_1V it can be denoted as $[(y, h_1g)]$ and since it is in h_2V as $[(y, h_2g)]$. Since in both notations we have $y \in U_\alpha$ in the disjoint union Σ, we must have $h_1 = h_2$. This proves that the action is properly discontinuous.

Define $p : Y \to X$ as the projection $[(x,g)] \mapsto x$, which clearly is well-defined. Also let $\psi : X \to Y/G$ be the map $x \mapsto [(x,e)]G$. This is a homeomorphism and $\psi \circ p([(x,g)]) = [(x,e)]G = [(x,g)]G$, so $\psi \circ p$ is just the projection $Y \to Y/G$. By lemma 4.2 this projection is a covering. Since ψ is a homeomorphism, p is also a covering and therefore a G-covering. It can easily be seen that p is trivial over each U_α, since $p^{-1}(U_\alpha) = q(U_\alpha \times G)$ and q restricts to an isomorphism of G-coverings $U_\alpha \times G \to q(U_\alpha \times G)$.

If we start with two cohomologous cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$, there are locally constant functions $\{h_\alpha : \alpha \in \mathcal{A}\}$ such that $g'_{\alpha\beta} = (h_\alpha)^{-1} g_{\alpha\beta} h_\beta$ on $U_\alpha \cap U_\beta$. We get two equivalence relations with corresponding qoutient maps on Σ. The first relation $\sim_R$ is given by $(x_\alpha, g) \sim_R (x_\beta, g \cdot g_{\alpha\beta}(x))$ and the second, $\sim_{R'}$, by $(x_\alpha, g) \sim_{R'} (x_\beta, g \cdot g'_{\alpha\beta}(x))$. We denote the two quotient maps as follows:

$$\begin{aligned} q : \Sigma &\to Y := \Sigma/\sim_R \\ (x,g) &\mapsto [(x,g)] \\ q' : \Sigma &\to Y' := \Sigma/\sim_{R'} \\ (x,g) &\mapsto \overline{(x,g)}. \end{aligned}$$

Now define $\phi : Y \to Y'$ as $[(x,g)] \mapsto \overline{(x, g \cdot h_\alpha(x))}$ for $x \in U_\alpha$. We first show this is well-defined. If $x \in U_\alpha \cap U_\beta$, then $[(x,g)] = [(x, g \cdot g_{\alpha\beta}(x))]$. We know

$$\begin{aligned} \phi([(x, g \cdot g_{\alpha\beta}(x))]) &= \overline{(x, g \cdot g_{\alpha\beta}(x) h_\beta(x))} \\ &= \overline{(x, g \cdot h_\alpha(x) g'_{\alpha\beta}(x))} \\ &= \overline{(x, g \cdot h_\alpha(x))} \\ &= \phi([(x,g)]), \end{aligned}$$

so ϕ is well-defined. It is also a homeomorphism since the h_α's are locally constant. If we take $g, \tilde{g} \in G$ then we clearly have $\phi(\tilde{g}[(x,g)]) = \tilde{g}\phi([(x,g)])$. Also, if p and p' are the G-coverings constructed from the two cocycles, then $p' \circ \phi = p$ since the coverings are just projections on X. So ϕ satisfies all the conditions for being an isomorphism of G-coverings. Thus cohomologous cocycles induce isomorphic G-coverings.

In summary, in this section we proved the following theorem.

Theorem 4.6. *The constructions from this section give well-defined maps*

$$\mathcal{M}_G(X;\mathcal{U}) \to H^1(\mathcal{U}, G) \quad \text{and} \quad H^1(\mathcal{U}, G) \to \mathcal{M}_G(X;\mathcal{U}).$$

Here, the set $\mathcal{M}_G(X;\mathcal{U})$ denotes the set of equivalence classes of G-coverings that are trivial over each U_α, where two G-coverings are equivalent if they are isomorphic.

These maps turn out to be inverses to each other, which will not be shown here. See [Fu95] for the proof.

4.3 Homotopy description

The main idea of this section is to construct a so-called universal G-covering. This means that we will define a covering space EG with corresponding base space BG such that for every space X there is a bijection from the homotopy classes of maps $X \to BG$ to the set of equivalence classes of G-coverings over X. Some parts of this section will sketch the steps involved rather than going into the details of the process. This is done in order to improve the legibility and because the detailed process won't be important for the rest of the chapters.

4.3.1 (Semi)simplicial sets

To define the spaces EG and BG, we first need the concept of (semi)simplicial spaces. For a reference for this subsection, see [Mü07].

Definition 4.7. *A* semisimplicial set *(SSS) $X_\star$ is a collection of spaces X_n, where $n \geq 0$, and face maps $\partial_i^{(n)} : X_n \to X_{n-1}$, where $n \geq 1$ and $0 \leq i \leq n$, such that*

$$\partial_i^{(n-1)} \circ \partial_j^{(n)} = \partial_{j-1}^{(n-1)} \circ \partial_i^{(n)}, \tag{4.1}$$

for all $i < j$. For two SSS's $X_\star$ and $Y_\star$ a collection of maps $m_n : X_n \to Y_n$, for $n \geq 0$ is called a morphism of SSS's *if the m_n's commute with the ∂_i's. That is, for $n \geq 1$ and $0 \leq i \leq n$ the following diagram commutes.*

$$\begin{array}{ccc} X_n & \xrightarrow{\partial_{X,i}} & X_{n-1} \\ \Big\downarrow{\scriptstyle m_n} & & \Big\downarrow{\scriptstyle m_{n-1}} \\ Y_n & \xrightarrow{\partial_{Y,i}} & Y_{n-1} \end{array} \tag{4.2}$$

This defines a category **SSS** *of SSS's.*

Note that in the diagram, the index above the face maps is omitted. This is a common convention, as it rarely causes confusion. In the following text we will adopt this convention as well.

