# Lectures on Loop Quantum Gravity

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## Abstract

Quantum General Relativity (QGR), sometimes called Loop Quantum Gravity, has matured over the past fifteen years to a mathematically rigorous candidate quantum field theory of the gravitational field. The features that distinguish it from other quantum gravity theories are 1) *background independence* and 2) *minimality of structures*.

Background independence means that this is a non-perturbative approach in which one does not perturb around a given, distinguished, classical background metric, rather arbitrary fluctuations are allowed, thus precisely encoding the quantum version of Einstein's radical perception that *gravity is geometry*.

Minimality here means that one explores the logical consequences of bringing together the two fundamental principles of modern physics, namely general covariance and quantum theory, without adding any experimentally unverified additional structures such as extra dimensions, extra symmetries or extra particle content beyond the standard model. While this is a very conservative approach and thus maybe not very attractive to many researchers, it has the advantage that pushing the theory to its logical frontiers will undoubtedly either result in a successful theory or derive exactly which extra structures are required, if necessary. Or put even more radically, it may show which basic principles of physics have to be given up and must be replaced by more fundamental ones.

QGR therefore is, by definition, not a unified theory of all interactions in the standard sense since such a theory would require a new symmetry principle. However, it unifies all presently known interactions in a new sense by quantum mechanically implementing their common symmetry group, the four-dimensional diffeomorphism group, which is almost completely broken in perturbative approaches.

# Contents

This is the expanded version of a talk given by the author at the 271st WE Heraeus Seminar "Aspects of Quantum Gravity: From Theory to Experimental Search", Bad Honnef, Germany, February 25th – March 1st 2002, http://www.uni-duesseldorf.de/QG-2002.

Basically, we summarize the present status of Canonical Quantum General Relativity (QGR), also known as "Loop Quantum Gravity". Our presentation tries to be precise and at the same time technically not too complicated by skipping the proofs of all the statements made. These many missing details, which are relevant to the serious reader, can be found in the notation used here in this overview e.g. in the recent, close to exhaustive review [1] and references therein. Of course, in order to be useful as a text for first reading we did not include all the relevant references here. We apologize for that to the researchers in the field but we hope that a close to complete list of their work can be found in [1]. Nice reports, treating complementary subjects of the field and more general aspects of quantum gravity can be found in [2].

The text is supplemented by numerous exercises of varying degree of difficulty whose purpose is to cut the length of the exposition and to arouse interest in further studies. Solving the problems is not at all mandatory for an undertsanding of the material, however, the exercises contain further information and thus should be looked at even on a first reading.

On the other hand, if one solves the problems then one should get a fairly good insight into the techniques that are important in QGR and in principle could serve as a preparation for a diploma thesis or a dissertation in this field. The problems sometimes involve mathematics that may be unfamiliar to students, however, this should not scare off but rather encourage the serious reader to learn the necessary mathematical background material. Here is a small list of mathematical texts, from the author's own favourites, geared at theoretical and mathematical physicists, that might be helpful:

- *General*
  A fairly good encyclopedia is
  Y. Choquet-Bruhat, C. DeWitt-Morette, "Analysis, Manifolds and Physics", North Holland, Amsterdam, 1989

- *General Topology*
  A nice text, adopting almost no prior knowledge is
  J. R. Munkres, "Toplogy: A First Course", Prentice Hall Inc., Englewood Cliffs (NJ), 1980

- *Differential and Algebraic Geometry*
  A modern exposition of this classical material can be found in
  M. Nakahara, "Geometry, Topology and Physics", Institute of Physics Publishing, Bristol, 1998

- *Functional Analysis*
  The number one, unbeatable and close to complete exposition is
  M. Reed, B. Simon, "Methods of Modern Mathematical Physics", vol. 1 – 4, Academic Press, New York, 1978
  especially volumes one and two.

- *Measure Theory*
  An elementary introduction to measure theory can be found in the beautiful book
  W. Rudin, "Real and Complex Analysis", McGraw-Hill, New York, 1987

- *Operator Algebras*
  Although we do not really make use of $C^*-$algebras in this review, we hint at the importance of the subject, so let us include
  O. Bratteli, D. W. Robinson, "Operator Algebras and Quantum Statistical Mechanics", vol. 1,2, Springer Verlag, Berlin, 1997

- *Harmonic Analysis on Groups*
  Although a bit old, it still contains a nice collection of the material around the Peter & Weyl theorem:
  N. J. Vilenkin, "Special Functions and the Theory of Group Representations", American Mathematical Society, Providence, Rhode Island, 1968

- *Mathematical General Relativity*
  The two leading texts on this subject are
  R. M. Wald, "General Relativity", The University of Chicago Press, Chicago, 1989
  S. Hawking, Ellis, "The Large Scale Structure of Spacetime", Cambridge University Press, Cambridge, 1989

- *Mathematical and Physical Foundations of Ordinary QFT*
  The most popular books on axiomatic, algebraic and constructive quantum field theory are
  R. F. Streater, A. S. Wightman, "PCT, Spin and Statistics, and all that", Benjamin, New York, 1964
  R. Haag, "Local Quantum Physics", 2nd ed., Springer Verlag, Berlin, 1996
  J. Glimm, A. Jaffe, "Quantum Physics", Springer-Verlag, New York, 1987

In the first part we motivate the particular approach to a quantum theory of gravity, called (Canonical) Quantum General Relativity, and develop the classical foundations of the theory as well as the goals of the quantization programme.

In the second part we list the solid results that have been obtained so far within QGR. Thus, we will apply step by step the quantization programme outlined at the end of section I.3 to the classical theory that we defined in section I.2. Up to now, these steps have been completed approximately until step vii) at least with respect to the Gauss – and the spatial diffeomorphism constraint. The analysis of the Hamiltonian constraint has also reached level vii) already, however, its classical limit is presently under little control which is why we discuss it in part three where current research topics are listed.

In the third part we discuss a selected number of active research areas. The topics that we will describe already have produced a large number of promising results, however, the analysis is in most cases not even close to being complete and therefore the results are less robust than those that we have obtained in the previous part.

Finally, in the fourth part we summarize and list the most important open problems that we faced during the discussion in this report.

# Part I

# Motivation and Introduction

## I.1 Motivation

### I.1.1 Why Quantum Gravity in the 21'stCentury ?

Students that plan to get involved in quantum gravity research should be aware of the fact that in our days, when financial resources for fundamental research are more and more cut and/or more and more absorbed by research that leads to practical apllications on short time scales, one should have a good justification for why tax payers should support any quantum gravity research at all. This seems to be difficult at first due to the fact that even at CERN's LHC we will be able to reach energies of at most $10^4$ GeV which is **fifteen orders of magnitude below the Planck scale** which is the energy scale at which quantum gravity is believed to become important. Therefore one could argue that quantum gravity research in the 21'st century is of purely academic interest only.

To be sure, it is a shame that one has to justify fundamental research at all, a situation unheard of in the beginning of the 20'th century which probably was part of the reason for why so many breakthroughs especially in fundamental physics have happened in that time. Fundamental research can work only in absence of any pressure to produce (mainstream) results, otherwise new, radical and independent thoughts are no longer produced. To see the time scale on which fundamental research leads to practical results, one has to be aware that General Relativity (GR) and Quantum Theory (QT) were discovered in the 20's and 30's already but it took some 70 years before quantum mechanics through, e.g. computers, mobile phones, the internet, electronic devices or general relativity through e.g. space travel or the global positioning system (GPS) became an integral part of life of a large fraction of the human population. Where would we be today if the independent thinkers of those times were forced to do practical physics due to lack of funding for analyzing their fundamental questions ?

Of course, in the beginning of the 20'th century, one could say that physics had come to some sort of crisis, so that there was urgent need for some revision of the fundamental concepts: Classical Newtonian mechanics, classical electrodynamics and thermodynamics were so well understood that Max Planck himself was advised not to study physics but engineering. However, although from a practical point of view all seemed well, there were subtle inconsistencies among these theories if one drove them to their logical frontiers. We mention only three of them:

1) Although the existence of atoms was by far not widely accepted at the end of the 19th century (even Max Planck denied them), if they existed then there was a serious flaw, namely, how should atoms be stable ? Acceralated charges radiate Bremsstrahlung according to Maxwell's theory, thus an electron should fall into the nucleus after a finite amount of time.

2) If Newton's theory of absolute space and time was correct then the speed of light should depend on the speed of the inertial observer. The fact that such velocity dependence was ruled out to quadratic order in $v/c$ in the famous Michelson-Morley experiment was explained by postulating an unknown medium, called ether, with increasingly (as measurement precision was refined) bizarre properties in order to conspire to a negative outcome of the interferometer experiment and to preserve Newton's notion of space and time.

3) The precession of mercury around the sun contradicted the ellipses that were predicted by Newton's theory of gravitation.

Today we easily resolve these problems by 1) quantum mechanics, 2) special relativity and 3) general relativity. Quantum mechanics does not allow for continuous radiation but predicts a discrete energy spectrum of the atom, special relativity removed the absolute notion of space and time and general relativity generalizes the static Minkowski metric underlying special relativity to a dynamical theory of a metric field which revolutionizes our understanding of gravity not as a force but as geometry. Geometry is curved at each point in a manifold proportional to the matter density at that point and in turn curvature tells matter what are the straightest lines (geodesics) along which to move. The ether became completely unnecessary by changing the foundation of physics and beautifully demonstrates that driving a theory to its logical frontiers can make extra structures redundant, what one had to change is the basic principles of physics.[1]

This historic digression brings us back to the motivation for studying quantum gravity in the beginning of the 21st century. The question is whether fundamental physics also today is in a kind of crisis. We will argue that indeed we are in a situation not unsimilar to that of the beginning of the 20th century:
Today we have very successful theories of all interactions. Gravitation is described by general relativity, matter interactions by the standard model of elementary particle physics. As classical theories, their dynamics is summarized in the classical Einstein equations. However, there are several problems with these theories, some of which we list below:

i) *Classical – Quantum Inconsistency*
The fundamental principles collide in the classical Einstein equations

Hence, we are enforced to quantize the metric itself, that is, we need a quantum theory of gravity resulting in the

The occurance of UV singularities is in deep conflict with general relativity due to the following reason: In perturbation theory, the divergences have their origin in Feynman loop integrals in momentum space where the inner loop 4-momentum $k = (E, P)$ can become arbitrarily large, see figure 1 for an example from QED (mass renormalization). Now such virtual (off-shell) particles with energy $E$ and momentum $P$ have a spatial extension of the order of the Compton radius $\lambda = \hbar/P$ and a mass of the order of $E/c^2$. Classical general relativity predicts that this lump of energy turns into a black hole once $\lambda$ reaches the Schwarzschild radius of the order of $r = GE/c^4$. In a Lorentz frame where $E \approx Pc$ this occurs at the Planck energy $E = E_P = \sqrt{\hbar/\kappa c} \approx 10^{19}$GeV or at the Planck length Compton radius $\ell_P = \sqrt{\hbar\kappa} \approx 10^{-33}$cm. However, when a (virtual) particle turns into a black hole it completely changes its properties. For instance, if the virtual particle is an electron then it is able to interact only electroweakly and thus can radiate only particles of the electrowak theory. However, once a black hole has formed, also Hawking processes are possible and now any kind of particles can be emitted, but at a different production rate. Of course, this is again an energy regime at which quantum gravity must be important and these qualitative pictures must be fundamnetally wrong, however, they show that there is a problem with integrating virtual loops into the UV regime. In fact, these qualitative thoughts suggest that gravity could serve as a *natural cut-off* because a black hole of Planck mass size $\ell_P$ should decay within a Planck time unit $t_P = \ell_P/c \approx 10^{-43}s$ so that one has to integrate $P$ only until $E_P/c$. Moreover, it indicates that spacetime geometry itself acquires possibly a *discrete structure* since arguments of this kind make it plausible that it is impossible to resolve spacetime distances smaller than $\ell_P$, basically because the spacetime behind an event horizon is in some sense "invisible". These are, of course, only hopes

[1]Notice, that the stability of atoms is still not satisfactorily understood even today because the full problem also treats the radiation field, the nucleus and the electron as quantum objects which ultimately results in a problem in QED, QFD and QCD for which we have no entirely satisfactory description today, see below.

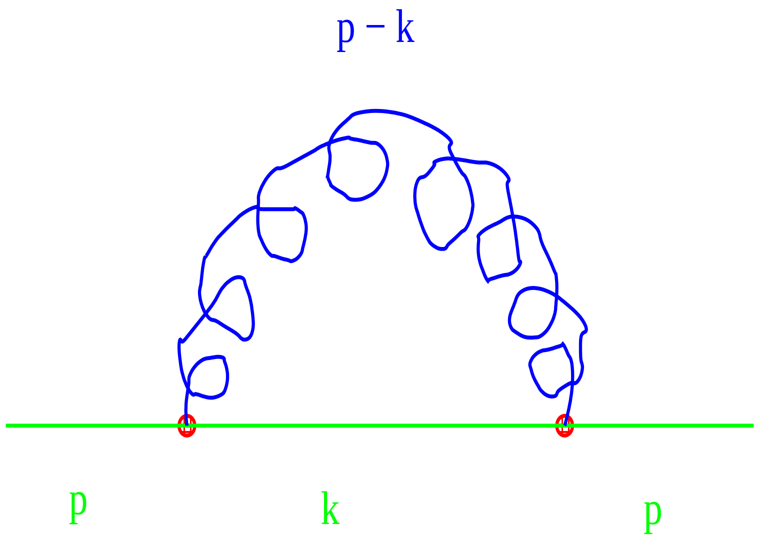


Figure 1: One loop correction to the electron propagator in QED

and must be demonstrated within a concrete theory. We will see that QGR is able to precisely do that and its fundamental discreteness is in particular responsible for why the Bekenstein Hawking entropy of black holes is finite, see sections II.2.2, III.1.1 and III.4.

So we see that there is indeed a fundamental inconsisteny within the current description of fundamental physics comparable to the time before the discovery of GR and QT and its resolution, Quantum Gravity, will revolutionize not only our understanding of nature but will also drive new kinds of technology that we do not even dare to dream of today.

### I.1.2 The Role of Background Independence

Given the fact that both QT and GR were discovered already more than 70 years ago and that people have certainly thought about quantizing GR since then and that matter interactions are more or less succesfully described by ordinary quantum field theories (QFT), it is somewhat surprising that we do not yet have a quantum gravity theory. Why is it so much harder to combine gravity with the principles of quantum mechanics than for the other interactions ? The short answer is that

**Ordinary QFT only incorporates Special Relativity.**

To see why, we just have to remember that ordinary QFT has an axiomatic definition, here for a scalar field for simplicity:

**WIGHTMAN AXIOMS** (Scalar Fields on Minkowsk Space)

- W1 **Poincaré Group $\mathcal{P}$**:
  $\exists$ continuous, unitary representation $\hat{U}$ of $\mathcal{P}$ on a Hilbert space $\mathcal{H}$.
- W2 Forward Lightcone **Spektral Condition**:
  For the generators $\hat{P}^\mu$ of the translation subgroup of $\mathcal{P}$ holds $\eta_{\mu\nu}\hat{P}^\mu\hat{P}^\nu \leq 0, \quad \hat{P}^0 \geq 0$.

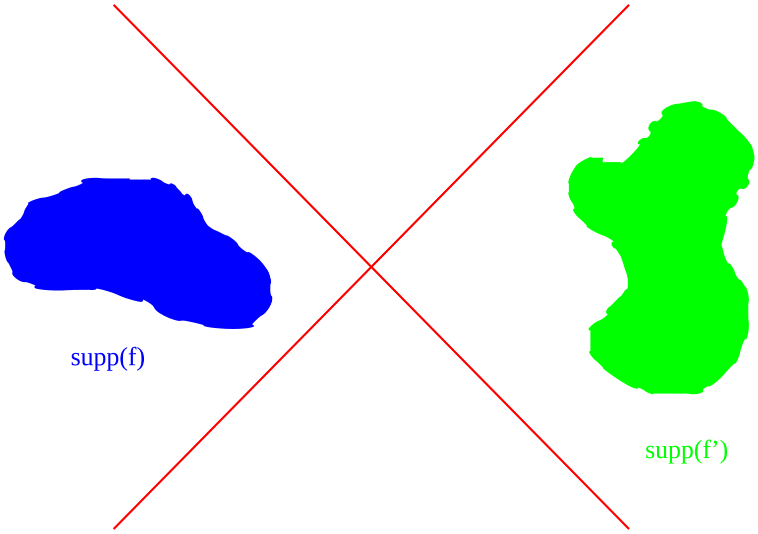


Figure 2: Spacelike separated regions in Minkowski space

W3 Existence and Uniqueness of a **$\mathcal{P}-$inariant Vacuum $\Omega$**:
$\hat{U}(p)\Omega = \Omega \ \ \forall p \in \mathcal{P}$.

W4 **$\mathcal{P}-$Covariance**:

$$\hat{\phi}(f) := \int d^{D+1}x f(x)\hat{\phi}(x), \ \ f \in \mathcal{S}(R^{D+1})$$

$$\hat{\phi}(f_1)..\hat{\phi}(f_n)\Omega \text{ dense in } \mathcal{H} \text{ and } \hat{U}(p)\hat{\phi}(f)\hat{U}(p)^{-1} = \hat{\phi}(f \circ p)$$

W5 **Locality (Causality):**
If supp$(f)$, supp$(f')$ **spacelike separated** (see figure 2), then $[\hat{\phi}(f), \hat{\phi}(f')] = 0$.

It is obvious that due to the presence of the Minkowski backgrounde metric $\eta$ we have available a large amount of structure which forms the fundament on which ordinary QFT is built. Roughly, we have the following scheme:

People have tried to rescue the framework of ordinary QFT by splitting the metric into a background piece and a fluctuation piece

At present only string theory has a chance to explain the matter content of our universe. The unification of symmetries is a strong guiding principle in physics as well and has been pushed also by Einstein in his programme of geometrization of physics attemting to unify electromagnetism and gravity in a five-dimensional Kaluza – Klein theory. The unification of the electromagnetic and the weak force in the electroweak theory is a prime example for the success of such ideas. However, unification of forces is an additional principle completely independent of background independence and is not necessarily what a *quantum theory* of gravity must achieve: Unification of forces can be

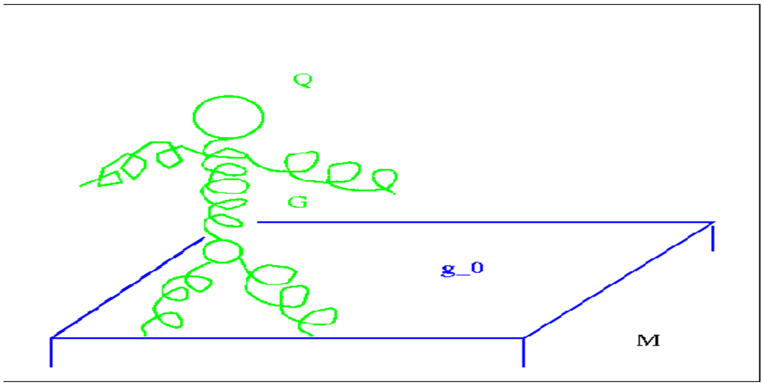


Figure 3: QFT on Background Spacetime $(M, g_0)$: Actor: **Matter**, Stage: **Geometry** + Manifold $M$.

analyzed at the purely *classical level*[2]. Thus, the only question is whether the theory can be quantized before unification or not (should unification of geometry and matter be realized in nature at all).

