

TOPOLOGICAL QUANTUM FIELD THEORIES

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ABSTRACT. Following my plenary lecture on ICMP2000 I review my results concerning two closely related topics: topological quantum field theories and the problem of quantization of gauge theories. I start with old results (first examples of topological quantum field theories were constructed in my papers in late seventies) and I come to some new results, that were not published yet.

0. Introduction.

I review my results concerning two closely related topics: topological quantum field theories and the problem of quantization of gauge theories. I'll start with old results (first examples of topological quantum field theories were constructed in my papers in late seventies) and I'll come to some new results, that were not published yet. But first of all I would like to give a short (and very incomplete) overview of these problems and of some related questions (see, for example , [8], [24] for more complete review).

Massive intervention of topology into quantum field theory was triggered by discovery of magnetic monopoles in $SU(2)$ gauge theory with scalar fields-Georgi-Glashow model (Polyakov [26], 'tHooft [19]). It was recognized very soon ([13], [14], [28], [25], [2]) that magnetic charge has topological nature and that simple topological considerations can be used to prove the existence of magnetic monopoles in a large class of gauge theories (including all grand unification theories).

Other topologically non-trivial field configurations were considered shortly .The most significant role was played by topologically non-trivial extremals of Yang-Mills Euclidean action-gauge instantons [7].

I started as a topologist in fifties, and it was very pleasant for me to find important applications of topological ideas to physics-homology and homotopy theory, characteristic classes, Atiyah-Singer index theory became common tools in quantum field theory. I was pleased even more when I found an idea permitting to apply quantum field theory to topology. The idea was very simple-if an action functional depends only on smooth structure of a manifold then corresponding physical quantities (in particular, the partition function) should have the same property. The simplest example is a functional

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$$(1) \quad S = \int_M A \wedge dA,$$

where A is a 1-form on three-dimensional compact manifold M . This functional is invariant with respect to gauge transformations $A \rightarrow A + d\lambda$, therefore to calculate its partition function one should impose gauge condition. The gauge condition cannot be invariant with respect to diffeomorphisms; it should involve some additional data, for example Riemannian metric. However, the answer should not depend on the choice of gauge condition. This logic is not flawless; formal arguments above can be destroyed by quantum anomalies. However, it is possible to give a rigorous theory of partition function of degenerate quadratic functionals and to relate the partition function of the functional (1) and of its multidimensional generalizations to Ray-Singer torsion-smooth version of Reidemeister torsion. (See [30], [32] and Sec.2)

The theory described by the action functional (1) is the simplest example of topological quantum field theory. It admits a non-abelian generalization-so called Chern-Simons action functional

$$(2) \quad S = \int_M 12 \operatorname{Tr} A \wedge dA + 13 \operatorname{Tr} A \wedge A \wedge A$$

where A is a matrix-valued 1-form on three-dimensional compact manifold M . (The functional (1) is called sometimes abelian Chern-Simons action.) It was conjectured in [34] that the action functional (2) leads to invariants of manifold M that are closely related to Jones polynomial of knots. I was not able to prove this conjecture. This was done a year later in remarkable paper by E. Witten [44], who found a way to calculate physical quantities associated with Chern-Simons action in terms of two-dimensional conformal field theory. It is difficult to overestimate the influence of this paper and of the papers [42], [43] that E. Witten has written almost at the same time. It became clear after these papers that using topological quantum field theories one can obtain very interesting mathematical results and that these theories are very useful not only in mathematics, but also in physics. The papers [42], [43] are closely related to earlier Witten's paper [41] and to work of Donaldson, Floer and Gromov [12], [16], [15], [18]. Donaldson and Floer used some ideas from physics to obtain beautiful mathematical results. Donaldson applied instanton moduli space, studied earlier in [29], [31], [5] to obtain very strong results about 4-manifolds. Floer's starting point was Witten's paper about supersymmetry and Morse theory [41]. Witten has shown that their constructions, as well as Gromov's invariants of symplectic manifolds, can be understood in the framework of topological quantum field theory. This understanding led to very important development in pure mathematics culminating in theory of Seiberg-Witten invariants of 4-manifolds based on results of [27], [45] and in enumerative geometry of (pseudo)holomorphic curves on symplectic manifolds [10], [22], [17]. After axiomatization of topological quantum field theories by Atiyah [4] these theories became a subject of extensive mathematical analysis; I'll stay mostly on Lagrangian viewpoint in this lecture.

I would like to mention a couple of important ideas that appeared in Witten's papers. One of them is to allow dependence of action functional S of metric $g_{\alpha\beta}$, but to require the energy- momentum tensor $T^{\alpha\beta} = \delta S / \delta g_{\alpha\beta}$ that governs the dependence of S of the metric to be BRST-trivial (See Sec.3 for introduction to BRST formalism.) Then expectation values of observables (of BRST closed functionals)

should be metric independent, i.e. the quantum field theory is topological. One calls theories of this kind topological quantum theories of Witten type, as opposed to topological theories of Schwarz type, when the action functional is metric independent. The next idea that in supersymmetric theories one can declare a supersymmetry generator that squares to zero to be a BRST operator. Then it follows from supersymmetry algebra that the translation operator is BRST-trivial; this means that correlation functions $\langle A_1(x_1) \dots A_n(x_n) \rangle$ of BRST closed observables A_1, \dots, A_n don't depend on x_1, \dots, x_n . One can say that BRST closed observables are topological observables: it is possible to construct a topological quantum field theory where correlation functions are equal to correlation functions of topological observables of supersymmetric theory. One says that topological theory is obtained from supersymmetric theory by means of twisting. Twisting $N = 2$ supersymmetric four-dimensional gauge theory one can obtain topological theory that is closely related to Donaldson invariants of four-dimensional manifolds. Two-dimensional sigma-model has $N = 2$ superconformal symmetry if the target space is a Kaehler manifold. There are two essentially different possibilities to choose BRST-operator and to twist the theory. (This is true for any $N = 2$ superconformal theory; see [39] for discussion in the framework of axiomatic quantum field theory.) In one case we obtain so called *A*-model; it is defined for every symplectic target and its correlation function can be expressed in terms of (pseudo)holomorphic curves (in terms of Gromov-Witten invariants). In another case we obtain *B*-model where one should assume that the target is a complex Calabi-Yau manifold. We'll not discuss remarkable theory relating *A*-model on one manifold with *B*-theory on another (mirror) manifold ([10], [22], [17]).

As I emphasized from the very beginning the development of topological quantum field theory was intertwined with the progress in our understanding of quantization of gauge theories. To analyze topological theories related to Ray-Singer torsion it was necessary to deal with so called reducible gauge theories; the analysis of this class of gauge theories was given in [32]. In this case the needs of TQFT led to progress in the problem of quantization. (See Sec.1 and 2) Later, as we have seen, BRST formalism in quantization of gauge theories was used to construct topological quantum theories of new type. It seems that Batalin-Vilkovisky version of BRST formalism is very convenient to construct action functionals of TQFT. I'll give a short exposition of Batalin-Vilkovisky formalism from geometric viewpoint following my papers [36], [37] [38]. This geometric approach will be used to construct BV topological sigma-model, that includes many interesting topological quantum field theories as particular cases [1]. In particular, it includes BV version of Chern-Simons action functional as well as its multidimensional generalization (Sec.4). It is interesting to notice that some of Witten type topological theories can be formulated as theories with metric independent action in BV formalism; maybe this is true for all theories of this kind. Sec.5 devoted to perturbation theory in BV-formalism. It seems that the version of perturbation expansion described in this section was not analyzed in previous publications although it was used implicitly in [8]. In Sec.6 I'll discuss recent results about quantum observables in BV formalism and their application to TQFT following [40].

