Institute for Theoretical Physics Heidelberg University

Quantum Field Theory I

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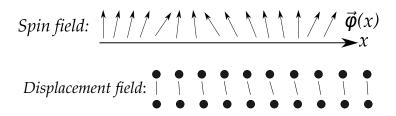
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1. Introduction

1.1 Motivation

- The world is quantum. Classicality is a certain limit.
- Fields are part of reality (e.g. $\vec{E}(\vec{x}, t)$, $A_{\mu}(\vec{x}, t)$).
 - \Rightarrow We need a quantum theory of fields.
- The photon and its interactions will be part of this theory.
- The electron will also emerge as a quantum of an appropriate field.
- All particles of the "Standard Model" are quanta of fields as above. (At sufficiently low energies even the graviton falls into that scheme.)
 - \Rightarrow QFT is *the* fundamental theory of this world (QM is its non-relativistic limit).
- QFT is also the most precisely tested theory we have.
- Furthermore: "Effective fields" are central in Condensed Matter Theory (CMT).
- e.g.



- Most dynamics of these fields is in the quantum regime, thus we need a QFT description.
- QFT is *the* modern language of CMT (including "cold atom physics"), even though it is just an effective description of many-body QM.

1.2 Symmetries (Poincaré transformations)

- As in all of physics, understanding the symmetries of the system we study will be essential.
- Our stage is space-time, $\mathbb{R}^4 \ni (t, \vec{x}) = \{x^{\mu}\}.$ First, we shall focus on space only: $\vec{x} = \{x^i\}$; i = 1, 2, 3.

Symmetries: Group of translations and "rotations" (actually rotations and reflections),

$$\vec{x} \longrightarrow \vec{x}' \quad \text{with} \quad x^i = R^i_{\ i} x^j + d^i.$$
 (1.1)

Focusing on Rotations we ask which matrices R are allowed.

 \Rightarrow The length of vectors should not change:

$$|\vec{x}|^2 = \sum_{i=1}^3 (x^i)^2 = x^i x^j \delta_{ij} \qquad \delta_{ij} = \text{Euclidean metric on } \mathbb{R}^3 \qquad (1.2)$$

$$x^{\prime i} x^{\prime j} \delta_{ij} \stackrel{!}{=} x^i x^j \delta_{ij} \tag{1.3}$$

$$\delta_{ij} R^i_k x^k R^j_l x^l = \delta_{ij} x^i x^j \qquad \forall x \in \mathbb{R}^3$$
 (1.4)

$$\Rightarrow \delta_{ij} R^i_{\ k} R^j_{\ l} = \delta_{kl} \qquad \Leftrightarrow \qquad R R^\top = \mathbb{1} \qquad \Rightarrow R \in O(3) \tag{1.5}$$

Summary: Space is \mathbb{R}^3 with euclidean metric.

Symmetries are translations and "rotations".

"Rotations" are defined by invariance of the metric.

• The generalization to space-time and Poincare invariance is straightforward:

$$\mathbb{R}^3 \ni \{x^i\} \longrightarrow \mathbb{R}^4 \ni \{x^\mu\} = (t, \vec{x}) \tag{1.6}$$

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{ij} \longrightarrow \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu}$$
(1.7)

Note: The overall sign is pure convention. The relative sign between time part and space part of η is deep physical reality. If you wish it is observational data!

$$|\vec{x}|^2 = x^i x^j \delta_{ij} \longrightarrow x^2 = x^\mu x^\nu \eta_{\mu\nu} = t^2 - \vec{x}^2$$
 (1.8)

Note: $c \equiv 1$ throughout this course

if
$$x^2 > 0 \rightarrow$$
 time-like separation if $x^2 < 0 \rightarrow$ space-like separation

$$R \in O(3) \text{ if } \delta_{ij} R^i_k R^j_l = \delta_{kl} \longrightarrow \Lambda \in O(1,3) \text{ if } \eta_{\mu\nu} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} = \eta_{\rho\sigma}$$
 (1.9)

Poincaré transformations

$$x^{\mu} \longrightarrow x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + d^{\mu} \tag{1.10}$$

• The Poincare group is the group of pairs (Λ, d) with $\Lambda \in O(1,3)$, $d \in \mathbb{R}^4$ and the composition law:

$$(\Lambda_1, d_1) \cdot (\Lambda_2, d_2) = (\Lambda_1 \cdot \Lambda_2, \Lambda_1 \cdot d_2 + d_1) \tag{1.11}$$

This will be our most important symmetry!

We will call \mathbb{R}^4 with this symmetry $\mathbb{R}^{1,3}$ or "Minkowski space".

• Clearly, rotations are a subgroup of Lorentz-transformations.

$$\Lambda = \begin{pmatrix} 1 & \vec{0}^{\top} \\ \vec{0} & R \end{pmatrix} \quad \text{with} \quad R \in SO(3) \text{ is a rotation in } O(1,3). \tag{1.12}$$

• The subgroup of "special Lorentz-transformations" is defined by demanding:

$$\det(\Lambda) = 1 \quad ; \quad \Lambda_0^0 > 0 \tag{1.13}$$

It is also the "identity component" $SO^+(1,3) \subset O(1,3)$.

• For $\mathbb{R}^{1,1}$, Λ is obviously just a 2 \times 2 Matrix. Hence we can be very explicit:

$$\Lambda = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \in SO(1, 1)$$
 (1.14)

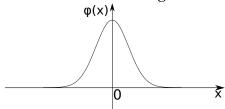
is the general group element,

• This is obviously a boost with $\beta = \frac{v}{c} = v$ and $\cosh \alpha = \frac{1}{\sqrt{1-\beta^2}}$; $\sinh \alpha = \frac{\beta}{\sqrt{1-\beta^2}}$.

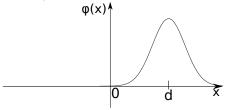
1.3 Symmetries acting on fields

- Consider some scalar field configuration, including its classical evolution in time. Mathematically, this is given by a function
 φ: ℝ⁴ → ℝ ; x → φ(x)
- For simplicity, let us first replace \mathbb{R}^4 by \mathbb{R} (i.e. 1D world, without time evolution).

• Consider a field configuration "localized" near zero:



• Perform a translation by *d* ("active point of view", not a coordinate transformation!)



• Clearly we need to demand that for

$$x \xrightarrow{d} x' = x + d$$
$$\varphi \xrightarrow{d} \varphi'$$

we find

$$\varphi'(x') = \varphi(x)$$
$$\varphi'(x+d) = \varphi(x)$$
$$\varphi'(x) = \varphi(x-d)$$

This makes sense: if φ had its maximum at x = 0, then φ' has its maximum at x = d.

Thus, φ' is defined by applying the inverse transformation to the argument.

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- The "real thing": $\mathbb{R}^{1,3}$ Poincaré transformations.
- For $\varphi \xrightarrow{(\Lambda,d)} \varphi'$, we have:

$$\varphi'(x') = \varphi(x)$$

$$\varphi'(\Lambda x + d) = \varphi(x) , \quad x = \Lambda^{-1} y$$

$$\varphi'(y + d) = \varphi(\Lambda^{-1} y) , \quad y = x - d$$

$$\Rightarrow \varphi'(x) = \varphi(\Lambda^{-1} (x - d)) \quad (\text{different } x)$$

$$(1.16)$$

• Let us look at the transformations of derivatives of fields:

$$\{\partial_{\mu}\varphi\} \equiv \left\{\frac{\partial}{\partial x^{\mu}}\varphi\right\} \equiv \left\{\frac{\partial}{\partial x^{0}}\varphi(x^{0},...,x^{3}),...,...\right\}$$
 (1.17)

$$\partial_{\mu}\varphi' = \frac{\partial}{\partial x^{\mu}}\varphi(y(x))$$

$$= \frac{\partial y^{\nu}}{\partial x^{\mu}}\frac{\partial}{\partial y^{\nu}}\varphi(y) = \frac{\partial}{\partial x^{\mu}}\left((\Lambda^{-1})^{\nu}_{\rho} x^{\rho}\right)\partial_{\nu}(\varphi(y))$$

$$= (\Lambda^{-1})^{\nu}_{\mu}(\partial_{\nu}\varphi)(y)$$
(1.18)

where $y = \Lambda^{-1}x$ and for simplicity d = 0.

- Simple mathematical excursion: (*Dual vector spaces ... inverse metric*) Let x^{μ} , $\mu = 0, ..., 3$ be an element of $\mathbb{R}^{1,3} = V$. Let the elements of V^* be given by y_{μ} , $\mu = 0, ..., 3$ such that $xy := x^{\mu}y_{\mu}$. The metric η provides a natural map $V \to V^* : x^{\mu} \mapsto \eta_{\mu\nu}x^{\nu}$. This map and its inverse are often referred to as "lowering/raising indices". For raising the inverse metric $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is used. (Note: $\eta^{\mu\nu}\eta_{\nu\rho} = \delta^{\mu}_{\ \rho} = \eta^{\mu}_{\ \rho}$)
- We already know that $\eta_{\mu\nu}\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma}=\eta_{\rho\sigma}$

$$\Rightarrow \Lambda^{T} \eta \Lambda = \eta$$

$$\Rightarrow \eta^{-1} \Lambda^{T} \eta = \Lambda^{-1}$$

$$\Rightarrow (\Lambda^{-1})^{\mu}_{\nu} = \eta^{\mu\rho} (\Lambda^{T})^{\sigma}_{\rho} \eta_{\sigma\nu} = \eta^{\mu\rho} \Lambda^{\sigma}_{\rho} \eta_{\sigma\nu} \equiv \Lambda^{\mu}_{\nu}$$
(1.19)

 Λ_{ν}^{μ} : "A with lowered/raised first/second index"

• Thus, for derivatives of fields we get:

$$\partial_{\mu}\varphi'(x) = \Lambda_{\mu}^{\nu}\partial_{\nu}\varphi(\Lambda^{-1}x) \tag{1.20}$$

The derivative transforms as an element of V^* , namely like $x_{\mu} \longrightarrow \Lambda_{\mu}^{\nu} x_{\nu}$ (cf. $x^{\mu} \longrightarrow \Lambda_{\nu}^{\mu} x^{\nu}$ for V).

2. Free Scalar Field

2.1 Classical theory - Lagrangian formulation

• You should already know from Electrodynamics:

$$S = \frac{1}{2} \int d^4x \ F_{\mu\nu} F^{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$$
 (2.1)

- The quantization of this theory is intricate due to A_{μ} having 4 components and the gauge invariance $A_{\mu} \rightarrow A_{\mu} + \partial_{\mu} \chi$.
- Hence, look first at toy model: $A_{\mu}(x) \rightarrow \varphi(x)$ (This is also relevant for Higgs & pion.)
- We formulate the theory in analogy to mechanics:

(1-dimensional) mechanics: Scalar FT:
$$x: t \mapsto x(t) \qquad \varphi: x \mapsto \varphi(x) \quad , \quad x \in \mathbb{R}^4$$
 (2.2)
$$S = S[x] = \int \mathrm{d}t \ L(x,\dot{x}) \qquad S = S[\varphi] = \int \mathrm{d}t \ L[\varphi(t,\vec{x}),\dot{\varphi}(t,\vec{x})]$$

We assume that *L* is local in \vec{x} , i.e.

$$L = \int d^3x \ \mathcal{L}(\varphi(x), \dot{\varphi}(x), \vec{\nabla}\varphi(x), \vec{\nabla}\dot{\varphi}(x), \dots), \tag{2.3}$$

with only finitely many higher derivatives (x meaning (t, \vec{x})).

• One could say we generalize the locality of S[x] of mechanics in t to locality in $x \in \mathbb{R}^{1,3}$.

$$\rightarrow S = \int d^4x \ \mathcal{L}(\varphi, \dot{\varphi}, \vec{\nabla}\varphi, \vec{\nabla}\dot{\varphi}, ...)$$
 (2.4)

• Next we define $V(\varphi) \equiv -\mathcal{L}(\varphi = \text{const.})$ and assume that V has a minimum at φ_0 .

Without loss of generality let $\varphi_0 = 0 \& V(\varphi_0) = 0$ such that:

$$V(\varphi) = \frac{1}{2}m^{2}\varphi^{2} + \dots$$
 (2.5)

- Let us focus on small excitations of vacuum/ground state: let φ be small (ignore higher terms in Taylor expansion).
- However, we need to allow $\dot{\varphi}$ to appear. To have a chance of Poincaré invariance, this means that the whole vector $\partial_u \varphi$ must appear.

• Fact: $(\partial_{\mu}\varphi)(\partial_{\nu}\varphi)\eta^{\mu\nu}$ is the lowest-order (in powers of φ) invariant term.

$$\Rightarrow S = \int d^4x \ \mathcal{L} = \int d^4x \ \left(\frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2}{2}\varphi^2\right)$$
 (2.6)

is the unique $\mathcal{O}(\varphi^2)$ Poincaré-invariant action. (Also a useful approximation to many interesting systems.)

• To gain some intuition, separate time and space $(\varphi(x) = \varphi(t, \vec{x}))$ and discretize the latter:

$$\int d^3x \longrightarrow \sum_{\vec{x}} \tag{2.7}$$

where $\vec{x} \in (3$ -dimensional lattice with spacing $\Delta)$. The Lagrangian

$$L = \int d^3x \left(\frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} (\nabla \varphi)^2 - \frac{m^2}{2} \varphi^2 \right)$$
 (2.8)

then reads:

$$L = \underbrace{\sum_{\vec{x}} \left\{ \frac{1}{2} \dot{\varphi}(t, \vec{x})^{2} - \underbrace{\frac{1}{2} \sum_{i=1}^{3} \left(\frac{\varphi(t, \vec{x} + \hat{e}_{i} \Delta) - \varphi(t, \vec{x})}{\Delta} \right)^{2} - \frac{m^{2}}{2} \varphi^{2} \right\}}_{V_{\text{class. mech.}}}$$
(2.9)

⇒ We have a "mechanical" system with infinitely many degrees of freedom. The free dynamics is simply that of a set of coupled harmonic oscillators. It will be easy to decouple them - see below.

• Equation of motion:

$$0 \stackrel{!}{=} \delta S = \delta \int d^4 x \left(\frac{1}{2} (\partial \varphi)^2 - \frac{m^2}{2} \varphi^2 \right)$$

$$= \int d^4 x \left((\partial_\mu \varphi) \eta^{\mu\nu} (\partial_\nu \delta \varphi) - m^2 \varphi \ \delta \varphi \right)$$

$$= \int d^4 x \left(-(\partial_\nu \partial_\mu \varphi) \eta^{\mu\nu} - m^2 \varphi \right) \delta \varphi$$
(2.10)

$$\Rightarrow (\partial^2 + m^2)\varphi = 0$$
 Klein-Gordon-equation (2.11)

where: $\partial^2 \equiv \partial_u \partial^{\mu}$.

• The solutions are plane waves, e.g.

$$\varphi(x) = \varphi_0 \sin(kx)$$
 with $k^2 - m^2 = 0$ (2.12)

Here $k=(k^0,\vec{k})$ corresponds to the 4-momentum of the particle/particles. Choosing $k=(k^0,\vec{0})$ corresponds to particles at rest. $k^0=m$ will turn out to be the particle mass.

2.2 Classical theory - Hamiltonian formulation

- Classical mechanics: $L(q_i, \dot{q}_i) \longrightarrow H(q_i, \pi_i) = \sum_i \pi_i \dot{q}_i L$ with $\pi_i = \frac{\partial L(q_i, \dot{q}_i)}{\partial \dot{q}_i}$
- Field Theory (FT) on lattice:

$$L(\varphi, \dot{\varphi}) = \frac{1}{2} \sum_{\vec{x}} \dot{\varphi}(\vec{x})^2 + \dots$$
 (2.13)

where \vec{x} plays the role of index i. For the conjugate momentum and the Hamilton function we obtain:

$$\pi(\vec{x}) = \frac{\partial L}{\partial \dot{\varphi}(\vec{x})} = \dot{\varphi}(\vec{x}) \text{ from: } \frac{\partial}{\partial \dot{\varphi}(\vec{x})} \left(\frac{1}{2} \sum_{\vec{y}} \dot{\varphi} \left(\vec{y} \right)^2 \right) = \dot{\varphi} \left(\vec{x} \right)$$

$$\Rightarrow H = \sum_{\vec{x}} \pi \left(\vec{x} \right) \dot{\varphi} \left(\vec{x} \right) - L$$

$$= \sum_{\vec{x}} \left(\pi^2 \left(\vec{x} \right) - \frac{1}{2} \pi^2 \left(\vec{x} \right) + \frac{1}{2} \sum_{i=1}^3 \left(\frac{\varphi \left(t, \vec{x} + \hat{e}_i \Delta \right) - \varphi \left(t, \vec{x} \right)}{\Delta} \right)^2 + \frac{m^2}{2} \varphi^2 \right)$$

$$= \sum_{\vec{x}} \frac{1}{2} \pi^2 \left(\vec{x} \right) + \frac{1}{2} \sum_{i=1}^3 \left(\frac{\varphi \left(t, \vec{x} + \hat{e}_i \Delta \right) - \varphi \left(t, \vec{x} \right)}{\Delta} \right)^2 + \frac{m^2}{2} \varphi^2$$

$$(2.14)$$

• Continuum Field Theory: We need to generalize $\frac{\partial L}{\partial q_i}$ to the case of continuous index *i*. Thus we need functional derivatives.

Mathematical interlude: let $F: f \mapsto \mathbb{R}$ be functional. The functional derivative $\frac{\delta F}{\delta f(x)}$ is defined by:

$$F[f + \epsilon] - F[f] = \int dx \, \frac{\delta F[f]}{\delta f(x)} \epsilon(x) + O(\epsilon^2)$$
 (2.15)

We also use the natural generalization

$$\sum_{i} \pi_{i} \, \dot{q}_{i} \longrightarrow \int d^{3}x \, \pi(\vec{x}) \, \dot{q}(\vec{x}) \tag{2.16}$$

Lagrangian/Hamiltonian-transition for continuous system

$$\pi(\vec{x}) = \frac{\delta}{\delta \dot{\varphi}(\vec{x})} L \left[\varphi, \dot{\varphi} \right]$$

$$H \left[\varphi, \pi \right] = \int d^3 x \ \pi \, \dot{\varphi} - L$$
(2.17)

• In full generality, we have: If F is given as: $F = \int dx A(f(x))$, then

$$\frac{\delta F}{\delta f(x)} = \frac{\partial A}{\partial f}(x) \tag{2.18}$$

• Hence: $\pi(\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}(\vec{x}) = \dot{\phi}(\vec{x})$

$$H\left[\varphi,\pi\right] = \frac{1}{2} \int d^3x \underbrace{\left(\pi^2 + \left(\vec{\nabla}\varphi\right)^2 + m^2\varphi^2\right)}_{\mathcal{H}} = \int d^3x \,\mathcal{H} \tag{2.19}$$

with \mathcal{H} the Hamiltonian density

2.3 Quantization (canonical)

- Usual: $[q_i, \pi_j] = i\delta_{ij}; \quad [q_i, q_j] = [\pi_i, \pi_j] = 0$
- Here: $[\varphi(\vec{x}), \pi(\vec{y})] = i \delta^3(\vec{x} \vec{y}); \quad [\varphi(\vec{x}), \varphi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0$
- With *H* given as in 2.19, the similarity to a set of harmonic oscillators is obvious
- They are coupled due to the " $\vec{\nabla}$ "-term and thus can be easily decoupled by Fourier transformation:

$$\varphi(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p}\vec{x}} \tilde{\varphi}(\vec{p})$$

$$\tilde{\varphi}(\vec{p}) = \int d^3 x \ e^{-i\vec{p}\vec{x}} \varphi(\vec{x})$$
(2.20)

(analogously for $\pi(\vec{x}) \longleftrightarrow \tilde{\pi}(\vec{p})$)

• The commutation relations for $\tilde{\varphi}$, $\tilde{\pi}$ read:

$$\begin{split} \left[\tilde{\varphi}(\vec{p}), \tilde{\pi}(\vec{q})\right] &= \int \mathrm{d}^3 x \ \mathrm{d}^3 y \, e^{-i\vec{p}\vec{x}} e^{-i\vec{q}\vec{y}} \left[\varphi(\vec{x}), \pi(\vec{y})\right] \\ &= i \int \mathrm{d}^3 x \ e^{-i(\vec{p}+\vec{q})\vec{x}} = i(2\pi)^3 \delta^3(\vec{p}+\vec{q}) \\ \left[\tilde{\varphi}(\vec{p}), \tilde{\varphi}(\vec{q})\right] &= \left[\tilde{\pi}(\vec{p}), \tilde{\pi}(\vec{q})\right] = 0 \end{split}$$

• Now, we express H through $\tilde{\varphi}, \tilde{\pi}$. We only focus on the "most interesting" term: $(\vec{\nabla}\varphi)^2$

$$\int d^3x \, (\vec{\nabla}\varphi)^2 = \int d^3x \int \frac{d^3p}{(2\pi)^3} (i\vec{p}) e^{i\vec{p}\vec{x}} \tilde{\phi}(\vec{p}) \int \frac{d^3q}{(2\pi)^3} (i\vec{q}) e^{i\vec{q}\vec{x}} \tilde{\phi}(\vec{q})$$

x integration $\Rightarrow (2\pi)^3 \delta^3(\vec{p} + \vec{q})$

$$\Rightarrow \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \vec{p}^2 \tilde{\varphi}(\vec{p}) \tilde{\varphi} (-\vec{p}) \tag{2.21}$$

• Note that $\overline{\varphi(\vec{x})} = \varphi(\vec{x})$ implies $\overline{\tilde{\varphi}(\vec{p})} = \tilde{\varphi}(-\vec{p})$ (this will also be true for the operators $\varphi(\vec{x})^{\dagger} = \varphi(\vec{x})$ implies $\tilde{\varphi}(\vec{p})^{\dagger} = \tilde{\varphi}(-\vec{p})$)

¹Reminder: We still have $\hbar = 1 = c$

• Hence:

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2} \left(|\tilde{\pi}|^2 + (\vec{p}^2 + m^2) |\tilde{\varphi}|^2 \right)$$
 (2.22)

where $|\tilde{\pi}|^2$ can be understood either literally or as $\tilde{\pi}\tilde{\pi}^{\dagger}$, depending on whether we are before or after quantization.

• To further perfect the similarity to harmonic oscillators, let

$$\omega_{\vec{p}} \equiv \sqrt{\vec{p}^2 + m^2} \tag{2.23}$$

$$\Rightarrow H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2} \left(|\tilde{\pi}|^2 + \omega_{\vec{p}}^2 |\tilde{\varphi}|^2 \right) \tag{2.24}$$

• Reminder from QM:

$$H = \frac{1}{2} \left(\pi^2 + \omega^2 q^2 \right) \qquad [q, \pi] = i$$

$$a = \frac{1}{2} \left(\sqrt{2\omega} q + i \sqrt{\frac{2}{\omega}} \pi \right) \qquad a^{\dagger} = \frac{1}{2} \left(\sqrt{2\omega} q - i \sqrt{\frac{2}{\omega}} \pi \right)$$

$$\Rightarrow H = \omega \left(a^{\dagger} a + \frac{1}{2} \right) \qquad \left[a, a^{\dagger} \right] = 1$$

• Motivated by this, let

$$a_{\vec{p}} = \frac{1}{2} \left(\sqrt{2\omega_{\vec{p}}} \, \tilde{\varphi}(\vec{p}) + i \sqrt{\frac{2}{\omega_{\vec{p}}}} \tilde{\pi}(\vec{p}) \right) \tag{2.25}$$

$$a_{\vec{p}}^{\dagger} = \frac{1}{2} \left(\sqrt{2\omega_{\vec{p}}} \, \tilde{\varphi}(-\vec{p}) - i \sqrt{\frac{2}{\omega_{\vec{p}}}} \tilde{\pi}(-\vec{p}) \right) \tag{2.26}$$

Comment: It is not yet totally clear that this will work since our analogy is not perfect: Unlike p,q our $\tilde{\varphi}$, $\tilde{\pi}$ are not real and they are not (quite) conjugate variables: $\delta^3(\vec{p}+\vec{q})=\delta^3(\vec{p}-(-\vec{q}))$, i.e. $\tilde{\varphi}(\vec{p})$ is conjugate to $\tilde{\pi}(-\vec{p})$. We could keep "massaging" our classical system into perfect agreement with a set of oscillators and only then introduce a,a^{\dagger} , but we would not gain much new information.

• It is easy to derive:

$$\begin{bmatrix} a_{\vec{p}}, a_{\vec{q}}^{\dagger} \end{bmatrix} = (2\pi)^3 \delta^3(\vec{p} - \vec{q})$$

$$\begin{bmatrix} a_{\vec{p}}, a_{\vec{q}} \end{bmatrix} = \begin{bmatrix} a_{\vec{p}}^{\dagger}, a_{\vec{q}}^{\dagger} \end{bmatrix} = 0$$
(2.27)

Furthermore

$$\tilde{\varphi}(\vec{p}) = \frac{1}{\sqrt{2\omega_{\vec{p}}}} (a_{\vec{p}} + a_{-\vec{p}}^{\dagger})$$

$$\tilde{\pi}(\vec{p}) = -i\sqrt{\frac{\omega_{\vec{p}}}{2}} (a_{\vec{p}} - a_{-\vec{p}}^{\dagger})$$
(2.28)

and hence:

$$H = \int \frac{\mathrm{d}^{3} p}{(2\pi)^{3}} \frac{1}{2} \left(\frac{\omega_{\vec{p}}}{2} (a_{\vec{p}} - a_{-\vec{p}}^{\dagger}) (a_{\vec{p}}^{\dagger} - a_{-\vec{p}}) + \frac{\omega_{\vec{p}}^{2}}{2\omega_{\vec{p}}} (a_{\vec{p}} + a_{-\vec{p}}^{\dagger}) (a_{\vec{p}}^{\dagger} + a_{-\vec{p}}) \right)$$
(2.29)

cross-terms like $a_{\vec{v}} \, a_{-\vec{v}}$ cancel, can use $\vec{p} \to -\vec{p}$ freely

$$\Rightarrow H = \int \frac{\mathrm{d}^{3} p}{(2\pi)^{3}} \frac{\omega_{\vec{p}}}{2} \left(a_{\vec{p}}^{\dagger} a_{\vec{p}} + a_{\vec{p}} a_{\vec{p}}^{\dagger} \right)$$

$$= \int \frac{\mathrm{d}^{3} p}{(2\pi)^{3}} \omega_{\vec{p}} \left(a_{\vec{p}}^{\dagger} a_{\vec{p}} + \frac{1}{2} \underline{\left[a_{\vec{p}}, a_{\vec{p}}^{\dagger} \right]} \right)$$

$$(2\pi)^{3} \delta^{3}(\vec{0})'' = \int \mathrm{d}^{3} x x e^{i\vec{0}\vec{x}''} = \text{``Vol}(\mathbb{R}^{3}) = V$$

$$= \int \frac{\mathrm{d}^{3} p}{(2\pi)^{3}} \omega_{\vec{p}} a_{\vec{p}}^{\dagger} a_{\vec{p}} + V \int \frac{\mathrm{d}^{3} p}{(2\pi)^{3}} \frac{1}{2} \omega_{\vec{p}}$$

$$(2.30)$$

V can be derived properly by using T^3 instead of \mathbb{R}^3 , where $\delta^3(\vec{p}-\vec{q})$ becomes $\delta_{p_iq_i}$. The last integral is truly divergent unless a "UV-cutoff" is introduced. This is due to contributions of zero-point energies of harmonic oscillators with arbitrarily high frequencies to the vacuum energy density.

Vacuum energy:

- Irrelevant for QFT in \mathbb{R}^3 since it can be absorbed in overall constant shift of H.
- Relevant if QFT is coupled to gravity since it then curves space-time (→ "Cosmological constant problem").
- In non-trivial geometries (e.g. QED with conducting plates) the energies of the low-lying modes can be manipulated by moving the plates. This leads to a finite effect (force on the plates) independently of the still present divergence of $|\vec{p}| \to \infty$ (\to Casimir energy/effect).
- We have:

$$\varphi(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} e^{i\vec{p}\vec{x}} (a_{\vec{p}} + a_{-\vec{p}}^{\dagger})$$
 (2.31)

$$\pi(\vec{x}) = -i \int \frac{d^3 p}{(2\pi)^3} \sqrt{\frac{\omega_{\vec{p}}}{2}} e^{i\vec{p}\vec{x}} (a_{\vec{p}} - a_{-\vec{p}}^{\dagger})$$
 (2.32)

Also : $[a_{\vec{p}}, a_{\vec{q}}^{\dagger}] = (2\pi)^3 \delta^3(\vec{p} - \vec{q})$

$$H = \int \frac{d^3 p}{(2\pi)^3} \omega_{\vec{p}} a_{\vec{p}}^{\dagger} a_{\vec{p}}$$
 (2.33)

At the moment, this is just an (operator) algebra with one distinguished operator *H*. Physics starts if we also provide a Hilbert-space representation of this algebra. To construct this representation:

• Postulate a vacuum state $|0\rangle$ such that

$$a_{\vec{p}} |0\rangle = 0 \qquad \forall \vec{p} \tag{2.34}$$

• One-particle states: $a_{\vec{p}}^{\dagger} |0\rangle \ \forall \vec{p}$. It is easy to calculate the energy

$$Ha_{\vec{p}}^{\dagger}|0\rangle = \int \frac{d^{3}k}{(2\pi)^{3}} \omega_{\vec{k}} a_{\vec{k}}^{\dagger} a_{\vec{k}} a_{\vec{p}}^{\dagger} |0\rangle$$

$$= \int \frac{d^{3}k}{(2\pi)^{3}} \omega_{\vec{k}} a_{\vec{k}}^{\dagger} (2\pi)^{3} \delta^{(3)} (\vec{k} - \vec{p}) |0\rangle$$

$$= \omega_{\vec{p}} a_{\vec{p}}^{\dagger} |0\rangle$$
(2.35)

giving the important result

$$Ha_{\vec{p}}^{\dagger}|0\rangle = \omega_{\vec{p}}a_{\vec{p}}^{\dagger}|0\rangle. \tag{2.36}$$

Indeed: $\omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$ is the energy (in SRT) of a particle with mass m and momentum \vec{p} .

• Two-particle states: $a_{\vec{p}}^{\dagger}a_{\vec{q}}^{\dagger}\ket{0}$ $\forall \vec{p}, \vec{q}$. It is easy to check

$$Ha_{\vec{p}}^{\dagger}a_{\vec{q}}^{\dagger}|0\rangle = (\omega_{\vec{p}} + \omega_{\vec{q}})a_{\vec{p}}^{\dagger}a_{\vec{q}}^{\dagger}|0\rangle, \qquad (2.37)$$

by commuting the annihilation operator $a_{\vec{k}}$ to the right until it hits the vacuum. Each time it passes one of the creation operators, we pick up a δ^3 distribution, and hence a $\omega_{\vec{k}}$. The term in brackets is the energy of two non-interacting particles.

- This extends to any number of particles and, in total, is called *Fock space*.
- Choose normalisation of vacuum:

$$||0\rangle|^2 = \langle 0|0\rangle = 1 \tag{2.38}$$

• One finds:

$$\left(a_{\vec{p}}^{\dagger}\left|0\right\rangle\right)\cdot\left(a_{\vec{q}}^{\dagger}\left|0\right\rangle\right) = \left\langle 0\right|a_{\vec{p}}a_{\vec{q}}^{\dagger}\left|0\right\rangle = (2\pi)^{3}\delta^{(3)}(\vec{p}-\vec{q}),\tag{2.39}$$

e.g. states with different numbers of particles are orthogonal.

• Convenient notation and normalization:

$$|\vec{p}\rangle = \sqrt{2\omega_{\vec{p}}} \, a_{\vec{p}}^{\dagger} \, |0\rangle$$

$$|\vec{p}\vec{q}\rangle = \sqrt{2\omega_{\vec{p}}} \, \sqrt{2\omega_{\vec{q}}} \, a_{\vec{p}}^{\dagger} \, a_{\vec{q}}^{\dagger} \, |0\rangle$$
(2.40)

. . .

This implies:

$$\langle \vec{p} | \vec{q} \rangle = 2\omega_{\vec{p}} (2\pi)^3 \delta^{(3)} (\vec{p} - \vec{q}) \tag{2.41}$$

Note that many books also absorb the $\omega_{\vec{p}}$ into the definition of the $a_{\vec{p}}$.

2.4 Complex scalar

We have seen that

$$\mathcal{L} = \eta^{\mu\nu}(\partial_{\mu}\phi)(\partial_{\nu}\overline{\phi}) - m^{2}\phi\overline{\phi} = |\partial\phi|^{2} - m^{2}|\phi|^{2}, \tag{2.42}$$

where the second notation is slightly sloppy but very common.

- Note the different normalization conventions compared to the real scalar. With $\phi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$, this goes over in precisely twice the Lagrangian of the type discussed before.
- Nevertheless, it is useful to quantize this without giving up the complex notation:

$$\delta S = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} = 0. \tag{2.43}$$

• Treating ϕ , $\overline{\phi}$ as independent variables, we get

$$-m^2\overline{\phi} - \Box\overline{\phi} = 0$$

$$-m^2\phi - \Box\phi = 0$$
(2.44)

where $\Box = \partial_{\mu} \partial^{\mu}$.

• The conjugate momenta take the form

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \overline{\dot{\phi}} \qquad \overline{\pi} = \frac{\partial \mathcal{L}}{\partial \overline{\dot{\phi}}} = \dot{\phi}.$$
(2.45)

We thus get

$$\mathcal{H} = \pi \dot{\phi} + \overline{\pi} \overline{\dot{\phi}} - \mathcal{L} = |\pi|^2 + \left| \vec{\nabla} \phi \right|^2 + m^2 |\phi|^2. \tag{2.46}$$

• Quantization:

$$[\phi(\vec{x}), \pi(\vec{y})] = \left[\phi^{\dagger}(\vec{x}), \pi^{\dagger}(\vec{q})\right] = i\delta^{(3)}(\vec{x} - \vec{y}). \tag{2.47}$$

Before we were successful using the ansatz

$$\varphi(\vec{x}) = \int \frac{d^{3}p}{(2\pi)^{3}} \frac{1}{\sqrt{2\omega_{\vec{p}}}} e^{i\vec{p}\vec{x}} (a_{\vec{p}} + a_{-\vec{p}}^{\dagger})
= \int \frac{d^{3}p}{(2\pi)^{3}} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}}^{\dagger} e^{-i\vec{p}\vec{x}} + a_{\vec{p}} e^{i\vec{p}\vec{x}} \right).$$
(2.48)

In this ansatz the actual reality is encoded in $(a_{\vec{p}}^{\dagger}) = (a_{\vec{p}})^{\dagger}$.

• We now generalize this and make the new ansatz

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{v}}}} \left(a_{\vec{p}}^{\dagger} e^{-i\vec{p}\vec{x}} + b_{\vec{p}} e^{i\vec{p}\vec{x}} \right). \tag{2.49}$$

With this ansatz one finds commutation relations for ϕ , π , ϕ^* , π^* , that are the commutation relations of two sets of harmonic oscillators $a_{\vec{p}}^{\dagger}$, $a_{\vec{p}}$ and $b_{\vec{p}}^{\dagger}$, $b_{\vec{p}}$ (see problem sheet).

$$H = \int \frac{d^3 p}{(2\pi)^3} \omega_{\vec{p}} \left(a_{\vec{p}}^{\dagger} a_{\vec{p}} + b_{\vec{p}}^{\dagger} b_{\vec{p}} \right)$$
 (2.50)

Fock space: Multi-particle states for *two* types of particles, with symmetry $\phi \rightarrow e^{i\alpha}\phi$ (we will see that this gives us a conserved charge). These will turn out to be particles and anti-particles later.