We are therefore again in a situation, similar to that before the discovery of special relativity, where we have the choice between a) preserving an old principle, here renormalizability of perturbative QFT on background spacetimes $(M, \eta)$, at the price of introducing extra structure (extra unification symmetry), or b) replacing the old principle by a new principle, here non-perturbative QFT on a differentiable manifold $M$, without new hypothetical structure. At this point it unclear which methodology has more chances for success, historically there is evidence for either of them (e.g. the unification of electromagnetism and the massive Fermi model is evidence for the former, the replacement of Newton's notion of spactime by special relativity is evidence for the latter) and it is quite possible that we actually need both ideas. In QGR we take the latter point of view to begin with since there maybe zillions of ways to unify forces and it is hard to judge whether there is a "natural one", therefore the approach is *purposely conservative* because we actually may be able to *derive* a natural way of unification, if necessary, if we drive the theory to its logical frontiers. Among the various non-perturbative approaches available we will choose the canonical one.

Pictorially, one could illustrate the deep difference between a background dependent QFT and background independent QFT as follows: In figure 3 we see matter in the form of QCD (notice the quark ($Q$) propagators, the quark-gluon vertices and the three – and four point gluon ($G$) vertices) displayed as an actor in green. Matter propagates on a fixed background spacetime $g_0$ according to well-defined rules, particles know exactly what timelike geodesics are etc. This fixed background spacetime $g_0$ is displayed as a firm stage in blue. This is the situation of a QFT on a *Background Spacetime.*

In contrast, in figure 4 the stage has evaporated, it has become itself an actor (notice the arbitrarily high valent graviton ($g$) vertices) displayed in blue as well. Both matter and geometry are now dynamical entities and interact as displayed by the red vertex. There are no light cones any longer,

[2]In fact, e.g. the unified electroweak $SU(2)_L \times U(1)$ theory with its massless gauge bosons can be perfectly described by a classical Lagrangean. The symmetry broken, massive $U(1)$ theory can be derived from it, also classically, by introducing a constant background Higgs field (Higgs mechanism) and expanding the symmetric Lagrangean around it. It is true that the search for a massless, symmetric theory was inspired by the fact that a theory with massive gauge bosons is not renormalizable (so the motivation comes from quantum theory) and, given the non-renormalizability of general relativity, many take this as an indication that one must unify gravity with matter, one incarnation of which is string theory. However, the argument obviously fails should it be possible to quantize gravity *non-perturbatively.*

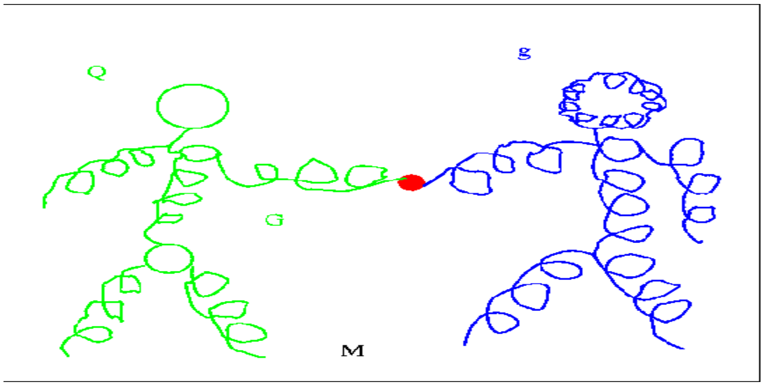


Figure 4: QGR on Differential Manifold $\boldsymbol{M}$: Actor: **Matter** + **Geometry**, Stage: Manifold $\boldsymbol{M}$

rather the causal structure is a semiclassical concept only. This is the situation of a QFT on a *Differential Manifold* and this is precisely what QGR aims to rigorously define.

It is clear from these figures that the passage from a QFT on a background spacetime to a QFT on a differential manifold is a very radical one: It is like removing the chair on which you sit and trying to find a new, yet unknown, mechanism that keeps you from falling down. We should mention here that for many researchers in quantum gravity even that picture is not yet radical enough, some proposals require not only to get rid of the background metric $\boldsymbol{g_0}$ but also of the differential manifold, allowing for topology change. This is also very desired in QGR but considered as a second step. In 3d QGR also this step could be completed and the final picture is completely combinatorial.

Let us finish this section by stating once more what we mean by Quantum General Relativity (QGR).

**DEFINITION**

**(Canonical) Quantum General Relativity (QGR) is an attempt to construct a mathematically rigorous, non – perturbative, background independent Quantum Field Theory of four – dimensional, Lorentzian General Relativity and all known matter in the continuum.**

**No additional, experimentally unverified structures are introduced. The fundamental principles of General Covariance and Quantum Theory are brought together and driven to their logical frontiers guided by mathematical consistency.**

**QGR is not a unified theory of all interactions in the standard sense since unification of gauge symmetry groups is not necessarily required in a non-perturbative approach. However, Geometry and Matter are unified in a non – standard sense by making them both transform covariantly under the Diffeomorphism Group at the quantum level.**

## I.2 Introduction: Classical Canonical Formulation of General Relativity

In this section we sketch the classical Hamiltonian formulation of general relativity in terms of Ashtekar's new variables. There are many ways to arrive at this new formulation and we will choose the one that is the most convenient one for our purposes.

The Hamiltonian formulation by definition requires some kind of split ot the spacetime variables into time and spatial variables. This seems to contradict the whole idea of general covariance, however, quantum meachanics as presently formulated requires a notion of time because we interpret expection values of operators as instantaneous measurement values averaged over a large number of measurements. In order to avoid this one has to "covariantize" the interpretation of quantum mechanics, in particular the measurement process, see e.g. [10] for a discussion. There are a number of proposals to make the canonical formulation more covariant, e.g.[3]: Multisymplectic Ansätze [13] in which there are multimomenta, one for each spacetime dimension, rather than just one for the time

[3] Path integrals [11] use the Lagrangean rather than the Hamiltonian and therefore seem to be better suited to a covariant formulation than the canonical one, however, usually the path integral is interpreted as some sort of propagator which makes use of instantaneous time Hilbert spaces again which therefore cannot be completely discarded

coordinate; Covariant phase space formulations [14] where one works on the space of solutions to the field equations rather than on the intial value instantaneous phase space; Peierl's bracket formulations [15] which covariantize the notion of the usual Poisson bracket; history bracket formulations [16], which grew out of the consistent history formulation of quantum mechanics [17], and which extends the usual spatial Poisson bracket to spacetime.

At the classical level all these formulations are equivalent. However, at the quantum level, one presently gets farthest within the the standard canonical formulation: The quantization of the multisymplectic approach is still in its beginning, see [18] for the most advanced results in this respect; The covariant phase space formulation is not only very implicit because one usually does not know the space of solutions to the classical field equations, but even if one manages to base a quantum theory on it, it will be too close to the classical theory since certainly the singularities of the classical theory are also built into the quantum theory; The Peierl's bracket also needs the explicit space of solutions to the classical field equations; Also the quantization of the history bracket formulation just has started, see [19] for first steps in that direction.

Given this present status of affairs, we will therefore proceed with the standard canonical quantization and see how far we get. Notice that there is no obvious problem with general covariance: For instance, standard Maxwell theory can be quantized canonically without any problem and one can show that the theory is Lorentz covariant although the spacetime split into space and time seems to break the Lorentz group down to the rotation group. *This is not at all the case* ! It is just that Lorentz covariance is not manifest, one has to do some work in order to establish Lorentz covariance. Indeed, as we will see, at least at the classical level we will explicitly recover the four-dimensional diffeomorphism group in the formalism, although it is admittedly deeply hiddeen in the canonical formalism.

With these cautionary remarks out of the way, we will thus assume that the four dimensional spacetime manifold has the topology $\mathbb{R}\times\sigma$, where $\sigma$ is a three dimensional manifold of arbitrary topology, in order to perform the $3+1$ split. This assumption about the topology of $M$ may seem rather restrictive, however, it is not due to the following reasons: (1) According to a theorem due to Geroch any globally hyperbolic manifold (roughly those that admit a smooth metric with everywhere Lorentzian signature) are necessarily of that topology. Since Lorentzian metrics are what we are interested in, at least classically the assumption about the topology of $M$ is forced on us. (2) Any four manifold $M$ has the topology of a countable disjoint union $\cup_\alpha I_\alpha \times \sigma_\alpha$ where either $I_\alpha$ are open intervals and $\sigma_\alpha$ is a three manifold or $I_\alpha$ is a one point set and $\sigma_\alpha$ is a two manifold (the latter are the intersections of the closures of the former). In this most generic situation we thus allow topology change between different three manifolds and it is even classically an open question how to make this compatible with the action principle. We take here a practical point of view and try to understand the quantum theory first for a single copy of the form $\mathbb{R}\times\sigma$ and later on worry how we glue the theories for different $\sigma's$ together.

### I.2.1 The ADM Formulation

In this nice situation the $3+1$ split is well known as the Arnowitt – Deser – Misner (ADM) formulation of general relativity, see e.g. [4] and we briefly sketch how this works. Since $M$ is diffeomorphic

with. At present, this connection with the canonical formulation is not very transparent, part of the reason being that the path integral is usually only defined in its Euclidean formulation, however the very notion of analytic continuation in time is not very meaningful in a theory where there is no distinguished choice of time, see however [12] for recent progress in this direction.

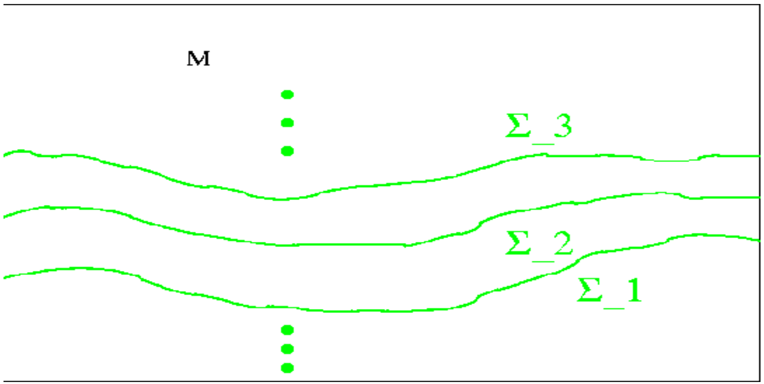


Figure 5: Foliation of $M$

to $\mathbb{R} \times \sigma$ we know that $M$ foliates into hypersurfaces $\Sigma_t,\ t \in \mathbb{R}$ as in figure 5, where $t$ labels the hypersurface and will play the role of our time coordinate. If we denote the four dimensional coordinates by $X^\mu,\ \mu = 0,1,2,3$ and the three dimensional coordinates by $x^a,\ a = 1,2,3$ then we know that there is a diffeomorphism $\varphi :\ \mathbb{R} \times \sigma \to M;\ (t,x) \mapsto X = \varphi(t,x)$ where $\Sigma_t = \varphi(t,\sigma)$. We stress that the four diffeomorphism $\varphi$ is *completely arbitrary* until this point and thus *the foliation of $M$ is not at all fixed*, in other words, the set of foliations *is in one to one correspondence with Diff$(M)$, the four dimensional diffeomorphism group.* Consider the tangential vector fields to $\Sigma_t$ given by

Here we are considering for simplicity only the case that $\sigma$ is compact, otherwise (??) would contain boundary terms. The end result is

The reason for the occurence of the Lagrange multipliers $\lambda, \lambda^a$ is that the Lagrangean (??) is singular, that is, one cannot solve all the velocities in terms of momenta and therefore one must use Dirac's procedure [20] for the Legendre transform of singular Lagrangeans. In this case the singularity structure is such that the momenta conjugate to $N, U^a$ vanish identically, whence the Lagrange multipliers which when varied give the equations of motion $P = P_a = 0$. The equations of motion with respect to the Hamiltonian (i.e. $\dot{F} := \{H, F\}$ for any functional $F$ of the canonical coordinates)

Now the equations of motion for $q_{ab}, P^{ab}$ imply the so-called *Dirac (or hypersurface deformation) algebra*

$$\begin{aligned}
\{V(U), V(U')\} &= \kappa V(\mathcal{L}_U U') \\
\{V(U), C(N)\} &= \kappa C(\mathcal{L}_U N) \\
\{C(N), C(N')\} &= \kappa V(q^{-1}(N dN' - N' dN))
\end{aligned} \qquad \text{(I.2.1.1)}$$

where e.g. $C(N) = \int d^3x NC$. These equations tell us that the condition $H = V_a = 0$ is preserved under evolution, in other words, the evolution is consistent ! This is a non-trivial result. One says, the Hamiltonian and vector constraint form a *first class constraint algebra*. This algebra is much more complicated than the more familiar Kac-Moody algebras due to the fact that it is not an (infinite) dimensional Lie algebra in the true sense of the word because the "structure constants"

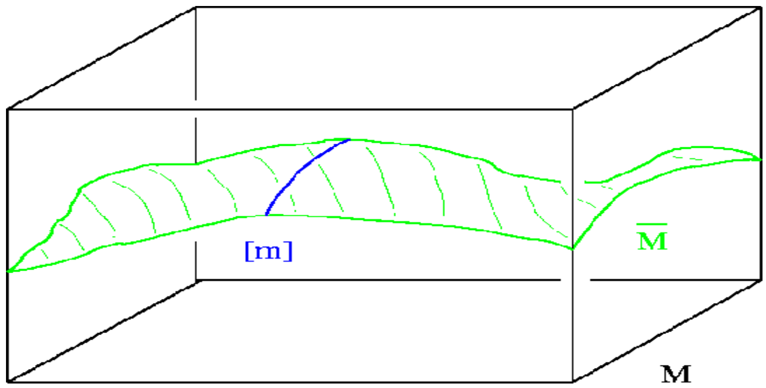


Figure 6: Constraint hypersurface $\overline{\mathcal{M}}$ and gauge orbit $[m]$ of $m \in \overline{\mathcal{M}}$ in $\mathcal{M}$

.

on the right hand side of the last line in (I.2.1.1) are not really constants, they depend on the phase space. Such algebras are open in the the terminology of BRST [21] and about their representation theory only very little is known.

**Exercise I.2.1.**
*Derive (I.2.1.1) from (??).*
*Hint:*
*Show first that the Poisson bracket between local functions which contain spatial derivatives is simply the spatial derivatives applied to the Poisson bracket. Since the Poisson bracket of local functions is distributional recall that derivatives of distributions are defined through an integration by parts.*

Since the variables $P, P_a$ drop out completely from the analysis and $N, U^a$ are Lagrange multipliers, we may replace (??) by

Let us summarize the gauge theory of GR in figure 6: The constraints $C = V_a = 0$ define a constraint hypersurface $\overline{\mathcal{M}}$ within the full phase space $\mathcal{M}$. The gauge motions are defined on all of $\mathcal{M}$ but they have the feature that they leave the constraint hypersurface invariant, and thus the orbit of a point $m$ in the hypersurface under gauge transformations will be a curve or gauge orbit $[m]$ entirely within it. The set of these curves defines the so-called reduced phase space and Dirac observables restricted to $\overline{\mathcal{M}}$ depend only on these orbits. Notice that as far as the counting is concerned we have twelve phase space coordinates $q_{ab}, P^{ab}$ to begin with. The four constraints $C, V_a$ can be solved to eliminate four of those and there are still identifications under four independent sets of motions among the remaining eight variables leaving us with only four Dirac observables. The corresponding so-called reduced phase space has therefore precisely the two configuration degrees of freedom of general relativity.

### I.2.2 Gauge Theory Formulation

We can now easily introduce the shift from the ADM variables $q_{ab}, P^{ab}$ to the connection variables introduced first by Ashtekar [25] and later somewhat generalized by Immirzi [26] and Barbero [27]. We introduce $su(2)$ indices $i, j, k, .. = 1, 2, 3$ and co-triad variables $e_a^j$ with inverse $e_j^a$ whose relation with $q_{ab}$ is given by

We then have the following theorem [25].

