Knots and the Maxwell's Equations

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Abstract

We are going to review the Rañada's field line solutions to the Maxwell's equations in vacuum which are topologically non-trivial and their relation with the knot theory. Also, we present the generalization of these solutions to the non-linear electrodynamics obtained recently in the literature.

Keywords: Maxwell's equations. Non-linear electrodynamics. Rañada solutions. Knot solutions.

1 Introduction

The discovery of the knot solutions of the Maxwell's equations in vacuum represents one of the most exciting results obtained recently in the modern classical electrodynamics. Since their presentation in the seminal papers [1, 2, 3, 4], many interesting properties, applications and generalizations of these fields have been appeared in the literature. The topological are described in terms of the field lines of the electric and magnetic components of the electromagnetic field tensor and their topology is given in terms of complex scalar fields that are interpreted as Hopf maps $S^3 \to S^2$ acting on the compactified space-like directions of the Minkowski space-time [4]-[6]. Among the properties studied thus far, one can cite the relationship between the linked and the knotted electromagnetic fields discussed in [7, 8]; the dynamics of the electric charges in topologically non-trivial electromagnetic backgrounds investigated in [10]-[13]; the topological quantization presented in [14]-[17]. Electromagnetic fields with a more general toric topology were found in [18]-[20]. Also, due to the importance of the electromagnetic interaction to a large spectrum of physical phenomena, topological solutions that have specific properties were shown to exist in fluid physics [22, 23], atmospheric physics [24], liquid crystals [25], plasma physics [26], optical vortices [27, 28] and superconductivity [29]. (For a review of the subject and some of its applications see [34] and the references therein). The topological solutions of the Maxwell's equations has been extended recently to the non-linear electrodynamics and fluid physics [21]-[23] as well as to the gravitational phenomena [30, 31, 32, 33]. As we can see, the topological electromagnetic fields have many theoretical and practical applications in different areas of physics as well as of mathematics and represents an active line of research.

In the present paper, we are going to revisit the construction of the knot solutions to the Maxwell's equations in the Ranãda's approach. Also, we are going to review briefly the different mathematical formulations of the Maxwell's equations and to present the construction of the Hopf maps in electrodynamics. Of utmost importance for the understanding of the knot solutions is the method introduced by Bateman to factorize the second rank forms. We will review how this method is applied in the topological electrodynamics. Finally, we are going to present the argument from [22, 23] that show that knot solutions are also present in the non-linear Born-Infeld electrodynamics and other non-linear generalizations of the Maxwell's electrodynamics that recover the standard theory in the weak field limit. We will confine our presentation to the case of the electromagnetic fields in flat space-time. The results reviewed here are not original and can be found in the original papers cited in the text. Other reviews are available too, most notably [34] to which we refer for an updated list of references. In the Appendix, we collect some basic properties of the Hopf mapping. Also, we adopt in this paper the natural units with $c = 1$.

2 Maxwell's equations

In this section we will briefly review the formulation of the Maxwell's electrodynamics in terms of differential forms in the three-dimensional space as well as in the four-dimensional space-time (covariant formulation). This is a well known material which can be found in standard texts on classical electrodynamics such as [36, 37].

2.1 Maxwell's equations in the three-dimensional formulation

The set of the Maxwell's equations in terms of differential forms or in the covariant formulation is equivalent to the set of the standard Maxwell's equations in the vector formulation that have the following well known form [36]

$$\nabla \cdot \mathbf{D} = \rho \,, \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \,, \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \,, \tag{3}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \,. \tag{4}$$

Here, $\mathbf{E}$ and $\mathbf{H}$ denotes the vector intensities of the electric and magnetic fields while $\mathbf{D}$ and $\mathbf{B}$ are the electric and magnetic flux densities, respectively. The sources of the electric and magnetic fields are the density of charge ρ and the density of vector current $\mathbf{J}$. All vectors are three dimensional and without further conditions they can sit ant any point of $\mathbb{R}^3$.

The Maxwell's equations (1) - (4) can be cast in terms of differential forms from either $\Omega^k(\mathbb{R}^3)$ or $\Omega^k(\mathbb{R}^{1,3})$. The advantage of this higher mathematical approach is two-fold: on one hand it provides an economical formulation of the relations of the electromagnetism and on the other hand it gives the mathematical framework necessary for the generalization of the classical electromagnetism to the relativistic case and to the curved space-time, respectively, (see e. g. [37]). The differential forms associated with the three-dimensional electromagnetic field are given in the Table 1.

Table 1: Field content of Maxwell's equations

Vector notation	**Form notation**	**Form rank**	**Field**
$\mathbf{E}$	$\mathcal{E}$	1 - form	electric intensity
$\mathbf{H}$	$\mathcal{H}$	1 - form	magnetic intensity
$\mathbf{D}$	$\mathcal{D}$	2 - form	electric flux density
$\mathbf{B}$	$\mathcal{B}$	2 - form	magnetic flux density
ρ	$\mathcal{Q}$	3 - form	charge density
$\mathbf{J}$	$\mathcal{J}$	2 - form	current density

The forms from the Table 1 can be expanded in terms of components in an orthonormal cartesian basis of $\mathbb{R}^3$ as given by the following relations

$$
\begin{aligned}
\mathcal{E} &= E_x dx + E_y dy + E_z dz \,, \\
\mathcal{H} &= H_x dx + H_y dy + H_z dz \,, \\
\mathcal{D} &= D_x dy \wedge dz + D_y dz \wedge dx + D_z dx \wedge dy \,, \\
\mathcal{B} &= B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \,, \\
\mathcal{Q} &= \rho \, dx \wedge dy \wedge dz \,, \\
\mathcal{J} &= J_x dy \wedge dz + J_y dz \wedge dx + J_z dx \wedge dy \,.
\end{aligned} \tag{5}
$$

Here, we have used the three-dimensional notation (x, y, z) for the spatial coordinates on $\mathbb{R}^3$. To these coordinates it is associated the canonical basis (dx, dy, dz) of 1-forms from $\Omega^1(\mathbb{R}^3)$. A canonical basis of 2-forms $(dx \wedge dy, dy \wedge dz, dz \wedge dx)$ can be constructed from (dx, dy, dz) by using the antisymmetric wedge product defined as follows

$$
\wedge : \Omega^k \times \Omega^s \to \Omega^{k+s} \,, \qquad (\omega, \sigma) \to \omega \wedge \sigma \,, \tag{6}
$$

where the base space has not been specified since the wedge product is defined in the same way on $\mathbb{R}^3$ as well as on $\mathbb{R}^{1,3}$.

