GENERAL RELATIVITY

M2 Theoretical physics

Correction ; October 16, 2019

1 Linearization of Einstein's equations

1-a) Let $k^{\mu\nu}$ be the perturbation of the contravariant quantity $g^{\mu\nu}=\eta^{\mu\nu}+k^{\mu\nu}$. By definition of the inverse, one has $g^{\mu\nu}g_{\nu\alpha}=\delta^{\mu}_{\alpha}$. At first order, it gives $\eta_{\eta_\beta}k^{\mu\nu}=-\eta^{\mu\nu}h_{\nu\beta}$. After multiplying by $\eta^{\alpha\beta}$, one finds

$$k^{\mu\nu}=-\eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}=-h^{\mu\nu},$$

so that $g^{\mu\nu}=\eta^{\mu\nu}-h^{\mu\nu}$.
Beware that $h^{\mu\nu}$ is then not the inverse of $h_{\mu\nu}$.
The manipulation from a covariant to a contravariant quantity reads

$$T^{\alpha}=g^{\alpha\beta}T_\beta=\left(\eta^{\alpha\beta}-h^{\alpha\beta}\right)T_\beta\approx\eta^{\alpha\beta}T_\beta.$$

This kind of computation is also true for the inverse operation. At first order, the indices can indeed be manipulated by η.

1-b) At first order, the Christoffel symbols are (one can use Cartesian coordinates so that the background terms vanish)

$$\Gamma^{\gamma}_{\alpha\beta}=\frac{1}{2}\eta^{\gamma\sigma}\left[\partial_\alpha h_{\sigma\beta}+\partial_\beta h_{\alpha\sigma}-\partial_\sigma h_{\alpha\beta}\right].$$

In the expression of the Ricci tensor, the terms in $\Gamma\Gamma$ are second order so that its expression reduces to $R_{\mu\nu}=\partial_\alpha\Gamma^{\alpha}_{\mu\nu}-\partial_\mu\Gamma^{\alpha}_{\alpha\nu}$ which gives

$$R_{\mu\nu}=\frac{1}{2}\eta^{\alpha\beta}\left[\partial_\alpha\partial_\nu h_{\mu\beta}+\partial_\mu\partial_\beta h_{\alpha\nu}-\partial_\alpha\partial_\beta h_{\mu\nu}-\partial_\mu\partial_\nu h_{\alpha\beta}\right].$$

The Ricci scalar is then

$$R=\eta^{\mu\nu}R_{\mu\nu}=\partial_\alpha\partial_\beta h^{\alpha\beta}-\eta^{\alpha\beta}\partial_\alpha\partial_\beta h.$$

Einstein's tensor is the given by $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}R\eta_{\mu\nu}$.

1-c) At first order, one has $\bar{h}=\eta^{\mu\nu}\bar{h}_{\mu\nu}=\eta^{\mu\nu}\left(h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h\right)=-h$.
Replacing the terms in $h_{\mu\nu}$ as a function of $\bar{h}_{\mu\nu}$ enables to remove all the terms involving the trace so that one has

$$2G_{\mu\nu}=-\eta^{\alpha\beta}\partial_\alpha\partial_\beta\bar{h}_{\mu\nu}+\eta^{\alpha\beta}\partial_\alpha\partial_\nu\bar{h}_{\mu\beta}+\eta^{\alpha\beta}\partial_\mu\partial_\beta\bar{h}_{\alpha\nu}-\eta_{\mu\nu}\partial_\alpha\partial_\beta\bar{h}^{\alpha\beta}.$$

The last three terms can be expressed as a function of V^α while the first one is the d'Alembertian so that one finds

$$2G_{\mu\nu}=-\Box\bar{h}_{\mu\nu}+\partial_\nu V_\mu+\partial_\mu V_\nu-\eta_{\mu\nu}\partial_\alpha V^\alpha.$$

2 Lorenz gauge

2-a) Let us compute $V'_\alpha = \eta^{\beta\gamma}\partial_\beta \bar{h}'_{\gamma\alpha}$. Replacing $\bar{h}'$ by its expression gives

$$V'_\alpha = V_\alpha + \Box\xi_\alpha.$$

In Einstein's tensor, replacing the $\bar{h}$ by their expression in terms of the $\bar{h}'$, one can show that $G_{\mu\nu} = G'_{\mu\nu}$. Indeed, the partial derivative commute with the operator $\Box$ so that all the terms containing the vector $\vec{\xi}$ vanish.

