GENERAL RELATIVITY

M2 Theoretical physics

Correction ; September 25, 2019

1 Auto-parallel and affine parameters

1-a) Consider a new parameter λ' along the curve. The tangent vector is such that

$$U^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda'} \times \frac{\mathrm{d}\lambda'}{\mathrm{d}\lambda}.$$

Let us define $f(\lambda) = \frac{\mathrm{d}\lambda'}{\mathrm{d}\lambda}$. Then one has $U^\alpha = f U'^\alpha$.

One wants that $U'^\alpha D_\alpha U'^\beta = 0$. Replacing $\vec{U}'$ by its expression in terms of $\vec{U}$, this condition gives

$$\begin{aligned} & \frac{1}{f} U^\alpha D_\alpha \left(\frac{1}{f} U^\beta\right) = 0 \\ \Longleftrightarrow \quad & \frac{1}{f} U^\alpha D_\alpha U^\beta - \frac{f'}{f^2} U^\beta = 0 \qquad \text{(Leibniz)} \\ \Longleftrightarrow \quad & U^\beta \left(C - \frac{f'}{f}\right) = 0 \qquad \text{(geodesic).} \end{aligned}$$

One will have the right property if and only if $\frac{f'}{f} = C(\lambda)$ that can be integrated as

$$f(\lambda) = f_0 \exp\left(\int_0^\lambda C(\lambda')\,\mathrm{d}\lambda'\right).$$

1-b) The two parameters verify the above condition so f_1 and f_2 have a constant ratio $f_1/f_2 = A$. One then gets $\mathrm{d}\lambda_1 = A\mathrm{d}\lambda_2$, which can be integrated as

$$\lambda_1 = A\lambda_2 + B.$$

This is an affine-law, hence the name affine parameters.

2 Classification of the auto-parallel curves

2-a) Let us write $n = U_\mu U^\mu$. The variation of the norm along the curve is given by

$$\begin{aligned} \frac{\mathrm{d}n}{\mathrm{d}\lambda} &= U^\alpha D_\alpha (U_\mu U^\mu) \\ &= 2U_\mu (U^\alpha D_\alpha U\mu) \quad \text{(Leibniz)} \\ &= 2C(\lambda)\, n \quad \text{(geodesic).} \end{aligned}$$

One can integrate this equation and show that

$$n = n_0 \exp\left(\int_0^{\lambda} 2C\left(\lambda'\right) \mathrm{d}\lambda'\right),$$

where it is explicit that n can not change sign.

2-b) For an affine parameter $C = 0$ and so n is constant.

By choosing a new affine parameter $\lambda' = a\lambda + b$, one modifies the tangent vector as $U'^{\alpha} = \frac{1}{a}U^{\alpha}$ (same thing for the covariant version). The norm is so modified as $n' = \frac{1}{a^2}n$. One can then choose a so that the norm is

- -1 for massive particles.
- 0 for massless particles (photons).
- $+1$ for space-like geodesics (which are not the trajectories of physical objects).

3 Variational calculus

3-a) Let us write Euler-Lagrange equations for $\mathcal{L} = \sqrt{-g_{\mu\nu}U^{\mu}U^{\nu}}$. The variables are the x^{μ} and are associated with $U^{\mu} = \dfrac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}$.

One shows that

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x^{\alpha}} &= -\frac{\partial_{\alpha} g_{\mu\nu}}{2\mathcal{L}} U^{\mu}U^{\nu} \\
\frac{\partial \mathcal{L}}{\partial U^{\alpha}} &= -\frac{g_{\mu\alpha}U^{\mu}}{\mathcal{L}} \\
\frac{\mathrm{d}\left(\frac{\partial \mathcal{L}}{\partial U^{\alpha}}\right)}{\mathrm{d}\lambda} &= -\frac{\partial_{\nu} g_{\mu\alpha} U^{\mu}U^{\nu}}{\mathcal{L}} - \frac{g_{\mu\alpha}\frac{\mathrm{d}U^{\mu}}{\mathrm{d}\lambda}}{\mathcal{L}} + \frac{g_{\mu\alpha}U^{\mu}}{\mathcal{L}^2}\left(\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\lambda}\right).
\end{aligned} \tag{1}$$

Euler-Lagrange equations are then (after multiplication by $g^{\alpha\beta}$ to isolate the term in $\frac{\mathrm{d}U^{\mu}}{\mathrm{d}\lambda}$) :

$$\frac{\mathrm{d}U^{\beta}}{\mathrm{d}\lambda} + \frac{1}{2} g^{\alpha\beta}\left(2\partial_{\nu} g_{\mu\alpha} - \partial_{\alpha} g_{\mu\nu}\right) U^{\mu}U^{\nu} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\lambda}\frac{U^{\beta}}{\mathcal{L}}.$$

3-b) For one geodesic with affine parameter, one has shown that $\mathcal{L}$ is constant and so the right-hand side vanishes. Moreover one can see the Christoffel symbol in the parenthesis (the expression being symmetric in μ and ν). One then finds that

$$\frac{\mathrm{d}U^{\beta}}{\mathrm{d}\lambda} + \Gamma^{\alpha}_{\mu\nu} U^{\mu}U^{\nu} = U^{\nu} D_{\nu} U^{\mu} = 0.$$

which is the geodesic equation.