In order to write the Maxwell's equations in terms of differential forms, one has to recall that the exterior derivative d is defined as the $\mathbb{R}$-linear map from $\Omega^k \to \Omega^{k+1}$ such that:

$$
df = \partial_i f \, dx^i \,, \tag{7}
$$

$$
d^2 \omega = 0 \,, \tag{8}
$$

$$
d\left(\omega \wedge \sigma\right) = d\left(\omega\right) \wedge \sigma + (-)^k \omega \wedge d\left(\sigma\right) \,, \tag{9}
$$

for all smooth functions f and all forms ω and σ, respectively. The exterior derivative is nilpotent $d^2 = 0$. Also, there is a map on the space of all differential forms given by the Hodge dual operation denoted by $\star$ which is defined as the map from $\Omega^{n-k} \to \Omega^k$, where n is the dimension of the base manifold. The dual map must satisfy the following defining relation

$$\omega \wedge (\star\sigma) = \langle \omega, \sigma \rangle \mathrm{n}\,. \tag{10}$$

Here, we have denoted by n an unitary vector and by $\langle \cdot, \cdot \rangle$ the scalar product. In particular, for any k-form the following action on components holds [37]

$$\omega = \frac{1}{k!}\omega_{i_1,\ldots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_{i_1<\cdots<i_k} \omega_{i_1,\ldots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}\,, \tag{11}$$

$$(\star\omega) = \frac{1}{(n-k)!}(\star\omega)_{i_{k+1},\ldots,i_n} dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_n}\,. \tag{12}$$

The above equations can be easily particularized to the three-dimensional case. For example, the basis that is Hodge dual to the 1-form canonical basis is given by the following relations

$$\star\, dx = dydz\,, \qquad \star dy = dzdx\,, \qquad \star dz = dxdy\,. \tag{13}$$

Similarly, the dual of the 2-form canonical basis is obtained from the equation (12) and it is given by the following relations

$$\star\, dydz = dx\,, \qquad \star dzdx = dy\,, \qquad \star dxdy = dz\,. \tag{14}$$

The equations (13) and (14) illustrate the more general property of involution $\star(\star\omega) = \omega$ in the three-dimensional Euclidean space. Using these general properties, one can show that the exterior derivative d has the following decomposition in the canonical basis

$$d = (\partial_x dx + \partial_y dy + \partial_z dz) \wedge\ . \tag{15}$$

By using the exterior derivative from the equation (15), one can write the Maxwell's equations in terms of the three-dimensional differential forms. The result is given by the following set of equations

$$d\mathcal{D} = \mathcal{Q}\,, \tag{16}$$

$$d\mathcal{B} = 0\,, \tag{17}$$

$$d\mathcal{E} = -\partial_t \mathcal{B}\,, \tag{18}$$

$$d\mathcal{H} = \mathcal{J} + \partial_t \mathcal{D}\,. \tag{19}$$

It is easy to recognize in the equations (16) - (19) the familiar laws of the electromagnetism. The above equations allow one to introduce the electromagnetic potentials Φ and $\mathcal{A}$. To this end, it is instrumental the *Poincar's* *theorem* that states that on a contractible manifolds all closed forms ($d\omega = 0$) are exact, i. e for any exact form ω there exists a form σ such that $\omega = d\sigma$ [37]. By applying this mathematical result, one can show that the electric and magnetic 1-forms can be written as

$$\mathcal{E} = -d\Phi - \partial_t \mathcal{A}\,, \tag{20}$$

$$\mathcal{B} = d\mathcal{A}\,. \tag{21}$$

Then it is an simple exercise to show that the Maxwell's equations (16) - (19) are invariant under the following gauge transformations

$$\Phi \to \Phi' = \Phi - \partial_t \Lambda\,, \tag{22}$$

$$\mathcal{A} \to \mathcal{A}' = \mathcal{A} + d\Lambda\,, \tag{23}$$

where Λ is the (arbitrary, smooth, scalar) gauge parameter.

The above equations (16) - (19) can be used to study the dynamics of the electromagnetic field in terms of the charge and current densities in the language of the differential forms. In order to study the dynamics of charges and currents, too, one needs to add to the above set of equations the Lorentz force. In terms of differential forms, the Lorentz force has the following expression

$$\mathcal{F}_L = \rho\mathcal{E} - \iota_{(\star\mathcal{J})}\mathcal{B}\,. \tag{24}$$

Here, we have used the interior product ι defined as the contraction between a form from Ω^k and a vector field X. The result is a Ω^{k-1} form. For example, for a 2-form $\omega \in \Omega^2$ the interior product with the vector field X is an 1-form $\iota_X \omega$ such that

$$\iota_X \omega(Y) = \omega(X, Y)\,, \tag{25}$$

for any arbitrary vector field Y. This completes the set of equations necessary to describe the classical electrodynamics in terms of differential forms. However, in what follows we are going to focus on the electromagnetic fields in vacuum, that is away from sources and we will ignore the Lorentz force.

2.2 Maxwell's equations in the covariant formulation

The formulation of the Maxwell's electrodynamics in terms of differential forms in the three dimensional space emphasizes the geometric character of the electromagnetic field. However, the invariance of the Maxwell's equations under the Lorentz transformations in the Minkowski space-time $M = \mathbb{R}^{1,3}$ is not apparent in the three-dimensional formulation. The Lorentz symmetry can be kept manifest by writing the Maxwell's equations in the covariant form in terms of differential forms on M. To this end, one has to augment the basis of the 1-forms to $\{dx^\mu\} = \{dx^0 = cdt, dx^i\}$ to include the time variable.[1] Then we introduce the electromagnetic 2-form field F by the following relation

$$F = B + E \wedge dx^0 \,. \tag{26}$$

In this notation, the electromagnetic rank-2 antisymmetric tensor $F_{\mu\nu}$ corresponds to the components of F in the basis $\{dx^\mu \wedge dx^\nu\}$, that is