There is a construction that turns a SSS into a topological space, called the geometric realisation, which we define below. As it turns out, this construction can be extended to a functor $\mathbf{SSS} \to \mathbf{Top}$. To define the geometric realization, first we briefly look at maps between standard n-simplices. For $0 \leq i \leq n$ define $d_i : \Delta_{n-1} \to \Delta_n$ by $(x_0, \dots, x_{n-1}) \mapsto (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1})$. This maps the standard $(n-1)$-simplex to a face of the standard n-simplex. Now we can define the geometric realization.

Definition 4.8. *For a SSS $X_\star$ let all the X_n be equipped with the discrete topology. On the space $\coprod_{n\geq 0} X_n \times \Delta_n$ define the equivalence relation $\sim$, where $\sim$ is generated by $(x, d_i(y)) \sim (\partial_i(x), y)$, for $x \in X_n$ and $y \in \Delta_{n-1}$. Define the* geometric realization *of $X_\star$ as*

$$|X_\star| = \left(\coprod_{n\geq 0} X_n \times \Delta_n \right) / \sim,$$

where we give $|X_\star|$ the quotient topology.

As the following theorem proves, a morphism of SSS's induces a continuous map between the geometric realizations of the involved SSS's, making $|\cdot|$ into a functor.

Theorem 4.9. *Let $m : X_\star \to Y_\star$ be a morphism of SSS's. Then there is a continuous map $|m| : |X_\star| \to |Y_\star|$.*

Proof. First define $|m|_0 : \coprod_{n\geq 0} X_n \times \Delta_n \to \coprod_{n\geq 0} Y_n \times \Delta_n$ by $(x,t) \mapsto (m_n(x), t)$ for $x \in X_n$ and $t \in \Delta_n$. Now $|m|_0(\partial_i(x), t) = (m_n\partial_i(x), t) = (\partial_i m_n(x), t)$, because m is a morphism of SSS's. Also $|m|_0(x, d_i(t)) = (m_n(x), d_i(t)) \sim (\partial_i m_n(x), t)$, so $|m|_0$ factors through the equivalence relation to give a function $|m| : |X_\star| \to |Y_\star|$. Since $|m|_0$ is continuous, the map $p_Y \circ |m|_0$ is too, where $p_Y : \coprod_{n\geq 0} Y_n \times \Delta_n \to |Y_\star|$ is the quotient map. So for $U \subseteq |Y_\star|$ open, $(p_Y \circ |m|_0)^{-1}(U)$ is open. Then $|m|^{-1}(U) = p_X((p_Y \circ |m|_0)^{-1}(U))$, where $p_X : \coprod_{n\geq 0} X_n \times \Delta_n \to |X_\star|$ is again the quotient map. Since p_X is an open map, $|m|^{-1}(U)$ is open and $|m|$ is continuous. □

The constructions above can be extended to simplicial sets. A simplicial set is a semisimplicial set that also has maps $\sigma_i : X_n \to X_{n+1}$ for $0 \leq i \leq n$, called the degeneracy maps. The degeneracy maps have to satisfy certain commutation relations with the face maps, that we won't discuss in detail, because they are in our case nearly trivial. The geometric realization of a simplicial set is the same as that of a semisimplicial set with an extra equivalence relation induced by the degeneracy maps and the maps $s_i : \Delta_{n+1} \to \Delta_n$ given by $s_i(t_0, \dots, t_{n+1}) = (t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1})$. For a more detailed introduction to simplicial sets, see [Ma99].

4.3.2 Classifying space of a group

The construction in the previous section can be applied to groups. This will form two spaces: EG and BG. We will give two descriptions of BG, namely as the geometric realization of a simplicial set and as the quotient EG/G. Most of this subsection is based on [Ma99].

Definition 4.10. *Define $E_n(G) = G^{n+1}$. This defines a simplicial set with face maps $\partial_i : E_n(G) \to E_{n-1}(G)$ and degeneracy maps $\sigma_i : E_n(G) \to E_{n+1}(G)$ defined as follows:*

$$\partial_i(g_1, \dots, g_{n+1}) = \begin{cases} (g_2, \dots, g_{n+1}) & i = 0 \\ (g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_{n+1}) & 1 \leq i \leq n \end{cases}$$
$$\sigma_i(g_1, \dots, g_{n+1}) = (g_1, \dots, g_{i-1}, e, g_i, \dots, g_{n+1}),$$

for $0 \leq i \leq n$. The space EG is now defined as the geometric realization of $E_\star(G)$. So $EG = \coprod_{n \geq 0}(G^{n+1} \times \Delta_n)/\sim$, where $\sim$ is generated by the two relations

$$(\zeta, d_i(t)) \sim (\partial_i(\zeta), t) \text{ for } \zeta \in G^{n+1} \text{ and } t \in \Delta_{n-1}$$
$$(\zeta, s_i(t')) \sim (\sigma_i(\zeta), t') \text{ for } \zeta \in G^{n+1} \text{ and } t' \in \Delta_{n+1}.$$

We define a group action on EG by $g \cdot [((g_i)_{i=1}^{n+1}, (t_i)_{i=0}^{n})] = [((g \cdot g_i)_{i=1}^{n+1}, (t_i)_{i=0}^{n})]$. Here $[(\zeta, t)]$ denotes the equivalence class of (ζ, t) in EG for $\zeta \in E_n(G)$ and $t \in \Delta_n$. So multiplication by $g \in G$, amounts to multiplying all the $g_i \in G$ that represent the element in $E_n(G)$ by g, while leaving the $t \in \Delta_n$ unchanged.

To see that we get a G-covering $EG \to EG/G$, consider the next theorem.