The variation of the action $\mathcal{L}'$ leads to the same equations, the computation being the same except for terms in $1/\mathcal{L}$.

3-c) One uses the action $\mathcal{L}' = R^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}^2 + \sin^2\theta \frac{\mathrm{d}\varphi}{\mathrm{d}\lambda}^2\right)$.

One then gets

$$
\begin{aligned}
\frac{\partial \mathcal{L}'}{\partial \theta} &= 2R^2 \sin\theta\cos\theta \frac{\mathrm{d}\varphi}{\mathrm{d}\lambda}^2 \\
\frac{\partial \mathcal{L}'}{\partial \varphi} &= 0 \\
\frac{\partial \mathcal{L}'}{\partial\left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)} &= 2R^2 \frac{\mathrm{d}\theta}{\mathrm{d}\lambda} \\
\frac{\partial \mathcal{L}'}{\partial\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda}\right)} &= 2R^2 \sin^2\theta \frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} \\
\frac{\mathrm{d}\left(\frac{\partial \mathcal{L}'}{\partial\left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)}\right)}{\mathrm{d}\lambda} &= 2R^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}\lambda^2} \\
\frac{\mathrm{d}\left(\frac{\partial \mathcal{L}'}{\partial\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda}\right)}\right)}{\mathrm{d}\lambda} &= 2R^2 \sin^2\theta \frac{\mathrm{d}^2\varphi}{\mathrm{d}\lambda^2} + 4R^2\sin\theta\cos\theta \frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda}. \qquad (2)
\end{aligned}
$$

Hence the two following equations

$$
\begin{aligned}
\frac{\mathrm{d}^2\theta}{\mathrm{d}\lambda^2} &= \sin\theta\cos\theta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda}\right)^2 \\
\frac{\mathrm{d}^2\varphi}{\mathrm{d}\lambda^2} &= -2\frac{\cos\theta}{\sin\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda}
\end{aligned}
$$

on which one can read the Christoffel symbols of the sphere $\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta$ and $\Gamma^{\varphi}_{\varphi\theta} = \frac{\cos\theta}{\sin\theta}$.

3-d) The variation of the action with respect to t is given by

$$
\begin{aligned}
\frac{\partial \mathcal{L}'}{\partial t} &= 2He^{(2Ht)}\left(\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right)^2\right) \\
\frac{\partial \mathcal{L}'}{\partial\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)} &= -2\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right) \\
\frac{\mathrm{d}\left(\frac{\partial \mathcal{L}'}{\partial\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)}\right)}{\mathrm{d}\lambda} &= -2\frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2},
\end{aligned}
$$

So that

$$
\begin{aligned}
\frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} &+ He^{(2Ht)}\left(\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right)^2\right) = 0 \\
\Gamma^t_{xx} &= \Gamma^t_{yy} = \Gamma^t_{zz} = He^{(2Ht)}.
\end{aligned}
$$

Varying with respect to x, one gets

$$
\begin{aligned}
\frac{\partial \mathcal{L}'}{\partial x} &= 0 \\
\frac{\partial \mathcal{L}'}{\partial \left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)} &= 2e^{(2Ht)}\frac{\mathrm{d}x}{\mathrm{d}\lambda} \\
\frac{\mathrm{d}\left(\frac{\partial \mathcal{L}'}{\partial\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)}\right)}{\mathrm{d}\lambda} &= 4He^{(2Ht)}\frac{\mathrm{d}x}{\mathrm{d}\lambda}\frac{\mathrm{d}t}{\mathrm{d}\lambda} + 2e^{(2Ht)}\frac{\mathrm{d}^2x}{\mathrm{d}\lambda^2},
\end{aligned}
$$

from which one gets

$$
\begin{aligned}
\frac{\mathrm{d}^2x}{\mathrm{d}\lambda^2} &+ 2H\frac{\mathrm{d}x}{\mathrm{d}\lambda}\frac{\mathrm{d}t}{\mathrm{d}\lambda} = 0 \\
\Gamma^x_{xt} &= \Gamma^y_{yt} = \Gamma^z_{zt} = H.
\end{aligned}
$$

4 Event horizon

4-a) For radial photons, one has $\mathrm{d}\Omega = 0$ and $\mathrm{d}s^2 = 0$ which gives

$$
2\mathrm{d}r\mathrm{d}v = \left(1 - \frac{2M}{r}\right)\mathrm{d}v^2.
$$

One solution is $\mathrm{d}v = 0$, so that $v = v_0$. This is the trajectory of ingoing photons (hence the name of the coordinates).