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \,. \tag{27}$$

The source of the electromagnetic field is the four-current $J^\mu = (\rho, \mathbf{J})$. Its components can be used to construct the 1-form $J = J_\mu dx^\mu$. Then the Maxwell's equations in terms of the differential forms on the space-time take are given by the following two equations

$$dF = 0 \,, \tag{28}$$

$$\star d \star F = J \,. \tag{29}$$

It is easy to show that the homogeneous Maxwell's equation (28) corresponds to the magnetic Gauss' law and the Faraday's law, while the inhomogeneous Maxwell's equation (29) corresponds to the electric Gauss' law and the Maxwell-Ampére law, respectively. From the geometrical point of view, the equation (28) defines the electromagnetic field as an closed 2-form. By using the fact that the exterior derivative is nilpotent $d^2 = 0$, we can derive from the equation (26)- (28) the following expression

$$dF = dB + dE \wedge dx^0 \,. \tag{30}$$

Then by using the following decomposition of the exterior derivative in M into the exterior derivative $\mathbf{d}$ on the spatial subspace $\mathbb{R}^3$ and dx^0 along the time direction $\mathbb{R}$

$$d = dx^0 \wedge \partial_0 + \mathbf{d} \,, \tag{31}$$

one can write the equation (28) with (30) as follows

$$\mathbf{d}E + \partial_0 B = 0 \,, \tag{32}$$

$$\mathbf{d}B = 0 \,. \tag{33}$$

These are the homogeneous Maxwell's equations. In order to derive the inhomogeneous equations, we note that the Hodge dual form $\star F$ can be obtained from F by making the following replacements of components

$$E_j \to -B_j \,, \qquad B_j \to E_j \,. \tag{34}$$

[1] In what follows, we are going to use the indices $\mu, \nu = 0, 1, 2, 3$ to denote the geometrical objects and their components on M. Recall that $c = 1$.

Also, let us introduce the notation $\star$ for the Hodge star operator on $\mathbb{R}^3$. Then one can decompose $\star F$ as follows

$$\star F = \star E - \star B \wedge dx^0 \,. \tag{35}$$

By taking the sequence of operations from the equation (29) one obtains the following set of equations

$$\star \mathbf{d} \star E = \rho \,, \tag{36}$$

$$\star \mathbf{d} \star B - \partial_0 E = \mathcal{J} \,. \tag{37}$$

This concludes the derivation of the three-dimensional Maxwell's equations from the four-dimensional formulation in terms of differential forms.

The vacuum is characterized by the absence of sources, therefore $J = 0$. Then it is easy to see that the Maxwell's equations have the following electromagnetic duality symmetry

$$F \leftrightarrow \star F \,. \tag{38}$$

From the equation (38) it follows that the electromagnetic 2-form F can be written as the sum of a self-dual form F_+ and an anti self-dual F_- form, respectively,

$$F = F_+ + F_- \,, \qquad \star F_\pm = \pm i F_\pm \,. \tag{39}$$

Note that the self-duality relation for 2-forms in the Minkwoski space-time is $\star \star \, \omega = -\omega$. All known results from the vector formulation of the electrodynamics can be formulated in terms of differential forms. Beside being more economical, this framework gives a deeper view of the electromagnetic field and its symmetries. An important information that is provided by the differential forms is about the geometric and topological properties of the electromagnetic field.

3 Knots in Maxwell's electrodynamics

In this section we are going to review the field line solutions of the Maxwell's equations and their relationship with the Hopf knots. These solutions that were discovered by Rañada and communicated in [2, 3]. A very good in depth review of these topics is [34] which we closely follow in the presentation of some key results.

3.1 Ranãda's solutions

The Ranãda solutions represent a particular class of *field line* solutions in which the electromagnetic field is specified in terms of its field lines rather than general functions. The importance of this description is that the geometric and topological character of the electric and magnetic fields are more transparent in three-dimensional space as well as in the four-dimensional space-time. As can be seen from the equations e. g. (28) and (29), the properties of the electromagnetic field either physical or geometrical, are determined by the properties of the sources. Therefore, the electric and magnetic field lines are more easily visualized in the vacuum $J = 0$ where the electromagnetic field also has its highest symmetry as discussed in the previous section. On the other hand, the topological fields are solutions of the Maxwell's equations in vacuum.

In order to define the electromagnetic field in terms of its field lines, one needs a formal description of the latter. That can be given in terms of two smooth scalar complex fields on M

$$\phi : M \to \mathbb{C}\,, \qquad \theta : M \to \mathbb{C}\,. \tag{40}$$

By introducing the above functions, one can express the electric and the magnetic field lines as level lines of θ and ϕ, respectively. In the covariant formulation, the corresponding electromagnetic fields take the following form [34]

$$F_{\mu\nu} = g(\bar{\phi},\phi)\left(\partial_\mu\bar{\phi}\,\partial_\nu\phi - \partial_\nu\bar{\phi}\,\partial_\mu\phi\right)\,, \tag{41}$$

$$\star F_{\mu\nu} = f(\bar{\theta},\theta)\left(\partial_\mu\bar{\theta}\,\partial_\nu\theta - \partial_\nu\bar{\theta}\,\partial_\mu\theta\right)\,, \tag{42}$$

where the functions g and f are smooth on their variables θ and ϕ and the dual electromagnetic field has the following form

$$\star\, F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\,. \tag{43}$$

From the above tensor formulation we can obtain the explicit vector fields **E** and **B**. The three formulations of the electromagnetism, namely in terms of differential three-dimensional forms, the covariant formulation and the formulation in terms of vector fields, are related by the following relations

$$F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu = -\varepsilon_{jkl}B_j dx^k \wedge dx^l + E_j dx^j \wedge dx^0\,, \tag{44}$$

$$\star F = \frac{1}{2}\star F_{\mu\nu}dx^\mu \wedge dx^\nu = \varepsilon_{jkl}E_j dx^k \wedge dx^l + B_j dx^j \wedge dx^0\,. \tag{45}$$

Note that the field line covariant solutions (41) and (42) do not display the geometry of the electromagnetic field in $\mathbb{R}^3$ explicitly. Moreover, they define a rather large class of solutions parametrized by the pair of functions (g, f). Therefore, in order to obtain a clear picture of the electric and magnetic field lines, one has to go to the vectorial representation given by the equations (44) and (45) above. Also, it is necessary to fix the functions (g, f) in order to obtain concrete geometries. The first solution that historically has achieved these goals is the Ranãda's solution which has the following electric and magnetic decomposition

$$E_j = \frac{\sqrt{a}}{2\pi i} \left(1 + |\theta|^2\right)^{-2} \varepsilon_{jkl}\, \partial_k \bar{\theta}\, \partial_l \theta \,, \tag{46}$$

$$B_j = \frac{\sqrt{a}}{2\pi i} \left(1 + |\phi|^2\right)^{-2} \varepsilon_{jkl}\, \partial_k \bar{\phi}\, \partial_l \phi \,. \tag{47}$$

