The Topology of Chaos

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The Topology of Chaos

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October 17, 2008

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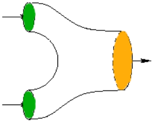

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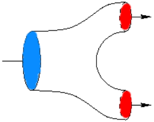

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October 23, 2008

Outline

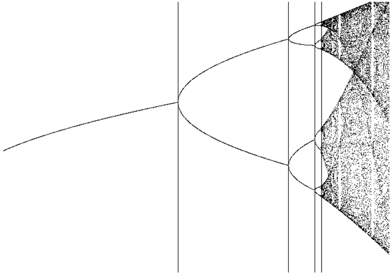
Background

J. R. Tredicce

Can you explain my data?

I dare you to explain my data!

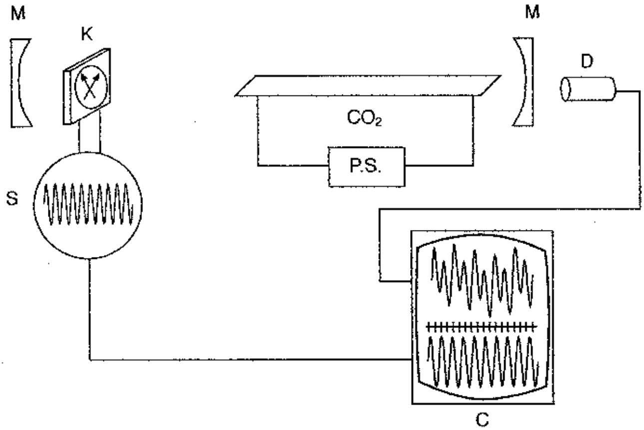
Where is Tredicce coming from?



Feigenbaum:

$$\alpha = \quad 4.66920\ 16091\ \ldots..$$
$$\delta = -2.50290\ 78750\ \ldots..$$

Laser with Modulated Losses Experimental Arrangement



Original Objectives

Construct a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

Result

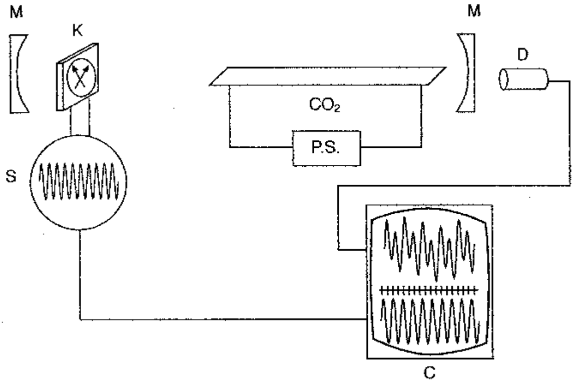
There is now a classification theory.

1. It is topological
2. It has a hierarchy of 4 levels
3. Each is discrete
4. There is rigidity and degrees of freedom
5. It is applicable to R^3 only — for now

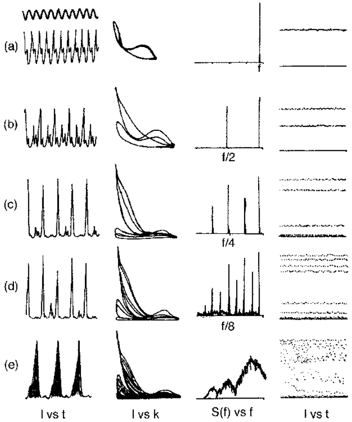
The 4 Levels of Structure

- **Basis Sets of Orbits**
- **Branched Manifolds**
- **Bounding Tori**
- **Extrinsic Embeddings**

