M2 ICFP 2019/2020

Introduction to AdS/CFT TD n°5

Thermodynamics in asymptotically AdS spacetimes

1 Planar AdS-Schwarzschild

The planar, Euclidean AdS-Schwarzschild (AdS-S) metric in $d+1=5$ is given by :

$$ds^2 = \frac{\ell^2}{z^2}\left[\frac{dz^2}{f(z)} + f(z)d\tau^2 + \delta_{ij}dx^i dx^j\right], \quad i,j = 1,2,3 \tag{1}$$

where

$$f(z) = 1 - \frac{z^4}{z_h^4} \tag{2}$$

and the coordinate $z \in (0, z_h)$. Eudlidean time is periodic, with period given by the inverse temperature :

$$\tau \sim \tau + \beta, \qquad \beta = \frac{1}{T} = \pi z_h$$

In the limit $z_h \to \infty$, $f(z) \to 1$ and the metric becomes that of Eucliean thermal AdS (EAdS),

$$ds^2 = \frac{\ell^2}{z^2}\left[dz^2 + d\tau^2 + \delta_{ij}dx^i dx^j\right], \tag{3}$$

which can be defined at any temperature (i.e. Euclidean time τ is still periodic but the period β is arbitrary).

Both metrics (1) and (3) have boundary topology $S^1 \times R^3$ and spatial Euclidean symmetry (3d space rotations + space tralsations), instead of the full conformal symmetry (thus, conformal invariance *and* Lorentz invariance are broken down to the Euclidean group in 3d).

1.1 Solving Einstein's equation

Compute the Ricci tensor of the metric (1) and show that it is a solution of Einstein's equation corresponding to the action

$$S_{bulk} = -M_p^2 \int d^5x \sqrt{g}\,(R - 2\Lambda) \qquad \Lambda = -\frac{6}{\ell^2} \tag{4}$$

1.2 Computing the on-shell action

We want to compute the free energy of the AdS-S solution by calculating the on-shell action. To do this, we need to add boundary terms to the action (4), for two reasons :

— In gereral relativity, on a manifold with a boundary, in order for the variational principle to be well-defined (i.e. to cancel boundary terms containing the derivative of the variation) we need to add the *Gibbons-Hawking term* to the action,

$$S_{GH} = -2M_p^3 \int_{bdr} d^4x \sqrt{\gamma} K \tag{5}$$

Here, $\gamma_{\mu\nu}$ is the induced metric on the boundary and K is the trace of the *extrinsic curvature* of the boundary,

$$K = \nabla_a n_a \tag{6}$$

where n_a is the unit normal vector to the boundary [1].

— We need to add a local boundary counterterm to the action to cancel the dirvegences coming from the near-boundary region of AdS. In our case, it is enough to add the counterterm :

$$S_{ct} = 6\frac{M_p^3}{\ell}\int_{bdr} d^4x\sqrt{\gamma}. \tag{7}$$

Therefore the full action is

$$S^{\epsilon} = S_{bulk} + S_{GH} + S_{ct} \tag{8}$$

where the ϵ subscript means that we have moved the boundary from $z = 0$ to $z = \epsilon$, so that the first term is an integral over $z \in (\epsilon, z_h)$ (with z_h possibly $= +\infty$ in the case of no black hole) and the second and third terms are evaluated at $z = \epsilon$. Notice that we do *not* add the GH term at z_h since this is not a boundary of the Euclidean spacetime.

The boundary terms do not affect the field equations (so that (1) is still a solution of the same Einstein's equation), but they affect the value of the on-shell action.

1. Compute each term of the on-shell action for the thermal AdS solution (3) and show that they add up to zero, for any temperature.
2. Repeat the same calculation for the black hole solution, and show that the result is finite. Highlight the dependence on z_h.
3. Obtain the free energy from the on-shell action and express it as a function of temperature, recalling that $T = (\pi z_h)^{-1}$. What is the lowest free energy solution at a given fixed temperature ?
4. Obtain the other thermodynamic quantities (pressure p, entropy density s, energy density ρ) from the first law in the canonical ensemble,

$$F = E - TS = -pV, \qquad dF = -pdV - SdT$$

   In particular, write the equation of state in the form $p = p(\rho)$.
5. Obtain the specific heat c_v and the speed of sound c_s of the fluid,

$$c_v = \frac{\partial \rho}{\partial T}, \qquad c_s^2 = \frac{dp}{d\rho}$$

   (the latter quantity is a linear hydrodynamics quantity but it can be computed at equilibrium from the equation of state $p = p(\rho)$)

2 The AdS soliton

Consider the five-dimensional AdS soliton geometry [2], obtained by Wick-rotating one of the spatial coordinates (say $x^3 \to it$) of the metric (1) (and renaming $\tau \to \theta$) :

$$ds^2 = \frac{\ell^2}{z^2}\left[\frac{dz^2}{f(z)} + f(z)d\theta^2 - dt^2 + \delta_{ij}dx^i dx^j\right], \quad i,j = 1,2 \tag{9}$$

where

$$f(z) = 1 - \frac{z^4}{z_0^4}, \qquad \theta \sim \theta + \beta_0, \qquad \beta_0 \equiv \pi z_0. \tag{10}$$

1. See the appendix of Wald's General Relativity book if you know where this comes from.

2. For simplicity we will consider the gravity theory in 5 dimensions. This means that the dual field theory will be living on $S^1_\times R^{1,2} \to R^{1,2}$ after dimensional reduction, i.e. we get a confining theory in $d = 3$. This way we can use the action and counterterms of the previous exercise. Going one dimension up to 4d does not change the essence, but only the details.