The electromagnetic duality imposes constraints on the fields θ and ϕ, respectively. These constraints can be easily found by substituting the functions $g(\bar{\phi}, \phi)$ and $f(\bar{\theta}, \theta)$ from the equations (46) and (47) into the equation (38). The result is given by the following equations [34]

$$\left(1 + |\phi|^2\right)^{-2} \varepsilon_{jmn} \partial_m \phi \partial_n \bar{\phi} = \left(1 + |\theta|^2\right)^{-2} \left(\partial_0 \bar{\theta} \partial_j \theta - \partial_0 \theta \partial_j \bar{\theta}\right) \,, \tag{48}$$

$$\left(1 + |\theta|^2\right)^{-2} \varepsilon_{jmn} \partial_m \bar{\theta} \partial_n \theta = \left(1 + |\phi|^2\right)^{-2} \left(\partial_0 \bar{\phi} \partial_j \phi - \partial_0 \phi \partial_j \bar{\phi}\right) \,. \tag{49}$$

The equations (48) and (49) represent a set of non-linear partial differential equations. Their solutions are the functions θ and ϕ that define the electric and magnetic field lines through their level curve equations.

As one can see from the equations (46) and (47), the Ranãda's fields correspond to particular values of the functions g and f. In order to understand their physical properties, recall that there are *null field* solutions to the Maxwell's equations in vacuum defined by the following equations

$$E_j B_k \delta_{jk} = 0 \,, \tag{50}$$

$$\delta_{jk} \left(E^j E^k - B^j B^k\right) = 0 \,. \tag{51}$$

It is easy to see that $\mathbf{E}$ and $\mathbf{B}$ satisfy the first null field equation (50) but not the second one. Thus, the electric and magnetic fields are orthogonal to each other and so are their field lines.

The parametrization of the electromagnetic 2-forms F and $\star F$ in terms of the functions ϕ and θ is not the only possible one. The are other parametrizations available are due to the geometric properties of the solutions given by the equations (46) and (47). One possibility is to use the so called *Clebsch parametrization.* Since the electromagnetic 2-forms F and $\star F$ satisfy the Darboux theorem [37] one can express F and $\star F$ in terms of canonical 1-forms $d\sigma^a$, $d\xi^a$, $d\rho^a$ and $d\zeta^a$ [34] as follows

$$F = \delta_{ab}\, d\sigma^a \wedge d\xi^b\,, \tag{52}$$

$$\star F = \delta_{ab}\, d\rho^a \wedge d\zeta^b\,, \tag{53}$$

$$d\sigma^a \wedge d\xi^a = d\rho^a \wedge d\zeta^a = -i\tau_2\,, \tag{54}$$

where the indices $a, b = 1, 2$ enumerate the number of 1-forms and τ_2 denotes the Pauli matrix. The Clebsch representation of the electromagnetic fields takes a simpler form for fields that satisfy the equation (50), namely

$$F = d\sigma \wedge d\xi\,, \tag{55}$$

$$\star F = d\rho \wedge d\zeta\,. \tag{56}$$

Another parametrization of the electromagnetic fields is given in terms of *Euler potentials* which are smooth scalar real fields on M [34]

$$\alpha_a : M \to \mathbb{C}\,, \qquad \beta_a : M \to \mathbb{C}\,, \qquad a = 1, 2\,, \tag{57}$$

By using these functions the electric and magnetic fields can be written as follows

$$E_j = \varepsilon_{jkl}\, \partial_k \beta_2\, \partial_l \beta_1\,, \tag{58}$$

$$B_j = \varepsilon_{jkl}\, \partial_k \alpha_2\, \partial_l \alpha_1\,. \tag{59}$$

By comparing the two sets of equations (46)-(47) and (58)-(59) with each other, one can easily find the following relation among the parameters of the Ranãda and Euler parametrizations, respectively,

$$\beta_1 = \left(1 + |\theta|^2\right)^{-1}\,, \qquad \beta_2 = \frac{1}{2\pi}\arg(\theta)\,, \tag{60}$$

$$\alpha_1 = \left(1 + |\phi|^2\right)^{-1}\,, \qquad \alpha_2 = \frac{1}{2\pi}\arg(\phi)\,. \tag{61}$$

These representations are useful to express different properties of the field line solutions. Other representations can be found for more general topological electromagnetic fields. But before discussing this case, let us take a closer look at the Ranãda's fields.

3.2 Electromagnetic knot fields

As we have seen above, the electromagnetic field characterized by the equations (41) and (42) is parametrized by the functions g and f. In particular, by choosing these functions as in the equations (46) and (47), the field line solutions display non-trivial topological properties. In order to see that, we note that the complex functions ϕ and θ should be chosen such that the electromagnetic field and the observables constructed from it, e. g. the energy, linear momentum and angular densities, respectively, be finite. Therefore, if the field is defined in the full $\mathbb{R}^3$ space, the regularity condition to be imposed is

$$|\phi(x)| \to 0\,, |\theta(x)| \to 0 \text{ if } |\mathbf{x}| \to \infty\,. \tag{62}$$

That implies that at any given value of time the defining domain of the complex fields ϕ and θ is the compactification $\mathbb{R}^3 \cup \{\infty\} = S^3$. Also, ϕ and θ take values in the compactification $\mathbb{C} \cup \{\infty\} = S^2$. Thus, ϕ and θ can be viewed as two families of one-parameter maps $\phi(x) = \{\phi_t(\mathbf{x})\}_{t\in\mathbb{R}}$ and $\theta(x) = \{\theta_t(\mathbf{x})\}_{t\in\mathbb{R}}$, respectively, from S^3 to S^2. Since the discussion concerns the magnetic and the electric field lines separately, it is convenient to use the vector notation.