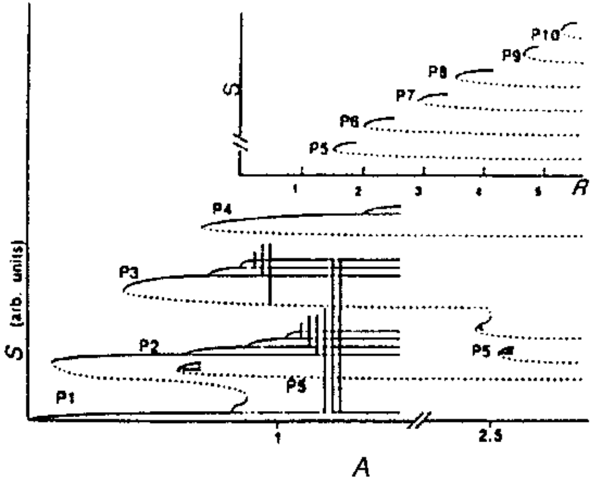
Laser Experimental Arrangement



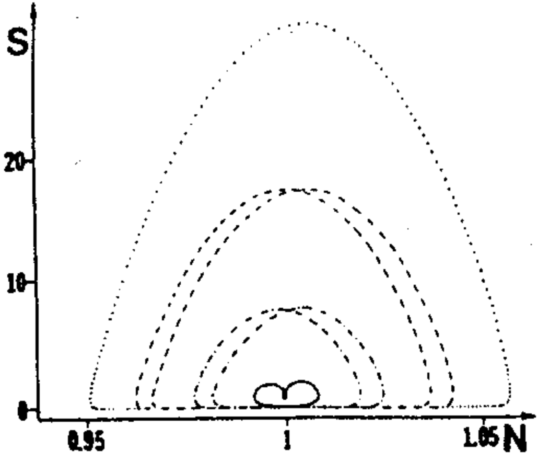
Oscilloscope Traces



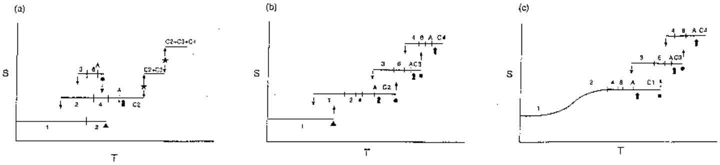
Bifurcation Schematics



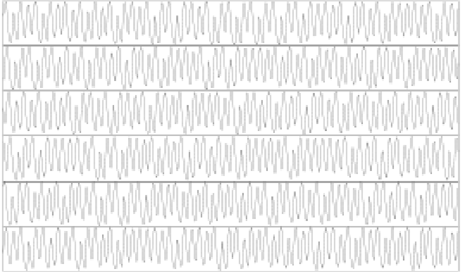
Coexisting Basins of Attraction

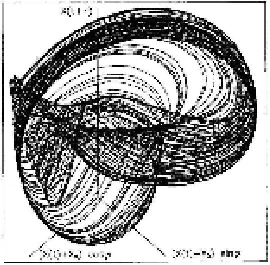


Bifurcation Perestroikas



Experimental Data: LSA





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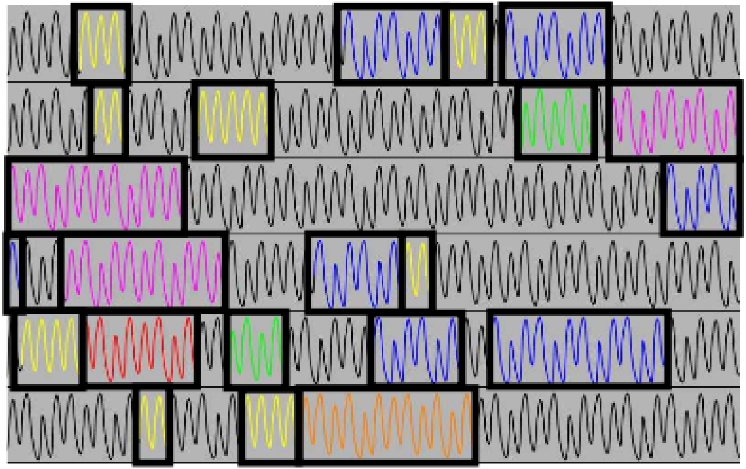
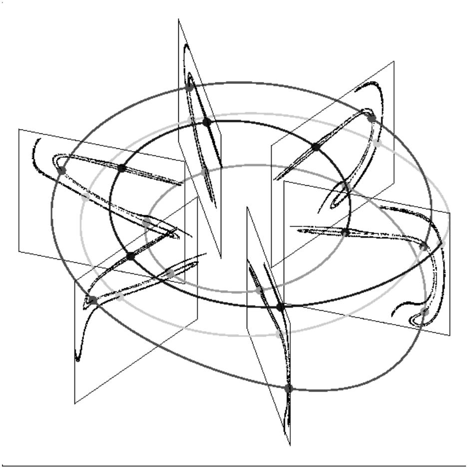
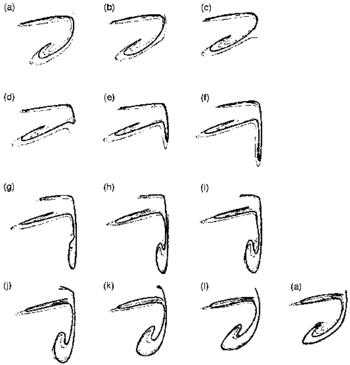
Experimental Data: LSA



Stretching & Squeezing in a Torus



Rotating the Poincaré Section around the axis of the torus



Rotating the Poincaré Section around the axis of the torus

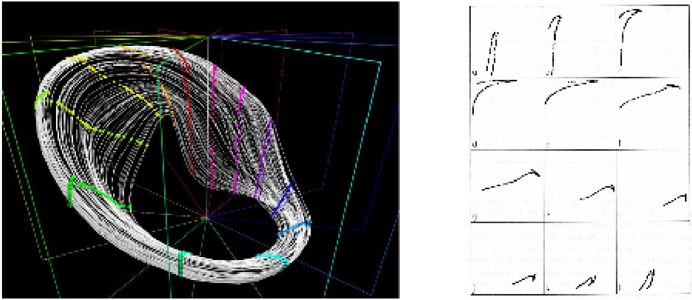


Figure 2. Left: Intersections of a chaotic attractor with a series of section planes are computed. Right: Their evolution from plane to plane shows the interplay of the stretching and squeezing mechanisms.

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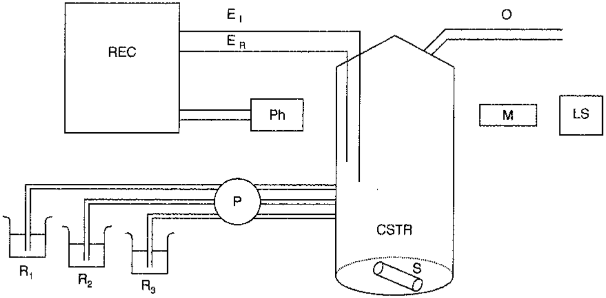
A Chemical Experiment

The Belousov-Zhabotinskii Reaction



Chaos

Motion that is

- **Deterministic:** $\frac{dx}{dt} = f(x)$
- **Recurrent**
- **Non Periodic**
- **Sensitive to Initial Conditions**

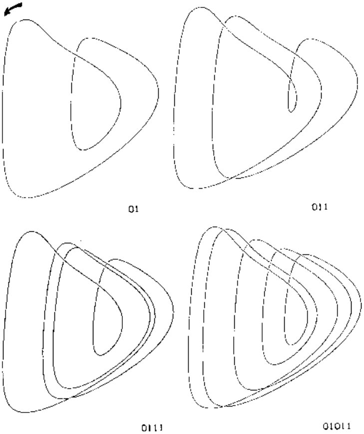
Strange Attractor

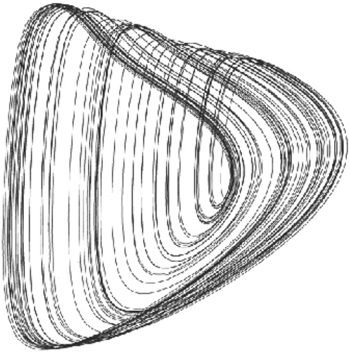
The Ω limit set of the flow. There are unstable periodic orbits "in" the strange attractor. They are

- **"Abundant"**
- **Outline the Strange Attractor**
- **Are the Skeleton of the Strange Attractor**

Skeletons

UPOs Outline Strange attractors





BZ reaction

UPOs Outline Strange attractors

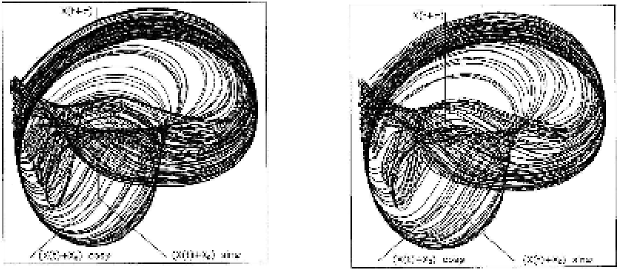


Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser ; Right : Superposition of 12 periodic orbits of periods from 1 to 10.