3. Noether Theorem

3.1 Formulation and Derivation in field theory

With every continous symmetry of the action comes a conserved current density (and a conserved charge). Note that this is very similar to the Noether theorem of mechanics. The crucial novelty is the current. *Derivation:*

By assumption we have a continous symmetry and thus can define an infinitesimal transformation as

$$\varphi(x) \to \varphi'(x) = \varphi(x) + \varepsilon \chi(x)$$
 (3.1)

• The change of φ ; $\delta_{\varepsilon}\varphi = \varphi' - \varphi$ induces a change of \mathcal{L} :

$$\delta_{\varepsilon} \mathcal{L} = \mathcal{L}' - \mathcal{L} = \mathcal{L}(\varphi', \partial \varphi') - \mathcal{L}(\varphi, \partial \varphi). \tag{3.2}$$

• *Symmetry* means that \mathcal{L} changes only by a total derivative:

$$\delta_{\varepsilon} \mathcal{L} = \varepsilon \partial_{u} F^{\mu}(\varphi, \partial \varphi, \partial \partial \varphi) \tag{3.3}$$

where *F* is some appropriately chosen vector field.

Indeed: Assume φ satisfies the equation of motion and check that φ' does so too: We need to check that $\delta S' = 0$ for any variation $\delta \varphi$ in a bounded region. But

$$\delta S' = \delta(\delta_{\varepsilon} S) + \delta S = \delta(\delta_{\varepsilon} S) = \int d^4 x \, \delta(\delta_{\varepsilon} \mathcal{L})$$

$$= \int d^4 x \, \varepsilon \partial_{\mu} \delta F^{\mu}(x) = 0$$
(3.4)

where the last equality is by Gauss law since $\delta \varphi$ and hence δF vanish outside a bounded region.

- Note: We did not demand that $\delta_{\varepsilon}S = 0$ or that $\delta_{\varepsilon}\varphi$ should vanish outside a bounded region.
- Simple calculation now leads to the statement of the theorem:

$$\varepsilon \partial_{\mu} F^{\mu} = \delta_{\varepsilon} \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} \delta_{\varepsilon} \varphi + \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \varphi)} \delta_{\varepsilon} \partial_{\mu} \varphi
= \frac{\partial \mathcal{L}}{\partial \varphi} \delta_{\varepsilon} \varphi + \partial_{\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \varphi)} \delta_{\varepsilon} \varphi \right) - \left(\partial_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \varphi)} \right) \delta_{\varepsilon} \varphi$$
(3.5)

Using the equation of motion

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \varphi)} = 0 \tag{3.6}$$

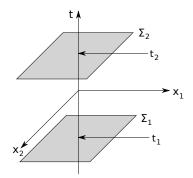


Figure 1: Charge conservation

and $\delta_{\varepsilon} \varphi = \varepsilon \chi$ gives

$$\varepsilon \partial_{\mu} F^{\mu} = \partial_{\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \varphi)} \varepsilon \chi \right) \Rightarrow j^{\mu} \equiv \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \varphi)} \chi - F^{\mu}$$
 (3.7)

The current j^{μ} is conserved, $\partial_{\mu}j^{\mu} = 0$.

• For many field configurations the following integral can be defined

$$Q(t) \equiv \int d^3x \, j^0(t, \vec{x}). \tag{3.8}$$

We name it the conserved charge since

$$Q(t_2) - Q(t_1) = \int_{\Sigma_2} df_{\mu} j^{\mu} - \int_{\Sigma_1} df_{\mu} j^{\mu} = \int_{\text{Vol}} dV \, \partial_{\mu} j^{\mu} = 0, \quad (3.9)$$

where the first equality holds due to the normal vectors being perpendicular, and the second follows from Gauß' law.

- Problems:
 - Derive $\dot{Q} = 0$ directly, i.e. without introducing a finite interval $\delta t = t_2 t_1$.
 - Apply our derivation to mechanics.

3.2 Energy-Momentum-Conservation

• Concerning energy-momentum-conservation, the relevant symmetry is:

$$x^{\mu} \to x'^{\mu} = x^{\mu} - \epsilon^{\mu} \tag{3.10}$$

i.e. a translation by " $-\epsilon$ ". Actually, there are four symmetries. We therefore use a "4-vector of ϵ 's".

• From this the transformation behavior of the field and the Lagrangian follows:

$$\varphi'(x) = \varphi(x + \epsilon) \Rightarrow \delta_{\epsilon} \varphi = \varphi(x + \epsilon) - \varphi(x) \approx \epsilon^{\nu} \partial_{\nu} \varphi$$

$$\mathcal{L}'(x) = \mathcal{L}(x + \epsilon) \Rightarrow \delta_{\epsilon} \mathcal{L} = \mathcal{L}(x + \epsilon) - \mathcal{L}(x) \approx \epsilon^{\mu} \partial_{\mu} \mathcal{L}$$

$$= \epsilon^{\nu} \partial_{\mu} (\delta^{\mu}_{\nu} \mathcal{L}) = \underbrace{\epsilon^{\nu} \partial_{\mu} (F^{\mu}_{\nu})}_{(*)}$$
(3.11)

where (*) is a linear superposition of four contributions of the type $\epsilon \partial_{\mu} F^{\mu}$, labelled by $\nu = 0, 1, 2, 3 \rightarrow \sum_{\nu} \epsilon^{\nu} \partial_{\mu} (F^{\mu}_{\nu})$

We defined:

$$F^{\mu}_{\ \nu} \equiv \delta^{\mu}_{\ \nu} \mathcal{L} \tag{3.12}$$

and furthermore, we identify:

$$\chi_{\nu} = \partial_{\nu} \varphi \tag{3.13}$$

• Then we apply the general formula, getting four conserved currents, labelled by ν :

$$j^{\mu}_{\ \nu} = \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \varphi)} \chi_{\nu} - F^{\mu}_{\ \nu} = \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \varphi)} \partial_{\nu} \varphi - \delta^{\mu}_{\ \nu} \mathcal{L}$$
 (3.14)

• By this we have found the *energy-momentum-tensor*:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi)} \partial^{\nu}\varphi - \eta^{\mu\nu}\mathcal{L}$$
 (3.15)

And by construction it is conserved:

$$\partial_{u}T^{\mu\nu} = 0 \tag{3.16}$$

• The name is indeed justified:

$$\int d^3x \, T^{00} = \int d^3x \, \left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \dot{\varphi} - \mathcal{L} \right) = H = P^0 \tag{3.17}$$

where P^0 is the first component of the 4-vector $\{P^{\mu}\}$ of the energy-momentum of the field configuration. More generally:

$$P^{\nu} = \int d^3x \, T^{0\nu} \tag{3.18}$$

Note: P is a 4-vector in spite of its apparently non-covariant definition. In particular P^{ν} does not change if the space-like hyperplane used in its definition is rotated.

$$P_{\nu} = \int d^3x \, T^{0\nu} = \int_{\Sigma} df_{\mu} \, T^{\mu\nu} = \int_{\Sigma'} df_{\mu} \, T^{\mu\nu} \tag{3.19}$$

• The spatial components of P read:

$$P^{i} = \int d^{3}x \, T^{0i} = \int d^{3}x \, \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \, \partial^{i} \varphi = -\int d^{3}x \, \pi \, \nabla_{i} \varphi \tag{3.20}$$

• Next, we rewrite π and φ in terms of \hat{a} and \hat{a}^{\dagger} where we use the "^" -sign to emphasize the operator-nature:

$$\hat{P}^{i} = \int \frac{\mathrm{d}^{3} q}{(2\pi)^{3}} q^{i} \hat{a}_{\vec{q}}^{\dagger} \hat{a}_{\vec{q}}$$

$$\mathrm{also} : \hat{P}^{\mu} | p \rangle = p^{\mu} | p \rangle$$
(3.21)

where $\hat{a}_{\vec{q}}^{\dagger} \hat{a}_{\vec{q}}$ is also known as the "particle number operator".

• The same analysis applied to the complex scalar gives:

$$\hat{P}^{\mu} = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \, q^{\mu} (\hat{a}_{\vec{q}}^{\dagger} \hat{a}_{\vec{q}} + \hat{b}_{\vec{q}}^{\dagger} \hat{b}_{\vec{q}}) \quad \text{here} : q^0 = \sqrt{\vec{q}^2 + m^2} = \omega_{\vec{q}}$$
 (3.22)

With this, our particle-interpretation of the Fock-space is fully justified.

- Comments concerning the energy-momentum-tensor:
 - From its definition, our "canonical" $T^{\mu\nu}$ does not necessarily have to be symmetric.
 - It happens to be symmetric for the scalar field

$$T^{\mu\nu} = \partial^{\mu}\varphi \,\partial^{\nu}\varphi - \eta^{\mu\nu}\mathcal{L} \tag{3.23}$$

but this fails already for QED.

- But: $T^{\mu\nu}$ can always be made symmetric by adding an independently conserved current, which also does not modify P^{ν} .
- In General Relativity, one uses the definition

$$T^{\mu\nu}(x) = \frac{-2}{\sqrt{-\det(g_{\rho\sigma})}} \frac{\delta S}{\delta g_{\mu\nu}(x)}.$$
 (3.24)

Here S is formulated using a general metric $g_{\mu\nu}$ instead of the Minkowski metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

- This directly gives the symmetric form of $T^{\mu\nu}$, which in fact is crucial since

$$\mathcal{L} = \dots + h_{\mu\nu} T^{\mu\nu} + \dots \quad \text{(with } g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\text{)}$$
 (3.25)

characterizes the coupling of QFT to gravity.

3.3 U(1)-symmetry and charge of complex scalar field

• Let us consider the following Lagrangian:

$$\mathcal{L} = \left| \partial_{\mu} \phi \right|^2 - m^2 |\phi|^2 \tag{3.26}$$

and the U(1)-symmetry group:

$$\phi \to \phi' = e^{i\epsilon}\phi = \phi + i\epsilon\phi + \dots$$

$$\phi^* \to {\phi'}^* = e^{-i\epsilon}\phi^* = \phi^* - i\epsilon\phi^* + \dots$$
(3.27)

Note: Treat ϕ , ϕ * as independent fields during the calculation. This is justified by the following:

We truly enlarge the field space from two to four dimensions:

$$\mathcal{L} = \eta^{\mu\nu}(\partial_{\mu}\phi)(\partial_{\nu}\psi) + \cdots \quad \phi, \psi \in \mathbb{C}$$
 (3.28)

and project on the real subspace, $\psi = \phi^*$, in the end.

• Our formula is

$$j^{\mu} = \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \varphi)} \chi - F^{\mu} \tag{3.29}$$

in the one-field case.

• It has the obvious multi-field generalization

$$j^{\mu} = \sum_{i} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \varphi^{i})} \chi^{i} - F^{\mu}$$
 (3.30)

in the case of several fields φ^i .

• In our case: $\mathcal{L} = \mathcal{L}' \Rightarrow F^{\mu} = 0$ and $\{\varphi^i\} = \{\phi, \phi^*\}$

$$\Rightarrow j^{\mu} = \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \chi + \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi^{*})} \chi^{*} \quad \text{with } \chi = i\phi, \quad \chi^{*} = -i\phi^{*}$$

$$= (\partial^{\mu} \phi^{*}) i\phi + (\partial^{\mu} \phi)(-i)\phi^{*} = -i(\phi^{*} \overset{\leftrightarrow}{\partial^{\mu}} \phi)$$
(3.31)

where we used the common notation $A \stackrel{\leftrightarrow}{\partial} B = A \partial B - (\partial A)B$.

• The conserved charge is then given by:

$$Q = \int d^3x \, j^0 = -i \int d^3x \, \phi^{\dagger} \stackrel{\leftrightarrow}{\partial_0} \phi = \int \frac{d^3p}{(2\pi)^3} \left(a_{\vec{p}}^{\dagger} a_{\vec{p}} - b_{\vec{p}}^{\dagger} b_{\vec{p}} \right)$$
(3.32)

 \Rightarrow We can think of the states created by a^{\dagger}/b^{\dagger} as particles/antiparticles with same mass but opposite charge.

4. Heisenberg picture, Causality, Covariance

4.1 Heisenberg picture

• So far, we dealt with the Schrödinger picture and the operators $\hat{\varphi}(\vec{x})$, $\hat{\pi}(\vec{x})$. In this picture they have no time dependence and we can think of them as arising from quantization at t = 0.

Therefore, our states $|0\rangle$, $\sqrt{2\omega_{\vec{p}}} \ a_{\vec{p}}^{\dagger} \ |0\rangle = |p\rangle$, etc. have to evolve in time:

$$|p_{t=0}\rangle \equiv |p_0\rangle \tag{4.1}$$

$$|p\rangle = \exp(-iHt)|p_0\rangle \tag{4.2}$$

- This is clearly not natural for a Poincaré-invariant theory since the time dependence sits in the states while the \vec{x} -dependence sits in the observables.
- Thus, let us convert from the

Schrödinger-picture
$$\longrightarrow$$
 O fix and $|\psi_t\rangle = e^{-iHt} |\psi_0\rangle$

to the

Heisenberg-picture
$$\longrightarrow O = O_t$$
 and $|\psi\rangle$ fix

where O denotes an operator.

• Concerning an operator *O*, the transition between the pictures can be done by the following physical requirement:

$$\langle \psi_t | O | \psi_t \rangle = \langle \psi | O_t | \psi \rangle$$
 (4.3)

This implies:

$$O_t = e^{iHt}O e^{-iHt} (4.4)$$

For us, the most interesting operator is:

$$\varphi(x) = \varphi(t, \vec{x}) = e^{iHt} \left(\int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} e^{i\vec{p}\vec{x}} + a_{\vec{p}}^{\dagger} e^{-i\vec{p}\vec{x}} \right) \right) e^{-iHt}$$
 (4.5)

• To simplify, we commute e^{iHt} through a, a^{\dagger} . This gives:

$$H a_{\vec{p}} = a_{\vec{p}}(H - \omega_{\vec{p}}) \tag{4.6}$$

$$H a_{\vec{p}}^{\dagger} = a_{\vec{p}}^{\dagger} (H + \omega_{\vec{p}}) \tag{4.7}$$

Note: This is clear. Consider $|\psi\rangle$ such that $H|\psi\rangle = E|\psi\rangle$:

$$H a_{\vec{v}}^{\dagger} |\psi\rangle = (E + \omega_{\vec{p}}) a_{\vec{v}}^{\dagger} |\psi\rangle = a_{\vec{v}}^{\dagger} (H + \omega_{\vec{p}}) |\psi\rangle$$
 (4.8)

• Since an exponential of an operator is defined as its Taylor series, we have:

$$e^{iHt} \, a_{\vec{v}} = a_{\vec{v}} \, e^{i(H - \omega_{\vec{v}})t} \tag{4.9}$$

$$e^{iHt} a_{\vec{p}}^{\dagger} = a_{\vec{p}}^{\dagger} e^{i(H - \omega_{\vec{p}})t} \tag{4.10}$$

• $p = \{p^0, \vec{p}\} = \{\omega_{\vec{p}}, \vec{p}\}$ yields:

$$p \, x = p^0 x^0 - \vec{p} \cdot \vec{x} \tag{4.11}$$

$$\varphi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \cdot \sqrt{2p^0}} \left(a_{\vec{p}} e^{-ipx} + a_{\vec{p}}^{\dagger} e^{+ipx} \right)$$
 (4.12)

Note: Define $\pi(x) = \pi(t, \vec{x})$ analogously, then you find

$$\pi(x) = \dot{\varphi}(x) \tag{4.13}$$

$$\left(\Box + m^2\right) \varphi(x) = 0. \tag{4.14}$$

4.2 Causality

Measurements of φ at x and y do not interfere if x and y are space-like separated, i.e.

$$[\varphi(x), \varphi(y)] = 0 \text{ for } (x - y)^2 < 0$$
 (4.15)

• *Note:* Equal time commutation relations $[\varphi(\vec{x}), \varphi(\vec{y})] = 0$ (in Schrödinger picture) do not directly imply causality after quantization.

$$\begin{split} \left[\varphi(x), \varphi(y) \right] &= \int \frac{\mathrm{d}^{3} p}{(2\pi)^{3} \sqrt{2p^{0}}} \int \frac{\mathrm{d}^{3} q}{(2\pi)^{3} \sqrt{2q^{0}}} \left\{ \left[a_{\vec{p}}, a_{\vec{q}}^{\dagger} \right] e^{-ipx + iqy} + \left[a_{\vec{p}}^{\dagger}, a_{\vec{q}} \right] e^{+ipx - iqy} \right\} \\ &= \int \frac{\mathrm{d}^{3} p}{(2\pi)^{3} 2p^{0}} e^{-ip(x-y)} - \int \frac{\mathrm{d}^{3} p}{(2\pi)^{3} 2p^{0}} e^{+ip(x-y)} \\ &= \int \frac{\mathrm{d}^{4} p}{(2\pi)^{3}} \delta \left(p^{2} - m^{2} \right) \Big|_{p^{0} > 0} e^{-ip(x-y)} \\ &- \int \frac{\mathrm{d}^{4} p}{(2\pi)^{3}} \delta \left(p^{2} - m^{2} \right) \Big|_{p^{0} > 0} e^{+ip(x-y)} \end{split} \tag{4.16}$$

(Use:
$$\delta(x^2 - a^2) = \frac{1}{2|a|} [\delta(x+a) + \delta(x-a)]$$
)

- This is invariant under $SO^+(1,3) \Leftrightarrow (\Lambda_0^0 > 0)$
- Using $SO^+(1,3)$ we can transform any space-like vector z ($z^2 < 0$) into -z. Hence transform (x - y) into, for example, $(0, \vec{l})$ $\Rightarrow [\varphi(x), \varphi(y)] = 0$

Note: A time-like vector $(l, \vec{0})$ does not vanish.

4.3 Covariance

- Of course, if $x^0 = y^0$, the claim $[\varphi(x), \varphi(y)] = 0$ immediately follows from canonical quantization. It is then tempting to claim the same for generic x y with $(x y)^2 < 0$ on the basis of Lorentz/Poincaré covariance of the theory. However, we have broken this symmetry in its manifest form during quantization and not yet established how it reappears at the quantum level.
- In QM \hat{H} and $\hat{\vec{P}}$ generate translations in t and \vec{x} , respectively. Similarly, in QFT we use the operator

$$e^{i\hat{P}^{\mu}\varepsilon_{\mu}} = e^{i\hat{H}\varepsilon_{0}} e^{-i\hat{\bar{P}}\,\bar{\epsilon}} \tag{4.17}$$

acting on states $|\psi\rangle$.

Note the sign of the exponents.

• In previous lectures, regarding Lorentz-rotations,

$$\Lambda^{\mu}_{\ \nu} = e^{i(\varepsilon^{\rho\sigma} M_{\rho\sigma})^{\mu}_{\ \nu}}
= \delta^{\mu}_{\ \nu} + i \left(\varepsilon^{\rho\sigma} M_{\rho\sigma}\right)^{\mu}_{\ \nu} + \dots$$
(4.18)

we used Noether's theorem to construct a conserved quantity by having employed symmetry, and promoted said quantity to an operator, e.g. \hat{P} . This gives $(\hat{M})_{\rho\sigma}$

$$\hat{T} = e^{i\delta^{\mu}\hat{P}_{\mu}} \qquad \text{translations}$$

$$\hat{\Lambda} = e^{i\epsilon^{\rho\sigma}M_{\rho\sigma}} \qquad \text{rotations}$$
(4.19)

- In QM, after having realized our symmetry group by unitary operators acting on the Hilbert space, we just need to check that they commute with *H* and we are done.
- Here, things are slightly different since, of course, boosts $\hat{\Lambda}$ do not commute with H (they change the energy of states).

• We require invariance of observables in the following sense; Measure the field value at some position $x \in \mathbb{R}^{1,3}$ in a state $|\psi\rangle$

$$\langle \psi | \varphi(x) | \psi \rangle$$
 (4.20)

If we 'rotate' the state by Λ and measure the field in the rotated position $x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu}$, we must get the same result:

$$\langle \psi | \varphi(x) | \psi \rangle = \langle \psi | \hat{\Lambda}^{\dagger} \varphi(\Lambda x) \hat{\Lambda} | \psi \rangle$$
 (4.21)

• Therefore we must demand that operators transform like

$$\hat{\Lambda} \, \varphi(x) \, \hat{\Lambda}^{\dagger} = \varphi(\Lambda x). \tag{4.22}$$

This can be checked explicitly by expressing $\hat{M}_{\rho\sigma}$ in a, a^{\dagger} (cf. problem set)

- Check consistency with group law and compare with the somewhat different relations for classical fields
- A: QF operators

1st transformation

$$\hat{\Lambda}_1 \, \varphi(x) \, \hat{\Lambda}_1^{\dagger} = \varphi(\Lambda_1 x) \tag{4.23}$$

2nd transformation

$$\hat{\Lambda}_{2} \left(\hat{\Lambda}_{1} \varphi(x) \hat{\Lambda}_{1}^{\dagger} \right) \hat{\Lambda}_{2}^{\dagger} = \hat{\Lambda}_{2} \left(\varphi(\Lambda_{1}x) \right) \hat{\Lambda}_{2}^{\dagger}$$

$$= \varphi(\Lambda_{2} \Lambda_{1}x)$$
(4.24)

• B: Classical Fields

1st transformation

$$(\Lambda_1 \varphi)(x) = \varphi(\Lambda_1^{-1} x) \tag{4.25}$$

2nd transformation

$$\Lambda_2 \left(\Lambda_1 \varphi \right) (x) = \left(\Lambda_1 \varphi \right) \left(\Lambda_2^{-1} x \right)$$

$$= \varphi(\Lambda_1^{-1} \Lambda_2^{-1} x) = \varphi((\Lambda_2 \Lambda_1)^{-1} x)$$
(4.26)

- Both cases are consistent, but note the difference in the argument.
- Finally we would like to have covariance of our Fock-space basis

$$\hat{\Lambda} |p\rangle = |p'\rangle \quad , \quad p'^{\mu} = \Lambda^{\mu}_{\nu} p^{\nu} \tag{4.27}$$

Due to the normalization ambiguity (our states are only δ -function-normalized and the prefactor of $\delta^3(\vec{p}-\vec{q})$ is, in principle, arbitrary) we cannot a priori be sure about the prefactor in this relation.

- One can of course work this out explicitly using a, a^{\dagger} .
- We will do a consistency check

$$\langle p'|q'\rangle = \langle p|\hat{\Lambda}^{\dagger}\hat{\Lambda}|q\rangle = \langle p|q\rangle$$
 (4.28)

where $p' = \Lambda p$, $q' = \Lambda q$.

$$2p_0'(2\pi)^3 \,\delta^3(\vec{p'} - \vec{q'}) = 2p_0(2\pi)^3 \,\delta^3(\vec{p} - \vec{q}) \tag{4.29}$$

It is obvious that both sides are non-zero only at the same point. So we just need to check the normalization, which we can do e.g. by integrating with an arbitrary smooth function in \vec{p} . We integrate over momentum space $\int \frac{\mathrm{d}^3 p}{2v^0} = \int \mathrm{d}^4 p \, \delta(p^2 - m^2).$

$$\int \frac{d^3 p}{2p^0} = \int d^4 p \left. \delta(p^2 - m^2) \right|_{p_0 > 0} \tag{4.30}$$

left hand side:

$$\int \frac{\mathrm{d}^{3} p}{2 p^{0}} 2 p_{0}' \delta^{3}(\vec{p'} - \vec{q'}) = \int \mathrm{d}^{4} p \left. \delta(p^{2} - m^{2}) \right|_{p_{0} > 0} 2 p_{0}' \delta^{3}(\vec{p'} - \vec{q'})$$

$$= \int \mathrm{d}^{4} p' \left. \delta(p'^{2} - m'^{2}) \right|_{p_{0} > 0} 2 p_{0}' \delta^{3}(\vec{p'} - \vec{q'}) \qquad (4.31)$$

$$= \int \frac{\mathrm{d}^{3} p'}{2 p'^{0}} 2 p_{0}' \delta^{3}(\vec{p'} - \vec{q'}) = 1$$

right hand side:

$$\int \frac{d^3 p}{2p^0} \, 2p_0 \, \delta^3(\vec{p} - \vec{q}) = 1 \tag{4.32}$$

Note: See in particular the books by Itzykson/Zuber (sec. 3.1.2)

5. Perturbation Theory - Leading Order Approach

5.1 S-Matrix

• Before we considered

$$V(\varphi) = \frac{m^2}{2} \varphi^2 \tag{5.1}$$

which was justified by a Taylor expansion. Now it is natural to ask for higher order terms in this expansion.

- It is convenient to impose the discrete symmetry $\varphi \rightarrow -\varphi$
- In that case the next term in the Taylor expansion becomes

$$\frac{\lambda}{4!}\varphi^4\tag{5.2}$$

And with this the Lagrangian and the Hamiltonian density can be split into

$$\mathcal{L}_{0} \to \mathcal{L}_{0} + \mathcal{L}_{int} \qquad \mathcal{L}_{int} = -\frac{\lambda}{4!} \varphi^{4}$$

$$\mathcal{H}_{0} \to \mathcal{H}_{0} + \mathcal{H}_{int} \qquad \mathcal{H}_{int} = \frac{\lambda}{4!} \varphi^{4}.$$
(5.3)

Notice the different signs for the Lagrangian and the Hamiltionian.

• If we would not impose the $\varphi \to -\varphi$ symmetry the next term would have been

$$V_{\rm int} = \frac{\lambda}{3!} \varphi^3 \tag{5.4}$$

One (weak) argument that is often brought forward against these terms is that they are unbound from below for $x \to -\infty$. However, this is not really crucial, as we are only interested in a perturbation theory around $\varphi = 0$. The true reason we start with φ^4 -theory lies in the ease of dealing with it.

• In the following discussion we are going to use the *interaction picture*. We keep the time evolution of the free operators as in the Heisenberg picture, but let the additional (interaction-induced) time-evolution act on the states.

Schrödinger picture
$$\rightarrow$$
 Interaction picture Operator O \rightarrow $O_t^I = e^{iH_0t} O e^{-iH_0t}$ State $|\psi_t\rangle = e^{-iHt} |\psi\rangle$ \rightarrow $|\psi_t^I\rangle = e^{iH_0t} e^{-iHt} |\psi\rangle$

• Analogously, we look at the evolution of $|\psi\rangle$ from 0 to t':

$$\left|\psi_{t'}^{I}\right\rangle = e^{iH_{0}t'}e^{-iHt'}\left|\psi\right\rangle \tag{5.5}$$

• We thus arrive at the important result

$$\left|\psi_{t'}^{I}\right\rangle = U(t',t)\left|\psi_{t}^{I}\right\rangle$$
with $U(t',t) = e^{iH_{0}t'}e^{-iH(t'-t)}e^{-iH_{0}t}$
(5.6)

Here U(t',t) is the unitary time evolution operator we are already familiar with from time-dependent perturbation theory in QM. It describes the time evolution of a state from time t to t' in the interaction picture.

• Let us split the time evolution from t to t' into n small steps $\Delta = \frac{t'-t}{n}$. Then U(t',t) becomes

$$U(t',t) = U(t',t'-\Delta) \cdot U(t'-\Delta,t'-2\Delta) \dots U(t+\Delta,t)$$
 (5.7)

• We now look at an individual step

$$U(t + \Delta, t) = e^{iH_0(t + \Delta)}e^{-iH\Delta}e^{-iH_0t} = e^{iH_0t}e^{iH_0\Delta}e^{-iH\Delta}e^{-iH_0t}$$

= $e^{iH_0t}e^{-iH_{\text{int}}\Delta}e^{-iH_0t} = e^{-iH_{\text{int}}^I(t)\Delta}$ (5.8)

where in the step from the first to the second line we dropped all commutator terms in the Baker-Campbell-Hausdorff formula as they are of second order in Δ :

$$e^{iH_0\Delta}e^{-iH\Delta} = e^{i(H_0 - H)\Delta + \mathcal{O}(\Delta^2)} \approx e^{-iH_{\text{int}}\Delta}.$$
 (5.9)

Above $H_{\text{int}}^{I}(t) = e^{iH_0t}H_{\text{int}}e^{-iH_0t}$ is the interaction Hamiltonian H_{int} transformed to the interaction picture.

• Thus, we have

$$U(t',t) = e^{-iH_{\text{int}}^{I}(t'-\Delta)\Delta} \cdot e^{-iH_{\text{int}}^{I}(t'-2\Delta)\Delta} \dots e^{-iH_{\text{int}}^{I}(t)\Delta}$$

$$= Te^{-iH_{\text{int}}^{I}(t'-\Delta)\Delta} \cdot e^{-iH_{\text{int}}^{I}(t'-2\Delta)\Delta} \dots e^{-iH_{\text{int}}^{I}(t)\Delta}$$

$$= Te^{-iH_{\text{int}}^{I}(t'-\Delta)\Delta - iH_{\text{int}}^{I}(t'-2\Delta)\Delta + \dots - iH_{\text{int}}^{I}(t)\Delta}$$

$$= T \exp\left(-i\int_{t}^{t'} d\tau H_{\text{int}}^{I}(\tau)\right) \text{ as } n \to \infty$$

$$(5.10)$$

where in the first step the time-ordering operator was defined by

$$T\varphi(t_1)\varphi(t_2) = \begin{cases} \varphi(t_1)\varphi(t_2) & \text{if } t_1 \ge t_2\\ \varphi(t_2)\varphi(t_1) & \text{if } t_2 > t_1 \end{cases}$$
 (5.11)

Strictly speaking it is not an operator – thus the name "time-ordering symbol" would be more accurate.

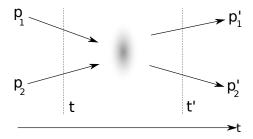


Figure 2: Incoming and outgoing momentum in 2-2-scattering. The two particles are separated in the initial and final state.

• Explicitly, we now have

$$H_{\text{int}}^{I}(t) = e^{iH_{0}t} \int d^{3}x \underbrace{\frac{\lambda}{4!} (\varphi(\vec{x}))^{4}}_{\mathcal{H}_{\text{int}}} e^{-iH_{0}t}$$

$$= \int d^{3}x \, \frac{\lambda}{4!} (\varphi^{I}(x))^{4}$$
(5.12)

where we obtained the Hamiltonian by integrating the Hamilton density. φ^I in the last term is the field in the interaction-picture. Notice that this is the Heisenberg-picture field of the free theory. We can thus apply the results we have already derived.

- Before calculating the S-matrix we first want to motivate scattering physically. For a pair of particles with four-momenta p_1 , p_2 , scattered into a state with momenta p_1' , p_2' , we consider the situation in figure 2. Since the interaction $\int \mathrm{d}^3 x \varphi^4$ includes terms of type $a_{\vec{p}_1}^{\dagger} a_{\vec{p}_2}^{\dagger} a_{\vec{p}_1} a_{\vec{p}_2}$ we expect it to induce 2-2-scattering at leading order. With higher orders of φ , in general, we could have 2-n-scattering (n > 2). For our approximation it is crucial that at times t and t' the particles are separated (as they are localized it will be necessary to model them as wave packets rather than plane waves). We are not interested in information at explicit times but rather in scattering cross sections and probabilities.
- Accordingly, we now define the *S-matrix*:

$$S = \lim_{\substack{t \to -\infty \\ t' \to \infty}} U(t', t) = T \exp\left(-i \int_{-\infty}^{\infty} dt \, H_{\text{int}}^{I}(t)\right)$$

$$= T \exp\left(i \int d^{4}x \, \mathcal{L}_{\text{int}}^{I}(x)\right)$$
(5.13)

and the S-matrix element

$$S_{\rm fi} = \langle p_1' p_2' | T \exp\left(i \int d^4 x \, \mathcal{L}_{\rm int}^I\right) | p_1 p_2 \rangle , \qquad (5.14)$$

where f and i denote the final and initial state, respectively.

• In some problems it is also useful to define the transition matrix or *T-matrix*

$$S = \mathbb{1} + iT \qquad S_{\rm fi} = \delta_{\rm fi} + iT_{\rm fi} \tag{5.15}$$

• Henceforth we change the normalization of a, a^{\dagger} to

$$(a_{\vec{p}})_{\text{new}} = \sqrt{2\omega_{\vec{p}}} (a_{\vec{p}})_{\text{old}}$$
 (5.16)

with $\omega_{\vec{p}} = p^0$, whereby covariance is being emphasized rather than the analogy to harmonic oscillators. This yields

$$\begin{bmatrix} a_{\vec{p}}, a_{\vec{q}}^{\dagger} \end{bmatrix} = 2p^{0}(2\pi)^{3}\delta^{3}(\vec{p} - \vec{q})
|p\rangle = a_{\vec{p}}^{\dagger} |0\rangle
\varphi^{I}(x) = \int \frac{d^{3}p}{(2\pi)^{3}2p^{0}} \left(a_{\vec{p}}e^{-ipx} + a_{\vec{p}}^{\dagger}e^{ipx} \right)$$
(5.17)

Note that requantization is not necessary here, the interaction does not change our quantization. Instead, we just choose a new normalization.

• With this normalization at leading order in λ we then have

$$iT_{fi} = \langle 0 | a_{\vec{p}_1'} a_{\vec{p}_2'} \left(-i \frac{\lambda}{4} \right) \int d^4 x \, (\varphi^I(x))^4 a_{\vec{p}_1}^{\dagger} a_{\vec{p}_2}^{\dagger} \, | 0 \rangle$$
 (5.18)

The intention of the previous manipulations was to express the matrix element in terms of *free* fields. Thus, the usual commutation relations for a, a^{\dagger} apply. With those it follows that

$$iT_{\rm fi} = -i\lambda(2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2')$$
 (5.19)

• The δ -distribution enforcing momentum conservation always arises in this context! Accordingly, one defines the *invariant matrix element* \mathcal{M}_{fi} by

$$S_{\rm fi} = \delta_{\rm fi} + i(2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2') \mathcal{M}_{\rm fi}. \tag{5.20}$$

• For 2-2-scattering in $\lambda \varphi^4$ -theory we found

$$i\mathcal{M}_{\rm fi} = -i\lambda.$$
 (5.21)

• This is also our first Feynman rule

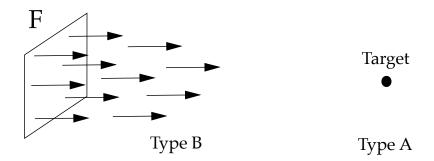
$$= -i\lambda. (5.22)$$

• 2-2-scattering in $\lambda \varphi^3$ -theory would yield a slightly more complicated result:

• Unsurprisingly, our result is singular $(\delta^4(p_{in}-p_{out}))$ as $t \to -\infty$ and $t' \to +\infty$. However, in the next section we are introducing a well-defined and finite quantity; the scattering cross section.

5.2 Scattering cross section

- Consider a "fixed target experiment" with a beam, which comprises N_B particles of type B, being spread out over a transverse area F and hitting a target being a single particle of type A.
- The scattering cross section σ for this experiment can be formulated as follows:



$$\frac{N_{\text{events}}}{N_B} = \frac{\sigma}{F} \rightarrow \sigma = \frac{N_{\text{events}}}{(N_B/F)},$$
 (5.24)

where N_B/F is the transverse beam density.

