GENERAL RELATIVITY

M2 Theoretical physics

Correction ; September 18, 2019

1 2-dimensional sphere

1-a) In the ambient space the distance is given by $\mathrm{d}s^2 = G_{AB}\mathrm{d}X^A\mathrm{d}X^B$.

Differentiating the equation of the hypersurface gives $\mathrm{d}X^A = \frac{\partial X^A}{\partial x^\mu}\mathrm{d}x^\mu$.

The distance between points on the hypersurface is then given by $\mathrm{d}s^2 = G_{AB}\frac{\partial X^A}{\partial x^\mu}\frac{\partial X^B}{\partial x^\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$.
So the induced metric reads

$$g_{\mu\nu} = G_{AB}\frac{\partial X^A}{\partial x^\mu}\frac{\partial X^B}{\partial x^\nu}.$$

For the Euclidean space, $G_{AB} = \delta_{AB}$ and the X are the Cartesian coordinates x, y, z.

For the sphere, one can choose the x^μ to be the angles θ, φ, which relate to the Cartesian coordinates by

$$\begin{aligned} x &= R\sin\theta\cos\varphi \\ y &= R\sin\theta\sin\varphi \\ z &= R\cos\theta \end{aligned}$$

The previous formulae enable to show that

$$\begin{aligned} g_{\theta\theta} &= \left(\frac{\partial x}{\partial\theta}\right)^2 + \left(\frac{\partial y}{\partial\theta}\right)^2 + \left(\frac{\partial z}{\partial\theta}\right)^2 = R^2 \\ g_{\theta\varphi} &= \frac{\partial x}{\partial\theta}\frac{\partial x}{\partial\varphi} + \frac{\partial y}{\partial\theta}\frac{\partial y}{\partial\varphi} = 0 \\ g_{\varphi\varphi} &= \left(\frac{\partial x}{\partial\varphi}\right)^2 + \left(\frac{\partial y}{\partial\varphi}\right)^2 = R^2\sin^2\theta. \end{aligned}$$

The metric on the sphere is then

$$\mathrm{d}s^2 = R^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right).$$

1-b) A direct computation of the Christoffel symbols shows that the non-vanishing ones are

$$\begin{aligned} \Gamma^\theta_{\varphi\varphi} &= -\sin\theta\cos\theta \\ \Gamma^\varphi_{\varphi\theta} &= \frac{\cos\theta}{\sin\theta}. \end{aligned}$$

The Ricci tensors then reads

$$\begin{aligned} R_{\theta\theta} &= 1 \\ R_{\theta\varphi} &= 0 \\ R_{\varphi\varphi} &= \sin^2\theta, \end{aligned}$$

and the Ricci scalar is $g^{\theta\theta}R_{\theta\theta} + g^{\varphi\varphi}R_{\varphi\varphi} = 2/R^2$.

1-c) The parameter along the curve can be chosen to be simply φ. Doing so, the tangent vector is $u^\mu = (0,1)$.

The equation governing the parallel transport is $u^\mu D_\mu T^\nu = 0$, which reduces to $D_\varphi T^\nu = 0$.

Given the value of the Christoffel symbols, one gets the following system for the two components of $\vec{T}$

$$\begin{aligned}
\partial_\varphi T^\theta - \sin\alpha\cos\alpha\, T^\varphi &= 0\\
\partial\varphi T^\varphi + \frac{\cos\alpha}{\sin\alpha} T^\theta &= 0.
\end{aligned}$$

Deriving the first equation and removing T^φ with the second one, one finds

$$\partial_\varphi^2 T^\theta + \cos^2\alpha\, T^\theta = 0.$$

α being constant, this can be integrated as :

$$T^\theta = C_1 \cos\left((\cos\alpha)\,\varphi\right) + C_2 \sin\left((\cos\alpha)\,\varphi\right).$$

This also gives

$$T^\varphi = -C_1 \frac{\sin\left((\cos\alpha)\,\varphi\right)}{\sin\alpha} + C_2 \frac{\cos\left((\cos\alpha)\,\varphi\right)}{\sin\alpha}.$$

The integration constants C_1 and C_2 are obtained by using the value of $\vec{T}$ at A and one gets

$$\begin{aligned}
C_1 &= -\sin(\alpha)\sin\left((\cos\alpha)\,\varphi_0\right)\\
C_2 &= \sin(\alpha)\cos\left((\cos\alpha)\,\varphi_0\right).
\end{aligned}$$

Putting all the pieces together, one finds the value of $\vec{T}$ at B

$$\begin{aligned}
T^\theta &= \sin(\alpha)\sin\left((\cos\alpha)\,\delta\right)\\
T^\varphi &= \cos\left((\cos\alpha)\,\delta\right).
\end{aligned}$$

After one round trip one finds that $\vec{T}$ does not go back to its starting value. This illustrates the fact that the curve $\theta = \alpha$ is not a geodesic of the sphere (except for $\alpha = \pi/2$).