Theorem 4.11. *The action of G on EG is properly discontinuous.*

Proof. Let $p := ((g_1, \dots, g_{n+1}), t) \in G^{n+1} \times \Delta_n$. Assume $t \in \partial\Delta_n$. Then there is a $t' \in \mathrm{int}(\Delta_{n+1})$ such that $s_i(t') = t$ and therefore $p \sim (\sigma_i(g_1, \dots, g_{n+1}), t')$. So without loss of generality we may take p such that $t \in \mathrm{int}(\Delta_n)$. Let $\bar{V} = \{(g_1, \dots, g_{n+1})\} \times \mathrm{int}(\Delta_n)$ and V be the corresponding set in EG. Then $[p] \in V$. First we prove that V is open. Observe that $\bar{V}$ is open, since G is discreet. Now in the proof of theorem 4.9, we saw that the quotient map from a (semi)simplicial set to its geometric realization is open and therefore V is open. Now we prove that V satisfies the condition from definition 4.1. Note that $EG = \bigcup_{q \geq 0} \left(\coprod_{0 \leq m \leq q}(G^{m+1} \times \Delta_m)/\sim\right)$. Clearly in $\coprod_{0 \leq m \leq n-1}(G^{m+1} \times \Delta_m)/\sim$ we have $gV \cap hV = \varnothing$ if $g, h \in G$ with $g \neq h$, because p is not equivalent to anything in this set. So also in EG, we must have $gV \cap hV = \varnothing$ if $g \neq h$. It follows that the action is properly discontinuous. $\square$

As a corollary of this theorem, we see that the map $\pi_G : EG \to EG/G$ is a G-covering. The base space EG/G will be denoted by BG and is called the classifying space of G. We define $\xi_G = (EG, \pi_G, BG)$ as this G-covering. It turns out that BG can also be defined as the geometric realization of a simplicial set. Since this description will be of use later, this way of constructing BG will also be covered here.

Definition 4.12. *Define $B_n(G) = G^n$, together with face maps $\partial_i : B_n(G) \to B_{n-1}(G)$*

and degeneracy maps $\sigma_i : B_n(G) \to B_{n+1}(G)$ *defined by:*

$$\partial_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & i = 0 \\ (g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n) & 1 \le i \le n-1 \\ (g_1, \dots, g_{n-1}) & i = n \end{cases}$$
$$\sigma_i(g_1, \dots, g_n) = (g_1, \dots, g_{i-1}, e, g_i, \dots, g_n),$$

for $0 \le i \le n$. *Then* $BG = |B_\star(G)|$ *is the geometric realization of this simplicial set.*

4.3.3 Homotopy classification

Now that the classifying space of a group has been defined, this space will be used to construct a bijection $[X, BG] \to \mathcal{M}_G(X)$, where $[X, BG]$ is the set of homotopy classes of maps $X \to BG$ and $\mathcal{M}_G(X)$ are the isomorphism classes of G-coverings over X. The construction that will be used for this bijection is called the *pull back*. See [Hus94] as a reference.

Definition 4.13. *Let* $p : E \to B$ *be a* G*-covering and* $f : X \to B$ *a map. Then we define the* pull back *of* f *as* $(f^*(E), f^*(p), X)$*, where*

$$f^*(E) = \{(x, e) \in X \times E : f(x) = p(e)\}$$

and $f^*(p) : f^*(E) \to X$ *is the projection* $(x, e) \mapsto x$. *The action of* G *on* $f^*(E)$ *is defined as multiplication with the second coordinate. This* G*-covering is often denoted* $f^*(\xi)$*, where* $\xi = (E, p, B)$. *There is a canonical map* $f_\xi : f^*(E) \to E$ *that satisfies* $p \circ f_\xi = f \circ f^*(p)$*, namely the projection on the second coordinate.*

As the reader can check, the pull back indeed defines a G-covering. Note that for $x \in X$, we can identify the fiber of $f^*(\xi)$ over x with the fiber of p over $f(x)$, since the former is given by $(f^*(E))_x = \{(x, e) \in X \times E : f(x) = p(e)\}$ and the latter by $E_{f(x)} = \{e \in E : p(e) = f(x)\}$. Moreover, two homotopic maps define isomorphic G-coverings as the next theorem states. The proof of this theorem is based on [C98].

Theorem 4.14. *Let* $\xi = (E, p, B)$ *be a* G*-covering. If two maps* $f, g : X \to B$ *are homotopic, then their induced* G*-coverings* $f^*(\xi)$ *and* $g^*(\xi)$ *are isomorphic.*

Proof. Since f and g are homotopic, there exists a homotopy $H : X \times I \to B$, with $H_0 = f$ and $H_1 = g$. Then $F := H \circ (f^*(p) \times \mathrm{Id}_I) : f^*(E) \times I \to B$ is also a homotopy and f_ξ is a lifting of F_0, so by theorem 2.4[2] there is a lifting of F to a homotopy $\tilde{F} : f^*(E) \times I \to E$, with $\tilde{F}_0 = f_\xi$. Now for $(x, t) \in X \times I$ on the level of fibers $\tilde{F}$ maps $(f^*(E) \times I)_{(x,t)}$ to $E_{H(x,t)} = (H^*(E))_{(x,t)}$. So $\tilde{F}$ gives rise to a map $\tilde{H} : f^*(E) \times I \to H^*(E)$ that satisfies

[2]Note that here a slightly more general version of the homotopy lifting lemma is used than the one described in the preliminaries. As a reference see [Hus94] or [C98]. This version imposes extra conditions on the spaces involved, which we will not ellaborate on as the spaces we are concerned with will always satisfy these conditions.

$H^*(p) \circ \tilde{H} = f^*(p) \times \mathrm{Id}_I$. Now by restricting this map to $f^*(E) \times \{1\} \subset f^*(E) \times I$, we get the following commutative diagram:

$$\begin{tikzcd}
f^*(E) \times \{1\} \arrow[rr, "\tilde{H}"] \arrow[dr, "f^*(p)"'] & & g^*(E) \times \{1\} \arrow[dl, "g^*(p)"] \\
& X \times \{1\} &
\end{tikzcd}.$$

The reader is invited to check that this map indeed defines an isomorphism of G-coverings. □

Recall that we defined a G-covering $\xi_G = (EG, \pi_G, BG)$. For a topological space X, a map $X \to BG$ then induces a G-covering over X. By the previous theorem, the map $\psi : [X, BG] \to \mathcal{M}_G(X)$ with $\psi([f]) = f^*(\xi_G)$ is well defined, where $[f]$ denotes the homotopy class of f. The main result of this section will be that ψ is a bijection. In the following paragraphs an inverse map will be constructed to sketch this proof. This method is based on [S68].