2-b) Starting from any gauge, one can consider the above transformation with a vector such that $\Box\xi_\alpha = -V_\alpha$. The transformation law for V_α then implies that $V'_\alpha = 0$.

In this gauge, Einstein's equations are simply written

$$\Box\bar{h}^{\mu\nu} = -16\pi T^{\mu\nu}.$$

2-c) In vacuum one has $T^{\mu\nu} = 0$. The wave must then verify two conditions :

- $V^\alpha = 0$ which reduces to $k_\mu A^{\mu\nu} = 0$. The wave is orthogonal to its direction of propagation.
- $\Box\bar{h}_{\mu\nu} = 0$, which gives $k_\alpha k^\alpha = 0$. This is the dispersion relation of the wave which shows that its velocity is the speed of light.

3 TT gauge (transverse and traceless)

3-a) Given that $k_\alpha k^\alpha = 0$, one has $\Box\xi^\alpha = 0$. All those transformations leave the V_α unchanged so that one stays in Lorenz gauge.

3-b) Given the expressions of $\bar{h}_{\mu\nu}$ and of ξ^α, one can find the transformation law for the amplitude $A_{\mu\nu}$ which is

$$A'_{\mu\nu} = A_{\mu\nu} + i\left(k_\mu B_\nu + k_\nu B_\mu - \eta_{\mu\nu} k_\tau B^\tau\right).$$

The traceless condition $\eta^{\mu\nu}A'_{\mu\nu} = 0$ then reads

$$k_\tau B^\tau = -\frac{i}{2}A.$$

Contracting the transversality condition with $\vec{u}$, one finds $u^\mu u^\nu A'_{\mu\nu} = 0$, which enables to compute the scalar product of $\vec{B}$ and $\vec{u}$:

$$u_\tau B^\tau = \frac{i}{2u_\tau k^\tau}\left[u^\alpha u^\beta A_{\alpha\beta} - \frac{A}{2}u_\sigma u^\sigma\right].$$

The transversality condition then reads $u^\mu A_{\mu\nu} + i\left(k_\tau u^\tau\right)B_\nu + i\left(B_\tau u^\tau\right)k_\nu - i\left(k_\tau B^\tau\right)u_\nu = 0$. Replacing $k_\tau B^\tau$ and $u_\tau B^\tau$ by the expressions found previously, one finds a unique value for $\vec{B}$:

$$B_\nu = \frac{i}{k_\tau u^\tau}\left[u^\mu\left(A_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}A\right) - \frac{k_\nu}{2k_\sigma u^\sigma}u^\alpha u^\beta\left(A_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}A\right)\right].$$

One can verify that this solution is indeed consistent with $k_\tau B^\tau = -\frac{i}{2}A$. The TT gauge thus fixes completely the coordinate system, once $\vec{u}$ is given.

3-c) The amplitude must fulfill :

- $k_\mu A^{\mu\nu} = 0$, that is 4 conditions.
- $\eta_{\mu\nu} A^{\mu\nu} = 0$, that is 1 condition.
- $u_\mu A^{\mu\nu} = 0$ that is a priori 4 conditions. However, the nullity in the direction of $\vec{k}$ is already accounted for (i.e. $k_\mu u_\nu A^{\mu\nu} = 0$), so that this last property corresponds to only 3 additional conditions.