In Sec.7 I'll formulate some new results about families of action functionals. It will be shown that using families of equivalent action functionals or families of gauge conditions one can construct some numbers generalizing expectation values

of observables. More precisely, one can consider a kind of moduli space corresponding to a given action functional and under certain conditions one can define closed differential forms on moduli space. Integrating these forms over cycles we obtain some interesting quantities. In particular, one can show that the application of this construction to two-dimensional topological quantum field theory gives string amplitudes. Applying it to multi-dimensional analog of Chern-Simons action functional we obtain cohomology classes of $\text{BDiff}(M)$ constructed by M.Kontsevich.

The results I know in this direction are not complete. Some of them were obtained in collaboration with M. Kontsevich. Several years ago we planned to work together on families of topological quantum field theories, however both of us found more urgent problems to study.

1. Quantization of gauge theories.

Let us consider a functional S defined on space \mathcal{E} ("space of fields"). We can regard S as classical action functional; Euler-Lagrange equations for stationary points of S are interpreted as equations of motion of corresponding classical mechanical system. We can also try to consider corresponding quantum system; this means that we should calculate the integral Z of $\exp(-S)$ over \mathcal{E} (the partition function) and the expressions of the form $Z_A = \int_{\mathcal{E}} A \exp(-S)$ where A is a function on \mathcal{E} . (One can regard Z_A/Z as an expectation value of the observable A). Notice, that in the above terminology we consider A as a Euclidean action functional.

In interesting cases the space \mathcal{E} is infinite-dimensional, therefore the integrals Z and Z_A are ill-defined; it is quite difficult to make sense of them. However, if \mathcal{E} is a vector space and S is represented as a sum of quadratic functional S_{qu} and polynomial functional V one can try to construct perturbation series for these integrals, considering V as a perturbation. This problem is much easier, but still it is not simple. (To solve it one should analyze the integrals of $AV^n \exp(-S_{qu})$ over \mathcal{E} .) If the functional S has a large symmetry group (gauge group), the quadratic part of S is degenerate and in addition to other problems we should deal with the divergence of gaussian integral $\int_{\mathcal{E}} \exp(-S_{qu})$.

The standard way to work with ill-defined infinite-dimensional integrals is to consider similar finite-dimensional integrals and to apply rigorously proven finite-dimensional formulas to infinite-dimensional case without any justification. (Such a justification is impossible because usually we don't have any rigorous definitions of integrals at hand. Sometimes it is useful to say that finite-dimensional theorems become definitions in infinite-dimensional case).

Let us consider a compact Lie group G that acts on finite-dimensional Riemannian manifold M preserving the Riemannian metric. Let us fix a G -invariant function f on M . If G acts freely on M and Σ is a subset of M having precisely one common point with every orbit of G , then we can reduce the integration over M to the integration over Σ . (In physics G plays the role of gauge group; replacing M by Σ we impose gauge condition.) More precisely, if the set Σ is singled out by means of equation $F(x) = 0$ where $F = (F^1, \dots, F^k)$ is a map of M into \mathbb{R}^k , then

$$\int_M \exp(-S) d\mu = \int_M \exp(-S) W_F(x) \delta(F(x)) d\mu$$

where $W_F(x)$ is specified by the formula

$$(4) \quad W_F(x) \cdot \int \delta(F(gx)) dg = 1.$$

(Here dg stands for invariant volume element on G normalized by the condition that the volume of G is equal to 1.)

If M is an infinite-dimensional manifold one can consider the right hand side of (3) as a definition of left hand side. This idea (Faddeev-Popov trick) is very useful in quantization of gauge theories. Of course, to apply this idea one should verify that the left hand side does not depend on the choice of Σ (on the choice of gauge condition).

The physical quantities are defined as functional integrals of $A \exp(-S)$ where both S and A are gauge invariant. In many cases Faddeev-Popov trick permits us to obtain a perturbation series for quantities of this kind.

We will use an invariant form of Faddeev-Popov trick that is based on the following finite-dimensional statement.

Let us consider again a compact group G of isometries of Riemannian manifold M and a G -invariant function f . Without loss of generality we can assume that stable subgroups H_x for all points $x \in M$ are conjugate (this fact is always true for almost all points). Then

$$(5) \quad \int_M f(x) d\mu = \int_{M/G} f(x) (D(x))^{1/2} (V(H_x))^{-1} d\nu$$

Here $d\nu$ is the volume element corresponding to the natural Riemannian metric on M/G , $V(H_x)$ stands for the volume of H_x in the metric induced by invariant Riemannian metric on G obeying $V(G) = 1$ and $D(x) = \det \widetilde{\mathcal{T}}_x^+ \widetilde{\mathcal{T}}_x$ where $\widetilde{\mathcal{T}}_x$ is a linear operator acting from $\text{Lie } G / \text{Lie } H_x$ into tangent space $T_x(M)$. (The action of G on M determines an operator $\mathcal{T}_x : \text{Lie } G \rightarrow T_x(M)$. This operator descends to $\widetilde{\mathcal{T}}_x : \text{Lie } G / \text{Lie } H_x \rightarrow T_x(M)$.) Let us say that the compact Lie groups G_0, \dots, G_N and homomorphisms $T_i : G_i \rightarrow G_{i-1}$ form a resolution of subgroup $H \subset G$ if $G = G_0, \text{Im } T_1 = H, \text{Im } T_{i+1} = \text{Ker } T_i$. We introduce an invariant Riemannian metric on G_i , and assume that it is normalized by the condition $V(G_i) = 1$. The homomorphism T_i generates a homomorphism \mathcal{T}_i of corresponding Lie algebras; it descends to a linear map $\widetilde{\mathcal{T}}_i : \text{Lie } G_i / \text{Ker } \mathcal{T}_i \rightarrow \text{Lie } G_{i-1}$. Using (5) it is easy to check that

$$(6) \quad \log V(H) = \sum 12(-1)^{i-1} \log \det \mathcal{T}_i^* \mathcal{T}_i.$$

Combining (5),(6) we obtain

$$(1) \quad \int_M f(x) d\mu = \int_{M/G} f(x) (\det \widetilde{\mathcal{T}}_x^* \widetilde{\mathcal{T}}_x)^{1/2} \prod_{1 \leq i \leq N} (\det \widetilde{\mathcal{T}}_i^* \widetilde{\mathcal{T}}_i)^{12(-1)^i} d\nu = 7$$

$$\int_{M/G} f(x) (\det \square_0(x))^{1/2} \prod_{1 \leq i \leq N} (\det \square_i)^{\sigma_i} d\nu$$

where $\square_0(x) = \mathcal{T}_x^* \mathcal{T}_x + \mathcal{T}_1 \mathcal{T}_1^*$, $\square_i = \mathcal{T}_i^* \mathcal{T}_i + \mathcal{T}_{i+1} \mathcal{T}_{i+1}^*$, $\sigma_i = 12(-1)^i / (i+1)$.

