ICFP M2 – SOFT MATTER PHYSICS
Tutorial 2. The $n = 0$ theorem

Jean-Marc Di Meglio and Thomas Salez

One of the important achievements of de Gennes was to draw a connexion between critical phenomena and polymer statistics. Here, we would like to understand the essence of this link – without questioning the mathematical legitimity of the invoked limit. We consider the following magnetic system : a d-dimension periodic lattice at temperature T, at each site i of which is located one n-component classical spin $\mathbf{S}_i$. Two spins $\mathbf{S}_i$ and $\mathbf{S}_j$ interact through the exchange energy $-J_{i,j}\,\mathbf{S}_i{\cdot}\mathbf{S}_j$, with $J_{i,j} > 0$ to promote alignment. We assume $J_{i,j} = k_\mathrm{B}T_J$ (where k_B is the Boltzmann constant) to be a finite constant when i and j correspond to neighbouring sites, and we impose $J_{i,j}$ to be zero otherwise. Finally, for practical purposes, we choose the following normalization : $\mathbf{S}_i^2 = n$.

I Preliminaries

1 Express the Hamiltonian H of the problem.

2 Write the associated canonical partition function Z as an integral over the solid angles Ω_i of all the spins $\mathbf{S}_i$.

3 By introducing $\Omega = \prod_i \Omega_i$, rewrite Z as an average (noted $\langle\cdot\rangle_0$) over all possible orientations of the spins.

4 Recall the definition of the average $\langle\cdot\rangle$ in the canonical ensemble, and relate it to $\langle\cdot\rangle_0$.

II The $n = 0$ characteristic function

We consider a given spin $\mathbf{S}_i \equiv \mathbf{S}$ as a uniformly distributed random vector.

1 Write the characteristic function $f(\mathbf{k})$, of wave vector $\mathbf{k}$, associated to $\mathbf{S}$.

2 Explain how f allows to generate the successive moments of the distribution of $\mathbf{S}$. Relate in particular $\langle S_\alpha S_\beta\rangle_0$ to f, where the α and β indexes indicate components of $\mathbf{S}$.

3 By invoking the normalization of $\mathbf{S}$, demonstrate that f satisfies the following partial differential equation :

$$\Delta f + nf = 0\ , \tag{1}$$

where Δ is the Laplacian operator in Fourier space.

4 By symmetry of the problem, how does the orientation of $\mathbf{k}$ matter ? Simplify Eq. (2) into the following ordinary differential equation :

$$f" + \frac{n-1}{k} f' + nf = 0 \ , \tag{2}$$

where the prime denotes the derivative with respect to the norm $k = ||\mathbf{k}||$ of $\mathbf{k}$.

5 What are the boundary conditions at $k = 0$?

6 When $n = 0$, solve for f and conclude about the moments of the distribution of $\mathbf{S}$.

III Loop expansion of the $n = 0$ partition function

1 Show that the partition function has the exact expansion at $n = 0$:

$$\frac{Z}{\Omega} = \lim_{n\to 0} \left\langle \prod_{(i<j)} \left[1 + \frac{T_J}{T} \sum_{\alpha} S_{i\alpha} S_{j\alpha} + \frac{1}{2} \left(\frac{T_J}{T} \right)^2 \sum_{\alpha,\beta} S_{i\alpha} S_{i\beta} S_{j\alpha} S_{j\beta} \right] \right\rangle_0 \tag{3}$$

where the notation $(i < j)$ indicates that $i < j$ together with sites i and j being nearest neighbours.

2 Propose a diagrammatic representation, involving bonds connecting sites on a lattice as well as "charges", of each term in the bracket $[\cdot]$.

3 Show that only closed loops made of the previous elementary diagrams contribute to Z. What about loops with self-intersections ?

4 Calculate Z using those considerations, and comment the obtained result.

IV The $n = 0$ theorem

1 Express the correlator $\langle S_{p\delta} S_{q\delta} \rangle$, for two given sites p and q and a component index δ, as an average $\langle \cdot \rangle_0$ over all possible orientations of all the spins.

2 As in Eq. (3), provide an exact expansion of $\langle S_{p\delta} S_{q\delta} \rangle$ in the $n = 0$ case.

3 What are the diagrams contributing to $\langle S_{p\delta} S_{q\delta} \rangle$?

4 Let $W_N(p, q)$ be the number of self-avoiding random walks linking sites p and q in exactly N steps. Show that for $n = 0$:

$$\langle S_{p\delta} S_{q\delta} \rangle = \sum_N W_N(p, q) \left(\frac{T_J}{T} \right)^N \ . \tag{4}$$

Comment this central result and link it with polymer physics.