Géométrie différentielle et théorie de jauge, 07/01/2020

1. For the sphere $S^n \subset \mathbb{R}^{n+1}$ write explicitly the two stereographic projections, and check that the transition between the two charts is the inversion $x \mapsto \frac{x}{|x|^2}$.

2. Prove that $\mathbb{C}P^1$ is diffeomorphic to S^2: denote by p_N the stereographic projection from the north pole, then consider the map $f : \mathbb{C}P^1 \to S^2$ defined by

$$\begin{cases} f([1:z]) & = p_N^{-1}(z), \\ f([0:1]) & = N. \end{cases}$$

Hint: prove that $f([z:1]) = p_S^{-1}(\bar{z})$.

3. Define the open set $U_i \subset \mathbb{R}P^n$ or $\mathbb{C}P^n$ $(i = 0, ..., n)$ by $U_i = \{[x^0 : \cdots : x^n], x^i \neq 0\}$, and the charts $\phi_i : U_i \to \mathbb{R}^n$ or $\mathbb{C}^n$ by $\phi_i[x^0 : \cdots : x^n] = (\frac{x^0}{x^i}, ..., \hat{1}, ..., \frac{x^n}{x^i})$. Prove that these charts give a manifold structure on $\mathbb{R}P^n$ and $\mathbb{C}P^n$, of respective (real) dimensions n and $2n$.

4. Prove the **theorem of submersions** : if $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ satisfies $d_x f$ surjective, then there exists an open neighbourhood V of x and a diffeomorphism $\phi : V \to W \subset \mathbb{R}^n$ such that

$$f \circ \phi^{-1}(x^1, \ldots, x^n) = (x^1, \ldots, x^k).$$

(Take a basis of $\mathbb{R}^n$ such that $\ker d_x f = \langle e_{k+1}, \ldots, e_n \rangle$ and apply the inverse function theorem to the map $y \mapsto (f(y), y^{k+1}, \ldots, y^n)$.

Deduce that if f is a submersion at any point of $f^{-1}(c)$, then $f^{-1}(c)$ is a submanifold of $\mathbb{R}^n$ of dimension $n-k$, with tangent space at x equal to $\ker d_x f$.

5. The example of the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ can be generalized in the following way. A *discrete* group Γ acts freely on M^n if for any $x \in M$, the isotropy group $\Gamma_x = \{g \in \Gamma, g \cdot x = x\}$ is trivial ; Γ acts properly if for any compact $K \subset M$, the set $\{g \in \Gamma, g(K) \cap K \neq \emptyset\}$ is finite. Suppose Γ acts freely and properly on M, and let $p : M \to M/\Gamma$ denote the projection. Prove that for each $x \in M$ there exists a small open set $U_x \ni x$ inside the domain of a chart ϕ_x such that $p|_{U_x}$ is injective. Deduce that the charts $(p(U_x), \phi_x \circ (p|_{U_x})^{-1})$ give the structure of a n-dimensional manifold on the quotient M/Γ, such that p is a local diffeomorphism.