2 Generalities on Killing vector fields

2-a) The transformation law of the metric tensor reads

$$g'_{\mu\nu}\left(x'^\rho\right) = \left(\frac{\partial x^\alpha}{\partial x'^\mu}\right)\left(\frac{\partial x^\beta}{\partial x'^\nu}\right) g_{\alpha\beta}\left(x^\rho\right).$$

Using $\dfrac{\partial x^\beta}{\partial x'^\nu} = \delta^\beta_\nu - \epsilon\partial_\nu\xi^\beta$ and the Taylor expansion $g_{\alpha\beta}\left(x^\rho = x'^\rho - \epsilon\xi^\rho\right) = g_{\alpha\beta}\left(x'^\rho\right) - \epsilon\xi^\sigma\partial_\sigma g_{\alpha\beta}\left(x'^\rho\right)$, one finds that, at first order :

$$g'_{\mu\nu}\left(x'^\rho\right) = g_{\mu\nu}\left(x'^\rho\right) - \epsilon\left(\xi^\alpha\partial_\alpha g_{\mu\nu} + g_{\alpha\nu}\partial_\mu\xi^\alpha + g_{\mu\alpha}\,\partial_\nu\xi^\alpha\right).$$

2-b) One simply has

$$\xi^\alpha\partial_\alpha g_{\mu\nu} + g_{\alpha\nu}\partial_\mu\xi^\alpha + g_{\mu\alpha}\,\partial_\nu\xi^\alpha = 0$$

where one can recognize the Lie derivative of the metric along ξ^i. The Killing equation is then

$$\mathcal{L}_\xi g_{\mu\nu} = 0.$$

2-c)
The covariant derivative of the metric being zero, one has

$$\frac{\partial g_{\mu\nu}}{\partial x^\alpha} = \Gamma^\beta_{\mu\alpha} g_{\beta\nu} + \Gamma^\beta_{\nu\alpha} g_{\beta\mu}.$$

The partial derivative involving $\vec{\xi}$ can be rewritten as

$$\frac{\partial \xi^\alpha}{\partial x^\mu} = D_\mu \xi^\alpha - \Gamma^\alpha_{\mu\beta} \xi^\beta.$$

By injecting those expressions in the Killing equations, one finds the following expression

$$D_\mu \xi_\nu + D_\nu \xi_\mu = 0.$$

2-d) If the metric does not depend on the n^{th} coordinate, it is easy to show that $\vec{\xi}$ such that $\xi^i = \delta^i_n$ is a Killing vector.

2-e) Given the fact that the Killing equation is linear (as is the Lie derivative), the first proposition is straightforward.

By replacing the partial derivative by the covariant ones (as in **2-e**), $\vec{\Psi}$ can be rewritten as

$$\Psi^\mu = \xi^\nu D_\nu \omega^\mu - \omega^\nu D_\nu \xi^\mu.$$

One can then compute $D_\mu \Psi_\nu + D_\nu \Psi_\mu$. By injecting the expression of $\vec{\Psi}$ and by using the Leibniz rule, one finds that

$$\begin{aligned} D_\mu \Psi_\nu + D_\nu \Psi_\mu \quad = \quad & \xi^\alpha \left(D_\mu D_\alpha \omega_\nu + D_\nu D_\alpha \omega_\nu \right) - \omega^\alpha \left(D_\mu D_\alpha \xi_\nu + D_\nu D_\alpha \xi_\nu \right) \\ & + D_\mu \xi^\alpha D_\alpha \omega_\nu - D_\mu \omega^\alpha D_\alpha \xi_\nu + D_\nu \xi^\alpha D_\alpha \omega_\mu - D_\nu \omega^\alpha D_\alpha \xi_\mu. \end{aligned}$$

Given that $\vec{\xi}$ and $\vec{\omega}$ are Killing vectors, one can permute the indices to show that

$$D_\mu \xi^\alpha D_\alpha \omega_\nu = D_\alpha \xi_\mu D_\nu \omega^\alpha$$

from which one deduces that the term involving first order derivative vanishes.