Definition 4.15. *Let X be a topological space and $\mathcal{U} = \{U_i : i \in I\}$ an open cover. Construct the following simplicial set:*

$$X_0 = \coprod_{i \in I} U_i$$
$$X_n = \coprod_{i_0, \dots, i_n \in I} (U_{i_0} \cap \dots \cap U_{i_n}),$$

with face maps $\partial_j : X_n \to X_{n-1}$ defined by $\partial_j(x_{i_0,\dots,i_n}) = x_{i_0,\dots,i_{j-1},i_{j+1},\dots,i_n}$ and degeneracy maps $\sigma_j : X_n \to X_{n+1}$ defined by $\sigma_j(x_{i_0,\dots,i_n}) = x_{i_0,\dots,i_j,i_j,\dots,i_n}$. Here, $x_{i_0,\dots,i_n}$ denotes the element x seen as an element of $U_{i_0} \cap \dots \cap U_{i_n}$ in the disjoint union.

As the reader may check, these maps satsify the properties for a simplicial space. From a G-covering we will construct a morphism of simplicial sets. As shown before, a G-covering induces a Čech cocycle. So for a G-covering over X, let $\mathcal{U}$ be an open covering such that the covering is trivial over $\mathcal{U}$. Then, let $X_\star$ be the simplicial set defined above and $\{g_{ij}\}_{i,j \in I}$ the cocycle over $\mathcal{U}$ induced by the covering. Now let $m_0 : X_0 \to B_0(G) = \{*\}$ be the unique map to a point space and $m_n : X_n \to B_n(G)$ be given by $m_n(x_{i_0,\dots,i_n}) = (g_{i_0 i_1}, \dots, g_{i_{n-1} i_n})$. An easy check confirms that these maps commute with the face and degeneracy maps and therefore define a morphism of simplicial spaces. Now by the extension of theorem 4.9 to simplicial spaces, this induces a continuous map from $|X_\star|$ to $|B_\star(G)| = BG$. Since $|X_\star|$ is homotopy equivalent to X (see [S68]), this defines a map $X \to BG$ up to homotopy. It turns out that this construction factors to a map $\mathcal{M}_G(X) \to [X, BG]$, which we won't prove here. Also, this map will be the inverse of the map ψ that was defined above. For a different approach to the proof that ψ is a bijection, see chapter 16 of [Ma99].

In summary, this chapter has introduced the concept of G-coverings and has given two ways of classifying the equivalence classes of G-coverings over a given space. In the next chapter these concepts will be used to define a TQFT.

5 Dijkgraaf-Witten theory

This chapter will cover the details of Dijkgraaf-Witten (DW) theory. We fix a finite group G throughout this chapter and write $\mathcal{M}(X) := \mathcal{M}_G(X)$ for the G-coverings over a topological space X. First, the simplest version of DW theory is introduced, which is called untwisted DW theory. Afterwards, a generalization is introduced using a fixed cohomology class in the cohomology of BG with coefficients in the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \subseteq \mathbb{C}^*$. The main concept behind DW theory is the counting of G-coverings over manifolds. To each G-covering a certain "mass" is associated, by counting the number of automorphisms of the covering. This brings us to the following defininition.

Definition 5.1. *Let M and N be manifolds (either with or without boundary) such that $N \subseteq M$. This induces a functor $r : \mathcal{G}(M) \to \mathcal{G}(N)$ by restriction of G-coverings over M to the subspace N. For a G-covering ξ in $\mathcal{G}(M)$ this functor maps automorphisms of ξ to automorphisms of $r\xi$. Define*

$$\mathrm{Aut}(\xi; r) := \{\varphi \in \mathrm{Hom}_{\mathcal{G}(M)}(\xi, \xi) : r\varphi = \mathrm{Id}_{r\xi}\}.$$

If for example $N = \partial M$ in the previous definition, then $\mathrm{Aut}(\xi; r)$ consists of all the automorphisms of ξ that are trivial on the boundary of M. With this definition in mind, we start with the untwisted version of DW theory.

5.1 Untwisted Dijkgraaf-Witten theory

This section is largely based on [Lu12]. For any manifold N with (possibly empty) boundary, we define the complex vector space

$$V(N) = \mathbb{C}[N, BG] = \{f : \mathcal{M}(N) \to \mathbb{C}\}. \tag{5.1}$$

First, we want to see that when N is compact, this defines a finite-dimensional space as the following lemma states.

Lemma 5.2. *The complex vector space $V(N)$ defined in equation 5.1 is finite dimensional for every compact manifold N.*

Proof. First, suppose N is connected. Then a G-covering $M \to N$ induces a homomorphism $\pi_1(N, n) \to G$, where n is a basepoint of N, in the following way. Fix a basepoint $m \in M$. A loop at n can be lifted to a path in M starting at m in a unique way. Since the endpoint m' of this path sits in the fiber over n there is a unique $g \in G$ such that $m' = g \cdot m$ and this group element is the same for homotopic loops. Changing the basepoint n or m amounts to a conjugation of the constructed homomorphism. If we let G act on $\mathrm{Hom}(\pi_1(N), G)$ by conjugation, we get an isomorphism $\mathcal{M}(N) \cong \mathrm{Hom}(\pi_1(N), G)/G$ [Fr92]. The fundamental group of a compact manifold is finitely generated[1], so there are only finitely many homomorphisms $\pi_1(N) \to G$. This means that $V(N)$ is a finite-dimensional vector space. If N is not connected then $N = \coprod_{i\in\mathcal{I}} N_i$, where the N_i are the components of N and $V(N) = V(\coprod_{i\in\mathcal{I}} N_i) \cong \bigotimes_{i\in\mathcal{I}} V(N_i)$ as will be shown in theorem 5.4. Here, the tensor product is taken over $\mathbb{C}$. This shows that $V(N)$ is still finite dimensional. □