• We require localized states and hence consider wave packets *f* only:

$$|f_{\vec{p}}\rangle := \int d\tilde{k} f_{\vec{p}}(\vec{k})|k\rangle, \qquad d\tilde{k} := \frac{d^3k}{(2\pi)^3 2k^0}$$
 (5.25)

• $f_{\vec{p}}(\vec{k})$ as a function of \vec{k} is peaked near $\vec{k} = \vec{p}$, e.g. $f_{\vec{p}}(\vec{k}) \approx \exp\left(-\alpha \left| \vec{k} - \vec{p} \right|^2\right)$. For normalization we propose,

$$\langle f_{\vec{p}}|f_{\vec{p}}\rangle = \int d\vec{k}d\vec{k}' \overline{f_{\vec{p}}(\vec{k})} f_{\vec{p}}(\vec{k}') \langle k|k'\rangle$$

$$= \int d\vec{k} \left| f_{\vec{p}}(\vec{k}) \right|^2 \stackrel{!}{=} 1$$
(5.26)

• Next, we check that $\left|f_{\vec{p}}\right\rangle$ is indeed localized in \mathbb{R}^3 for an appropriate choice of $f_{\vec{p}}$.

Using the by now rather familiar expression for $\varphi^{I}(x)$ in terms of a and a^{\dagger} one can readily check that

$$a_{\vec{k}}^{\dagger} = -i \int d^{3}x e^{-ikx} \stackrel{\leftrightarrow}{\partial_{0}} \varphi^{I}(x) \Big|_{x^{0}=0}.$$

$$\rightarrow |f_{\vec{p}}\rangle = \int d\tilde{k} f_{\vec{p}}(\vec{k}) a_{\vec{k}}^{\dagger} |0\rangle$$

$$= \int d^{3}x \left\{ \int d\tilde{k} e^{i\vec{k}\vec{x}} \left(k_{0} \varphi^{I}(\vec{x}) - i \dot{\varphi}^{I}(\vec{x}) \right) f_{\vec{p}}(\vec{k}) \right\} |0\rangle$$
(5.27)

We identify expressions:

$$\int d^{3}x \varphi^{I}(\vec{x}) \left(\int d\tilde{k} e^{i\vec{k}\vec{x}} f_{\vec{p}}(\vec{k}) k^{0} \right) |0\rangle$$

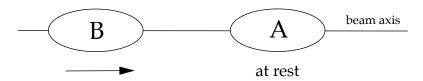
$$\int d^{3}x \dot{\varphi}^{I}(\vec{x}) \left(\int d\tilde{k} e^{i\vec{k}\vec{x}} f_{\vec{p}}(\vec{k}) \right) |0\rangle$$
(5.28)

Now with $f_{\vec{p}}(k)$ and $k^0 f_{\vec{p}}(k)$ being smooth and localized in \vec{k} their Fourier transforms will be localized in \vec{x} . $\Rightarrow |f_{\vec{p}}\rangle$ is created by $\varphi^I(x^0=0,\vec{x})$ and $\dot{\varphi}^I(x^0=0,\vec{x})$ acting on vacuum in a localized region near $\vec{x}=0$.

• Thus, the state

$$|i\rangle = \int d\tilde{k}_A d\tilde{k}_B f_{\vec{p}_A}(\vec{k}_A) f_{\vec{p}_B}(\vec{k}_B) |k_A k_B\rangle$$
 (5.29)

describes the following situation: such that for appropriate p_A , p_B and f's the



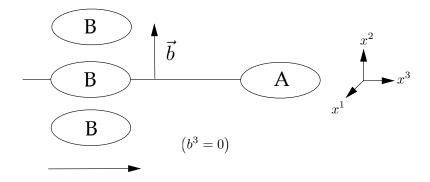
two particles A, B are in the regions overlapping at $t = x^0 = 0$.

- Next, we need to account for the transverse spread of incoming particles ("type B"). In other words: we must allow for a non-zero impact parameter \vec{b} .
- As in QM, the operator \hat{P} generates shifts. Thus, the incoming particle with $\vec{b} \neq 0$ is given by:

$$e^{-i\hat{\vec{P}}\vec{b}} \int d\tilde{k} f_{\vec{p}_B}(\vec{k}_B) |k_B\rangle \tag{5.30}$$

Since $\hat{\vec{P}}|k_B\rangle = \vec{k}_B|k_B\rangle$, our initial state for given \vec{b} reads as

$$|\mathbf{i}_b\rangle = \int d\tilde{k}_A d\tilde{k}_B f_{\vec{p}_A}(\vec{k}_A) f_{\vec{p}_B}(\vec{k}_B) e^{-i\vec{k}_B \vec{b}} |k_A k_B\rangle.$$
 (5.31)



• Starting with $|i_b\rangle$ at $t=-\infty$, when A and B are certainly far apart, we evolve in time up to the point where A and B make contact up to the impact parameter \vec{b} . Subsequently, we project on the desired, scattered state at $t=-\infty$. The probability for scattering into our final state $|p_1p_2\rangle$ is:

$$\left| \langle p_1 p_2 | S | \mathbf{i}_b \rangle \right|^2 \tag{5.32}$$

• Summing over a set of particles with different \vec{b} gives

$$N_{\text{events}} = \sum_{\vec{b}} |\langle p_1 p_2 | S | \mathbf{i}_b \rangle|^2.$$
 (5.33)

• For a homogeneous, transverse distribution of N_B particles in an area F we can approximate the sum by an integral:

$$\Rightarrow N_{\text{events}} = \frac{N_B}{F} \int_F d^2b |\langle p_1 p_2 | S | \mathbf{i}_b \rangle|^2$$

$$\Rightarrow \sigma(\vec{p}_1, \vec{p}_2) = \frac{N_{\text{events}}}{(N_B/F)} = \int_F d^2b |\langle p_1 p_2 | S | \mathbf{i}_b \rangle|^2$$
(5.34)

• Clearly, this is too naive as we cannot ask for a specific point of final-state momenta $(p_1, p_2) \in \mathbb{R}^6$. This also clashes with the finite precision of any real detector. Instead, let us define σ for a finite region of phase-space $V_f \subset \mathbb{R}^6$.

$$\rightarrow \sigma(V_f) = \int_{V_f} d\tilde{p}_1 d\tilde{p}_2 \sigma(\vec{p}_1, \vec{p}_2)$$
 (5.35)

• The correctness of the proposed measure $d\tilde{p}_1d\tilde{p}_2$ follows immediately by considering the free theory (S=1) and integration over the whole phase space, assuming for simplicity that the two particle types are distinguishable:

$$1 \stackrel{!}{=} \int d\tilde{p}_1 d\tilde{p}_2 \left| \langle p_1 p_2 | f_{\vec{p}_A} f_{\vec{p}_B} \rangle \right|^2 = \int d\tilde{p}_1 d\tilde{p}_2 \left| f_{\vec{p}_A} (\vec{p}_1) \right|^2 \left| f_{\vec{p}_B} (\vec{p}_2) \right|^2$$
(5.36)

This holds because of the normalization of f.

• All of the above generalizes to *n* final-state particles. Also, we actually want "differential" cross sections

$$d\sigma = \prod_{j=1}^{n} d\tilde{p}_{j} \int_{F} d^{2}b |\langle p_{1}...p_{n}|S|i_{b}\rangle|^{2}.$$
 (5.37)

• Utilize $S_{f\,i}=\delta_{f\,i}+i(2\pi)^4\delta^4(p_f-p_i)\mathcal{M}_{f\,i}$ as well as $|i_b\rangle$ as given above. Only the $\mathcal{M}_{f\,i}$ term contributes unless f=i.

$$d\sigma = \prod_{j=1}^{n} d\tilde{p}_{j} \int d^{2}b \int d\tilde{k}_{A} d\tilde{k}_{B} f_{\vec{p}_{A}}(\vec{k}_{A}) f_{\vec{p}_{B}}(\vec{k}_{B}) \int d\tilde{k}'_{A} d\tilde{k}'_{B} \overline{f}_{\vec{p}_{A}}(\vec{k}'_{A}) \overline{f}_{\vec{p}_{B}}(\vec{k}'_{B})$$

$$= e^{i\vec{b}(\vec{k}'_{B} - \vec{k}_{B})} |\mathcal{M}_{fi}|^{2} (2\pi)^{4} \delta^{4} (p_{f} - k_{i}) \delta^{4} (p_{f} - k'_{i}),$$
(5.38)

where $p_f = \sum_{j=1}^n p_j$, $k_i = k_A + k_B$ and $k_i' = k_A' + k_B'$. In the following calculation we only write down the parts of the equation we are interested in and do not insert the obtained results:

$$\to \int d^2b e^{i\vec{b}(\vec{k}_B' - \vec{k}_B)} \dots = (2\pi)^2 \delta^2 \left(k_{B_{\perp}}' - k_{B_{\perp}} \right) \dots$$
 (5.39)

- 1) Obviously $k'_{B_{\perp}}=k_{B_{\perp}}$, less obviously $k'_{A_{\perp}}=k_{A_{\perp}}$. The latter follows from $\delta^2\left(p_{\mathrm{f}_\perp}-k_{A_\perp}'-k_{B_\perp}'\right)$ and $\delta^2\left(p_{\mathrm{f}_\perp}-k_{A_\perp}-k_{B_\perp}\right)$ which are "hidden" in the dots.
- 2) Obviously $k_B^{3\prime} = p_f^3 k_A^{3\prime}$.
- 3) With $(\vec{k}'_A)^2 = (k_{A_\perp})^2 + (k_A^{3\prime})^2$, it follows that $(\vec{k}'_B)^2 = (k_{B_\perp})^2 + (p_f^3 k_A^{3\prime})^2$.
- 4) Use $k_{A,B} \simeq p_{A,B}$ as induced by $f_{\vec{p}_A}(\vec{k}_A)$ and $f_{\vec{p}_B}(\vec{k}_B)$, respectively.

Note: Initially, we talked about "fixed target". However, nothing in our analysis depended on $v_a = 0$, so we might as well keep it general.

• Now, we have completely carried out the $k'_{A,B}$ integrations, implementing:

$$k'_{A_{\perp}} = k_{A_{\perp}}$$

$$k'_{B_{\perp}} = k_{B_{\perp}}$$

$$k''_{B} + k''_{A} = p_{f}^{3}$$

$$k''_{B} + k''_{A} = p_{f}^{0} \iff \left(m_{B}^{2} + (k_{B_{\perp}})^{2} + (k_{B}^{3\prime})^{2}\right)^{\frac{1}{2}} + \left(m_{A}^{2} + (k_{A_{\perp}})^{2} + (k_{A}^{3\prime})^{2}\right)^{\frac{1}{2}} = p_{f}^{0}$$
(5.41)

- Of these 4 relations, view the two last ones as fixing the two variables $k_A^{3\prime}, k_B^{3\prime}$ But note: Two analogous relations hold for k_A^3 , k_B^3 due to the yet unused deltafunction $\delta^4(p_{\rm f}-k_A-k_B)$
- Thus, we have implemented $\vec{k}_A = \vec{k}_A'$ and $\vec{k}_B = \vec{k}_B'$
- So far, we have:

$$d\sigma = \prod_{j} d\tilde{p}_{j} \int d\tilde{k}_{A} d\tilde{k}_{B} \left| f_{\vec{p}_{A}} \left(\vec{k}_{A} \right) \right|^{2} \left| f_{\vec{p}_{B}} \left(\vec{k}_{B} \right) \right|^{2} \left| \mathcal{M}_{fi} \right|^{2}$$

$$\cdot \frac{(2\pi)^{4} \delta(p_{f} - k_{A} - k_{B})}{4k_{A}^{0} k_{B}^{0} \left| v_{A} - v_{B} \right|}$$
(5.42)

• We can now view the two $|f|^2$ as effective δ -functions ensuring $\vec{k}_A = \vec{p}_A$, $\vec{k}_B = \vec{p}_B$. We then get:

$$d\sigma = \underbrace{\frac{|\mathcal{M}_{fi}|^2}{4 p_A^0 p_B^0 |v_A - v_B|}}_{(1)} \underbrace{(2\pi)^4 \delta^4 (p_f - p_i) \prod_j^n \frac{d^3 p_j}{(2\pi)^3 2 p_j^0}}_{(2)}$$
(5.43)

(1) This pre-factor is invariant under boosts along x^3 -direction. It can be written as follows:

$$\frac{\left|\mathcal{M}_{\rm fi}\right|^2}{4\,p_A^0\,p_B^0|v_A - v_B|} = \frac{1}{2\,w\left(s, m_A^2, m_B^2\right)} \tag{5.44}$$

where:

$$w(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 2xy - 2xz - 2yz}$$

$$s = (p_A + p_B)^2 (\sqrt{s} := \text{center-of-mass energy})$$
(5.45)

(2) This part is the "n-particle phase space".

At the moment, the highly relativistic case will be the most relevant for us.
 Thus, let:

$$p_A^0 = m_A; \quad \vec{p}_A = 0; \quad m_A, m_B \ll \sqrt{s}$$
 (5.46)

Moreover, we get:

$$|v_A - v_B| = |v_B| = c = 1 (5.47)$$

and

$$4 p_A^0 p_B^0 = 2(p_A + p_B)^2 = 2s (5.48)$$

In the first equality it has been used that $p_A^2 \approx p_B^2 \approx 0$, which follows directly from (5.46) and the fact that p_B is approximately light-like.

• Thus, in the highly relativistic case we have:

$$d\sigma = \frac{1}{2s} |\mathcal{M}_{fi}|^2 dX^{(n)}; \quad dX^{(n)} = (2\pi)^4 \delta^4(p_f - p_i) d\tilde{p}_1 \dots d\tilde{p}_n$$
 (5.49)

5.3 2-particle phase-space & a simple example

- Let us look at the event " $A + B \longrightarrow 1 + 2$ " in $\lambda \varphi^4$ -theory.
- We now focus on phase space first:

$$\int dX^{(2)} = \int (2\pi)^4 \,\delta^4(p_1 + p_2 - p_A - p_B) \,\frac{\mathrm{d}^3 p_1}{(2\pi)^3 \, 2p_1^0} \frac{\mathrm{d}^3 p_2}{(2\pi)^3 \, 2p_2^0} \tag{5.50}$$

• Carrying out the d^3p_1 -integration trivially and using $\vec{p}_1 = -\vec{p}_2$ (i.e. choosing the center-of-mass frame), as well as $p_2^2 = (p_2^0)^2 - \vec{p}_2^2 = m^2$ (≈ 0 here), the following can be obtained:

$$\int \frac{\mathrm{d}^3 p_2}{(2\pi)^2 4 p_1^0 p_2^0} \,\delta\left(p_1^0 + p_2^0 - \sqrt{s}\right) = \int \frac{\mathrm{d}^3 p_2}{(2\pi)^2 4 |\vec{p}_2|^2} \,\delta\left(2|\vec{p}_2| - \sqrt{s}\right) \tag{5.51}$$

• Now, we switch to spherical coordinates:

$$d^{3}p_{2} = d\Omega |\vec{p}_{2}|^{2} d|\vec{p}_{2}|; \quad d\Omega = d\varphi \sin\theta d\theta$$
 (5.52)

• In these coordinates the integration can be carried out:

$$\int d|\vec{p}_2| \,\delta\left(2|\vec{p}_2| - \sqrt{s}\right) \,\frac{d\Omega}{16\pi^2} = \frac{d\Omega}{32\pi^2} \tag{5.53}$$

Note: The integral " \int " was interpreted as not containing the angular integration.

• Eventually, we obtain the *differential cross section*:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\left|\mathcal{M}_{\mathrm{fi}}\right|^2}{64\pi^2 s} = \frac{\lambda^2}{64\pi^2 s} \tag{5.54}$$

Note: $\frac{\lambda^2}{s}$ could have been argued without any calculations. The scattering amplitude is proportional to λ and therefore, it has to appear quadratic in the differential cross section. Cross sections have units of [Length]² corresponding to [Energy]⁻². Hence, we have to divide λ^2 by the energy scale squared: s.

6. LSZ-FORMALISM

(Lehmann, Symanzik, Zimmermann; 1955) In this chapter the presentation is similar to the approach taken in Peskin and Schröder which in turn is similar to Weinberg's approach, but differs from a lot of other books.

• General idea:

S-matrix-elements
$$\longleftrightarrow$$
 Correlation functions or Green-functions

$$\underbrace{\operatorname{out} \langle p'_1 \dots p'_n | p_1 \dots p_m \rangle_{\operatorname{in}}}_{\operatorname{needed for cross section}} \longleftrightarrow \underbrace{\langle 0 | T \varphi(x_1) \dots \varphi(x_{n+m}) | 0 \rangle}_{\operatorname{easily calculable in pert. theory}}$$

6.1 Spectral density & Z-factors

• In the following we will use the Heisenberg picture:

$$\varphi(x) = e^{iHt} \varphi_S(\vec{x}) e^{-iHt}; \quad H = H_0 + H_{int}$$

$$|\psi\rangle = |\psi_S(t=0)\rangle - \text{(time independent)}$$
 (6.1)

• Consider the correlation function:

$$\langle 0 | \varphi(x)\varphi(y) | 0 \rangle \tag{6.2}$$

This is interpretable as the amplitude for a particle to propagate from \vec{y} at time y^0 to \vec{x} at time x^0 .

• For a free field φ_0 with mass m_0 we have:

$$\langle 0 | \varphi_{0}(x)\varphi_{0}(y) | 0 \rangle = \int d\tilde{p} d\tilde{q} \langle 0 | \hat{a}_{\vec{p}} \hat{a}_{\vec{q}}^{\dagger} | 0 \rangle e^{-ipx+iqy}$$

$$= \int \frac{d^{3}p}{(2\pi)^{3} 2p^{0}} e^{-ip(x-y)} = \int \frac{d^{4}p}{(2\pi)^{3}} e^{-ip(x-y)} \delta(p^{2} - m_{0}^{2}) \Theta(p^{0})$$

$$\equiv D(x - y, m_{0}^{2})$$
(6.3)

where Θ denotes the Heaviside function.

• Now we consider the general case:

$$\langle 0 | \varphi(x)\varphi(y) | 0 \rangle = \sum_{\alpha} \langle 0 | \varphi(x) | \alpha \rangle \langle \alpha | \varphi(y) | 0 \rangle \tag{6.4}$$

where the sum includes *all* states, also multi-particle states. As this parameter α is continuous this "sum" is not a proper discrete sum but rather to be understood symbolically.

• By using the following two relations

$$\varphi(x) = e^{i\hat{p}x}\varphi(0)e^{-i\hat{p}x} \tag{6.5}$$

$$e^{-i\hat{p}x} |\alpha\rangle = e^{-ip_{\alpha}x} |\alpha\rangle \tag{6.6}$$

we get:

$$\langle 0 | \varphi(x)\varphi(y) | 0 \rangle = \sum_{\alpha} e^{-ip_{\alpha}(x-y)} |\langle 0 | \varphi(0) | \alpha \rangle|^{2}$$

$$= \int d^{4}q \sum_{\alpha} e^{-iq(x-y)} |\langle 0 | \varphi(0) | \alpha \rangle|^{2} \delta^{4}(q - p_{\alpha})$$

$$= \int \frac{d^{4}q}{(2\pi)^{3}} e^{-iq(x-y)} \rho(q)$$
(6.7)

In this we defined:

$$\rho(q) \equiv (2\pi)^3 \sum_{\alpha} \delta^4(q - p_{\alpha}) |\langle 0| \varphi(0) |\alpha \rangle|^2$$
 (6.8)

Note that $\rho(q)$ is manifestly SO⁺(1, 3)-invariant and vanishes for $q^0 < 0$.

• We can then introduce the *spectral density* $\sigma(q^2)$:

$$\rho(q) \equiv \Theta(q^0) \, \sigma(q^2) \tag{6.9}$$

 σ quantifies the contribution of the intermediate states $|\alpha\rangle$ with $p_{\alpha}^2 = q^2$.

• Further rewriting yields:

$$\langle 0 | \varphi(x)\varphi(y) | 0 \rangle = \int_{0}^{\infty} d(M^{2}) \int \frac{d^{4}q}{(2\pi)^{3}} e^{-iq(x-y)} \delta(q^{2} - M^{2}) \Theta(q^{0}) \sigma(M^{2})$$

$$= \int_{0}^{\infty} d(M^{2}) D(x - y, M^{2}) \sigma(M^{2})$$
(6.10)

It can be seen easily that by choosing $\sigma(q^2) = \delta(q^2 - m_0^2)$ we obtain the result (6.3) of the free field case.

• More generally the cross section approximately looks like the graph in Figure 3. Note that we assume that the vacuum does not contribute as an intermediate state, $\langle 0 | \varphi(x) | 0 \rangle$. This can be ensured by redefinition $\varphi \to \varphi$ + const. Thus:

$$\sigma(q^2) = Z \, \delta(q^2 - \underbrace{m^2}_{\neq m_0^2}) + \dots$$
 (6.11)

where the *Z*-factor is a normalization factor. We get further non-zero contributions (indicated by "...") for $q^2 > M_t^2$ (= multi-particle threshold).

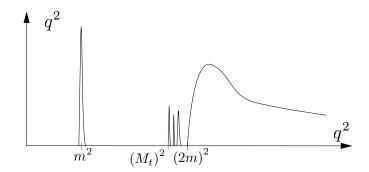


Figure 3: The peaks shortly before $(2m)^2$ are multi-particle bound states or resonances, starting at the multi-particle-threshold M_t^2

• So if we rewrite the "sum" \sum_{α} into a single-particle part and a separate multiparticle one ("...")

$$\sum_{\alpha} |\alpha\rangle \langle \alpha| = \int d\tilde{p} |p\rangle \langle p| + \dots$$
 (6.12)

we obtain:

$$\langle 0 | \varphi(x)\varphi(y) | 0 \rangle = \int d\tilde{p} \langle 0 | \varphi(x) | p \rangle \langle p | \varphi(y) | 0 \rangle + \dots$$

$$= \int d\tilde{p} e^{-ip(x-y)} \underbrace{|\langle 0 | \varphi(0) | p \rangle|^{2}}_{Z} + \dots$$

$$= D(x - y, m^{2}) Z + \dots$$
(6.13)

In the last step we used the fact that Z does not depend on p.

• Finally, we have:

$$\langle 0 | \varphi(x)\varphi(y) | 0 \rangle = Z D(x - y, m^2) + \int_{M_t^2}^{\infty} d(M^2) \sigma(M^2) D(x - y, M^2)$$
multi-particle contribution (6.14)

• Subtracting the same equation with $x \leftrightarrow y$ yields

$$\langle 0|[\varphi(x), \varphi(y)]|0\rangle = Z \Delta(x - y, m^2) + \int_{M_t^2}^{\infty} d(M^2) \sigma(M^2) \Delta(x - y, M^2)$$
 (6.15)

with

$$\Delta(x - y, M^2) \equiv \langle 0 | [\varphi_0(x), \varphi_0(y)] | 0 \rangle. \tag{6.16}$$

• Next we apply $\left. \frac{\partial}{\partial y^0} \right|_{y^0 = x^0}$ to the above equation $(\dot{\varphi} = \pi)$ and obtain

$$\left[\varphi(x^0, \vec{x}), \pi(x^0, \vec{y})\right] = i\,\delta^3\left(\vec{x} - \vec{y}\right) \tag{6.17}$$

both for the interacting and free theory.

• Thus:

$$1 = Z + \int_{M_t^2}^{\infty} d(M^2) \, \sigma(M^2)$$
(6.18)

 \Rightarrow Z \leq 1 and Z = 1 precisely for the free theory.

The size of 1 - Z accounts for the overlap of $\varphi(0) |0\rangle$ with multi-particle states.

• Finally, in complete analogy, we can derive with the help of $T\varphi(x)\varphi(y)$ instead of $[\varphi(x), \varphi(y)]$:

$$\langle 0|T\varphi(x)\varphi(y)|0\rangle = Z\,D_F(x-y,\,m^2) + \int_{M_t^2}^{\infty} \mathrm{d}(M^2)\,\sigma(M^2)\,D_F(x-y,\,M^2) \quad (6.19)$$

• The "Feynman propagator" $D_F(x-y, m_0^2) \equiv \langle 0|T\varphi_0(x)\varphi_0(y)|0\rangle$ will be of particular interest in the following sections.

6.2 LSZ reduction formula

• Now we will relate time-ordered correlation functions to scattering amplitudes. We start by considering the Fourier transform:

$$\int d^{4}x \ e^{ipx} \langle 0|T\varphi(x)\varphi(z_{1})\dots\varphi(z_{n})|0\rangle = \int_{-\infty}^{T_{-}} dx^{0} \left(\int d^{3}x \ e^{ipx} \langle 0|T\varphi(x)\varphi(z_{1})\dots\varphi(z_{n})|0\rangle\right)$$

$$+ \int_{T_{-}}^{T_{+}} dx^{0} \left(\int d^{3}x \ e^{ipx} \langle 0|T\varphi(x)\varphi(z_{1})\dots\varphi(z_{n})|0\rangle\right)$$

$$+ \int_{T_{+}}^{\infty} dx^{0} \left(\int d^{3}x \ e^{ipx} \langle 0|T\varphi(x)\varphi(z_{1})\dots\varphi(z_{n})|0\rangle\right)$$

$$(6.20)$$

• We will only need *the pole structure in* p^0 of this expression (view it as a function of the complex variable p^0). In particular, we claim that there is a pole at $p^0 = \sqrt{\vec{p}^2 + m^2}$ and we determine its residue. First focus on integration region III:

$$\int_{T_{+}}^{\infty} dx^{0} (\ldots) = \int_{T_{+}}^{\infty} dx^{0} \int d^{3}x \ e^{ipx} \sum_{\alpha} \langle 0 | \varphi(x) | \alpha \rangle \langle \alpha | T \varphi(z_{1}) \ldots \varphi(z_{n}) | 0 \rangle$$

$$= \sum_{\alpha} \int_{T_{+}}^{\infty} dx^{0} \int d^{3}x \ e^{ix^{0}(p^{0} - q_{\alpha}^{0}) - i\vec{x}(\vec{p} - \vec{q}_{\alpha})} \langle 0 | \varphi(0) | \alpha \rangle \langle \alpha | T \varphi(z_{1}) \ldots \varphi(z_{n}) | 0 \rangle$$
(6.21)

- A pole in p^0 will arise if the oscillating exponent vanishes, i.e. if $p^0 q_\alpha^0 = 0$. Thus, a pole at $p^0 = \omega_{\vec{p}}$ can only come from 1-particle-states ($\int d^3x$ forces $\vec{q}_\alpha = \vec{p}$ and only 1-particle-states then have $q_\alpha^0 = \omega_{\vec{p}}$)
- Thus we can replace $\sum_{\alpha} |\alpha\rangle\langle\alpha| \to \int d\tilde{q}_{\alpha} |q_{\alpha}\rangle\langle q_{\alpha}|$
- \Rightarrow (~ means 'equal up to finite terms')

$$\cdots \sim \int_{T_{+}}^{\infty} dx^{0} \int d^{3}x \int d\tilde{q}_{\alpha} e^{ix^{0}(p^{0}-q_{\alpha}^{0})-i\vec{x}(\vec{p}-\vec{q}_{\alpha})} \langle 0| \varphi(0) |q\rangle \langle q| T\varphi(z_{1}) \dots \varphi(z_{n}) |0\rangle$$

$$\sim \frac{\sqrt{Z}}{2\omega_{\vec{p}}} \int_{T_{+}}^{\infty} dx^{0} e^{ix^{0}(p^{0}-\omega_{\vec{p}})} \langle p| T\varphi(z_{1}) \dots \varphi(z_{n}) |0\rangle$$

$$(6.22)$$

Note: Allow for a small, positive imaginary part of p^0 whereby the integral is well-defined at $x^0 \to \infty$, then analytically continue back to real axis. With $p^0 \to p^0 + i\varepsilon$ we get $-x^0p^0$ in the exponent.

$$\sim \frac{\sqrt{Z}}{2\omega_{\vec{p}}} \cdot \frac{1}{i(p^{0} - \omega_{\vec{p}})} \left(0 - e^{ix^{0}(p^{0} - \omega_{\vec{p}})} \Big|_{x^{0} = T_{+}} \right) \langle p | T \varphi(z_{1}) \dots \varphi(z_{n}) | 0 \rangle$$

$$\sim \frac{i\sqrt{Z}}{p^{2} - m^{2}} \langle p | T \varphi(z_{1}) \dots \varphi(z_{n}) | 0 \rangle \quad \text{desired pole-structure at } p^{0} \to \omega_{\vec{p}}.$$
(6.23)

• Now: integration region I $\int_{-\infty}^{T_{-}} dx^{0} \left(\int d^{3}x \ e^{ipx} \langle 0 | T \varphi(x) \varphi(z_{1}) \dots \varphi(z_{n}) | 0 \rangle \right)$

$$\int_{-\infty}^{T_{-}} dx^{0}(\ldots) \to \sum_{\alpha} \langle \ldots | \alpha \rangle \langle \alpha | \varphi(x) | 0 \rangle$$
 (6.24)

 $\Rightarrow e^{ix^0(p^0+\omega_{\vec{p}})} \Rightarrow$ pole at $p^0=-\omega_{\vec{p}}$, not relevant. *Note:* The above formula does in fact continue to hold, but here we only care about $p^0\sim\omega_{\vec{p}}$. Sign flip due to $\langle 0|\varphi(x)|\alpha\rangle \rightarrow \langle \alpha|\varphi(x)|0\rangle$

- Integration region II: Finite, hence analytic in p^0 , no poles.
- Preliminary Result: We managed to trade $\varphi(x)$ for outgoing particle $\langle p|$.
- Completely analogously for incoming particles (note the different sign in the exponent!):

$$\int d^4x \ e^{-ipx} \langle 0|T\varphi(x)\varphi(z_1)\dots\varphi(z_n)|0\rangle \sim \frac{i\sqrt{Z}}{k^2 - m^2} \langle 0|T\varphi(z_1)\dots\varphi(z_n)|k\rangle \quad (6.25)$$

• Finally we need to be able to do several such manipulations at once, e.g. to derive

$$\int_{x_{1}} \int_{x_{2}} d^{4}x_{1} d^{4}x_{2} e^{ip_{1}x_{1}+ip_{2}x_{2}} \langle 0|T\varphi(x_{1})\varphi(x_{2})\varphi(z_{1})\dots\varphi(z_{n})|0\rangle
\sim \frac{i\sqrt{Z}}{p_{1}^{2}-m_{1}^{2}} \cdot \frac{i\sqrt{Z}}{p_{2}^{2}-m_{2}^{2}} \langle p_{1}p_{2}|T\varphi(z_{1})\dots\varphi(z_{n})|0\rangle.$$
(6.26)

• The derivation will go through if the following holds:

$$\langle 0|T\varphi(x_1)\varphi(x_2)|p_1p_2\rangle = \langle 0|\varphi(x_1)|p_1\rangle \langle 0|\varphi(x_2)|p_2\rangle \tag{6.27}$$

• This will be true if \vec{x}_1 , \vec{x}_2 are always far apart, when x_1^0 , $x_2^0 \to \infty$. For a double Fourier-transform in x_1, x_2 this is certainly not true. Hence we need wave packets:

$$\int d^4x \ e^{ipx} \qquad \to \qquad \underbrace{\int d^4x \ \int d\tilde{k} \ f_{\vec{p}}(\vec{k})e^{ikx}}_{\text{(6.28)}}$$

function of \vec{x} which is localized near zero at x^0 and correspondingly in other regions of space at other times.

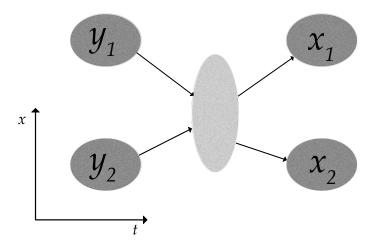


Figure 4: Visualization of wave packets, $\varphi(x_2)$ only contributes for x_2 in the borttom right region and analogously.

- \Rightarrow Convolve $\langle 0|T\varphi(z_1)\dots\varphi(z_n)|0\rangle$ with appropriate wave packets, show that the result is some linear combination of poles. The residues are the desired matrix elements. Take the limit $f_{\vec{p}}(\vec{k}) \to \delta(\vec{k} \vec{p})$ at the end (Doing this last limit rigorously is non-trivial but if it works, the the following result is clear).
- \Rightarrow LSZ-Reduction Formula:

$$\prod_{l=1}^{n} \int d^{4}x \ e^{-ip_{l}x_{l}} \prod_{j=1}^{m} \int d^{4}y \ e^{-ik_{j}y_{j}} \left\langle 0 | T\varphi(x_{1}) \dots \varphi(x_{n})\varphi(y_{1}) \dots \varphi(y_{m}) | 0 \right\rangle$$

$$\sim \prod_{l=1}^{n} \frac{i\sqrt{Z}}{p_{l}^{2} - m_{l}^{2}} \prod_{j=1}^{m} \frac{i\sqrt{Z}}{k_{j}^{2} - m_{j}^{2}} \underbrace{\text{out} \left\langle p_{1} \dots p_{n} | k_{1} \dots k_{m} \right\rangle_{\text{in}}}_{\equiv S_{6}} \tag{6.29}$$

- *Crucial:* $|k_1k_2\rangle_{\text{in}}$ means a state in the Heisenberg picture of the fully interaction theory which physically corresponds to two incoming wave-packets of particles $|k_1\rangle$, $|k_2\rangle$ (separated in \vec{x}) at $x^0 \to -\infty$ (analogously for out $\langle p_1 \dots p_n | \rangle$).
- We will calculate S_{fi} 's by Fourier-transforming time-ordered correlation functions and extracting the relevant residue.

6.3 Calculating time-ordered correlation functions

• We are interested in calculating

$$\langle 0 | T \varphi(x_1) \cdots \varphi(x_n) | 0 \rangle$$
 (6.30)

for any n.

- Let now n = 2 and for notational simplicity, $x_1 = x'$, $x_2 = x$ (our argument will go through identically for n > 2).
- We get

$$\langle 0| T\varphi(x')\varphi(x) |0\rangle = \langle 0| \varphi(x')\varphi(x) |0\rangle = \langle 0| \varphi(t')\varphi(t) |0\rangle$$
 (6.31)

where in the first step we assumed $x'^0 > x^0$ without loss of generality, and in the second we suppressed \vec{x}, \vec{x}' in the notation, understanding that the \vec{x}, \vec{x}' dependence is kept implicitly throughout the following derivation. Inserting multiple unities we get

$$\langle 0|\varphi(t')\varphi(t)|0\rangle = \langle 0|e^{iHt'}\varphi(0)e^{-iH(t'-t)}\varphi(0)e^{-iHt}|0\rangle$$

$$= \langle 0|\underbrace{e^{iHt'}e^{-iH_0t'}}_{U(0,t')}\underbrace{e^{iH_0t'}\varphi(0)e^{-iH_0t'}}_{\varphi_I(t')}\underbrace{e^{iH_0t'}e^{-iH(t'-t)}e^{-iH_0t}}_{U(t',t)}\underbrace{e^{iH_0t}\varphi(0)e^{-iH_0t}}_{\varphi_I(t)}\underbrace{e^{iH_0t}e^{-iHt}}_{U(t,0)}|0\rangle$$

$$= \langle 0|U(0,t')\varphi_I(t')U(t',t)\varphi_I(t)U(t,0)|0\rangle$$

$$= \langle 0|U(0,\infty)U(\infty,t')\varphi_I(t')U(t',t)\varphi_I(t)U(t,-\infty)U(-\infty,0)|0\rangle$$
(6.32)

where $U(t, -\infty)$ is understood as U(t, T), the unitary operator evolving states in the interaction picture, in the limit $T \to -\infty$.