Let us consider the case when the function $f(x)$ is defined on vector space \mathcal{E} and has the form $f(x) = \exp(-\mathcal{S}(x))$ where $\mathcal{S}(x)$ is a quadratic functional. If \mathcal{E} is equipped with inner product we can represent $\mathcal{S}(x)$ in the form

$$\mathcal{S}(x) = \langle Sx, x \rangle = \langle x, Sx \rangle.$$

In finite-dimensional case

$$(8) \quad \int_{\mathcal{E}} e^{-\langle Sx, x \rangle} dx = (\det S)^{-1/2}$$

for appropriate normalization of the volume element on \mathcal{E} . If S is non-degenerate we accept the right hand side of (8) as a definition of infinite-dimensional Gaussian integral. However, to apply this definition we should have a definition of infinite-dimensional determinant. One of possible approaches is based on the notion of zeta-function:

$$\log \det S = 12 \log \det S^* S = -12 \zeta'_{S^* S}(0).$$

(One can define the zeta-function of nonnegative operator A by the formula $\zeta_A(s) = \sum \lambda_i^{-s}$, where λ_i runs over positive eigenvalues of A . In the case when A is an elliptic operator on compact manifold the series converges for $s \gg 0$, however one can define $\zeta'_A(0) = d\zeta_A(s)/ds|_{s=0}$ by means of analytic continuation. Notice that this definition can be applied also to operators having zero modes, because zero eigenvalues don't enter the expression for zeta-function.)

If the functional \mathcal{S} is degenerate, one can try to define the Gaussian functional integral (the partition function corresponding to \mathcal{S}) by the formula (8). However, such an attempt does not lead to interesting results; we need additional structure to give a reasonable definition.

Let us consider a quadratic functional \mathcal{S} on the space $\mathcal{E} = \mathcal{E}_0$, vector spaces $\mathcal{E}_1, \dots, \mathcal{E}_n$ and operators $T_i : \mathcal{E}_i \rightarrow \mathcal{E}_{i-1}$, obeying $T_{i-1} \cdot T_i = 0$, $\mathcal{S}(x + T_1 y) = \mathcal{S}(x)$. We will assume that spaces \mathcal{E}_i are equipped with Hermitian inner product; this means that we can represent $\mathcal{S}(x)$ in the form $\mathcal{S}(x) = \langle Sx, x \rangle = \langle x, Sx \rangle$ and consider adjoint operators T_i^* .

We will say that the spaces \mathcal{E}_i and operators T_i constitute an elliptic resolution of the functional \mathcal{S} if the space \mathcal{E}_i , $i = 0, \dots, N$, can be considered as a space of smooth sections of a vector bundle with compact base and the operators $\square_0 = S^2 + T_1 T_1^*$, $\square_i = T_i^* T_i + T_{i+1} T_{i+1}^*$ are elliptic operators. We define the partition function of the functional \mathcal{S} with respect to elliptic resolution by the formula:

$$(9) \quad Z = \prod_{0 \leq i \leq N} (\det \square_i)^{(-1)^{i+1} (2i+1)/4}.$$

(We can come to this definition applying formally Eqn (7)). In general, Z depends on the choice of inner products on $\mathcal{E}_0, \dots, \mathcal{E}_N$. However, it is possible to calculate the variation of Z when these inner products vary. Let us suppose that we have a family \langle, \rangle_i^u of inner products on \mathcal{E}_i depending on parameter u denote the operators governing infinitesimal variation of inner product by B_i^u :

$$ddu \langle f, g \rangle_i^u = \langle B_i^u f, g \rangle_i^u = \langle f, B_i^u g \rangle_i^u.$$

Then

$$d \log Z(u) du = 12 \sum_{0 \leq i \leq n} (-1)^i \Psi_0(B_i^u | \square_i^u).$$

(The Seeley coefficients $\Psi_k(R|A)$ are defined by means of asymptotic expansion:

$$Tr(R e^{-At}) = \sum \Psi_k(R|A) t^{-k}$$

for $t \rightarrow 0$). We assume that the operators \square_i do not have zero modes (the resolution is an exact sequence); to take zero modes into account one should subtract the trace of B_i on the space of zero modes from Ψ_0 .

If \square_i are differential operators then the Seeley coefficients are given by local formulas. If the operators \square_i act on a vector bundle with odd dimensional base then Ψ_0 vanishes and Z does not depend on u . In other words, we don't have a quantum anomaly in this case. (One speaks about a quantum anomaly if something that is true for classical theory is violated at the quantum level. Classical theory is determined by the functional \mathcal{S} in our case; it does not depend on inner products. Therefore dependence of inner products can be characterized as a quantum anomaly.)

Notice that in this statement we assumed that the operators \square_i don't have zero modes. In general we have to consider a vector space $\mathcal{H} = \sum \mathcal{H}_i = \sum \text{Ker} T_i / \text{Im} T_{i+1} \approx \sum \text{Ker} \Delta_i$ (homology of the resolution (\mathcal{E}_i, T_i)). The partition function can be regarded as a measure on linear superspace \mathcal{H} (natural \mathbb{Z}_2 -grading determines a structure of superspace on \mathcal{H}). If the relevant Seeley coefficients vanish the partition function does not depend on the choice of inner products on \mathcal{E}_i (see[32] [33]).

2. Topological gauge theories.

Let us consider an action functional

$$(10) \quad \mathcal{S} = \int_M A \wedge dA$$

where M is a compact $(2n+1)$ -dimensional manifold, A is an n -form on M . This functional is invariant with respect to transformations $A \rightarrow A + d\lambda$, where λ is an $(n-1)$ -form on M . Denoting the space of smooth k -forms on M by Ω^k we can say that \mathcal{S} is a degenerate quadratic functional on Ω^n and that Ω^{n-1} can be considered as symmetry group of (10). In the case $n=1$ we can use the Faddeev-Popov trick to calculate the corresponding partition function. If $n > 1$ then the map $d : \Omega^{n-1} \rightarrow \Omega^n$ has an infinite-dimensional kernel; in this case we can use the notion of elliptic resolution to define the partition function of (10). Namely, the spaces $\Omega^n, \Omega^{n-1}, \dots, \Omega^0$ and operators $d : \Omega^{k-1} \rightarrow \Omega^k$ provide us with elliptic resolution of (10).

Let us fix a Riemannian metric on M . This metric induces an inner product on spaces Ω^k ; we can use this inner product to calculate the partition function. Using (9) we obtain an expression of the partition function Z in terms of $\det \Delta_k$, where $\Delta_k = d^*d + dd^*$ stands for the Laplace operator on the space of k -forms.

The same construction can be repeated in the case when we allow forms with coefficients in a local system (i.e. forms taking values in fibres of a vector bundle equipped with a flat connection).