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Organization of UPOs in R^3: Gauss Linking Number

$$LN(A,B) = \frac{1}{4\pi} \oint \oint \frac{(\mathbf{r}_A - \mathbf{r}_B) \cdot d\mathbf{r}_A \times d\mathbf{r}_B}{|\mathbf{r}_A - \mathbf{r}_B|^3}$$

$\#$ Interpretations of LN $\simeq$ $\#$ Mathematicians in World

Linking Number of Two UPOs

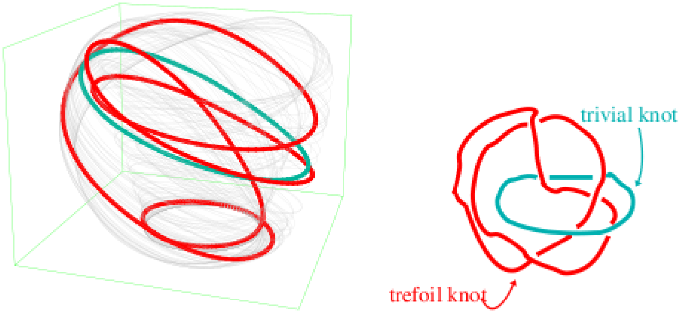

Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

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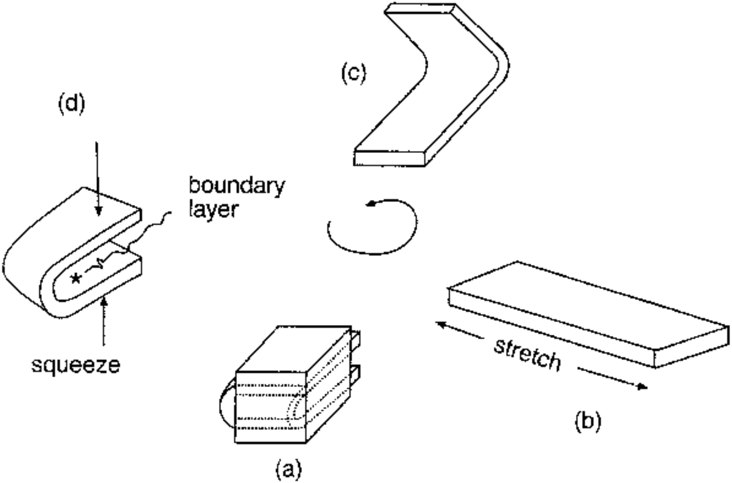
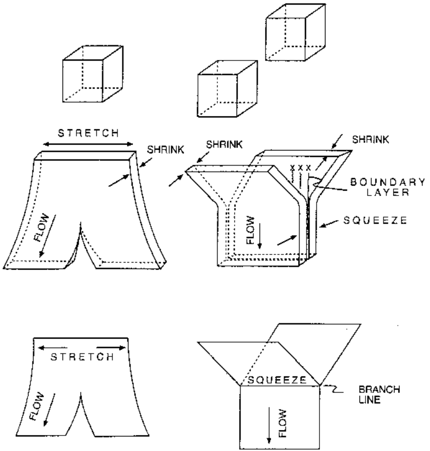
One Stretch-&-Squeeze Mechanism



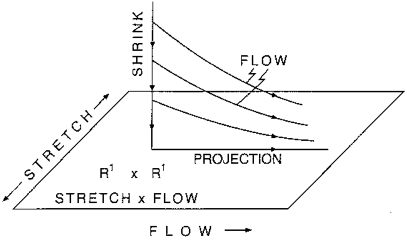
Stretching — Squeezing



Birman - Williams Projection

Identify x and y if

$$\lim_{t\to\infty} |x(t) - y(t)| \rightarrow 0$$



Birman - Williams Theorem

If:

Then:

Fundamental Theorem

Birman - Williams Theorem

If: Certain Assumptions

Then:

Birman - Williams Theorem

If: **Certain Assumptions**

Then: **Specific Conclusions**

Assumptions, B-W Theorem

A flow $\Phi_t(x)$

- **on** R^n **is dissipative,** $\underline{n = 3}$**, so that** $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0.$
- **Generates a** **<u>hyperbolic</u>** **strange attractor** $\mathcal{SA}$

IMPORTANT: The underlined assumptions can be relaxed.

Conclusions, B-W Theorem

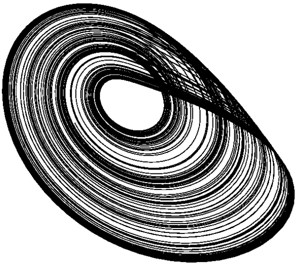
- **The projection maps the strange attractor $\mathcal{SA}$ onto a 2-dimensional branched manifold $\mathcal{BM}$ and the flow $\Phi_t(x)$ on $\mathcal{SA}$ to a semiflow $\overline{\Phi}(x)_t$ on $\mathcal{BM}$.**
- **UPOs of $\Phi_t(x)$ on $\mathcal{SA}$ are in 1-1 correspondence with UPOs of $\overline{\Phi}(x)_t$ on $\mathcal{BM}$. Moreover, every link of UPOs of $(\Phi_t(x), \mathcal{SA})$ is isotopic to the correspond link of UPOs of $(\overline{\Phi}(x)_t, \mathcal{BM})$.**

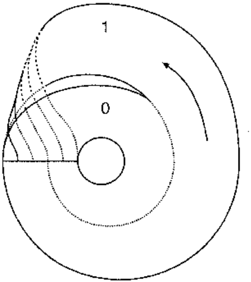
Remark: "One of the few theorems useful to experimentalists."

