M2 ICFP 2019/2020

Introduction to AdS/CFT **TD n°3**

Fields in Anti-de Sitter spacetime

Throughout, the AdS radius is denoted by ℓ. The metric in Poincaré coordinates is :

$$ds^2 = \frac{\ell^2}{z^2}\eta_{ab}dx^a dx^b = \frac{\ell^2}{z^2}\left[dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right] \qquad 0 < z < +\infty$$

Throughout the text, we denote $(d{+}1)$-dimensional (bulk) indices by $a, b, \ldots$ and d-dimensional (boundary) indices by $\mu, \nu \ldots$.

1 Scalar Fields in AdS

1.1 Hamiltonian analysis and the BF bound

The action for a minimally coupled massive scalar field in AdS is :

$$S = -\frac{1}{2\ell^{d-1}}\int dz d^d x\sqrt{-g}\left(g^{ab}\partial_a\phi\partial_b\phi + m^2\phi^2\right).$$

where $a, b = 0 \ldots d$, and the scalar ϕ is taken to be dimensionless.

1. Show that the Klein-Gordon equation in AdS_{d+1} stemming from the action S is

$$\phi''(x,z) - \frac{(d-1)}{z}\phi' + \partial^\mu\partial_\mu\phi - \frac{m^2\ell^2}{z^2}\phi = 0$$

   where $' \equiv \partial_z$.

2. Consider a plane-wave ansatz of the form $\phi(z,x) = e^{ip_\mu x^\mu} z^{\frac{d-1}{2}}\psi(z)$. Show that the equation for $\psi(z)$ becomes equivalent to a one-dimensional Schrodinger problem on the half-line $z > 0$,

$$H\psi = E\psi \qquad H = -\frac{d^2}{dz^2} + V(z)$$

   where the potential is given by

$$V(z) = \left(m^2 + \frac{d^2-1}{4}\right)\frac{1}{z^2}, \qquad E = -p^\mu p_\mu$$

3. Argue that the absence of instabilities in the original problem is equivalent to the requirement that the Hamiltonian above has no negative eigenvalues (i.e. all eigenfunction -normalizable or plane-wave normalizable- of H have $E > 0$).
   [For example, pick $p_\mu = (\omega, 0, 0, 0)$ and look at the time-dependence of the plane-wave solution].

4. Find the asymptotic behavior of ψ close to the boundary and as $z \to +\infty$. In particular, show that close to the boundary the two independent solutions behave as

$$\psi_\pm(z) \simeq b_\pm z^{\alpha_\pm}, \quad z \to 0^+, \qquad (\text{take } \alpha_- < \alpha_+)$$

   where $\alpha_\pm$ are two (complex) parameters which depend on m^2 and $b_\pm$ are constants. What is the corresponding behavior of $\phi(z)$?

5. Show that, if we require that the action S is *finite*[1] as $z \to 0^+$, then only the α_+ solutions are allowed. Show that the corresponding wave-function $\psi_+(z)$ is normalizable in the usual L^2-sense.
6. Show that, on the subspace of normalizable wavefunctions $\psi(z)$, satisfying the boundary condition $b_- = 0$, the Hamiltonian is hermitian *and* positive as long as :

$$m^2\ell^2 > -\frac{d^2}{4} \tag{1}$$

[Positivity should be easy to see for $m^2\ell^2 > -d^2/4 + 1/4$. For $-d^2/4 < m^2\ell^2 < -d^2/4+1/4$, you have to work harder. For example, you can try to show that $H = P^\dagger P$ for an appropriate operator P.]
This shows in particular that, in this mass range, fluctuations around AdS are stable (cfr. question 3), even if $m^2 < 0$, which would normally signal an instability if we were in flat space.
7. Show that, if (1) is violated, then whatever the choice of boundary condition, the Hamiltonian has negative eigenvalues (and, worse, it is unbounded below).
[For this, it might be helpful to have Landau-Lifshitz vol. 3 at hand]
8. What is the value m^2 which corresponds to conformally coupling the scalar filed to gravity (cfr. Problem set 1, ex. 3.3.ii.) ? Is it in the allowed range ? Could we have guessed the wavefunctions in this case without doing any calculation ?

Among other things, this exercise established the *stability bound* (BF bound, after Breitenlhoner and Freedman who found it in 1982) (1) for the stability of a scalar field on an AdS background.

1.2 What happened to the unitarity bound ?

For a scalar field of mass m^2, the two independent solutions have a near-boundary behavior of the form

$$\phi(z,x) \simeq z^{\Delta_-}\phi_-(x),\ z^{\Delta_+}\phi_+(x)$$

where

$$\Delta_\pm = \frac{d}{2} \pm \frac{1}{2}\sqrt{d^2 + 4m^2\ell^2}$$

If we identify the leading term $\phi_-(x)$ with the source for an operator on the field theory side, then the source has dimension Δ_- and the operator has dimension $\Delta = \Delta_+$. Now, it is clear that

$$\Delta_+ \geq \frac{d}{2}.$$

The ϕ_+ solutions corresponds to the normalizable wave-functions of problem 1.

On the other hand, the unitarity bound in a CFT is $\Delta > d/2-1$. How do we get operators of dimension $d/2 - 1 < \Delta \leq d/2$?

1. Show that in the range of masses

$$-\frac{d^2}{4} < m^2 < -\frac{d^2}{4} + 1, \tag{2}$$

both types of eigenfunctions $\psi_\pm(z)$ of H (defined as in the previous section and associated to the two asymptotic behaviors for ϕ as $z \to 0$) are normalizable in the Schrödinger sense.

1. In the Euclidean signature this means that the configuration has a finite "energy."

2. Show that the *alternative* action

$$S_{alt} = S - \int_{z=0^+} d^d x \sqrt{\gamma}\phi(x)\partial_n \phi(x)$$

is finite at $z \to 0$, in the mass range (2), for both ϕ_+ and ϕ_- solutions (and it gives the same field equations as S). Here $\gamma_{\mu\nu}$ is the induced metric on the boundary, γ is its determinant, and n is the unit normal vector to the boundary.

3. Show that the alternative boundary condition $b_+ = 0$ *also* makes the Hamiltonian hermitian (where b_+ is defined in part 1.1.4). *[This is called alternative quantization]*.

4. Give an alternative AdS/CFT prescription to identify sources and operators in the dual field theory, such that now we can obtain operator dimensions below $d/2$ (but always above the unitarity bound).