Note that a functor $r : \mathcal{G}(M) \to \mathcal{G}(N)$ induces a function $\bar{r} : \mathcal{M}(M) \to \mathcal{M}(N)$, since a functor preserves ismorphisms. This in turn defines a linear map $\bar{r}^* : V(N) \to V(M)$ by $f \mapsto f \circ \bar{r}$, called the *pull-back*. It will be needed to also have a map $\bar{r}_! : V(M) \to V(N)$ going in the other direction. There is a natural way of defining this map, sometimes referred to as *integration over fibers*. In what follows, we will not make a notational difference between a G-covering in $\mathcal{G}(N)$ and its isomorphism class in $\mathcal{M}(N)$, in order to make the formulas look somewhat more clear. For $f : \mathcal{M}(M) \to \mathbb{C}$ in $V(M)$ and $\xi \in \mathcal{M}(N)$ define

$$\bar{r}_!(f)(\xi) = \sum_{\substack{\nu\in\mathcal{M}(M)\\ \bar{r}\nu=\xi}} f(\nu)\frac{1}{\#\mathrm{Aut}(\nu; r)}.$$

This means that $\bar{r}_!(f)$ sums the values of f on all isomorphism classes of G-coverings that restrict to the G-covering of its input with the appropriate mass as discussed in definition 5.1.

Now that we have access to these different construction of maps, we can associate a linear map to a cobordism $M : \Sigma_0 \to \Sigma_1$. Recall from the definition of a cobordism (definition 3.1) that M comes with inclusions $\Sigma_i \hookrightarrow M$ for $i = 0, 1$. As before, these inclusions give rise to functors $r_i : \mathcal{G}(M) \to \mathcal{G}(\Sigma_i)$. Define $Z_M : V(\Sigma_0) \to V(\Sigma_1)$ as $Z_M = (\bar{r}_1)_! \circ \bar{r}_0^*$. This clearly defines a linear map, since both $(\bar{r}_1)_!$ and $\bar{r}_0^*$ are linear. The formula for Z_M is:

$$Z_M(f)(\sigma) = \sum_{\substack{\nu\in\mathcal{M}(M)\\ \bar{r}_1\nu=\sigma}} f \circ \bar{r}_0(\nu)\frac{1}{\#\mathrm{Aut}(\nu; r_1)}, \tag{5.2}$$

where $f \in V(\Sigma_0)$ and $\sigma \in \mathcal{M}(\Sigma_1)$.

Theorem 5.3. *Define* $\mathcal{A} : \mathbf{nCob} \to \mathbf{Vect}_{\mathbb{C}}$ *as*

$$\Sigma \mapsto V(\Sigma);$$
$$M \mapsto Z_M,$$

for Σ *an object in* **nCob** *and* M *a morphism. Then* $\mathcal{A}$ *is a functor.*

[1] See exercise 4 on page 500 of [Mu00]. The statement follows from the fact that every manifold is semilocally simply connected.

Proof. First note that when $M \sim M' : \Sigma_0 \to \Sigma_1$ as in definition 3.2, then $Z_M = Z_{M'}$. So $\mathcal{A}$ is well-defined. Now we check that $\mathcal{A}$ behaves well under composition. Take two cobordisms $M_0; \Sigma' \to \Sigma$ and $M_1 : \Sigma \to \Sigma''$. This gives rise to the following diagram:

$$\begin{tikzcd}
 & & \mathcal{G}(M_0 \coprod_\Sigma M_1) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & & \\
 & \mathcal{G}(M_0) \arrow[dl, "p'"'] \arrow[dr, "p"] & & \mathcal{G}(M_1) \arrow[dl, "q"'] \arrow[dr, "q''"] & \\
\mathcal{G}(\Sigma') & & \mathcal{G}(\Sigma) & & \mathcal{G}(\Sigma'').
\end{tikzcd}$$

Here, all the functors are restriction to a subspace. Note that the diagram commutes: $p \circ r_0 = q \circ r_1$. Computing $Z_{M_0 \coprod_\Sigma M_1}$ amounts to pulling back along p' and r_0 and then integrating over fibers along r_1 and q'', whereas $Z_{M_1} \circ Z_{M_0} = \bar{q}''_! \circ \bar{q}^* \circ \bar{p}_! \circ (\bar{p}')^*$. So to check that $Z_{M_0 \coprod_\Sigma M_1} = Z_{M_1} \circ Z_{M_0}$, we have to verify $\bar{q}^* \circ \bar{p}_! = (\bar{r}_1)_! \circ \bar{r}_0^*$. Take $f \in V(M_0)$ and $\xi \in \mathcal{M}(M_1)$. Computing both sides of the equation, we get:

$$\bar{q}^* \circ \bar{p}_!(f)(\xi) = \sum_{\substack{\mu \in \mathcal{M}(M_0) \\ \bar{p}\mu = \bar{q}\xi}} f(\mu) \frac{1}{\#\mathrm{Aut}(\mu; p)} \tag{5.3}$$

$$\begin{aligned}
(\bar{r}_1)_! \circ \bar{r}_0^*(f)(\xi) &= \sum_{\substack{\eta \in \mathcal{M}(M_0 \coprod_\Sigma M_1) \\ \bar{r}_1\eta = \xi}} f \circ r_0(\eta) \frac{1}{\#\mathrm{Aut}(\eta; r_1)} \qquad (5.4) \\
&= \sum_{\substack{\mu \in \mathcal{M}(M_0) \\ \bar{p}\mu = \bar{q}\xi}} f(\mu) \sum_{\substack{\eta \in \mathcal{M}(M_0 \coprod_\Sigma M_1) \\ \bar{r}_1\eta = \xi \\ \bar{r}_0\eta = \mu}} \frac{1}{\#\mathrm{Aut}(\eta; r_1)}.
\end{aligned}$$