A Very Common Mechanism

Rössler:

Attractor **Branched Manifold**

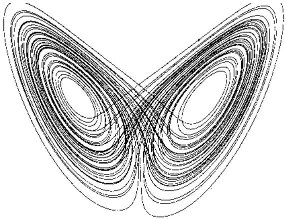


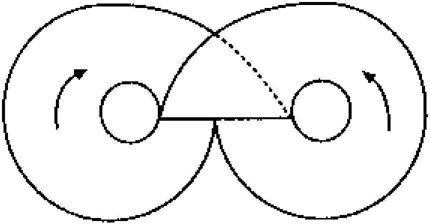


A Mechanism with Symmetry

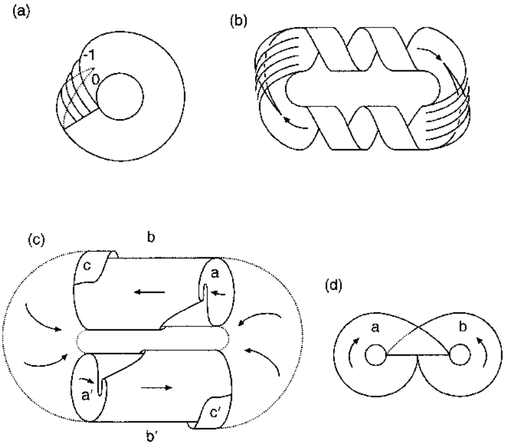
Lorenz:

Attractor **Branched Manifold**



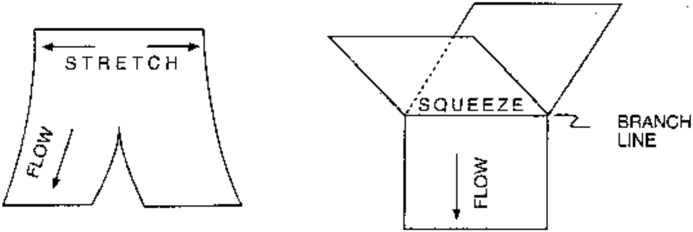


Inequivalent Branched Manifolds



Aufbau Princip for Branched Manifolds

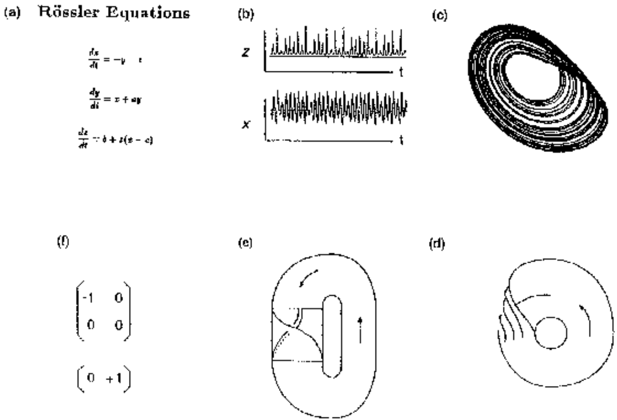
Any branched manifold can be built up from stretching and squeezing units



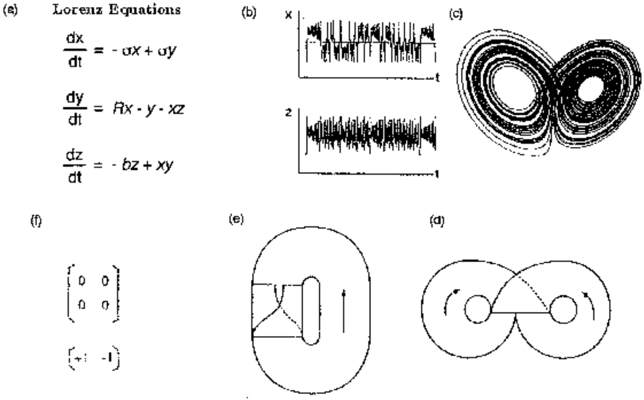
subject to the conditions:

- **Outputs to Inputs**
- **No Free Ends**

Rossler System



Lorenz System



Poincaré Smiles at Us in R^3

- **Determine organization of UPOs** $\Rightarrow$
- **Determine branched manifold** $\Rightarrow$
- **Determine equivalence class of** $\mathcal{SA}$

Topological Analysis Program

Locate Periodic Orbits

Create an Embedding

Determine Topological Invariants (LN)

Identify a Branched Manifold

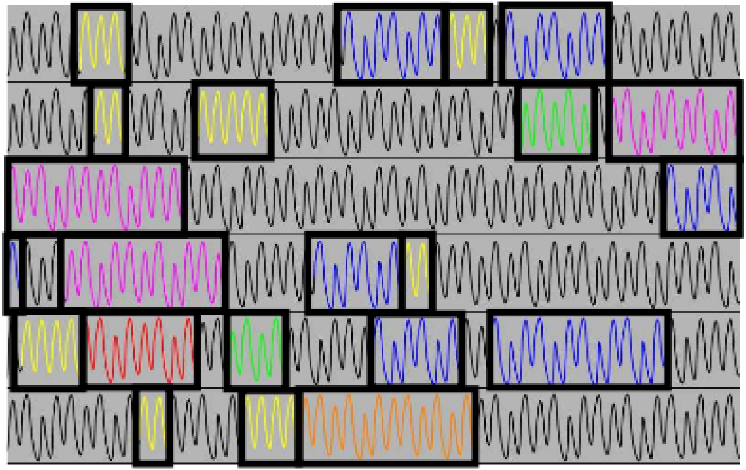
Verify the Branched Manifold

Model the Dynamics

Validate the Model

Locate UPOs

Method of Close Returns



Embeddings

Many Methods: Time Delay, Differential, Hilbert Transforms, SVD, Mixtures, ...

