Géométrie différentielle et théorie de jauge, 11/02/2020
Connections and curvature

1. Let E be a rank r vector bundle with connection ∇. In a local trivialisation $(s_1, \dots, s_r)$ of E, one has $\nabla = d + a_i dx^i$. Prove that in the trivialization $(s_1 \wedge \cdots \wedge s_r)$ of $\det(E) := \Lambda^r E$, the induced connection is $\nabla^{\det(E)} = d + \operatorname{tr}(a_i) dx^i$.

We define a $SL(r, \mathbb{C})$- bundle as a rank r bundle E with a given isomorphism $\det(E) \simeq \mathbb{C}$, and a $SL(r, \mathbb{C})$-connection on E as a connection ∇ on E such that $\nabla^{\det(E)} = d$ the trivial connection on $\mathbb{C}$. This means that in any trivialization $(s_1, \dots, s_r)$ of E such that $s_1 \wedge \cdots \wedge s_r = 1$ one has $\nabla = d + a_i dx^i$ with $\operatorname{tr}(a_i) = 0$.

2. Calculate the curvature of the bundle $\mathscr{O}(-1)$ on $\mathbb{C}P^1$ and check that $\frac{i}{2\pi} \int_{\mathbb{C}P^1} F = -1$. Check that this result actually does not depend on the chosen connection, it is the *first Chern number* of $\mathscr{O}(-1)$. In particular, the bundle $\mathscr{O}(-1)$ is topologically non trivial.

3. Prove that the curvature $F^{\operatorname{End}(E)}$ of the endomorphism bundle of E is given in terms of F^E by $F^{\operatorname{End}(E)}(u) = [F^E, u]$.

4. Let us calculate the curvature of the sphere S^n in the following way. We take standard coordinates x^0,...,x^n on $\mathbb{R}^{n+1}$. Near the north pole, we take coordinates $y = (y^1, \dots, y^n)$ on S^n parametrizing the point $f(y) = (\sqrt{1 - |y|^2}, y) \in S^n \subset \mathbb{R}^{n+1}$. Prove the development

$$d_y f\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial}{\partial x^i} - y^i \frac{\partial}{\partial x^0} + O(|y|^2),$$

and deduce a development at the origin of the Christoffel symbols $\Gamma_{ij}^k = \delta_{ij} y^k + O(|y|^2)$. Deduce that the curvature at $y = 0$ is given by $R_{ijk}^\ell = \delta_i^\ell \delta_{jk} - \delta_{ik} \delta_j^\ell$. Prove that the sphere has constant sectional curvature equal to $+1$.

5. (i) Let (M, g) be a Riemannian manifold, and E be a Hermitian bundle with unitary connection $\nabla : \Gamma(E) \to \Omega^1(E)$. Prove that if $\alpha \in \Omega^1(E)$ then $\nabla^* \alpha = -\operatorname{tr}^g(\nabla \alpha)$, that is $\nabla^* \alpha = -\sum i(e_i) \nabla_{e_i} \alpha$ for any orthonormal basis (e_i) of TM.

Hint. Take an orthonormal frame (e_i) of TM such that at the point p we have $\nabla e_i(p) = 0$, and show that $(d^\nabla * (\alpha_i e^i))(p) = \sum_1^n (\nabla_{e_i} \alpha_i)(p) e^1 \wedge \cdots \wedge e^n$.

(ii) For $d^\nabla : \Omega^p(E) \to \Omega^{p+1}(E)$, deduce from the Maurer-Cartan formula the identity

$$d^\nabla \alpha(X_0, \dots, X_p) = \sum_0^p (\nabla_{X_i} \alpha)(X_0, \dots, \widehat{X_i}, \dots, X_p).$$

This means that $d^\nabla \alpha = \frac{1}{p+1} \mathbf{a}(\nabla \alpha)$, where $\mathbf{a}$ is the antisymmetrization $\Omega^1 \otimes \Omega^p \to \Omega^{p+1}$.

(iii) Be careful that the norms on Ω^p and $\otimes^p \Omega^1$ do not coincide, $|\cdot|_{\Omega^p}^2 = p! |\cdot|_{\otimes^p \Omega^1}^2$ (think for example that $e^1 \wedge e^2 = e^1 \otimes e^2 - e^2 \otimes e^1$). Deduce from (ii) that $(d^\nabla)^* : \Omega^{p+1}(E) \to \Omega^p(E)$ is the restriction of ∇^* to antisymmetric tensors $\Omega^{p+1}(E) \subset \Omega^1 \Omega^p(E)$. Apply (i) to get for any p the formula

$$(d^\nabla)^* \alpha = -\sum i(e_i) \nabla_{e_i} \alpha.$$

6. Let (M, g) be a Riemannian manifold (or Lorentzian, etc., but let's say Riemannian for simplicity). We define the divergence $\delta \alpha$ of a symmetric 2-tensor α by $\delta \alpha = -\sum i(e_i) \nabla_{e_i} \alpha$ for any orthonormal basis of TM (equivalently, $(\delta \alpha)_k = -g^{ij} \alpha_{ik,j}$). Prove that the Einstein tensor $r = \operatorname{Ric} - \frac{1}{2} \operatorname{Scal} g$ is divergence free: $\delta r = 0$. (Calculate at a point p in a basis such that $\nabla e_i(p) = 0$ and use Bianchi differential identity).