 Let us now assume that the interactions are adiabatically switched off at t → ±∞. This can be realised by transforming the Hamiltonian in the following way

$$H_{int}(t) \rightarrow f(t)H_{int}(t)$$
 (6.33)

with an appropriately chosen f(t). Under this transformation the quantum-mechanical time evolution is still unitary, quantum mechanics remains unchanged.

- By adiabaticity, the interacting vacuum now evolves into the *free vacuum* in the limit $t \to \pm \infty$. This free vacuum takes the form $|0\rangle_I$, which is a state annihilated by all $a_{\vec{p}}$ in the familiar way. The $a_{\vec{p}}$ are related to φ_I as before.
- With this we can conclude that

$$U(t, -\infty)U(-\infty, 0) |0\rangle = U(t, -\infty) |0\rangle_{II} \langle 0| U(-\infty, 0) |0\rangle$$
(6.34)

and from this

$$\langle 0 | T\varphi(t')\varphi(t) | 0 \rangle = \frac{I \langle 0 | U(\infty, t')\varphi_I(t')U(t', t)\varphi_I(t)U(t, -\infty) | 0 \rangle_I}{(\langle 0 | U(0, \infty) | 0 \rangle_{I,I} \langle 0 | U(-\infty, 0 | 0 \rangle)^{-1}}$$
(6.35)

Here by unitarity the denominator is a product of two phases. Thus its inverse just is the complex conjugate $(\cdot \cdot \cdot)^{-1} = \overline{(\cdot \cdot \cdot)}$, which is true for phases.

• With this the denominator reads

$$\overline{\langle 0| U(0,\infty) |0\rangle_{II} \langle 0| U(-\infty,0) |0\rangle} = {}_{I} \langle 0| U(\infty,0) |0\rangle \langle 0| U(0,-\infty) |0\rangle_{I}
= {}_{I} \langle 0| U(\infty,-\infty) |0\rangle_{I}$$
(6.36)

• The next step is to remember the definition of U(t, t') 5.10 and inserting this into numerator and denominator of the previous expression. First we notice that for the denominator everything is time ordered. For the numerator we can write

Since under the time ordering the order of operators does not matter – they will be ordered by *T* anyways – we can combine the three exponents into one.

• With these results and generalizing to $n \ge 2$ we get

$$\langle 0|T\varphi(x_{1})\cdots\varphi(x_{n})|0\rangle$$

$$=\frac{I\langle 0|T\varphi_{I}(x_{1})\cdots\varphi_{I}(x_{n})\exp\left(-i\int_{-\infty}^{\infty}d\tau\,H_{int}(\varphi_{I}(\tau,\vec{x}))\right)|0\rangle_{I}}{I\langle 0|\exp\left(-i\int_{-\infty}^{\infty}d\tau\,H_{int}(\varphi_{I}(\tau,\vec{x}))\right)|0\rangle_{I}}$$
(6.38)

This is an enormous progress as now we are – at least in principle – able to evaluate this expression just using free field commutation relations.

7. Wick-Theorem and Feynman Rules

7.1 Time ordering and normal ordering

• In general we need to work out expressions like

$$_{I}\langle 0| T\varphi_{I}(x_{1})\cdots\varphi_{I}(x_{n}) \exp\left(i\int d^{4}x \mathcal{L}_{int}(\varphi_{I}(x))\right)|0\rangle_{I}$$
 (7.1)

Here we used $-i \int d\tau H_{int} = i \int d^4x \mathcal{L}_{int}$.

• We now want to study these expressions in more detail. As this whole section will mostly treat free fields we drop the index *I* for the interaction picture and instead we write

$$\varphi_I \to \varphi \qquad |0\rangle_I \to |0\rangle.$$
(7.2)

• Since \mathcal{L}_{int} is a polynomial in φ , we can expand the exponential and reduce our last expression to a sum of *free-field correlation functions*

$$\langle 0 | T \varphi(x_1) \cdots \varphi(x_n) | 0 \rangle$$
 with φ free. (7.3)

• We have already given a name to the case n = 2:

$$\langle 0|T\varphi(x)\varphi(y)|0\rangle = D_F(x-y) = \text{Feynman propagator}$$
 (7.4)

and we could easily evaluate it since we know $\langle 0 | \varphi(x)\varphi(y) | 0 \rangle$.

• To be able to generalize to the multi-field case it turns out to be useful to split the field into a creation part φ^c and an annihilation part φ^a

$$\varphi(x) = \int d\tilde{k} \left(a_{\vec{k}} e^{-ikx} + a_{\vec{k}}^{\dagger} e^{ikx} \right) = \varphi^{a}(x) + \varphi^{c}(x)$$
 (7.5)

• Next we define the *normal-ordered* form of any operator:

$$: \left(a_{\vec{k}_1} a_{\vec{k}_2}^{\dagger} a_{\vec{k}_3}^{\dagger} a_{\vec{k}_4} \cdots a_{\vec{k}_n}^{\dagger} \right) := a_{\vec{k}_2}^{\dagger} a_{\vec{k}_3}^{\dagger} \cdots a_{\vec{k}_n}^{\dagger} a_{\vec{k}_1} a_{\vec{k}_4} \cdots$$
 (7.6)

with all creation operators on the left side and all annihilation operators on the right side.

• Since φ is a linear combination of a, a^{\dagger} this definition extends to any product of φ . In particular one gets

$$: \varphi^{a}(x)\varphi^{c}(y) := \varphi^{c}(y)\varphi^{a}(x) \tag{7.7}$$

• Note the following two properties of normal ordering: First for any operator \hat{O} that is a polynomial or series in a, a^{\dagger} without a constant term we get

$$\langle 0|:\hat{O}:|0\rangle = 0. \tag{7.8}$$

Secondly, our prescription for dropping the vacuum energy amounts to saying:

$$H_0 = : \frac{1}{2} \int d^3x \left(\pi^2 + (\vec{\nabla}\varphi)^2 + m^2\varphi^2 \right) : \tag{7.9}$$

• For two fields, $\varphi(x)\varphi(y)$ and the normal ordered expression : $\varphi(x)\varphi(y)$: only differ by a number since the commutator of two fields is just a number.

$$\varphi(x)\varphi(y) = (\varphi_x^a + \varphi_x^c)(\varphi_y^a + \varphi_y^c) = \varphi_x^a \varphi_y^a + \varphi_y^c \varphi_x^a + \varphi_x^c \varphi_y^a + \varphi_x^c \varphi_y^c + \underbrace{\left[\varphi_x^a, \varphi_y^c\right]}_{\text{"number"}}$$

$$= : \varphi(x)\varphi(y) : +\text{"number"}$$

$$(7.10)$$

And accordingly

$$\langle 0 | \varphi(x)\varphi(y) | 0 \rangle = \text{"number"}$$

$$\Rightarrow \varphi(x)\varphi(y) =: \varphi(x)\varphi(y) :+ \langle 0 | \varphi(x)\varphi(y) | 0 \rangle$$

$$\Rightarrow T\varphi(x)\varphi(y) =: \varphi(x)\varphi(y) :+ \langle 0 | T\varphi(x)\varphi(y) | 0 \rangle$$
(7.11)

We now introduce the convenient notation

$$\langle 0 | T\varphi(x)\varphi(y) | 0 \rangle = \varphi(x)\varphi(y) \tag{7.12}$$

and call this a contraction. Hence, we get

$$T\varphi(x)\varphi(y) = : \varphi(x)\varphi(y) : +\varphi(x)\varphi(y) \tag{7.13}$$

7.2 Wick theorem

• Theorem:

$$T\varphi(x_1)\cdots\varphi(x_n)=:\varphi(x_1)\cdots\varphi(x_n):+\text{ all contractions of }:\varphi(x_1)\cdots\varphi(x_n):$$
(7.14)

We have to sum over all terms that arise by contracting one or more pairs of fields.

• Example: Let $\varphi_i = \varphi(x_i)$.

$$T\varphi_{1}\varphi_{2}\varphi_{3}\varphi_{4} = : \varphi_{1}\varphi_{2}\varphi_{3}\varphi_{4} :$$

$$+ (: \varphi_{1}\varphi_{2}\varphi_{3}\varphi_{4} : + 5 \text{ analogous terms})$$

$$+ (: \varphi_{1}\varphi_{2}\varphi_{3}\varphi_{4} : + : \varphi_{1}\varphi_{2}\varphi_{3}\varphi_{4} : + : \varphi_{1}\varphi_{2}\varphi_{3}\varphi_{4} :)$$

$$(7.15)$$

The normal ordering in the last term can be dropped, as due to the contractions we are just dealing with numbers. We can now identify the doubly contracted terms with the Feynman propagator

$$(: \varphi_{1}\varphi_{2}\varphi_{3}\varphi_{4} : + : \varphi_{1}\varphi_{2}\varphi_{3}\varphi_{4} : + : \varphi_{1}\varphi_{2}\varphi_{3}\varphi_{4} :)$$

$$= D_{F}(x_{1} - x_{2})D_{F}(x_{3} - x_{4}) + D_{F}(x_{1} - x_{4})D_{F}(x_{2} - x_{3})$$

$$+ D_{F}(x_{1} - x_{3})D_{F}(x_{2} - x_{4})$$

$$(7.16)$$

- Relevance: After taking the vacuum expectation value $\langle 0| ... |0 \rangle$, only the total contraction survives \Rightarrow Full solution in terms of D_F
- Proof: By induction

n = 1: trivial

n = 2: see section 7.1

Step n to n+1 (Without loss of generality, the (n+1)st field can be taken at latest time $x^0 \ge x_i^0 \ \forall i$)

$$T\varphi\varphi_1\cdots\varphi_n = \varphi T\varphi_1\cdots\varphi_n$$

$$= \varphi: \varphi_1\cdots\varphi_n: +\varphi \text{ (all contractions)}$$
(7.17)

The claim follows with the following Lemma:

• *Lemma*: Let $x^0 \ge x_i^0 \ \forall i \in \{1, ..., n\}$, Then:

$$\varphi: \varphi_1 \cdots \varphi_n := : \varphi \varphi_1 \cdots \varphi_n : + : \varphi \varphi_1 \cdots \varphi_n : + : \varphi \varphi_1 \varphi_2 \cdots \varphi_n : + \dots + : \varphi \varphi_1 \cdots \varphi_n :$$

$$(7.18)$$

Proof:

$$\varphi: \varphi_{1} \cdots \varphi_{n} := \varphi^{c}: \varphi_{1} \cdots \varphi_{n} : + \varphi^{a}: \varphi_{1} \cdots \varphi_{n} :$$

$$= \varphi^{c}: \varphi_{1} \cdots \varphi_{n} : + : \varphi_{1} \cdots \varphi_{n} : \varphi^{a} + [\varphi^{a}, : \varphi_{1} \cdots \varphi_{n} :]$$

$$= : \varphi\varphi_{1} \cdots \varphi_{n} : + [\varphi^{a}, : \varphi_{1} \cdots \varphi_{n} :]$$

$$(7.19)$$

Using the two facts

$$[A, B_1 \cdots B_n] = [A, B_1] B_2 \cdots B_n + B_1 [A, B_2] B_3 \cdots B_n + \dots + B_1 \cdots B_{n-1} [A, B_n]$$
$$[\varphi^a, \varphi_i] = [\varphi^a, \varphi_i^c] = \langle 0 | \varphi \varphi_i | 0 \rangle = \langle 0 | T \varphi \varphi_i | 0 \rangle = \varphi \varphi_i$$

*) for $x^0 \ge x_i^0 \, \forall i \in \{1, ..., n\}$

the lemma follows immediately.

7.3 The Feynman propagator

• Claim

$$D_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \exp\{-ip(x - y)\} \bigg|_{\epsilon \to 0}$$
 (7.20)

Derivation:

Let $x^0 > y^0$; perform p^0 integration first. To do that, write the denominator as:

$$\left(p^0 - \left(p^0\right)_1\right) \cdot \left(p^0 - \left(p^0\right)_2\right) \tag{7.21}$$

where
$$(p^0)_{1,2} = \pm \sqrt{\vec{p}^2 + m^2 - i\epsilon} = \pm \left(\sqrt{\vec{p}^2 + m^2} - i\epsilon'\right)$$
.

View p^0 -integration in complex p^0 - plane and close the integral-contour such that the integrand is suppressed. Since for $p^0 \to -i\infty$, we have $-ip^0(x^0 - y^0) \to -\infty$, we close the contour in the lower half plane.

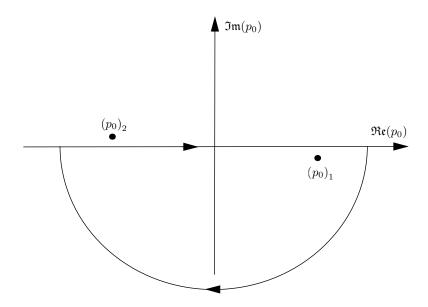


Figure 5: Illustration of the contour integration

Pick up residue and compare to $\langle 0 | \varphi(x)\varphi(y) | 0 \rangle$. For $x^0 < y^0$ close the contour in the upper half plane.

7.4 Feynman rules

Feynman rules allow for systematically writing down mathematical expressions for terms in the perturbative expansion of correlation functions. Consider:

$$\left\langle T\varphi_1\varphi_2\varphi_3\varphi_4\exp\left(i\int d^4x\,\mathcal{L}_{int}(\varphi)\right)\right\rangle$$

and work it out order-by-order in λ

•
$$\mathcal{O}(\lambda^0)$$
:

where $_{1} \bullet _{2} \equiv D_{F}(x_{1}, x_{2}).$

•
$$\mathcal{O}(\lambda^1)$$
:

$$\left\langle T\varphi_{1}\cdots\varphi_{4}\left(-\frac{i\lambda}{4!}\int d^{4}x\,\varphi(x)^{4}\right)\right\rangle$$
$$=-\frac{i\lambda}{4!}\int d^{4}x\,\varphi_{1}^{\Box}\varphi\varphi_{2}^{\Box}\varphi\varphi_{3}^{\Box}\varphi\varphi_{4}^{\Box}\varphi\cdot4!$$

+ ...terms in which $\varphi \varphi$ appears once

+ ...terms in which $\varphi \varphi \varphi \varphi \varphi$ appear

• Here, we did not comment on the prefactors.

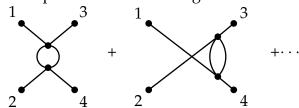
• $\mathcal{O}(\lambda^2)$:

$$\left\langle T\varphi_{1}\cdots\varphi_{4}\frac{1}{2!}\left(-\frac{i\lambda}{4!}\int d^{4}x\,\varphi(x)^{4}\right)\left(-\frac{i\lambda}{4!}\int d^{4}x\,\varphi(x)^{4}\right)\right\rangle$$

$$=\frac{1}{2!}\left(-\frac{i\lambda}{4!}\right)^{2}\int d^{4}x\int d^{4}y\,\overline{\varphi_{1}}\overline{\varphi_{x}}\overline{\varphi_{2}}\overline{\varphi_{x}}\overline{\varphi_{y}}\overline{\varphi_{y}}\overline{\varphi_{y}}\overline{\varphi_{y}}\overline{\varphi_{y}}\overline{\varphi_{y}}\overline{\varphi_{4}}\cdot\#$$
+ other full contractions that give another Feynman graph (7.24)

where # is a number. The naive expectation is: 4! from reshuffling φ_x , 4! from reshuffling φ_y and 2! from $\varphi_x \leftrightarrow \varphi_y$. But it is actually slightly different in this particular case.

- The dots stand for all other Feynman diagrams (truly different pictures!) which can be built from 6 propagators and two vertices.
- Rules for drawing:
- Each end of each propagator attaches either to an external point or to a vertex
- Each external point accepts one and each vertex accepts four ends of a propagator
- Examples for the other diagrams are:



• Feynman rules:

$$y = D_F(x - y)$$

$$= (-i\lambda) \int d^4x$$
(7.25)

- Prefactors: Generally, the prefactor will be unity by blindly applying the Feynman rules to a given picture (Reason: 1/4!'s from L_{int} and 1/n!'s from the expansion precisely compensate the number of combinatorial possibilities)
 Unfortunately, many diagrams are non-generic, in the sense of having symmetries and hence non-trivial prefactors, so called symmetry factors.
- Example for such a symmetry factor: Consider the following Feynman diagram:

Write down the fields $(\varphi_1\varphi_3\varphi_2\varphi\varphi_4\varphi\varphi\varphi)$, apply Wick's theorem and count the number of possibilities:

$$\rightarrow \underbrace{\varphi_1 \varphi_3}_{1} \underbrace{\varphi_2 \varphi}_{4} \underbrace{\varphi_4 \varphi}_{3} \underbrace{\varphi \varphi}_{1} \Rightarrow \frac{4 \cdot 3}{4!} = \frac{1}{2}$$

Thus, this diagram has a symmetry factor of 1/2. This symmetry factor is associated with the fact that the diagram does not change if the two downward-pointing ends of the vertex are swapped.

Note: General formulae and computer programs for this exist ("FeynArts").

• So far we have:

$$\langle 0|T\varphi_{1}\dots\varphi_{n}\cdot e^{iS_{int}}|0\rangle = \sum_{\text{all contractions}}\varphi_{1}\dots\varphi_{n}\cdot \exp\left\{-i\lambda\int\varphi^{4}\right\}$$

$$= \begin{cases} \text{Sum over all Feynman diagrams} \\ \text{(including symmetry factors)} \end{cases}$$
(7.26)

• For any diagram one can "split off" the so called "vacuum bubbles" (Diagrams without external lines).

• A quite 'plausible' claim is the following:

$$\left\{ \begin{array}{c} \text{sum over all} \\ \text{Feynman diagrams} \\ \text{(with certain external lines)} \end{array} \right\} = \left\{ \begin{array}{c} \text{sum over all} \\ \text{Feynman diagrams} \\ \text{without vacuum bubbles} \end{array} \right\} \cdot \left\{ \begin{array}{c} \text{sum over all} \\ \text{vaccum bubbles} \end{array} \right\}$$
 (7.31)

• Clearly

$$\langle T \exp(iS_{int}) \rangle = \begin{cases} \text{sum over all} \\ \text{vacuum bubbles} \end{cases}$$
 (7.32)

• Thus we finally have:

$${}_{I}\langle 0|T\varphi_{1}^{H}\dots\varphi_{n}^{H}|0\rangle_{I} = \frac{\langle 0|T\varphi_{1}\dots\varphi_{n}e^{iS_{int}}|0\rangle}{\langle 0|e^{iS_{int}}|0\rangle}$$

$$= \begin{cases} \text{sum over all} \\ \text{Feynman diagrams} \\ \text{without vacuum bubbles} \end{cases}$$
(7.33)

where the vacuum on the left of the equality is the interacting vacuum.

7.5 Feynman Rules in Momentum Space

- According to LSZ, scattering amplitudes are determined by residues of poles of Fourier-transformed, time-ordered correlation functions.
- Thus we have to translate our Feynman rules to momentum space.
- With $G(x_1,...,x_n) \equiv \langle T\varphi_1^H...\varphi_n^H \rangle$ let us define

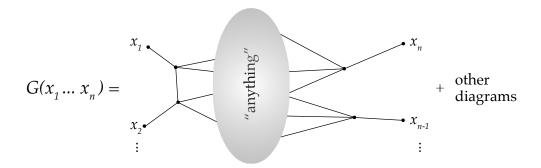
$$\tilde{G}(p_1...,p_n) \equiv \int d^4x_1 \ e^{-ip_1x_1}...\int d^4x_n \ e^{+ip_nx_n} G(x_1,...,x_n).$$
 (7.34)

Note: The minus sign in the exponent is associated to incoming particles while the plus sign in the exponent is associated to outgoing particles. This is consistent with our discussion of LSZ.

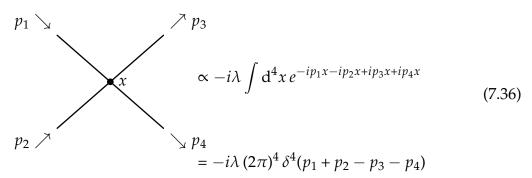
Recall that

$$x \bullet \longrightarrow y \equiv D_F(x - y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^3} \frac{i}{p^2 - m_0^2 + i\varepsilon} e^{-ip(x - y)}$$

$$\equiv -i \lambda \int \mathrm{d}^4 x$$
(7.35)



- Clearly in \tilde{G} the momentum of every external line is fixed to the external momentum (the appropriate argument of \tilde{G}). This is due to the d^4x_i integration followed by d^4p_i integration.
- Next, each d^4x integration of a vertex enforces momentum conservation at that vertex:



Whereas incoming is meant from the perspective of the vertex, outgoing is meant from the perspective of the attached propagator.

- Since each propagator is either at a vertex or external, all $e^{\pm ipx}$ factors of the D_F 's are now gone / used up.
- Momentum conservation at all vertices is enforced and "kills" many of the d⁴p integrations.
- All those propagator momenta which are *not* fixed by the δ^4 -function from external or vertex $\mathrm{d}^4 x$ integration are still integrated over with $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$ ("loop momenta")

Example 1:

$$p_{1} \to \bullet \longrightarrow p_{2} = \int d^{4}x_{1} e^{-ip_{1}x_{1}} \int d^{4}x_{2} e^{+ip_{2}x_{2}} D_{F}(x_{2} - x_{1})$$

$$= \frac{i}{p_{1}^{2} - m_{0}^{2} + i\varepsilon} (2\pi)^{4} \delta^{4}(p_{1} - p_{2})$$
(7.37)

Example 2:

$$p_1
ightarrow egin{pmatrix} q & & & \\ &
ightarrow & & \\ & &
ightarrow & \\ & &
ightarrow & p_2 & \\ & &
ightarrow & & \\ & &
ightarrow & p_2 & \\ & &
ightarrow & & \\ &
ight$$

$$= \int d^4x_1 \ e^{-ip_1x_1} \int d^4x_2 \int d^4x \ e^{+ip_2x_2} D_F(x_2 - x) D_F(x - x_1) D_F(x - x)$$

$$= \left(\frac{i}{p_1^2 - m_0^2 + i\varepsilon}\right)^2 (2\pi)^4 \delta^4(p_2 - p_1) (-i\lambda) \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m_0^2 + i\varepsilon}$$
(7.38)

Note:
$$\delta^4(p_2 + q - p_1 - q) = \delta^4(p_2 - p_1)$$
 and $D_F(x - x) = \int \frac{i \, d^4q}{q^2 - m_0^2 + i\varepsilon}$

Example 3:

$$p_{1} \searrow p_{3}$$

$$p_{2} \nearrow p_{4}$$

$$= \int d^{4}x_{1} e^{-ip_{1}x_{1}} \int d^{4}x_{2} e^{-ip_{2}x_{2}} \int d^{4}x_{3} e^{+ip_{3}x_{3}} \int d^{4}x_{4} e^{+ip_{4}x_{4}}$$

$$\cdot (-i\lambda) \int d^{4}x D_{F}(x - x_{1}) D_{F}(x - x_{2}) D_{F}(x_{3} - x) D_{F}(x_{4} - x)$$

$$= \left(\frac{i}{p_{1}^{2} - m_{0}^{2} + i\varepsilon}\right) \left(\frac{i}{p_{2}^{2} - m_{0}^{2} + i\varepsilon}\right) \left(\frac{i}{p_{3}^{2} - m_{0}^{2} + i\varepsilon}\right) \left(\frac{i}{p_{4}^{2} - m_{0}^{2} + i\varepsilon}\right)$$

$$\cdot (2\pi)^{4} \delta^{4}(p_{3} + p_{4} - p_{1} - p_{2}) (-i\lambda)$$

$$(7.39)$$

Example 4:

$$p_{1} \searrow q - p_{1} - p_{2} \searrow p_{4}$$

$$= \left(\frac{i}{p_{1}^{2} - m_{0}^{2} + i\varepsilon}\right) \left(\frac{i}{p_{2}^{2} - m_{0}^{2} + i\varepsilon}\right) \left(\frac{i}{p_{3}^{2} - m_{0}^{2} + i\varepsilon}\right) \left(\frac{i}{p_{4}^{2} - m_{0}^{2} + i\varepsilon}\right) \left(\frac{i}{p_{4}^{2} - m_{0}^{2} + i\varepsilon}\right)$$

$$\cdot (2\pi)^{4} \delta^{4}(p_{3} + p_{4} - p_{1} - p_{2})(-i\lambda)^{2}$$

$$\cdot \int \frac{d^{4}q}{(2\pi)^{4}} \frac{i}{q^{2} - m_{0}^{2} + i\varepsilon} \frac{i}{(q - p_{1} - p_{2})^{2} - m_{0}^{2} + i\varepsilon} \cdot \{\text{symm. factor}\}$$

$$\text{"loop integral"}$$

Summary:

$$\begin{array}{ccc}
\bullet & \longrightarrow & = \frac{i}{p^2 - m_0^2 + i\varepsilon} & (7.41)
\end{array}$$

$$=-i\lambda \tag{7.42}$$

Additionally the following rules apply:

- Assign momenta at each vertex such that momentum conservation is ensured.
- Multiply an overall factor of $(2\pi)^4 \delta^4(p_f p_i)$.
- Multiply by $\int \frac{d^4p}{(2\pi)^4}$ for each closed loop.

7.6 Calculating the Z-Factor and the Physical Mass

• Recall

$$\langle 0|T\varphi^{H}(x)\varphi^{H}(y)|0\rangle = Z\,D_{F}(x-y,\,m^{2}) + \int_{M_{t}^{2}}^{\infty} \mathrm{d}(M^{2})\,\sigma(M^{2})\,D_{F}(x-y,\,m^{2}) \quad (7.43)$$

• Perform a Fourier-transform and drop overall δ -function

$$p - p = \frac{iZ}{p^2 - m^2 + i\varepsilon} + \int_{M_t^2}^{\infty} dM^2 \ \sigma(M^2) \frac{i}{p^2 - m^2}$$
 (7.44)

Where $p \leftarrow p$ denotes all Feynman diagrams with two external

lines in momentum space without vacuum bubbles and without an overall δ -function. Furthermore we drop the ε prescription.

- Let us introduce the notation for those diagrams which *do not* fall apart upon cutting any internal line.
- Clearly

Note: _____ are the "one-particle irreducible diagrams" (1PI)

• Now we *define* the "self-energy" $\Pi(p^2)$ by the following expression:

$$\bullet = \frac{i}{p^2 - m_0^2} \left(-i \Pi(p^2) \right) \frac{i}{p^2 - m_0^2}$$
 (7.47)

• We obtain:

$$\bullet \bullet \bullet \bullet = \frac{i}{p^2 - m_0^2} + \frac{i}{p^2 - m_0^2} \left(-i \Pi(p^2) \right) \frac{i}{p^2 - m_0^2} \\
+ \frac{i}{p^2 - m_0^2} \left(-i \Pi(p^2) \right) \frac{i}{p^2 - m_0^2} \left(-i \Pi(p^2) \right) \frac{i}{p^2 - m_0^2} + \dots \\
= \frac{i}{p^2 - m_0^2} \left(\frac{1}{1 - \frac{(-i \Pi(p^2))i}{p^2 - m_0^2}} \right) \\
= \frac{i}{p^2 - m_0^2 - \Pi(p^2)}$$
(7.48)

• We want to have matching poles as well as matching residues at $p^2 \approx m^2$:

$$\frac{i}{p^2 - m_0^2 - \Pi(p^2)} = \frac{iZ}{p^2 - m^2} + \underbrace{\int_{M_t^2}^{\infty} dM^2 \, \sigma(M)^2 \frac{i}{p^2 - M^2}}_{\text{no poles at } p^2 \approx m^2}$$
(7.49)

Matching poles:

$$p^2 - m_0^2 - \Pi(p^2) = 0$$
 at $p^2 = m^2$
 $\Rightarrow m^2 = m_0^2 + \Pi(m^2)$ (7.50)

Matching residues:

$$\frac{p^2 - m^2}{p^2 - m_0^2 - \Pi(p^2)} \longrightarrow Z \quad \text{as} \quad p^2 \to m^2$$
 (7.51)

If we Taylor expand $\Pi(p^2)$ around $p^2 = m^2$ we get:

$$\frac{p^2 - m_0^2 - \{\Pi(m^2) + \Pi'(m^2)(p^2 - m^2)\}}{p^2 - m^2} \longrightarrow Z^{-1}$$
 (7.52)

$$\Rightarrow 1 - \Pi'(m^2) \stackrel{!}{=} Z^{-1} \tag{7.53}$$

This (7.53) gives our wave function renormalization: $Z^{-1} = 1 - \Pi'(m^2)$

7.7 Feynman rules for scattering amplitudes

• We already know:

$$G(p_1,\ldots,p_n,k_1,\ldots,k_n) \sim \prod_j \frac{iZ}{p_j^2 - m^2} \prod_i \frac{iZ}{p_i^2 - m^2} \operatorname{out} \langle p_1,\ldots | k_1,\ldots \rangle_{\operatorname{in}} \quad (7.54)$$

$$G(p_1,\ldots,p_n,k_1,\ldots,k_n) = \bigcup_{k_n}^{k_1} \bigcap_{k_n}^{p_n} p_n$$

where "A" stands for "amputated diagram", i.e. contains only diagrams which, after cutting any internal line, will not fall apart in such a way that precisely one external line is gone. (By definition, no external lines (i.e. propagators) are present in "A".)

Note: "A" is not exactly the same as 1 PI:

For example: The following diagram is a legitimate part of "A":

• Furthermore we also know that:

$$\bullet \quad \sim \frac{iZ}{p^2 - m^2} \tag{7.57}$$

• Thus, we find that:

$$_{\text{out}}\langle p_1, \dots, p_n | k_1, \dots, k_m \rangle_{\text{in}} = \left(Z^{\frac{1}{2}} \right)^{n+m} \cdot \left\{ \begin{matrix} k_n & P_n \\ k_m & P_m \end{matrix} \right\}$$
 (7.58)

Or, equivalently we can formulate the *Feynman rules* for $i\mathcal{M}_{fi}$:

 $i\mathcal{M}_{fi}$ = Sum of all amputated, connected* diagrams without vacuum bubbles & without overall δ -function built from:

$$= \frac{i}{p^2 - m_0^2 + i\epsilon}; \qquad = -i\lambda; \qquad \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \text{ for each loop}$$

ternal particles.)

* connected means that diagrams like

$$(7.59)$$

are *not* part of $2 \rightarrow 4$ scattering amplitudes, which is obvious anyway since one of the incoming particles does not change its momentum in this particular contribution.

• *Note*: Now, we could calculate cross-sections at loop level. But we would have trouble dealing with divergent diagrams. To solve this, one could perform a "wick rotation" $p_0 \to i p_0$ & demand $p_E^2 < \Lambda^2$ for the euclidean momentum p_E . The procedure of removing this cutoff (i.e. let $\Lambda \to \infty$) is called *renormalization*. This will be easier to understand in QED.

8. The Electromagnetic Field

8.1 Gauge invariance

• Let us consider the Lagrangian for the complex scalar field again:

$$\mathcal{L} = \partial_{\mu}\phi \partial^{\mu}\phi^* - m^2\phi\phi^* \tag{8.1}$$

- We have already seen that it has a (global) U(1)-symmetry: $\phi(x) \to e^{i\alpha}\phi(x)$. In this context, global means that this phase α is the same for the "whole world".
- Following our "locality paradigm" we would like to promote this to a "local" or "gauge" symmetry:

$$\phi(x) \to e^{i\alpha(x)}\phi(x)$$
 (8.2)

• To figure out the transformation behaviour of \mathcal{L} we analyze how the derivative transforms:

$$\partial_{\mu}\phi \rightarrow \partial_{\mu}\left(e^{i\alpha}\phi\right) = e^{i\alpha}\left(\partial_{\mu}\phi + i(\partial_{\mu}\alpha)\phi\right)$$
 (8.3)

- This is *not* equal to $e^{i\alpha}\partial_{\mu}\phi$. Hence, unlike the global case, the phase does not drop out and \mathcal{L} is not invariant.
- We observe that $\partial_{\mu}\phi$ does not transform just with a phase.
- One introduces a "gauge connection" $A_{\mu}(x)$ and defines the "covariant derivative":

$$D_{\mu} \equiv \partial_{\mu} + iA_{\mu} \tag{8.4}$$

• It transforms as follows:

$$D_{\mu}\phi \to D'_{\mu}\phi' = \left(\partial_{\mu} + iA'_{\mu}\right)e^{i\alpha}\phi = e^{i\alpha}\left(\partial_{\mu}\phi + i(\partial_{\mu}\alpha)\phi + iA'_{\mu}\phi\right)$$

$$\stackrel{!}{=} e^{i\alpha}D_{\mu}\phi = e^{i\alpha}\left(\partial_{\mu}\phi + iA_{\mu}\phi\right)$$
(8.5)

• Therefore, we must demand:

$$A'_{\mu} = A_{\mu} - \partial_{\mu}\alpha \tag{8.6}$$

- Now, $D_{\mu}\phi$ transforms with an overall phase factor and \mathcal{L} is invariant.
- Comment:
 - A conventional derivative is defined as:

$$n^{\mu}\partial_{\mu}\phi = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\phi(x + \epsilon n) - \phi(x) \right) \tag{8.7}$$

- In the presence of (local) gauge symmetry this does not make sense since phases of $\phi(x)$ and $\phi(x + \epsilon n)$ are independent. Consequently, we cannot "compare" these two quantities.
- Hence, we want to "parallel transport" ϕ :

$$\phi(x) \to \underbrace{U(y, x)}_{\in U(1)} \phi(x) \tag{8.8}$$

such that:

$$U'(y,x) = e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)}$$
(8.9)

Only then $U(y, x)\phi(x)$ will transform just like a field at y.

- Now we can compare $\phi(x)$ and $\phi(y)$ sensibly:

$$n^{\mu}D_{\mu}\phi = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\phi(x + \epsilon n) - U(x + \epsilon n, x)\phi(x) \right)$$
 (8.10)

– If we assume *U* to be smooth and U(x, x) = 1, we have

$$U(x + \epsilon n, x) \equiv 1 - i\epsilon n^{\mu} A_{\mu}(x) + \dots, \qquad (8.11)$$

where A_{μ} has been *defined* as the linear coefficient of the Taylor expansion of U.

- This resulting D_{μ} and the transformation properties of A_{μ} agree with our earlier definition.
- Furthermore, given some $A_{\mu}(x)$, we can define:

$$U(y,x) = \exp\left(i\int_{x}^{y} A_{\mu} dx^{\mu}\right)$$
 (8.12)

with, for example, the straight line connecting x and y chosen as the integration path (however, this is not a unique choice). The argument of the exponential function is called "Wilson line" (or the exponential function itself). The name "gauge connection" should now be more clear.

- Having introduced A_{μ} , we *must* specify its dynamics, i.e. a gauge invariant action.
- To do so, we observe that the differential operator D_{μ} transforms as:

$$D_{\mu} \stackrel{\alpha}{\to} D_{\mu}' = e^{i\alpha} D_{\mu} e^{-i\alpha} \tag{8.13}$$

We may check the equality of the two differential operators explicitly:

$$e^{i\alpha}(\partial_{\mu} + iA_{\mu})e^{-i\alpha} = e^{i\alpha}(\partial_{\mu}e^{-i\alpha}) + \partial_{\mu} + iA_{\mu} = \partial_{\mu} + iA'_{\mu}$$
(8.14)

• Hence, we see how the commutator transforms:

$$[D_{\mu}, D_{\nu}] \longrightarrow e^{i\alpha} [D_{\mu}, D_{\nu}] e^{-i\alpha}$$
 (8.15)

• At the same time, we find:

$$[D_{\mu}, D_{\nu}] = \partial_{\mu}\partial_{\nu} + \partial_{\mu}iA_{\nu} + iA_{\mu}\partial_{\nu} - A_{\mu}A_{\nu} - \underbrace{\{\mu \leftrightarrow \nu\}}_{(*)}$$

$$= i(\partial_{\mu}A_{\nu}) + iA_{\nu}\partial_{\mu} + iA_{\mu}\partial_{\nu} - \{\mu \leftrightarrow \nu\}$$

$$= iF_{\mu\nu}$$
(8.16)

with:

$$F_{\mu\nu} \equiv \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \tag{8.17}$$

and: (*) = same terms as before with μ and ν exchanged. Some of the terms cancel due to their counterpart in (*).