**Theorem I.2.1.**
*Consider the phase space $\mathcal{M}$ coordinatized by $(A_a^j, E_j^b)$ with Poisson brackets*

# I.3 Canonical Quantization Programme for Theories with Constraints

## I.3.1 Refined Algebraic Quantization (RAQ)

*As we have seen, GR can be formulated as a constrained Hamiltonian system with first class constraints. The quantization of such systems has been considered first by Dirac [20] and was later refined by a number of authors. It is now known under the name refined algebraic quantization (RAQ). We will briefly sketch the main ideas following [29].*

i) *Phase Space and Constraints*
*The starting point is a phase space $(\mathcal{M}, \{.,.\})$ together with a set of first class constraints $C_I$ and possibly a Hamiltonian $H$.*

ii) *Choice of Polarization*
*In order to quantize the phase space we must choose a polarization, that is, a Lagrangean submanifold $\mathcal{C}$ of $\mathcal{M}$ which is called configuration space. The coordinates of $\mathcal{C}$ have vanishing Poisson brackets among themselves. If $\mathcal{M}$ is a cotangent bundle, that is, $\mathcal{M} = T^*\mathcal{Q}$ then it is natural to choose $\mathcal{Q} = \mathcal{C}$ and we will assume this to be the case in what follows. For more general cases, e.g. compact phases spaces one needs ideas from geometrical quantization, see e.g. [30]. The idea is that (generalized, see below) points of $\mathcal{C}$ serve as arguments of the vectors of the Hilbert space to be constructed.*

iii) *Preferred Kinematical Poisson Subalgebra*
*Consider the space $C^\infty(\mathcal{C})$ of smooth functions on $\mathcal{C}$ and the space $V^\infty(\mathcal{C})$ of smooth vector fields on $C$. The vertical polarization of $\mathcal{M}$, that is, the space of fibre coordinates called momentum space, generates preferred elements of $V^\infty(\mathcal{C})$ through $(v_p[f])(q) := (\{p, f\})(q)$ where we have denoted configuration and momentum coordinates by $q, p$ respectively and $v[f]$ denotes the action of a vector field on a function. The pair $C^\infty(\mathcal{C}) \times \mathcal{V}^\infty(\mathcal{C})$ forms a Lie algebra defined by $[(f, v), (f', v')] = (v[f'] - v'[f], [v, v'])$ of which the algebra $\mathcal{B}$ generated by elements of the form $(f, v_p)$ forms a subalgebra. We assume that $\mathcal{B}$ is closed under complex conjugation which becomes its $^*-$operation (involution).*

iv) *Representation Theory of the Corresponding Abstract $^*-$Algebra*
*We are looking for all irreducible $^*-$representations $\pi : \mathcal{B} \to \mathcal{L}(\mathcal{H}_{kin})$ of $\mathcal{B}$ as linear operators on a kinematical Hilbert space $\mathcal{H}_{kin}$ such that the $^*-$relations becomes the operator adjoint and such that the canonical commutation relations are implemented, that is, for all $a, b \in \mathcal{B}$*

$$\begin{aligned} \pi(a)^\dagger &= \pi(a^*) \\ [\pi(a), \pi(b)] &= i\hbar\pi([a, b]) \end{aligned} \tag{I.3.1.1}$$

*Strictly speaking, (I.3.1.1) is to be supplemented by the domains on which the operators are defined. In order to avoid this one will work with the subalgebra of $C^\infty(\mathcal{C})$ formed by bounded functions, say of compact support and one will deal with exponentiated vector fields in order to obtain bounded operators. Irreducibility is a physically meaningful requirement because we are not interested in Hilbert spaces with superselection sectors and the reason for why we do not require the full Poisson algebra to be faithfully represented is that this is almost always impossible in irreducible representations as stated in the famous Groenewald – van Hove theorem. The Hilbert space that one gets can usually be described in the form $L_2(\overline{\mathcal{C}}, d\mu)$ where $\overline{\mathcal{C}}$ is a distributional extension of $\mathcal{C}$ and $\mu$ is a probability measure thereon. A well-known example is the case of free scalar fields on Minkowski space where $\mathcal{C}$ is some space of smooth scalar fields on $\mathbb{R}^3$ vanishing at spatial infinity while $\overline{\mathcal{C}}$ is the space of tempered distributions on $\mathbb{R}^3$ and $\mu$ is a normalized Gaussian measure on $\overline{\mathcal{C}}$.*

v) *Selection of Suitable Kinematical Representations*
*Certainly we want a representation which supports also the constraints and the Hamiltonian as operators which usually will limit the number of available representations to a small number, if possible at all. The constraints usually are not in $\mathcal{B}$ unless linear in momentum and the expressions $\hat{C}_I := \pi(C_I), \hat{H} = \pi(H)$ will involve factor ordering ambiguities as well as regularizationand renormalization processes in the case of field theories. In the generic case, $\hat{C}_I, \hat{H}$ will not be bounded and $\hat{C}_I$ will not be symmetric. We will require that $\hat{H}$ is symmetric and that the constraints are at least closable, that is, they are densely defined together with their adjoints. It is then usually not too difficult to find a dense domain $\mathcal{D}_{kin} \subset \mathcal{H}_{kin}$ on which all these operators and their adjoints are defined and which they leave invariant. Typically $\mathcal{D}_{kin}$ will be a space of smooth functions of rapid decrease so that arbitray derivatives and polynomials of the configuration variables are defined on them and such spaces naturally come with their own topology which is finer than the subspace topology induced from $\mathcal{H}_{kin}$ whence we have a topological inclusion $\mathcal{D}_{kin} \hookrightarrow \mathcal{H}_{kin}$.*

vi) *Imposition of the Constraints*
*The two step process in the classical theory of solving the constraints $C_I = 0$ and looking for the gauge orbits is replaced by a one step process in the quantum theory, namely looking for solutions $l$ of the equations $\hat{C}_I l = 0$. This is because it is obviously solves the constraint at the quantum level (in the corresponding representation on the solution space the constraints are replaced by the zero operator) and it simultaneously looks for states that are gauge invariant because $\hat{C}_I$ is the quantum generator of gauge transformations.*

*Now, unless the point $\{0\}$ is in the common point spectrum of all the $\hat{C}_I$, solutions $l$ to the equations $\hat{C}_I l = 0 \ \forall \ I$ do not lie in $\mathcal{H}_{kin}$, rather they are distributions. Here one has several options, one could look for solutions in the space $\mathcal{D}'_{kin}$ of continuous linear functionals on $\mathcal{D}_{kin}$ (topological dual) or in the space $\mathcal{D}^*_{kin}$ of linear functionals on $\mathcal{D}_{kin}$ with the topology of pointwise convergence (algebraic dual). Since certainly $\mathcal{H}_{kin} \subset \mathcal{D}'_{kin} \subset \mathcal{D}^*_{kin}$ let us choose the latter option for the sake of more generality. The topology on $\mathcal{H}_{kin}$ is again finer than the subspace topology induced from $\mathcal{D}^*_{kin}$ so that we obtain a Gel'fand triple or Rigged Hilbert Space*

*We are now looking for a subspace $\mathcal{D}^*_{phys} \subset \mathcal{D}^*_{kin}$ such that for its elements $l$ holds*

*A physical inner product on a subset $\mathcal{H}_{phys} \subset \mathcal{D}^*_{phys}$ is a positive definite sesquilinear form $<.,.>_{phys}$ with respect to which the $\hat{O}'$ become self-adjoint operators, that is, $\hat{O}' = (\hat{O}')^\star$ where the adjoint on $\mathcal{H}_{phys}$ is denoted by $\star$. Notice that $[\hat{O}'_1, \hat{O}'_2] = ([\hat{O}_1, \hat{O}_2])'$ so that commutation relations on $\mathcal{H}_{kin}$ are automatically transferred to $\mathcal{H}_{phys}$ which then carries a proper $^*-$representation of the physical observables. The observables themselves will only be defined on a dense domain $\mathcal{D}_{phys} \subset \mathcal{H}_{phys}$ and we get a second Gel'fand triple*

### I.3.2 Selected Examples with First Class Constraints

*In the case that a theory has only first class constraints, Dirac's algorithm [20] boils down to the following four steps:*
*1)*
*Define the momentum $p_a$ conjugate to the configuration variable $q^a$ by (Legendre transform)*

*In this report we will only deal with theories which have no second class constraints, so this algorithm is all we need.*

**Exercise I.3.1.**
*Perform the quantization programme for a couple of simple systems in order to get a feeling for the formalism:*

1. *Momentum Constraint*
   $\mathcal{M} = T^*\mathbb{R}^2$ *with standard Poisson brackets among* $q^a, p_a;\ a = 1, 2$ *and constraint* $C := p_1$. *Choose* $\mathcal{H}_{kin} = L_2(\mathbb{R}^2, d^2x),\ \mathcal{D}_{kin} = \mathcal{S}(\mathbb{R}^2),\ \mathcal{D}^*_{kin} = \mathcal{S}'(\mathbb{R}^2)$ *(spaces of functions of rapid decrease and tempered distributions respectively).*
   *Solution:*
   *Dirac observables are the conjugate pair* $q^2, p_2$, $\mathcal{H}_{phys} = L_2(\mathbb{R}, dx_2)$.
   *Hint: Work in the momentum representation and conclude that the general solution is of the form* $l_f(p_1, p_2) = \delta(p_1)f(p_2)$ *for* $f \in \mathcal{S}'(\mathbb{R})$.
2. *Angular Momentum Constraint*
   $\mathcal{M} = T^*\mathbb{R}^3$ *with standard Poisson brackets among* $q^a, p_a;\ a = 1, 2, 3$ *and constraints* $C_a := \epsilon_{abc}x^b p_c$. *Check the first class property and choose the kinematical spaces as above with* $\mathbb{R}^2$ *replaced by* $\mathbb{R}^3$.
   *Solution:*
   *Dirac observables are the conjugate pair* $r := \sqrt{\delta_{ab}q^a q^b} \geq 0,\ p_r = \delta_{ab}q^a p_b/r$, *the physical phase space is* $T^*\mathbb{R}_+$ *and* $\mathcal{H}_{phys} = L_2(\mathbb{R}_+, r^2 dr)$ *where* $\hat{r}$ *is a multiplication operator and* $\hat{p}_r = i\hbar\frac{1}{r}\frac{d}{dr}r$ *with dense domain of symmetry given by the square integrable functions* $f$ *such that* $f$ *is regular at* $r = 0$.
   *Hint:*
   *Introduce polar coordinates and decompose kinematical wave functions into spherical*

*harmonics. Conclude that the physical Hilbert space this time is just the restriction of the kinematical Hilbert space to the zero angular momentum subspace, that is, $\mathcal{H}_{phys} \subset \mathcal{H}_{kin}$. The reason is of course that the spectrum of the $\hat{C}_a$ is pure point (discrete).*

3. *Relativistic Particle*
   *Consider the Lagrangean $L = -m\sqrt{-\eta_{\mu\nu}\dot{q}^\mu\dot{q}^\nu}$ where $m$ is a mass parameter, $\eta$ is the Minkowski metric and $\mu = 0, 1, .., D$. Verify that the Lagrangean is singular, that is, the velocities $\dot{q}^\mu$ cannot be expressed in terms of the momenta $p_\mu = \partial L/\partial\dot{q}^\mu$ which gives rise to the mass shell constraint $C = m^2 + \eta^{\mu\nu}p_\mu p_\nu$. Verify that this happens because the corresponding action is invariant under Diff($\mathbb{R}$), that is, reparameterizations $t \mapsto \varphi(t),\ \dot{\varphi}(t) > 0$. Perform the Dirac analysis for constraints and conclude that the system has no Hamiltonian, just the Hamiltonian constraint $C$ which generates reparameterizations on the kinematical phase space $\mathcal{M} = T^*\mathbb{R}^{D+1}$ with standard Poisson brackets. Now choose kinematical spaces as in 1. with $\mathbb{R}^2$ replaced by $\mathbb{R}^{D+1}$.*
   *Solution:*
   *Conjugate Dirac observables are $Q^a = q^a - \frac{q^0 p_a}{\sqrt{m^2+\delta^{ab}p_a p_b}}$ and $\mathcal{H}_{phys} = L_2(\mathbb{R}^D, d^D p)$ on which $\hat{q}^0 = 0$.*
   *Hint:*
   *Work in the momentum representation and conclude that the general solution to the constraints is of the form $l_f = \delta(C)f(p_0, \vec{p})$. Now notice that the $\delta-$distribution can be written as a sum of two $\delta-$distribution corresponding to the positive and negative mass shell and choose $f$ to have support in the former.*
   *This example has features rather close to those of general relativity.*

4. *Maxwell Theory*
   *Consider the action for free Maxwell-theory on Minkowski space and perform the Legendre transform. Conclude that there is a first class constraint $C = \partial_a E^a$ (Gauss constraint) with Lagrange multiplier $A_0$ and a Hamiltonian $H = \int_{\mathbb{R}^3} d^3x[E^aE^b + B^aB^b]/2$ where $E^a = \dot{A}_a - \partial_a A_0$ is the electric field and $B^a = \epsilon^{abc}\partial_b A_c$ the magnetic one. Verify that the Gauss constraint generates $U(1)$ gauge transformations $A \mapsto A - df$ while $E^a$ is gauge invariant. Choose $\mathcal{H}_{kin}$ to be the standard Fock space for three massless, free scalar fields $A_a$ and as $\mathcal{D}_{kin},\ \mathcal{D}^*_{kin}$ the finite linear span of $n-$particle states and its algebraic dual respectively.*
   *Solution:*
   *Conjugate Dirac observables are the transversal parts of $A, E$ respectively, e.g. $E^a_\perp = E^a - \partial_a\frac{1}{\Delta}\partial_b E^b$ where $\Delta$ is the Laplacian on $\mathbb{R}^3$. The physical Hilbert space is the standard Fock space for two free, masless scalar fields corresponding to these transversal degrees of freedom.*
   *Hint:*
   *Fourier transform the fields and compute the standard annihilation and creation operators $\hat{z}_a(k), \hat{z}^\dagger_a(k)$ with canonical commutation relations. Express the Gauss constraint operator in terms of them and conclude that the gauge invariant part satisfies $\hat{z}_a(k)k^a = 0$. Introduce $\hat{z}_I(k) = \hat{z}_a(k)e^a_I(k)$ where $\vec{e}_1(k), \vec{e}_2(k), \vec{e}_3(k) := \vec{k}/||k||$ form an oriented orthonormal basis. Conclude that physical states are states without longitudonal excitations and build the Fock space generated by the $\hat{z}^\dagger_1(k), \hat{z}^\dagger_2(k)$ from the kinematical vacuum state.*

# Part II

# Mathematical and Physical Foundations of Quantum General Relativity

## II.1 Mathematical Foundations

### II.1.1 Polarization and Preferred Poisson Algebra $\mathcal{B}$

*The first two steps of the quantization programme were already completed in section I.2: The phase space* $\mathcal{M}$ *is coordinatized by canonically conjugate pairs* $(A_a^j, E_j^a)$ *where* $A$ *is an* $SU(2)$ *connection over* $\sigma$ *while* $E$ *is a* $su(2)-$*valued vector density of weight one over* $\sigma$ *and the Poisson brackets were displayed in (??). Strictly speaking, since* $\mathcal{M}$ *is an infinite dimensional space, one must supply* $\mathcal{M}$ *with a manifold structure modelled on some Banach space but we will skip these functional analytic niceties here, see [1] for further information. Also we must specify the principal fibre bundle of which* $A$ *is the pull-back by local sections of a globally defined connection, and we must specify the vector bundle associated to that principal bundle under the adjoint representation of which* $E$ *is the pull-back by local sections. Again, in order not to dive too deeply into fibre bundle theoretic subtleties, we will assume that the principal fibre bundle is trivial so that* $A, E$ *are actually globally defined. In fact, for the case of* $G = SU(2)$ *and* $\dim(\sigma) = 3$ *one can show that the fibre bundle is necessarily trivial but for the generalization to the generic case we again refer the reader to [1].*

*With these remrks out of the way we may begin by defining a polarization. The fact that GR has been casted into the language of a gauge theory suggests the choice* $\mathcal{C} = \mathcal{A}$*, the space of smooth* $SU(2)$ *connections over* $\sigma$*.*

*The next question then is how to choose the space* $C^\infty(\mathcal{A})$*. Since we are dealing with a field theory, it is not clear a priori what smooth or even differentiable means. In order to give precise meaning to this, one really has to equip* $\mathcal{A}$ *with a manifold structure modelled on a Banach space. This is because one usually says that a function* $F : \mathcal{A} \to \mathbb{C}$ *is differentiable at* $A_0 \in \mathcal{A}$ *provided that there exists a bounded linear functional* $DF_{A_0} : T_{A_0}(\mathcal{A}) \to \mathbb{C}$ *such that* $F[A_0 + \delta A] - F[A_0] - DF_{A_0} \cdot \delta A$ *vanishes "faster than linearly" for arbitrary tangent vectors* $\delta A \in T_{A_0}(\mathcal{A})$ *at* $A_0$*. (The proper way of saying this is using the natural Banach norm on* $T(\mathcal{A})$*.) Of course, in physicist's notation the differential* $DF_{A_0} = (\delta F/\delta A)(A_0)$ *is nothing else than the functional derivative. Using this definition it is clear that polynomials in* $A_a^j(x)$ *are not differentiable because their functional derivative is proportional to a* $\delta-$*distribution as it is clear from (??). Thus we see that the smooth functions of* $A$ *have to involve some kind of smearing of* $A$ *with test functions, which is generic in field theories.*

*Now this smearing should be done in a judicious way. The function* $F[A] := \int_\sigma d^3x F_j^a(x) A_a^j(x)$ *for some smooth test function* $F_j^a$ *of compact support is certainly smooth in the above sense, its functional derivative being equal to* $F_a^j$ *(which is a bounded operator if* $F$ *is e.g. an* $L_2$ *function on* $\sigma$ *and the norm on the tangent spaces is an* $L_2$ *norm). However, this function does not transform nicely under* $SU(2)$ *gauge transformations which will make it very hard to construct* $SU(2)$ *invariant functions from them. Here it helps to look up how physicists have dealt with this problem in ordinary canonical quantum Yang-Mills gauge theories and they found the following, more or less unique solution [31]:*
*Given a curve* $c : [0,1] \to \sigma$ *in* $\sigma$ *and a point* $A \in \mathcal{A}$ *we define the holonomy or parallel transport* $A(c) := h_{c,A}(1) \in SU(2)$ *as the unique solution to the following ordinary differential equation for functions* $h_{c,A} : [0,1] \to SU(2)$

*The fact that the holonomy smears* $A$ *only one-dimensionally is nice due to the above reasons but it is also alarming because its functional derivative is certainly distributional and thus does not exist in* *the strict mathematical sense. However, in order to obtain a well-defined Poisson algebra it is not necessary to have smooth functions of $A$, it is only sufficient. The key idea idea is that if we smear also the electric fields $E$ then we might get a non-distributional Poisson algebra. By inspection from (??) it is clear that $E$ has to be smeared in at least two dimensions in order to achieve this. Now again background independence comes to our help: Let $\epsilon_{abc}$ be the totally skew, background independent tensor density of weight $-1$, that is, $\epsilon_{abc} = \delta^1_{[a}\delta^2_b\delta^3_{c]}$ where $[..]$ denotes total antisymmetrization. Then $(*E)^j_{ab} := E^j_{ab} := E^c_j\epsilon_{abc}$ is a 2-form of density weight $0$. Therefore $E$ is naturally smeared in two dimensions. Notice that the smearing dimensions of momenta and configuration variables add up to the dimension of $\sigma$, they are dual to each other which is a generic phenomenon for any canonical theory in any dimension. We are therefore led to consider the electric fluxes*

*Let us see whether the Poisson bracket between an electric flux and a holonomy is well-defined. Actually, let us be slightly more general and introduce the following notion: Let us losely think for the moment of a graph $\gamma$ as a collection of a finite number of smooth, compactly supported, oriented curves, called edges $e$, which intersect at most in their end points, which are called vertices $v$. We denote by $E(\gamma),\ V(\gamma)$ the edge and vertex set of $\gamma$ respectively. A precise definition will be given in section II.1.2.*

**Definition II.1.1.**
*Given a graph $\gamma$ we define*

### II.1.2 Representation Theory of $\mathcal{B}$ and Suitable Kinematical Representations

*The representation Theory of $\mathcal{B}$ has been considered only rather recently [35] and the analysis is not yet complete. However, if one sticks to irreducible representations for which 1) the flux operators are well-defined and self-adjoint (in other words, the corresponding one parameter unitary groups are weakly continuous) and 2) the representation is spatially diffeomorphism invariant, then the unique solution to the representation problem is the representation which we describe in this section.*

*This representation is of the form $\mathcal{H}_0 = L_2(\overline{\mathcal{A}}, d\mu_0)$ where $\overline{\mathcal{A}}$ is a certain distributional extension of $\mathcal{A}$ and $\mu_0$ is a probability measure thereon. The most elegant description of this Hilbert space uses the theory of $C^*-$algebras [36] but fortunately there is a purely geometric description available [37] which is easier to access for the beginner. In what follows we assume for simplicity that $\sigma$ is an oriented, connected, simply connected smooth manifold. One can show that each smooth manifold admits at least one analytic structure (i.e. the atlas of charts consists of real analytic maps) and we assume to have picked one once and for all.*

#### II.1.2.1 Curves, Paths, Graphs and Groupoids

**Definition II.1.2.**
*i)*
*By a curve $c$ we mean a map $c:\ [0,1] \to \sigma$ which is piecewise analytic, continuous, oriented and an embedding (does not come arbitrarily close to itself). It is automatically compactly supported. The*

*set of curves is denoted $\mathcal{C}$ in what follows.*
*ii)*
*On $\mathcal{C}$ we define maps $\circ,\ (.)^{-1}$ called composition and inversion respectively by*

#### II.1.2.2 Topology on $\overline{\mathcal{A}}$

*So far $\overline{\mathcal{A}}$ is just a set. We now equip it with a topology. The idea is actually quite simple. Recall the maps (??) which easily extend from $\mathcal{A}$ to $\overline{\mathcal{A}}$ and maps $\overline{\mathcal{A}}$ into $SU(2)^{|E(\gamma)|}$. Now $SU(2)^{|E(\gamma)|}$ is a compact Hausdorff topological group*[4] *in its natural manifold topology and we would like to exploit that. Thus we are motivated to consider the spaces $X_\gamma :=$ Hom$(\gamma, SU(2)^{|E(\gamma)|})$ where $\gamma$ is considered as a subgroupoid of $\Gamma$ with objects $V(\gamma)$ and morphisms generated by the $e \in E(\gamma)$. The map*

*Now another standard result from topology now tells us that $\overline{X}$, being the closed subset of a compact Hausdorff space, is a compact Hausdorff space in the subspace topology and the question arises whether*

#### II.1.2.3 Measures on $\overline{\mathcal{A}}$

*A powerful theorem due to Riesz and Markov, sometimews called the Riesz representation theorem, tells us that there is a one – to – one correspondence between the positive linear functionals $\Lambda$ on the algebra $C(\overline{\mathcal{A}})$ of continuous functions on a compact Hausdorff space $\overline{\mathcal{A}}$ and (regular, Borel) probability measures $\mu$ thereon through the simple formula*

*In order to specify the measure $\mu_0$ that we are interested in, it is therefore enough to specify a positive linear functional $\Lambda_0$. The most elegant way of defining $\Lambda_0$ is through the following definition.*

**Definition II.1.3.**
*i)*
*Given a graph $\gamma$, label each edge $e \in E(\gamma)$ with a triple of numbers $(j_e, m_e, n_e)$ where $j_e \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, ..\}$ is a half-integral spin quantum number and $m_e, n_e \in \{-j_e, -j_e+1, .., j_e\}$ are magnetic quantum numbers. A quadruple*

#### II.1.2.4 Representation Property

*So far we did not verify that $\mathcal{H}_{kin}$ is a representation space for our $^*-$algebra $\mathcal{B}$ of basic operators. This will be done in the present section. Indeed, until today no other irreducible representation of the holonomy – flux algebra has been found (except if one allows also infinite graphs [42]).*

[4]Here it is crucial that $G = SU(2)$ is compact and thus for non-real Immirzi parameter all of what follows would be false.