In the vacuum, one can express the fields $\mathbf{E}$ and $\mathbf{B}$ in terms of electromagnetic potentials $\mathbf{C}$ and $\mathbf{A}$ in a symmetric way

$$E_j = \varepsilon_{jkl}\partial_k C_l\,, \qquad B_j = \varepsilon_{jkl}\partial_k A_l\,. \tag{63}$$

Note that $\mathbf{C}$ is dependent on $\mathbf{A}$ as they are related by the following equation

$$\varepsilon_{jmn}\partial_m C_n = -\partial_0 A_j\,. \tag{64}$$

However, it is convenient to keep $\mathbf{C}$ explicit as it makes the electric-magnetic duality more symmetric. The helicities of the electromagnetic field are defined by the Chern-Simons integrals associated to the electric and magnetic fields and their potentials as follows

$$H_{ee} = \int d^3x\, \delta_{ij} E_i C_j = \int d^3x\, \varepsilon_{jkl} C_j \partial_k C_l\,, \tag{65}$$

$$H_{mm} = \int d^3x\, \delta_{ij} B_i A_j = \int d^3x\, \varepsilon_{jkl} A_j \partial_k A_l\,, \tag{66}$$

$$H_{em} = \int d^3x\, \delta_{ij} B_i C_j = \int d^3x\, \varepsilon_{jkl} C_j \partial_k A_l\,, \tag{67}$$

$$H_{mm} = \int d^3x\, \delta_{ij} E_i A_j = \int d^3x\, \varepsilon_{jkl} A_j \partial_k C_l\,. \tag{68}$$

As is reviewed in the Appendix, the pure electric and magnetic helicites given above are the Hopf's indexes of the corresponding electric and magnetic field lines. If one substitutes the equations (63) into the Maxwell's equations (1)

- (4) in vacuum, one obtains the following set of equations

$$\varepsilon_{jmn}\partial_m\left(\frac{\partial A_n}{\partial x^0}+\varepsilon_{nrs}\partial_r C_s\right)=0\,, \tag{69}$$

$$\varepsilon_{jmn}\partial_m\left(\frac{\partial C_n}{\partial x^0}-\varepsilon_{nrs}\partial_r A_s\right)=0\,. \tag{70}$$

From the vector calculus we conclude that there are two scalar functions κ_1 and κ_2 such that

$$\frac{\partial A_j}{\partial x^0}+\varepsilon_{jmn}\partial_m C_n=\partial_j\kappa_1\,, \tag{71}$$

$$\frac{\partial C_j}{\partial x^0}-\varepsilon_{jmn}\partial_m A_n=\partial_j\kappa_2\,. \tag{72}$$

By using the equations above, one can prove that

$$\delta_{mn}\frac{\partial(A_mB_n)}{\partial x^0}+2\delta_{mn}E_mB_n-\partial_k\left(\varepsilon_{krs}A_kE_s-\kappa_1B_k\right)=0\,. \tag{73}$$

$$\delta_{mn}\frac{\partial(C_mE_n)}{\partial x^0}+2\delta_{mn}E_mB_n+\partial_k\left(\varepsilon_{krs}C_kB_s+\kappa_2E_k\right)=0\,. \tag{74}$$

From these, it follows that

$$\frac{\partial(H_{mm}-H_{ee})}{\partial x^0}+4\int d^3x\delta_{mn}E_mB_n=0\,, \tag{75}$$

$$\frac{\partial(H_{mm}+H_{ee})}{\partial x^0}=0\,. \tag{76}$$

The above equations (75) and (76) imply that the pure electric and magnetic heliticities are conserved if the fields satisfy the first null field condition (50).

Let us quote here the explicit Hopfion solution obtained in [6]. The scalar fields ϕ and θ are given by the following equations

$$\phi=\frac{(ax_1-x_0x_3)+i(ax_2+x_0(a-1)}{(ax_3+x_0x_1)+i[a(a-1)-x_0x_1]}\,, \tag{77}$$

$$\theta=\frac{[ax_2+x_0(a-1)]+i(ax_3+x_0x_1)}{(ax_1-x_0x_3)+i[a(a-1)-x_0x_2]}\,, \tag{78}$$

where the x_μ's are dimensionless coordinates and

$$a=\frac{r^2-x_0^2+1}{2}\,,\qquad r=\sqrt{\delta_{mn}x_m^2x_n^2}\,. \tag{79}$$

The electric and magnetic fields derived from the above scalars have the following expression

$$\mathbf{E} = \frac{c}{\pi} \frac{q\mathbf{H}_1 - p\mathbf{H}_2}{(a^2 + x_0^2)^3} , \tag{80}$$

$$\mathbf{B} = \frac{1}{\pi} \frac{q\mathbf{H}_1 + p\mathbf{H}_2}{(a^2 + x_0^2)^3} , \tag{81}$$

where

$$\begin{aligned} \mathbf{H}_1 &= (x_2 + x_0 - x_1 x_3)\, \mathbf{e}_1 - [x_1 + (x_2 + x_0)\, x_3]\, \mathbf{e}_2 \\ &\quad + \frac{1}{2} \left[-1 - x_3^2 + x_1^2 + (x_2 + x_0)^2\right] \mathbf{e}_3 \end{aligned} \tag{82}$$

$$\begin{aligned} \mathbf{H}_2 &= +\frac{1}{2} \left[1 + x_1^2 - x_3^2 - (x_2 + x_0)^2\right] \mathbf{e}_1 + [-x_3 + (x_2 + x_0)\, x_1]\, \mathbf{e}_2 \\ &\quad + (x_2 + x_0 + x_1 x_3)\, \mathbf{e}_3 , \end{aligned} \tag{83}$$

and

$$p = x_0 \left(x_0^2 - 3a^2\right) , \qquad q = a \left(a^2 - 3x_0^2\right) . \tag{84}$$

Here, we have denoted by $\{\mathbf{e}_l\}$ the unit vectors of the cartesian basis that spans the spatial directions. The above Hopfion solution has the Hopf's indices $H(\phi) = H(\theta) = 1$. For a thorough review of its properties we refer to [34].

