

## Conformal Field Theory

### 1. Conformal Algebra

The generators of infinitesimal conformal transformations on functions of  $x^\mu$  are :

$$P_\mu = -i\partial_\mu; \quad L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu);$$

$$D = -ix^\mu\partial_\mu; \quad K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^\nu x_\nu\partial_\mu)$$

Verify the commutation relations :

$$[D, P_\mu] = iP_\mu, \quad [D, K_\mu] = -iK_\mu.$$

### 2. Special Conformal Transformations

We want to construct finite special conformal transformation, starting from the infinitesimal ones :

$$x^{\mu'} = 2(b^\nu x_\nu)x^\mu - b^\mu x^2 \quad (x^2 \equiv x^\mu x_\mu).$$

1. Show that the *inversion*

$$I : x^\mu \rightarrow x^{\mu'} = \frac{x^\mu}{x^2}$$

is a conformal transformation. Can we obtain  $I$  from an infinitesimal transformation ?

2. Show that the infinitesimal special conformal generators  $K$  are related to the translation generators  $P$  by

$$K_\mu = -IP_\mu I$$

[use the action of the transformations on the coordinates]

3. From the previous result and the fact that  $I^2 = 1$ , show that

$$e^{ib^\mu K_\mu} = I e^{-ib^\mu P_\mu} I.$$

Use this identity to find the form of the finite special conformal transformation with parameter  $b^\mu$ .

### 3. Noether currents

Consider a classical field theory for a scalar field  $\phi(x)$  and an infinitesimal transformation, with parameter  $\epsilon$ , which we assume linear in  $\epsilon$  (but not necessarily in  $\phi$ ) :

$$\delta\phi \equiv \phi'(x) - \phi(x) = X(\phi)\epsilon$$

If such a transformation leaves the action invariant, or equivalently if the functional variation of the Lagrangian is a total divergence,

$$\delta\mathcal{L} = \epsilon(\partial^\mu F_\mu),$$

then Noether's theorem ensures the existence of a current  $J^\mu$ , conserved on the equations of motion, and given by

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} X(\phi) - F^\mu.$$

Below we consider the two kinds of transformations corresponding to translation and dilatation of the coordinates, and the corresponding action on a scalar field of conformal weight  $\Delta$ ,

$$x'_\mu = x_\mu - \epsilon_\mu, \quad \phi'(x') = \phi(x)$$

and

$$x'_\mu = x_\mu - \epsilon x_\mu, \quad \phi'(x') = (1 + \Delta\epsilon)\phi(x)$$

1. Write the functional variation  $\delta\phi(x)$  of the field under the two transformations
2. Write functional variation of the Lagrangian, and verify that it is a total divergence in both cases (assuming the classical theory is scale invariant). This will lead you to identify the quantity  $F^\mu$ . [Recall that for a CFT, the Lagrangian is a scalar field of weight  $d$ ]
3. Construct the Noether currents for translation and dilatation using the general formula, and check they are given by the following expressions :

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L}, \quad J_\mu^D = x^\nu T_{\mu\nu} + \Delta\phi\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}.$$

#### 4. Improved stress tensor and conformal coupling

Consider the theory of a real scalar field in  $d$  dimensions with action

$$S = - \int d^d x \left[ \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{\lambda}{n} \phi^n \right]$$

##### 3.1. Canonical stress tensor

- i. What must be the conformal weight  $\Delta$  of the field  $\phi$  such that the kinetic term has the appropriate scaling dimension ?
- ii. Write down the explicit form of the canonical (i.e. Noether) stress tensor  $T_{\mu\nu}$ , of its trace, and of the dilatation Noether current  $J_\mu^D$ . For which value(s) of  $n$  is the dilatation current conserved (on a solution of the equations of motion) ? Is the stress tensor traceless in this case ?

##### 3.2. Improved stress tensor

We want to construct an *improved* stress tensor  $\Theta_{\mu\nu}$  and an improved dilatation current  $\tilde{J}_\mu^D$  such that 1)  $\Theta_{\mu\nu}$  it is conserved 2) it is traceless iff the theory is conformal.

- i. Define :

$$\Theta_{\mu\nu} = T_{\mu\nu} + \xi (\partial_\mu \partial_\nu - \eta_{\mu\nu} \partial^\rho \partial_\rho) \phi^2$$

Show that it is conserved and that the associated Noether charges  $P_\mu$  coincide with those associated to  $T_{\mu\nu}$ ,

$$P_\mu \equiv \int d^3 x T_{0\mu} = \int d^3 x \Theta_{0\mu}$$

- ii. Show, for an appropriate value of  $\xi$  (which depends on  $d$ ),  $\Theta_{\mu\nu}$  is traceless (on solutions of the EOM) for the appropriate value of  $n$  (which you have found in part 1) for which the theory is conformally invariant.
- iii. Define the *improved dilatation current*  $\tilde{J}_\mu^D$  as

$$\tilde{J}_\mu^D = x^\nu \Theta_{\mu\nu}.$$

Show that tracelessness of  $\Theta_{\mu\nu}$  implies conservation of  $\tilde{J}_\mu^D$ , and that the associated dilatation charge  $D$  is the same as the one obtained from  $J_\mu^D$ , i.e.

$$D \equiv \int d^3 x J_0^D = \int d^3 x \tilde{J}_0^D.$$

The conserved charges constructed from  $\Theta_{\mu\nu}$  and  $\tilde{J}_\mu^D$  are the same as those obtained from  $T_{\mu\nu}$  and  $J_\mu^D$ , respectively. Therefore the improved conserved currents are associated to the same symmetries as the old (Noether) currents.

### 3.3. Covariant version

Now consider the same scalar field theory minimally coupled to a background metric  $g_{\mu\nu}$ ,

$$S = - \int d^d x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{\lambda}{n} \phi^n \right]$$

- i. Show that the definition

$$T_{\mu\nu} = - \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}$$

gives back the canonical stress tensor (after setting  $g_{\mu\nu} = \eta_{\mu\nu}$ )

- ii. Can you find a (covariant) term such that, when added to the action, gives the improved stress tensor of part 2 as

$$\Theta_{\mu\nu} = - \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} ?$$

*[Hint : recall how the Ricci scalar changes under a variation of the metric. Remember to set  $g_{\mu\nu} = \eta_{\mu\nu}$  at the end of the calculation, since we want to reproduce the flat space result of part 2.]*

The resulting action is the action of a *conformally coupled* scalar field. It is conformal in any background, i.e. it is invariant under Weyl transformations (which are *not* coordinate transformations)

$$g_{\mu\nu} \rightarrow e^{2\omega(x)} g_{\mu\nu}, \quad \phi \rightarrow e^{-\Delta\omega(x)} \phi$$

for an arbitrary function  $\omega(x)$ .