In the acyclic case (in the case when the operators Δ_k have no zero modes) it follows from the results of Sec.1 that Z does not depend on Riemannian metric on M . This means that Z is invariant with respect to diffeomorphisms. It is easy to check that Z coincides with Ray-Singer torsion (smooth version of Reidemeister torsion) and therefore is a topological invariant. In general case Z is a topologically invariant measure on superspace $H(M)$ (direct sum of cohomology groups of M).

In the case $\dim M = 3$ one can generalize the action functional (10) to the case when A is a 1-form taking values in Lie algebra \mathcal{G} equipped with invariant inner product. Such a form can be considered as a connection (gauge field) in a trivial vector bundle over M and one can modify (10) to get a functional that is invariant with respect to infinitesimal gauge transformations $\delta A = d\gamma + [\gamma, A]$ where γ is a \mathcal{G} -valued function. This functional (Chern-Simons action functional) has the form

$$(11) \quad S(A) = \int_M (12A \wedge dA + 13A \wedge A \wedge A)$$

where $A \wedge dA$ stands for $h_{ik}A^i \wedge (dA)^k$ and $A \wedge A \wedge A$ stands for $f_{ijk}A^i \wedge A^j \wedge A^k$. (We denote by A^i components of A with respect to a basis in \mathcal{G} ; f_{ijk} are structure constants of \mathcal{G} and h_{ik} is the metric tensor of \mathcal{G} in this basis.)

The Chern-Simons functional depends only on smooth structure of M therefore one can hope it gives invariants of M . If Γ is a closed curve in M one can construct a gauge invariant expression $W_\Gamma(A)$ as trace of monodromy of the connection A in some representation of the group G corresponding to the Lie algebra \mathcal{G} . Integral of $W_\Gamma(A) \exp(-kS(A))$ over infinite-dimensional space of all gauge fields (or, better to say, over the space of gauge classes of gauge fields) depends only on isotopy class of Γ considered as a knot in M and on topology of M . There exist two ways to obtain well defined invariants from this ill-defined integral: to use perturbation theory or to calculate this integral precisely in terms of two-dimensional conformal theory. Direct application of Faddeev-Popov procedure leads to very complicated expressions that I have written down in 1987, but was not able to analyze rigorously. Mathematical analysis of perturbation series was performed much later in [6], [21] on the base of diagram technique that can be obtained from Batalin-Vilkovisky formalism (see Sec.7). Some remarks about Witten's explicit solution are contained in Sec.8.

3. Gauge theories in BRST-BFV formalism.

Let us consider a \mathbb{Z}_2 -graded vector space \mathcal{E} equipped with an odd operator $\hat{\Omega}$ obeying $\hat{\Omega}^2 = 0$ (in mathematical terminology $\hat{\Omega}$ is a differential, in physics $\hat{\Omega}$ is called BRST operator; BRST stands for Becchi-Rouet-Stora-Tyutin). An operator $A : \mathcal{E} \rightarrow \mathcal{E}$ commuting with $\hat{\Omega}$ is called quantum observable; such an operator descends to an operator $\tilde{A} : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}}$ acting on homology $\tilde{\mathcal{E}} = \text{Ker } \hat{\Omega} / \text{Im } \hat{\Omega}$. It is easy to check that $\text{Tr } A = \text{Tr } \tilde{A}$, where Tr stands for supertrace. (This fact is used in topology in the derivation of Lefschetz fixed point formula.)

If A can be represented as a (super)commutator of Ω with some operator B we can say that the observable A is trivial: $\text{Tr } A = 0$. Quantum observables are called also BRST-closed operators, trivial observables are BRST-exact. (Observables are related to the homology of the space of linear operators on \mathcal{E} where Ω acts by the formula $A \rightarrow [\Omega, A]$, where $[\cdot, \cdot]$ stands for supercommutator.) If A and H both commute with $\hat{\Omega}$ we obtain that

$$\text{Tr } \tilde{A} \exp(-\tilde{H}\beta) = \text{Tr } A \exp(-H\beta).$$

This formula shows that at the level of expectation values of observables the theory with Hamiltonian H on \mathcal{E} is equivalent to the theory with Hamiltonian \tilde{H} on $\tilde{\mathcal{E}}$. This observation permits us to replace a theory with complicated space $\tilde{\mathcal{E}}$ by a theory with simple space \mathcal{E} at the price of introducing additional degrees of freedom and BRST operator (see [35] for more detail). One can say that BRST formalism is a version of standard mathematical idea of resolution, when a complicated module is replaced with a complex of simple modules.

Let us consider for example operators $T_\alpha : E \rightarrow E, \alpha = 1, \dots, n$, acting on space E and generating a Lie algebra \mathcal{G} (i.e. $[T_\alpha, T_\beta] = f_{\alpha\beta}^\gamma T_\gamma$). Let us denote by \mathcal{E}

the space of all E -valued functions depending on anticommuting variables c^1, \dots, c^n (the space of cochains of Lie algebra \mathcal{G}). The operator

$$\widehat{\Omega} = T_\alpha c^\alpha + 12f_{\alpha\beta}^\gamma c^\alpha c^\beta \partial \partial c^\gamma$$

obeys $\widehat{\Omega}^2 = 0$; corresponding homology $\widetilde{\mathcal{E}} = H(\mathcal{G}, E)$ are called Lie algebra cohomology. The space \mathcal{E} , and, therefore, $\widetilde{\mathcal{E}}$ have natural \mathbb{Z} -grading with $\deg c^\alpha = 1$. It is easy to see that $\widetilde{\mathcal{E}}^0 = H^0(\mathcal{G}, E)$ is the \mathcal{G} -invariant subspace of E (i.e. $\widetilde{\mathcal{E}}^0 = \{x \in E : T_\alpha x = 0\}$). Let H be a \mathcal{G} -invariant hamiltonian on E . Then can restrict it to $\widetilde{\mathcal{E}}^0$ (i.e we can introduce constraints $T_\alpha x = 0$.) We will assume that $\widetilde{\mathcal{E}}^i = H^i(\mathcal{G}, E) = 0$ for $i > 0$ (the cohomology is concentrated in degree 0). Then the physics described by the Hamiltonian H restricted to $\widetilde{\mathcal{E}}^0$ is equivalent to the physics of H extended to \mathcal{E} if we are interested in expectation values of BRST-closed operators. This is the easiest way to take into account constraints: instead of restricting the space we enhance it including ghosts.

The quantum consideration above has a classical counterpart. In Hamiltonian approach we should work with symplectic supermanifold M (i.e. with a supermanifold equipped with even close nondegenerate 2-form). An analog of a BRST-operator is a function Ω on M obeying $\{\Omega, \Omega\} = 0$ where $\{, \}$ stands for Poisson bracket. Classical observables are associated with homology of operator $\widehat{Q} : A \rightarrow \{A, \Omega\}$ acting on the space of functions on M .

The operator \widehat{Q} , obeying $\widehat{Q}^2 = 0$, can be considered as an odd vector field on M . We say that such a vector field specifies a structure of Q -manifold on M . Notice, that this structure is compatible with symplectic structure (the Lie derivative of symplectic form with respect to Q vanishes).