Another example is given by the Hopf's map given by the following scalar fields

$$\phi = \frac{2(x_1 + ix_2)}{2x_3^2 + i(\delta_{mn} x^m x^n - 1)} , \tag{85}$$

$$\theta = \bar{\phi} . \tag{86}$$

The electromagnetic field corresponding to these scalars has the following components

$$E_m = \frac{1}{\sqrt{(2\pi)^3}} \int d^3k \left[P_m(k_j) \cos\left(\eta_{\mu\nu} k^\mu x^\nu\right) - Q_m(k_j) \sin\left(\eta_{\mu\nu} k^\mu x^\nu\right)\right] , \tag{87}$$

$$B_m = \frac{1}{\sqrt{(2\pi)^3}} \int d^3k \left[P_m(k_j) \cos\left(\eta_{\mu\nu} k^\mu x^\nu\right) + Q_m(k_j) \sin\left(\eta_{\mu\nu} k^\mu x^\nu\right)\right] , \tag{88}$$

where

$$\mathbf{P} = \frac{e^{-k_0}}{\sqrt{2\pi}} \left(-\frac{k_1 k_3}{k_0}, \frac{k_0 k_2 + k_2^2 + k_3^2}{k_0}, -\frac{k_0 k_1 + k_1 k_2}{k_0}\right) , \tag{89}$$

$$\mathbf{Q} = \frac{e^{-k_0}}{\sqrt{2\pi}}\left(-\frac{k_0k_2 + k_1^2 + k_2^2}{k_0}, \frac{k_1k_3}{k_0}, \frac{k_0k_3 + k_2k_3}{k_0}\right) . \tag{90}$$

The solution (87) and (88) describe a particular wave packet that travels along the x_3 axis and has the following energy, linear momentum and angular momentum densities, respectively,

$$E = 2\,, \qquad \mathbf{p} = (0, 0, 1)\ , \qquad \mathbf{L} = (0, 0, 1)\ . \tag{91}$$

The above relations are valid in a dimensionless parametrization of space-time coordinates. From the equations (91) it follows that the mass of the wave packet is finite and it has the value $m^2 = 3$.

4 Electric and magnetic knots in the Bateman's parametrization

As we have seen in the previous section, there are several possible parametrizations of the field line solutions. In this section we present the factorized parametrization of the self-dual electromagnetic fields in terms of 1-forms constructed from scalar complex functions given by Bateman in [39]. This representation is useful for the generalization of the field line solutions to the gravitating electromagnetic fields [31] as well as to the non-linear electromagnetism [22, 23]. For extensive reviews of the properties of the Hopfions in the Bateman's representation see [20],[34], [40].

Let us recall the covariant form of the Maxwell's equation in vacuum given by the equations (28) and (29), namely

$$dF = 0\,, \tag{92}$$

$$d \star F = 0\,. \tag{93}$$

The equation (92) states that the electromagnetic 2-form field is closed. Then F can be written in terms of two scalar complex functions

$$\alpha : M \to \mathbb{C}\,, \qquad \beta : M \to \mathbb{C}\,, \tag{94}$$

as follows

$$F = d\alpha \wedge d\beta\,. \tag{95}$$

It is relevant to write the above equation in terms of the electric and magnetic fields. To this end, we use the components of the electromagnetic 2-form F which are the same as the components of the electromagnetic tensor $F_{\mu\nu}$. It follows from the equation (95) that $F_{\mu\nu}$ satisfies the following relation

$$i\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} - 2F^{\mu\nu} = 2\varepsilon^{\mu\nu\rho\sigma}\partial_\rho\alpha\partial_\sigma\beta\,. \tag{96}$$

By definition, the electric and magnetic fields in terms of $F_{\mu\nu}$ are given by the following identification

$$- E^m = F^{0m} \, , \qquad B^m = \frac{1}{2} \varepsilon^{mnp} F_{np} \, . \tag{97}$$

Then after a short algebra, one can derive from the equation (96) the following relation between the complex scalars

$$B_m - iE_m = i \left(\partial_0 \alpha \partial_m \beta - \partial_0 \beta \partial_m \alpha \right) \, . \tag{98}$$

Since the electromagnetic field in vacuum is either self-dual or anti self-dual according to the eigenvalues of the $\star$ operation as given by the equation (39), it follows that the functions α and β should obey the following relation

$$\nabla \alpha \times \nabla \beta = \pm i \left(\partial_0 \alpha \nabla \beta - \partial_0 \beta \nabla \alpha \right) \, . \tag{99}$$

In order to understand the geometry of the electric and magnetic fields on the spatial $\mathbf{R}^3$ manifold, it is useful to introduce the following Riemann-Silberstein vector

$$\mathbf{F} = \mathbf{B} \pm i \mathbf{E} \, , \tag{100}$$

where $\mathbf{E}$ and $\mathbf{B}$ could be possibly complex. Then it is required that $\mathbf{F}$ be a solution of the Bateman's equation

$$\delta_{mn} F_m F_n = 0 \, . \tag{101}$$

Note that in general the norm of the field $\mathbf{F}$ is non-zero

$$\delta_{mn} \bar{F}_m F_n \neq 0 \, , \tag{102}$$

where the bar stands for the complex conjugate. One can easily write the equations (101) and (102) on components. The result is given by the following equations

$$\delta_{mn} \left(B_m B_n - E_m E_n \right) \pm 2i \delta_{mn} E_m B_n = 0 \, , \tag{103}$$

$$\delta_{mn} \left(\bar{B}_m B_n + \bar{E}_m E_n \right) = 0 \, . \tag{104}$$

If the vector fields $\mathbf{E}$ and $\mathbf{B}$ are real, it follows that the left-hand side of the equation (103) is an invariants of the electromagnetic field and the left-hand side of the equation (104) corresponds to the energy density of the electromagnetic field. The equations (103) and (104) define the so called *null fields* [19].

It is easy to verify that the Bateman's solutions conserve the energy, momentum and the angular momentum of the electromagnetic field due to the Lorentz symmetry. Also, the $U(1)$ symmetry of these equations implies that there is a conserved four-current whose components are given by the following equations

$$\rho = \frac{1}{2}\delta_{mn}\left(\frac{1}{c^2}E_m E_n + B_m B_n\right), \tag{105}$$

$$J_k = \varepsilon_{kmn} E_m B_n \,. \tag{106}$$

Besides the above known conserved quantities that are the result of the symmetries of the Maxwell's equations, there are new topologically conserved charges that are the consequence of the topology of the Bateman's solutions represented by the helicities given by the equations (65)-(66).