$A^{\mu\nu}$ must fulfill 8 independent conditions for 10 components. One has 2 degrees of freedom.

3-d) The plane wave propagates along z so that the vector $\vec{k}$ is simply $k_\alpha = (\omega, 0, 0, \pm\omega)$ where one has taken into account $k_\alpha k^\alpha = 0$ and where the sign corresponds to the two possible directions of propagation. This gives $k_\alpha x^\alpha = \omega\,(t \pm z)$.

By hypothesis, one is in the TT gauge with $u^\alpha = (1, 0, 0, 0)$. The condition $u^\alpha A_{\alpha\beta} = 0$ then gives $A_{0\beta} = 0$.

Moreover $k^\alpha A_{\alpha\beta} = 0$ implies that $A_{z\beta} = 0$.

Last, the trace of A is zero so that $A_{xx} = -A_{yy}$.

The only non-vanishing parts of $A_{\mu\nu}$ are

$$\begin{aligned} A_{xx} &= -A_{yy} = A_+ \\ A_{xy} &= A_{yx} = A_\times . \end{aligned}$$

4 Action on matter

4-a) The point mass being at rest

$$u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = (1, 0, 0, 0)$$

which gives $x^\alpha = (\tau, 0, 0, 0)$.

When the wave reaches the particle, the trajectory is modified as $x^\alpha = (\tau + \delta_\tau, \delta_x, \delta_y, \delta_z)$. One then sees that u^0 is of order 0 whereas the u^i are of order 1.

This implies that, at the leading order, the geodesic equation is

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{00} = 0.$$

The spatial components then vary as

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} + \Gamma^i_{00} = 0.$$

Moreover, in the TT gauge, one can simply see that $\Gamma^i_{00} = 0$ and so : $\dfrac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = 0$.

The particle stays at the same spatial position. This result is true only in the TT gauge.

4-b) For a photon $\mathrm{d}s^2 = 0$, which gives

$$\mathrm{d}t^2 = (\delta_{ij} + h_{ij})\,\mathrm{d}x^i \mathrm{d}x^j .$$

By hypothesis, the distances are small with respect to the wavelength so that one obtains the integrated form

$$\Delta t^2 = (\delta_{ij} + h_{ij})\, x_B^i x_B^i.$$

The spatial coordinates of A being constants, its proper time coincides with the coordinate time and the length measured by A is then

$$L^2 = (\delta_{ij} + h_{ij})\, x_B^i x_B^i.$$

By posing $L = L_0 + \delta L$ and $x_B^i = L_0 n^i$,the result can be written as

$$\frac{\delta L}{L_0} = \frac{1}{2} h_{ij} n^i n^j.$$

4-c) The metric does not depend on the position so that

$$\mathrm{d}x'^i = \mathrm{d}x^i + \frac{1}{2}\delta_k^i h_{km}\mathrm{d}x^m + \frac{1}{2}\delta^{ik} x^j \partial_t h_{ij}\mathrm{d}t.$$

The term in $\mathrm{d}t$ is of second order because x^i and $\partial_t h_{ij}$ are small. At first order, one then has

$$\delta_{ij}\mathrm{d}x'^i \mathrm{d}x'^j = (\delta_{ij} + h_{ij})\, \mathrm{d}x^i \mathrm{d}x^j.$$

Fermi coordinates are purely local, which shows in the fact that one needs to consider that the x^i are small.

4-d) In the TT gauge, the spatial coordinates are constant so that : $x'^i = x_0^i + \frac{1}{2} h_{ij} x_0^j$.