Note that when $\bar{p}\mu = \bar{q}\xi$, we can glue μ and ξ to give an element $\eta \in \mathcal{M}(M_0 \coprod_\Sigma M_1)$ in exactly one way up to isomorphism, so the second sum in the last line of equation 5.4 only contains one term. So to check that equations 5.3 and 5.4 are equal, we must prove that $\#\mathrm{Aut}(\mu; p) = \#\mathrm{Aut}(\eta; r_1)$. To see this, note the map $\mathrm{Aut}(\eta; r_1) \to \mathrm{Aut}(\mu; p) : \varphi \mapsto r_0\varphi$ is a bijection with inverse given by gluing an automorphism of μ that leaves $p\mu$ intact with the identity on ξ to give an automorphism of η that leaves ξ intact. This proves $\bar{q}^* \circ \bar{p}_! = (\bar{r}_1)_! \circ \bar{r}_0^*$, which in turn shows $Z_{M_0 \coprod_\Sigma M_1} = Z_{M_1} \circ Z_{M_0}$. Left to check is that $Z_{\Sigma \times I} = \mathrm{Id}_{V(\Sigma)}$. To see this, take a manifold M with Σ as its in-boundary. Then by the previous arguments $Z_M \circ Z_{\Sigma \times I} = Z_{(\Sigma \times I) \coprod_\Sigma M} = Z_M$, where the last equality follows from the fact that M and $(\Sigma \times I) \coprod_\Sigma M$ are equivalent. The same reasoning shows that $Z_{\Sigma \times I} \circ Z_M = Z_M$, when Σ is an out-boundary of M. This proves that $\mathcal{A}$ is a functor. $\square$

As a final theorem of this section we prove that $\mathcal{A}$ defines a TQFT.

Theorem 5.4. *The functor $\mathcal{A} : \mathbf{nCob} \to \mathbf{Vect}_{\mathbb{C}}$ from theorem 5.3 is a TQFT.*

Proof. The two conditions from definition 3.5 need to be checked. The second one is trivial, since there is only one map $\varnothing \to BG$, so $V(\varnothing) = \mathbb{C}$. For the first condition, note that $[\Sigma \coprod \Sigma', BG] = [\Sigma, BG] \times [\Sigma', BG]$. Then the map $\mathbb{C}([\Sigma, BG] \times [\Sigma', BG]) \to \mathbb{C}[\Sigma, BG] \otimes \mathbb{C}[\Sigma', BG]$ given by $(f, f') \mapsto f \otimes f'$ is easily seen to be a linear bijection (see [Q91]). This proves the theorem. $\square$

This concludes the exposition of untwisted DW theory. As a remark, note that this TQFT does not ¨see¨ orientation. The orientation of the manifold M is not used in the definition of Z_M, when M is closed. This will change when the twist is added in the next section.

5.2 Twisted Dijkgraaf-Witten theory

After having explained untwisted DW theory in much detail, the twisted version will merely be sketched. This section is mainly for the purpose of describing the concepts involved and not to give rigorous proofs. References will be given for more detailed texts.

To define twisted DW theory, one fixes a cohomology class $[\alpha] \in H^n(BG; \mathbb{T})$. For a manifold M, a G-covering over M is given by a map $\nu : M \to BG$ up to homotopy as covered in chapter 4. Therefore, we can pull back the cohomology class $[\alpha]$ to a cohomology class $\nu^*[\alpha] \in H^n(M; \mathbb{T})$, where $v^*[\alpha]([z]) = [\alpha][\nu_\# z]$ for a cycle z in M. The main difference between untwisted and twisted DW theory is that formula 5.2 gets an extra wait factor of $W(\nu)$ for every G-covering ν. For a closed manifold M, this weight is defined as $\nu^*[\alpha]([M])$ [DW90], where $[M]$ is the fundamental class of M (see definition 2.5). For manifolds with boundary, this definition is more complicated, since the fundamental class becomes a relative class in $H_n(M, \partial M)$, so it is not in the domain of $\nu^*[\alpha]$. There are different approaches to solving this problem. In [DW90] this problem is solved for a four-dimensional TQFT by seeing M as a union of oriented tetrahedra and defining the weight W as the product of the weights of the tetrahedra. Another approach is given in [Q91], where representatives α for the cohomology class, $\hat{\nu}$ for the classifying map and $m \in C_n(M)$ are chosen, such that $i(m)$ represents $[M]$, where $i : C_n(M) \to C_n(M, \partial M)$. Then $\hat{\nu}^*\alpha(m)$ is a well-defined complex number. Lemma 5.3 from [Q91] states that for a different choice $\hat{\nu}'$ such that $\hat{\nu} \simeq \hat{\nu}'$ rel ∂M and m' such that $\partial m' = \partial m$ the outcome stays the same. For a more detailed explanation of these two solutions we refer to the original articles.

The addition of this weight factor complicates the definition of the vector spaces that get assigned to closed $(n-1)$-manifolds. Several different solutions can be given for this problem. In [T16], a complex number $\chi_\alpha(\varphi)$ is defined for every automorphism φ of a G-covering. Then $V(\Sigma)$ is taken to be the direct sum of $\mathbb{C}$ over all coverings that have $\chi_\alpha(\varphi) = 1$ for all automorphisms φ:

$$V(\Sigma) := \bigoplus_{\substack{\nu \in \mathcal{M}(\Sigma) \\ \chi_\alpha(\mathrm{Aut}(\nu)) = \{1\}}} \mathbb{C}.$$

Note that this is the same space as defined in equation 5.1, except for the fact that the direct sum runs over a smaller set. For different approaches to define the vector spaces, see for example [Q91].

In conclusion, we have seen that DW theories come from cocycles in $H^n(BG;\mathbb{T})$, where the untwisted version corresponds to the case where the cocycle is trivial. Referring back to section 1.2, these cocycles therefore provide information about the phases of certain kinds of matter. In the next chapter we will see how DW theory fits into a slightly different framework, namely that of quantum field theory, comparing the results with those obtained in this chapter.