This is a Lorentzian solution at zero temperature, with full Poincarè symmetry in (2+1)d but no conformal symmetry (e.g. scale symmetry is broken by the function $f(z)$). The boundary geometry is $S^1_\theta \times R^{1,2}$. At large distances ($\gg z_0$) this is therefore dual to the vacuum state of a 2+1 dimensional theory in Minkowski spacetime (see footnote 2).

2.1 Wilson loop

Here we want to perform the holographic calculation of the Wilson loop and show it exhibits an area law.

For this, we cansider a string attached on a rectangle on the boundary, $(t, x) \in [0, \Delta t] \times [-L/2, L/2]$, and a string wordlsheet attached on the boundary of the rectagle and extending in the bulk from $z = 0$ to $z = z_*$. As for the AdS computation, we take the embedding of the string to depend only on $x \equiv x^1$. Denoting by $\sigma^\alpha \equiv (\sigma, \eta)$ the worldsheet coordinates, we take

$$z = z(\sigma), \; t = \eta, \; x = \sigma, \; y = 0, \; \tau = 0.$$

The worldsheet is determined by extremizing the 2d Nambu-Goto action,

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\eta d\sigma \sqrt{detG} \qquad G_{\alpha\beta} \equiv g_{ab} \frac{\partial X^a}{\partial \sigma^\alpha} \frac{\partial X^b}{\partial \sigma^\beta} \tag{11}$$

where g_{ab} is the bulk 5d metric and $X^a = (z, \tau, x^i, t)$. To make contact with the computation in AdS, it is convenient to change coordinates in the bulk to $u = 1/z$ and introduce the parameter (with dimensions of mass) $u_0 = 1/z_0$.

1. Write the metric in the new coordinates (u, τ, x^i, t).
2. Write the Nambu-Goto action and obtain the 1-d action for the dynamical variable $u(\sigma)$.
3. Write the expression of the conserved "Hamiltonian " H, and relate it to the value of the turning point coordinate $u = u_*$, characterized by $u'(\sigma) = 0$ (which by $x \to -x$ symmetry will occur at $x = \sigma = 0$).
4. Using the conservation of H, reduce the problem to solving a first order differential equation, and write the solution for $u(\sigma)$ in implicit form in terms of an integral function, and write the integral relation between L and the turning point u_* by evaluating the integrals between appropriate extrema.
5. Show that for $L \ll u_0^{-1}$ we recover the relation $u_* \propto L^{-1}$ as in the conformal case.
6. Dervive the corresponding relation for $L \gg u_0^{-1}$.
7. Evaluate the Nambu-Goto action on the solution, and show that it behaves as

$$S_{NG}(L) \propto \begin{cases} \Delta t \, L^{-1} & L \ll u_0^{-1} \\ \\ \Delta t \, L & L \gg u_0^{-1} \end{cases}$$

8. Conclude that we have area law for large L, and give the confining string tension σ_c (up to a dimensionless numerical constant).

2.2 Deconfinement transition

Now we want to consider the same theory at finite temperature, so we go to periodic Euclidean time $t \to -i\tau$, $\tau \sim \tau + \beta$. The boundary manifold is now Euclidean $S^1_\theta \times S^1_\tau \times R^2$.

For all values of β, there are two distinct solutions with the same topology : the first one is the Euclidean version of (9) with compact time,

$$ds_c^2 = \frac{\ell^2}{z^2}\left[\frac{dz^2}{f_0(z)} + f_0(z)d\theta^2 + d\tau^2 + \delta_{ij}dx^i dx^j\right], \quad f_0(z) = 1 - \frac{z^4}{z_0^4}, \quad 0 < z < z_0 \tag{12}$$

The second one is obtained by putting the blackening function f in front of τ instead of θ :

$$ds_d^2 = \frac{\ell^2}{z^2}\left[\frac{dz^2}{f_h(z)} + d\theta^2 + f_h(z)d\tau^2 + \delta_{ij}dx^i dx^j\right], \quad f_h(z) = 1 - \frac{z^4}{z_h^4}, \quad 0 < z < z_h \tag{13}$$

For both metrics, the two circles have the periods :

$$\theta \sim \theta + \beta_0, \qquad \tau \sim \tau + \beta$$

with $\beta_0 = \pi z_0$ and $\beta = \pi z_h$, therefore both are regular at the respective endpoints [3]. Locally, they are the same solution except for the different value of $z_h \neq z_0$ in f and a relabeling of the coordinates $\theta \leftrightarrow \tau$. Therefore, both metric solve the same Einstein equation (this fact is independent of the value of z_h).

1. Show that in the solution (13) the confining string tension vanishes, i.e. the dual field theory is deconfined.
2. Compute the free energy of both solutions (12) and (13) , by the same calculation of Problem 1.2.2.
3. Write the free energy difference $\Delta F = F_c - F_d$ and show that there is a phase transition. Find the corresponding critical temperature.
4. Show that the phase transition is first order, and find the corresponding latent heat.

3. Notice that only z_h has to do with temperature, not z_0.