Problem. The (non commutative) field of quaternions is defined as the real algebra

$$\mathbb{H} = \{q := x_0 + x_1 i + x_2 j + x_3 k, x_i \in \mathbb{R}\}$$

where $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. One can realize $\mathbb{H}$ as a space of 2x2 matrices, with

$$1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad i = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right), \quad j = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \quad k = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right).$$

There is an identification $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ by writing $q = (x_0 + ix_1) + (x_2 + ix_3)j$.
One has a conjugation $\bar{q} = x_0 - x_1 i - x_2 j - x_3 k$. One has $q\bar{q} = \bar{q}q = x_0^2 + \cdots + x_3^2 = |q|^2$, and therefore $q^{-1} = q/|q|^2$. One can define a real part $\mathfrak{R}(q) = \frac{q+\bar{q}}{2} = x_0$ and an imaginary part $\mathfrak{I}(q) = \frac{q-\bar{q}}{2} = x_1 i + x_2 i + x_3 k$. As is clear from the matrix description, the space $\mathfrak{I}(\mathbb{H})$ of imaginary quaternions, equipped with the bracket $[q, q'] = qq' - q'q$, is a Lie algebra isomorphic to $\mathfrak{su}(2)$.
The projective space $\mathbb{H}P^1$ of quaternionic lines in $\mathbb{H}^2$ has homogeneous coordinates $[q_0 : q_1] = [q_0\lambda : q_1\lambda]$ for any $\lambda \in \mathbb{H}$. Similarly to $\mathbb{C}P^1 = S^2$, one has $\mathbb{H}P^1 = S^4$, with $[q : 1] \mapsto q \in \mathbb{H} = \mathbb{R}^4$.
There is a tautological quaternionic line bundle E over $\mathbb{H}P^1$, given by $E_{[q_0:q_1]} = (q_0, q_1)\mathbb{H}$. This is in particular a Hermitian rank 2 vector bundle (the complex structure comes from the right multiplication by i). By orthogonal projection, E gets a unitary connection that we will now calculate:
— Prove that the orthogonal projection of (x_0, x_1) on $(q_0, q_1)\mathbb{H}$, where $|q_0|^2 + |q_1|^2 = 1$, is $(q_0, q_1)(\bar{q}_0 x_0 + \bar{q}_1 x_1)$.
— For $x \in \mathbb{H} = \mathbb{R}^4$ we consider the (normed) section of E given by $s(x) = \frac{(x,1)}{\sqrt{1+|x|^2}}$ (this gives a complex orthonormal trivialization (s, sj)). Local sections of E are given by $s(x)\lambda(x)$ for $\lambda(x) \in \mathbb{H}$. Prove that the connection on E is given by

$$\nabla(s\lambda) = s(d\lambda + A\lambda), \quad A = \frac{\mathfrak{I}(\bar{x}\, dx)}{1 + |x|^2}.$$

Since $\mathfrak{I}(\mathbb{H}) = \mathfrak{su}(2)$, the 1-form A with values in $\mathfrak{I}(\mathbb{H})$ is a $SU(2)$ connection matrix. Be careful here that the bundle is quaternionic for the right multiplication by $\mathbb{H}$, but the $\mathfrak{su}(2)$ is given by *left* multiplication on the coefficient λ (this of course depends on the choice of basis s).
— Deduce the curvature

$$F = \frac{d\bar{x} \wedge dx}{(1 + |x|^2)^2}.$$

Check that F is actually an antiselfdual form on $\mathbb{R}^4$, that is $*F = -F$. Therefore the connection is an *instanton* on $\mathbb{R}^4$.
— Check that $\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{tr}(F \wedge F) = 1$ (the *instanton number*).
— We say that two Riemannian metrics g and g' are conformally equivalent if there exists a function $f > 0$ such that $g' = fg$. Prove that if we have two conformally equivalent metrics g and g' on M^{2n} and $\alpha, \beta \in \Omega^n(M)$, then $\langle\alpha, \beta\rangle_{g'} \mathrm{vol}_{g'} = \langle\alpha, \beta\rangle_g \mathrm{vol}_g$, and deduce that $*_{g'}\alpha = *_g\alpha$.
— Prove that in the $\mathbb{R}^4$ chart given by stereographic projection one has $g_{S^4} = \frac{4|dx|^2}{(1+|x|^2)^2}$ which is conformal to $g_{\mathbb{R}^4}$. Deduce that the connection A is also an instanton on E over S^4, that is the connection has antiselfdual curvature.