Gauge theories in Hamiltonian formalism are systems with constraints. The simplest way to study Hamiltonian constrained systems is to introduce ghosts as we have explained. However, usually it is easier to work in Lagrangian formalism. Lagrangian analogs of above constructions will be described in the next section.

4. Batalin-Vilkovisky formalism.

Let us consider an $(n | n)$ -dimensional supermanifold M equipped with an odd non-degenerate closed 2-form $\omega = \omega_{ij} dz^i dz^j$. We'll say that M is an odd symplectic manifold (a P -manifold). In appropriate local coordinates ω has the form $\omega = \sum dx^i d\xi_i$, $i = 1, \dots, n$; in other words M can be pasted together from $(n | n)$ -dimensional superdomains by means of transformations preserving $\sum dx^i d\xi_i$.

In the same way as on even symplectic manifold we can define Poisson bracket $\{f, g\}$ on a P -manifold. For every function H on P -manifold M we define first order differential operator \widehat{K}_H (a vector field K_H) by the formula $\widehat{K}_H(f) = \{f, H\}$. It is easy to check that K_H preserves odd symplectic structure (i.e. Lie derivative L_K of ω with respect to K_H vanishes). Conversely, if a vector field K preserves ω , it can be represented as K_H at least locally. If a P -manifold M is equipped with a volume element we can define an odd second order differential operator Δ on M by the formula $\Delta f = \text{div} K_f$. If the operator Δ obeys $\Delta^2 = 0$ we say that M is an SP -manifold. One can prove that an SP -manifold can be pasted together from $(n | n)$ -dimensional superdomains by means of transformations preserving

$\omega = \sum dx^i d\xi_i$ and volume element; the operator Δ is equal to $2\partial^2/\partial x^i \partial \xi_i$ in the coordinates $(x^1, \dots, x^n, \xi_1, \dots, \xi_n)$.

Let us consider a function A defined on a compact SP -manifold M and obeying $\Delta A = 0$. One can prove the following statement :

The expression

$$(12) \quad \int_L A d\nu$$

where L is a Lagrangian submanifold of M does not change by continuous variation of L ; moreover, L can be replaced by any other Lagrangian submanifold L' belonging to the same homology class. (The notion of Lagrangian submanifold of odd symplectic manifold can be defined as in even case; a Lagrangian submanifold of L of an SP -manifold can be equipped naturally by a volume element $d\nu$.) In the case when $A = \Delta B$ the integral (12) vanishes; this means that (12) determines a functional on $\text{Ker}\Delta/\text{Im}\Delta$. If the function A is represented in the form $A = \exp(\hbar^{-1}S)$ the equation $\Delta A = 0$ is equivalent to the following equation for S

$$(13) \quad \hbar\Delta S + \{S, S\} = 0.$$

This equation is known as quantum master equation; it plays an important role in BV quantization procedure. Namely, in this procedure we can take as a starting point classical action functional and construct a solution to (13); the physical quantities are obtained as integrals of the form

$$\int_L e^{-1\hbar S} d\nu;$$

the choice of Lagrangian submanifold L corresponds to the choice of gauge condition. Of course, in the quantization problem we should consider ill-defined infinite-dimensional integrals; statements about the integral (12) are proved rigorously in finite-dimensional case, but don't have any precise meaning in infinite-dimensional situation. Moreover, it is difficult even to define the notion of infinite-dimensional SP -manifold and to construct the operator Δ . Nevertheless, one can use the framework of perturbation theory to quantize gauge theories in BV formalism.

Let us emphasize that the notion of P -manifold and the equation

$$(14) \quad \{S, S\} = 0$$

(classical master equation) make sense in the infinite-dimensional case. It is natural to say that a solution of (14) specifies a classical mechanical system in BV formalism. We will show that this viewpoint permits us to give a very simple construction of topological quantum field theories.

First of all we should give a geometric interpretation of the solution to classical master equation. Let us denote by Q an odd vector field corresponding to S . It follows from $\{S, S\} = 0$ that $\{Q, Q\} = 0$ (in other words the first order differential operator \hat{Q} defined by the formula $\hat{Q}\Phi = \{\Phi, S\}$ obeys $\hat{Q}^2 = 0$). We'll say that

a supermanifold equipped with an odd vector field Q obeying $\{Q, Q\} = 0$ is a Q -manifold. We see that a solution of classical master equation on a P -manifold M specifies a structure of a Q -manifold on M ; these two structures are compatible (the odd symplectic structure is Q -invariant; i.e. the Lie derivative of odd symplectic form with respect to Q vanishes). It is easy to check that, conversely, every QP -manifold (i.e. Q -manifold equipped with Q -invariant odd symplectic structure) can be obtained from a solution of classical master equation.

One can obtain many examples of QP -manifolds by means of simple geometric constructions. We'll show how to construct topological quantum field theories this way.

Notice, first of all that the space $\{\Sigma \rightarrow X\}$ of maps of a Q -manifold Σ into a Q -manifold X can be considered as a Q -manifold. Let us take as Σ the manifold ΠTM where M is a d -dimensional manifold (the symbol Π stands for parity change; one obtains the supermanifold ΠTM from the tangent bundle TM reversing parity of the fibers). The functions on ΠTM can be regarded as differential forms on M ; de Rham differential can be interpreted as an odd vector field Q on ΠTM obeying $\{Q, Q\} = 0$. This means that ΠTM can be considered as a Q -manifold. The natural volume element on ΠTM is Q -invariant (i.e. $\text{div} Q = 0$). Notice that the volume element on ΠTM is odd if M is odd-dimensional and even if M is even-dimensional.

To introduce a symplectic structure on the space of maps $\{\Sigma \rightarrow X\}$ we need a volume element on Σ and a symplectic structure on X . Then the symplectic form on the space of maps can be defined as an integral

$$\tilde{\omega}(\delta_1 f, \delta_2 f) = \int_{\Sigma} \omega(\delta_1 f(\sigma), \delta_2 f(\sigma)) d\sigma$$

where ω stands for symplectic form on X and $d\sigma$ for volume element on Σ . To obtain an odd symplectic structure on $\{\Sigma \rightarrow X\}$ we should assume that the parity of symplectic structure on X is opposite to the parity of volume element on Σ .

Now we can say that $\{\Sigma \rightarrow X\}$ is a QP -manifold if Σ is a Q -manifold equipped with even Q -invariant volume element and X is a QP -manifold. If Σ is a Q -manifold with odd Q -invariant volume element, then we can introduce a structure of a QP -manifold in $\{\Sigma \rightarrow X\}$ in the case when X is a Q -manifold with Q -invariant even symplectic structure. The functional S on $\{\Sigma \rightarrow X\}$ corresponding to the vector field Q obeys $\{S, S\} = 0$ and specifies a classical mechanical system, that can be called BV sigma-model. In the case when $\Sigma = \Pi TM$ corresponding sigma-model can be considered as topological field theory (the action functional depends only on smooth structure of the manifold M and therefore corresponding physical quantities should provide diffeomorphism invariants of M). This general construction leads to many interesting TQFTs. In particular, we can obtain (generalized) Chern-Simons theory in the following way. Let us take as X a linear $\Pi \mathcal{G}$ where \mathcal{G} is a Lie algebra equipped with invariant inner product. One can consider $\Pi \mathcal{G}$ as a Q -manifold equipped with Q -invariant symplectic structure. Functions on X can be interpreted as cochains of Lie algebra \mathcal{G} ; the differential acting on cochains can be considered as vector field Q on X . The symmetric inner product on \mathcal{G} specifies an even symplectic structure on $X = \Pi \mathcal{G}$.

