Wilson Loops in the Large N Limit at Finite Temperature¹

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Abstract

Using a proposal of Maldacena we compute in the framework of the supergravity description of N coincident D3 branes the energy of a quark anti-quark pair in the large N limit of U(N) $\mathcal{N}=4$ SYM in four dimensions at finite temperature.

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Recently, Maldacena conjectured [1] that the large N limit of certain super-conformal theories is dual to M/string theory on a background of AdS times a sphere. This remarkable conjecture was studied further in a large number of papers in the last couple of months with promising results [2]-[30]. In particular, a way how to compute the Wilson line in four dimensional SYM via supergravity was suggested in [31] and [32].

In the present work we want to study the finite temperature effects on the Wilson line. We will concentrate on $\mathcal{N}=4$ SYM in four dimensions. The difference between the zero temperature case treated in [31, 32] and the finite temperature case is that now the relevant solution is the near extremal solution which has the following form ² [1]

$$ds^{2} = \alpha' \left\{ \frac{U^{2}}{R^{2}} [-f(U)dt^{2} + dx_{i}^{2}] + R^{2}f(U)^{-1} \frac{dU^{2}}{U^{2}} + R^{2}d\Omega_{5}^{2} \right\},$$

$$f(U) = 1 - U_{T}^{4}/U^{4},$$

$$R^{2} = \sqrt{4\pi gN}, \qquad U_{T}^{4} = \frac{2^{7}}{3}\pi^{4}g^{2}\mu,$$

$$(1)$$

where μ is the energy density above extremality on the brane and the Hawking temperature derived from the Euclidean metric is $T = U_T/(\pi R^2)$. Thus what one has to do is simply to follow the line of arguments in [31, 32] but with the metric (2) as a starting point. However, as we shall see, new ingredients will appear. Note that for large R at the region outside the horizon the curvature in string units is small and hence one can trust the supergravity solution in that region [33]. At U = 0 there is a curvature singularity. However, the region inside the horizon plays no role here.

The action for the string worldsheet is just the usual Nambu-Goto action

$$S = \frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{h},\tag{2}$$

where h is the induced metric on the string worldsheet. Using the Euclidean form of the metric (2) as the background we obtain (in static gauge) the following action

$$S = \frac{T}{2\pi} \int dx \sqrt{(\partial_x U)^2 + (U^4 - U_T^4)/R^4} \quad . \tag{3}$$

The action does not depend on x explicitly thus the Hamiltonian in the x direction is a constant of motion. Namely,

$$\frac{U^4 - U_T^4}{\sqrt{(\partial_x U)^2 + (U^4 - U_T^4)/R^4}} = \text{const.} = R^2 \sqrt{U_0^4 - U_T^4},\tag{4}$$

² By Wick rotation the metric is transformed into the Euclidean solution which has a periodicity along the time direction. The periodicity is simply the inverse of the Hawking temperature of the near extremal solution. As a result the two point correlation function for scalars computed in [15, 11] is easily generalized to the finite temperature case by super-position of "free" T=0 propagators. The superposition is chosen to ensure the periodicity.

where the integration constant U_0 is the minimal value of U which occurs at x = 0. This allows us to express x as a function of U

$$x = \frac{R^2}{U_0^3} \sqrt{U_0^4 - U_T^4} \int_1^{U/U_0} \frac{dy}{\sqrt{(y^4 - 1)(y^4 - U_T^4/U_0^4)}} . \tag{5}$$

The integration constant U_0 can, therefore, be related to L, the distance between the quark and the anti-quark.

$$L = 2\frac{R^2}{U_0} \sqrt{\varepsilon} \int_1^\infty \frac{dy}{\sqrt{(y^4 - 1)(y^4 - 1 + \varepsilon)}}$$
 (6)

where $\varepsilon = 1 - U_T^4/U_0^4$.

The calculation of the energy proceeds as explained in [31]. To obtain a finite result from (3) we have to subtract the (infinite) mass of the W-boson which corresponds to a string stretched between the brane at $U=\infty$ and the N branes. In the presence of a finite energy the string ends at the horizon, $U = U_T$, and not at U = 0. As we shall see this point is crucial for our discussion. There are several arguments for this. The first argument is due to D-branes probing black holes. In [34] it was shown that in the case of finite temperature the coordinates of the supergravity solution are not identical to the coordinates of the field theory living on the D-brane. A coordinate transformation is needed to match them. This transformation is such that from the point of view of the field theory living on the brane (at the one-loop order) the horizon is the origin. Another argument is that the Euclidean solution (which is obtained by wick rotation of (2)) contains only the region outside the horizon. Our last argument is due to Hawking radiation. As is well known, due to the red shift-effect, the local temperature close to the horizon is very high. In fact it is so high that any static particle/string will burn. In our case, by comparing the local temperature, $T_{loc} \sim T_{Haw} \sqrt{g^{tt}}$ to the string mass, $M_s = 1/\sqrt{\alpha'}$ we find that the minimal distance for the string not to burn is $U_T(1+1/R^2)$. Since the supergravity description is valid for large R we conclude that the ends are at $U=U_T$. Integrating from the horizon we obtain a finite result for the static energy

$$E = \frac{1}{\pi} \left\{ U_0 \int_1^\infty \left(\frac{\sqrt{y^4 - 1 + \varepsilon}}{\sqrt{y^4 - 1}} - 1 \right) - U_0 + U_T \right\}.$$
 (7)