Terms containing the second order derivatives can be rewritten by inverting the order of the derivative, which makes appear the Riemann tensor :

$$D_\mu D_\alpha \omega_\nu + D_\nu D_\alpha \omega_\mu = D_\alpha \left(D_\mu \omega_\nu + D_\nu \omega_\mu \right) - R^\beta{}_{\mu\nu\alpha} \omega_\beta - R^\beta{}_{\nu\mu\alpha} \omega_\beta$$

where the first term is zero by application of the Killing equation.

One is then left with

$$\begin{aligned} D_\mu \Psi_\nu + D_\nu \Psi_\mu \quad & = \quad -R^\beta{}_{\mu\nu\alpha} \omega_\beta \xi^\alpha - R^\beta{}_{\nu\mu\alpha} \omega_\beta \xi^\alpha + R^\beta{}_{\mu\nu\alpha} \omega^\alpha \xi_\beta + R^\beta{}_{\nu\mu\alpha} \omega^\alpha \xi_\beta \\ & = \quad \left(-R_{\beta\mu\nu\alpha} - R_{\beta\nu\mu\alpha} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\mu\beta} \right) \omega^\beta \xi^\alpha \end{aligned}$$

where the last equality is a simple manipulation of the indices.

Last, by making use of the symmetries of the Riemann tensor, the term in the parenthesis vanishes, so that $D_\mu \Psi_\nu + D_\nu \Psi_\mu = 0$ which demonstrates the proposition.

2-f) Consider the vector $I^\mu = \xi_\alpha T^{\alpha\mu}$. Using Leibniz, one finds that

$$D_\mu I^\mu = T^{\alpha\mu} D_\mu \xi_\alpha + \xi_\alpha D_\mu T^{\alpha\mu}.$$

The first term vanishes because $T^{\alpha\mu}$ is symmetric and $D_\mu \xi_\alpha$ antisymmetric (by the Killing equation). The second term is also zero because $T^{\mu\nu}$ is conserved. $\vec{I}$ is a conserved vector.

$$\begin{aligned}
\partial_\mu \hat{I}^\mu &= \sqrt{-g}\partial_\mu I^\mu + I^\mu \partial_\mu \sqrt{-g} \\
&= \sqrt{-g}\left(\partial \mu I^\mu + \Gamma^\nu_{\mu\nu} I^\mu\right) \\
&= \sqrt{-g} D_\mu I^\mu = 0
\end{aligned}$$

where one first uses the definition $\vec{\hat{I}}$ and Leibniz rule, then the fact that $\Gamma^\nu_{\mu\nu} = \partial_\mu \log\left(\sqrt{-g}\right)$ and last the definition of the covariant derivative.

Let us compute the time derivative of Q.

$$\partial_0 Q = \int \partial_0 \hat{I}^0 \mathrm{d}x^3 = -\int \partial_i \hat{I}^i \mathrm{d}x^3$$

where one uses the fact that $\partial_\mu \hat{I}^\mu = 0$. The volume integral can be replaced by a surface integral at infinity by Gauss formula to get:

$$\partial_0 Q = \int_\infty \hat{I}^i \mathrm{d}\Sigma_i = 0$$

which vanishes, at least if $\vec{\hat{I}}$ decreases fast enough (for instance faster than $1/r^2$ in "usual" spacetime).

3 Maximally symmetric spaces

3-a) The idea is to start from $D_\mu D_\nu \xi_\rho$ and to switch the indices of the derivatives and the vector.

Each time one switch the indices of the covariant derivatives, one makes appear the Riemann tensor by $D_\mu D_\nu \xi_\rho - D_\nu D_\mu \xi_\rho = -R^\sigma_{\rho\mu\nu}\xi_\sigma$. When one switches indices between a derivative and the vector, by using Killing, one gets $D_\mu \xi_\nu = -D_\nu \xi_\mu$.