*By theorem (II.1.1) the subspace of finite linear combinations of SNWF's is dense in $\mathcal{H}_{kin}$ with respect to the $L_2$ norm. On the other hand, we notice that the definition of Cyl$^\infty(\mathcal{A})$ simply extends to Cyl$^\infty(\overline{\mathcal{A}})$ and that finite linear combinations of SNWF's form a subspace of Cyl$^\infty(\overline{\mathcal{A}})$. Thus, we may choose $\mathcal{D}_{kin} :=$ Cyl$^\infty(\overline{\mathcal{A}})$ and obtain a dense, invariant domain of $\mathfrak{B}$ as we will see shortly. We define a representation of the holonomy – flux algebra by ($f' \in$ Cyl$^\infty(\mathcal{A}),\ f \in$ Cyl$^\infty(\overline{\mathcal{A}}),\ A \in \overline{\mathcal{A}}$)*

$$
\begin{array}{rcl}
[\pi(f)\cdot f'](A) & := & (f'f)(A) \\
{[\pi(v^j_S)\cdot f](A)} & = & [\pi(v^j_S)\pi(f)\cdot 1](A) = [([\pi(v^j_S),\pi(f)] + \pi(f)\pi(E_j(S)))\cdot 1](A) \\
 & := & i\hbar[\pi(v^j_S[f])\cdot 1](A) = i\hbar(v^j_S[f])(A)
\end{array}
\tag{II.1.2.1}
$$

*Thus $\pi(f)$ is a multiplication operator while $\pi(v^j_S)$ is a true derivative operator, i.e. it annihilates constants. Notice that the canonical commutation relations are already obeyed by construction, thus we only need to verify the $^*-$relations and the fact that $\pi(v^j_S)$ annihilates constants will be crucial for that.*

*The $\pi(f)$ are bounded multiplication operators (recall that smooth, i.e. in particular continuous, functions on copmpact spaces are uniformly bounded, that is, have a sup – norm) so that the adjoint is complex conjugation, therefore there is nothing to check. As for $\pi(v^j_S)$ we notice that given two smooth cylindrical functions on $\overline{\mathcal{A}}$ we always find a graph $\gamma$ over which both of them are cylindrical and which is already adapted to $S$.*

**Exercise II.1.1.**
*Let $f$ be cylindrical over $\gamma$. Verify that*

*Finally, let us verify that the representation is irreducible. By definition, a representation is irreducible if every vector is cyclic and a vector $\Omega$ is cyclic if the set of vectors $\pi(a)\Omega,\ a \in \mathfrak{B}$ is dense. Now the vector $\Omega = 1$ is cyclic because the vectors $\pi(f)\Omega = f,\ f \in$ Cyl$^\infty$ are already dense. Given an arbitrary element $\psi \in \mathcal{H}_{kin}$ we know that it is a Cauchy limit of finite linear combinations of spin network functions. Thus, if we can show that we find a sequence $a_n \in \mathfrak{B}$ such that $\pi(a_n)\psi$ converges to $\Omega$, then we are done. It is easy to see (exercise) that this problem is equivalent to showing that any $F \in L_2(F, d\mu_H)$ can be mapped by the algebra formed out of right invariant vector fields and smooth functions on $SU(2)$ to the constant function.*

**Exercise II.1.2.**
*Check that this is indeed the case.*
*Hint:*
*Show first that it is sufficient to establish that any polynomial $p$ of the $a, b, c, d,\ ad - bc = 1$ for $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ can be mapped to the constant function. Show then that suitable linear combinations of the $R^j,\ j = 1, 2, 3$ with coefficients in $C^\infty(SU(2))$ produce the derivatives $\partial_a, \partial_b, \partial_c$ and convince yourself that $a^N p$ is a polynomial in $a, b, c$ for sufficiently large $N$.*

*Let us collect these results in the following theorem [43].*

**Theorem II.1.1.**
*The relations (II.1.2.1) define an irreducible representation of $\mathfrak{B}$ on $\mathcal{H}_{kin}$.*

*Thus, the representation space $\mathcal{H}_{kin}$ constructed satisfies all the requirements that qualify it as a good kinematical starting point for solving the quantum constraints. Moreover, the measure $\mu_0$ is spatially diffeomorphism invariant as we will see shortly and together with the uniqueness result quoted at the beginning of this section, this is the only representation with that property. There are, however, doubts on physical grounds whether one should insist on a spatially diffeomorphism invariant representation because the smooth and even analytic structure of $\sigma$ which is encoded in the spatial diffeomorphism group should not play a fundamental role at short scales if Planck scale physics is fundamentally discrete. In fact, as we will see later, QGR predicts a discrete Planck scale structure and therefore the fact that we started with analytic data and ended up with discrete (discontinuous) spectra of operators looks awkward. Therefore, on the one hand we should keep in mind that other representations are possibly better suited in the final picture, on the other hand there is no logical contradiction within the present formulation and in fact in 2+1 gravity one has a final combinatorical description while one started with analytical structures as well.*

# II.2 Quantum Kinematics

*In this section we discuss the complete solution of the Gauss and Vector constraint as well as the quantization of kinematical, geometrical operators that measure the length, area and volume of coordinate curves, surfaces and regions respectively. We call these results kinematical because the Gauss and Vector constraint do not generate dynamics, this is the role of the Hamiltonian constraint which we will discuss in the third part. Moreover, the kinematical geometrical operators do not commute with the Vector constraint or the Hamiltonian constraint and are therefore not Dirac observables. However, as we will show, one can turn these operators easily into Dirac observables, at least with respect to the Vector constraint, and the fact that the spectrum is discrete is robust under those changes.*

## II.2.1 The Space of Solutions to the Gauss and Spatial Diffeomorphism Constraint

*Recall the transformation behaviour of classical connections $A \in \mathcal{A}$ under $SU(2)$ gauge transformations and spatial diffeomorphisms (??). These equations trivially lift from $\mathcal{A}$ to $\overline{\mathcal{A}}$ and we may construct corresponding operators as follows: Let $\overline{\mathcal{G}} := Fun(\Sigma, SU(2))$ be the set of local gauge transformations without continuity requirement and consider the set $Diff^{\omega}(\Sigma)$ of analytic diffeomorphisms. We are forced to consider analytic diffeomorphisms as otherwise we would destroy the analyticity of the elements of $\Gamma$. These two groups have a natural semi – direct product structure that has its origin in the algebra (??) and is given by*

*One may ask whether one should already define a physical inner product with respect to the Gauss and spatial Diffeomorphism constraint and then solve the Hamiltonian constraint in a second, separate step on that already partly physical Hilbert space . While such a Hilbert space can indeed be constructed [43] it is of no use for QGR because the Hamiltonian constraint cannot leave that Hilbert space invariant as we see from the second equation in (I.2.1.1) and we must construct the physical inner product from the full solution space to all constraints. However, at least with respect to the kinematical* *constraints the full quantization programme including the question of observables has already been completed except for the analysis of the classical limit.*

### II.2.2 Kinematical Geometrical Operators

*We will restrict ourselves to the description of the area operator the classical expression of which we already wrote in (**??**) and (**??**).*

*In order to quantize* $Ar(\boldsymbol{S})$ *one starts from (**??**) and decomposes the analytical, compactly supported and oriented surface* $\boldsymbol{S}$ *or, equivalently, its preimage* $\boldsymbol{D}$ *under* $\boldsymbol{X_S}$ *into small pieces* $\boldsymbol{S_I}$*. Then the exact area functional is approximated by the Riemann sum*

# Part III

# Selected Areas of Current Research

## III.1 Quantum Dynamics

*The Hamiltonian constraint* $C$ *of QGR is, arguably, the holy grail of this approach to quantum gravity, therefore we will devote a substantial amount of space to this subject. In fact, unless one can quantize the Hamiltonian constraint, literally no further progress can be made so that it is important to know what its status is. From the explicit, non-polynomial expression (??) it is clear that a well-defined operator version of this object will be extremely hard to obtain and in fact this had been the major obstacle in the whole approach until the mid 90's. In particular, within the original ADM formulation only formal results were available. However, since with the new connection formulation also the non-polynomial kinematical operators of the previous section could be constructed, chances might be better.*

*At this point we include a brief account of the historical development of the subject in order to avoid confusion as one looks at older papers:*
*Originally one chose the Immirzi parameter as* $\beta = \pm i$ *and considered* $\tilde{C} = \sqrt{\det(q)}$ *rather than* $C$ *because then* $\tilde{C}$ *is actually a simple polynomial of only fourth order (the "More" term disappears). Polynomiality was considered as mandatory. There were three problems with this idea:*
*1) The non-polynomiality was shifted from* $C$ *into the reality conditions* $A + \bar{A} = 2\Gamma(E)$ *where the spin connection* $\Gamma$ *is now a highly non-polynomial function of* $E$*. The operator version of this equation should be very hard to implement.*
*2) If* $A$ *is complex, then we are dealing with an* $SL(2,\mathbb{C})$ *bundle rather than an* $SU(2)$ *bundle. Since* $SL(2,\mathbb{C})$ *is not compact, the mathematical apparatus of section II is blown away.*
*3) Even formal trials to quantize* $\tilde{C}$ *resulted in either divergent, or background dependent operators. In [27] it was suggested to keep* $\beta$ *real which solves problems 1) and 2), however, then* $\tilde{C}$ *becomes even more complicated and anyway problem 3) is not cured. Finally in [48] it was shown that non-polynomiality is not necessarily an obstacle, even better, it is actually required in order to arrive at a well-defined operator: It was established that the reason for problem 3) is that* $\tilde{C}$ *is a scalar density of weight two while it was shown that only density weight one scalars have a chance to be quantized rigorously and background independently. Therefore the currently accepted point of view is that* $\beta$ *should be real and that one uses the original unrescaled* $C$ *rather than* $\tilde{C}$*.*

### III.1.1 A Possible New Mechanism for Avoiding UV Singularities in Background Independent Quantum Field Theories

*Before we go into more details concerning [48], let us give a heuristic explanation just why it happens that QGR may cure UV problems of QFT, making the connection with the issue of the density weight just mentioned. Consider classical Einstein – Maxwell theory on* $M = \mathbb{R} \times \sigma$ *in its canonical formulation, then the Hamiltonian constraint gains an extra matter piece given for unit lapse* $N = 1$ *by*

*We can now quantize (??) in two ways:*
*1)*
*In the first version we notice that if* $g = \eta$ *is the Minkowski metric, that is,* $q_{ab} = \delta_{ab}$ *then (??) reduces to the ordinary Maxwell Hamiltonian on Minkowski space. Thus we apply the formalism of QFT on a background spacetime, in this case Minkowski space, because we have fixed* $q_{ab}$ *to the non-dynamical* $\mathbb{C}-$*number field* $\delta_{ab}$ *which is not quantized at all.*

2)
*In the second version we keep $q_{ab}$ dynamical and quantize it as well. Thus we apply QGR, a background independent quantization. Now $q_{ab}$ becomes a field operator $\hat{q}_{ab}$ and the statement that the metric is flat can at most have a semiclassical meaning, that is, the expectation value of $\hat{q}_{ab}$ in a gravitational state is close to $\delta_{ab}$.*
*Let us now sketch how these two different quantizations are performed and exactly pin-point how it happens that the first quantization is divergent while the second is finite.*

1) *QFT on a background spacetime*
*As we have said, the metric $q_{ab} = \delta_{ab}$ is now no longer a dynamical entitity but just becomes a complex number. What we get is the usual Maxwell Hamiltonian operator*

*These heuristic arguments can of course be made precise: (??) is quantized on the Fock space $\mathcal{H}_{Fock}$ and one obtains*

*Now something very beautiful happens, which is not put in by hand but rather is a derived result: A priori the states $T_s, T'_c$ may live on different graphs, however, unless the graphs are identical, the operator automatically (??) annihilates $T_s \otimes T'_c$ [49]. This is the mathematical manifestation of the following deep physical statement: Matter can only exist where geometry is excited. Indeed, if we have a gravitational state which has no excitations in a coordinate region $R$ then the volume of that region as measured by the volume operator is identically zero. However, if a coordinate region has zero volume, then it is physically simply not there, it is empty space. Summarizing, the operator (??) is non-trivial only if $\gamma(s) = \gamma(c)$.*
*With this being understood, let us then sketch the action of (??) on our basis. One finds heuristically*

$$\hat{H}_{EM} T_s \otimes T'_c = m_P \sum_{v \in V(\gamma)} \sum_{e,e' \in E(\gamma), e \cap e' = v} \times$$
$$\times \quad \int_\Sigma d^3x [\frac{\hat{q}_{e,e'}}{\hat{V}_v} \underbrace{\underbrace{\frac{1}{\delta(x,v)}}_{\uparrow} T_s] \otimes [\underbrace{\delta(x,v)}_{\uparrow}}_{Cancellation} \delta(y,v) Y^e Y^{e'} T'_c]_{x=y}] \qquad \text{(III.1.1.1)}$$

*where $m_P = \sqrt{\hbar/\kappa}$ is the Planck mass. Here $\hat{q}_{e,e'}$ and $Y^e$ are well-defined, dimensionfree operators (not distribution valued !) on $\mathcal{H}_{kin}$ and $\mathcal{H}'_{kin}$ respectively built from the right invariant vector fields $R^j_e, R_e$ that enter the definition of the flux operators as in (??) and its analog for $U(1)$. The product of $\delta-$distributions in the numerator of (III.1.1.1) has its origin again in the fact that the matter operator has density weight $+2$ certainly also in this representation and therefore has to be there, so nothing is swept under the rug! The $\delta-$distribution in the denominator comes from (??) and correctly accounts for the fact that the geometry operator has density weight $-1$. Again we have a coincidence limit $x = y$ which comes from a point splitting regularization and which in the background dependent quantization gave rise to the UV singularity. Now we see what happens: One of the $\delta-$distributions in the numerator gets precisely cancelled by the one in the denominator leaving us with only one $\delta-$distribution correctly accounting for the fact that the net density weight is $+1$. The integrand is* *then well-defined and the integral can be performed resulting in the finite expression*

$$\hat{H}_{EM} T_s \otimes T'_c = \sum_{v \in V(\gamma)} \sum_{e,e' \in E(\gamma), e \cap e' = v} \times$$
$$\times \quad [\frac{\hat{q}_{e,e'}}{\hat{V}_v} T_s] \otimes [Y^e Y^{e'} T'_c]_{x=y} \tag{III.1.1.2}$$

*Notice that finite here means non-perturbatively finite, that is, not only finite order by order in perturbation theory (notice that in coupling gravity we have a highly interacting theory in front of us). Thus, comparing our non-perturbative result to perturbation theory the result obtained is comparable to showing that the perturbation series converges ! Notice also that for non-compact $\sigma$ the expression (III.1.1.2) possibly has the physically correct IR divergence coming from a sum over an infinite number of vertices.*

**Exercise III.1.1.**
*Recall the Fock space quantization of the Maxwell field and verify (**??**).*

*This ends our heuristic discussion about the origin of UV finiteness in QGR. The crucial point is obviously the density weight of the operator in question which should be precisely $+1$ in order to arrive at a well-defined, background independent result: Higher density weight obviously leads to more and more divergent expressions, lower density weight ends in zero operators.*

### III.1.2 Sketch of a Possible Quantization of the Hamiltonian Constraint

*We now understand intuitively why the rescaled Hamiltonian constraint $\tilde{C}$ had no chance to be well-defined in the quantum theory: It is similar to (**??**) due to its density weight $+2$. The same factor $1/\sqrt{\det(q)}$ that was responsible for making (**??**) finite also makes the original, non-polynomial, unrescaled Hamiltonian constraint $C = \tilde{C}/\sqrt{\det(q)}$ finite. We will now proceed to some details how this is done, avoiding intermediate divergent expressions such as in (III.1.1.1).*
*The essential steps can already be explained for the first term in (**??**) so let us drop the "More" term and consider only the integrated first term*

*As we have said, the loop assignment is to a very large extent arbitrary at the level of $\mathcal{H}_{kin}$ and represents a serious quantization ambiguity, it cannot even be specified precisely because we are using the axiom of choice. However, at the level of $\mathcal{H}_{phys}$ this ambiguity evaporates to a large extent because all choices that are related by a diffeomorphism result in the same solution space to all constraints defined by elements $l \in \mathcal{D}^*_{Diff}$ which satisfy in addition*