What we are after is the static energy E(L,T) between the "quark" and the "antiquark". To obtain this expression we have to eliminate U_0 between eqs. (7) and (6). This can be done only numerically. Instead we shall find the qualitative behavior by looking at (7) and (6) and the corresponding numerical integration depicted in fig. 1 and fig. 2. We note that $\epsilon \approx 1$ corresponds to the low temperature region, $TL \ll 1$, while $\epsilon \approx 0$ corresponds to the high temperature region $TL \gg 1$.

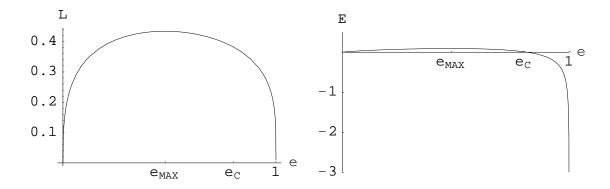


Figure 1: The distance L between the quark-anti-quark pair as a function of $\epsilon = \epsilon(T, \tilde{U})$. Note that L has a maximal value L_{max} .

Figure 2: The energy of the quark-antiquark pair relative to the "free" quark situation as a function of $\epsilon = \epsilon(T, \tilde{U})$. Note that E = 0 is achieved at $\epsilon_C < \epsilon_{max}$ corresponding to $L_C < L_{max}$.

For small temperature the behavior is roughly $E \sim -1/L$ as in the zero temperature case [31]. Taking into account the lowest corrections in U_0 the expression is

$$E = -\frac{2\sqrt{2}\pi^{3/2}(4\pi g_{YM}^2 N)^{1/2}}{\Gamma(1/4)^4} \frac{1}{L} [1 + c(TL)^4]$$
(8)

where c is a positive numerical constant which does not depend on R. The underlying conformal nature of the theory reveals itself in the fact that EL can depend on T only through the combination TL.

The behavior of L as a function of ϵ seems a priori puzzling since it indicates the existence of a maximum distance L_{max} . Indeed, if we assume that the behavior depicted in fig. 1 and fig. 2 always holds we will run into strange double valued behavior of E(L,T) for $L > L_{max}$. Fortunately physics tells us to believe the result only in the region $0 < L < L_C$ where $L_C < L_{max}$. The existence of L_C is seen in fig. 2. Starting from the low temperature region $\epsilon \sim 1$ we reach ϵ_C at which E = 0. At this point the energy associated with our string configuration (fig. 3) is the same as the energy of a pair of free quark and anti-quark with asymptotically zero force between them (fig. 4). It is important to note that ϵ_C is reached before L reaches its maximal value L_{max} (fig. 1). Once we reach L_C our string configuration (fig. 3) does not correspond to the lowest energy configuration and we should stop to trust eqs. (6) and (7).

The physical picture which emerges is quite reasonable and simple. For a given temperature T we encounter two regions with different behavior. For $L \ll 1/T$ we observe a Coulomb like behavior while for $L \gg 1/T$ the quarks become free due to screening by

the thermal bath.

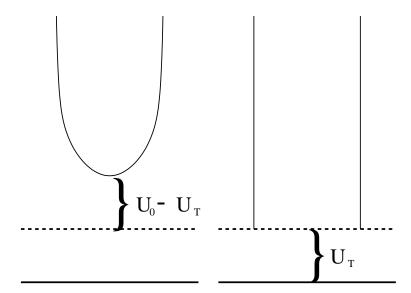


Figure 3: The energetically favorable configuration for $L < L_C$.

Figure 4: The energetically favorable configuration for $L > L_C$.

In fig 5. we have plotted E = E(L) for a given T by eliminating ϵ between eqs. (6) and (7) and trusting the result up to L_C .

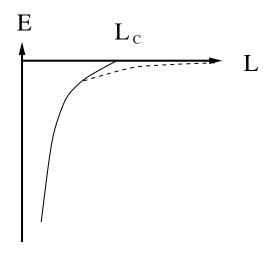


Figure 5: The energy E of the quark-anti-quark pair as a function of L for a given T. The solid line corresponds to the numerical calculation up to L_C , the dashed line indicates the expected behavior for large L.

The non-conformal theories, studied in [7] from the supergravity point of view, contain

a length scale (which is related to g_{YM}) and as such a phase transition might take place. It should be interesting to study these phase transitions and their relation to the transitions between the supergravity description and the perturbative/conformal field theory description discussed in [7].

Our result agrees with observations made in [35] about the conformal theories, namely, that there are no phase transitions at finite temperature. This is only true if the theory lives on a non-compact space which is the case in our paper. In [35] it was shown that the same theory on S^3 shows a confinement/deconfinement phase transition because the radius of the sphere introduces a scale and breaks conformal invariance.

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