Tests for Embeddings: Geometric, Dynamic, Topological†

None Good

We Demand a 3 Dimensional Embedding

An Embedding and Periodic Orbits

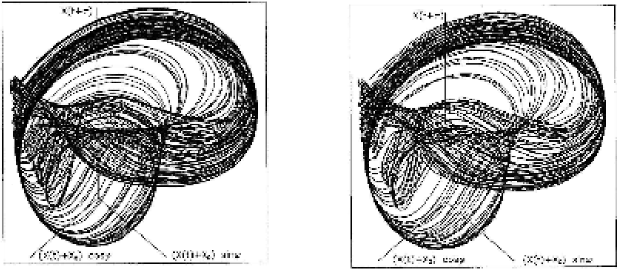


Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser ; Right : Superposition of 12 periodic orbits of periods from 1 to 10.

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Linking Number of Orbit Pairs

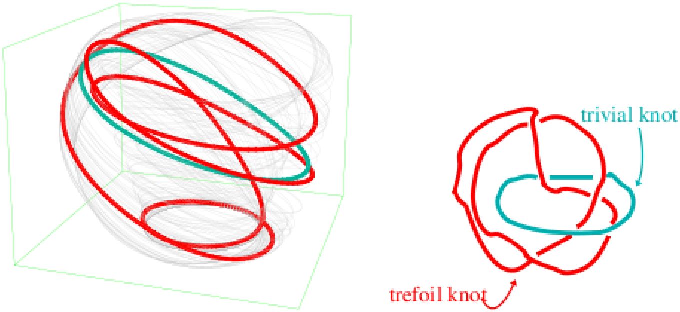

Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

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Compute Table of Expt'l LN

Table 7.2 Linking numbers for all the surrogate periodic orbits, to period 8, extracted from Belousov–Zhabotinskii data[a]

Orbit	Symbolics	1	2	3	4	5	6	7	8a	8b
1	1	0	1	1	2	2	2	3	4	3
2	01	1	1	2	3	4	4	5	6	6
3	011	1	2	2	4	5	6	7	8	8
4	0111	2	3	4	5	8	8	11	13	12
5	01 011	2	4	5	8	8	10	13	16	15
6	011 0M1	2	4	6	8	10	9	14	16	16
7	01 01 011	3	5	7	11	13	14	16	21	21
8a	01 01 0111	4	6	8	13	16	16	21	23	24
8b	01 011 011	3	6	8	12	15	16	21	24	21

[a] All indices are negative.

Compare w. LN From Various $\mathcal{BM}$

Table 2.1 Linking numbers for orbits to period five in Smale horseshoe dynamics.

	$1s$	$1f$	2_1	$3f$	$3s$	4_1	4_2f	4_2s	5_3f	5_3s	5_2f	5_2s	5_1f	5_1s
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	2	1	1	1	1	2	2	2	2
01	0	1	1	2	2	3	2	2	2	2	3	3	4	4
001	0	1	2	2	3	4	3	3	3	3	4	4	5	5
011	0	1	2	3	2	4	3	3	3	3	5	5	5	5
0111	0	2	3	4	4	5	4	4	4	4	7	7	8	8
0001	0	1	2	3	3	4	3	4	4	4	5	5	5	5
0011	0	1	2	3	3	4	4	3	4	4	5	5	5	5
00001	0	1	2	3	3	4	4	4	4	5	5	5	5	5
00011	0	1	2	3	3	4	4	4	5	4	5	5	5	5
00111	0	2	3	4	5	7	5	5	5	5	6	7	8	9
00101	0	2	3	4	5	7	5	5	5	5	7	6	8	9
01101	0	2	4	5	5	8	5	5	5	5	8	8	8	10
01111	0	2	4	5	5	8	5	5	5	5	9	9	10	8

Guess Branched Manifold

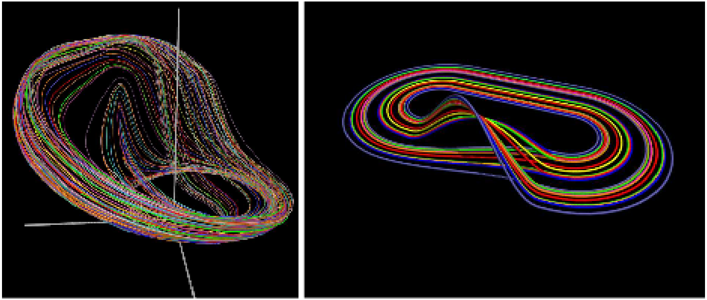

Figure 7. "Combing" the intertwined periodic orbits (left) reveals their systematic organization (right) created by the stretching and squeezing mechanisms.

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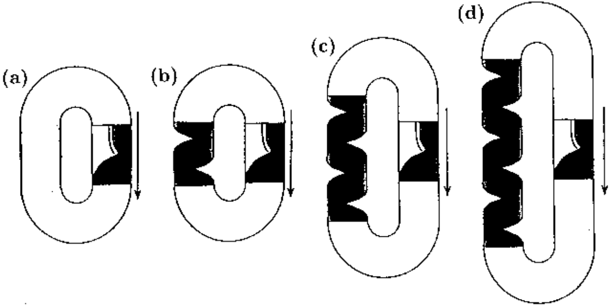
Identification & ‘Confirmation’

- $\mathcal{BM}$ Identified by LN of small number of orbits
- Table of LN GROSSLY overdetermined
- Predict LN of additional orbits
- Rejection criterion

What Do We Learn?

- $\mathcal{BM}$ Depends on Embedding
- Some things depend on embedding, some don't
- Depends on Embedding: Global Torsion, Parity, ..
- Independent of Embedding: Mechanism



Evolution Under Parameter Change

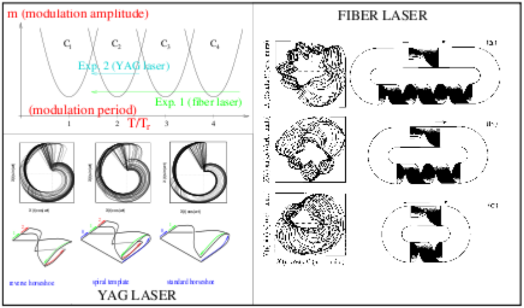


Figure 11. Various templates observed in two laser experiments. Top left: schematic representation of the parameter space of forced nonlinear oscillators showing resonance tongues. Right: templates observed in the fiber laser experiment: global torsion increases systematically from one tongue to the next [40]. Bottom left: templates observed in the YAG laser experiment (only the branches are shown): there is a variation in the topological organization across one chaotic tongue [39, 41].