• Thus, contrary to its appearance, $[D_{\mu}, D_{\nu}]$ is *not* a differential operator (since all "derivatives acting right" have dropped out).

$$\Rightarrow \left(\left[D_{\mu}, D_{\nu} \right] \to e^{i\alpha} \left[D_{\mu}, D_{\nu} \right] e^{-i\alpha} \quad \text{implies} \quad F_{\mu\nu} \to F_{\mu\nu} \right) \tag{8.18}$$

• Therefore, $F_{\mu\nu}$ is gauge invariant and we can propose the *scalar QED Lag-rangian*

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \left| D_{\mu} \phi \right|^2 - m^2 |\phi|^2, \tag{8.19}$$

which is invariant under $\phi \to \phi e^{i\alpha(x)}$ and $A_{\mu} \to A_{\mu} - \partial_{\mu}\alpha$.

• It is also Poincaré-invariant:

Shift
$$d^{\mu}: A'_{\mu}(x) = A_{\mu}(x - d)$$

Lorentz-transformation $\Lambda: A'_{\mu}(x) = \Lambda_{\mu}^{\nu} A_{\nu}(\Lambda^{-1}x)$ (8.20)

We saw earlier that the vector field $\partial_{\mu} \varphi$ constructed from φ transforms in this way. Here, we declare A_{μ} to be a *fundamental vector field* and to have this property by definition.

• A possibly more familiar form of the Lagrangian is obtained by the field redefinition $A_{\mu} \rightarrow e A_{\mu}$ ($A_{\mu} \equiv e B_{\mu}$, let us now rename *B* into *A*):

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_{\mu}\phi|^2 - m^2|\phi|^2; \quad D_{\mu} = \partial_{\mu} + ieA_{\mu}$$
 (8.21)

In this form, it is more apparent that *e* is a coupling constant.

• Advanced comment:

Using the language of differential forms:

A is a 1-form:
$$A = A_{\mu} dx^{\mu}$$

 α is a 0-form: (scalar) (8.22)
gauge-transformation: $A \to A + d\alpha$; $F = dA$ is 2-form

the action is:

$$\int \mathcal{L} \sim \int F \wedge \star F \qquad \text{with "} \star \text{" the } Hodge \ operator$$
 (8.23)

8.2 Gupta-Bleuler Quantization

We focus on the free theory first,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},\tag{8.24}$$

and attempt to quantize the system in the familiar way (treat all four components independently):

$$\pi^{\mu} = \frac{\partial \mathcal{L}}{\partial \dot{A}_{\mu}} = \frac{\partial}{\partial (\partial_{0} A_{\mu})} \left(-\frac{1}{4} F_{\rho \nu} F_{\sigma \tau} \eta^{\rho \sigma} \eta^{\nu \tau} \right)$$

$$= -\frac{1}{2} F_{\rho \nu} \eta^{\rho \sigma} \eta^{\nu \tau} \frac{\partial}{\partial (\partial_{0} A_{\mu})} \left(\partial_{\sigma} A_{\tau} - \partial_{\tau} A_{\sigma} \right)$$

$$= -\frac{1}{2} F_{\rho \nu} \left(\eta^{\rho 0} \eta^{\nu \mu} - \eta^{\rho \mu} \eta^{\nu 0} \right) = F^{\mu 0}$$
(8.25)

- In particular we find $\pi^0 = 0$ as $F^{\mu\nu}$ is anti-symmetric. This is a problem. (For a deeper understanding of this problem see e.g. Dirac's famous "Lectures on Quantum Mechanics" or Kugo, "Eichtheorie" on the quantization of systems with constraints.)
- To overcome this problem we proceed 'naively' by fixing the gauge.
- We choose Lorentz (or invariant) gauge and demand

$$\partial A \equiv \partial_{\mu} A^{\mu} = 0. \tag{8.26}$$

We now use the new Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\lambda}{2}(\partial A)^2. \tag{8.27}$$

This does not change the equation of motion as the above Lagrangian contains only second order terms in ∂A . Thus, its variation will still be first order in ∂A . Under Lorentz gauge all terms of this form vanish. Accordingly, the equations of motion remain unchanged.

• Setting $\lambda = 1$ for simplicity we obtain for the Lagrangian:

$$\mathcal{L} = -\frac{1}{4} \left(2(\partial_{\mu} A_{\nu})(\partial^{\mu} A^{\nu}) - 2(\partial_{\mu} A_{\nu})(\partial^{\nu} A^{\mu}) \right) - \frac{1}{2}(\partial A)^{2}$$
integrate by parts and drop total derivatives
$$= -\frac{1}{2} (\partial_{\mu} A_{\nu})(\partial^{\mu} A^{\nu}) + \frac{1}{2} (\partial A)^{2} - \frac{1}{2} (\partial A)^{2}$$

$$= \frac{1}{2} (\partial_{\mu} A_{\nu})(\partial^{\mu} A_{\rho})(-\eta^{\nu\rho})$$
(8.28)

This is the result for four real scalars. Merely, the overall sign of A_0 is wrong.

• By explicit calculation or by analogy to the real scalar case, we see that

$$\pi^{\mu} = -\dot{A}^{\mu} = (-\eta^{\mu\nu})\dot{A}_{\nu},\tag{8.29}$$

which gives the wrong sign for A_0 . Accordingly, we need to take care of this sign flip in our quantization.

• We quantize by demanding:

$$[A, A] = [\pi, \pi] = 0$$

$$[A_{\mu}(\vec{x}), \pi^{\nu}(\vec{y})] = i\eta_{\mu}^{\nu} \delta^{(3)}(\vec{x} - \vec{y})$$

$$= i\delta_{\mu}^{\nu} \delta^{(3)}(\vec{x} - \vec{y})$$
(8.30)

• If we now Fourier transform, introduce a and a^{\dagger} as linear combinations of the transformed fields, determine their commutation relations and go to Heisenberg fields everything works as before. Doing this one obtains the result

$$A_{\mu}(x) = \int d\vec{k} \left(a_{\vec{k},\mu} e^{-ikx} + a_{\vec{k},\mu}^{\dagger} e^{ikx} \right)$$

$$[a,a] = \left[a^{\dagger}, a^{\dagger} \right] = 0$$

$$\left[a_{\vec{k},\mu}, a_{\vec{k}',\nu}^{\dagger} \right] = -\eta_{\mu\nu} 2k^{0} (2\pi)^{3} \delta^{(3)} (\vec{k} - \vec{k}'),$$
(8.31)

where it was used that $k^0 = |\vec{k}|$ as m = 0 for a photon. Note that again the sign of a_0, a_0^{\dagger} is wrong.

- The logical next step is to define $|0\rangle$ as the state annihilated by all $a_{\vec{k},\mu} \forall \vec{k}, \mu$ and to construct the Fock space basis by applying all types of a^{\dagger} . However, this leads to two problems:
 - Problem 1: The wrong sign of the a_0 , a_0^{\dagger} commutator renders our Hilbert space metric non-positive-definite. Consider a harmonic oscillator and an algebra with the only non-trivial commutation relation $[a, a^{\dagger}] = -1$. Focusing on the first excited state

$$||a^{\dagger}|0\rangle||^{2} = \langle 0|aa^{\dagger}|0\rangle = \langle 0|(a^{\dagger}a - 1)|0\rangle = -1.$$
 (8.32)

We get a result that is physically unacceptable in QM. Switching the roles of a and a^{\dagger} does not allow us to solve this problem, since there is a relative sign change between $[a_0, a_0^{\dagger}]$ and $[a_i, a_i^{\dagger}]$ which is enforced by Lorentz-symmetry.

- Problem 2: We cannot impose $\partial A = 0$ at operator level:

$$[A_0, \partial A] = [A_0, \partial_0 A^0 + \partial_i A^i] = [A_0, \dot{A}_0] \neq 0,$$
 (8.33)

where the commutator with the spatial derivatives vanishes due to $[A_{\mu}(\vec{x}), A_{\nu}(\vec{y})] = 0$ for $\mu \neq \nu$. Obviously this is a contradiction to our gauge condition.

 To avoid these problems in line with Gupta and Bleuler we construct the following solution: Let *F* be the Fock space constructed above. We now define *F*_{physical} ⊂ *F* by

$$\partial A^a |\Psi\rangle = 0 \quad \Leftrightarrow \quad |\Psi\rangle \in F_{\text{physical}}$$
 (8.34)

where the superscript a marks the annihilation part of A. This definition implies

$$\langle \Psi | \partial A | \Psi \rangle = \langle \Psi | \partial A^{a} + \partial A^{c} | \Psi \rangle$$

$$= \langle \Psi | \partial A^{a} + (\partial A^{a})^{\dagger} | \Psi \rangle = 0 \quad \text{if} \quad | \Psi \rangle \in F_{\text{physical}}$$
(8.35)

where $(\partial A)^{\dagger}$ vanishes acting to the left. Thus, our gauge condition is satisfied for all physical states.

• It will turn out that F_{physical} is positive-semi-definite, i.e. contains no negative-norm-states, but still zero-norm states. We define

$$F_0 \equiv \{ |\Psi\rangle \in F_{\text{physical}} : \| |\Psi\rangle\| = 0 \}$$
 (8.36)

as the zero-norm subspace.

• With this we can define our Hilbert space \mathcal{H} as the space of equivalence classes

$$\mathcal{H} = F_{\text{physical}}/F_0 \tag{8.37}$$

with the equivalence relation \sim

$$|\Psi\rangle \sim |\Psi\rangle' \quad \Leftrightarrow \quad |||\Psi\rangle - |\Psi\rangle'|| = 0.$$
 (8.38)

- For a more detailed account of this it is convenient to treat polarization. Thus, a small interlude on polarization follows:
 - A general 1-photon state is a linear combination of states $a_{\vec{k},\mu}^{\dagger} | 0 \rangle$ with $\mu = 0, \dots, 3$, given by

$$-\varepsilon^{\mu}(k)a_{\vec{k},\mu}^{\dagger}\left|0\right\rangle \tag{8.39}$$

where \vec{k} is fixed.

- Now there are four independent polarizations for any given k and many possible basis choices. It is convenient to demand covariant orthogonality:

$$\varepsilon_{\mu}^{(\lambda)}(k)\overline{(\varepsilon^{(\lambda')\mu}(k))} = \eta^{\lambda\lambda'}$$
 (8.40)

 In the problems it is shown that orthonormality implies completeness in the sense that

$$\sum_{\lambda\lambda'} \eta_{\lambda\lambda'} \varepsilon_{\mu}^{(\lambda)}(k) \overline{\left(\varepsilon_{\nu}^{(\lambda')}(k)\right)} = \eta_{\mu\nu} \tag{8.41}$$

– To make one concrete choice, we introduce some arbitrary but fixed unit vector $n = \{n^{\mu}\}$ with $n^2 = 1$, $n_0 > 0$ and demand

$$\varepsilon^{(0)} = n$$
 $\varepsilon^{(i)} \cdot n = 0$ $\varepsilon^{(1)} \cdot k = \varepsilon^{(2)} \cdot k = 0.$ (8.42)

 The consistency of this requirement is most easily seen by going to the coordinate system in which

$$n = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad k = |\vec{k}| \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{8.43}$$

Recall that $k^2 = m^2 = 0$. Thus, we effectively fix the ε 's.

- In this coordinate system all of the above conditions are met by

$$\varepsilon^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \varepsilon^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \varepsilon^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \varepsilon^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$
(8.44)

which is unambiguous up to rotations in the *x-y*-plane.

– As a different common choice, we can choose an n with $n^2 = 0$ (light-like) and demand

$$\varepsilon^{u} = n \qquad \varepsilon^{L} \sim k \qquad \varepsilon^{(1)}, \varepsilon^{(2)} \text{ orthogonal to } n, k$$
 (8.45)

which forms a basis but does not obey orthonormality.

- The previous orthonormality relation is replaced by

$$(\varepsilon^{u})^{2} = (\varepsilon^{L})^{2} = 0$$

$$\varepsilon^{u} \cdot \varepsilon^{(i)} = \varepsilon^{L} \cdot \varepsilon^{(i)} = 0$$

$$\varepsilon^{(i)} \cdot \varepsilon^{(j)} = -\delta_{ij}$$

$$\varepsilon^{u} \cdot \varepsilon^{L} = 1$$
(8.46)

where the second and third relation are the same as before.

- If we now go to a coordinate system such that

$$n = \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\0\\0\\-1 \end{pmatrix}, \quad k = |\vec{k}| \begin{pmatrix} 1\\0\\0\\1 \end{pmatrix}$$
 (8.47)

we get

$$\varepsilon^{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\0\\0\\-1 \end{pmatrix}, \quad \varepsilon^{L} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\0\\0\\1 \end{pmatrix}, \quad \varepsilon^{(1)} = \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \quad \varepsilon^{(2)} = \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}. \quad (8.48)$$

- Finally we can write

$$\varepsilon^{\mu}(k)a_{\vec{k},\mu}^{\dagger}|0\rangle = |\varepsilon,k\rangle \tag{8.49}$$

and with this

$$\langle \varepsilon', k' | \varepsilon, k \rangle = \overline{\varepsilon'^{\mu}(k')} \varepsilon^{\nu}(k) \langle 0 | a_{\vec{k}', \mu} a_{\vec{k}, \nu}^{\dagger} | 0 \rangle$$

$$= \overline{\varepsilon'^{\mu}(k')} \varepsilon^{\nu}(k) \left(-\eta_{\mu\nu} 2k^{0} (2\pi)^{3} \delta^{(3)} (\vec{k} - \vec{k}') \right)$$

$$= -(\varepsilon' \cdot \varepsilon) 2k^{0} (2\pi)^{3} \delta^{(3)} (\vec{k} - \vec{k}')$$
(8.50)

Thus, $(-\varepsilon' \cdot \varepsilon)$ measures the overlap of states, and $-\varepsilon^2$ measures the norm of a single state. With this we are now ready to continue our discussion.

• With the above we can now reformulate our physical-state condition $\partial A^a |\Psi\rangle = 0$ in terms of polarizations

$$\partial A^{a} | \varepsilon, q \rangle = 0 \qquad \Leftrightarrow \quad k_{\mu} a_{\vec{k}}^{\mu} | \varepsilon, q \rangle = 0$$

$$\Leftrightarrow \quad k_{\mu} a_{\vec{k}}^{\mu} a_{\vec{q}}^{\nu \dagger} \varepsilon_{\nu}(q) | 0 \rangle \quad \Leftrightarrow \quad k \cdot \varepsilon(k) = 0$$
(8.51)

- This condition is only violated for ε^{u} (hence "unphysical").
- Let us perform a linear transformation on the space of creation/annihilation operators, defining

$$\alpha_{\vec{k},(u,L,1,2)}^{\dagger} \equiv \varepsilon^{(u,L,1,2)\mu}(k) a_{\vec{k},\mu}^{\dagger}$$
 (8.52)

We think of *F* as built by applying the four α^{\dagger} to $|0\rangle$ (this is of course still the same *F*).

• Fact: F_{physical} is the subspace of F built by using only $\alpha_{\vec{k},(L,1,2)}^{\dagger}$. This is clearly true, since

$$(k \cdot a_{\vec{k}})$$
 (products of various $\varepsilon_{\vec{k}}^{(L,1,2)} \cdot a_{\vec{k}}^{\dagger}$) $|0\rangle = 0$ (8.53)

 \Rightarrow F_{physical} is spanned by vectors of the type

$$|\psi\rangle = \left(\text{products of }\alpha^{\dagger}_{(L,1,2)}\right)|0\rangle$$
 (8.54)

- Fact: $||\psi\rangle|| = 0$ if and only if at least one α_L^{\dagger} appears in this product (in this case also $|\psi\rangle \sim 0$).
- Fact: The presence of such zero-norm states does not affect observables.

$$\langle \psi' | O | \psi' \rangle = \langle \psi | O | \psi \rangle \text{ if } | \psi' \rangle = | \psi \rangle + (\cdots \alpha_L^{\dagger} \cdots) | 0 \rangle$$
 (8.55)

The relies on gauge-invariance of observables. We will only give an illustrative example:

$$H = : \int d^{3}x \left(\pi^{\mu} \dot{A}_{\mu} - \mathcal{L} \right) := \dots = \int d\tilde{k} k_{0} \left(-a_{\vec{k},\mu}^{\dagger} a_{\vec{k}}^{\mu} \right)$$

$$= \int d\tilde{k} k_{0} \left(\sum_{i=1}^{2} \alpha_{\vec{k},i}^{\dagger} \alpha_{\vec{k},i} - \underbrace{\left[\alpha_{\vec{k},u}^{\dagger} \alpha_{\vec{k},L} + \alpha_{\vec{k},L}^{\dagger} \alpha_{\vec{k},u} \right]}_{(1)} \right)$$

$$(8.56)$$

- (1) vanishes "inside" any physical state $\langle \psi | \cdots | \psi \rangle$. States with α_L^{\dagger} excitations do not contribute to $\langle H \rangle$
- Summary:
 - $|\psi\rangle \in F \Rightarrow |\psi\rangle = \sum \left(\alpha_{\vec{k},i}^{\dagger} \alpha_{\vec{p},u}^{\dagger} \alpha_{\vec{q},L}^{\dagger} \cdots \right) |0\rangle$
 - $|\psi\rangle$ ∈ F_{physical} ⇒ No terms involving α_u^{\dagger} are allowed.
 - $|\psi\rangle \in F_0 \Rightarrow$ Each term in the sum involves at least one α_L^{\dagger} factor (hence F_0 is a linear subspace).
 - $\mathcal{H} = F_{\text{physical}}/F_0 = F_{\text{physical}}/\sim$, where \sim is an equivalence relation. Two states are equivalent if they differ only by terms with α_L^{\dagger} .
- Note: The freedom of adding states from F_0 (recall ε_L is parallel to \vec{k}) corresponds to residual gauge freedom of the classical theory:

$$A_{\mu} \to A_{\mu} + \partial_{\mu} \chi$$
Fourierspace : $\tilde{A}_{\mu} \to \tilde{A}\mu + ik_{\mu}\tilde{\chi}$
corresponds to ε_{L} -polarizations (8.57)

Residual means, that the new field still obeys $k^{\mu}\tilde{A}_{\mu} = 0$.

• F, F_{physical} and \mathcal{H} were defined abstractly before a specific choice for n. Hence, they do not depend on n (only our concrete realization is constructed using n).

8.3 Photon Propagator

In the left A field of the photon propagator only a, in the right A field only a^{\dagger} is relevant. We thus get

$$\langle 0 | A_{\mu}(x) A_{\nu}(y) | 0 \rangle = \langle 0 | \int d\tilde{k} d\tilde{k'} e^{-ikx + ik'y} a_{\vec{k},\mu} a_{\vec{k'},\nu}^{\dagger} | 0 \rangle$$

$$= -\eta_{\mu\nu} \int d\tilde{k} e^{-ik(x-y)}$$
(8.58)

$$\langle 0| TA_{\mu}(x)A_{\nu}(y) |0\rangle = \Theta\left(x^{0} - y^{0}\right) \langle 0| A_{\mu}(x)A_{\nu}(y) |0\rangle + \{x \leftrightarrow y\}$$

$$= \Theta\left(x^{0} - y^{0}\right) \left(-\eta_{\mu\nu} \int d\tilde{k}e^{-ik(x-y)}\right) + \{x \leftrightarrow y\}$$

$$= -\eta_{\mu\nu} \left(\Theta\left(x^{0} - y^{0}\right) \langle 0| \varphi(x)\varphi(y) |0\rangle + \{x \leftrightarrow y\}\right)$$

$$= -\eta_{\mu\nu} \langle 0| T\varphi(x)\varphi(y) |0\rangle = -\eta_{\mu\nu} D_{F}(x - y, m^{2} = 0)$$
(8.59)

• For a more general gauge $(\lambda \neq 1)$, we obtain:

$$\langle 0|TA_{\mu}(x)A_{\nu}(y)|0\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}(-i)\left(\frac{\eta_{\mu\nu}}{k^2 + i\varepsilon} + \frac{1 - \lambda}{\lambda}\frac{k_{\mu}k_{\nu}}{(k^2 + i\varepsilon)^2}\right)e^{-ik(x-y)}$$
(8.60)

This will be derived with the path integral approach.

8.4 Feynman rules for scalar QED

• As a warm-up, consider first a model with *N* different real scalars:

$$\mathcal{L} = \sum_{i=1}^{N} \frac{1}{2} \left(\partial \varphi^{i} \right)^{2} - m^{2} \left(\varphi^{i} \right)^{2} - \frac{\lambda}{8} \underbrace{\left(\sum_{i=1}^{N} \left(\varphi^{i} \right)^{2} \right)^{2}}$$
(8.61)

This type of interaction respects the O(n)—symmetry of the free part.

• The propagator is:

$$\left\langle T\varphi_1^i\varphi_2^j\right\rangle = \overline{\varphi_1^i\varphi_2^j} = \delta^{ij}D_F(x_1 - x_2) = \mathbf{i}$$
 (8.62)

• Vertex: We consider the simplest non trivial four-point function

$$\left\langle T\varphi_1^i \varphi_2^j \varphi_3^k \varphi_4^l \int d^4 x \left(-\frac{i\lambda}{8} \right) \left(\delta_{mn} \varphi_x^m \varphi_x^n \right) \left(\delta_{pq} \varphi_x^p \varphi_x^q \right) \right\rangle \tag{8.63}$$

We only consider the fully connected part



• This means that φ_1 , φ_2 , φ_3 and φ_4 must each be contracted with one of the φ_x . First, contract as follows

$$\varphi_1^i \varphi_x^m \varphi_2^j \varphi_x^n \varphi_3^k \varphi_x^p \varphi_4^l \varphi_x^q \Rightarrow -\frac{i\lambda}{8} \delta^{ij} \delta^{kl} D_F(x_1 - x) D_F(x_2 - x) D_F(x_3 - x) D_F(x_4 - x)$$
(8.65)

We see that from this we obtain a contribution of

$$-\frac{i\lambda}{8}\delta^{ij}\delta^{kl} \tag{8.66}$$

to the vertex.

The exact same contribution arises by exchanging $\varphi_x^m \leftrightarrow \varphi_x^n$ or $\varphi_x^p \leftrightarrow \varphi_x^q$ or $\varphi_x^m \varphi_x^n \leftrightarrow \varphi_x^p \varphi_x^q \Rightarrow$ factor 1/8 disappears.

• One can think of this contribution as arising from "pairing up" i with j and k with l. There are two more such pairings: (ik)(jl) and (il)(jk). Hence:

$$\begin{vmatrix}
i & 1 \\
 & = -i\lambda \left(\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} \right) \\
i & k
\end{vmatrix} (8.67)$$

(As a check note that $8 \cdot 3 = 4!$, so we did not forget anything.)

 As a former warmup consider a single complex scalar: Recall:

$$\phi(x) = \int d\tilde{k} \left(a_{\vec{k}}^{\dagger} e^{ikx} + b_{\vec{k}} e^{-ikx} \right)$$

$$\Rightarrow \phi_x \phi_y = 0, \quad \phi_x^{\dagger} \phi_y = \phi_x \phi_y^{\dagger} = D_F(x - y)$$
(8.68)

• Since ϕ and ϕ^{\dagger} are truely different, we can assign a direction to the corresponding line in the Feynman rule:

$$\phi^{\dagger}(x) \longrightarrow \phi(y) = D_F(x - y) \tag{8.69}$$

Here the arrow gives the direction of a b-particle.

After these preliminaries we simply state the Feynman rules of scalar QED. We will then give a partial derivation, which will be completed in the tutorials.

Scalar QED:

$$= \frac{i}{k^2 - m^2 + i\varepsilon}; \quad \mu \sim \nu = \frac{-i\eta^{\mu\nu}}{k^2 + i\varepsilon}$$
 (8.70)

$$\rightarrow p \stackrel{k}{\longrightarrow} \stackrel{\mu}{=} -ie \left(p_{\mu} + p'_{\mu} \right) \quad \text{with } p' = p - k$$

$$\searrow p'$$
(8.71)

$$\mu \nwarrow k' \qquad \nu \nearrow k$$

$$\rightarrow p \qquad \rightarrow p' = 2ie^2 \eta^{\mu\nu} \quad \text{with } p' = p - k - k' \qquad (8.72)$$

$$=Z_{\phi}^{1/2}$$
 external scalar particle (8.73)

$$= Z_A^{1/2} \epsilon_{\mu}(k)$$
 incoming photon (ϵ^* for outgoing photon)

(8.74)

- In principle, a proper derivation of this requires to go through the whole procedure of the last sections (Green functions, LSZ-formula, etc.) with our new theory replacing the real scalar $\lambda \varphi^4$ model.
- As an example, consider the process: $\gamma + \phi^{\dagger} \rightarrow \phi^{\dagger}$ i.e.



Note: As this diagram alone does not satisfy the energy-momentum conservation this can be seen as part of the following process:



Let us calculate the amplitude for the above process (8.75), which was also considered in the problems:

$$\langle 0 | b_{\vec{p}'} \left(i \int d^4 x \, \mathcal{L}_{int} \right) b_{\vec{p}}^{\dagger} \epsilon^{\mu}(k) \, a_{\vec{k},\mu}^{\dagger} \, | 0 \rangle = i \mathcal{M}_{fi}(2\pi)^4 \delta^4(p' - k - p) \tag{8.77}$$

Here we just need to consider the cubic part of the interaction Lagrangian \mathcal{L}_{int} , meaning the part of $|D_{\mu}\phi|^2(\supset |\partial_{\mu}\phi|^2)$ which is cubic in the fields A^{μ} , ϕ^{\dagger} & ϕ :

$$-ieA^{\nu}\phi^{\dagger}\partial_{\nu}\phi + \text{ a second similar term}$$
 (8.78)

We obtain the following contributions:

$$A^{\nu} \to \int d\tilde{q} \, a_{\vec{q}}^{\nu} e^{-iqx} \text{ acting on } a_{\vec{k},\mu}^{\dagger} |0\rangle$$

$$\Rightarrow a_{\vec{q}}^{\nu} a_{\vec{k},\mu}^{\dagger} = -\eta_{\mu}^{\nu} (2\pi)^{3} 2k_{0} \, \delta^{3}(\vec{k} - \vec{q})$$

$$\partial_{\nu} \to -ip_{\nu}$$

$$\phi \to \int d\tilde{q}' \, b_{\vec{q}'} e^{-iq'x} \text{ acting on } b_{\vec{p}}^{\dagger} |0\rangle$$

$$(8.79)$$

Collecting everything we find:

$$i\mathcal{M}_{fi} = -iep^{\mu} \epsilon_{\mu}(k) \underbrace{-iep'^{\mu} \epsilon_{\mu}(k)}_{\text{from the second term in (8.78)}} \underbrace{-ie(p'^{\mu} + p^{\mu})}_{\text{incoming photon}} \epsilon_{\mu}(k)$$
(8.80)

• *Important comment:*

Since the interactions involve A, the canonical momentum of A receives a contribution from \mathcal{L}_{int} . Thus, $\mathcal{H}_{int} = -\mathcal{L}_{int}$ is not true any more and since \mathcal{H}_{int} is the crucial quantity in perturbation theory, our derivation above is not correct. However, at the same time, it is too naive to assume that ∂_{μ} commutes with contractions. For example:

$$\left\langle \mathrm{T}\,\partial_{\mu}\varphi_{x}\partial_{\nu}\varphi_{y}^{\dagger}\right\rangle = \frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}}\left\langle \mathrm{T}\varphi_{x}\varphi_{y}\right\rangle - i\eta_{\mu0}\eta_{\nu0}\,\delta^{4}(x-y) \tag{8.81}$$

This effect also corrects the Feynman rules precisely compensating the "error" we made by assuming $\mathcal{H}_{int} = -\mathcal{L}_{int}$. In fact, this *had* to be the case to ensure that the final result is Poincaré-invariant. In summary, our naive analysis gave the correct result. For details see Itzykson/Zuber; Section: "Scalar Electrodynamics".

9. Spinors

9.1 Fields and representations

We already know three types of fields with different transformation properties:

$$\varphi(x) \to \varphi(\Lambda^{-1}x)$$

$$A^{\mu}(x) \to \Lambda^{\mu}_{\ \nu} A^{\nu}(\Lambda^{-1}x)$$

$$F^{\mu\nu}(x) \to \Lambda^{\mu}_{\ \rho} \Lambda^{\nu}_{\ \sigma} F^{\rho\sigma}(\Lambda^{-1}x)$$

$$(9.1)$$

with the last one not being elementary. In this, we could go on to higher tensors.

• Now we want to think about this in a more abstract way: Let us define our field as a map

$$\mathbb{R}^4 \longrightarrow V; \quad x \mapsto \{\phi^i(x)\}$$
 (9.2)

where V denotes a vector space.

- It can be observed that a Lorentz-transformation acts in two ways:
 - 1) On the argument (or on \mathbb{R}^4): it acts always in the same way.
 - 2) On the field value (or on V): it acts differently from field to field.
- \Rightarrow The field is characterized by a vector space V and a representation of SO(1,3) on V.
 - Reminder:

For any group G a representation R is a map $G \xrightarrow{R} GL(V)$ (general linear transformations on V) such that:

$$R(1) = 1$$

$$R(gh) = R(g)R(h)$$
(9.3)

- Our examples above have:
 - scalar: $V = \mathbb{R}$ or \mathbb{C} ; $R(\Lambda) = 1$ (trivial transformation)
 - vector: $V = \mathbb{R}^4$; $R(\Lambda) = \Lambda$ (fundamental representation)
 - tensor: $V = \mathbb{R}^4 \otimes \mathbb{R}^4$; $R(\Lambda) = \Lambda \otimes \Lambda$ (antisymmetric tensor representation)
- $\Rightarrow F^{\mu\nu}$ or, more correctly, $F^{\mu\nu}\hat{e}_{\mu}\otimes\hat{e}_{\nu}$ is in the antisymmetric subspace of $\mathbb{R}^4\otimes\mathbb{R}^4$.
 - *Note:* If we want to fit the tensor (e.g. our $F^{\mu\nu}$) into the general $\{\phi^i\}$ -notation, then the index "i" runs over all pairs of distinct indices $\mu \& \nu$: $\{\mu\nu\} = \{i\}$.
- \Rightarrow Our representation has $(4 \times 4 4) \div 2 = 6$ dimensions.

$$SO(1,3) \supset SO(3); \quad "6 = 3 + 3"$$

9.2 Remarks on Lie Groups & Lie Algebras

- Lie groups are groups which are also manifolds, such that the group operation is a diffeomorphism. (If manifolds are not yet known, the reader might think of smooth subspaces of \mathbb{R}^4 and the group operation being differentiable.)
- The prime examples are:
 - O(n) orthogonal
 - U(n) unitary
 - Sp(n) symplectic
- With a Lie group always comes a Lie algebra Lie(G). A Lie algebra is a vector space \mathfrak{g} with a bilinear, antisymmetric map: $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(a, b) \mapsto [a, b]$ satisfying the Jacobi-identity:

$$[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$$
 (9.4)

- We will only need *matrix groups & Lie algebras*. In this case, the relation between them can be understood elementary: (Here we will not give any proofs.)
 - The map "exp" is a diffeomorphism of a small neighborhood of O & 1 in $M_n(\mathbb{R})$ and $Gl(n,\mathbb{R})$: $(M_n(\mathbb{R})$ denoting all $n \times n$ matrices)

$$\exp(a) = g$$
 (for *a* being near *O*, *g* is near 1)
 $\exp(0) = 1$ (9.5)

- *Lie*(*G*) is the linear subspace of M_n(ℝ) generated by $\exp^{-1}(O_1)$, where O_1 is a neighborhood of $1 \in G \subset M_n(\mathbb{R})$ (cf. Figure 6)
- − *In other words*: $\exp(a) = g$ with $a \in Lie(G)$, $g \in G$ maps (at least) a small neighborhood/patch of Lie(G) near O to a small patch of G near $\mathbb{1}$.
- *Example*: G = SO(3), $Lie(G) = \{antisymmetric <math>3 \times 3 \text{ matrices} \}$ *Indeed*: If R = exp(T) then

$$RR^{T} = \exp(T) \exp(T)^{T} = \exp(T) \exp(-T) = 1$$
 (9.6)

since *T* is antisymmetric (the S of SO is not visible at the Lie-algebra level).

• It is illuminating to see in general that, if $a,b \in Lie(g)$, then $\exp [a,b] \in G$, where $[\cdot,\cdot]$ now is a simple commutator: Let $A = \exp(\epsilon a)$; $B = \exp(\epsilon b)$. (ϵ denotes a small number)

Clearly, $ABA^{-1}B^{-1} = C \in G$. Therefore we get:

$$C = (\mathbb{1} + \epsilon a + \frac{\epsilon^2}{2}a^2)(\mathbb{1} + \epsilon b + \frac{\epsilon^2}{2}b^2)(\mathbb{1} - \epsilon a + \frac{\epsilon^2}{2}a^2)(\mathbb{1} - \epsilon b + \frac{\epsilon^2}{2}b^2) + \mathcal{O}(\epsilon^3)$$

$$= \mathbb{1} + \epsilon^2 [a, b] + \mathcal{O}(\epsilon^3)$$

$$\Rightarrow C^{1/\epsilon^2} = \exp[a, b]$$
(9.7)

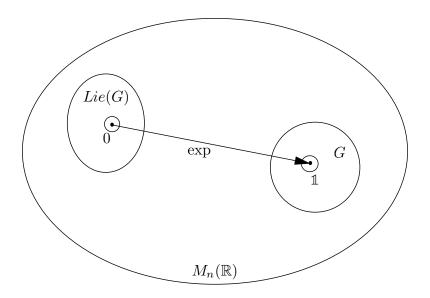


Figure 6: Illustration of group G and its Lie group on a manifold

• In analogy to groups, we also have the concept of representations of Lie-Algebras:

$$Lie(G) \xrightarrow{R} M(n)$$

$$a \mapsto R(a)$$
(9.8)

with: R(0) = 0, R([a, b]) = R(a)R(b) - R(b)R(a) = [R(a), R(b)]

• Crucial fact:

Given some representation R of a Lie-algebra Lie(G), we can always construct an associated representation of G (which we will call R by abuse of notation), such that:

$$R(A) = \exp(R(a)) \tag{9.9}$$

if $A = \exp(a)$.