*Let us list without proof some of the properties of this operator:*

*i) Matter Coupling*
*Similar Techniques can be applied to the case of (possibly supersymmetric) matter coupled to GR [48].*

*ii) Anomaly – Freeness*
*The constraint algebra of the Hamiltonian constraint with the spatial diffeomorphism constraint and among each other is mathematically consistent. From the*

*classical constraint algebra* $\{V, C\} \propto C$ *we expect that* $\hat{V}(\varphi)\hat{C}_E^\dagger(N)\hat{V}(\varphi)^{-1} = \hat{C}_E(\varphi^* N)$ *for all diffeomorphisms* $\varphi$*. However, this is just the statement of the loop assignment being diffeomorphism covariant which can be achieved by making use of the axiom of choice. Next, from* $\{C, C\} \propto V$ *we expect that the dual of* $[\hat{C}_E^\dagger(N), \hat{C}_E^\dagger(N')] = [\hat{C}_E(N'), \hat{C}_E(N)]^\dagger$ *annihilates the elements of* $\mathcal{D}^*_{Diff}$*. This can be explicitly verified [48]. We stress that* $[\hat{C}_E^\dagger(N), \hat{C}_E^\dagger(N')]$ *is not zero, the algebra of Hamiltonian constraints is not Abelean as it is sometimes misleadingly stated in the literature. The commutator is in fact explicitly proportional to a diffeomorphism.*

*iii) Physical States*
*There is a rich space of rigorous solutions to (??) and a precise algorithm for their construction has been developed [48].*

*iv) Intuitive Picture*
*The Hamiltonian constraint acts by annihilating and creating spin degrees of freedom and therefore the dynamical theory obtained could be called "Quantum Spin Dynamics (QSD)" in analogy to "Quantum Chromodynamics (QCD)" in which the Hamiltonian acts by creating and annihilating colour degrees of freedom. In fact we could draw a crude analogy to Fock space terminology as follows: The (perturbative) excitations of QCD carry a continuous label, the mode number* $k \in \mathbb{R}^3$ *and a discrete label, the occupation number* $n \in \mathbb{N}$ *(and others). In QSD the continuous labels are the edges* $e$ *and the discrete ones are spins* $j$ *(and others). So we have something like a non – linear Fock representation in front of us.*

*Next, when solving the Hamiltonian constraint, that is, when integrating the Quantum Einstein Equations, one realizes that one is not dealing with a (functional) partial differential equation but rather with a (functional) partial difference equation. Therefore, when understanding coordinate time as measured how for instance volumes change, we conclude that also time evolution is necessarily discrete. Such discrete time evolution steps driven by the Hamiltonian constraint assemble themselves into what nowadays is known as a spin foam. A spin foam is a four dimensional complex of two dimensional surfaces where each surface is to be thought of as the world sheet of an edge of a SNW and it carries the spin that the edge was carrying before it was evolved*[5]*.*

*Another way of saying this is that a spin foam is a complex of two-surfaces labelled by spins and when cutting a spin foam with a spatial three-surface* $\Sigma$ *one obtains a SNW. If one uses two such surfaces* $\Sigma_t, \Sigma_{t+T_P}$ *where* $T_P = \ell_P/c$ *is the Planck time then one rediscovers the discrete time evolution of the Hamiltonian constraint. These words are summarized in figure 7.*

*While these facts constitute a promising hint that the Hilbert space* $\mathcal{H}_{kin}$ *could in fact support the quantum dynamics of GR, there are well-taken concerns about the physical correctness of the operator* $\hat{C}_E^\dagger(N)$*:*

*The problem is that one would like to see more than that the commutator of two dual Hamiltonian constraints annihilates diffeomorphism invariant states, one would like to see something of the*

[5]Thus, a spin foam model can be thought of as a background independent string theory !

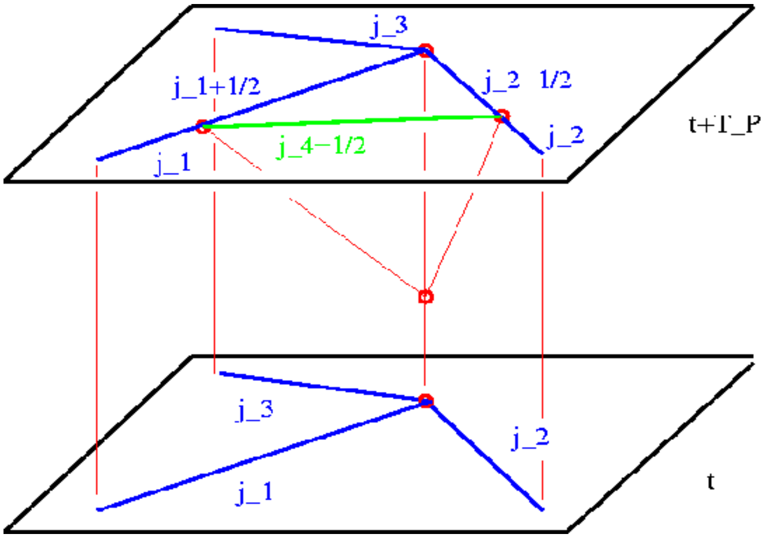


Figure 7: Emergence of a spin foam from a SNW by the action of the Hamiltonian constraint

*kind*

*Another way to get confidence in the quantization method applied to the Hamiltonian constraint is to study model systems for which the answer is known. This has been done for 2+1 gravity [48] and for quantum cosmology to which we turn in the next section.*

# III.2 Loop Quantum Cosmology

## III.2.1 A New Approach To Quantum Cosmology

*The traditional approach to quantum cosmology consists in a so-called mini – superspace quantization, that is, one imposes certain spacetime Killing symmetries on the metric, plugs the symmetric metric into the Einstein Hilbert action and obtains an effective action which depends only on a finite number of degrees of freedom. Then one canonically quantizes this action. Thus one symmetrizes before quantization. These models are of constant interest and have natural connections to inflation. See e.g. [52] for recent reviews.*

*What is not perfect about these models is that 1) not only do they switch off all but an infinite number of degrees of freedom, but 2) also the quantization method applied to the reduced model usually is quite independent from that applied to the full theory. A fundamental approach to quantum cosmology will be within the full theory and presumably involves the construction of semiclassical physical states whose probability amplitude is concentrated on, say a Friedmann – Robertson – Walker (FRW) universe. This would cure both drawbacks 1) and 2). At the moment we cannot really carry out such a programme since the construction of the full theory is not yet complete. However, one can take a more modest, hybrid approach, where while dealing only with a finite number of degrees of freedom one takes over all the quantization machinery from the full theory ! Roughly speaking, one works on the space* $\boldsymbol{\mathcal{H}_{kin}}$ *of the full theory but considers only states therein which satisfy the Killing symmetry. Hence one symmetrizes after quantization which amounts to considering only a finite subset of holonomies and fluxes. This has the advantage of leading to a solvable model*

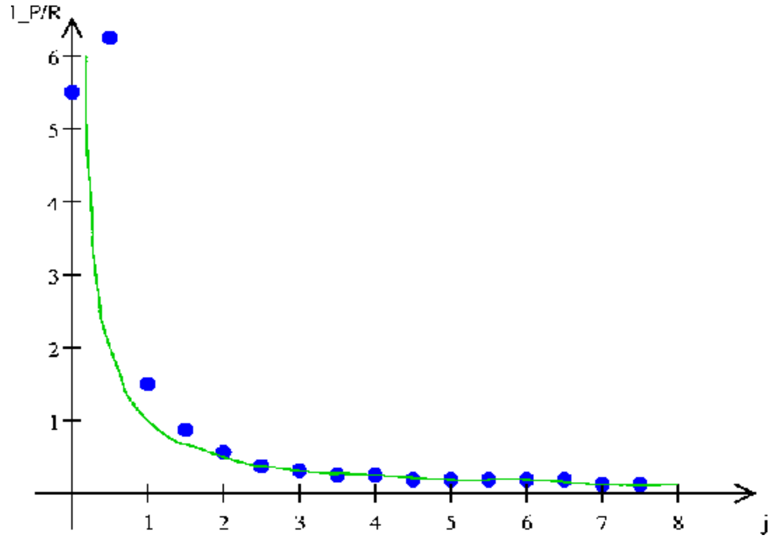


Figure 8: Spectrum of the inverse scale factor

*while preserving pivotal structures of the full theory, e.g. the volume operator applied to symmetric states will still have a discrete spectrum as in the full theory while in the traditional approaches it is continuous. Such a programme has been carried out in great detail by Bojowald in a remarkable series of papers [53] and his findings are indeed spectacular, should they extend to the full theory: It turns out that the details of the quantum theory are drastically different from the traditional minisuperspace approach. In what follows we will briefly describe some of these results, skipping many of the technical details.*

### III.2.2 Spectacular Results

*Consider the FRW line element (in suitable coordinates)*

*We are interested in whether the curvature singularity $1/R \to \infty$ exists also in the quantum theory. To study this we notice that for (??) $\det(q) = R^6 \det(q^0)$. Hence, up to a numerical factor this question is equivalent to the question whether the operator corresponding to $1/\sqrt[6]{\det(q)}$, when applied to symmetric states, is singular or not. However, we saw in the previous section that one can trade a negative power of $\det(q)$ by a Poisson bracket with the volume operator. In [53] precisely this, for the Hamiltonian constraint, essential quantization technique is applied which is why this model tests some aspects of the quantization of the Hamiltonian constraint. Now it turns out that this operator, applied to symmetric states, leads to an operator $\widehat{\frac{1}{R}}$ which is diagonalized by (symmetric) SNWF's and the spectrum is bounded ! In figure 8 we plot the qualitative behaviour of the eigenvalues $\ell_P \lambda_j$ as a function of $j$ where $j$ is the spin label of a gauge invariant SNWF with a graph consisting of one loop only (that only such states are left follows from a systematic analysis which defines what a symmetric SNWF is). One can also quantize the operator $\hat{R}$ and one sees that its eigenvalues are essentially given by $j\ell_P$ up to a numerical factor. Thus the classical singularity corresponds to $j = 0$ and one expects the points $\lambda_j \ell_p$ at the values $1/j$ on the curve $\ell_P/R$. Evidently the spectrum is discrete (pure point) and bounded, at the classical singularity it is finite. In other words, the quantum universe never decreases to zero size. For larger $j$, in fact already for $R$ of the order of ten Planck* *lengths and above, the spectrum follows the classical curve rather closely hinting at a well-behaved classical limit (correspondence principle).*

*Even more is true: One can in fact quantize the Hamiltonian constraint by the methods of the previous section and solve it exactly. One obtains an eighth order difference equation (in $j$). The solution therefore depends non-trivially on the initial condition. What is surprising, however, is the fact that only one set of initial conditions leads to the correct classical limit, thus in loop quantum cosmology initial conditions are derived rather than guessed. One can even propagate the quantum Einstein equations through the classical singularity and arrives at the picture of a bouncing universe.*

*Finally one may wonder whether these results are qualitatively affected by the operator ordering ambiguities of the Hamiltonian constraint. First of all one finds that these results hold only if one orders the loop in (??) to the left of the volume operator as written there. However, one is not forced to work with the holonomy around that loop in the fundamental representation of $SU(2)$, there is some flexibility [51] and one can choose a different one, say $j_0$. It turns out that the value $j_0$ influences the onset of classical behaviour, that is, the higher $j_0$ the higher the value $j(j_0)$ from which on the spectrum in figure 8 lies on the curve $1/j$. Now this is important when one copuples, say scalar matter because the operator $\widehat{\frac{1}{R}}$ enters the matter part of the Hamiltonian constraint and modifies the resulting effective equation for $R(t)$ in the very early phase of the universe and leads to a quantum gravity driven inflationary period whose duration gets larger with larger $j_0$ !*

*Thus, loop quantum cosmology not only confirms aspects of the quantization of the Hamiltonian constraint but also predicts astonishing deviations from standard quantum cosmology which one should rederive in the full theory.*

## III.3 Path Integral Formulation: Spin Foam Models

### III.3.1 Spin Foams from the Canonical Theory

*Spin Foam models are the fusion of ideas from topological quantum field theories and loop quantum gravity, see e.g. [54] for a review, especially the latest, most updated one by Perez. The idea that connects these theories is actually quite simple to explain at an heuristic level:*

*If we forget about 1) all functional analytic details, 2) the fact that the operator valued distributions corresponding to the Hamiltonian constraint $\hat{C}(x)$ do not mutually commute for different $x \in \sigma$ and 3) that the Hamiltonian constraint operators $\hat{C}(N)$ are certainly not self-adjoint, at least as presently formulated, then we can formally write down the complete space of solutions to the Hamiltonian constraint as a so-called "rigging map" (see e.g. [1])*

*Now by the usual formal manipulations that allow us to express a unitary operator $e^{i(t_2-t_1)\hat{H}}$ as a path integral over the classical pase space $\mathcal{M}$ (the rigorous version of which is the Feynman – Kac formula, e.g. [56]) one can rewrite (??) as*

### III.3.2 Spin Foams and BF – Theory

*Thus, it is formally possible to write the inner product between physical states as a covariant path integral for the classical canonical action and using only the kinematical inner product, thus providing* *a bridge between the covariant and canonical formalism. However, this bridge is far from being rigorously established as we had to perform many formal, unjustified manipulations. Now rather than justifying the steps that lead from $\hat{C}$ to (??) one can turn the logic upside down and start from a manifestly covariant formulation and derive the canonical formultion. This is the attitude that one takes among people working actively on spin foam models. Thus, let us forget about the topological term in (??) and consider only the Palatini term. Then the Palatini action has precisely the form of a BF – action*

*Let us now discuss how one formulates the path integral corresponding to the action (??). It is formally given by*

*Now what one does is a certain jump, whose physical implication is still not understood: Instead of performing perturbation theory in $S_I$ one argues that formally integrating over $\Phi$ and thus imposing the simplicity constraint is equivalent to the restriction of the direct integral that enters the $\delta-$distributions to simple representations, that is, representations for which either $n = 0$ or $\rho = 0$. In other words, one says that the triangulated Palatini path integral is the same as the triangulated BF path integral restricted to simple representations. To motivate this argument, one notices that upon canonical quantization of BF theory on a triangulated manifold the $B$ field is the momentum conjugate to $\omega$ and if one quantizes on a Hilbert space based on $sl(2,\mathbb{C})$ connections using the Haar measure (similar as we have done for $SU(2)$ for a fixed graph), its corresponding flux operator $\hat{B}_{IJ}(S)$ becomes a linear combination of right invariant vector fields $R^{IJ}$ on $SL(2,\mathbb{C})$. Now the simplicity constraint becomes the condition that the second Casimir operator $R^{IJ}R^{KL}\epsilon_{IJKL}$ vanishes. However, on irreducible representations this operator is diagonal with eigenvalues $n\rho/4$. While this is a strong motivation, it is certainly not sufficient justification for this way of implementing the simplicity constraint in the path integral because it is not clear how this is related to integrating over $\Phi$.*
*In any case, if one does this then one arrives at (some version of) the Lorentzian Barrett – Crane model [59]. Surprisingly, for a large class of triangulations $\tau$ the amplitudes*

*There is still an open issue, namely how one should get rid of the regulator (or triangulation) dependence. Since BF – theory is a topological QFT, the amplitudes are automatically triangulation independent, however, this is certainly not the case with GR. One possibility is to sum over triangulations and a concrete proposal of how to weigh the contributions from different triangulations comes from the so-called field theory formulation of the theory [61]. Here one reformulates the BF – theory path integral as the path integral for a scalar field on a group manifold which in this case is a certain power of $SL(2,\mathbb{C})$. The action for that scalar field has a free piece and an interaction piece and performing the perturbation theory (Feynman graphs !) for that field theory is equivalent to the sum over BF – theory amplitudes for all triangulated manifolds with precisely defined weights. This idea can be straightforwardly applied also to our context where the restriction to simple representations is realized by imposing corresponding restrictions (projections) on the scalar field.*

*Summarizing, spin foam models are a serious attempt to arrive at a covariant formulation of QGR but many issues are still unsettled, e.g.:*
*1)*

*There is no clean equivalence with the Hamiltonian formulation as we have seen. Without that it is unlcear how to interpret the spin foam model amplitude and whether it has the correct classical limit. In order to make progress on the issue of the classical limit, model independent techniques for constructing "causal spin foams" [62] with a built in notion of quantum causality and renormalization methods [63], which should allow in principle the derivation of a low energy effective action, have been developed.*
*2)*
*The physical correctness of the Barrett – Crane model is unclear. This is emphasized by recent results within the Euclidean formulation [64] which suggest that the classical limit is far off GR since the amplitudes are dominated by spin values close to zero. This was to be expected because in the definition of the Barrett – Crane model there is a certain flexibility concerning the choice of the measure that replaces* $[D\omega\ DB]$ *at the triangulated level and the result [64] indicates that one must gain more control on that choice.*
*3)*
*It is not even clear that these models are four – dimensionally covariant: One usually defines that the amplitudes for a fixed triangulation are the same for any four - diffeomorphic triangulation. However, recent results [65] show that this natural definition could result nevertheless in anomalies. This problem is again related to the choice of the measure just mentioned.*

*Thus, substantially more work is required in order to fill in the present gaps but the results already obtained are very promising indeed.*

# III.4 Quantum Black Holes

## III.4.1 Isolated Horizons

*Any theory of quantum gravity must face the question whether it can reproduce the celebrated result due to Bekenstein and Hawking [66] that a black hole in a spacetime* $(M, g)$ *should account for a quantum statisical entropy given by*

*In [67] the authors performed a bold computation: For any surface* $S$ *and any positive number* $A_0$ *they asked the question how many SNW states there are in QGR such that the area operators eigenvalues lie within the interval* $[A_0 - \ell_P^2, A_0 + \ell_P^2]$*. This answer is certainly infinite because a SNW can intersect* $S$ *in an uncountably infinite number of different positions without changing the eigenvalues. This divergence can be made less severe by moding out by spatial diffeomorphisms which we can use to map these different SNW onto each other in the vicinity of the surface. However, since there are still an infinite number of non-spatially difeomorphic states which look the same in the vicinity of the surface but different away from it, the answer is still divergent. Therefore, one has to argue that one must not count information off the surface, maybe invoking the Hamiltonian constraint or using the information that* $S = H$ *is not an arbitrary surface but actually the horizon of a black hole. Given this assumption, the result of the, actually quite simple counting problem came rather close to (**??**) with the correct factor of* $1/4$*.*

*Thus the task left is to justify the assumptions made and to make the entropy counting water – tight by invoking the information that* $H$ *is a black hole horizon. The outcome of this analysis created a whole industry of its own, known under the name "isolated horizons", which to large part is a beautiful*

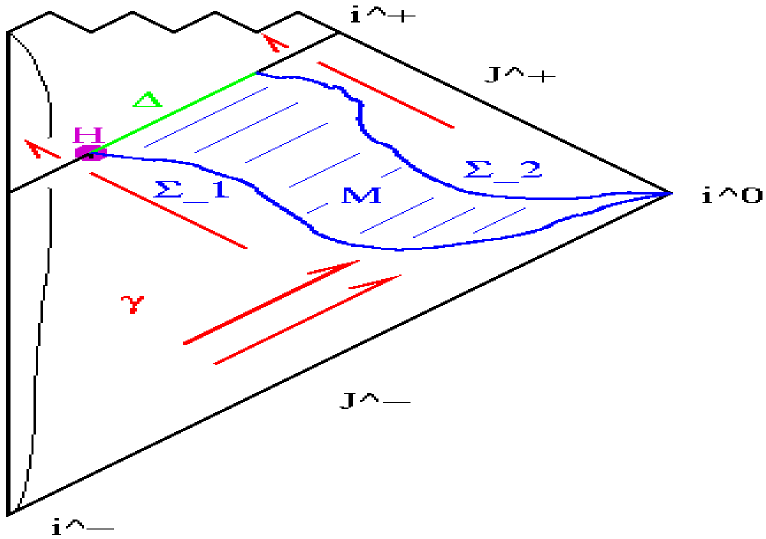


Figure 9: An isolated horizon $\boldsymbol{\Delta}$ boundary of a piece $\boldsymbol{M}$ (shaded) of spacetime also bounded by spacelike hypersurfaces $\boldsymbol{\Sigma_1}, \boldsymbol{\Sigma_2}$. Radiation $\boldsymbol{\gamma}$ may enter or leave $\boldsymbol{M}$ and propagate into the singularity before or after the isolated horizon has formed but must not cross $\boldsymbol{\Delta}$. An intersection of a spacelike hypersurface $\boldsymbol{\Sigma}$ with $\boldsymbol{\Delta}$ is denoted by $\boldsymbol{H}$ which has the topology of a sphere.