A map of $\Sigma = \Pi TM$ into $X = \Pi \mathcal{G}$ can be considered as \mathcal{G} -valued differential form A on M . To obtain a structure of a QP -manifold on $\{\Pi TM \rightarrow \Pi \mathcal{G}\}$ we

assume that M is an odd-dimensional manifold. Above arguments lead to the following action functional:

$$(15) \quad S(A) = \int_M (12A \wedge dA + 13A \wedge A \wedge A)$$

In the case $\dim M = 3$ we obtain a BV version of Chern-Simons action functional.

Let us consider the space $\{\Pi T M \rightarrow X\}$ where M is an odd-dimensional manifold and X is an even symplectic manifold equipped with trivial Q -structure ($Q = 0$). We will analyze in detail the case when X is a symplectic vector space with symplectic form having constant coefficients $\omega_{\alpha\beta}$. In this case maps $\Pi T M \rightarrow X$ can be identified with vector valued forms A^α on M ($\alpha = 1, \dots, \dim X$) and the action functional is quadratic.

$$(16) \quad S = \int_M \omega_{\alpha\beta} A^\alpha \wedge A^\beta.$$

We obtain a BV-version of action functional (10); corresponding partition function is related to Ray-Singer torsion.

For every manifold Y we can construct a structure of a Q -manifold on $\Pi T Y$ and a structure of a P -manifold on $\Pi T^* Y$. If Y is an even symplectic manifold we can identify $\Pi T Y$ and $\Pi T^* Y$ and obtain a structure of a QP -manifold on $\Pi T^* Y$; corresponding solution to classical master equation can be written in the form $s = \omega^{\alpha\beta}(y)\eta_\alpha\eta_\beta$, where $y \in Y$, η_α are odd coordinates in the fibres of $\Pi T^* Y$ and $\omega^{\alpha\beta}$ stands for bivector that is inverse to symplectic form $\omega_{\alpha\beta}$. It is easy to verify that this construction specifies a structure of a QP -manifold on $\Pi T^* Y$ in more general case when Y is a Poisson manifold ($s = \omega^{\alpha\beta}(y)\eta_\alpha\eta_\beta$ obeys classical master equation iff $\omega^{\alpha\beta}$ determines Poisson structure on Y). We can consider now topological BV sigma-model on the space of maps $\{\Pi T M \rightarrow \Pi T^* Y\}$ where M is a two-dimensional (or, more generally, even-dimensional) manifold and Y is a Poisson manifold. In the case when M is a disk this sigma-model was used by Kontsevich in his famous work about formal quantization of Poisson manifolds [23], see also [11]. Taking a symplectic manifold equipped with an almost complex structure as Y we can single out a Lagrangian submanifold L of $\{\Pi T M \rightarrow \Pi T^* Y\}$ in such a way that restriction functional to L leads so called A -model [1].

5. Perturbation theory in BV-formalism.

Let us begin with finite-dimensional case. We consider a functional S defined on linear SP -manifold \mathcal{E} . (Every manifold of this kind is isomorphic to $\mathbb{R}^{n|n}$ equipped with standard odd symplectic form and volume element.) We represent S as $S_0 + V$ where S_0 consists of quadratic terms in Taylor series at stationary point x_0 of S ; without loss of generality we can take $x_0 = 0$. We assume that S obeys both quantum and classical master equations (i.e. $\{S, S\} = 0$, $\Delta S = 0$). Then S_0 and V also have this property: $\{S_0, S_0\} = 0$, $\Delta S_0 = 0$, $\{V, V\} = 0$, $\Delta V = 0$; therefore $\{S_0, V\} = 0$. Let us consider a linear Lagrangian subspace $L \subset \mathcal{E}$; we suppose that

S_0 is nondegenerate on L and the integral

$$Z(\lambda) = \int_L e^{-(S_0 + \lambda V)} dv,$$

representing the partition function of action functional $S_0 + \lambda V$, converges. We can apply standard methods to get a perturbative expansion of $Z(\lambda)/Z(0)$ with respect to λ . The series we obtain does not depend on the choice of L ; therefore one can describe the answer in a form where L is not involved. We'll prove that one can use the standard Feynman diagram technique where vertices are governed by V and the propagator p is a bivector on \mathcal{E} that obeys

$$(17) \quad q_\gamma^\alpha p^{\gamma\beta} - p^{\alpha\gamma} q_\gamma^\beta = \omega^{\alpha\beta}$$

Here $\omega^{\alpha\beta}$ is the bivector that is inverse to the matrix $\omega_{\alpha\beta}$ of odd symplectic form, $q_\beta^\alpha = \omega^{\alpha\gamma} s_{\gamma\beta}$ where $s_{\alpha\beta}$ stands for the matrix of quadratic form S_0 . In more invariant form we can say that quadratic form S_0 generates a vector field Q on \mathcal{E} . The coordinates of Q are linear functions on \mathcal{E} , therefore we can construct a linear operator q acting on \mathcal{E} ; it follows from $\{S_0, S_0\} = 0$ that $q^2 = 0$. This means that we can regard q as a differential; it follows from our assumptions that corresponding homology is trivial ($\text{Ker } q = \text{Im } q$). The operator q generates a differential in the space of bivectors; we'll use the same symbol for it. The condition (17) means that $qp = \omega$ where ω stands for bivector, that is inverse to symplectic form. Notice that for two bivectors p' and p that obey (17) we have $q(p' - p) = 0$. It follows from acyclicity of q that there exists an odd bivector u satisfying $p' - p = qu$. One can use this remark to show that diagrams constructed by means of propagator p' coincide with diagrams with propagator p .

The above prescription can be justified in the following way. One can introduce such a coordinate system $x^1, \dots, x^n, \xi_1, \dots, \xi_n$ on \mathcal{E} that S_0 depends only on ξ_1, \dots, ξ_n ; the parity of ξ_i is opposite to the parity of x^i , the odd symplectic form is equal to $\sum dx^i d\xi_i$ and the volume element is standard (see [37]). Then the partition function can be written as integral of $\exp(-(S_0 + \lambda V))$ over Lagrangian submanifold $x = X$, where X^1, \dots, X^n is a fixed vector. This integral can be converted into an integral over \mathcal{E} :

$$(18) \quad Z = \text{const} \int e^{-(S_0 + \sigma) - \lambda V} d\xi dx$$

(we included $\delta(x - X)$ into the integrand, multiplied it by $\exp(-\sigma(X))$ and integrated over X . Here $\sigma(X)$ stands for nondegenerate quadratic form; we require convergence of the integral of $\exp(-\sigma(X))$ over X^1, \dots, X^n).