• *Sketch of proof:*

Take the above definition of the group representation *R*

$$R(A) \equiv \exp\left(R\left(\exp^{-1}(A)\right)\right)$$
 (9.10) $(A \in G \text{ and always "near" 1})$

One has to show that *R* defined as above is really a representation of *G*:

$$R(A) R(B) = R(A \cdot B) \tag{9.11}$$

Given

$$A \equiv e^a$$
 ; $B \equiv e^b$; $C \equiv e^c \equiv A \cdot B$ (9.12)

We need to show:

$$e^{R(a)} e^{R(b)} = e^{R(c)}$$
 (9.13)

We already know:

$$e^a e^b = e^c (9.14)$$

and

$$e^{a} e^{b} = e^{Z(a,b)}$$

$$\Rightarrow c = Z(a,b)$$
(9.15)

with
$$Z(a,b) = a + b + \frac{1}{2}[a,b] + \frac{1}{12}[a,[a,b]] - \frac{1}{12}[b,[a,b]] + \dots$$
 (9.16) (Baker-Campell-Hausdorff)

This implies:

$$e^{R(a)} e^{R(b)} = e^{Z(R(a),R(b))} = e^{R(Z(a,b))} = e^{R(c)}$$
(9.17)

9.3 The spinor representation of SO(1,3)

• Let us first understand $Lie(SO(1,3)) = \mathfrak{so}(1,3)$

$$\Lambda = e^{iT} \longrightarrow \Lambda_{\mu}^{\nu} = \delta_{\mu}^{\nu} + i T_{\mu}^{\nu} \tag{9.18}$$

Note: The *i* in the exponent is a conventional definition by physicists. T is "small".

• Recall

$$\Lambda_{\mu}^{\nu} \Lambda_{\rho}^{\sigma} \eta_{\nu\sigma} = \eta_{\mu\rho} \tag{9.19}$$

and see what it implies for *T*:

$$\left(\delta_{\mu}^{\nu} + i T_{\mu}^{\nu}\right) \left(\delta_{\rho}^{\sigma} + i T_{\rho}^{\sigma}\right) \eta_{\nu\sigma} = \eta_{\mu\rho} + \mathcal{O}\left(T^{2}\right)
\Rightarrow T_{\mu\rho} + T_{\rho\mu} = 0$$
(9.20)

After lowering the second index SO(1,3) generators are antisymmetric.

• We need a canonical basis: Let $\left(M_{\rho\sigma}\right)_{\mu}^{\ \nu}$ be our canonical basis

$$\Rightarrow T_{\mu}^{\nu} = t^{\rho\sigma} \left(\mathbf{M}_{\rho\sigma} \right)_{\mu}^{\nu} \tag{9.21}$$

Both *t* and M are antisymmetric which implies six linear independent elements.

• Problem: Explicitly define the basis matrices $\{M_{\mu\nu}\}$ such that each $M_{\mu\nu}$ generates rotations in the μ - ν -plane and

$$[\mathbf{M}_{\mu\nu}, \mathbf{M}_{\rho\sigma}] = i \left(\eta_{\nu\rho} \mathbf{M}_{\mu\sigma} - \eta_{\mu\rho} \mathbf{M}_{\nu\sigma} - \eta_{\nu\sigma} \mathbf{M}_{\mu\rho} + \eta_{\mu\sigma} \mathbf{M}_{\nu\rho} \right). \tag{9.22}$$

Any $\Lambda \in SO^+(1,3)$ can be written as $\Lambda = \exp(it^{\mu\nu}M_{\mu\nu})$. The proof is omitted.

• To define the spinor representation, we first introduce the **Clifford algebra** which is generated by 1 and four elements γ^{μ} which satisfy

$$\{\gamma^{\mu}, \gamma^{\nu}\} \equiv \gamma^{\mu} \gamma^{\nu} + \gamma^{\nu} \gamma^{\mu} = 2\eta^{\mu\nu} \mathbb{1}$$
(9.23)

where \cdot , \cdot is the anti-commutator.

- Thus the Clifford algebra is the vector space generated by $\mathbb{1}$, γ^0 , γ^1 , γ^2 , γ^3 , $\gamma^0\gamma^1$, $\gamma^1\gamma^2$, $\gamma^0\gamma^1\gamma^2$,... with the relation $\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}$ imposed.
- We will see that this algebra is finite-dimensional.
- Much more could be done at this abstract level.
 Nevertheless we want to use an explicit representation

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \overline{\sigma}^{\mu} & 0 \end{pmatrix}, \tag{9.24}$$

whereby every entry denotes a 2×2 matrix and

$$\sigma^{\mu} = (\sigma^{0}, \sigma^{i}) = \left\{ \mathbb{1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

$$\overline{\sigma}^{\mu} = (\sigma^{0}, -\sigma^{i}). \tag{9.25}$$

Furthermore we will use that

$$\gamma_{\mu} = \eta_{\mu\nu} \gamma^{\nu}. \tag{9.26}$$

 We have to check whether those 4 × 4 matrices represent the Clifford algebra indeed:

$$\left\{\gamma^{\mu}, \gamma^{\nu}\right\} = \left\{ \begin{pmatrix} 0 & \sigma^{\mu} \\ \overline{\sigma}^{\mu} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^{\nu} \\ \overline{\sigma}^{\nu} & 0 \end{pmatrix} \right\} = \begin{pmatrix} \sigma^{\mu} \overline{\sigma}^{\nu} + \sigma^{\nu} \overline{\sigma}^{\mu} & 0 \\ 0 & \overline{\sigma}^{\mu} \sigma^{\nu} + \overline{\sigma}^{\nu} \sigma^{\mu} \end{pmatrix} \tag{9.27}$$

Analyze cases separately: $\mu, \nu = 0, 0 / 0, i / i, j$ and also use $\{\sigma^i, \sigma^j\} = 2\delta^{i,j} \mathbb{1}$

- *Problem:* Show that $\mathcal{M}_{\mu\nu} \equiv \frac{i}{4} [\gamma_{\mu}, \gamma_{\nu}]$ satisfies the same commutator relations as the $M_{\mu\nu}$ introduced earlier.
- Thus the $\mathcal{M}_{\mu\nu}$ represent $\mathfrak{so}(1,3)$ and we can construct a corresponding representation of SO(1,3) at least near 1.

Write $\Lambda \in SO(1,3)$ as

$$\Lambda = \exp(i t^{\mu \nu} \mathbf{M}_{\mu \nu}) \tag{9.28}$$

Define an action on \mathbb{C}^4 as:

$$\psi_D \xrightarrow{\Lambda} S(\Lambda) \cdot \psi_D$$

$$S(\Lambda) = \exp\left(i \, t^{\mu\nu} \mathcal{M}_{\mu\nu}\right)$$
(9.29)

• A "Dirac spinor" is a set of fields

$$(\psi_D)_a(x)$$
 , $a = 1, 2, 3, 4$ (9.30)

transforming as

$$(\psi_D)_a(x) \xrightarrow{\Lambda} S(\Lambda)_a^b(\psi_D)_b(\Lambda^{-1}x). \tag{9.31}$$

• *Note:* $\Lambda \longrightarrow S(\Lambda)$ is not defined globally on $SO^+(1,3)$. To see this chose some arbitrary rotation axis and some corresponding generator $T = t^{\mu\nu} M_{\mu\nu}$. The rotation is given by $\Lambda(\varphi) = \exp(i\varphi T)$. Naturally,

$$\Lambda(2\pi) = 1. \tag{9.32}$$

However, for $T_S = t^{\mu\nu} \mathcal{M}_{\mu\nu}$ one finds

$$S(2\pi) = \exp(i2\pi T_S) = -1. \tag{9.33}$$

• *Resolution:* The group Spin(1, 3), generated by $\mathcal{M}_{\mu\nu}$'s, is the fundamental symmetry group of nature. The map $\mathcal{M}_{\mu\nu} \mapsto M_{\mu\nu}$ leads to an associated representation of this group acting on vectors:

$$\Lambda = \Lambda(S)$$

Spin(1,3) is the "double cover" of $SO^+(1,3)$. Visualize:



• The representation of SO(1,3) (more correctly Spin(1,3)) on Dirac spinors is reducible

$$\mathcal{M}_{\mu\nu} = \frac{i}{4} \begin{bmatrix} \gamma_{\mu}, \gamma_{\nu} \end{bmatrix} = \frac{i}{4} \begin{bmatrix} \begin{pmatrix} 0 & \sigma_{\mu} \\ \bar{\sigma}_{\mu} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_{\nu} \\ \bar{\sigma}_{\nu} & 0 \end{pmatrix} \end{bmatrix}$$
$$= \begin{pmatrix} \sigma_{\mu}\bar{\sigma}_{\nu} - \sigma_{\nu}\bar{\sigma}_{\mu} & 0 \\ 0 & \bar{\sigma}_{\mu}\sigma_{\nu} - \bar{\sigma}_{\nu}\sigma_{\mu} \end{pmatrix}.$$
 (9.34)

This matrix is block-diagonal. (Note that exp retains this property.)

• We thus can write

$$\psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{9.35}$$

where the indices α , $\dot{\alpha}$ run over 1,2. Here the *Weyl spinor* ψ (and the complex conjugate Weyl spinor $\bar{\chi}$) transform independently.

- The decomposition of ψ_D in two independent parts can also be understood abstractly (i.e. without using our explicit representation of the γ 's):
 - First we introduce

$$\gamma^5 \equiv i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \tag{9.36}$$

– Now due to the obvious property $\gamma^5\gamma^\mu$ = $-\gamma^\mu\gamma^5$ it follows that

$$\gamma^5 \mathcal{M}_{\mu\nu} = \mathcal{M}_{\mu\nu} \gamma^5. \tag{9.37}$$

- Additionally one finds

$$(\gamma^5)^2 = 1 \tag{9.38}$$

- We now define

$$P_L \equiv \frac{1}{2} \left(\mathbb{1} - \gamma^5 \right) \qquad P_R \equiv \frac{1}{2} \left(\mathbb{1} + \gamma^5 \right). \tag{9.39}$$

With this definition the following properties follow immediately

$$P_L^2 = P_L P_R^2 = P_R P_L + P_R = 1 P_L P_R = 0.$$
 (9.40)

These properties make P_L and P_R projection operators. They induce a decomposition of the space on which they act such that

$$V = V_L \oplus V_R \equiv \operatorname{Im}(P_L) \oplus \operatorname{Im}(P_R) \tag{9.41}$$

- It now follows that $\psi_{D,L} \equiv P_L \psi_D$ and $\psi_{D,R} \equiv P_R \psi_D$ transform independently, since P_L and P_R commute with $\mathcal{M}_{\mu\nu}$. This is the result from above.
- We will call $\psi_{D,L}$ and $\psi_{D,R}$ the *left-handed* and *right-handed* Dirac spinors.

• In our explicit representation these take the form

$$\gamma^{5} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P_{L} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad P_{R} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \qquad \psi_{D,L} = \begin{pmatrix} \psi \\ 0 \end{pmatrix} \qquad \psi_{D,R} = \begin{pmatrix} 0 \\ \bar{\chi} \end{pmatrix}$$

$$(9.42)$$

Thus the Weyl-spinor is a two-dimensional object, containing the information of two complex numbers. The Dirac spinors are four component vectors. Both contain the same information.

- All of the above works for any even number of dimensions d. In this case the Dirac spinor has dimension $2^{\frac{d}{2}}$. For an odd number of dimensions one uses (d-1) dimensional γ 's and adds $\gamma^d \propto \gamma^0 \gamma^1 \dots \gamma^{d-2}$. This γ^d would now not commute with the $\mathcal{M}_{\mu\nu}$. Accordingly L,R or the Weyl-spinors do not exist for an odd number of dimensions. (For a more detailed account see Polchinski 'String Theory', vol. II, Appendix 'Spinors in various dimensions')
- An interesting and useful fact special to d = 4 is that

$$Spin(1,3) = SL(2,\mathbb{C}) \tag{9.43}$$

where $SL(2, \mathbb{C})$ are the 2 × 2 matrices M with det M = 1.

• Using this the 2:1 map from Spin(1,3) to SO(1,3) can be given explicitly: Let $M \in SL(2,\mathbb{C}); v \in \mathbb{R}^4; \hat{v} \equiv v_\mu \sigma^\mu$. Since $\{\sigma^\mu\}$ is a basis of hermitian 2×2 matrices, \hat{v} is a generic hermitian matrix.

Next we define $\hat{v}' = M\hat{v}M^{\dagger}$. Following that we define the vector v' implicitly by $\hat{v}' = v'_{\mu}\sigma^{\mu}$. With this definition we can calculate

$$(v')^{2} = (v'_{0})^{2} - (\vec{v}')^{2} = \det \begin{pmatrix} v'_{0} + v'_{3} & v'_{1} - iv'_{2} \\ v'_{1} + iv'_{2} & v'_{0} - v'_{3} \end{pmatrix}$$

$$= \det(\hat{v}') = \det(\hat{v}) = \det \begin{pmatrix} v_{0} + v_{3} & v_{1} - iv_{2} \\ v_{1} + iv_{2} & v_{0} - v_{3} \end{pmatrix} = v^{2}$$

$$(9.44)$$

Thus any $M \in SL(2,\mathbb{C})$ defines a map $\hat{v} \mapsto \hat{v}' \equiv M\hat{v}M^{\dagger}$ on hermitian 2×2 matrices. Hence it also defines a map $v_{\mu} \mapsto v'_{\mu}$ that preserves the length. Accordingly there exists $\Lambda = \Lambda(M) \in SO(1,3)$ such that $v'_{\mu} = \Lambda_{\mu}^{\ \nu} v_{\nu}$. Obviously $\Lambda(M) = \Lambda(-M)$, which gives us the 2:1 mapping.

• Our Weyl spinor ψ_{α} transforms as

$$\psi_{\alpha} \to M_{\alpha}^{\ \beta} \psi_{\beta} \qquad M \in \mathrm{SL}(2, \mathbb{C})$$
 (9.45)

The other spinor is $\bar{\chi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{\dot{\beta}}$ with $\bar{\chi}_{\dot{\beta}}$ transforming as

$$\bar{\chi}_{\dot{\beta}} \to \bar{M}_{\dot{\beta}}^{\dot{\alpha}} \bar{\chi}_{\dot{\beta}}$$
 (9.46)

which is the complex conjugate of $\chi_{\alpha} \to M_{\alpha}^{\ \beta} \chi_{\beta}$.

These claims could be checked explicitly using our definition $\psi_D = \begin{pmatrix} \psi \\ \bar{\chi} \end{pmatrix}$. We thus could also say that SL(2, $\mathbb C$) is the fundamental symmetry group of space-time. Note that the relation between

$$SU(2) \subset SL(2,\mathbb{C})$$

 $SO(3) \subset SO(1,3)$ (9.47)

and thus spinors in non-relativistic quantum mechanics work very similar.

9.4 Invariants involving spinors and Lagrangian Equations of motion

- For brevity in this subsection let $\psi_D \to \psi$.
- To write Lagrangians we need invariants/Lorentz-singlets built from ψ .
- Our first step in constructing such singlets is to recall that for the unitary representation of the symmetry group $v \to Uv$, $U \in U(n)$ the object $v^{\dagger}v \equiv \sum_i \bar{v}_i v_i$ would be invariant

$$v'^{\dagger}v' = (Uv)^{\dagger}Uv = v^{\dagger}U^{\dagger}Uv = v^{\dagger}v. \tag{9.48}$$

This can also be written infinitesimally as

$$v'^{\dagger}v' \approx ((\mathbb{1} + iT)v)^{\dagger} (\mathbb{1} + iT)v = v^{\dagger} (\mathbb{1} - iT^{\dagger})(\mathbb{1} + iT)v \approx v^{\dagger} (\mathbb{1} + i(T - T^{\dagger}))v.$$
(9.49)

Thus our claim follows from $T = T^{\dagger}$.

• In our case ψ transforms as

$$\psi \to \left(\mathbb{1} + it^{\mu\nu} \mathcal{M}_{\mu\nu}\right) \psi \tag{9.50}$$

From $(\gamma^0)^{\dagger} = \gamma^0$ and $(\gamma^i)^{\dagger} = -\gamma^i$ we can now conclude that

$$\mathcal{M}_{0i}^{\dagger} = -\mathcal{M}_{0i} \qquad \mathcal{M}_{ij}^{\dagger} = \mathcal{M}_{ij}$$
 (9.51)

Thus $\psi^{\dagger}\psi$ is *not* an invariant. We thus need to find a different object to construct our Lagrangian from. (Note: The mathematical reason for this is, that SO(1,3) is non-compact, implying that no finite-dimensional unitary representation exists.)

To construct an invariant we note that

$$\gamma^0 \gamma^\mu \gamma^0 = (\gamma^\mu)^\dagger \quad \gamma^0 \mathcal{M}_{\mu\nu} \gamma^0 = \mathcal{M}^\dagger_{\mu\nu} \quad \gamma^0 \mathcal{M}^\dagger_{\mu\nu} \gamma^0 = \mathcal{M}_{\mu\nu}$$
 (9.52)

• Hence it follows that

$$\psi^{\dagger} \gamma^{0} \rightarrow \psi^{\dagger} (\mathbb{1} + it^{\mu\nu} \mathcal{M}_{\mu\nu})^{\dagger} \gamma^{0}$$

$$= \psi^{\dagger} (\mathbb{1} - it^{\mu\nu} \mathcal{M}_{\mu\nu}^{\dagger}) \gamma^{0}$$

$$= \psi^{\dagger} \gamma^{0} (\mathbb{1} - it^{\mu\nu} \mathcal{M}_{\mu\nu})$$

$$(9.53)$$

and thus $\psi^{\dagger} \gamma^0 \psi$ is an invariant.

• With the definition

$$\bar{\psi} \equiv \psi^{\dagger} \gamma^0 \tag{9.54}$$

we write

$$\bar{\psi}\psi$$
 (9.55)

for the above invariant.

• For the second step in constructing Lorentz-singlets we apply the following corollary to a task on one of the last exercise sheets:

$$\left[\mathcal{M}_{\mu\nu}, \gamma_{\rho}\right] = -(M_{\mu\nu})_{\rho}^{\ \sigma} \gamma_{\sigma} \tag{9.56}$$

Using this corollary it follows that

$$(\mathbb{1} + it^{\mu\nu}\mathcal{M}_{\mu\nu})\gamma_{\rho}(\mathbb{1} - it^{\mu\nu}\mathcal{M}_{\mu\nu}) = (\mathbb{1} - it^{\mu\nu}\mathcal{M}_{\mu\nu})_{\rho}^{\sigma}\gamma_{\sigma}$$
(9.57)

or after exponentiation

$$S(\Lambda)\gamma_{\rho}S(\Lambda)^{-1} = (\Lambda^{-1})_{\rho}^{\sigma}\gamma_{\sigma}. \tag{9.58}$$

If we now multiply by Λ from left and make the spinor representation explicit we get

$$\Lambda_{\rho}^{\sigma}(S(\Lambda))_{a}^{b}(\gamma_{\sigma})_{b}^{c}(S(\Lambda)^{-1})_{c}^{d} = (\gamma_{\rho})_{a}^{d}. \tag{9.59}$$

• We have learned that $(\gamma_{\sigma})_a^b$ is an invariant tensor of SO(1,3) where σ is the vector index, a is the spinor index and b is the (upper or inverse) spinor index, similar to the index of $\bar{\psi}^a$. Hence $\bar{\psi}\gamma^{\mu}\psi$ is a vector and $\bar{\psi}\gamma^{\mu}\psi v_{\mu}$ is a scalar. For the Lagrangian we can thus write

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi \tag{9.60}$$

- This is the lowest order Lagrangian in fields and derivatives. In the problems it will be shown that the *i* is needed for *S* to be real.
- Equations of Motion: Treat ψ , $\bar{\psi}$ as independent variables and use the notation $\gamma^{\mu}x_{\mu} \equiv x$:

$$0 \stackrel{!}{=} \delta S = \int d^4 x \delta \mathcal{L} = \int d^4 x \left[\delta \bar{\psi} (i \partial \!\!\!/ - m) \psi + (i \partial \!\!\!/ \bar{\psi} - \bar{\psi} m) \delta \psi \right]$$

$$\Rightarrow (i \partial \!\!\!/ - m) \psi = 0 \quad \text{Dirac equation}$$
(9.61)

• *Important fact:* If ψ solves the Dirac equation $\Rightarrow \psi$ solves the Klein-Gordon-equation.

(Note: For any vector p, we have:

$$p^{2} = \gamma_{\mu}\gamma_{\nu}p^{\mu}p^{\nu} = \frac{1}{2}(\gamma_{\mu}\gamma_{\nu} + \gamma_{\nu}\gamma_{\mu})p^{\mu}p^{\nu} = \frac{1}{2}\{\gamma_{\mu}, \gamma_{\nu}\}p^{\mu}p^{\nu} = \frac{1}{2}2\eta_{\mu\nu}p^{\mu}p^{\nu} = p^{2}$$

$$0 = (-i\partial - m)(i\partial - m)\psi$$

$$= (\partial^2 + m^2)\psi$$

$$\Rightarrow (\partial^2 + m^2)\psi = 0$$
(9.62)

9.5 Solutions of the Dirac equation

• Ansatz (completly general): $\psi(x) = u(p)e^{-ipx}$ with $p^0 > 0$, $p^2 = m^2$

$$(i\partial \!\!\!/ - m)\psi = 0 \Rightarrow (\not \!\!p - m)u(p) = 0 \tag{9.63}$$

• Choose a frame where $p = (m, \vec{0})$

$$\Rightarrow m(\gamma^0 - 1)u(p) = 0 \tag{9.64}$$

$$\Rightarrow \begin{pmatrix} -1 & 1\\ 1 & -1 \end{pmatrix} u(p) = 0 \tag{9.65}$$

- Ansatz: $u(p) = \begin{pmatrix} \xi \\ \xi' \end{pmatrix} \Rightarrow \xi \xi' = 0$
- Hence, we have two independent solutions. We write them as:

$$u_s \sim \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix} \qquad s = 1, 2 \tag{9.66}$$

with
$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
 and $\xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

• It will be convenient to choose the following normalization

$$u_s(p) \equiv \sqrt{m} \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}$$
 in a frame where $p = (m, \vec{0})$ (9.67)

(Notice: Once defined in this particular frame, the solution is defined in any other frame by transformations with $S(\Lambda)$)

- *Comment:* In the frame $p = (m, \vec{0})$ the relation for non-relativistic spinors is particular obvious:
 - Let's exclude boosts → we are in $SO(3) \subset SO(1,3)$

- This gives the following restriction

$$t^{\mu\nu}\mathcal{M}_{\mu\nu} \to t^{jk}\mathcal{M}_{jk} = t^{jk}\frac{i}{4} \begin{pmatrix} \sigma_j \bar{\sigma_k} - \sigma_k \bar{\sigma_j} & 0\\ 0 & \bar{\sigma_j} \sigma_k - \bar{\sigma_k} \sigma_j \end{pmatrix}$$
(9.68)

and using $[\sigma_j, \sigma_k] = 2i\varepsilon_{jkl}\sigma_l$

$$t^{jk}\mathcal{M}_{jk} = t^{jk} \frac{1}{2} \begin{pmatrix} \varepsilon_{jkl} \sigma_l & 0\\ 0 & \varepsilon_{jkl} \sigma_l \end{pmatrix}. \tag{9.69}$$

We see that both, upper and lower two-component solutions rotate as in Quantum Mechanics. Even more explicitly: For rotating around the 3-axis by an angle φ we must pick $t^{jk} = \frac{1}{2} \varepsilon^{jkl} (\hat{e}_3)_l \varphi$ and find:

$$\Rightarrow \exp\left(it^{\mu\nu}\mathcal{M}_{\mu\nu}\right) = \begin{pmatrix} \exp\left(i\varphi\frac{1}{2}\sigma_3\right) & 0\\ 0 & \exp\left(i\varphi\frac{1}{2}\sigma_3\right) \end{pmatrix} \tag{9.70}$$

- − This is also consistent with $SU(2) \subset SL(2,\mathbb{C})$ and the $SL(2,\mathbb{C})$ -action on spinors described earlier. It also shows that our Dirac spinors are going to describe spin- $\frac{1}{2}$ -particles.
- A second set of so called *negative frequency* solutions exist: $\psi(x) = v(p)e^{+ipx}$ with $p^0 > 0$, $p^2 = m^2$

$$(i\partial \!\!\!/ - m)\psi = 0 \Rightarrow (\not p + m)v(p) = 0 \tag{9.71}$$

Choose $p = (m, \vec{0})$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v(p) = 0 \tag{9.72}$$

$$\Rightarrow v_s(p) = \sqrt{m} \begin{pmatrix} \eta_s \\ -\eta_s \end{pmatrix} \quad s = 1, 2 \tag{9.73}$$

with
$$\eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
 and $\eta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

• Our basis and norm choice is sucht that:

$$\bar{u}_r(p)u_s(p) = 2m\delta_{rs} \qquad \bar{u}_r(p)v_s(p) = 0$$

$$\bar{v}_r(p)v_s(p) = -2m\delta_{rs} \qquad \bar{v}_r(p)u_s(p) = 0$$
(9.74)

This holds in all frames by Lorentz invariance.

• In addition to these "orthonormality" relations, there is also a form of a "completeness" relation:

$$\sum_{s=1}^{2} (u_s(p))_a (\bar{u}_s(p))^b = (p + m)_a^b$$

$$\sum_{s=1}^{2} v_s(p)\bar{v}_s(p) = p - m$$
(9.75)

Derivation of the first equation: Let both sides of the equation act on the basis $\{u_s(p), v_s(p)\}$ of the "spinor space" \mathbb{C}^4 and use the relations 9.74

LHS:
$$\left(\sum_{s=1}^{2} u_{s}(p)\bar{u}_{s}(p)\right) u_{r}(p) = \sum_{s=1}^{2} u_{s}(p)2m\delta_{rs} = 2mu_{r}(p)$$

 $\left(\sum_{s=1}^{2} u_{s}(p)\bar{u}_{s}(p)\right) v_{r}(p) = 0$

$$(9.76)$$

RHS:
$$(p + m)u_r(p) = \underbrace{(p - m)u_r(p)}_{=0} + 2mu_r(p) = 2mu_r(p)$$

$$(p + m)v_r(p) = 0$$
(9.77)

Analogous for the other equation.

• *Final comment:* It is easy to remember the signs, as the equations must be consistent with the Dirac equation:

$$(p - m) \sum_{s} u_{s}(p) \bar{u}_{s}(p) = (p - m)(p + m) = 0$$
 (9.78)

10. Quantization of Spinors

10.1 Hamiltonian

 To the get from the Lagrangian to the Hamiltonian picture we compute the canonical momentum

$$\mathcal{L} = \bar{\psi}(i\partial \!\!\!/ - m)\psi \quad \text{with } \psi = \{\psi_a\}$$

$$\Rightarrow \pi^a = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_a} = \frac{\partial}{\partial \dot{\psi}_a}(i\psi^{\dagger}\gamma^0\gamma^0\dot{\psi}) = i(\psi^{\dagger})^a$$

$$\Rightarrow \pi = i\psi^{\dagger}$$
(10.1)

with π interpreted as a row vector.

• Notice: Lagrangian (and hence Hamiltonian) can be given just by using ψ , $\pi \sim \psi^{\dagger}$. There is no need for a canonical momentum corresponding to ψ^{\dagger} . This is different from the complex scalar case, where both ϕ , π and ϕ^{\dagger} , π^{\dagger} were needed.

Deep reason: There is an effective reduction of the number of degrees of freedom due to equation of motion being first order in t (for more details, see Weinberg Chapter 7).

$$\mathcal{H} = \pi \dot{\psi} - \mathcal{L} = i \psi^{\dagger} \dot{\psi} - \psi^{\dagger} \gamma^{0} (i \partial - m) \psi$$

$$= -\psi^{\dagger} \gamma^{0} (i \gamma^{i} \partial_{i} - m) \psi$$

$$= i \pi \gamma^{0} (i \gamma^{i} \partial_{i} - m) \psi$$
(10.2)

10.2 Quantization attempts with commutators

• We could attempt to define:

$$[\psi(\vec{x}), \pi(\vec{y})] = \left[\psi(\vec{x}), i\psi^{\dagger}(\vec{y})\right] = i\delta^{3}(\vec{x} - \vec{y})\mathbb{1}$$
(10.3)

• Skipping the familiar intermediate steps, we directly jump to the expression for the free fields in terms of creation and annihilation operators.

$$\psi(x) = \int d\tilde{p} (a_{\vec{p}}^s u_s(p) e^{-ipx} + b_{\vec{p}}^{s \dagger} v_s(p) e^{ipx}), \tag{10.4}$$

where

$$\left[a_{\vec{p}}^{s}, a_{\vec{q}}^{r\dagger}\right] = (2\pi)^{3} \delta^{3}(\vec{p} - \vec{q}) \delta^{rs} 2p^{0} = \left[b_{\vec{p}}^{s}, b_{\vec{q}}^{r\dagger}\right]. \tag{10.5}$$

• We need to check the consistency with the original commutation relations (at

$$x^0 = y^0$$
):

$$\begin{split} \left[\psi(\vec{x}), \psi^{\dagger}(\vec{y})\right] &= \int d\tilde{p} d\tilde{q} \left(e^{i\vec{p}\vec{x}-i\vec{q}\vec{y}}u_{s}(p)u_{r}^{\dagger}(q)\left[a_{\vec{p}}^{s}, a_{\vec{q}}^{r\dagger}\right]\right) \\ &+ e^{-i\vec{p}\vec{x}+i\vec{q}\vec{y}}v_{s}(p)v_{r}^{\dagger}(q)\left[b_{\vec{p}}^{s\dagger}, b_{\vec{q}}^{r}\right]\right) \gamma^{0} \gamma^{0} \\ &= \int d\tilde{p} \left(e^{i\vec{p}(\vec{x}-\vec{y})}(\not p+m) - \underbrace{e^{-i\vec{p}(\vec{x}-\vec{y})}(\not p-m)}_{\vec{p}\to -\vec{p}}\right) \gamma^{0} \\ &= \int d\tilde{p} e^{i\vec{p}(\vec{x}-\vec{y})} \left(p_{0}\gamma^{0} + p_{i}\gamma^{i} + m - \left(p_{0}\gamma^{0} - p_{i}\gamma^{i} - m\right)\right) \gamma^{0} \end{split} \tag{10.6}$$

This cannot work since the p^0 term required to cancel the $\frac{1}{p^0}$ from $d\tilde{p}$ drops out.

So let us try to assume $[b, b^{\dagger}] = -1$, effectively exchanging the roles of b and b^{\dagger} .

• This appears to work at first glance:

$$\Rightarrow \dots = \int d\tilde{p} e^{i\vec{p}(\vec{x}-\vec{y})} 2p^0 \gamma_0 \gamma^0$$
$$= \delta^3 (\vec{x} - \vec{y}) \mathbb{1}$$
 (10.7)

• However, straightforward calculations yield:

$$H = \int \mathcal{H} \, d^3 x = \int d\tilde{p} \, p^0 \sum_s \left(a_{\vec{p}}^{s \dagger} a_{\vec{p}}^s - b_{\vec{p}}^{s \dagger} b_{\vec{p}}^s \right)$$
 (10.8)

The relative sign makes the vacuum unstable, i.e. energy is unbounded below. There is no easy cure known!

10.3 Quantization with anticommutators

- The only known cure is a fundamental change of the quantization procedure. It can be proven rigorously that this must be done for all fields with half-integer spin. This is known as the *spin-statistics theorem* for all half-integer spin fields (cf. Streater & Wightman (2000). *PCT, spin and statistics, and all that*).
- We basically repeat the previous subsection with new postulates introducing the anti-commutator $\{a,b\} = ab + ba$.

$$\{\psi(\vec{x}), \pi(\vec{y})\} = \{\psi(\vec{x}), i \psi^{\dagger}(\vec{y})\} = i\delta^{3}(\vec{x} - \vec{y})\mathbb{1}$$

$$\Leftrightarrow \{a_{\vec{p}}^{r}, a_{\vec{q}}^{s\dagger}\} = \{b_{\vec{p}}^{r}, b_{\vec{q}}^{s\dagger}\} = (2\pi)^{3} 2p^{0} \delta^{3}(\vec{p} - \vec{q}) \delta^{rs}$$

$$(10.9)$$

All other anti-commutators vanish.

• Thus we obtain, solving our previous problem of an unbound Hamiltonian:

$$H = \int d\tilde{p} \, p^{0} \sum_{s} \left(a_{\vec{p}}^{s\dagger} a_{\vec{p}}^{s} - b_{\vec{p}}^{s} b_{\vec{p}}^{s\dagger} \right)$$

$$= \int d\tilde{p} \, p^{0} \sum_{s} \left(a_{\vec{p}}^{s\dagger} a_{\vec{p}}^{s} + b_{\vec{p}}^{s\dagger} b_{\vec{p}}^{s} \right) + \underbrace{\left(\propto 1 \right)}_{\text{omitted like in the scalar case}}$$

$$(10.10)$$

• Now let us have a look at the calculational details:

$$H = \int d^3x \ \bar{\psi} \left(-i\vec{\gamma}\vec{\nabla} + m \right) \psi \tag{10.11}$$

$$\psi(\vec{x}) = \int d\vec{p} \left(a_{\vec{p}}^s u_s(p) e^{+i\vec{p}\vec{x}} + b_{\vec{p}}^{s\dagger} v_s(p) e^{-i\vec{p}\vec{x}} \right)$$
(10.12)

$$\bar{\psi}(\vec{x}) = \int d\tilde{p}' \left(a_{\vec{p}'}^{s'} \bar{u}_{s'}(p') e^{-i\vec{p}'\vec{x}} + b_{\vec{p}'}^{s'} \bar{v}_{s'}(p') e^{+i\vec{p}'\vec{x}} \right)$$
(10.13)

$$\int d^3x \ e^{i\vec{p}\vec{x}\pm i\vec{p}'\vec{x}} = (2\pi)^3 \,\delta^3(\vec{p}\pm\vec{p}') \tag{10.14}$$

• We find four terms with $a^{\dagger}a$, $a^{\dagger}b^{\dagger}$, ba, and bb^{\dagger} respectively:

1.

$$H_{a^{\dagger}a} = \int \frac{\mathrm{d}\vec{p}}{2p^{0}} a_{\vec{p}'}^{s'\dagger} a_{\vec{p}}^{s} \,\bar{u}_{s'}(p) \left(\vec{\gamma}\vec{p} + m\right) u_{s}(p) \tag{10.15}$$

Furthermore use:

$$0 = (p - m)u(p) = (\gamma^{0}p^{0} - \vec{\gamma}\vec{p} - m)u(p)$$

$$(\vec{\gamma}\vec{p} + m)u(p) = \gamma^{0}p^{0}u(p)$$
(10.16)

$$H_{a^{\dagger}a} = \int \frac{\mathrm{d}\tilde{p}}{2} a_{\vec{p}'}^{s'\dagger} a_{\vec{p}}^{s} \,\bar{u}_{s'}(p) \gamma^{0} u_{s}(p) \tag{10.17}$$

noting that $\vec{p} = \{p^i\}$, $\vec{\nabla} = \{\frac{\partial}{\partial x^i}\}$. The scalar product of two three-component-vectors here uses the Euclidean metric.

A useful relation:

$$\bar{u}_r(p)\gamma^0 u_s(p) = \bar{v}_r(p)\gamma^0 v_s(p) = 2p^0 \delta_{rs}$$
 (10.18)

Proof:

$$(p - m)u(p) = 0$$

$$0 = u^{\dagger}(p)(p^{\dagger} - m) = u^{\dagger}(p)(p^{\dagger} - m)\gamma^{0}$$

$$= \bar{u}(p)(p - m)$$
(10.19)

$$\bar{u}_{r}(p)\gamma^{0}u_{s}(p) = \frac{1}{2m}\bar{u}_{r}(p)\left\{m,\gamma^{0}\right\}u_{s}(p)$$

$$= \frac{1}{2m}\bar{u}_{r}(p)\left\{p - m + m,\gamma^{0}\right\}u_{s}(p)$$

$$= \frac{1}{2m}\bar{u}_{r}(p)\left\{p,\gamma^{0}\right\}u_{s}(p)$$

$$= \frac{p^{0}}{m}\bar{u}_{r}(p)u_{s}(p) = \frac{p^{0}}{m}2m\delta_{rs} = 2p^{0}\delta_{rs}$$
(10.20)

This works analogously for v and \bar{v} . With this relation we get

$$H_{a^{\dagger}a} = \int d\tilde{p} \ p^0 \, a_{\vec{p}}^{s\dagger} a_{\vec{p}}^s. \tag{10.21}$$

2.

$$H_{a^{\dagger}b^{\dagger}} = 0$$
 since $u_s^{\dagger}(p^0, -\vec{p}) v_r(p^0, \vec{p}) = 0$ (10.22) $v_s^{\dagger}(p^0, -\vec{p}) u_r(p^0, \vec{p}) = 0$

which will be shown in the problems.