*new chapter in classical GR. In what follows we will focus only on a tiny fraction of the framework, mostly concentrating on the ingredients essential for the quantum formulation. For reviews see [68] which also contain a complete list of references on the more classical aspects of this programme, the pivotal papers concerning the quantum applications are [69].*

*By definition, an event horizon is the external boundary of the part of $\boldsymbol{M}$ that does not lie in the past of null future infinity $\boldsymbol{J}^+$ in a Penrose diagramme. From an operational point of view, this definition makes little sense because in order to determine whether a candidate is an event horizon, one must know the whole spacetime $(\boldsymbol{M}, \boldsymbol{g})$ which is never possible by measurements which are necessarily loacal in spacetime (what looks like an eternal black hole now could capture some dust later and the horizon would change its location). Thus one looks for some local substitute of the notion of an event horizon which captures the idea that the black hole has come to some equilibrium state at least for some amount of time. This is roughly what an isolated horizon $\boldsymbol{\Delta}$ is, illustrated in figure 9. More in technical details we have the following.*

**Definition III.4.1.**
*A part $\boldsymbol{\Delta}$ of the boundary $\boldsymbol{\partial M}$ of a spacetime $(\boldsymbol{M}, \boldsymbol{g})$ is called an isolated horizon, provided that*
*1) $\boldsymbol{\Delta} \equiv \mathbb{R} \times \boldsymbol{S^2}$ is a null hypersurface and has zero shear and expansion[6].*
*2) The field equations and matter energy conditions hold at $\boldsymbol{\Delta}$.*
*3) $\boldsymbol{g}$ is Lie derived by the null generator $\boldsymbol{l}$ of $\boldsymbol{\Delta}$ at $\boldsymbol{\Delta}$.*

*The canonical formulation of a field theory on a manifold $\boldsymbol{M}$ with boundary $\boldsymbol{\Delta}$ must involve boundary conditions at $\boldsymbol{\Delta}$ in order that the variation principle be well-defined (the action must be functionally differentiable). Such boundary conditions usually give birth to boundary degrees of freedom [70] which would normally be absent but now come into being because (part of the) gauge transformations are forced to become trivial at $\boldsymbol{\Delta}$. In the present situation what happens is that the boundary term is*

[6] Recall the notions of shear, expansion and twist of a congruence of vector fields in connection with Raychaudhuri's equation.

*actually a $U(1)$ Chern-Simons action*[7]

### III.4.2 Entropy Counting

*One now has to quantize the system. This consists of several steps whose details are complicated and which we will only sketch in what follows.*

*i) Kinematical Hilbert Space*
*The bulk and boundary degrees of freedom are independent of each other, therefore we choose $\mathcal{H}_{kin} = \mathcal{H}^{\Sigma}_{kin} \otimes \mathcal{H}^{H}_{kin}$ where both spaces are of the form $L_2(\overline{\mathcal{A}}, d\mu_0)$ just that the first factor is for an $SU(2)$ bundle over $\Sigma$ while the second is for an $U(1)$ bundle over $H$.*

*ii) Quantum Boundary conditions*
*Equation (??) implies, in particular, that in quantum theory we must have schematically*

*Consider now SNWF's $T_s$ of the bulk theory and those of the boundary theory $T'_c$. Then $[\widehat{X^*_H(*E)_j}]r_j]$ acts on $T_s$ like the $z-$component of the angular momentum operator and will have distributional eigenvalues proportional to the magnetic quantum numbers $m_e$ of the edges with punctures $p = e \cap H$ and spin $j_e$ where $m_e \in \{-j_e, -j_e + 1, .., j_e\}$.*

*iii) Implementation of Quantum Dynamics at $\Delta$*
*It turns out that $\widehat{X^*_H(*E)_j}]r_j$ and $\widehat{dW}$ are the generators of residual $SU(2)$ gauge transformations close to $X_H(H)$ and of $U(1)$ on $H$ respectively. Now these residual $SU(2)$ transformations are frozen to $U(1)$ transformations by $r_j$ and the most general situation in order for a state to be gauge invariant is that these residual $SU(2)$ transformations of the bulk theory and the $U(1)$ transformations of the boundary theory precisely cancel each other. It turns out that this cancellation condition is precisely given by the quantum boundary condition (??). Thus the states that solve the Gauss constraint are linear combinations of states of the form $T_s \otimes T'_c$ where the boundary data of these states are punctures $p \in \mathcal{P}$ where $p \in \gamma(s) \cap H$, the spins $j_p = j_{e_p}$ of edges $e \in E(\gamma(s))$ with $e_p \cap H = p$ and their magnetic quantum numbers $m_p = m_{e_p}$. However, due to the specific features of the geometrical quantization of Chern-Simons theories [72] the $m_p$ cannot be specified freely, they have to satisfy the constraint*

*Next, the spatial diffeomorphism constraint of the bulk theory tells us that the position of the punctures on $H$ are not important, important is only their number.*

*Finally, one of the boundary conditions at $\Delta$ implies that the lapse becomes trivial $N = 0$ at $H$ if $\hat{C}(N)$ is to generate an infinitesimal time reparameterization*[8]*. Thus, luckily we can escape the open issues with the Hamiltonian constraint as far as the quantum dynamics at $H$ is concerned.*

[7] It was observed first in [71] that general relativity in terms of connection variables and in the presence of boundaries leads to Chern − Simons boundary terms.

[8] This does not mean that the lapse of a classical isolated horizon solution must vanish at $S$, rather there is a subtle difference between gauge motions and symmetries for field theories with boundaries [70] where in this case symmetries map solutions to gauge inequivalent or equivalent ones respectively, if $N_{|H} \neq 0$ or $N_{|H} = 0$ respectively.

*We can now come to the issue of entropy counting. First of all we notice that* $Ar(H)$ *is a Dirac observable because* $H$ *is invariant under* $Diff(H)$ *and* $N = 0$ *at* $H$*. Given* $n$ *punctures with spins* $j_l,\ l = 1,..,N$ *the area eigenvalue for* $H$ *is*

## III.5 Semiclassical Analysis

### III.5.1 The Complexifier Machinery for Generating Coherent States

*Let us first specify what we mean by semiclassical states.*

**Definition III.5.1.**
*Let be given a phase space* $\mathcal{M},\ \{.,.\}$ *with preferred Poisson subalgebra* $\mathcal{O}$ *of* $C^\infty(\mathcal{M})$ *and a Hilbert space* $\mathcal{H},\ [.,.]$ *together with an operator subalgebra* $\hat{\mathcal{O}}$ *of* $\mathcal{L}(\mathcal{H})$*. The triple* $\mathcal{M},\ \{.,.\}, \mathcal{O}$ *is said to be a classical limit of the triple* $\mathcal{H},\ [.,.], \hat{\mathcal{O}}$ *provided that there exists an (over)complete set of states* $\{\psi_m\}_{m\in\mathcal{M}}$ *such that for all* $O, O' \in \mathcal{O}$ *the infinitesimal Ehrenfest property*

$$
\begin{aligned}
&|\frac{<\hat{O}>_m}{O(m)} - 1| \ll 1 \\
&|\frac{<[\hat{O},\hat{O}']>_m}{i\hbar\{O,O'\}(m)} - 1| \ll 1
\end{aligned}
\tag{III.5.1.1}
$$

*and the small fluctuation property*

*The key question then is how to construct semiclassical states. Fortunately, for phase spaces which have a cotangent bundle structure as is the case with QGR, a rather general construction guideline is available [74], the so-called Complexifier Method, which we will now sketch:*

*Let* $(\mathcal{M}, \{.,.\})$ *be a phase space with (strong) symplectic structure* $\{.,.\}$ *(notice that* $\mathcal{M}$ *is allowed to be infinite dimensional). We will assume that* $\mathcal{M} = T^*\mathcal{C}$ *is a cotangent bundle. Let us then choose a real polarization of* $\mathcal{M}$*, that is, a real Lagrangean submanifold* $\mathcal{C}$ *which will play the role of our configuration space. Then a loose definition of a complexifier is as follows:*

**Definition III.5.2.** *A complexifier is a positive definite function* [9] $C$ *on* $\mathcal{M}$ *with the dimension of an action, which is smooth a.e. (with respect to the Liouville measure induced from* $\{.,.\}$*) and whose Hamiltonian vector field is everywhere non-vanishing on* $\mathcal{C}$*. Moreover, for each point* $q \in \mathcal{C}$ *the function* $p \mapsto C_q(p) = C(q,p)$ *grows stronger than linearly with* $||p||_q$ *where* $p$ *is a local momentum coordinate and* $||.||_q$ *is a suitable norm on* $T^*_q(\mathcal{C})$*.*

*In the course of our discussion we will motivate all of these requirements.*

*The reason for the name complexifier is that* $C$ *enables us to generate a complex polarization of* $\mathcal{M}$ *from* $\mathcal{C}$ *as follows: If we denote by* $q$ *local coordinates of* $\mathcal{C}$ *(we do not display any discrete or continuous labels but we assume that local fields have been properly smeared with test functions) then*

[9]For the rest of this section $C$ will denote a complexifier function and not the Hamiltonian constraint.

*The importance of this observation is that either of $z, \bar{z}$ are coordinates of a Lagrangean submanifold of the complexification $\mathcal{M}^{\mathbb{C}}$, i.e. a complex polarization and thus may serve to define a Bargmann-Segal representation of the quantum theory (wave functions are holomorphic functions of $z$). The diffeomorphism $\mathcal{M} \to \mathcal{C}^{\mathbb{C}};\ m \mapsto z(m)$ shows that we may think of $\mathcal{M}$ either as a symplectic manifold or as a complex manifold (complexification of the configuration space). Indeed, the polarization is usually a positive Kähler polarization with respect to the natural $\{.,.\}$-compatible complex structure on a cotangent bundle defined by local Darboux coordinates, if we choose the complexifier to be a function of $p$ only. These facts make the associated Segal-Bargmann representation especially attractive.*

*We now apply the rules of canonical quantization: a suitable Poisson algebra $\mathcal{O}$ of functions $O$ on $\mathcal{M}$ is promoted to an algebra $\hat{\mathcal{O}}$ of operators $\hat{O}$ on a Hilbert space $\mathcal{H}$ subject to the condition that Poisson brackets turn into commutators divided by $i\hbar$ and that reality conditions are reflected as adjointness relations, that is,*

*The fact that $C$ is positive motivates to quantize it in such a way that it becomes a self-adjoint, positive definite operator. We will assume this to be the case in what follows. Applying then the quantization rules to the functions $z$ in (??) we arrive at*

*Let now $q \mapsto \delta_{q'}(q)$ be the $\delta$-distribution with respect to $\mu$ with support at $q = q'$. (More in mathematical terms, consider the complex probability measure, denoted as $\delta_{q'}d\mu$, which is defined by $\int \delta_{q'}d\mu f = f(q')$ for measurable $f$). Notice that since $C$ is non-negative and necessarily depends non-trivially on momenta (which will turn into (functional) derivative operators in the quantum theory), the operator $e^{-\hat{C}/\hbar}$ is a smoothening operator. Therefore, although $\delta_{q'}$ is certainly not square integrable, the complex measure (which is probability if $\hat{C} \cdot 1 = 0$)*

*We will see in a moment that (??) qualifies as a candidate coherent state if we are able to analytically extend (??) to complex values $z$ of $q'$ where the label $z$ in $\psi_z$ will play the role of the point in $\mathcal{M}$ at which the coherent state is peaked. In order that this is possible (and in order that the extended function is still square integrable), (??) should be entire analytic. Now $\delta_{q'}(q)$ roughly has an integral kernel of the form $e^{i(k,(q-q'))}$ (with some pairing $< .,. >$ between tangential and cotangential vectors) which is analytic in $q'$ but the integral over $k$, after applying $e^{-\hat{C}/\hbar}$, will produce an entire analytic function only if there is a damping factor which decreases faster than exponentially. This provides the intuitive explanation for the growth requirement in definition III.5.2. Notice that the $\psi_z$ are not necessarily normalized.*
*Let us then assume that*

*Next, let us verify that $\psi_m$ indeed has a chance to be peaked at $m$. To see this, let us consider the self-adjoint (modulo domain questions) combinations*

*Now we compute by similar methods that*

*The infinitesimal Ehrenfest property*

*For the same reason one expects that the peakedness property*

*Finally one wants coherent states to be overcomplete in order that every state in $\mathcal{H}$ can be expanded in terms of them. This has to be checked on a case by case analysis but the fact that our complexifier coherent states are for real $z$ nothing else than regularized $\delta$ distributions which in turn provide a (generalized) basis makes this property plausible to hold.*

**Exercise III.5.1.**
*Consider the phase space: $\mathcal{M} = T^*\mathbb{R} = \mathbb{R}^2$ with standard Poisson brackets $\{q,q\} = \{p,p\} = 0,\ \{p,q\} = 1$ and configuration space $\mathcal{C} = \mathbb{R}$. Consider the complexifier $C = p^2/(2\sigma)$ where $\sigma$ is a dimensionful constant such that $C/\hbar$ is dimensionfree. Check that it meets all the requirements of definition III.5.2 and perform the coherent state construction displayed above.*
*Hint:*
*Up to a phase, the resulting, normalized coherent states are the usual ones for the harmonic oscillator with Hamiltonian $H = (p^2/m + m\omega^2 q^2)/2$ with $\sigma = m\omega$. Verify that the states $\psi_m$ are Gaussian peaked in the configuration representation with width $\sqrt{\hbar/\sigma}$ around $q = q_0$ and in the momentum representation around $p = p_0$ with width $\sqrt{\hbar\sigma}$ where $m = (p_0, q_0)$.*

*As it has become clear from the discussion, the complexifier method gives a rough guideline, but no algorithm, in order to arrive at a satisfactory family of coherent states, there are things to be checked on a case by case basis. On the other hand, what is nice is that given only one input, namely the complexifier $C$, it is possible to arrive at a definite and constructive framework for a semiclassical analysis. It is important to know what the classical limit of $\hat{C}$ is, otherwise, if we have just an abstract operator $\hat{C}$ then the map $m \mapsto z(m)$ is unknown and an interprtation of the states in terms of $\mathcal{M}$ is lost.*

### III.5.2 Application to QGR

*Let us now apply these ideas to QGR. Usually the choice of $C$ is stronly motivated by a Hamiltonian, but in QGR we have none. Therefore, at the moment the best we can do is to play with various proposals for $\hat{C}$ and to explore the properties of the resulting states. For the simplest choice of $\hat{C}$ [75] those properties have been worked out more or less completely and we will briefly describe them below.*
*The operator $\hat{C}$ is defined by its action on cylindrical functions $f = p_\gamma^* f_\gamma$ by*

*The $\delta-$distribution with respect to the measure $\mu_0$ can be written as the sum over all SNW's (exercise !)*

*In order to study the semiclassical properties of these states we consider their cut-offs $\psi_{A^{\mathbb{C}},\gamma}$ for each graph $\gamma$ defined on cylindrical functions $f = p_\gamma^* f_\gamma$ by*

*In the following graphic we display as an example the peakedness properties of the analog of (??) for the simpler case of the gauge group $U(1)$, the case of $SU(2)$ is similar but requires more plots because of the higher dimensionality of $SU(2)$. Thus $g_0 = e^p h_0 \in U(1)^{\mathbb{C}} = \mathbb{C} - \{0\},\ p \in \mathbb{R},\ h_0 \in U(1)$ and $u \in U(1)$ where we parameterize $u = e^{i\phi},\ \phi \in [-\pi, \pi)$. Similarly, $g = e^{p_1} u,\ p_1 \in \mathbb{R}$. We consider in figure ?? the peakedness in the configuration representation given by the probability amplitude*

## III.6 Gravitons

### III.6.1 The Isomorphism

*The reader with a strong background in ordinary QFT and/or string theory will have wondered throughout these lectures where in QGR the graviton, which plays such a prominent role in the perturbative, background dependent approaches to quantum gravity, resides. In fact, if one understands the graviton, as usually, as an excitation of the quantum metric around Minkowski space, then there is a clear connection with the semiclassical analysis of the previous section: One should construct a suitable coherent state which is peaked on the gauge invariant phase space point characterizing Minkowski space and identify suitable excitations thereof as gravitons. It is clear that at the moment such graviton states from full QGR cannot be constructed, because we would need first to solve the Hamiltonian constraint.*

*However, one can arrive at an approximate notion of gravitons through the quantization of linearized gravity: Linearized gravity is nothing else than the expansion of the full GR action around the gauge variant initial data $(E^0)^a_j = \delta^a_j,\ (A^0)^j_a = 0$ to second order in $E - E^0, A$ which results in a free, classical field theory with constraints. In fact, the usual notion of gravitons is precisely the ordinary Fock space quantization of that classical, free field theory [79]. In order to see whether QGR can possibly accomodate these graviton states, Varadarajan in a beautiful series of papers [80] has carried out a polymer like quantization of that free field theory on a Hilbert space $\mathcal{H}_{kin}$ which is in complete analogy to that for full QGR, the only difference being that the gauge group $SU(2)$ is replaced by the gauge group $U(1)^3$. While there are certainly large differences between the highly interacting QGR theory and linearized gravity, one should at least be able to gain some insight into the the answer to the question, how a Hilbert space in which the excitations are one dimensional can possibly describe the Fock space excitations (which are three dimensional).*