This gives

$$\begin{aligned}
D_\mu D_\nu \xi_\sigma &= D_\nu D_\mu \xi_\rho - R^\sigma_{\rho\mu\nu}\xi_\sigma \qquad \text{(Riemann)} \\
&= -D_\nu D_\rho \xi_\mu - R^\sigma_{\rho\mu\nu}\xi_\sigma \qquad \text{(Killing)} \\
&= -D_\rho D_\nu \xi_\mu - R^\sigma_{\rho\mu\nu}\xi_\sigma + R^\sigma_{\mu\nu\rho}\xi_\sigma \qquad \text{(Riemann)} \\
&= D_\rho D_\mu \xi_\nu - R^\sigma_{\rho\mu\nu}\xi_\sigma + R^\sigma_{\mu\nu\rho}\xi_\sigma \qquad \text{(Killing)} \\
&= D_\mu D_\rho \xi_\nu - R^\sigma_{\rho\mu\nu}\xi_\sigma + R^\sigma_{\mu\nu\rho}\xi_\sigma - R^\sigma_{\nu\rho\mu}\xi_\sigma \qquad \text{(Riemann)} \\
&= -D_\mu D_\nu \xi_\rho - R^\sigma_{\rho\mu\nu}\xi_\sigma + R^\sigma_{\mu\nu\rho}\xi_\sigma - R^\sigma_{\nu\rho\mu}\xi_\sigma \qquad \text{(Killing).}
\end{aligned}$$

The same term appears on both sides of the equation so that

$$2D_\mu D_\nu \xi_\rho = \left(-R^\sigma_{\rho\mu\nu} + R^\sigma_{\mu\nu\rho} - R^\sigma_{\nu\rho\mu}\right)\xi_\sigma.$$

The term in the parenthesis can be simplified by using Bianchi first identity. One then gets the result

$$D_\mu D_\nu \xi_\rho = R^\sigma_{\mu\nu\rho}\xi_\sigma.$$

3-b) The equation is second order in term of the parameter λ along the curve. So one needs to give the value of the field ξ_α and its derivative along the curve $\dfrac{\mathrm{d}\xi_\alpha}{\mathrm{d}\lambda}$.

Considering all the possible curves starting from A one can prescribe the values of $D_\beta\xi_\alpha$. It correspond to n values for ξ_α and the $n\,(n-1)\,/2$ values of $D_\beta\xi_\alpha$ (this quantity must be antisymmetric because of the Killing equation). The equation being linear, one has at most $n\,(n+1)\,/2$ independent solutions.

So a manifold of dimension n admits at most $n\,(n+1)\,/2$ linearly independent Killing vector fields. Let us mention that, in general (i.e. for non symmetric spaces), the above procedure does not permit the construction of a true Killing vector field, because the result of the integration would depend on the curve chosen to join A and B.

3-c) The metric of the sphere found in **1**) does not depend on φ so that $(0,1)$ in the natural basis $(\partial_\theta, \partial_\varphi)$, is a Killing vector field. It can be expressed in Cartesian coordinates, in the 3-dimensional ambient space by using the formula:

$$T'^i = \frac{\partial x'^i}{\partial x^j} T^j,$$

where the x^j are θ, φ and the x'^i are x, y, z.

One then gets the vector $(-y, x, 0)$. The two other Killing vectors are obtained by permutation of the coordinates and one gets $(z, 0, -x)$ and $(0, -z, y)$. The sphere is maximally symmetric.

3-d) Using Cartesian coordinates, the metric is constant so that translations in space and time are Killing vectors: $T^i = \delta^i_n$ with $n = 0, 1, 2, 3$.

The three rotations found in **3-c)** are also Killing vectors.

The last three Killing vectors are the three independent boosts. One can get them by looking at infinitesimal Lorentz transformations or by invoking Lorentz rotations. One can easily verify that a boost in the x direction is associated with the Killing vector $B^x = (x, t, 0, 0)$. The two other ones are obtained by permuting x, y and z.

Minkowski spacetime admits 10 independent Killing vectors and so is maximally symmetric.

4 de Sitter spacetime

4-a) By differentiating one finds

$$\mathrm{d}X^1 = He^{(Ht)}x^1\mathrm{d}t + e^{(Ht)}\mathrm{d}x^1$$

from which one deduces that

$$\left(\mathrm{d}X^1\right)^2 + \left(\mathrm{d}X^2\right)^2 + \left(\mathrm{d}X^3\right)^2 = H^2e^{(2Ht)}r^2\mathrm{d}t^2 + 2He^{(2Ht)}r\mathrm{d}r\mathrm{d}t + e^{(2Ht)}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)$$

The definition of t implies that $\mathrm{d}X^0 - \mathrm{d}X^4 = 2He^{(Ht)}\mathrm{d}t$.