3.

$$H_{ba} = 0$$
 (10.23)

Same reasoning as above.

4.

$$H_{bb^{\dagger}} = \int \frac{\mathrm{d}\tilde{p}}{2p^{0}} b_{\vec{p}'}^{s'} b_{\vec{p}}^{s\dagger} \, \bar{v}_{s'}(p) (-\vec{\gamma}\vec{p} + m) v_{s}(p)$$

$$= \int \frac{\mathrm{d}\tilde{p}}{2p^{0}} b_{\vec{p}'}^{s'} b_{\vec{p}}^{s\dagger} \, \bar{v}_{s'}(p) (-\gamma^{0} p_{0}) v_{s}(p)$$

$$= \int \mathrm{d}\tilde{p} \, p^{0} \left(-b_{\vec{p}}^{s} b_{\vec{p}}^{s\dagger} \right)$$
(10.24)

Thus we get

$$H = \int d\tilde{p} \ p^0 \left(a_{\vec{p}}^{s\dagger} a_{\vec{p}}^s + b_{\vec{p}}^s b_{\vec{p}}^{s\dagger} \right) + \text{irrelevant constant.}$$
 (10.25)

• Define Fock space just like in the bosonic case:

$$a_{\vec{p}}^{s\dagger} |0\rangle , b_{\vec{p}}^{s\dagger} |0\rangle , a_{\vec{p}}^{s\dagger} a_{\vec{q}}^{s\dagger} |0\rangle , \dots$$
 (10.26)

$$a_{\vec{p}}^s |0\rangle = b_{\vec{p}}^s |0\rangle = 0 \quad \forall \, \vec{p}, \, s \tag{10.27}$$

• Crucial difference:

Due to anti-commutation relations

$$\left(a_{\vec{p}}^{s\dagger}\right)^{2}|0\rangle = 0,\tag{10.28}$$

i.e. multiple particle states with identical quantum numbers never occur. Our particles are fermions. Hence the name "spin-statistics theorem". Actually this never occurs for plane waves with p=p' anyway. A rough explanation is provided when changing to a finite volume where \vec{p} is discrete and $\left(a_{\vec{p}}^{s\dagger}\right)^2|0\rangle=0$ makes sense in a more straightforward way.

10.4 Time ordering, Green's functions, Dirac propagators

• The basic object of interest is

$$\langle 0|T(\text{product of }\psi'\text{s and }\bar{\psi}'\text{s})|0\rangle$$
. (10.29)

• Due to the anti-commutation relations the definition of *T* has changed:

$$T\psi_{a_1}(x_1)\dots\psi_{a_n}(x_n)$$

$$\equiv \operatorname{sgn}(\sigma)\psi_{a_{\sigma(1)}}(x_{\sigma(1)})\dots\psi_{a_{\sigma(n)}}(x_{\sigma(n)})$$
(10.30)

whereas $\{\sigma(1)...\sigma(n)\}$ is a permutation " σ " of $\{1...n\}$ such that $x^0_{\sigma(1)} \ge ... \ge x^0_{\sigma(n)}$ and $\operatorname{sgn}(\sigma) = \pm 1$ for even and odd σ , respectively. This is defined analogously for $\bar{\psi}$'s or combinations of ψ , $\bar{\psi}$.

- With this definition of *T* at hand the LSZ-formula still holds (up to a possible overall sign, which we ignore here).
- The relation between time-ordered Green's functions for interacting and free fields still holds as well (this is where $\exp(i S_{int})$ enters).
- Finally, in the last step towards the Feynman rules the Wick theorem is profoundly affected.

$$T(\Pi \psi_i \bar{\psi}_j) = : (\Pi \psi_i \bar{\psi}_j + \text{all possible contractions}):$$
 (10.31)

By definition "contraction" includes a factor of (-1) for each exchange of neighboring ψ , $\bar{\psi}$ required to place contracted pairs next to each other. In fact, witout the above adjustment in the definition of T we would not be able to derive a Wick-theorem at all.

Example:

$$: \overline{\psi_1 \psi_2 \bar{\psi}_3 \bar{\psi}_4} := -\overline{\psi_1 \bar{\psi}_3} : \psi_2 \bar{\psi}_4 : \tag{10.32}$$

• A contraction is defined in the same way as in the bosonic case:

$$\overline{\psi_a(x)}\overline{\psi}^b(y) \equiv \left\langle \mathrm{T}\psi_a(x)\overline{\psi}^b(y) \right\rangle \equiv S_F(x-y)_a^b \tag{10.33}$$

Contractions of $\psi\psi$ or $\bar{\psi}\bar{\psi}$ vanish or, if you wish, do not exist.

• The Dirac popagator (where the index "F" denotes the Feynman-*ie*-prescription) then reads:

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(p+m)}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$
 (10.34)

The derivation of this is analogous to the bosonic case. (The " $\psi + m$ "-factor is obtained from the u's/v's in the field decomposition.)

- An alternative derivation is as follows: (The fact that the following is a proper derivation will only become clear in the path integral approach which we will encounter in Quantum Field Theory II)
 - Recall the scalar case:

$$-\left(\Box_{x}+m^{2}\right)D(x-y)\equiv i\delta^{4}(x-y)\tag{10.35}$$

which makes D a Green's function.

In particular:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$
 (10.36)

This denominator is just $-(\Box_x + m^2)$ in Fourier space

Depending on the pole-prescription used, we get the Feynman, retarded or advanced Green's function.

- Analogously:

$$(i\partial_x - m) S(x - y) \equiv \mathbb{1} i\delta^4(x - y) \tag{10.37}$$

with

$$S_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(\not p + m)}{v^2 - m^2 + i\epsilon} e^{-ip(x-y)}.$$
 (10.38)

- Here, the crucial piece of algebra is

$$(i\partial - m) \to (p - m)$$
 & $(p - m)\frac{p + m}{p^2 - m^2} = 1$ (10.39)

where we have used $(p + m)(p - m) = (p^2 - m^2) \mathbb{1}$.

- Equivalently:

$$\frac{p + m}{p^2 - m^2} = \frac{1}{p - m} \tag{10.40}$$

10.5 U(1)-Symmetry of the Dirac-Lagrangian

• Consider the free Dirac-Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\partial \!\!\!/ - m)\psi \tag{10.41}$$

We find the *global* symmetry:

$$\psi \to e^{-i\varepsilon} \psi \tag{10.42}$$

• Recall Noether's theorem:

$$j^{\mu} = \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \varphi)} \chi - F^{\mu} \tag{10.43}$$

is a conserved current, where:

- (1) the symmetry transformation is $\varphi \rightarrow \varphi + \epsilon \chi$.
- (2) \mathcal{L} transforms as $\mathcal{L} \to \mathcal{L} + \epsilon \partial_{\mu} F^{\mu}$.
- Here, we obtain:

$$j^{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\psi)}(-i\psi) = \bar{\psi}i\gamma^{\mu}(-i\psi) = \bar{\psi}\gamma^{\mu}\psi \tag{10.44}$$

where we have used that $F^{\mu}=0$ and $\chi=-i\psi$. (Here $\psi\to e^{-i\epsilon}\psi\approx\psi+\epsilon(-i\psi)$ infinitesimally.)

This will become the electromagnetic current after gauging or "making" the U(1)-symmetry "local".

Also, we obtain a charge Q:

$$Q = \int d^3x \ j^0 = \int d^3x \ \psi^{\dagger} \psi = \int d\tilde{p} \ \sum_{s} \left(a_{\vec{p}}^{s\dagger} a_{\vec{p}}^s - b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s \right)$$
 (10.45)

10.6 Yukawa theory

• The arguably simplest interacting theory with fermions is the Yukawa theory which is described by the following lagrangian:

$$\mathcal{L} = \underbrace{\bar{\psi}(i\partial \!\!\!/ - m)\psi}_{\text{free}} + \underbrace{\frac{1}{2}(\partial \varphi)^2 - \frac{\mu^2}{2}\varphi^2}_{\text{free}} + \underbrace{\lambda \varphi \bar{\psi}\psi}_{\text{interactions}}$$
(10.46)

This plays a fundamental role in the fermion-mass generation in the Standard Model. It also plays a phenomenological (i.e. "effective") role in nuclear physics.

- For details see "Christmas problem"...
- In the following, we turn to the more complicated and structurally more interesting *gauge interactions* of fermions.

11. Quantum Electrodynamics

11.1 Lagrangian

- The logic behind finding the QED Lagrangian is the same as in "scalar QED".
 - Promote the global U(1)-symmetry of $\mathcal{L} = \bar{\psi}(i\partial \!\!\!/ m)\psi$ to a local symmetry: $\psi \to e^{-i\alpha(x)}\psi$.
 - In order to maintain the gauge-invariance we also need to promote ∂_{μ} to $D_{\mu} = \partial_{\mu} + iA_{\mu}$.
 - We also need to add a kinetic term for A_{μ} .
- Hence the Lagrangian reads:

$$\mathcal{L}_{\rm QED} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i \not \!\!\!D - m) \psi \tag{11.1}$$

where $\not D \equiv \gamma^{\mu} D_{\mu}$.

Or, by redefining the gauge field *A*:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i \not \!\!\!D - m) \psi \qquad (11.2)$$

with $D_{\mu} = \partial_{\mu} + ieA_{\mu}$.

• Let us check the gauge invariance explicitly:

$$D_{\mu}\psi \to D'_{\mu}\psi' = (\partial_{\mu} + ieA'_{\mu})e^{-ie\alpha(x)}\psi = e^{-ie\alpha(x)}(\partial_{\mu} - ie\partial_{\mu}\alpha + ieA'_{\mu})\psi$$
$$= e^{-ie\alpha(x)}D_{\mu}\psi$$
(11.3)

if $A'_{\mu} = A_{\mu} + \partial_{\mu}\alpha(x)$. The exponential factor drops out when being combined with its counterpart from the other spinor in the Lagrangian.

11.2 Feynman rules

• We can split the Lagrangian in two parts:

$$\mathcal{L}_{\text{QED}} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$$
 (11.4)

with the free Lagrangian

$$\mathcal{L}_{\text{free}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i \partial \!\!\!/ - m) \psi \tag{11.5}$$

and the interaction Lagrangian

$$\mathcal{L}_{\text{int}} = -e\bar{\psi}A\psi. \tag{11.6}$$

Here, the interaction Lagrangian contains the "minimal coupling" that is needed to ensure gauge-invariance.

• Side remark:

A non-minimal coupling would be e.g.

$$\mathcal{L}_{\text{int}} = \frac{1}{\Lambda} F_{\mu\nu} \bar{\psi} \gamma^{\mu} \gamma^{\nu} \psi \tag{11.7}$$

This would be non-renormalizable (we will learn more about this later) and it is less important if Λ is large.

• Note:

$$[\varphi] = [A_{\mu}] = [\partial_{\mu}] = E^{1} \tag{11.8}$$

where $[\cdot]$ denotes the "mass" or "energy dimension" of φ . Recall that we have set $\hbar = c = 1$.

• We find:

$$[\psi] = E^{3/2} \tag{11.9}$$

• Reason:

$$[S] = [\hbar] = 1;$$
 $\left[d^4 x \right] = E^{-4} \quad \Rightarrow \quad [\mathcal{L}] = E^4 \quad (11.10)$

From this and the equations (11.5) & (11.8) we obtain (11.9).

• Also, from (11.7), one can find:

$$[\Lambda] = E \tag{11.11}$$

Typically, Λ is the energy scale at which some "new physics" appears. This new physics might "generate" the "non-minimal coupling".

 \equiv higher-dimension operator i.e. energy dimension bigger than four (here $[\cdots] = E^5$)

• $\mathcal{L}_{\text{free}}$ implies the following Feynman rules:

$$a \longrightarrow b = \left(\frac{i(p+m)}{p^2 - m^2 + i\epsilon}\right)^b_a \tag{11.12}$$

As before, this is just the Fourier-space expression for $\langle T\psi_a(x)\bar{\psi}^b(y)\rangle$.

• Comment: We also have:

$$\frac{i}{\psi - m + i\epsilon} = \frac{i(\psi + m - i\epsilon)}{(\psi + m - i\epsilon)(\psi - m + i\epsilon)} = \frac{i(\psi + m - i\epsilon)}{p^2 - (m - i\epsilon)^2}
= \frac{i(\psi + m - i\epsilon)}{p^2 - m^2 + 2mi\epsilon + \epsilon^2} = \frac{i(\psi + m)}{p^2 - m^2 + i\epsilon'}$$
(11.13)

In the last step we neglected the ϵ -prescription in the numerator as we are only interested in the poles. Furthermore, in the small ϵ -limit we can neglect the ϵ^2 -term. By redefining $\epsilon \to \epsilon' = 2m\epsilon$ we obtain the already known expression for the propagator in Fourier space.

• In our simplest gauge choice (Feynman gauge) the propagation of the gauge field yields:

$$\mu \sim p \rightarrow \nu = \frac{-i\eta^{\mu\nu}}{p^2 + i\epsilon}$$
 (11.14)

• \mathcal{L}_{int} implies:

$$\mu = ie(\gamma_{\mu})_b^{\ a} \tag{11.15}$$

Up to the sign this is clear: The vertex is just the coefficient of the 3-field term in \mathcal{L} .

- The last Feynman rule ("the vertex") can be derived from the imagined process: $e^+ + \gamma \rightarrow e^+$ (momenta: p + k = p').
- Thus:

$$\langle 0 | a_{\vec{p}'}^{s'} \left(i \int d^4 x \, \mathcal{L}_{int} \right) a_{\vec{p}}^{s\dagger} a_{\vec{k}}^{\mu\dagger} | 0 \rangle \, \epsilon_{\mu}(k) = (2\pi)^4 \delta^4(\dots) i \mathcal{M}_{fi}$$
 (11.16)

where the annihilation/creation operators account for:

 $a_{\vec{p}'}^{s'}$: outgoing positron with spin s'.

 $a_{\vec{p}}^{s+}$: incoming positron with spin s.

 $a_{\vec{k}}^{\mu\dagger}$: Incoming photon with polarization $\epsilon_{\mu}(k)$.

Plugging in the interaction Lagrangian yields:

$$\langle 0 | a_{\vec{p}'}^{s'}(-ie) \left(\int d^4x \, \bar{\psi}(x) \gamma_{\nu} A^{\nu}(x) \psi(x) \right) a_{\vec{p}}^{s\dagger} a_{\vec{k}}^{\mu\dagger} | 0 \rangle \, \epsilon_{\mu}(k) \tag{11.17}$$

• Recall:

$$\psi(x) = \int d\vec{q} \, a_{\vec{q}}^{r} \, u_{r}(q) e^{-iqx} + \dots$$

$$\bar{\psi}(x) = \int d\vec{q} \, a_{\vec{q}}^{r} \, \bar{u}_{r}(q) e^{iqx} + \dots$$

$$A^{\mu}(x) = \int d\vec{q} \, a_{\vec{q}}^{\mu} e^{-iqx} + \dots$$

$$\left\{ a_{\vec{q}}^{r}, a_{\vec{p}}^{s\dagger} \right\} = 2p_{0}(2\pi)^{3} \delta^{3}(\vec{p} - \vec{q}) \delta^{rs}$$

$$\left[a_{\vec{q}}^{\nu}, a_{\vec{k}}^{\dagger \mu} \right] = -2k_{0}(2\pi)^{3} \delta^{3}(\vec{k} - \vec{q}) \eta^{\nu \mu}$$
(11.18)

• Plugging in and carrying out the integrations one gets

$$i\mathcal{M}_{fi} = \underline{\bar{u}}_{s'}(p')\underbrace{(ie\gamma^{\mu})}_{\text{Outgoing state}} \underbrace{u_s(p)\varepsilon_{\mu}(k)}_{\text{Incoming state}}.$$
 (11.19)

Thus, we confirm the vertex Feynman-rule found above. In addition we find

$$p, s \rightarrow$$
 = $(...)u_s(p)_a$ incoming positron

$$\rightarrow p, s = \bar{u}_s(p)^a(\dots)$$
 outgoing positron (11.20)

$$k \rightarrow \sim \sim = \epsilon_{\mu}(k)$$
 incoming photon

$$\longrightarrow k = \epsilon_{\mu}^*(k)$$
 outgoing photon

Connection of vertex and propagators

- Above we considered how a vertex connects to incoming and outgoing states. Now we consider how it connects to internal lines, i.e. propagators.
- Consider $e^+\gamma \to e^+\gamma$: with time flowing from left to right.

$$\langle 0 | a^{s'} a^{\mu'} T \left[\left(\int_{x} \bar{\psi}(-ie\gamma_{\nu} A^{\nu}) \psi \right) \left(\int_{y} \bar{\psi}(-ie\gamma_{\rho} A^{\rho}) \psi \right) \right] a^{s\dagger} a^{\mu\dagger} | 0 \rangle$$
 (11.21)

where contractions involving a's are meant as an informal symbol for producing a non-zero number by commutation relations, similar to contractions. A detailed calculation gives

$$\begin{array}{ccc}
k \\
p \\
p'
\end{array} = \underbrace{\varepsilon_{\mu}^{*}(k')\bar{u}_{s'}(p')(ie\gamma^{\mu})}_{\text{Read from right to left}} \frac{i}{q - m + i\varepsilon}(ie\gamma^{\nu})u_{s}(p)\varepsilon_{\nu}(k) \\
\end{array} (11.22)$$

where the matrix notation is contracted along the fermion lines.

- Important conclusion: Our definiton of the propagator (arrow corresponds to going from $\bar{\psi}$ to ψ) is consistent with the way the arrow was introduced for external particles.
- Finally we look at external antiparticles, in this case electrons. We consider $e^-\gamma \to e^-$ with momenta p+k=p'. For the matrix element we get

$$\langle 0 | \overrightarrow{b_{s'}} \left(\int \overline{\psi} (-ie \cancel{A}) \psi \right) \overrightarrow{b_s} a_{\mu}^{\dagger} | 0 \rangle$$
 (11.23)

and thus

$$p \longrightarrow p' = \underbrace{\varepsilon_{\mu}(k)\bar{v}_{s}(p)(ie\gamma^{\mu})v_{s'}(p')}_{\text{time flows also from left to right}}$$

$$(11.24)$$

• Summary: In the matrix notation for a fermion line, time flows from right to left for particles and from left to right for antiparticles.

Summary of QED Feynman rules

• We summarize the QED Feynman rules

$$= \frac{i}{\not k - m + i\epsilon} \tag{11.25}$$

$$= \frac{-i\eta^{\mu\nu}}{k^2 + i\epsilon} \tag{11.26}$$

$$b = ie(\gamma^{\mu})_b^{\ a} \tag{11.27}$$

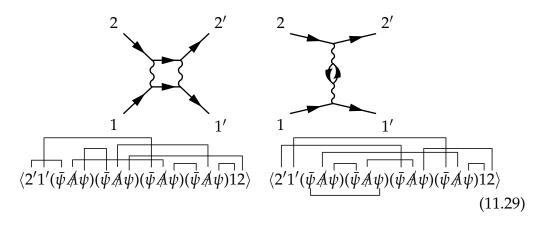
$$p \rightarrow -$$
 = $(...)u(p)$ incoming particle

$$p \rightarrow - \overline{v}(p)(...)$$
 incoming antiparticle (11.28)

$$\rightarrow$$
 $\rightarrow p = \bar{u}(p)(...)$ outgoing particle

$$\rightarrow p = (...)v(p)$$
 outgoing antiparticle

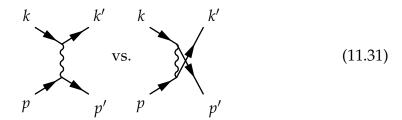
- Additional rules: A diagramm receives a relative minus sign for
 - 1. Every closed fermion line (loop).
 - 2. The exchange of two external fermion lines (relative to another diagram).
- In the following, we briefly illustrate the origin of these rules:
 - 1. Consider the following diagrams for $e^+e^+ \rightarrow e^+e^+$, which are of order e^4 for the amplitude, or NLO next to leading order.



- Each diagramm acquired two minus signs from "intersecting fermion contractions". In addition, the right-hand side has a contraction $\bar{\psi}\psi = -\psi\bar{\psi} = -S_F$. Such an extra minus always occurs if there is a closed fermion loop.
- 2. Here, the minus comes from

$$\langle 0 | a_{s'} a_s = -\langle 0 | a_s a_{s'} \tag{11.30}$$

An example would be



11.3 Elementary processes

• Compton scattering: $e^- \gamma \rightarrow e^- \gamma$

resulting in the Klein-Nishina formula.

• Møller scattering: $e^-e^- \rightarrow e^-e^-$ (or $e^+e^+ \rightarrow e^+e^+$)

Two limiting cases are distinguished: The first case is the case of a non-relativistic target and a highly relativistic projectile, e.g. a e^+ scattered off a μ^+ or a nucleus. This is Coulomb scattering, here the main result is the Mott formula (the name implies scattering off a static Coulomb field). The second case is the case of a non-relativistic projectile and a non-relativistic target. This is called Rutherford scattering, with the Rutherford formula emerging as a main result.

• Pair annihilation to photons: $e^+e^- \rightarrow \gamma\gamma$

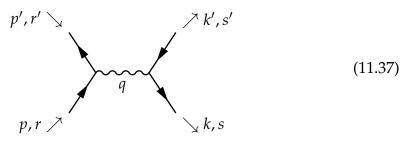
• Bhabha scattering: $e^+e^- \rightarrow e^+e^-$

• Light-by-light scattering: $\gamma \gamma \rightarrow \gamma \gamma$



which does not occur as a tree-level process, but only at loop-order.

• Our example will be the scattering process $e^+e^- \to \mu^+\mu^-$. This is even simpler than Bhabha scattering, as only the following diagramm is involved



$$q = p + p' = k + k'$$

$$q^2 \equiv s$$
(11.38)

$$i\mathcal{M} = \bar{u}_s(k)ie\gamma_\mu v_{s'}(k')\frac{-i\eta^{\mu\nu}}{q^2 + i\varepsilon}\bar{v}_{r'}(p')ie\gamma_\nu u_r(p) \tag{11.39}$$

As in section 5.3 and assuming that $\sqrt{s} \gg m_e, m_\mu$, the differential cross section reads:

$$d\sigma = \frac{1}{2s} |\mathcal{M}|^2 dX^{(2)} = \frac{1}{64\pi^2 s} |\mathcal{M}|^2 d\Omega$$
 (11.40)

Assume that we have an unpolarized beam and the spin of the outgoing particle is not measured:

$$\Rightarrow |\mathcal{M}|^{2} \to \underbrace{\frac{1}{2} \sum_{r} \frac{1}{2} \sum_{s'} \sum_{s'} \left| \mathcal{M}(r, r', s, s') \right|^{2}}_{\text{average}}$$

$$= \frac{e^{4}}{4s^{2}} \sum_{s,s'} \underbrace{\left(\bar{u}_{s}(k) \gamma_{\mu} v_{s'}(k')\right) \overline{\left(\bar{u}_{s}(k) \gamma_{\nu} v_{s'}(k')\right)}}_{\equiv A_{\mu\nu}} \cdot \underbrace{\sum_{r,r'} \underbrace{\left(\bar{v}_{r'}(p') \gamma^{\mu} u_{r}(p)\right) \overline{\left(\bar{v}_{r'}(p') \gamma^{\nu} u_{r}(p)\right)}}_{\equiv B^{\mu\nu}}$$

$$(11.41)$$

$$A_{\mu\nu} = \sum_{s,s'} \operatorname{tr} \left[\bar{u}_{s}(k) \gamma_{\mu} v_{s'}(k') \bar{v}_{s'}(k') \gamma_{\nu} u_{s}(k) \right]$$

$$= \sum_{s,s'} \operatorname{tr} \left[u_{s}(k) \bar{u}_{s}(k) \gamma_{\mu} v_{s'}(k') \bar{v}_{s'}(k') \gamma_{\nu} \right]$$

$$= \operatorname{tr} \left[(\not k + m_{\mu}) \gamma_{\mu} (\not k' - m_{\mu}) \gamma_{\nu} \right]$$

$$= \operatorname{tr} \left[\not k \gamma_{\mu} \not k' \gamma_{\nu} \right] - m_{\mu}^{2} \operatorname{tr} \left[\gamma_{\mu} \gamma_{\nu} \right]$$

$$= 4(k_{\mu} k'_{\nu} + k'_{\mu} k_{\nu} - (k \cdot k') \eta_{\mu\nu} - m_{\mu}^{2} \eta_{\mu\nu})$$

$$(11.42)$$

In the first line we interpreted the scalar expression as a 1×1 matrix, so the matrix is identical to its trace. To get to the second line we made use of the cyclicality of the trace and then used the relations 9.75 as well as the trace identities of gamma matrices:

$$\operatorname{tr}\left[\gamma_{\mu}\gamma_{\nu}\gamma_{\rho}\gamma_{\sigma}\right] = 4\left(\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma}\right) \tag{11.43}$$

From now on we neglect the m^2 -terms in $A_{\mu\nu}$ and $B^{\mu\nu}$, since m_{μ} , $m_e \ll \sqrt{s}$. We find:

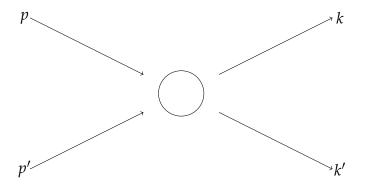
$$\frac{1}{4} \left(\sum \right)^{4} |\mathcal{M}|^{2}$$

$$= \frac{e^{4}}{4s^{2}} 16 \left(k_{\mu} k'_{\nu} + k'_{\mu} k_{\nu} - \eta_{\mu\nu} (k \cdot k') \right) \left(p^{\mu} p'^{\nu} + p'^{\mu} p^{\nu} - \eta^{\mu\nu} (p \cdot p') \right)$$

$$= \frac{8e^{4}}{s^{2}} \left((k \cdot p)(k' \cdot p') + (k \cdot p')(k' \cdot p) \right)$$

$$= 2e^{4} \frac{t^{2} + u^{2}}{s^{2}}$$
(11.44)

In the last line we have used the *Mandelstam variables s*, *t*, *u*:



We define $s = (p + p')^2$, $t = (p - k)^2$ and $u = (p - k')^2$ where $s + t + u = \sum_{i=1}^4 m_i^2$. In the massless case we have:

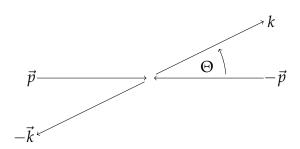
$$p^{2} = p'^{2} = k^{2} = k'^{2} = 0$$

$$s = 2pp' = 2kk'$$

$$t = -2kp = -2k'p'$$

$$u = -2pk' = -2kp'$$
(11.45)

• Let us go to the center of mass system (cms) and express this through scattering angle.



$$\Rightarrow t = -2kp = -2(k_0p_0 - \vec{k}\vec{p})$$

$$= -2k_0p_0(1 - \cos\Theta)$$

$$= -2\left(\frac{\sqrt{s}}{2}\right)^2(1 - \cos\Theta)$$

$$= -\frac{s}{2}(1 - \cos\Theta)$$
(11.46)

$$u = -s - t = -\frac{s}{2}(1 + \cos\Theta) \tag{11.47}$$

Inserting this in equation 11.44 we find:

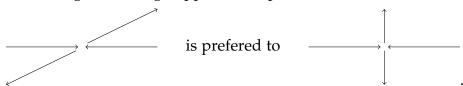
$$\Rightarrow \frac{1}{4} \left(\sum \right)^4 |\mathcal{M}|^2 = e^4 (1 + \cos^2 \Theta) \tag{11.48}$$

Hence for the cross section:

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{1}{4} \left(\sum\right)^4 \left|\mathcal{M}\right|^2 = \frac{\alpha^2}{4s} (1 + \cos^2 \Theta), \tag{11.49}$$

where $\alpha \equiv \frac{e^2}{4\pi}$.

• Something interesting happens: The process



This can be understood as follows:

• Write $\sum |\mathcal{M}|^2$ as

$$\underbrace{\sum_{\text{spins}} (\bar{v}\gamma^{\mu}u) \overline{(\bar{v}\gamma^{\nu}u)} \eta^{\mu\mu'} \eta^{\nu\nu'}}_{\propto B^{\mu\nu}} \underbrace{\sum_{\text{spins}} (\bar{u}\gamma^{\mu'}v) \overline{(\bar{u}\gamma^{\nu'}v)}}_{\propto A^{\mu'\nu'}}$$
(11.50)

Replace $\eta^{\mu\mu'} \to \eta^{\mu\mu'} - \frac{q^{\mu}q^{\mu'}}{q^2}$ (same for $\eta^{\nu\nu'}$), since q = p + p' and e.g. $\psi u(p) = 0$ etc. In the center-of-mass system these expression become $\delta^{ii'}\delta^{jj'}$.

 \Rightarrow We can restrict our 'density matrices' A and B to their spatial components:

$$\rho_{\text{ini}}^{ij} \sim B^{ij}, \, \rho_{\text{fin}}^{ij} \sim A^{ij} \Rightarrow \sum |\mathcal{M}|^2 \sim \rho_{\text{ini}}^{ij} \delta^{ii'} \delta^{jj'} \rho_{\text{fin}}^{i'j'}$$
(11.51)

(All indices i, j correspond to the three physical polarizations of the intermediate, massive photon.)

In a frame where
$$\vec{p} \sim \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
 from $B^{\mu\nu} \sim p^{\mu}p'^{\nu} + p'^{\mu}p^{\nu} - \eta^{\mu\nu}(pp')$ we get:

$$\rho_{\text{ini}}^{ij} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}^{ij} \sim \delta^{ij} - \hat{p}^{i}\hat{p}^{j},$$
where $\hat{p} \equiv \vec{p}/|\vec{p}|$

$$\rho_{\text{fin}}^{ij} \sim \delta^{ij} - \hat{k}^{i}\hat{k}^{j}$$
(11.52)

We find:

$$\sum |\mathcal{M}|^2 \sim \text{tr}(\rho_{\text{ini}}\rho_{\text{fin}}) \sim 3 - 1 - 1 + (\hat{k}\hat{p})^2 = 1 + \cos^2\Theta$$
 (11.53)

• Crucial physical point: As we have seen, two incoming spin-1/2 particles always produce a photon with spin ± 1 (not spin 0) along the 3-axis:

$$\begin{array}{cccc} & \longleftarrow & 1 \\ & \longrightarrow & -1 \\ & \longrightarrow & \longleftarrow \end{array}$$

This is also true for the coupling of the decay products of the photon. Hence the correlation between beam and decay axis.

Why is +1/2 + (-1/2) = 0 not realized? It would also be a perfectly good

process for producing a massive intermediate photon.

The reason is that we have neglected the fermion mass. Thus we are dealing with two independent types of fields

$$\bar{\psi}_{L} \not D \psi_{L} + \bar{\psi}_{R} \not D \psi_{R} + \underbrace{(m \bar{\psi}_{L} \psi_{R})}_{\text{neglected}}$$

$$e_{L}^{+}, e_{L}^{-} \text{ can anihilate} \qquad p \longrightarrow \qquad p' \\
s \longrightarrow \qquad s'$$

$$e_{R}^{+}, e_{R}^{-} \text{ can anihilate} \qquad p \longrightarrow \qquad p' \\
s \longleftarrow \qquad p' \\
s \longleftarrow \qquad p' \\
s \longrightarrow \qquad p' \\
s \longrightarrow \qquad p'$$

$$e_{R}^{+}, e_{L}^{-} \text{ cannot!} \qquad p \longrightarrow \qquad p' \\
s \longrightarrow \qquad p' \\
s \longrightarrow \qquad p'$$

No spin-0 (along 3-axis) photon can be produced. "No heliticity flip in absence of mass-term!"

12. RENORMALIZATION

12.1 Concept

• Let us call $\{Q_j, j = 1, 2, ...\}$ the set of quantities we would like to calculate in a given QFT.

 $(\{Q_i\}\supset \{\text{cross-sections};\,\text{Greens-functions};\,\text{self-energies};\,...\}\supset \{\text{Observables}\})$

• In perturbation theory, at higher orders, divergent loop integrals appear, e.g. 2-2-scattering in $\lambda \varphi^4$ -theory:

$$+ \downarrow \downarrow + \dots$$

$$\int d^4k \frac{1}{k^2 - m^2 + i\varepsilon} \cdot \frac{1}{(k+q)^2 - m^2 + i\varepsilon}$$
(12.1)

- Let us, for the moment, regularize by analytically continuing to Euclidean space $(k^2 \to k_E^2 = k_0^2 + k_1^2 + k_2^2 + k_3^2)$ and introduce a *cutoff*: $|k_E| < \Lambda$
- As a result, we have $Q_i = Q_i(\Lambda)$ and naively the limit $\Lambda \to \infty$ cannot be taken. Renormalization is the method to properly taking this limit nevertheless.
- Our example here will be QED:

$$\mathcal{L} = -\frac{1}{4} F_{0,\mu\nu} F_0^{\mu\nu} + \bar{\psi}_0 (i(\partial + ie_0 A_0) - m_0) \psi_0$$
 (12.2)

where $F_0^{\mu\nu}=\partial^\mu A_0^\nu-\partial^\nu A_0^\mu.$ The index 0 denotes "bare" fields and couplings

• We rewrite \mathcal{L} in terms of renormalized quantities

$$A_0^{\mu} = Z_A^{1/2} A^{\mu}; \ \psi_0 = Z_{\psi}^{1/2} \psi; \ e_0 = Z_e e; \ m_0 = Z_m m$$

$$\Rightarrow \mathcal{L} = -\frac{1}{4} Z_A F_{\mu\nu} F^{\mu\nu} + Z_{\psi} \bar{\psi} (i(\partial + i Z_e Z_A^{1/2} e A) - Z_m m) \psi$$
(12.3)

• *Idea*: Choose Z_i to be specific functions of Λ : $Z_i(\Lambda)$, such that $Q_i = Q_i(e, m, \Lambda, Z_i(\Lambda))$ have a well-defined (finite) limit as $\Lambda \to \infty$

$$Q_j^{\infty} = \lim_{\Lambda \to \infty} Q_j(e, m, \Lambda, Z_i(\Lambda))$$
 (12.4)

- If that is possible, our QFT is called *renormalizable*.
- This is non-trivial since $j = 1, ..., \infty$ while the number of Z_i 's is finite (here: $i = 1, \ldots, 4$).

- Even if it is possible, this procedure is non-unique (since one can always move finite factors between renormalized quantities and the Z_i 's).
- ⇒ Renormalization conditions are needed.
 (E.g. fix positions of poles and residues in the propagators and as many cross sections as independent couplings in terms of the renormalized parameters m, e.)
 - Only *further* cross sections and correlation functions will then be predictions of the theory.
 - Comment: If one accepts that correlation functions diverge as $\Lambda \to \infty$, one can restrict oneself to the Z-factors $Z_e = Z_e(\Lambda)$ and $Z_m = Z_m(\Lambda)$. That is sufficient to make all observables *finite*.