The other family is given by

$$
2\frac{\mathrm{d}r}{\mathrm{d}v} = \left(1 - \frac{2M}{r}\right)
$$

which can be integrated as

$$
r - r_0 + 2M\ln\left(\left|\frac{r-2M}{r_0-2M}\right|\right) = \frac{1}{2}v.
$$

4-b) For $r > 2M$, r is an increasing function of v. Its smallest value then corresponds to $v = -\infty$ and is, given the expression found in 4-a) $r = 2M^+$. So the trajectories always lie at $r > 2M$. When $v = +\infty$ one can check that $r = +\infty$.

4-c) For $r < 2M$, r is a decreasing function of v. Its maximal value is obtained for $v = -\infty$ and corresponds to $r = 2M^-$. The smallest value of r is the origin, reached for $v = -2r_0 + 4M\ln\left(\left|\frac{2M}{2M-r_0}\right|\right)$.

4-d) If one starts at $r = 2M$ then one has $\frac{\mathrm{d}r}{\mathrm{d}v} = 0$ so that the trajectory is simply $r = 2M$.

5 Geodesic and conserved quantities

5-a) Let us consider the scalar $\xi_\nu U^\nu$. Its variation along a geodesic is given by

$$
U^\mu D_\mu\left(\xi_\nu U^\nu\right) = U^\mu U^\nu D_\mu \xi_\nu + \xi_\nu\left(U^\mu D_\mu U^\nu\right) \qquad \text{(Leibniz)} \tag{3}
$$

The first term vanishes because it is a contraction of the symmetric tensor $U^\mu U^\nu$ with the anti-symmetric one $D_\mu \xi_\nu$ (Killing). The last term vanishes because of the geodesic equation so that $U^\nu D_\nu (\xi_\nu U^\nu) = 0$. The quantity $\xi_\nu U^\nu$ is conserved along the geodesic.

5-b) Given the symmetries of the angular part, one can restrict ourselves to $\theta = \pi/2$. The metric being independent of t and φ, the vectors $\vec{T} = (1,0,0,0)$ and $\vec{P} = (0,0,0,1)$ are Killing vectors.

Consider the tangent vector $\vec{U} = (U^t, U^r, 0, U^\varphi)$ (θ being constant) and E and L the two constants associated with the conservation of $T_\mu U^\mu$ and $P_\mu U^\mu$, respectively. One then gets

$$g_{tt} U^t = E \iff U^t = \frac{-E}{\left(1 - \frac{2M}{r}\right)}$$

$$g_{\varphi\varphi} U^\varphi = L \iff U^\varphi = \frac{L}{r^2}.$$

As seen in **2-b**, one can choose the parameter so that the norm of $\vec{U}$ is $\epsilon = -1, 0, +1$. Doing so, one finds that $g_{rr} (U^r)^2 + g_{tt} (U^t)^2 + g_{\varphi\varphi} (U^\varphi)^2 = \epsilon$. Replacing U^t and U^φ by their expression in terms of E and L one gets

$$(U^r)^2 + \left(\frac{L^2}{r^2} - \epsilon\right)\left(1 - \frac{2M}{r}\right) = E^2.$$

5-c) The above equation can be rewritten as $\dot{r}^2 + V(r) = K$, which is the analogous of an energy conservation in Newtonian dynamics (actually taking the non-relativistic limit of this equation gives exactly the energy conservation-law in the Newtonian regime).

The effective potential can be written as

$$V = \frac{2M}{r} + \frac{L^2}{r^2} - \frac{2ML^2}{r^3}.$$

Thus the equilibrium position are given by the extrema of V (the minima being the stable ones).

5-d) The equation $\partial_r V = 0$ gives

$$r^2 - 2\left(\frac{L^2}{2M}\right) r + 3L^2 = 0$$

which admits solutions only if $\Delta' = \frac{L^4}{4M^2} - 3L^2$ is positive, that is only if

$$L > L_{\text{crit.}} = \sqrt{3}\,(2M).$$

One then easily verifies that one solution corresponds to a stable one and that the other one is unstable.