It follows from (18) that $Z(\lambda)/Z(0)$ can be represented by means of Feynman diagrams where propagator is inverse to $S_0 + \sigma$. It is easy to check that this recipe coincides with the above prescription for specific choice of bivector p obeying (17). This means that we can use any bivector satisfying (17). (We mentioned already that diagrams don't depend on the choice of propagator as long it obeys (17).)

We can apply the diagram technique developed in finite-dimensional case to infinite-dimensional situation.

Let us consider for example the BV-formalism of Chern-Simons action functional (i.e. functional (15) for $\dim M = 3$). Then we obtain precisely diagrams constructed by M. Kontsevich [21]. It is necessary to emphasize, however, that our considerations in infinite-dimensional case are heuristic. To obtain rigorous results one should analyze the convergence of integrals representing the diagrams, etc. (see [21]).

6. Observables .

Let us consider an SP -manifold M and a quantum system corresponding to a solution S of quantum master equation (13). We say that a function A on M is a quantum observable if it satisfies the equation

$$(19) \quad \hbar \Delta A + 2\{A, S\} = 0$$

It is important to notice that A is not necessarily an even function. The expression

$$\int_L A e^{S/\hbar} d\nu$$

where L is Lagrangian submanifold of L , has the meaning of the expectation value of A . This expression depends only on homology class of L . For even A this fact immediately follows from the remark that the equation (17) holds that iff $S + \varepsilon A$ where ε is an infinitesimal parameter obeys quantum master equation; analogous statement is true for odd A . If a quantum observable can be represented in the form $A = \hbar \Delta B + 2\{B, S\}$ then its expectation value vanishes; we say that such an observable is trivial.

The above remarks show that observables can be studied in the framework of families of quantum systems; expectation values of observables govern the variation of partition function by the infinitesimal change of parameters.

One can prove the following statements:

a) If A and B are quantum observables, then $\{A, B\}$ is also a quantum observable. (In other words, quantum observables constitute a Lie superalgebra.)

b) Let us suppose that quantum observables T_α span a Lie (super)algebra \mathcal{G} (i.e. $\{T_\alpha, T_\beta\} = f_{\alpha\beta}^\gamma T_\gamma$) and that the antisymmetric tensor $c^{\alpha_1 \dots \alpha_k}$ represents a homology class of \mathcal{G} . Then $c^{\alpha_1 \dots \alpha_k} T_{\alpha_1} \dots T_{\alpha_k}$ is also a quantum observable. If the tensor $c^{\alpha_1 \dots \alpha_k}$ belongs to the trivial homology class, the corresponding observable is also trivial.

One can derive these statements by means of straightforward calculations based on the definition of homology, on the relation

$$\{S, KL\} = \{S, K\} \cdot L + (-1)^{\varepsilon(K)} \{S, L\} \cdot K$$

and on the formula

$$\Delta(KL) = \Delta K \cdot L + (-1)^{\varepsilon(K)} K \cdot \Delta L + (-1)^{\varepsilon(K)} \{K, L\}.$$

Taking $\hbar = 0$ in the definition of quantum observable we obtain a definition of classical observable. We can also consider functionals that verify the condition (19) for all \hbar . One can say that these functionals are quantum and classical observables at the same time; we will omit the adjective talking about observables of this kind. These observables are related to infinitesimal variations of action functionals obeying quantum and classical master equations simultaneously: $\Delta S = 0, \{S, S\} = 0$. As we emphasized, in infinite-dimensional case the operator Δ is ill-defined, therefore it is difficult to work with quantum master equation. It is much easier to make sense of equation $\Delta S = 0$; it can be written in the form $\text{div} Q = 0$ and means that the odd vector field Q corresponding to the functional S is "volume preserving."

In Sec. 7 we'll discuss how to obtain topological invariants in the framework of perturbation theory taking as a starting point the BV version of Chern-Simons action functional (15). Notice that it is possible to obtain perturbative Chern-Simons invariants also from quadratic action functional (16) considering non-trivial quantum observables. Lie algebra \mathcal{H} of polynomial Hamiltonian vector fields on X can be considered as an algebra of symmetries of the functional (16). This means that we can associate a quantum observable with every homology class of this Lie algebra. Corresponding expectation values are topological invariants of M . Kontsevich constructed a graph complex having homology closely related to the homology of Lie algebra \mathcal{H} (see [21]). It follows from this result that one can associate topological invariants of M with homology of graph complex. It is shown in [40] that invariants obtained this way coincide with invariants derived in [21] from the analysis of perturbative Chern-Simons theory. One can say that these observations give physical explanation of some results of [21].

7. Families of action functionals.

Let us consider a smooth family S_λ of functionals defined on manifold M and labeled by points $\lambda \in \Lambda$. We assume that these functionals obey quantum and classical master equations for every $\lambda \in \Lambda$:

$$\Delta S_\lambda = 0, \{S_\lambda, S_\lambda\} = 0.$$

Let V stand for a vector on Λ . Then the variation of S in the direction V is governed by observable $T(V)$:

$$\widehat{V}S = T(V), \Delta T(V) = 0, \{T(V), S\} = 0.$$

We will assume that there exists functional $B(V)$ on M obeying

$$(20) \quad T(V) = \{B(V), S\}, \Delta B(V) = 0.$$

Then the observable $T(V)$ is trivial, because

$$\widehat{V}e^S = T(V)e^S = \{B(V), e^S\} = \Delta(B(V)e^S)$$

This means that the partition function corresponding to the action functional S_λ does not depend on λ . It seems that the consideration of the family S_λ is superfluous and we can restrict ourselves to one of the members of this family. We'll see that this is not the case. The functional $T(V)$ is defined for every vector V on (super)manifold Λ . For definiteness we consider only even vectors; then $T(V)$ is an even functional and $B(V)$ is an odd functional. We will use the notation $B(V)$ also in the case when V is a vector field; then $B(V)$ is a function depending on $x \in M$ and $\lambda \in \Lambda$. It is easy to check that

$$\widehat{V}_1 \widehat{V}_2 e^S = \widehat{V}_1 \{B(V_2), e^S\} = \{\widehat{V}_1 B(V_2), e^S\} + \{B(V_2), \{B(V_1), e^S\}\}.$$

Comparing this formula with

$$[\widehat{V}_1, \widehat{V}_2] e^S = \{B([V_1, V_2]), e^S\}$$

we obtain that

$$B([V_1, V_2]) = \widehat{V}_1 B(V_2) - \widehat{V}_2 B(V_1) - \{B(V_1), B(V_2)\} + \xi(V_1, V_2)$$

where $\Delta \xi(V_1, V_2) = 0, \{\xi(V_1, V_2), S\} = 0$.

We see that $\xi(V_1, V_2)$ is an observable.