12.2 Renormalization conditions

- (1) For completeness (though not needed in QED), let us start with mass and field renormalization of the scalar field.
 - Recall the results we have already derived in section 7.6:

$$= \frac{i}{p^2 - m^2 - \Pi(p^2)}$$
 (12.6)

Note that m denotes the Lagrangian (renormalized) mass parameter. (This quantity has been called m_0 before, but now we used that symbol for the bare mass.)

The physical mass then reads:

$$m_{\rm phys}^2 = m^2 + \Pi(m_{\rm phys}^2)$$
 (12.7)

Again this result has been derived before, in section 7.6. Our proposed choice of condition is:

$$m^2 = m_{\text{phys}}^2 \tag{12.8}$$

or, equivalently:

$$\Pi(m^2) = 0 \tag{12.9}$$

• Field (or "wave-function") renormalization:

$$Z^{-1} = 1 - \Pi'(m_{\text{phys}}^2) \tag{12.10}$$

Here, our choice is:

$$Z = 1 \tag{12.11}$$

or, equivalently:

$$\Pi'(m^2) = 0 (12.12)$$

• Excursion into the scalar case. The Lagrangian is given as follows:

$$\mathcal{L} = \frac{1}{2} (\partial \varphi)^2 Z_{\varphi} - \frac{1}{2} \varphi^2 m^2 Z_{\varphi} Z_m - \frac{\lambda}{4!} \varphi^4 Z_{\lambda} Z_{\varphi}^2$$
 (12.13)

In the case of free fields (i.e. no interaction) we expect the Z's to become unity, therefore:

$$Z_{\varphi} = 1 + \delta Z_{\varphi}$$

$$Z_{m} = 1 + \delta Z_{m}$$
(12.14)

 \Rightarrow With this, \mathcal{L} becomes:

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}\varphi^2m^2}_{\text{free}} + \underbrace{\frac{1}{2}(\partial\varphi)^2\delta Z_{\varphi} - \frac{1}{2}\varphi^2m^2(\delta Z_{\varphi} + \delta Z_m)}_{(*)} + \dots$$
(12.15)

- (*) This part is $\mathcal{O}(\lambda)$ and hence treated like an interaction.
- \Rightarrow Feynman rule:

$$\longrightarrow$$
 (12.16)

For the mass term, we have:

$$= im^2(\delta Z_{\varphi} + \delta Z_m) + \dots$$
 (12.17)

$$i\Pi(p^2) = \underbrace{\qquad}_{\sim \Lambda^2} + \underbrace{\qquad}_{\sim \Lambda^2} \text{ at } \mathcal{O}(\lambda)$$
 (12.18)

• We see: by imposing e.g. $\Pi(m^2) = 0$ we get conditions of the type $\delta Z_i \sim \lambda f(\Lambda)$ with $f(\Lambda) \to \infty$ as $\Lambda \to \infty$. The logic in perturbation theory is to *always* take the limit $\lambda \to 0$ (by definition of perturbation theory) more seriously than $\Lambda \to \infty$.

In other words: δZ is a small correction in spite of Λ being potentially large. Only after all Λ -dependence has disappeared we are allowed to give λ its "measured" physical value.

(2) Mass and field normalization for the *electron*:

a —
$$b = -i \Sigma(p)_a^b$$
 ; (12.19)

$$\underline{\qquad} = \frac{i}{p - m - \Sigma(p)} \tag{12.20}$$

Here the self-energy Σ is defined in complete analogy to the scalar case. It is, of course, a matrix since the field has four components. Also, it is not a Lorentz-invariant. Hence, it can depend on p more generally then through p^2 . This is accounted for by giving Σ the argument p.

• Our renormalization conditions are (in analogy to the scalar case): For the mass:

$$m = m_{\text{phys}} \to \Sigma(m) = 0 \tag{12.21}$$

For the field:

$$Z = 1 \to \Sigma'(m) = 0 \to \Sigma'(m) = 0$$
 (12.22)

• We now want to confirm that once $\Sigma(m) = \Sigma'(m) = 0$ holds, we really get a pole at $p^2 = m^2$ with canonical residue.

Taylor-expansion of the function Σ around m:

$$\Sigma(p) = \Sigma(m) + \Sigma'(m)(p - m) + \frac{1}{2}\Sigma''(m)(p - m)^2 + \dots$$
 (12.23)

$$\Rightarrow \frac{i}{\not p - m - \Sigma(\not p)} = \frac{i}{(\not p - m)\left(1 - \frac{1}{2}\Sigma''(m)(\not p - m) - \dots\right)}$$

$$= \frac{i(\not p + m)}{(p^2 - m^2)\left(1 - \frac{1}{2}\Sigma''(m)(\not p - m) - \dots\right)}$$

$$= \frac{i(\not p + m)\left(1 + \frac{1}{2}\Sigma''(m)(\not p - m) + \dots\right)}{(p^2 - m^2)}$$

$$= \frac{i(\not p + m)}{p^2 - m^2} + i\left(\frac{1}{2}\Sigma''(m) + \dots\right)$$
pole at $p^2 = m^2$ analytical at $p^2 = m^2$ analytical at $p^2 = m^2$

We see that the first contribution has a pole at $p^2 = m^2$ as in the free case. The second part does not give any contributions because it is analytical at $p^2 = m^2$. Overall, the residue is the same as in the free theory.

(3) Field normalization for *photon*: The photon mass should stay zero automatically by the structure of the theory.

$$\mu \sim \nu = i\Pi_{\mu\nu}(q) \tag{12.25}$$

This is also called *vacuum polarization*. The term on the right-hand side can be thought of as a 4×4 -matrix which we will call " Π^{M} ".

$$= i\Pi^{M};$$

$$= \frac{i}{-q^{2}\eta} + \frac{i}{-q^{2}\eta} i\Pi^{M} \frac{i}{-q^{2}\eta} + \dots = \frac{i}{-q^{2}\eta + \Pi^{M}(q)}$$
(12.26)

Here, η denotes the matrix $\eta^{\mu\nu}$.

- We know that $\Pi^M(0) = 0$ must be maintained to keep the photon massless. It will be assumed that our regularization respects this.
- By covariance, it holds:

$$\Pi_{\mu\nu}(q) = \eta_{\mu\nu}A(q^2) + q_{\mu}q_{\nu}B(q^2)$$
 (12.27)

• Gauge-invariance enforces: $A = -B \cdot q^2$, i.e.

$$\Pi_{\mu\nu}(q) = (\eta_{\mu\nu}q^2 - q_{\mu}q_{\nu})\Pi(q^2)$$
 (12.28)

For this, a general argument will be given later. A quick argument is formulated in the following.

Consider $2 \rightarrow n$ photon scattering:

$$n$$
 outgoing photons (12.29)

Let us change one polarization vector by gauge transformation

$$\epsilon^{\mu}(k) \to \epsilon^{\mu}(k) + \alpha k^{\mu}.$$
(12.30)

⇒ The amplitude should not change!

Our case is just like the special case of only two photon lines, hence:

$$\mu \underset{k}{\sim} \underset{\nu}{\sim} k^{\nu} = 0 \tag{12.31}$$

$$\Rightarrow \Pi_{\mu\nu}(q) \, q^{\nu} = 0 \Rightarrow A = -B \cdot q^2 \tag{12.32}$$

• Thus, we have:

$$= \frac{i}{-\eta q^2 (1 - \Pi(q^2)) - \underbrace{(q \otimes q)\Pi(q^2)}_{(*)}}$$
 (12.33)

where (*) does not contribute to physical polarizations since

$$\epsilon^{\mu}(q) \, q_{\mu} = 0. \tag{12.34}$$

 \Rightarrow Our condition is:

$$Z = 1 \to \Pi(0) = 0$$
 (12.35)

(where
$$\mu \sim = i\Pi_{\mu\nu}(q^2) \equiv i(\eta_{\mu\nu}q^2 - q_{\mu}q_{\nu})\Pi(q^2)$$
)

Note: Due to the extracted factor q^2 , $\Pi(0) = 0$ for the photon is analogous to $\Pi'(m^2) = 0$ for the scalar.

(4) Vertex normalization

This is technically simpler but in spirit analogous to fixing some specific cross section.

$$= ie\Gamma^{\mu}(p, p')_b^{\ a}$$
 (12.36)

where p belongs to the upper fermion line and p' to the lower, both are onshell.

Our condition: For $q \to 0$ we should find tree-level result. Hence: $\Gamma^{\mu}(p,p) = \gamma^{\mu}$

- It is interesting to see how many conditions we actually impose this is non-trivial since Γ^{μ} is in general a matrix.
- In full generality:

$$\Gamma^{\mu} = \gamma^{\mu} \cdot A(p, p') + (p'^{\mu} + p^{\mu}) B(p, p') + (p'^{\mu} - p^{\mu}) C(p, p')$$
 (12.37)

since p, p' are the only independent vectorial arguments.

• Take advantage of the fact that Γ always appears between $\bar{u}(p')$ and u(p) (or \bar{v}, v) and

$$pu(p) = mu(p'), \quad \bar{u}(p) p' = \bar{u}(p') m$$
 (12.38)

to argue that w.l.o.g. A, B, C are numbers $\times 1$.

$$\Rightarrow \Gamma^{\mu} = \gamma^{\mu} A(q^2) + \left(p'^{\mu} + p^{\mu} \right) B(q^2) + \left(p'^{\mu} - p^{\mu} \right) C(q^2)$$
 (12.39)

where q^2 is the only kinematic invariant.

• As before using gauge invariance

$$\bar{u}(p') q^{\mu} \Gamma_{\mu} u(p) = 0$$
 (12.40)

even if $q^2 \neq 0$ since the gauge parameter within the propagator could vary and should not affect the result:



$$q_{\mu} \left(p'^{\mu} - p^{\mu} \right) = q^{2} \neq 0 \qquad \Rightarrow C = 0$$

$$q_{\mu} \left(p'^{\mu} + p^{\mu} \right) = p'^{2} - p^{2} = m^{2} - m^{2} = 0 \qquad \Rightarrow \quad \text{B unconstrained}$$

$$(12.41)$$

$$\Rightarrow \quad \Gamma^{\mu} = \gamma^{\mu} A(q^2) + \left(p'^{\mu} + p^{\mu} \right) B(q^2) \tag{12.42}$$

• Using $\sigma^{\mu\nu} \equiv \frac{i}{2} [\gamma^{\mu}, \gamma^{\nu}]$ and external $\bar{u}(p')$, u(p) one can show that the following expression is equivalent.

$$\Gamma^{\mu} = \gamma^{\mu} F_1(q^2) + \frac{i\sigma^{\mu\nu} q_{\nu}}{2m^2} F_2(q^2)$$
 (12.43)

 $F_1(q^2)$ and $F_2(q^2)$ are so called "form factors". Our single constraint is $F_1(0)=1$.

12.3 QED β-function

- At this point in the course we *could* perform numerous worthwhile calculations:
 - put above conditions together and additionally calculate two cross-sections (one to fix the coupling *e*, the second as a prediction of our theory).
 - in fact one typically picks for the "first" cross-section $e\gamma \to e\gamma$ at $q_\gamma \to 0$.
 - With very little effort one would find that our coupling "e" is indeed the classical electric charge.
 - Nevertheless, we need at least to calculate the cross-sections of

- After all this is too much to fit in this course.
- However, there is a very important physical quantity (observable) which we can obtain with the help of only one diagram. The β -function determines the Λ -dependency of $e_0(\Lambda)$. Let us recall that our renormalized coupling e is by definition independent of Λ . As

$$e_0 = e_0(\Lambda) = Z_e(\Lambda) e, \tag{12.44}$$

we have

$$\frac{d}{d \ln(\Lambda)} e_0 = e \frac{d}{d \ln(\Lambda)} Z_e(\Lambda)$$

$$= e \frac{d}{d \ln(\Lambda)} (1 + \delta Z_e(\Lambda))$$

$$\simeq e \frac{d}{d \ln(\Lambda)} (1 + c e^2 \ln(\Lambda))$$

$$= c e^3,$$
(12.45)

where the second last step is made possible by the claim Z has a logarithmic Λ -dependency and we are in leading order. c is the coefficient of the 1-loop divergence in δZ_e .

• Since $e \simeq e_0$ at leading order we can discard higher order terms on the right hand side to find:

$$\frac{\mathrm{d}}{\mathrm{d}\ln(\Lambda)}e_0(\Lambda) = c\,e_0^3(\Lambda) \tag{12.46}$$

• Now let us define the β -function:

$$\beta(e_0(\Lambda)) \equiv \frac{\mathrm{d}}{\mathrm{d}\ln(\Lambda)} e_0(\Lambda) \tag{12.47}$$

Hence equation (12.46) supplies us with the leading order β -function for the bare coupling of QED.

- *Note:* Given $\beta(e_0)$ the above differential equation (as a matter of fact a "RGE" or renormalization group equation) allows us to find e_0 for any Λ provided some "boundary condition".
- Why is this "running" of the bare coupling of any physical relevance?
- Let us calculate a cross-section at a given energy \sqrt{s} at leading order:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{c_1 \, e_0^4(\Lambda)}{s} \qquad \left(s \gg m^2\right) \tag{12.48}$$

Here we used the bare coupling which is alright since the higher order difference is small as long as $\ln(\Lambda/\sqrt{s})$ is not large (cf. $\delta Z \simeq e^2 \ln(\Lambda/\sqrt{s})$).

• Let us define a "scale-dependent physical coupling" by

$$e^{4}(\mu) \equiv \frac{s}{c_{1}} \left. \frac{d\sigma}{d\Omega} \right|_{s=u^{2}} \tag{12.49}$$

The idea is that this coupling governs the proper strength of interaction at an energy corresponding to μ .

• We see:

$$e(\mu) \simeq e_0(\Lambda)$$
 at $\mu \lesssim \Lambda$ (12.50)

• Hence:

$$\frac{d}{d \ln(\mu)} e(\mu) = \beta (e(\mu))$$
 ; $\beta(e) = c e^3$ (12.51)

with *c* defined by $\delta Z_e = c e^2 \ln(\Lambda)$.

• *Caution:* Only at leading order the β -function and the various couplings are the same!

- The structure $\partial_{\mu} + ieA_{\mu}$ will be unchanged under renormalization if $Z_e \sqrt{Z_A} = 1$. This is precisely true and it can be shown (cf. "Ward-Takahashi identity" as well as later in this course).
- *Remark:* Many books use a different notation:

$$Z_A = Z_3$$
 ; $Z_{\psi} = Z_2$; $Z_e Z_{\psi} \sqrt{Z_A} = Z_1$ (12.52)

Then

$$Z_e \sqrt{Z_A} = 1 \quad \Leftrightarrow \quad Z_1 = Z_2 \tag{12.53}$$

Now we know that

$$c = -\frac{1}{2} \frac{1}{e^2} \frac{d}{d \ln(\Lambda)} Z_A.$$
 (12.54)

- We need a $ln(\Lambda)$ -term in δZ_A .
- δZ_A corresponds to a term in \mathcal{L} :

$$\mathcal{L} \supset -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \delta Z_A$$

$$= \frac{1}{2} \left(A_{\mu} \partial^2 A^{\mu} - A_{\mu} \partial^{\mu} \partial^{\nu} A_{\nu} \right) \delta Z_A$$
(12.55)

This gives a "vertex"

$$= i \left(-\eta^{\mu\nu} p^2 + p^{\mu} p^{\nu} \right) \delta Z_A. \tag{12.56}$$

• Hence

$$\underbrace{i\Pi_{\mu\nu}}_{=i(q^{2}\eta^{\mu\nu}-q^{\mu}q^{\nu})\Pi(q^{2})} = + \underbrace{\Xi\Pi_{(1)\mu\nu}(q^{2})}_{=i(q^{2}\eta_{\mu\nu}-q_{\mu}q_{\nu})\Pi_{(1)}(q^{2})}$$
(12.57)

self-energy = counter term + 1-loop term

• Hence we can conclude using renormalization condition on $\Pi(q^2)$:

$$c = -\frac{1}{2e^2} \cdot \left\{ \text{coeff. of } \ln(\Lambda) \text{-term in } \Pi_{(1)}(q^2) \right\}$$
 (12.58)

12.4 Vacuum polarization in dimensional regularization

• Let us embed our theoretical considerations in a *d* dimensional space-time.

$$i \Pi_{(1)}^{\mu\nu}(q) = -(-1)(ie)^2 \int \frac{d^dk}{(2\pi)^d} tr \left[\gamma^{\mu} \frac{i}{\not k - m} \gamma^{\nu} \frac{i}{\not k + \not q - m} \right]$$
 (12.59)

• Jumping ahead: The idea of dimensional regularization

$$\int^{\Lambda} \frac{d^4 k_E}{(k_E^2 + m^2)^2} = \Omega_3 \int_{0}^{\Lambda} \frac{d|k_E| |k_E|^3}{(|k_E|^2 + m^2)} \simeq \Omega_3 \ln\left(\frac{\Lambda}{m}\right)$$
(12.60)

• By dimensional regularization $d = 4 - \varepsilon$

$$\simeq \Omega_3 \int_{m}^{\infty} \frac{\mathrm{d}|k_E| |k_E|^{3-\varepsilon}}{|k_E|^4} \simeq \Omega_3 m^{-\varepsilon} \int_{1}^{\infty} \frac{\mathrm{d}x}{x^{1+\varepsilon}} = \Omega_3 m^{-\varepsilon} \frac{1}{\varepsilon}$$
 (12.61)

- Thus the poles in ε track the physical log-divergence. This can be made rigorous, see e.g. Collins "Renormalization".
- It is crucial that even for a modified number of dimensions Poincaré and gauge invariance are fully preserved.

$$\Pi_{\mu\nu} = \left(q^2 \eta_{\mu\nu} - q_{\mu} q_{\nu}\right) \Pi$$

$$\Pi_{\mu}^{\mu} = \left(q^2 d - q^2\right) \Pi$$

$$\Rightarrow \Pi = \frac{1}{(d-1)q^2} \Pi_{\mu}^{\mu}$$
(12.62)

• For the trace in the integral we get

$$\operatorname{tr}\left[\gamma^{\mu} \frac{i}{\not k - m} \gamma_{\mu} \frac{i}{\not k + \not q - m}\right]$$

$$= -\frac{\operatorname{tr}\left[\gamma^{\mu} (\not k + m) \gamma_{\mu} (\not k + \not q + m)\right]}{(k^{2} - m^{2})((k + \not q)^{2} - m^{2})}$$

$$= -\frac{\operatorname{tr}\left[((2 - d) \not k + m d) \left(\not k + \not q + m\right)\right]}{(k^{2} - m^{2})((k + \not q)^{2} - m^{2})}$$

$$= 4\frac{(d - 2)k(k + \not q) - m^{2}d}{(k^{2} - m^{2})((k + \not q)^{2} - m^{2})}$$
(12.63)

where we used the Clifford algebra in d dimensions with

$$\gamma_{\mu}\gamma^{\mu}=d\cdot\mathbb{1},\quad \gamma^{\mu}k\!\!/\gamma_{\mu}=2k\!\!/-\gamma^{\mu}\gamma_{\mu}k\!\!/=(2-d)k\!\!/,\quad \text{etc.} \eqno(12.64)$$

- Note that here the convention is tr(1) = 4, which is an unimportant overall factor.
- The integral in d dimensions over k of the above expression lets one expect a quadratic divergence in d=4. In dimensional regularization this corresponds to a pole at d=2. But at d=2, the coefficient of the k^2 -term vanishes. Accordingly $\Pi_{\mu}^{\ \nu}$ is not quadratically divergent in d=4. That is someting which is not as easily seen if one simply introduces a cutoff.
- For convenience, we next introduce the so-called Feynman parameter

$$\frac{1}{AB} = \int_0^1 \frac{1}{(xA + (1-x)B)^2}$$
 (12.65)

With this we can write

$$i\Pi_{(1)\mu}^{\ \mu} = 4e^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^1 \mathrm{d}x \, \frac{(d-2)k(k+q) - m^2 d}{\left[(1-x)(k^2 - m^2) + x((k+q)^2 - m^2) \right]^2} \tag{12.66}$$

• Now we change the order of integration and substitute the integration variable k' = k - xq and then rename k' to k to get in the denominator

$$\left[k^{2} + \underbrace{x(1-x)q^{2} - m^{2}}_{\equiv -\Delta}\right]^{2}$$
 (12.67)

where we introduced Δ as a convenient abbreviation. For the numerator we get

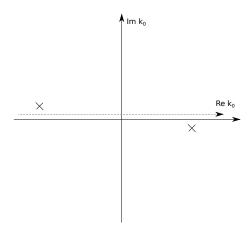
$$(d-2)\left(k^2 + (1-2x)kq - x(1-x)q^2\right) - m^2d\tag{12.68}$$

In this term we are able to drop the term (1-2x)kq as it is odd under $k \to -k$ and the denominator is even.

Together this yields

$$i\Pi_{(1)\mu}^{\ \mu} = 4e^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(d-2)\left(k^2 - x(1-x)q^2\right) - md^2}{(k^2 - \Delta)^2}$$
(12.69)

• In the above calculation we supressed $i\varepsilon$ for brevity. But in fact we always had $m^2 \to m^2 - i\varepsilon$. This determines the pole structure in the k^0 -plane.



• We can perform a so-called Wick rotation and rotate the integration contour as

$$k_0 \to i k_0 \implies dk_0 \to i dk_0; \quad k^2 = k_0^2 - \vec{k}^2 \to -k_E^2 = -(k_0^2 + \vec{k}^2)$$
 (12.70)

where the index E denotes the change to a Euclidean metric. With this we get

$$i\Pi_{(1)\mu}^{\mu} = 4ie^2 \int_0^1 dx \int \frac{d^d k_E}{(2\pi)^d} \frac{(d-2)\left(-k_E^2 + x(1-x)q^2\right) - m^2 d}{(k_E^2 + \Delta)^2}$$
(12.71)

• Next we split the fraction using

$$\frac{k_E^2 + \Delta - \Delta}{(k_F^2 + \Delta)^2} = \frac{1}{k_F^2 + \Delta} - \frac{\Delta}{(k_F^2 + \Delta)^2}$$
(12.72)

and with this are able to calculate

$$\int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{1}{(k_E + \Delta)^n} = \int \frac{\mathrm{d}\Omega_{d-1}}{(2\pi)^d} \int_0^\infty \mathrm{d}k_E \frac{k_E^{d-1}}{(k_E^2 + \Delta)^n}$$
(12.73)

Here the first integration is well-defined for all d > 1. It can easily be promoted to an analytic function of d with poles using

$$\int d\Omega_{d-1} = \frac{(2\pi)^{d/2}}{\Gamma(d/2)}$$
 (12.74)

The second Integral is well-defined for all d < 2n and can easily be promoted to an analytic function of both d and Δ , yielding the result

$$\int_0^\infty dk_e \, \frac{k_e^{d-1}}{(k_E^2 + \Delta)^n} = \frac{\Gamma(\frac{d}{2})\Gamma(n - \frac{d}{2})}{2\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2}} \tag{12.75}$$

• Crucially for n = 2 and $d = 4 - \varepsilon$ we get a pole in ε :

$$\Gamma(n - \frac{d}{2}) = \Gamma(\frac{\varepsilon}{2}) = \frac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon)$$
 (12.76)

where γ is the Euler-Mascheroni constant, $\gamma \approx 0.577$.

• Also crucially, we do not get a pole in d = 2 from the k_E^2 term

$$(d-2)\frac{\Gamma(1-\frac{d}{2})}{\Gamma(1)} \left(\frac{1}{d}\right)^{(1-\frac{d}{2})} = -2\frac{(1-\frac{d}{2})\Gamma(1-\frac{d}{2})}{1} \Delta^{\frac{\varepsilon}{2}}$$

$$= -2\Gamma(2-\frac{d}{2})\Delta^{\frac{\varepsilon}{2}}$$
(12.77)

The pole also appears in d = 4.

• Now we use equations (12.71) and (12.73), focus on the $\varepsilon \to 0$ limit and perform the x integration to get

$$\Pi_{(1)}(q^2) = \frac{1}{(d-1)q^2} \Pi_{(1)\mu}^{\ \mu} = -\frac{e^2}{6\pi^2 \varepsilon}$$
 (12.78)

This gives us

$$c = \frac{1}{12\pi^2} \tag{12.79}$$

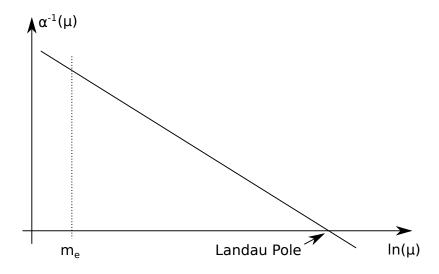
and thus the β factor at 1-loop level as

$$\beta(e) = \frac{e^3}{12\pi^2} \tag{12.80}$$

• Note what this implies for the interaction strength:

$$\frac{d}{d\ln(\mu)} \left(\frac{1}{e^2}\right) = -\frac{2}{e^3} \beta(e) = -\frac{2}{12\pi^2}$$
 (12.81)

We thus get a so-called "Landau pole". This occurs at very high energy scales (way above the Planck scale) and is another sign for our QFT being an effective theory for low energies.



12.5 The Ward-Takahashi identity

The Ward-Takahashi identity is an identity between amplitutes or Green's functions which relies on gauge invariance.

- We start with an illustrative example calculation.
- We consider a leading order three-point function with the γ -propagator amputated but the fermion propagator present, and contract this with k^{μ} :

$$k^{\mu} \cdot \left(k, \mu, \rightarrow \qquad p \right) = \frac{i}{(\not p + \not k) - m} i e \not k \frac{i}{\not p - m} = (-e) \left\{ \frac{i}{\not p - m} - \frac{i}{(\not p + \not k) - m} \right\}$$

$$(12.82)$$

Pictorially this can be understood as

$$k_{\mu} \cdot \frac{k}{p} = (-e) \left\{ k_{\beta} \right\} p - \left\{ p + k_{\beta} \right\}$$

- In words an external γ -line contracted with k_{μ} can be removed and the vertex replaced by the difference of the two propagators before and after the vertex (multiplied by (-e)).
- We can also amputate the external fermion propagators by multiplying with

$$-((p + k) - m) \{...\} (p - m)$$
 (12.83)

which yields

$$-ek_{\mu}\Gamma^{\mu} = i\left\{ ((p + k) - m) - (p - m) \right\}$$
 (12.84)

where in our leading order case $\Gamma^{\mu} = \gamma^{\mu}$.

• Further, we can look at the actual matrix element by putting p and p + k onshell and multiplying with $\bar{u}(p + k) \dots u(p)$:

$$\bar{u}(p+k)k_{\mu}\Gamma^{\mu}u(p) = 0 \tag{12.85}$$

which is just a special case of the general gauge-invariance based claim $k_{\mu}M^{\mu}$ = 0 for any physical amplituted with external photon.

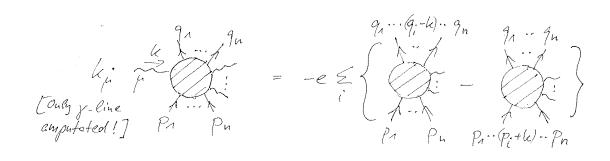
• The key interest is in the simple generalization of our argument:

• In the complete sum, all terms except the last and the first cancel pairwise. We thus get

• The same argument can be made for closed loops:

$$\sum_{\text{all insertions}} k_{\mu} \cdot \mu \longrightarrow_{\vec{k}} = 0 \qquad (12.86)$$

*) need also to shift integration variable at the end *General Theorem:*



- 1) *Corollary:* Amputate fermion lines (multiplying with q_i , $p_i \forall i$), go on shell, multiply with external spinors ($\bar{u}(q_i)$, $u(p_i)$)
 - ⇒ Find zero at RHS due to factors like

$$\bar{u}(q_j)iq_j\frac{i}{q_j-k}=0$$

 $\Rightarrow k_{\mu}\mathcal{M}^{\mu} = 0$ for \mathcal{M}^{μ} a physical amplitude, i.e. $\mathcal{M} = \varepsilon_{\mu}\mathcal{M}^{\mu}$ (12.87)

2) Corollary: look at 3-point-case

$$k_{\mu}$$
 $\begin{pmatrix} k & P+k \\ P & P \end{pmatrix} = (-e) \begin{pmatrix} P & P+k \\ P & P+k \end{pmatrix}$

Use S(p):

$$S(p) = \frac{i}{\not p - m - \Sigma(p)}$$

$$\Rightarrow S(p+k) \left(iek_{\mu} \Gamma^{\mu}(p+k,p) \right) S(p) = (-e) \left(S(p) - S(p+k) \right)$$

$$\Leftrightarrow -ik_{\mu} \Gamma^{\mu}(p+k,p) = S^{-1}(p+k) - S^{-1}(p)$$
(12.88)

- \Rightarrow Divergence in $k_{\mu}\Gamma^{\mu}$ at $k \to 0$ is the same as divergence in Σ'
- \Rightarrow We can (must to keep gauge invariance) choose $Z_{\psi}Z_A^{1/2}Z_e=Z_{\psi}$ (i.e. $Z_1=Z_2$)
- 12.6 Sketch of an operator derivation of the Ward Takahashi identity

- Missing -

13. Non-abelian Gauge Theory and Standard Model

13.1 Non-abelian gauge theory

Remember: $\mathcal{L} = (\partial_{\mu}\phi)\overline{(\partial^{\mu}\phi)} - m^2\overline{\phi}\phi$

is invariant under $\phi(x) \to e^{i\alpha(x)}\phi(x) \Rightarrow U(1)$ -gauge-theory

Now let $\phi(x) \in V$ (vector space), such that \mathcal{L} is invariant under a group G acting on V through a representation R:

$$\phi(x) \to R(g) \cdot \phi(x)$$
, want $g = g(x)$ (13.1)

- To be more concrete, focus on G = SU(n), R is the fundamental representation
- In this case:

$$\mathcal{L} = \overline{(\partial^{\mu}\phi)_{j}}(\partial_{\mu}\phi)^{j} - m^{2}\overline{\phi}_{j}\phi^{j}$$

$$= (\partial_{\mu}\phi)^{\dagger}(\partial^{\mu}\phi) - m^{2}\phi^{\dagger}\phi$$
(13.2)

This is obviously invariant under $\phi \to U\phi$, $U \in SU(n)$, as $U^{\dagger}U = 1$

• Let us, as in the U(1) case introduce:

$$D_{\mu}\phi = \partial_{\mu}\phi + iA_{\mu}\phi,\tag{13.3}$$

where A_{μ} is a matrix and demand

$$D_{\mu}\phi \to UD_{\mu}\phi,$$
 (13.4)

even if U = U(x) (This will ensure the invariance of \mathcal{L}). This will be true if:

$$D'_{\mu} \stackrel{!}{=} UD_{\mu}U^{\dagger}$$

$$\Leftrightarrow \mathbb{1}\partial_{\mu} + iA'_{\mu} \stackrel{!}{=} U \left(\mathbb{1}\partial_{\mu} + iA_{\mu}\right) U^{\dagger}$$

$$\Leftrightarrow \mathbb{1}\partial_{\mu} + iA'_{\mu} \stackrel{!}{=} \partial_{\mu} + U(\partial_{\mu}U^{\dagger}) + UiA_{\mu}U^{\dagger}$$

$$\Rightarrow A'_{\mu} \stackrel{!}{=} UA_{\mu}U^{\dagger} - iU(\partial_{\mu}U^{\dagger})$$

$$(13.5)$$

• Let us look at the infinitesimal version $U = e^{iT}$, $T \in \mathfrak{g}$ is small.

$$\stackrel{LO \ in \ T}{\Rightarrow} \delta A_{\mu} = i \left[T, A_{\mu} \right] \underbrace{-\partial_{\mu} T}_{*} + \mathcal{O} \left(T^{2} \right)$$
 (13.6)

*) We also have this in abelian gauge theory. But note: for abelian case [.,.] vanishes, so the expression holds.

• We see that if $A_{\mu} \in \mathfrak{g}$, it will stay so! This completes our construction:

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{matter}}$$

$$= \underbrace{-\frac{1}{2g^2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu}}_{\text{gauge}} + \underbrace{\left(D_{\mu} \psi\right)^{\dagger} \left(D^{\mu} \psi\right) - m^2 \psi^{\dagger} \psi}_{\text{matter}}, \tag{13.7}$$

where $F_{\mu\nu} = -i[D_{\mu}, D_{\nu}]$, $D_{\mu} = \partial_{\mu} + iA_{\mu}$ and $A_{\mu} \in \mathfrak{g}$.

• Invariance straight forwardly follows from

$$D_{\mu}' = UD_{\mu}U^{\dagger} \tag{13.8}$$

• In particular, $F_{\mu\nu}$ is defined as a differential operator but happens to be just a matrix. Also:

$$F'_{\mu\nu} = UF_{\mu\nu}U^{\dagger}$$
tr F^2 is invariant (13.9)

• *Sometimes:* Convenient to choose a basis $T^a \in \mathfrak{g}$, $a = 1, \dots, \dim(G)$. Then:

$$A_{\mu} = A_{\mu}^{a} T^{a}; \qquad F_{\mu\nu} = F_{\mu\nu}^{a} T^{a}$$
 (13.10)

Consider:

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + i[A_{\mu}, A_{\nu}]$$
 (13.11)

Decompose both sides in components, use the definition of structure constants of Lie algebras $[T^a, T^b] = if^{abc}T^c$

$$\Rightarrow F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f^{abc} A_\mu^b A_\nu^c \tag{13.12}$$

Also if we choose a basis where $tr(T^aT^b) = \frac{1}{2}\delta^{ab}$, then

$$tr(F_{\mu\nu}F^{\mu\nu}) = \frac{1}{2}F^a_{\mu\nu}F^{\mu\nu\,a}$$
 (13.13)

• Generalization to fermions:

$$\mathcal{L}_{\text{matter}} = \bar{\psi}(i\not D - m)\psi$$

$$= (\bar{\psi}_k)^a \left[i(\gamma^\mu)_a^b \left(\partial_\mu \delta_j^k + iA_\mu^\alpha (\underbrace{T^\alpha}_*)_j^k \right) - m\delta_a^b \delta_j^k \right] (\psi^j)_b$$
(13.14)

*) possibly $R(T^{\alpha})$ for other representation.

13.2 Standard Model

• We have:

$$\mathcal{L}_{\text{gauge}} = -\sum_{i=1}^{3} \frac{1}{2g_i^2} \operatorname{tr} \left(F_{\mu\nu}^{(i)} F^{(i)\mu\nu} \right), \tag{13.15}$$

where the group is $G = SU(3) \times SU(2) \times U(1)$ and i = 1, 2, 3 corresponds to U(1), SU(2), SU(3), respectively

• Fermions come in 3 generations:

$$\mathcal{L}_{\text{matter}} = \sum_{a=1}^{3} \bar{\psi}_{L}^{a} i \not \!\! D \psi_{L}^{a} \quad \text{all left handed}$$
 (13.16)

where for each *a*, we have the fields:

$$\psi_L^a = \{ Q^a, (u^c)^a, (d^c)^a, L^a, (e^c)^a \}$$

$$= \underbrace{(3,2)_{1/3}}_{*} + (\bar{3},1)_{-4/3} + (\bar{3},1)_{2/3} + (1,2)_{-1} + (1,1)_{2}$$
(13.17)

*) 3 stands for the fundamental representation of SU(3), 2 for the fundamental representation of SU(2) and the index gives the charge under U(1) (here: $\psi \rightarrow e^{i\alpha/3}\psi$).

The second line in 13.17 fixes all the couplings in \mathcal{L}_{matter}