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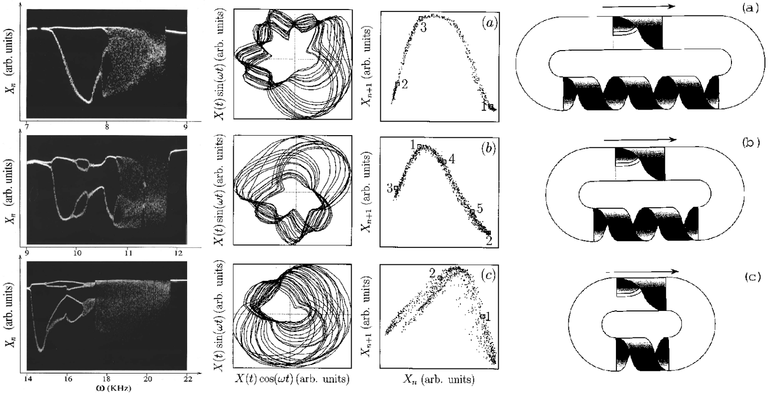
Evolution Under Parameter Change



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Analysis of Nonstationary Data

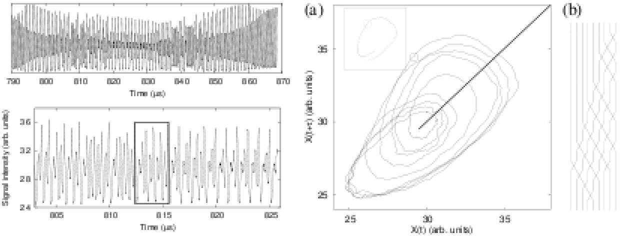

Figure 16. Top left: time series from an optical parametric oscillator showing a burst of irregular behavior. Bottom left: segment of the time series containing a periodic orbit of period 9. Right: embedding of the periodic orbit in a reconstructed phase space and representation of the braid realized by the orbit. The braid entropy is $h_T = 0.377$, showing that the underlying dynamics is chaotic. Reprinted from [61].

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Compare with Original Objectives

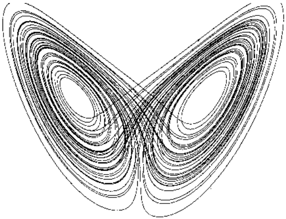
Construct a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

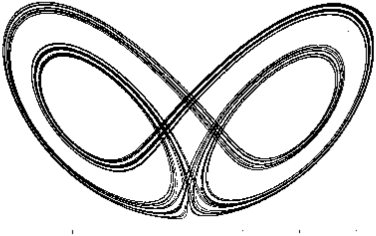
from experimental signals.

Orbits Can be “Pruned”

There Are Some Missing Orbits

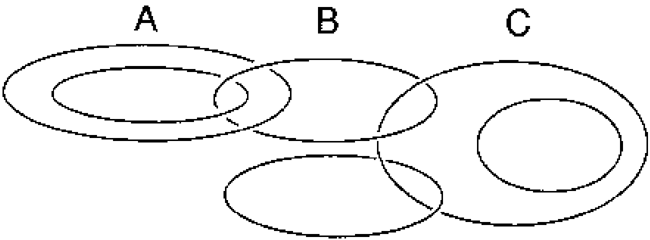


Lorenz



Shimizu-Morioka

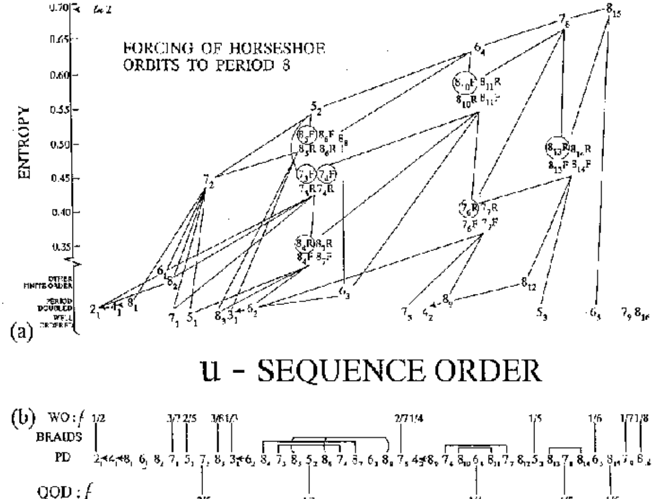
Orbit Forcing



$A \Rightarrow B \qquad B \Rightarrow C$

$A \Rightarrow C$

Forcing Diagram - Horseshoe



Status of Problem

- Horseshoe organization - active
- More folding - barely begun
- Circle forcing - even less known
- Higher genus - new ideas required

The Topology of Chaos

Robert Gilmore

Introduction-01

Introduction-02

Overview-01

Overview-02

Overview-03

Overview-04

Overview-05

Overview-06

Experimental-01

Experimental-02

Experimental-03

Rotating the Attractor

$$\frac{d}{dt}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} F_1(X,Y) \\ F_2(X,Y) \end{bmatrix} + \begin{bmatrix} a_1 \sin(\omega_d t + \phi_1) \\ a_2 \sin(\omega_d t + \phi_2) \end{bmatrix}$$

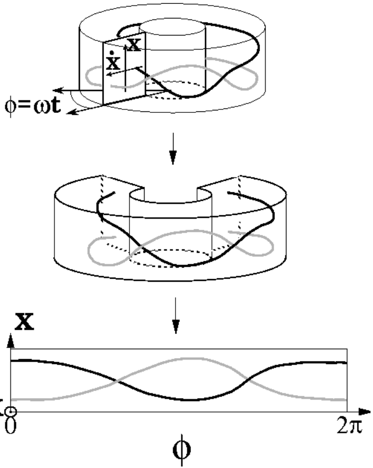
$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \cos\Omega t & -\sin\Omega t \\ \sin\Omega t & \cos\Omega t \end{bmatrix}\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = R\mathbf{F}(R^{-1}\mathbf{u}) + R\mathbf{t} + \Omega\begin{bmatrix} -v \\ +u \end{bmatrix}$$