Finally one can put the definition of the hypersurface on the form

$$X^4 + X^0 = \frac{-e^{(-Ht)}}{2H^2} + \frac{e^{(Ht)}r^2}{2}$$

from which one gets

$$\mathrm{d}X^0 + \mathrm{d}X^4 = \left(\frac{e^{(-Ht)}}{2H} + \frac{He^{(Ht)}r^2}{2}\right)\mathrm{d}t + e^{(Ht)}r\mathrm{d}r.$$

Given that $\mathrm{d}s^2 = \left(\mathrm{d}X^1\right)^2 + \left(\mathrm{d}X^2\right)^2 + \left(\mathrm{d}X^3\right)^2 - \left(\mathrm{d}X^0 + \mathrm{d}X^4\right)\left(\mathrm{d}X^0 - \mathrm{d}X^4\right)$, one gets, all computations being done,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{(2Ht)}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right),$$

which is of the right form with $a(t) = e^{(Ht)}$.

The coordinates thus defined cover only the part of the hypersurface where $X^0 > X^4$.

4-b) A direct computation shows that the only non-vanishing Christoffel symbols are

$$\begin{aligned} \Gamma^t_{xx} &= \Gamma^t_{yy} = \Gamma^t_{zz} = He^{(2Ht)} \\ \Gamma^x_{tx} &= \Gamma^y_{ty} = \Gamma^z_{tz} = H. \end{aligned}$$

The non-vanishing components o the Ricci tensor are

$$\begin{aligned} R_{xx} &= R_{yy} = R_{zz} = 3H^2e^{(2Ht)} \\ R_{tt} &= -3H^2. \end{aligned}$$

So the Ricci scalar is $R = 12H^2$. It is constant which shows that de Sitter spacetime is maximally symmetric.

4-c) Einstein tensor is $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$. Given the expressions computed previously one can show that

$$G_{ij} + 3H^2g_{ij} = 0.$$

de sitter spacetime is a solution of the vacuum Einstein equations with a positive cosmological constant $\Lambda = 3H^2$.

4-d) A direct computation shows that $-\left(X^0\right)^2 + \left(X^1\right)^2 + \left(X^2\right)^2 + \left(X^3\right)^2 + \left(X^4\right)^2 = 1/H^2$. The differentiation of the equations gives

$$\begin{aligned} \mathrm{d}X^0 &= \left(\cosh\left(H\bar{t}\right)\sqrt{1 - H^2\bar{r}^2}\right)\mathrm{d}\bar{t} - \frac{H\bar{r}\sinh\left(H\bar{t}\right)}{\sqrt{1 - H^2\bar{r}^2}}\mathrm{d}\bar{r} \\ \mathrm{d}X^4 &= \left(\sinh\left(H\bar{t}\right)\sqrt{1 - H^2\bar{r}^2}\right)\mathrm{d}\bar{t} - \frac{H\bar{r}\cosh\left(H\bar{t}\right)}{\sqrt{1 - H^2\bar{r}^2}}\mathrm{d}\bar{r} \\ \mathrm{d}X^i &= \mathrm{d}x^i \quad i = 1, 2, 3. \end{aligned}$$

The metric takes the following form

$$\mathrm{d}s^2 = -\sqrt{1 - H^2\bar{r}^2}\mathrm{d}\bar{t}^2 + \frac{1}{1 - H^2\bar{r}^2}\mathrm{d}\bar{r}^2 + \bar{r}^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right).$$

In those coordinates, the metric does not depend on $\bar{t}$ which means that this time coordinate is adapted to the temporal Killing vector of this spacetime.

5 Anti de Sitter spacetime

5-a) The equation of the hypersurface is

$$\left(X^1\right)^2+\left(X^2\right)^2+\left(X^3\right)^2-\left(X^0\right)^2-\left(X^4\right)^2=-\frac{1}{H^2}.$$