The geometry and the topology of the electric and magnetic fields can be determined by constructing explicit solutions. The Bateman's equations are satisfied by the electromagnetic Hopf knots or Hopfions, as in the other parametrizations discussed above, and also by tori. One advantage of the Bateman's representation is that if one solution is known, new solutions can be constructed due to the following property [28]:

Let α and β be two smooth complex scalar fields on M that satisfy the Bateman's relation (99). Then for any two arbitrary smooth complex functions f and g defined on $\mathbb{C}^2$ one can construct the following 2-form

$$\mathcal{F} := df(\alpha,\beta) \wedge dg(\alpha,\beta)\,, \tag{107}$$

and $\mathcal{F}$ has the following properties

$$d\mathcal{F} = 0\,, \tag{108}$$

$$\star\mathcal{F} = \pm i\mathcal{F}\,. \tag{109}$$

There are several known solutions constructed by using the Bateman's method in the literature. Let us cite here the Hopfion obtained in [19] whose complex functions are given by the following relations

$$\alpha = \frac{-x_0^2 + \delta_{mn}x^m x^n - 1 + 2ix_3}{-x_0^2 + \delta_{mn}x^m x^n + 1 + 2ix_0}\,, \tag{110}$$

$$\beta = \frac{x_1 - ix_2}{-x_0^2 + \delta_{mn}x^m x^n + 1 + 2ix_0}\,, \tag{111}$$

The complex functions α and β satisfy the following relation

$$|\alpha|^2 + |\beta|^2 = 1\,. \tag{112}$$

The electromagnetic vector $\mathbf{F}$ corresponding to the functions from the equations (110) and (111) has the form

$$\mathbf{F} = \frac{4}{(-x_0^2 + \delta_{mn}x^m x^n + 1 + 2ix_0)^3} \\ \times \begin{bmatrix} (x_0 - x_1 - x_3 + i(x_2 - 1))(x_0 + x_1 - x_3 - i(x_2 + 1)) \\ -i(x_0 - x_2 - x_3 - i(x_1 + 1))(x_0 + x_2 - x_3 + i(x_1 - 1)) \\ 2(x_1 - ix_2)(x_0 - x_3 - i) \end{bmatrix}, \quad (113)$$

where we have written $\mathbf{F}$ as a column vector. The general properties of the field (113) are discussed in [19, 20].

A method to obtain more Hopfions in the Bateman's representation is by performing infinitesimal conformal deformations or a subgroup of them on the functions α and β of a given solution. In general, it is known that the scalars α and β transform as follows under the infinitesimal coordinate trasformations [20]

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu \quad (114)$$

$$\alpha(x) \to \alpha'(x') = \alpha(x) + \xi^\nu \partial_\nu \alpha(x), \quad (115)$$

$$\beta(x) \to \beta'(x') = \beta(x) + \xi^\nu \partial_\nu \beta(x), \quad (116)$$

where $\xi = \xi^\mu \partial_\mu$ is an arbitrary infinitesimal smooth vector field on M. The equation (99) is invariant under the transformations (114) - (116) if the following condition is satisfied [20]

$$\epsilon_{rmn}\left[\delta_{jr}\left(\partial_0\xi_0 - \partial_s\xi_s\right) + i\epsilon_{jrs}\left(-\partial_0\xi_s + \partial_s\xi_0\right) + \partial_j\xi_r + \partial_r\xi_j\right]\partial_m\alpha\partial_n\beta = 0\,. \quad (117)$$

The equation (117) is satisfied by the generators of the special conformal transformations

$$\xi_\mu = a_\mu \eta_{\mu\nu} x^{\mu\nu} - 2a_\nu x^\nu x^\mu\,. \quad (118)$$

This fact allowed the authors of [20] to construct new solutions characterized by integer powers p and q of the scalar functions α and β, respectively. An example of electromagnetic field suitable for treatment with the conformal deformations is the plane wave solution characterized by

$$\alpha = e^{\imath(x_3 - x_0)}\,, \qquad \beta = x_1 + ix_2, \quad (119)$$

from which it was obtained the following Hopfion

$$\alpha = \exp\left[-1 + \frac{i(x_0 + x_3 - i)}{1 - x_0^2 + \delta_{mn}x^m x^n + 2ix_0}\right], \quad (120)$$

$$\beta = \frac{x_1 + ix_2}{1 - x_0^2 + \delta_{mn}x^m x^n + 2ix_0}\,. \tag{121}$$

The electric and magnetic fields obtained from the scalars given by the equations (120) and (121) have a *toric topology*. A generalization of this method to more complex knotted electromagnetic fields was given in [41]. The same method was applied to the study of the physical properties of the optical vorticies in [42, 28].

5 Knots in nonlinear electrodynamics

The Bateman method presented above allows one to construct topologically non-trivial solutions to the equations of motion in the non-linear electrodynamics that has as its weak field limit the Maxwell's electrodynamics as discussed in [22, 23]. In these works, the authors showed that there are knot solutions of the equations of motion in any non-linear extension of the Maxwell's electrodynamics that satisfies this requirement. In what follows, we are going to consider in some detail the Born-Infeld electrodynamics. The non-linear Born-Infeld action functional can be written in the following form

$$S_{BI} = -\alpha^2 \int d^4x \left(\sqrt{1 + F - P^2} - 1\right)\,, \tag{122}$$

where α is a constant of dimension 2 and L and P are the usual Lorentz invariants. In terms of the three-dimensional electric and magnetic vector fields, these have the following expressions

$$L = \alpha^{-2}\delta_{mn}\left(B^m B^n - E^m E^n\right)\,, \tag{123}$$

$$P = \alpha^{-2}\delta_{mn}E^m B^n\,. \tag{124}$$

The invariants L and P are zero for the null-fields that satisfy the equations (50) and (51). In order to prove that the theory described by (122) admits Hopfion solutions, the following argument has been developed in [22, 23]. Starting from the action (122), define new vector fields $\mathbf{H}$ and $\mathbf{D}$ whose components can be written in a notation analogous to the Maxwell's electrodynamics and are given by the following relations

$$H_m = -\frac{\partial \mathcal{L}_{BI}}{\partial B^m}\,, \qquad D_m = \frac{\partial \mathcal{L}_{BI}}{\partial E^m}\,. \tag{125}$$

Here, $\mathcal{L}_{BI}$ is the Lagrangian density from the action (122). By using it, one obtains the explicit form of H_m and D_m as follows

$$H_m = -\frac{1}{\sqrt{1 + F - P^2}}\left(B_m - PE_m\right)\,, \tag{126}$$

$$D_m = -\frac{1}{\sqrt{1+F-P^2}}\left(E_m - PB_m\right) . \tag{127}$$

By applying the variational principle to the action (122) one obtains the following equations of motion

$$\partial_m D_m = 0\,, \tag{128}$$

$$\partial_m B_m = 0\,, \tag{129}$$

$$\varepsilon_{mnr}\partial_n E_r = -\partial_0 B_m\,, \tag{130}$$

$$\varepsilon_{mnr}\partial_n H_r = \partial_0 D_m\,. \tag{131}$$

The key observation now is that the null fields that obey the equations (50) and (51) produce a theory in which the equations of motion are of Maxwellian type. Therefore, by using the results presented in the Section 2, one concludes that there are Hopfion solutions in the non-linear Born-Infeld electrodynamics for null fields.