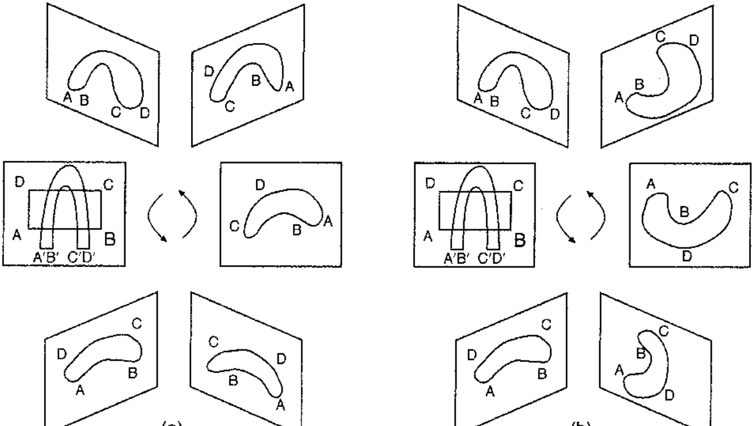
$\Omega = n\ \omega_d$ $\qquad$ $q\ \Omega = p\ \omega_d$

Global Diffeomorphisms $\qquad$ Local Diffeomorphisms (p-fold covers)

Cutting Open a Torus



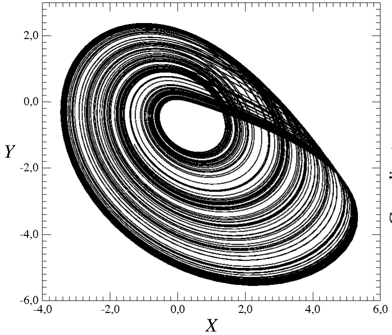
Global Torsion

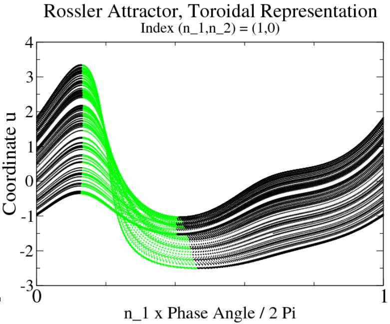


(a) (b)

Rossler Attractor: Two Representations

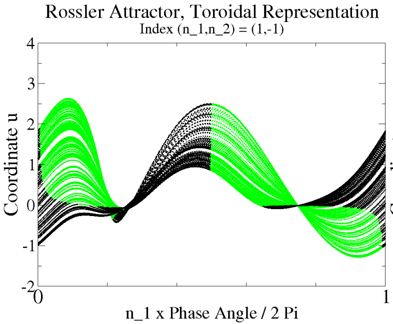
R^3 | $D^2 \times S^1$

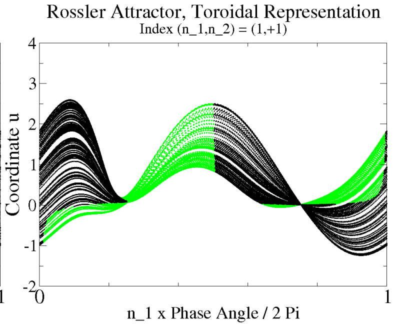




Rossler Attractor:

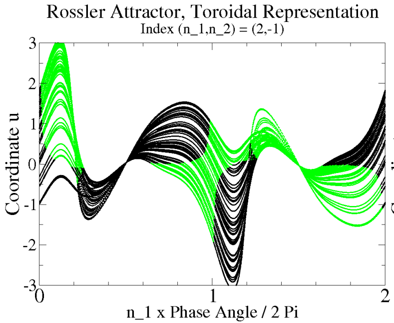
Two More Representations with $n = \pm 1$

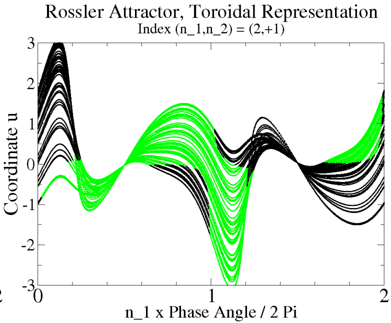




Rossler Attractor:

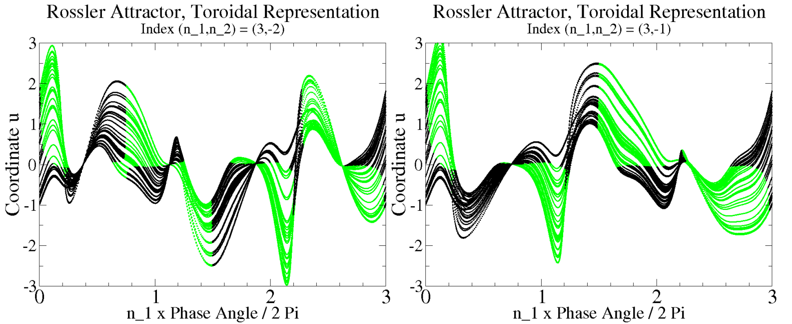
Two Two-Fold Covers with $p/q = \pm 1/2$





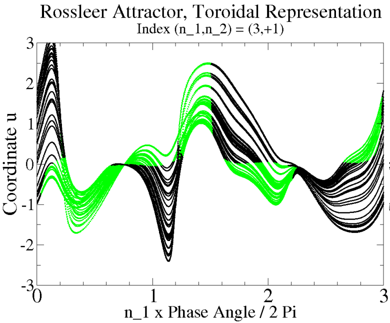
Rossler Attractor:

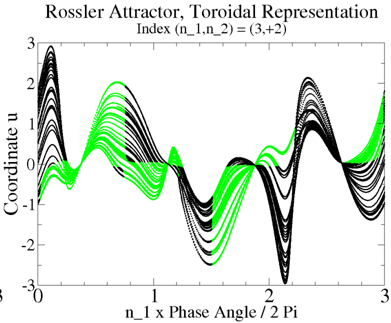
Two Three-Fold Covers with $p/q = -2/3, -1/3$



Rossler Attractor:

And Even More Covers (with $p/q = +1/3, +2/3$)





Angular Momentum and Energy

$$L(0) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} X dY - Y dX \qquad K(0) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} \frac{1}{2}(\dot{X}^2 + \dot{Y}^2) dt$$

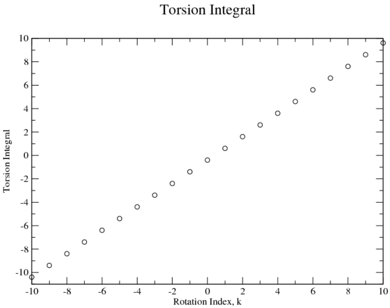
$$L(\Omega) = \langle u\dot{v} - v\dot{u} \rangle \qquad K(\Omega) = \langle \frac{1}{2}(\dot{u}^2 + \dot{v}^2) \rangle$$

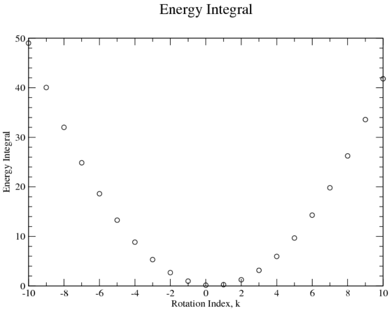
$$= L(0) + \Omega \langle R^2 \rangle \qquad = K(0) + \Omega L(0) + \frac{1}{2}\Omega^2 \langle R^2 \rangle$$

$$\langle R^2 \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} (X^2 + Y^2) dt = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} (u^2 + v^2) dt$$

Energy and Angular Momentum

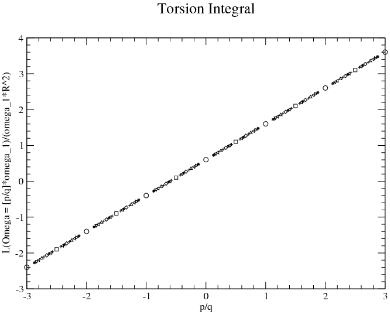
Diffeomorphic, Quantum Number n

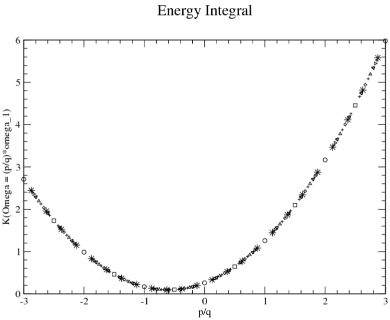




Energy and Angular Momentum

Subharmonics, Quantum Numbers p/q





Summary

1 Question Answered $\Rightarrow$

2 Questions Raised

We must be on the right track !

Original Objectives Achieved

There is now a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

Result

There is now a classification theory for low-dimensional strange attractors.

1. It is topological
2. It has a hierarchy of 4 levels
3. Each is discrete
4. There is rigidity and degrees of freedom
5. It is applicable to R^3 only — for now

The Classification Theory has 4 Levels of Structure

The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits

The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits
2. Branched Manifolds

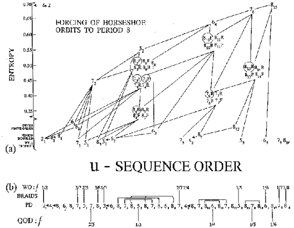
The Classification Theory has 4 Levels of Structure

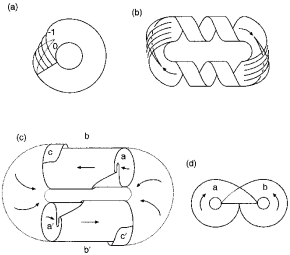
1. Basis Sets of Orbits
2. Branched Manifolds
3. Bounding Tori

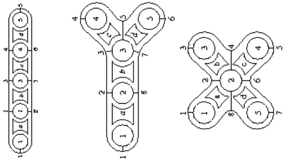
The Classification Theory has 4 Levels of Structure

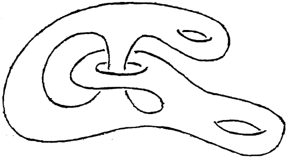
1. Basis Sets of Orbits
2. Branched Manifolds
3. Bounding Tori
4. Extrinsic Embeddings

Four Levels of Structure









Poetic Organization

LINKS OF PERIODIC ORBITS

organize

BOUNDING TORI

organize

BRANCHED MANIFOLDS

organize

LINKS OF PERIODIC ORBITS

Some Unexpected Results

- Perestroikas of orbits constrained by branched manifolds
- Routes to Chaos = Paths through orbit forcing diagram
- Perestroikas of branched manifolds constrained by bounding tori
- Global Poincaré section = union of $g-1$ disks
- Systematic methods for cover - image relations
- Existence of topological indices (cover/image)
- Universal image dynamical systems
- NLD version of Cartan's Theorem for Lie Groups
- Topological Continuation – Group Continuuation
- Cauchy-Riemann symmetries
- Quantizing Chaos
- Representation labels for inequivalent embeddings
- Representation Theory for Strange Attractors

Unanswered Questions

We hope to find:

- Robust topological invariants for R^N, $N > 3$
- A Birman-Williams type theorem for higher dimensions
- An algorithm for irreducible embeddings
- Embeddings: better methods and tests
- Analog of χ^2 test for NLD
- Better forcing results: Smale horseshoe, $D^2 \rightarrow D^2$, $n \times D^2 \rightarrow n \times D^2$ (e.g., Lorenz), $D^N \rightarrow D^N$, $N > 2$
- Representation theory: complete
- Singularity Theory: Branched manifolds, splitting points (0 dim.), branch lines (1 dim).
- Singularities as obstructions to isotopy