5-e) When $L = L_{\text{crit.}}$ the two extrema coincide (inflexion point). The last stable orbit corresponds to this solution and is found for $L = L_{\text{crit.}}$. This gives $r = 6M$ which is outside the radius of the horizon but of the same order of magnitude.

6 Geodesic deviation

6-a) Let us compute the two quantities

$$
\begin{aligned}
S^\mu D_\mu U^\nu &= \left(\frac{\partial x^\mu}{\partial s}\right)\left[\partial_\mu\left(\frac{\partial x^\nu}{\partial \lambda}\right)+\Gamma^\nu_{\mu\alpha}\frac{\partial x^\alpha}{\partial \lambda}\right] \\
&= \frac{\partial^2 x^\nu}{\partial s\partial\lambda}+\Gamma^\nu_{\mu\alpha}\frac{\partial x^\alpha}{\partial\lambda}\frac{\partial x^\mu}{\partial s}.
\end{aligned}
$$

$$
\begin{aligned}
U^\mu D_\mu S^\nu &= \left(\frac{\partial x^\mu}{\partial \lambda}\right)\left[\partial_\mu\left(\frac{\partial x^\nu}{\partial s}\right)+\Gamma^\nu_{\mu\alpha}\frac{\partial x^\alpha}{\partial s}\right] \\
&= \frac{\partial^2 x^\nu}{\partial\lambda\partial s}+\Gamma^\nu_{\mu\alpha}\frac{\partial x^\alpha}{\partial\lambda}\frac{\partial x^\mu}{\partial s}.
\end{aligned}
$$

They are equal because the partial derivatives commute.

6-b) The computation makes use of the Leibniz rule, the definition of the Riemann tensor and of the property demonstrated in **6-a**.

$$
\begin{aligned}
A^\mu &= U^\alpha D_\alpha\left(U^\beta D_\beta S^\mu\right) \\
&= U^\alpha D_\alpha\left(S^\beta D_\beta U^\mu\right) \qquad (6-\mathrm{a}) \\
&= \left(U^\alpha D_\alpha S^\beta\right)(D_\beta U^\mu)+U^\alpha S^\beta D_\alpha D_\beta U^\mu \qquad \text{(Leibniz)} \\
&= \left(U^\alpha D_\alpha S^\beta\right)(D_\beta U^\mu)+U^\alpha S^\beta\left(D_\beta D_\alpha U^\mu+R^\mu_{\gamma\alpha\beta}U^\gamma\right) \qquad \text{(Riemann)} \\
&= S^\beta D_\beta\left(U^\alpha D_\alpha U^\mu\right)+R^\mu_{\gamma\alpha\beta}U^\gamma U^\alpha S^\beta \qquad (\text{Leibniz}+6-\mathrm{a}).
\end{aligned}
\tag{4}
$$

The first term vanishes because of the geodesic equation and one then finds

$$A^\mu = R^\mu_{\gamma\alpha\beta}U^\gamma U^\alpha S^\beta.$$

7 Raychaudhuri equation

7-a) Let us first show that P applied to a vector $\vec{V}$ is orthogonal to $\vec{U}$.

$$
\begin{aligned}
U_\alpha\left(P^\alpha_\mu V^\mu\right) &= U_\alpha\left(V^\alpha+U_\mu U^\alpha V^\mu\right) \\
&= U_\alpha V^\alpha - U_\mu V^\mu = 0 \qquad \text{(norm)}.
\end{aligned}
$$

One also needs to show that P is idempotent, meaning

$$
\begin{aligned}
P^\alpha_\beta P^\beta_\gamma V^\gamma &= \left(\delta^\alpha_\beta+U^\alpha U_\beta\right)\left(\delta^\beta_\gamma+U^\beta U_\gamma\right)V^\gamma \\
&= \left(\delta^\alpha_\gamma+U^\alpha U_\gamma\right)V^\gamma = P^\alpha_\gamma V^\gamma,
\end{aligned}
$$

which demonstrates the proposal.