For a plane wave along the z direction, one has $h_{xx} = -h_{yy} = h_+$ and $h_{xy} = h_{yx} = h_\times$ and one finds that

$$\begin{aligned} x' &= x_0 + \frac{1}{2}h_+ x_0 + \frac{1}{2}h_\times y_0 \\ y' &= y_0 + \frac{1}{2}h_\times x_0 - \frac{1}{2}h_+ y_0. \end{aligned}$$

5 Sources of gravitational waves

5-a) Each derivative of Q makes appear the inverse of a time, that is a factor v/R. So that one has $\dfrac{\mathrm{d}^3 Q}{dt^3} = \varepsilon \dfrac{M}{R} v^3$. The emitted power is of order

$$\mathcal{P} = \varepsilon^2 \left(\frac{M}{R}\right)^2 v^6.$$

A good source must be compact (M/R big), typically a black hole or a neutron star and move at relativistic speeds. Moreover, the geometry must be such that one has a varying quadrupole momentum (no axisymmetric systems for instance).

6 Binary system

6-a) Let us recall some relations valid in Newtonian dynamics

$$\begin{aligned} \omega^2 &= \frac{2m}{d^3} \\ E &= -\frac{m^2}{2d} \\ L^2 &= \frac{m^3 d}{2}. \end{aligned} \tag{1}$$

In the plane xy (the momentum is along z), the coordinates of the two masses are

$$M = \left(\pm \frac{d}{2} \cos(\omega t), \pm \frac{d}{2} \sin(\omega t) \right).$$

One then needs to compute the quadrupole. The xx component reads (in the case of point masses, the integral is replaced by a sum) :

$$Q_{xx} = 2m \left(x^2 - \frac{1}{3}\left(x^2 + y^2\right) \right) = m \frac{d^2}{2} \left(\cos^2(\omega t) - \frac{1}{3} \right).$$

By keeping only the varying parts, one finds that

$$\begin{aligned} Q_{xx} &= \frac{md^2}{4} \cos(2\omega t) \\ Q_{xy} &= \frac{md^2}{4} \sin(2\omega t) \\ Q_{yy} &= -\frac{md^2}{4} \cos(2\omega t). \end{aligned}$$

The emitted power is then $\mathcal{P} = \frac{1}{5} \left((Q'''_{xx})^2 + 2\left(Q'''_{xy}\right)^2 + \left(Q'''_{yy}\right)^2 \right)$, which reads

$$\mathcal{P} = \frac{8}{5} m^2 d^4 \omega^6 = \frac{64}{5} \frac{m^5}{d^5}.$$

In the same manner, the momentum varies as $L' = -\frac{2}{5}\left(Q''_{xx} Q'''_{xy} + Q''_{xy} Q'''_{yy} - Q''_{xy} Q'''_{xx} - Q''_{yy} Q'''_{yx} \right)$ which gives

$$L' = -\frac{8}{5} m^2 d^4 \omega^5 = -\frac{32}{5} \frac{m^4}{d^3} \sqrt{\frac{2m}{d}}.$$

6-b) In Newtonian dynamics $EL^2 = -m^5/4$ is constant for all circular orbits. Conversely, the orbit will stay circular if the emission is such that this quantity remains constant. Using logarithmic derivatives, this means that one must have

$$\frac{\mathrm{d}E}{E} + 2\frac{\mathrm{d}L}{L} = 0.$$

Given the quantities found previously one has :

$$\begin{aligned} \frac{\mathrm{d}E}{E} &= \frac{128}{5}\frac{m^3}{d^5} \\ \frac{\mathrm{d}L}{L} &= -\frac{64}{5}\frac{m^3}{d^5} \end{aligned}$$

which validates the proposal. In fact one could have shown that the emission of gravitational waves tend to circularize initially elliptic orbits.

6-c) Differentiating the expression of the energy with respect to time one finds $E' = \dfrac{m^2}{2d^2}d'$. If this variation is only due to gravitational radiation, one finds a differential equation for the separation

$$d^3 d' = -\frac{128}{5}m^3$$

which can be integrated as

$$d\left(t\right) = \left(d_0^4 - \frac{512}{5}m^3 t\right)^{1/4}.$$

The coalescence occurs when $d = 0$, that is for a time

$$T = \frac{5d_0^4}{512m^3}.$$