Géométrie différentielle et théorie de jauge, 21/01/2020

1. Calculate the volume form of S^2 in angular coordinates (ϕ, θ), where ϕ is the angle with the vertical axis, and θ the angle in the horizontal plane. Calculate the area of S^2.

2. Use Stokes formula and the inequality $\int_0^L x^2(s)ds \leq \frac{L^2}{(2\pi)^2}\int_0^L \dot{x}^2(s)ds$ for any L-periodic function such that $\int_0^L x(s)ds = 0$ (this inequality follows from decomposition in Fourier series), to prove the isoperimetric inequality for a smooth domain $D \subset \mathbb{R}^2$: $\mathrm{area}(D) \leq \frac{1}{4\pi}\mathrm{length}(\partial D)^2$.

3. Show that the projective space $\mathbb{R}P^n$ is orientable if and only if n is odd (use the projection $p : S^n \to \mathbb{R}P^n$ and use $\tau^* p^* \omega = p^* \omega$, where τ is the antipodal map of S^n). Show that the Möbius band $(S^1 \times \mathbb{R})/\{(z, x) \mapsto (-z, -x)\}$ is not orientable.

4. Let X be a vector field generating the flow of diffeomorphism $(\phi_t)_{t\in\mathbb{R}}$, defined by $\phi_0(x) = x$ and $\frac{d}{dt}\phi_t(x) = X(\phi_t(x))$. For a p-form α define the p-form $\mathscr{L}_X\alpha = \frac{d}{dt}\Big|_{t=0}\phi_t^*\alpha$. Prove that $\mathscr{L}_X \circ d = d \circ \mathscr{L}_X$, $\mathscr{L}_X(\alpha \wedge \beta) = (\mathscr{L}_X\alpha) \wedge \beta + \alpha \wedge \mathscr{L}_X\beta$. Deduce the **Cartan formula**

$$\mathscr{L}_X\alpha = i_X d\alpha + d(i_X\alpha).$$

5. Let M^n be an oriented compact manifold with boundary ∂M. Prove that there is no retraction of M on its boundary, that is no smooth map $r : M \to \partial M$ such that $r(x) = x$ if $x \in \partial M$. (Apply Stokes theorem to $r^*\omega$, where ω is a $(n-1)$-form on ∂M of nonzero integral).

6. Show that if $i : Y^k \hookrightarrow M$ is an oriented compact submanifold of M, then $\alpha \mapsto \int_Y i^*\alpha$ defines a linear form on $H^k(M)$.

7. A theorem of De Rham says that if $\alpha \in \Omega^k M$ is closed and satisfies $\int_Y \alpha = 0$ for all k-dimensional compact oriented submanifolds $Y \subset M$, then $[\alpha] = 0$. Using this theorem, give a proof that $H^k(S^n) = 0$ for $0 < k < n$. (Use the fact that $H^k(\mathbb{R}^n) = 0$ for $k > 0$).

8. Show that $H^1(\mathbb{R}^2 \setminus \mathbb{Z})$ is infinite dimensional : look at what happens when we integrate along small circles along each point of $\mathbb{Z}$.

9. Let X be a vector field on M, generating a flow (ϕ_t) of diffeomorphisms. Let α be a closed form on M. Using $\phi_{t+t'} = \phi_t \circ \phi_{t'}$, prove that

$$\frac{d}{dt}\phi_t^*\alpha = \phi_t^*\mathscr{L}_X\alpha.$$

Deduce from the Cartan formula (above) that the cohomology class $[\phi_t^*\alpha]$ is constant. Prove that if a *connected* Lie group G acts on M, then the action of G on $H(M)$ is trivial.

10. Prove the formulas given in the lecture :

$$*^2 = (-1)^{p(n-p)} \text{ on } \Omega^p$$

$$*(\alpha_i e^i) = (-1)^{j-1}\sqrt{\det(g_{ij})}\,g^{ij}\alpha_i e^1 \wedge \cdots \wedge \widehat{e^j} \wedge \cdots \wedge e^n.$$

11. Show that on the torus $\mathbb{T}^n$, for a form $\alpha = \alpha_I dx^I$, one has $\Delta\alpha = (\Delta\alpha_I)dx^I$. Deduce that harmonic forms on the torus have constant coefficients, and calculate the cohomology of the torus.

12. Let M^n be a compact oriented Riemannian manifold. Fix a cohomology class $c \in H^k(M)$. Show that the minimum of $\{\|\alpha\|^2, [\alpha] = c\}$ is attained exactly once, for the harmonic representative of c.

13. On a compact oriented Riemannian manifold M^n, the Laplacian Δ on forms can be diagonalized: there are eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$ with $\lambda_k \to \infty$ with corresponding unit eigenforms α_k which is a Hilbert basis of L^2-forms: any form α can be decomposed as $\alpha = \sum_1^\infty \langle \alpha, \alpha_k \rangle \alpha_k$.
Consider p-forms on a Riemannian product $M \times N$, the bundle of p-forms decomposes as $\Lambda^p T^*(M \times N) = \oplus_0^P \Lambda^i T^* M \otimes \Lambda^{k-i} T^* N$. Show that $\Delta^{M \times N} = \Delta^M \otimes 1_N + 1_M \otimes \Delta^N$, and from the above spectral decomposition deduce the Künneth formula

$$H^p(M \times N) = \oplus_0^p H^i(M) \otimes H^{k-i}(N).$$