We consider $\xi(V_1, V_2)$ as a function on M and two-form on Λ ; one can check that this two-form is closed. In most interesting cases this form can be represented as a differential of one-form η obeying $\Delta \eta = 0, \{\eta(V), S\} = 0$. If this exactness condition is satisfied we can replace $B(V)$ with $B(V) - \eta(V)$ preserving relations (20); with this new definition of $B(V)$ the form $\xi(V_1, V_2)$ vanishes, i.e.

$$(21) \quad B([V_1, V_2]) = \widehat{V}_1 B(V_2) - \widehat{V}_2 B(V_1) - \{B(V_1), B(V_2)\}.$$

We'll assume that (21) is satisfied. Then one can prove that the n -form $\Delta \omega_n = \Delta(B(V_1) \dots B(V_n) e^S)$ is a differential of $(n-1)$ -form $\omega_{n-1} = B(V_1) \dots B(V_{n-1}) e^S$ on Λ . This means that for every n -cycle Γ on Λ we have

$$\Delta \int_{\Gamma} B(V_1) \dots B(V_n) e^S = 0.$$

Integrating $\int_{\Gamma} B(V_1) \dots B(V_n) e^S$ over a Lagrangian submanifold $L \subset M$ we obtain a number that depends only on homology classes of Γ and L .

The above statement can be generalized to the case when $\xi(V_1, V_2)$ does not vanish. In this case it is convenient to consider an inhomogeneous form $\omega = \sum \omega_n$. One can prove, that

$$\Delta \omega = (d + \xi) \omega.$$

Notice that instead of family $S(\lambda)$ of equivalent action functionals we can consider a functional S obeying $\Delta S = 0, \{S, S\} = 0$, and a family of Lagrangian submanifolds L_λ labeled by $\lambda \in \Lambda$.

Let us denote by \mathcal{L} the infinite-dimensional manifold of all Lagrangian submanifolds of M . (If $L \subset M$ is a Lagrangian submanifold we can identify a neighborhood of L in M with ΠT^*L . Using this identification we can construct for every odd function Ψ on L a Lagrangian submanifold by means of the formula $\xi_i = \partial\Psi\partial x^i$ where x^i are coordinates on L , ξ_i are coordinates on the fibers. This construction gives a parametrization of a subset of \mathcal{L} in terms of functions on L ; we see that \mathcal{L} can be considered as infinite-dimensional manifold. Lie algebra of even vector fields preserving odd symplectic structure and volume element on M acts on \mathcal{L} in natural way; it is easy to check that this action is Lie algebra of odd functionals B on M obeying $\Delta B = 0$. This means that to every B obeying $\Delta B = 0$ and every $L \in \mathcal{L}$ corresponds a vector $V \in T_L(\mathcal{L})$ (a tangent vector at the point $L \in \mathcal{L}$) and that this map is surjective. The inverse map is multivalued, but one can fix one-valued smooth branch $B(V)$; we'll use the same notation when V is a vector field on \mathcal{L} . We'll assume that

$$B([V_1, V_2]) = \widehat{V}_1 B(V_2) - \widehat{V}_2 B(V_1) - \{B(V_1), B(V_2)\}.$$

as in the case of family of action functionals considered above.

Let us consider an n -form on \mathcal{L} defined by the formula

$$\omega_n = \int_L B(V_1) \dots B(V_n) e^S d\nu$$

One can prove that this form is closed. The proof is based on the relation

$$\widehat{V} \int_L \varphi d\nu = \int_L \{\varphi, B(V)\} d\nu$$

(We consider infinitesimal transformation of M preserving volume element and odd symplectic structure. To calculate the variation of $\int_L \varphi d\nu$ by the variation of L we use the fact that instead of changing L we can change the integrand.)

Notice that we can modify the definition of the forms ω_n including an observable A into the integrand. The forms remain closed after such a modification (this follows immediately from the remark that observables are related to infinitesimal variations of action functional).

One can consider \mathcal{L} or the space of equivalent action functionals as a kind of moduli space for the problem at hand. We obtained under certain conditions closed forms on this space. Integrating these forms over cycles in moduli space we obtain numbers that generalize expectation values of observables (these expectation values correspond to 0-forms).

Let us apply the above consideration to the case of topological theories. In this case every metric on the worldsheet determines one of equivalent action functionals (in Witten's approach) or a gauge condition (a Lagrangian submanifold in BV approach). Under certain conditions closed forms on the space of metrics are equivariant with respect to diffeomorphisms of worldsheet M and therefore descend to the quotient space $\mathcal{M} = \{\text{metrics}\} / \{\text{diffeomorphisms}\}$. The space of metrics is contractible hence the quotient space is closely related to the classifying space $B\text{Diff}(M)$

of diffeomorphism group. In the case of multidimensional version of BV Chern-Simons functional the differential forms constructed above are related to forms on $\text{BDiff}(M)$ considered in [8]. (Notice that Kontsevich modifies the space $\text{BDiff}(M)$ to get rid of quantum anomalies connected with zero-dimensional homology of M .) In the case of two-dimensional topological theory the numbers obtained by means of integration of differential forms on moduli space over cycles coincide with string amplitudes. Notice, that in two-dimensional case the moduli space of metrics \mathcal{M} is homotopy equivalent to the moduli space of conformal structures on the world-sheet (=moduli space of complex curves with given topology). The appearance of so called Deligne-Mumford compactification of the moduli space of complex curves is related to the fact that one can obtain reasonable gauge conditions allowing metrics with some mild singularities.

8. Chern-Simons theory and topological sigma-model.

Chern-Simons theory is closely related to so called G/G model. This two-dimensional topological model can be considered as gauged WZNW model and can be solved either by conformal field theory methods or directly. To establish the relation between of Chern-Simons theory and G/G model one can use WZNW model as an intermediate step (see [44]); there exists also more direct way found in [9].

All these approaches are based on the remark that 1-form A satisfying the equations of motion corresponding to Chern-Simons action functional (11) can be considered as a flat connection on trivial vector bundle. In topologically trivial situation all flat connections are gauge equivalent; this means that every flat connection can be represented in the form

$$(21) \quad A = g^{-1}(x)dg(x)$$

where $g(x)$ is a function taking values in the group G . (We suppose that \mathcal{G} is a Lie algebra of the group G .) Another method uses BV-formalism; this approach can be applied also to multidimensional generalization (15) of Chern-Simons action functional. It is based on the remark that equation of motion corresponding to the action functional (15) can be easily solved. These equations have the form

$$(22) \quad dA + A \wedge A = 0$$

where A is $\Pi\mathcal{G}$ -valued function on ΠTM (\mathcal{G} -valued inhomogeneous form on M).

To every G -valued function $g(x)$ on ΠTM we can assign a $\Pi\mathcal{G}$ -valued function on ΠTM by formula $A = g^{-1}(x)(Qg)(x)$ or, more precisely,

$$(23) \quad A = (g^{-1}(x))_*(Qg)(x).$$

Recall that ΠTM is a Q -manifold. The vector field Q on ΠTM determines a vector field on the space of maps $\{\Pi TM \rightarrow G\}$ that is denoted by the same letter. The symbol $(g^{-1}(x))_*$ stands for the map of tangent space $g(x)$ of the space

$\{\Pi TM \rightarrow G\}$ at the point into tangent space at the point $g(x) = 1$ that is induced by left multiplication: $h(x) \rightarrow g^{-1}(x)h(x)$.

It is easy to check that (23) satisfies Eqn (22) and that in topologically trivial situation all solutions to (22) can be obtained this way.

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