Géométrie différentielle et théorie de jauge
Solutions to exercises

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3. Cohomology of $\mathbb{R}^2 \setminus \mathbb{Z}$. Consider the point $p_n = (n, 0)$, where $n \in \mathbb{Z}$, and a small circle $C_n = \partial D_n$ centered at p_n, boundary of a disk D_n. In polar coordinates centered at the point $(n, 0)$, we define the angular form $d\theta_n = \frac{(x-n)dy - ydx}{(x-n)^2+y^2}$ which is defined on $\mathbb{R}^2 \setminus \{p_n\}$. Then, by Stokes theorem, if $p \neq n$,

$$\int_{C_p} d\theta_n = \int_{D_p} d(d\theta_n) = 0.$$

This does not work for $p = n$, since $d\theta_n$ is not defined on the whole D_n, and we have on the contrary

$$\int_{C_n} d\theta_n = 2\pi.$$

Since $\int_{C_n}$ is a linear form on $H^1(\mathbb{R}^2 \setminus \mathbb{Z})$, it follows that $([d\theta_n])_{n\in\mathbb{Z}}$ is a free family in $\mathbb{R}^2 \setminus \mathbb{Z}$, therefore $\dim H^1(\mathbb{R}^2 \setminus \mathbb{Z}) = \infty$.

4. Cohomology of $\mathbb{R}P^n$. Since $p \circ \tau = p$, for a form α on $\mathbb{R}P^n$ we have $p^*\alpha = (p\circ\tau)^*\alpha = \tau^*(p^*\alpha)$. Conversely, if $\tilde\alpha \in \Omega^k(S^n)$ satisfies $\tau^*\tilde\alpha = \tilde\alpha$, then, since p is a local diffeomorphism, near any $x \in \mathbb{R}P^n$, noting $x_1, x_2 = \tau(x_1) \in S^n$ the two points such that $p(x_i) = x$, we can find small open sets $U \ni x$, $U_1 \ni x_1$ and $U_2 = \tau(U_1) \ni x_2$ such that $p|_{U_1} : U_1 \to U$ and $p|_{U_2} : U_2 \to U$ are diffeomorphisms. Observe that $(p|_{U_1})_*(\tilde\alpha|_{U_1}) = (p|_{U_1})_*(\tau_*\tilde\alpha|_{U_2}) = (p|_{U_1}\circ\tau)_*(\alpha|_{U_2}) = (p|_{U_2})_*(\tilde\alpha|_{U_2})$. Therefore we have no choice but to define α on $U \ni x$ by $\alpha = (p|_{U_i})_*(\tilde\alpha|_{U_i})$ for $i = 1$ or 2, and we have $\tilde\alpha|_{p^{-1}(U)} = p^*\alpha|_U$. By uniqueness of α the various α thus constructed on small open sets match on intersections, and we have a global form α on $\mathbb{R}P^n$ such that $\tilde\alpha = p^*\alpha$.

Suppose $\alpha \in \Omega^k(\mathbb{R}P^n)$ satisfies $p^*\alpha = d\tilde\beta$. Since $\tau^*(\tilde\beta + \tau^*\tilde\beta) = \tilde\beta + \tau^*\tilde\beta$, there exists $\beta \in \Omega^{k-1}(\mathbb{R}P^n)$ such that $\frac{\tilde\beta+\tau^*\tilde\beta}{2} = p^*\beta$. Then $p^*\alpha = \tau^*p^*\alpha = \tau^*d\tilde\beta = d(\tau^*\tilde\beta) = d(\frac{\tilde\beta+\tau^*\tilde\beta}{2}) = dp^*\beta = p^*d\beta$, which implies $\alpha = d\beta$.

Therefore, if a closed form $\alpha \in \Omega^k\mathbb{R}P^n$ is such that $[p^*\alpha] = 0$ on S^n, then there exists $\beta \in \Omega^{k-1}\mathbb{R}P^n$ such that $\alpha = d\beta$, that is $[\alpha] = 0$. It follows that the application

$$[p^*] : H^k(\mathbb{R}P^n) \longrightarrow H^k(S^n)$$

is injective. Since $\tau^*p^*\alpha = p^*\alpha$, its image is contained in $\{c \in H^k(S^n), [\tau^*]c = c\}$; it is actually equal to this later space, since if $[\tau^*][\tilde\alpha] = [\tilde\alpha]$, then $\tau^*\tilde\alpha - \tilde\alpha = d\tilde\beta$, and therefore $[\tilde\alpha] = [\frac{\tilde\alpha+\tau^*\tilde\alpha}{2}] + [\frac{\tilde\alpha-\tau^*\tilde\alpha}{2}] = [\frac{\tilde\alpha+\tau^*\tilde\alpha}{2}]$; taking $\alpha \in \Omega^k(\mathbb{R}P^n)$ such that $p^*\alpha = \frac{\tilde\alpha+\tau^*\tilde\alpha}{2}$, we obtain $[\tilde\alpha] = [p^*\alpha] = [p^*][\alpha]$.

It follows that $[p^*]$ is an isomorphism on $\{c \in H^k(S^n), [\tau^*]c = c\}$. So $H^0(\mathbb{R}P^n) = \mathbb{R}$, $H^k(\mathbb{R}P^n) = 0$ for $0 < k < n$, and $H^n(\mathbb{R}P^n) = \mathbb{R}$ is n is odd or 0 if n is even (indeed, $H^n(S^n)$ is represented by the volume form Ω and we have seen that $\tau^*\Omega = (-1)^{n-1}\Omega$).

6. Action of a connected group on the cohomology. If a group G acts on M, each element $g \in G$ induces a diffeomorphism of M and therefore a linear action $[g^*] : H^k(M) \to H^k(M)$. We will see that this action is always trivial is G connected (this is not true if G is not connected, see the previous example with the group $\mathbb{Z}_2$ acting on S^n).

We have $\phi^*_{t+t'}\alpha = \phi^*_t(\phi^*_{t'}\alpha)$. Differentiating at $t' = 0$ gives

$$\frac{d}{dt}\phi^*_t\alpha = \phi^*_t\mathscr{L}_X\alpha = \phi^*_t\Big(di(X)\alpha + i(X)d\alpha\Big).$$

If α is closed ($d\alpha = 0$), then $\frac{d}{dt}\phi^*_t\alpha = \phi^*_t(di(X)\alpha) = d(\phi^*_t(i(X)\alpha))$, and therefore $\frac{d}{dt}[\phi^*_t\alpha] = 0$, that is the cohomology class $[\phi^*_t\alpha]$ is constant.

If the group G is connected, then we deduce that for any X in the Lie algebra $\mathfrak{g}$, we have $[(e^{tX})^*\alpha]$ is constant. Since the set $(e^X)_{X\in\mathfrak{g}}$ covers an open neighbourhood U of $1 \in G$, we have for any $g \in U$ the identity $[g^*\alpha] = [\alpha]$. If $g, g' \in U$ we have $[(gg')^*\alpha] = [(g')^*][\alpha] = [\alpha]$ and therefore $[g^*\alpha] = [\alpha]$ for all elements g in the group generated by U. Since U is an open neighbourhood of 1 and G is connected, U generates G itself and it follows that $[g^*\alpha] = [\alpha]$ for any $g \in G$. Therefore the action of G on $H(M)$ is trivial.