6 Dijkgraaf-Witten as QFT

Up to this point, we solely discussed topological quantum field theory, mostly from a mathematical point of view. However, in physics one is mostly interested in quantum field theory, which is usually more complicated. This chapter will cover axioms of both classical and quantum field theory and discuss how Dijkgraaf-Witten theory fits into this picture. Most of this chapter is based on [Fr92].

6.1 Field theory

6.1.1 Classical field theory

To describe an n-dimensional field theory, two ingredients are needed. To every n-manifold M (the space-time), a *space of fields* C_M is assigned. In addition, there is a function $S_M : C_M \to \mathbb{R}$ for every M, called the *action*. The space-times can be equipped with some extra structure. For example they can be oriented or have a metric. The fields and actions are required to satisfy the following axioms.

1. Let $f : M' \to M$ be a diffeomorphism between two space-times that preserves the extra structure. Then there is a map $f^* : C_M \to C_{M'}$ such that
$$S_{M'}(f^*\nu) = S_M(\nu),$$
for all $\nu \in C_M$.

2. If $\bar{M}$ denotes M with opposite orientation[1], then $C_{\bar{M}} = C_M$ and $S_{\bar{M}}(\nu) = -S_M(\nu)$, for all $\nu \in C_M$. This axiom reflects the unitarity of the theory. There are also nonunitary field theories in which this axiom does not have to be true.

3. If M and M' are two space-times then $C_{M \coprod M'} \cong C_M \times C_{M'}$ and furthermore
$$S_{M \coprod M'}(\nu \sqcup \nu') = S_M(\nu) + S_{M'}(\nu'),$$
for all $\nu \in C_M$ and $\nu' \in C_{M'}$.

4. If $\Sigma \subseteq M$ is a submanifold of codimension one, M can be ¨cut¨ along Σ to obtain a new manifold M^{cut}. For every $\nu \in C_M$ with corresponding $\nu^{\text{cut}} \in C_{M^{\text{cut}}}$
$$S_M(\nu) = S_{M^{\text{cut}}}(\nu^{\text{cut}}).$$

[1] Note that not all field theories must have oriented space-times. This axiom is only applicable for theories that do.

6.1.2 From classical to quantum

To make a classical field theory into a quantum field theory one extra ingredient is needed: a measure μ_M on the space of fields C_M. Then for a closed space-time M, one can define the *partition function*

$$Z_M := \int_{C_M} e^{iS_M(\nu)}\, d\mu_M(\nu).$$

For space-times M with boundary ∂M, Z_M becomes a function of the fields on ∂M. Define $C_M(\sigma) = \{\nu \in C_M : \partial\nu = \sigma\}$ for fields $\sigma \in C_{\partial M}$. By abuse of notation, the measure on this new space of fields is still denoted as μ_M. Then Z_M is defined as

$$Z_M(\sigma) := \int_{C_M(\sigma)} e^{iS_M(\nu)}\, d\mu_M(\nu).$$

6.2 Untwisted Dijkgraaf-Witten theory

Now that field theories have been introduced, we will describe DW theory as a field theory, starting with untwisted DW theory. For a space-time M the space of fields is defined as $C_M = \mathcal{M}_G(M) = [M; BG]$. The definition of the action is very simple. We simply take $S_M : C_M \to \mathbb{R}$ to be the zero function. This action clearly satisfies all the axioms given in subsection 6.1.1. Left to check is that the space of field does too. Axioms 2 and 4 are trivially checked. If $f : M' \to M$ is a diffeomorphism of space-times then $f^* : C_M \to C_{M'}$ is defined as $f^*([\nu]) := [\nu \circ f]$ for $\nu : M \to BG$. This is clearly well-defined, proving axiom 1. Finally, for two space-times M and M' $[M \coprod M'; BG] \cong [M; BG] \times [M'; BG]$, which proves axiom 3. So these definitions make untwisted DW theory into a classical field theory.

Now, a measure on the space of fields is needed, in order to define a quantum field theory. We start by taking M to be a closed space-time. Define $\mu_M(\nu) = \frac{1}{\#\mathrm{Aut}(\nu)}$ as defined in the previous chapter. Then by definition

$$Z_M = \int_{\mathcal{M}_G(M)} d\mu_M = \sum_{\nu \in \mathcal{M}_G(M)} \frac{1}{\#\mathrm{Aut}(\nu)},$$

where the integral turns into a finite sum. For space-times M with boundary ∂M define $r : \mathcal{G}(M) \to \mathcal{G}(\partial M)$ to be the functor that restricts G-coverings to the boundary as in the previous chapter. Then $C_M(\sigma) = \{\nu \in C_M : \bar{r}\nu = \sigma\}$ for $\sigma \in \mathcal{M}_G(\partial M)$ and $\mu_M(\nu) = \frac{1}{\#\mathrm{Aut}(\nu; r)}$. Now

$$Z_M(\sigma) = \int_{C_M(\sigma)} d\mu_M = \sum_{\substack{\nu \in \mathcal{M}_G(M) \\ \bar{r}\nu = \sigma}} \frac{1}{\#\mathrm{Aut}(\nu; r)}.$$

Note the similarity in both notation as in formulas between the Z_M from this chapter and from the previous chapter (equation 5.2). Of course, this similarity is no coincidence. They both are essentially the same object, written in a different way.

6.3 Twisted Dijkgraaf-Witten theory

In untwisted DW theory, the action was trivial. This changes when moving to twisted DW theory. Fix a cohomology class $[\alpha] \in H^n(BG; \mathbb{T})$. Instead of looking at the action, it is easier in this case to consider the function $W_M = e^{iS_M} : C_M \to \mathbb{T}$. For twisted DW theory, this is defined as $W_M(\nu) = \nu^*[\alpha]([M])$ for closed M and $\nu \in C_M$, just like in the previous chapter. Again we face the problem of defining W_M for space-times with boundary, which is solved in the same way as described there. For the case of closed space-times we will check that W_M satisfies axioms 1, 2 and 3. Axiom 4 and the case where the space-time has a boundary will not be covered here, as a more detailed solutions to defining W_M for these space-times is needed for that.