*The problem of describing gravitons within linearized gravity by polymer like excitations is mathematically equivalent to the simpler problem of describing the photons of the ordinary Fock Hilbert space $\mathcal{H}_F$ of Maxwell theory by polymer like excitations within a Hilbert space $\mathcal{H}_P = L_2(\overline{\mathcal{A}}, d\mu)$ where $\overline{\mathcal{A}}$ is again a space of generalized $U(1)$ connections with some measure $\mu$ thereon. Thus, we decribe the latter problem in some detail since it requires less space and has the same educational value.*

*The crucial observation is the following isomorphism $\mathcal{I}$ between two different Poisson subalgebras of the Poisson algebra on the phase space $\mathcal{M}$ of Maxwell theory coordinatized by a canonical pair*

*$(E, A)$ defined by a $U(1)$ connection $A$ and a conjugate electric field $E$: Consider a one-parameter family of test functions of rapid decrease which are regularizations of the $\delta-$distribution, for instance*

## III.6.2 Induced Fock Representation With Polymer – Excitations

*Now we know that the unsmeared holonmy algebra is well represented on the Hilbert space $\mathcal{H}_{kin} = L_2(\overline{\mathcal{A}}, d\mu_0)$ while the smeared holonomy algebra is well represented on the Fock Hilbert space $\mathcal{H}_F = L_2(\mathcal{S}', d\mu_F)$ where $\mathcal{S}'$ denotes the space of divergence free, tempered distributions and $\mu_F$ is the Maxwell-Fock measure. These measures are completely characterized by their generating functional*

*Since $\omega_F$ defines a positive linear functional we may define a new representation of the algebra $A(p), E^a_r$ by*

*Since $\omega_r$ is a positive linear functional on $C(\overline{\mathcal{A}})$ by construction there exists is a measure $\mu_r$ on $\overline{\mathcal{A}}$ that represents $\omega_r$ in the sense of the Riesz representation theorem (recall ??). In [81] Velhinho showed that the one – parameter family of measures $\mu_r$ are expectedly mutually singular with respect to each other and with respect to the uniform measure $\mu_0$ (that is, the support of one measure is a measure zero set with respect to the other and vice versa).*

*Result 1: There is a unitary transformation between any of the Hilbert spaces $\mathcal{H}_r$ and their images under $\mathcal{I}_r$ in the usual Fock space $\mathcal{H}_F$. Since finite linear combinations of the $A_r(p)$ for fixed $r$ are still dense in $\mathcal{H}_F$ [80], there exists indeed a polymer like description of the usual $n$-photon states.*

*Recall that the Fock vacuum $\Omega_F$ is defined to be the zero eigenvalue coherent state, that is, it is annihilated by the annihilation operators*

# Part IV

# Selection of Open Research Problems

*Let us summarize the most important open research problems that have come up during the discussion in these lectures.*

i) *Hamiltonian Constraint and Semiclassical States*
*The unsettled correctness of the quantum dynamics is the major roadblock to completing the quantization programme of QGR. In order to make progress a better understanding of the kinematical semiclassical sector of the theory is necessary.*

ii) *Physical Inner Product*
*Even if we had the correct Hamiltonian constraint and the complete space of solutions, at the moment there is no really good idea available of how to construct a corresponding physical inner product because the constraint algebra is not a Lie algebra but an open algebra in the BRST sense so that techniques from rigged Hilbert spaces are not available. A framework for such open algebras must be developed so that an inner product can be constructed at least in principle.*

iii) *Dirac Observables*
*Not even in classical general relativity do we know enough Dirac observables. For QGR they are mandatory for instance in order to select an inner product by adjointness conditions and in order to arrive at an interpretation of the final theory. A framework of how to define Dirac observables, at least in principle, even at the classical level, would be an extremely important contribution.*

iv) *Covariant Formulation*
*The connection between the Hamiltonian and the Spin Foam formulation is poorly understood. Without such a connection e.g. a proof of covariance of the canonical formulation on the one hand and a proof for the correct classical limit of the spin foam formulation on the other cannot be obtained using the respective other formulation. One should prove a rigorous Feynman – Kac like formula that allows to switch between these complementary descriptions.*

v) *QFT on CST's and Hawking Effect from First Principles*
*The low energy limit of the theory in connection with the the construction of semiclassical states must be better understood. Once this is done, fundamental issues such as whether the Hawking effect is merely an artefact of an invalid description by QFT's on CST's while a quantum theory of gravity should be used or whether it is a robust result can be answered. Similar remarks apply to the information paradoxon associated with black holes etc.*

vi) *Combinatorial Formulation of the Theory*
*The description of a theory in terms of smooth and even analytic structures curves, surfaces etc. at all scales in which the spectra of geometrical operators are discrete at Planck scales is awkward and cannot be the most adequate language. There should be a purely combinatorical formulation in which notions such as topology, differential structure etc. can only have a semiclassical meaning.*

vii) *Avoidance of Classical and UV Singularities*
*That certain classical singularities are absent in loop quantum cosmology and that certain operators come out finite in the full theory while in the usual perturbative formulation they would suffer from UV singularities are promising results, but they must be better understood. If one could make contact with perturbative formulations and pin – point exactly why in QGR the usual perturbative UV singularities are absent then the theory would gain a lot more respect in other communities of high energy physicists. There must be some analog of the renormalization group and the running of coupling constants that one usually finds in QFT's and CST's. Similar remarks apply to the generalization of the loop quantum cosmology result to the full theory.*

viii) *Contact with String (M) – Theory*
*If there is any valid perturbative description of quantum gravity then it is almost certainly string the-*

*ory. It is conceivable that both string theory and loop quantum gravity are complementary descriptions but by themselves incomplete and that only a fusion of both can reach the status of a fundamental theory. To explore these possibilities, Smolin has launched an ambitious programme [82] which to our mind so far did not raise the interest that it deserves*[10]*. The contact arises through Chern – Simons theory which is part of both Loop Quantum Gravity and M – Theory [83] (when considered as the high energy limit of 11 dimensional Supergravity). Another obvious starting point is the definition of M – Theory as the quantum supermembrane in 11 dimensions [84], a theory that could be obtained as the quantization of the classical supermembrane by our non-perturbative methods. Finally, a maybe even more obvious connection could be found through the so-called* **Pohlmeyer String** *[85] which appears to be a method to quantize the string non-perturbatively, without supersymmetry, anomalies or extra dimensions, by working directly at the level of Dirac observables which are indeed possible to construct explicitly in this case.*

*We hope to have convinced the reader that Loop Quantum Gravity is an active and lively approach to a quantum theory of gravity which has produced already many non-trivial results and will continue to do so in the future. There are still a huge number of hard but fascinating problems to be solved of which the above list is at most the tip of an iceberg. If at least a tiny fraction of the readers would decide to dive into this challenging area and help in this endeavour, then these lectures would have been successful.*

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[10] That we did not devote a section to this topic in this review is due to the fact that we would need to include an introduction to M – Theory into these lectures which would require too much space. The interested reader is referred to the literature cited.

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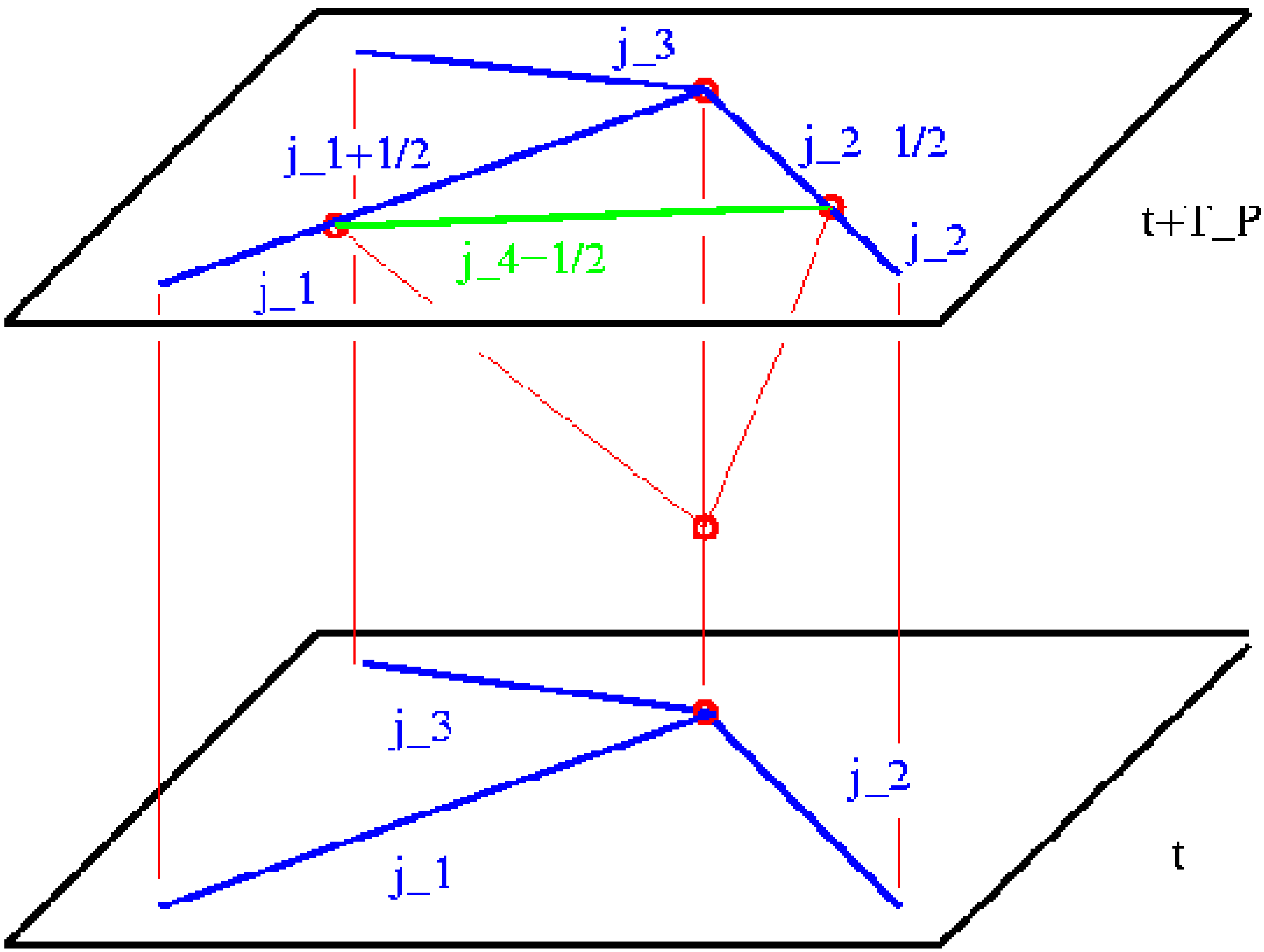

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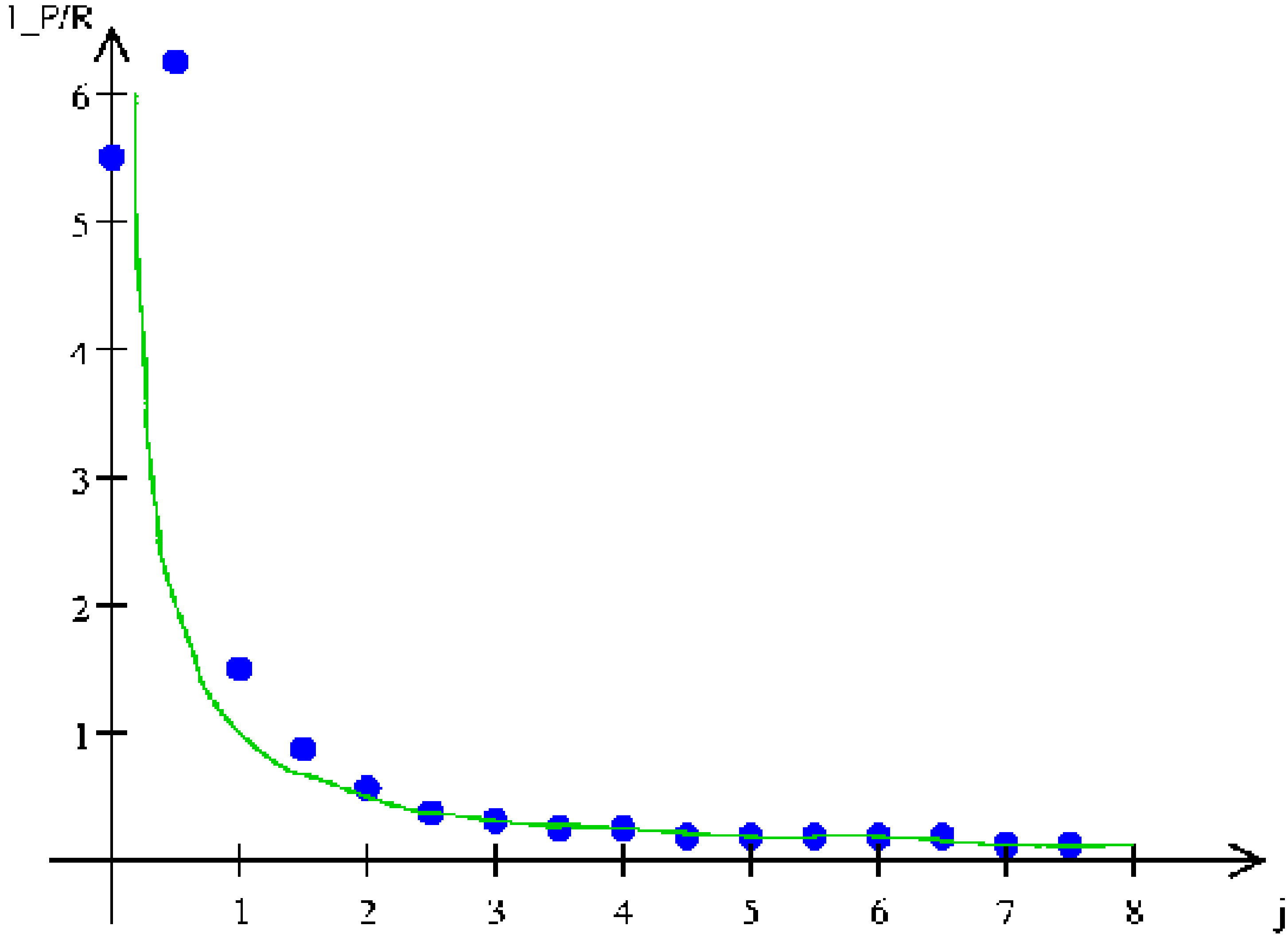

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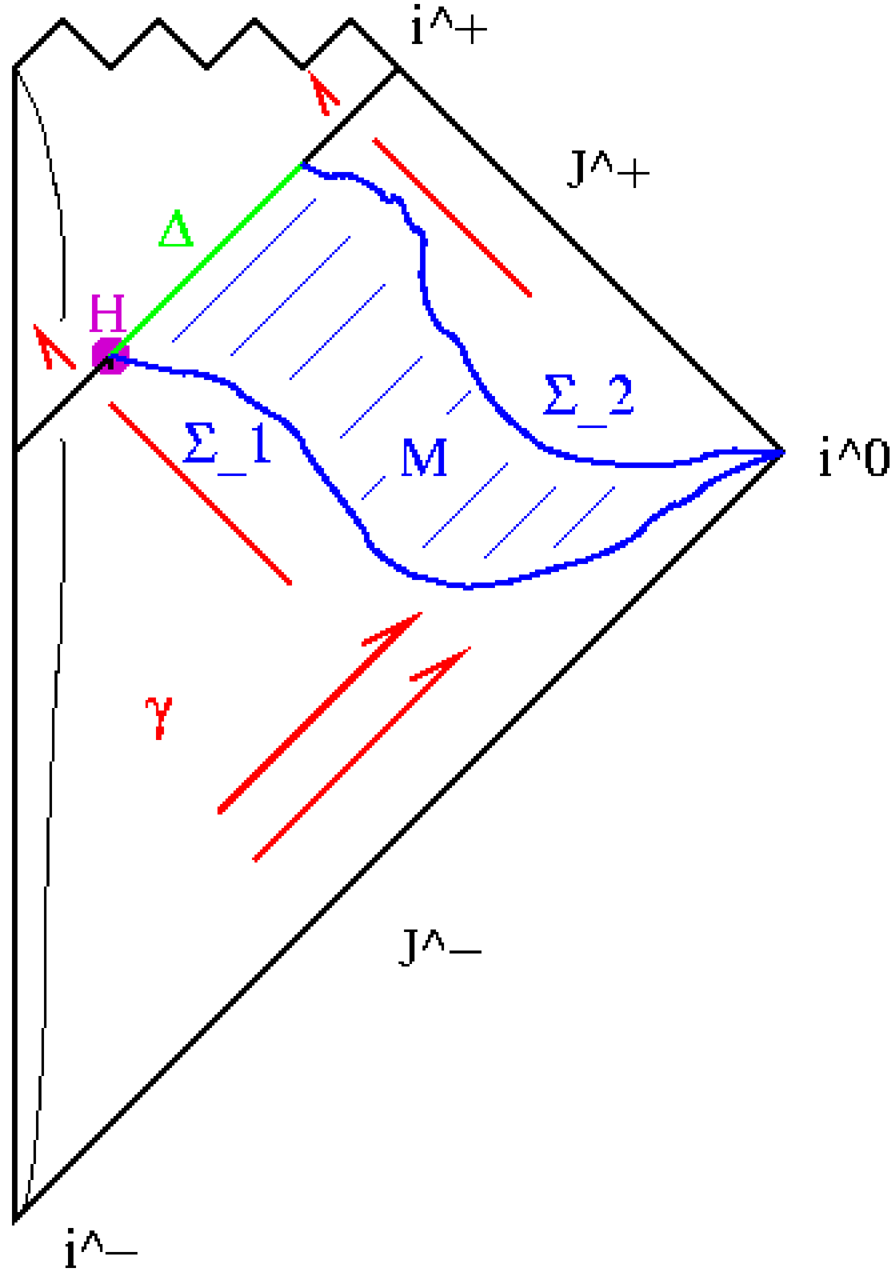