The definition of the coordinates give

$$\begin{aligned}
\mathrm{d}X^0 &= \cos\left(Ht\right)\cosh\left(r\right)\mathrm{d}t+\frac{1}{H}\sin\left(Ht\right)\sinh\left(r\right)\mathrm{d}r\\
\mathrm{d}X^4 &= -\sin\left(Ht\right)\cosh\left(r\right)\mathrm{d}t+\frac{1}{H}\cos\left(Ht\right)\sinh\left(r\right)\mathrm{d}r\\
\mathrm{d}X^1 &= \frac{1}{H}\cosh\left(r\right)\sin\left(\theta\right)\cos\left(\varphi\right)\mathrm{d}r+\frac{1}{H}\sinh\left(r\right)\cos\left(\theta\right)\cos\left(\varphi\right)\mathrm{d}\theta-\frac{1}{H}\sinh\left(r\right)\sin\left(\theta\right)\sin\left(\varphi\right)\mathrm{d}\varphi\\
\mathrm{d}X^2 &= \frac{1}{H}\cosh\left(r\right)\sin\left(\theta\right)\sin\left(\varphi\right)\mathrm{d}r+\frac{1}{H}\sinh\left(r\right)\cos\left(\theta\right)\sin\left(\varphi\right)\mathrm{d}\theta+\frac{1}{H}\sinh\left(r\right)\sin\left(\theta\right)\cos\left(\varphi\right)\mathrm{d}\varphi\\
\mathrm{d}X^3 &= \frac{1}{H}\cosh\left(r\right)\cos\left(\theta\right)\mathrm{d}r-\frac{1}{H}\sinh\left(r\right)\sin\left(\theta\right)\mathrm{d}\theta.
\end{aligned}$$

Putting that into the expression of the distance one finds

$$\mathrm{d}s^2=-\cosh^2\left(r\right)\mathrm{d}t^2+\frac{1}{H^2}\left(\mathrm{d}r^2+\sinh^2\left(r\right)\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2\right)\right).$$

The non-vanishing Christoffel symbols are :

$$\begin{aligned}
\Gamma^t_{rt} &= \frac{\sinh r}{\cosh r}\\
\Gamma^r_{tt} &= H^2\sinh r\cosh r\\
\Gamma^r_{\theta\theta} &= -\sinh r\cosh r\\
\Gamma^r_{\varphi\varphi} &= -\sinh r\cosh r\sin^2\theta\\
\Gamma^\theta_{\theta r} &= \frac{\cosh r}{\sinh r}\\
\Gamma^\theta_{\varphi\varphi} &= -\sin\theta\cos\theta\\
\Gamma^\varphi_{\varphi r} &= \frac{\cosh r}{\sinh r}\\
\Gamma^\varphi_{\varphi\theta} &= \frac{\cos\theta}{\sin\theta}.
\end{aligned}$$

5-b) The non-zero components of the Ricci tensor are

$$\begin{aligned}
R_{tt} &= 3H^2\cosh^2 r\\
R_{rr} &= -3\\
R_{\theta\theta} &= -3\sinh^2 r\\
R_{\varphi\varphi} &= -3\sinh^2 r\sin^2\theta.
\end{aligned}$$

The Ricci scalar is $R=-12H^2$. It is constant so that anti de Sitter spacetime is maximally symmetric.

5-c) Einstein tensor is

$$G_{ij} = 3H^2 g_{ij}$$

anti de Sitter spacetime verifies Einstein equations for the vacuum, with a negative cosmological constant $\Lambda = -3H^2$.

6 Surface gravity

6-a) $k^\alpha = (1, 0, 0, 0)$ is obviously a Killing vector field. Its norm is $k_\alpha k^\alpha = g_{tt}$ which vanishes on the horizon. The horizon is a Killing horizon.

6-b) Given the expression of the metric, and the fact that g_{tt} vanishes on the horizon, the only non-zero terms (on the horizon) are :

$$g^{tr} = 1 \quad ; \quad g^{\theta\theta} = 1/r^2 \quad ; \quad g^{\varphi\varphi} = 1/(r^2 \sin^2 \theta). \tag{1}$$

6-c) Given the expression of k^α, one finds that : $k^\alpha D_\alpha k^\beta = \Gamma^\beta_{tt}$.
An explicit computation shows that

$$\Gamma^\beta_{tt} = -\frac{1}{2} g^{\beta r} \partial_r g_{tt} \tag{2}$$

Given the values of the inverse metric on the horizon, it indeed shows that $k^\alpha D_\alpha k^\beta$ is proportional to k^β. The proportionality factor in κ and one finds that

$$\kappa = \frac{1}{4M}. \tag{3}$$

6-d) The procedure is the same up to the computation of Γ^β_{tt} where undetermined quantities appear. This is related to the fact that Schwarzschild coordinates are singular on the horizon.