7-b) One only needs to do the explicit calculation

$$\begin{aligned} P^\alpha_\mu B_{\alpha\nu} &= \left(\delta^\alpha_\mu + U^\alpha U_\mu\right)(D_\nu U_\alpha) \\ &= D_\nu U_\mu + U_\mu (U^\alpha D_\nu U_\alpha) \\ &= D_\nu U_\mu = B_{\mu\nu} \qquad \text{(norm)}. \end{aligned}$$

$$\begin{aligned} P^\alpha_\mu B_{\nu\alpha} &= \left(\delta^\alpha_\mu + U^\alpha U_\mu\right)(D_\alpha U_\nu) \\ &= D_\mu U_\nu + U_\mu U^\alpha D_\alpha U\nu \\ &= D_\mu U_\nu = B_{\nu\mu} \qquad \text{(geodesic)}. \end{aligned}$$

7-c) The computation proceeds as follows

$$\begin{aligned} U^\alpha D_\alpha B_{\mu\nu} &= U^\alpha D_\alpha D_\nu U_\mu \\ &= U^\alpha D_\nu D_\alpha U_\mu - R^\beta_{\mu\alpha\nu} U_\beta U^\alpha \qquad \text{(Riemann)} \\ &= D_\nu (U^\alpha D_\alpha U_\mu) - (D_\nu U^\alpha)(D_\alpha U_\mu) - R^\beta_{\mu\alpha\nu} U_\beta U^\alpha \qquad \text{(Leibniz)} \\ &= -B^\alpha{}_\nu B_{\mu\alpha} - R^\beta_{\mu\alpha\nu} U_\beta U^\alpha \qquad \text{(geodesic)}. \end{aligned}$$

7-d) One simply has

$$\begin{aligned} P^{\mu\nu} B_{\mu\nu} &= g^{\mu\alpha} P^\nu_\alpha B_{\mu\nu} \\ &= g^{\mu\alpha} (\delta^\nu_\alpha + U_\alpha U^\nu) B_{\mu\nu} \\ &= g^{\mu\nu} B_{\mu\nu} = \Theta \quad \text{because}\, U^\mu B_{\mu\nu} = 0. \end{aligned}$$

One can also verify that $P^{\mu\nu} P_{\mu\nu} = 3$, which justifies the decomposition of $B_{\mu\nu}$. Using the decomposition and the fact that $P^{\mu\nu}\omega_{\mu\nu} = 0$ (symmetric contacted with anti-symmetric), one does find that

$$P^{\mu\nu}\sigma_{\mu\nu} = 0.$$

7-e) The various terms involved in the equations are

$$g^{\mu\nu} U^\alpha D_\alpha B_{\mu\nu} = U^\alpha D_\alpha \Theta = \frac{\mathrm{d}\Theta}{\mathrm{d}\lambda}$$

$$\begin{aligned} g^{\mu\nu} R^\beta_{\mu\alpha\nu} U_\beta U^\alpha &= g^{\mu\nu} R_{\beta\mu\alpha\nu} U^\beta U^\alpha \\ &= R_{\alpha\beta} U^\alpha U^\beta. \end{aligned} \tag{5}$$

$$\begin{aligned} g^{\mu\nu} B^\alpha{}_\nu B_{\mu\alpha} = B^{\alpha\mu} B_{\mu\alpha} &= \left(\frac{1}{3}\Theta P^{\alpha\mu} + \sigma^{\alpha\mu} + \omega^{\alpha\mu}\right)\left(\frac{1}{3}\Theta P_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\mu\alpha}\right) \\ &= \frac{1}{3}\Theta^2 + \sigma^{\alpha\mu}\sigma_{\alpha\mu} - \omega^{\mu\alpha}\omega_{\mu\alpha} \end{aligned} \tag{6}$$

because of the contractions of symmetric and anti-symmetric tensors vanish. One also uses the fact that $P^{\mu\nu}\sigma_{\mu\nu} = 0$.

Putting all the pieces together one then gets Raychaudhuri equation

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\lambda} = -\frac{1}{3}\Theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - R_{\alpha\beta}U^\alpha U^\beta.$$

7-f) Einstein equation reads $G_{\mu\nu} = R_{\mu\nu} - 1/2Rg_{\mu\nu} = 8\pi T_{\mu\nu}$. Taking its trace leads to $R = -8\pi T$, so that one gets

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right).$$

A direct application of the strong energy condition to the vector U^α (which is time-like), thus gives

$$R_{\mu\nu}U^\mu U^\nu \geq 0.$$

7-g) Under the given assumptions and the strong energy condition, a direct examination of the Raychaudhuri equation gives the result

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\lambda} \leq -\frac{1}{3}\Theta^2.$$

7-h) The equation found in **7-g)** can be integrated between $\lambda = 0$ and λ and one gets

$$\frac{1}{\Theta} \geq \frac{1}{\Theta_0} + \frac{1}{3}\lambda$$

So for a parameter $\lambda \leq \lambda_c = -3/\Theta_0$, $1/\Theta$ will vanish, causing Θ to diverge. This is known as the focusing theorem and is related to the crossing of the geodesics (caustic). One has in particular shown that matter can only accelerate this effect and not suppress it.