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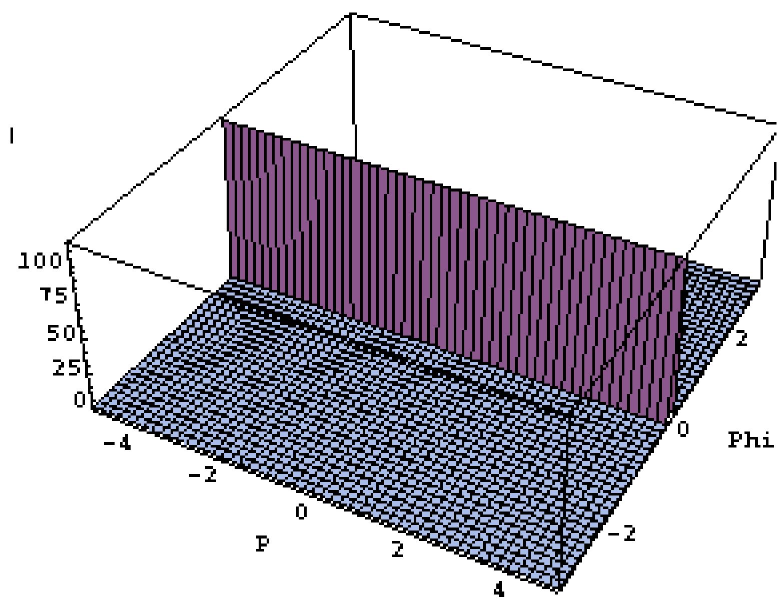
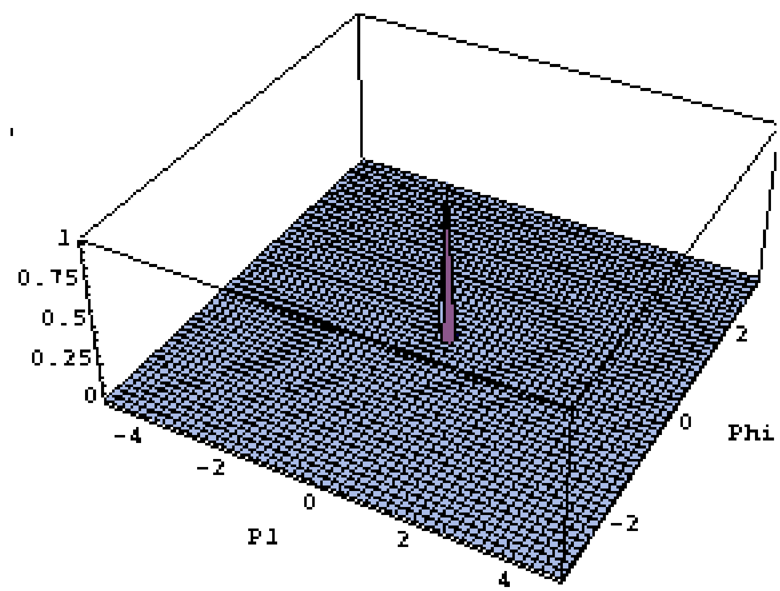
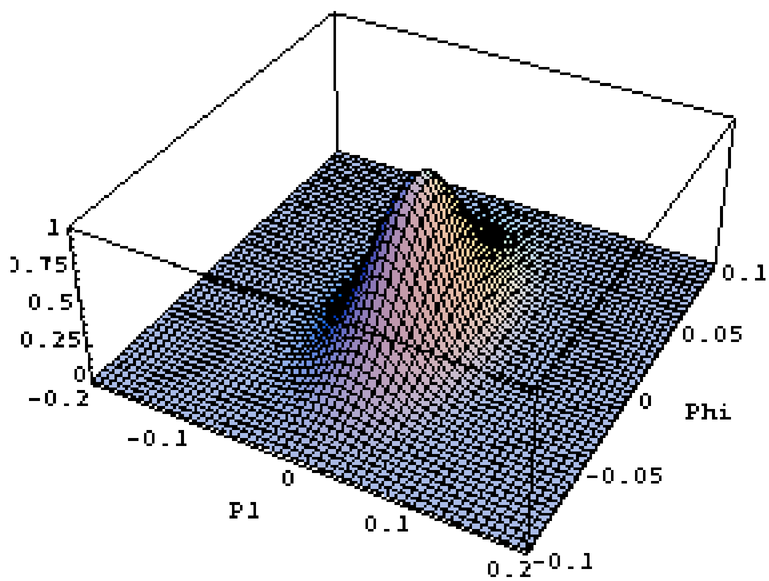
Σ_2
Σ_1
M
i^0
γ
J^−
i^−

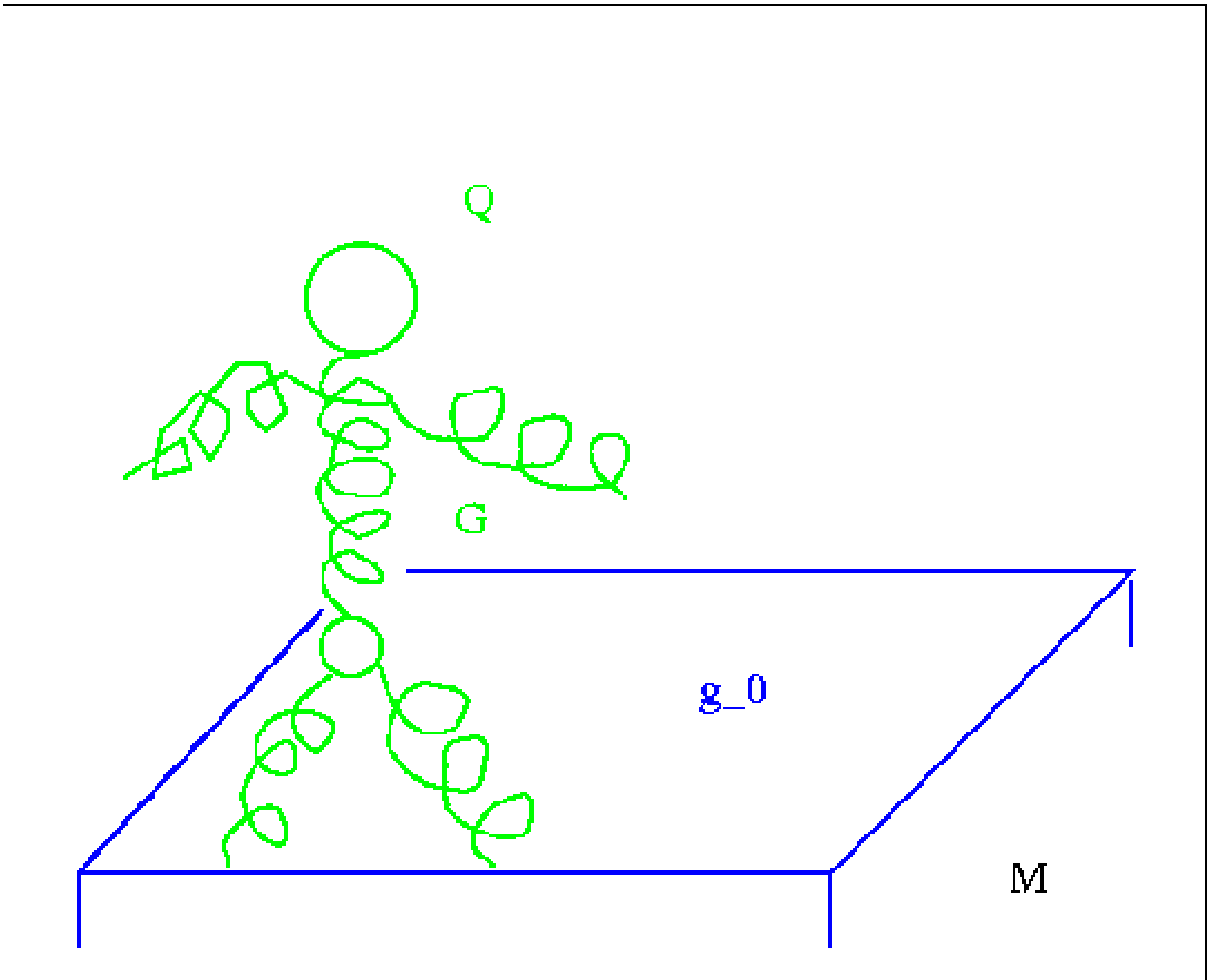

100
75
50
25
0
-4
-2
0
P
2
4
-2
0
2
Phi

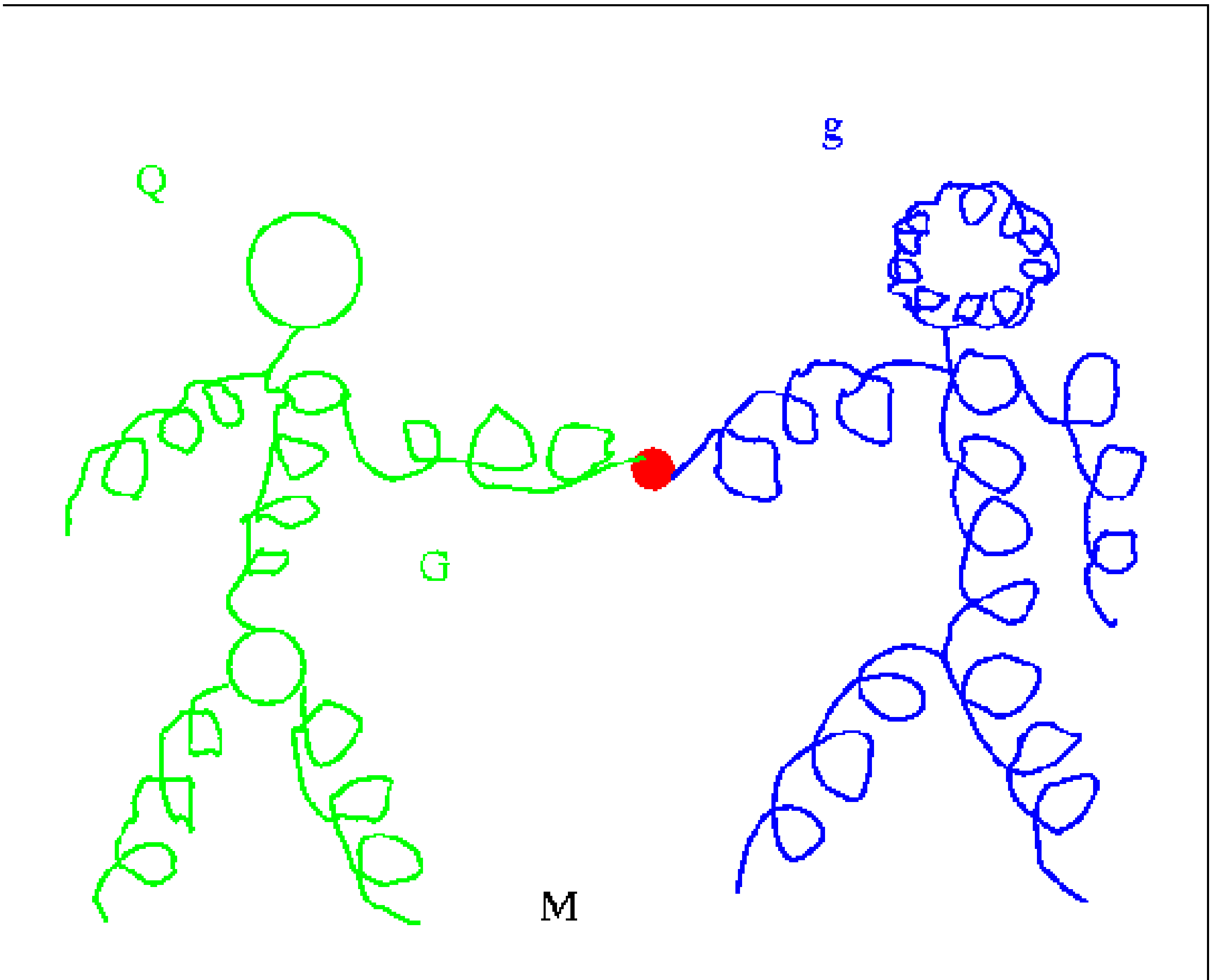


1
0.75
0.5
0.25
0
-4
-2
0
P1
2
4
2
0
Phi
-2

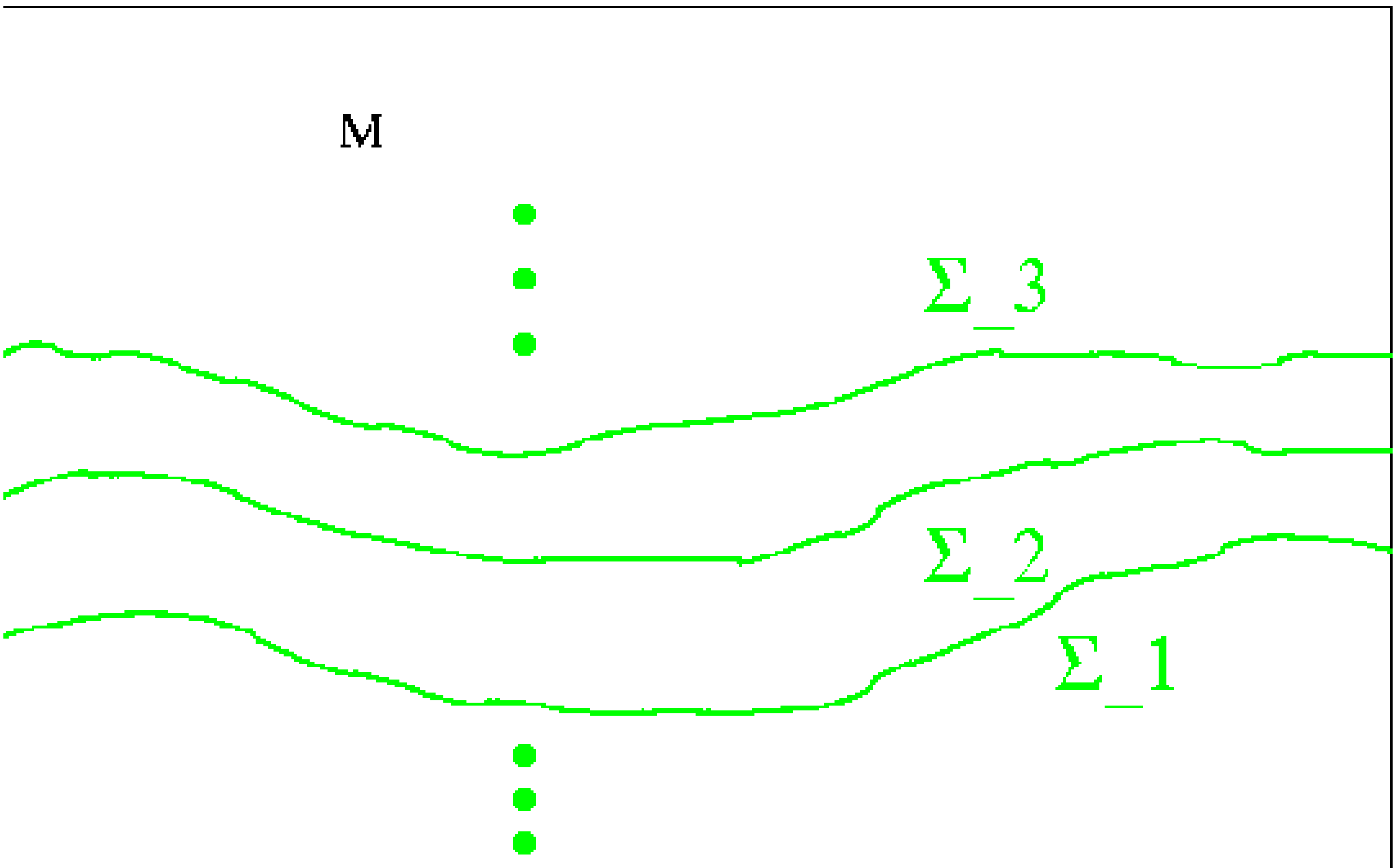


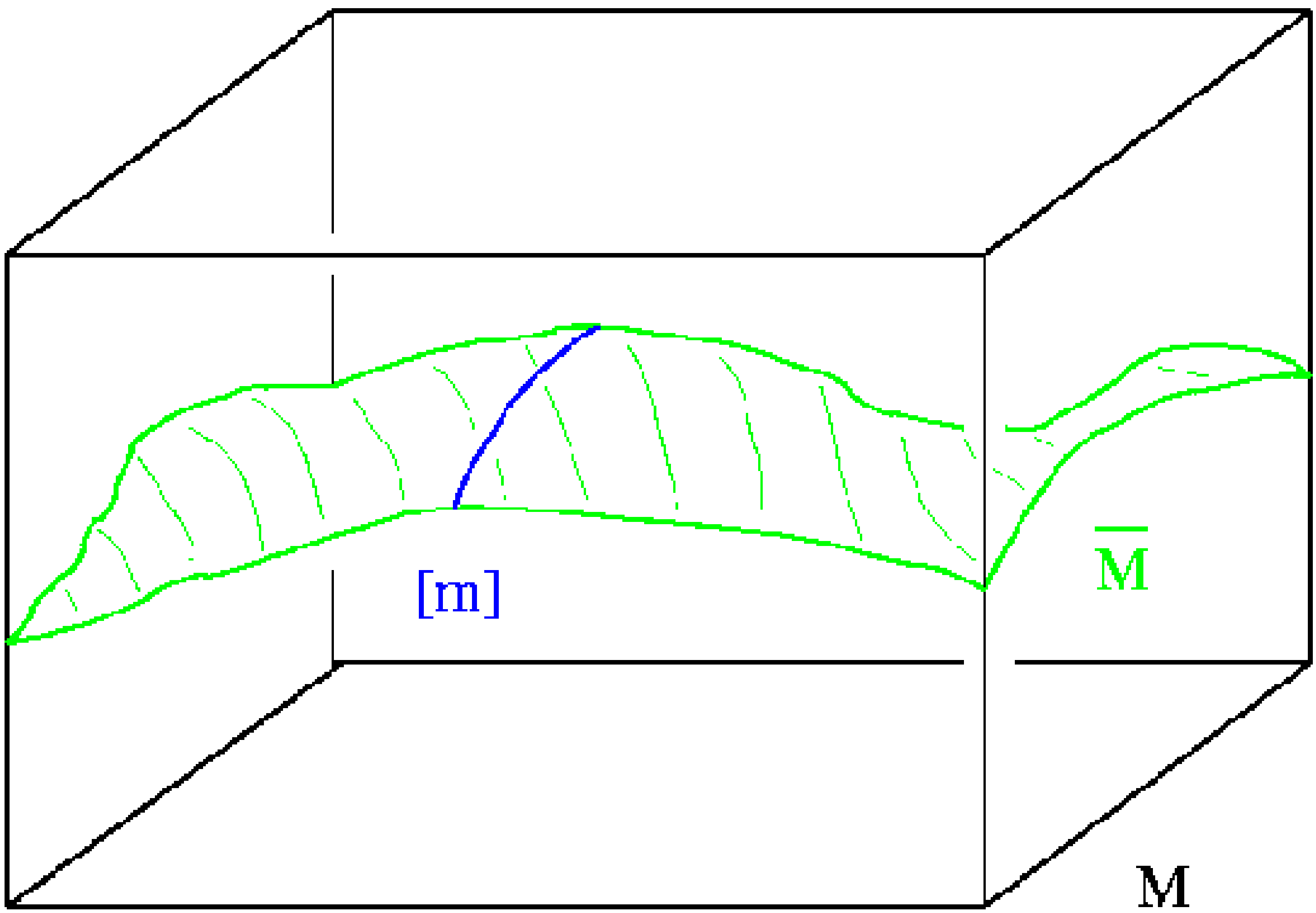
1
0.75
0.5
0.25
0
-0.2
-0.1
0
P1
0.1
0.2
-0.1
-0.05
0
Phi
0.05
0.1


Q
G
g_0
M



Q
g
G
M


M
Σ_3
Σ_2
Σ_1


[m]
$\overline{M}$
M