The same argument can be generalized to all non-linear actions that contain in the weak field limit the Maxwell's electrodynamics. Also, due to the similarity between the structure of the electrodynamics and the fluid mechanics, Hopfion solutions of the fluid flow lines were shown to exist in [22, 23].

Appendix - The Hopf's map

In this Appendix we briefly review the definition and some basic properties of the Hopf's map that have been used in the text. A classical reference on this topic is [38] in which the Hopf's theory is presented from a geometrical point of view.

Consider an n-dimensional unit sphere S^n defined as usual by the following relation

$$S^n = \{\, \mathbf{x} \in R^{n+1} : \delta_{ij}x_i x_j = 1,\, i,j = 0,1,\ldots,n \,\}\,. \tag{132}$$

The *Hopf fibration* is the mapping $h : S^3 \to S^2 \simeq \mathbb{CP}^1$ defined by the following *Hopf map*

$$h(x_0,x_1,x_2,x_3) = \left(x_0^2 + x_1^2 - x_2^2 - x_3^2\,, 2(x_0x_3 + x_1x_2)\,, 2(x_1x_2 - x_0x_3)\right)\,. \tag{133}$$

Alternatively, the Hopf map can be written in terms of complex coordinates. To this end, one defines the complex coordinates on the spheres S^3 and S^2

$$S^2 = \{(x,z) \in \mathbb{R}\times\mathbb{C} : x^2 + |z|^2 = 1\}\,, \tag{134}$$

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \, : \, |z_1|^2 + |z_2|^2 = 1\} \tag{135}$$

Then the Hopf map takes the following form

$$h(z_1, z_2) = \left(2z_1\bar{z}_2, |z_1|^2 - |z_2|^2\right) \, . \tag{136}$$

Since the projective space is a quotient set $\mathbb{CP}^1 \simeq S^3/U(1)$, the action of the $U(1)$ on S^3 defines the fibres over S^3. These are mapped into the fibres over S^2 as follows. Consider the equivalence class of points of S^3 defined by the following relation

$$(z'_1, z'_2) \sim (z_1, z_2) \, : \, \text{if} \, \exists \, \lambda \in \mathbb{C} \text{ such that } (z'_1, z'_2) = (\lambda z_1, \lambda z_2) \, . \tag{137}$$

Then the Hopf map of (z'_1, z'_2) satisfies the following relation

$$h(z'_1, z'_2) = |\lambda|^2 h(z_1, z_2) \, . \tag{138}$$

The points (z'_1, z'_2) and (z_1, z_2) of S^3 belong to a fibre over S^2 if they are mapped onto the same point of S^2. This represents a constraint on the parameter λ which is satisfied by any λ of the form $\lambda = \exp(i\vartheta)$, where $\vartheta \in [0, 2\pi)$. Thus, λ belong to the defining representation of $U(1)$.

From the geometrical point of view, the fibre of S^3 is the great circle that contains (z_1, z_2). An useful parametrization of fibres is given by the following relations

$$z_1 = \exp\left(i\xi + \frac{\varphi}{2}\right) \sin\left(\frac{\vartheta}{2}\right) \, , \qquad z_2 = \exp\left(i\xi - \frac{\varphi}{2}\right) \cos\left(\frac{\vartheta}{2}\right) \, . \tag{139}$$

It is easy to see that the Hopf's map h takes the points from the fibres of S^3 to the following points on the $S^2 \subset \mathbb{R}^3$

$$x_1 = 2|z_1||z_2| \cos(\varphi) \, , \tag{140}$$
$$x_2 = 2|z_1||z_2| \sin(\varphi) \, , \tag{141}$$
$$x_3 = |z_1|^2 - |z_2|^2 \, . \tag{142}$$

The stereographic projections follow from the very definition. Possible parametrizations of them are given by the equations

$$\pi_2(x_1, x_2, x_3) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}\right) \tag{143}$$
$$\pi_3(x_0, x_1, x_2, x_3) = \left(\frac{x_1}{1 - x_0}, \frac{x_2}{1 - x_0}, \frac{x_3}{1 - x_0}\right) \, . \tag{144}$$

The inverse of the Hopf map $\gamma = h^{-1}$ takes points (x,z) from S^2 into loops on S^3. In general, if the points are different $(x,z) \neq (x',z')$ so are the corresponding loops $\gamma \neq \gamma'$. However, there is an object associated to the loops called *Hopf's invariant* of h that is the (entire) linking number of the two loops (also called the *Hopf's index*) and which has the following general form

$$H(h) = l(\gamma, \gamma')\,. \tag{145}$$

$H(h)$ is an homotopy invariant that characterizes the Hopf's map. It is useful to write the Hopf's invariant in terms of a Chern-Simons integral [35] of vector fields. This can be done as follows. Consider an unit vector field $\mathbf{U}(\mathbf{x})$ with the following properties

$$\delta_{ij} U_i(\mathbf{x}) U_j(\mathbf{x}) = 1\,, \qquad |\mathbf{U}(\mathbf{x})| \to \mathbf{u}\,, \text{ if } |\mathbf{x}| \to \infty\,, \tag{146}$$

where $\mathbf{u}$ is a constant unit vector. Then define the field $\mathbf{F}(\mathbf{x})$ by giving its components as in the equation below

$$F_j(\mathbf{x}) = \varepsilon_{jmn} \varepsilon_{prs} U_p(\mathbf{x}) \partial_m U_r(\mathbf{x}) \partial_n U_s(\mathbf{x})\,. \tag{147}$$

The field defined above can be written in terms of a potential $\mathbf{A}(\mathbf{x})$ as follows

$$F_j(\mathbf{x}) = \varepsilon_{jmn} \partial_m A_n(\mathbf{x})\,. \tag{148}$$

Then the Hopf's index for the field $\mathbf{U}(\mathbf{x})$ is given by the following equalities

$$H(\mathbf{U}) = \int d^3x \delta_{ij} F_i(\mathbf{x}) A_j(\mathbf{x}) = \int d^3x \varepsilon_{jmn} A_j(\mathbf{x}) \partial_m A_n(\mathbf{x})\,. \tag{149}$$

We recognize in the last equality above the explicit form of the Chern-Simons integral for the potential $\mathbf{A}(\mathbf{x})$.

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