Let M and M' be two closed space-times and let $f : M' \to M$ be an orientation-preserving diffeomorphism. This entails that f sends the fundamental class of M' to that of M. In the untwisted version of DW theory, we saw that f^* sends a field $[\nu]$ to $[\nu \circ f]$. For notational simplicity, from now on we will write ν for both the map $M \to BG$ and the homotopy class of this map. The first axiom states that $W_{M'}(f^*\nu) = W_M(\nu)$. This can be seen in the following way.

$$\begin{aligned} W_{M'}(f^*\nu) &= W_{M'}(\nu \circ f) \\ &= (\nu \circ f)^*[\alpha]([M']) \\ &= [\alpha][(\nu \circ f)_\# M'] \\ &= [\alpha][\nu_\# M] \\ &= \nu^*[\alpha]([M]) \\ &= W_M(\nu). \end{aligned}$$

For the second axiom, note that the fundamental class of $\bar{M}$ equals $-[M]$. So $W_{\bar{M}}(\nu) = \nu^*[\alpha](-[M]) = (\nu^*[\alpha]([M]))^{-1}$, since α is multiplicative. So $W_{\bar{M}}(\nu) = (W_M(\nu))^{-1}$, which is exactly the statement of axiom 2 after taking the complex exponent. Left to prove is axiom 3, which turns into the statement that $W_{M \coprod M'}(\nu \sqcup \nu') = W_M(\nu) W_{M'}(\nu')$, for closed space-times M and M' and fields $\nu : M \to BG$ and $\nu' : M' \to BG$. First, note that the fundamental class of $M \coprod M'$ is $[M] + [M']$. So

$$\begin{aligned} W_{M \coprod M'}(\nu \sqcup \nu') &= (\nu \sqcup \nu')^*[\alpha]([M] + [M']) \\ &= [\alpha][(\nu \sqcup \nu')_\#(M + M')] \\ &= [\alpha][\nu_\# M + \nu'_\# M'] \\ &= ([\alpha][\nu_\# M])\,([\alpha][\nu'_\# M']) \\ &= W_M(\nu) W_{M'}(\nu'). \end{aligned}$$

This shows that at least for closed space-times, W_M satisfies the axioms. Now for such

M, the partition function can be calculated. By definition:

$$\begin{aligned} Z_M &:= \int_{C_M} W_M \, d\mu_M \\ &= \sum_{\nu \in \mathcal{M}_G(M)} \frac{\nu^*[\alpha]([M])}{\#\mathrm{Aut}(\nu)}. \end{aligned}$$

This concludes the description of DW theory as a QFT.

Popular summary

When you are walking on the surface of the earth, you get the impression that you are walking on a flat surface. However, as everybody knows, the earth is actually round. Therefore, the surface of the earth is an object that is not flat when looked at as a whole, but seems flat when you zoom in enough. In mathematics there is a name for these kind of objects that are "locally flat": *manifolds*. Other examples of manifolds are a circle and a donut (see figure 6.1). A circle, for example, is obviously not flat, but if you zoom

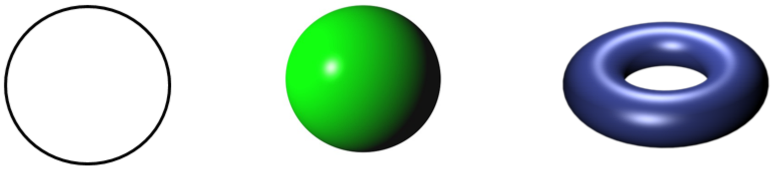

Figure 6.1: Three examples of manifolds: a circle, a sphere and a donut.

in enough on its edge, it will almost look like a straight line. A donut will look just like a flat surface if you zoom in enough. Manifolds have a certain dimension. For example, the dimension of the surface of the earth is two and that of the edge of a circle is one. You can see this the following way. If you are walking on the surface of the earth, you can walk forwards and backwards as well as sidewards, whereas when you're walking along the edge of a circle you can only walk forwards and backwards.

In physics, manifolds are used to model real-world systems. For example, you could have an experiment where a particle can move only along the edge of a circle. As many people know, the space we live in is three-dimensional and time is viewed as the fourth dimension. A physical model of our space would therefore be a three-dimensional manifold and a 4-dimensional manifold (which might be hard to envision) could be a model of the so-called *space-time*. Here, space-time means the four-dimensional space we live in, consisting of three space-dimensions and one extra dimension for the time. If we look at the example at the beginning of this section, the particle moves in one dimension (along a circle), so the space-time would be two-dimensional.

When looking at a system, physicists want to determine the *state* the system is in. For the example of the experiment with the particle that moves along a circle, a state of the system can be labelled by, amongst others, the position and velocity of the particle. Other states of physical systems can involve the temperature, pressure and accelaration of the system. By considering all of the states a system can be in, you get a whole collection of different states, that together we call the *state space*. Now, because of our identification of manifolds with physical systems, we can couple a manifold to the state space of the corresponding system. This is what a *Topological Quantum Field Theory* (TQFT) does. It takes a manifold as input and produces a state space as output. For the space we live in, 4-dimensional TQFT's are most useful, since we live in 4 dimensions, including the time dimension. What happens, is that this 4-dimensional TQFT actually takes 3-dimensional spaces as input (for example the space we live in) to give the state space as output. If we put a 4-dimensional space-time as input, it gives as output how the state space changes over time.

The construction of TQFT's is very important in physics, because it gives a lot of insight into the processes of determining the state space of a system. Also, mathematicians are interested in TQFT's, since they give information about the manifolds that you give as input. In short, TQFT's are very important to multiple branches of science, providing information about both mathematical and physical concepts.

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