2020 16th International Conference on
Control, Automation, Robotics and Vision (ICARCV)
December 13-15, 2020. Shenzhen, China

Predefined Finite-time Output Containment of Nonlinear Multi-Agent Systems with Undirected Topology

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Abstract— **This paper focuses on the finite-time output containment problem for a kind of nonlinear multi-agent systems with multiple dynamic leaders. Firstly, considering the topological structure among the followers, a kind of adaptive distributed observer is designed to estimate the whole states of all the leaders. By utilizing common Lyapunov theory, the finite-time convergence of proposed distributed observer is proved. On the basis of this conclusion, a containment control protocol including the desired convex combinations of the leaders is developed for each follower by using the given weights. With the help of the output regulation theory, the finite-time output containment criterion for nonlinear multi-agent systems is derived. Finally, a numerical example is presented to demonstrate the effectiveness of the theoretical results.**

I. Introduction

In recent years, the cooperative control problem of multi-agent systems (MASs) has attracted increasing attention due to its widespread application in various areas (e.g. flocking, formation control [1], and distributed sensor [2]). As a fundamental research topic in cooperative control of MASs, the consensus control problem has been widely investigated by lots of researchers. According to whether there is a leader, the consensus problem can be divided into leader-follower consensus [3] and leaderless consensus [4]. If there are multiple leaders in MASs, the containment problem arises. The main target of containment control is to develop appropriate control protocol or algorithm such that all the followers can converge to the convex hull formed by the leaders. Until now, many interesting and significant results about the containment control of MASs have been derived in [6], [7]. By utilizing the time-delayed protocols, Dong et al. [7] investigated the containment analysis and design problems for a high-order linear swarm systems.

As we all know, in addition to other agents, the dynamics of each agent is also subjected to its own intrinsic dynamics, which can't be described simply by linear equation. Apparently, it is of great significance to further investigate the containment control problem of the MAS with nonlinear dynamics, and several effective and important containment control results have been obtained in recent years [8], [10], [9]. By utilizing the distributed adaptive control protocol and designed sliding mode estimator, Ma et al. [8] derived several distributed containment criteria for a kind of nonlinear and directed second-order MASs with unknown parameters. With the help of the sliding mode control technique and backstepping control method, the authors [10] developed a distributed adaptive control scheme for guaranteeing a nonlinear MASs subject to input quantization can reach sate containment effectively.

Note that all the above references considered the state containment control of first-order or second-order MASs. However, in lots of practical systems, only a fraction of state variables of each agent is available. Moreover, some cases only require outputs of followers to converge to the convex hull formed by the outputs of leaders. Thus, it is more meaningful to study the output containment control problem for MASs, and some interesting results about the output containment control have been derived recently [11], [12], [13]. Based on the internal model principles, the authors [11] discussed the output containment control problem for a linear heterogeneous MAS by using designed full-state feedback and static output-feedback control protocols, respectively. In [12], the authors not only derived several sufficient conditions to guarantee the output containment for the heterogeneous linear MAS by employing a dynamic controller, but also solved a optimal containment control problem for proposed system with the help of reinforcement learning method. However, these results on containment control mainly focus on the linear system. To our knowledge, very few authors have studied the output containment control problem for nonlinear MASs in recent years [14]. Using developed observer and backstepping method, output feedback controllers for each followers are proposed in [14] such that the output containment can be achieved by high-order nonlinear MASs.

On the other hand, in many practical scenes, the MASs are expected to realize state or output containment in a finite time. In addition to the faster convergence rate, finite-time containment offers many benefits including better disturbance rejection, robustness against uncertainties. Thus, it is very meaningful to investigate the finite-time containment in MASs. Recently, some authors have paid their attention to investigating the finite-time containment problem of MASs, and derived several important results [15], [16], [17]. In [15], several finite-time containment criteria for a second-order system in the presence of multiple leaders and external disturbances were put proposed by employing designed finite-time containment control algorithms. Fu et al. [16] studied

This work was supported by the Science and Technology Innovation 2030-Key Project of "New Generation Artificial Intelligence" under Grant 2018AAA0102303, the National Natural Science Foundation of China under Grants 61922008, 61973013, 61873011 and 61803014, the Innovation Zone Project under Grant 18-163-00-TS-001-001-34, the Defense Industrial Technology Development Program under Grant JCKY2019601C106, the Beijing Natural Science Foundation under Grant 4182035, and the Special Research Project of Chinese Civil Aircraft.

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the robust finite-time containment problem for a linear MASs with directed topology by means of observer-based controller. Based on the nonlinear feedback controllers, several sufficient conditions to ensure the finite-time containment were obtained at any preset time for MASs with static and dynamic leaders in [17]. Because the dynamics of the agents are nonlinear in practical cases, achieving finite-time containment for nonlinear MASs imposes significant challenge [18]. Based on the form of interaction among agents, Wang et al. [18] studied the finite-time containment of nonlinear MASs by constructing a new symmetric Laplacian matrix and neural adaptive control methods. Unfortunately, to our knowledge, the finite-time output containment problem for linear or nonlinear MASs has not yet been discussed.

Motivated by the above discussion, this paper respectively discusses the output containment problem for a nonlinear MASs with multiple dynamic leaders. The main contributions of this paper can be summarized as follows. Firstly, a kind of adaptive distributed observer is designed to estimate the whole states of all the leaders. We apply inequality techniques and common Lyapunov theory to prove the finite-time convergence of proposed distributed observer. Then, a predefined finite-time containment control protocol for each follower is proposed, where the expected convex combination of the multiple leaders is specified by several given weights. With the help of the output regulation theory, the finite-time output containment criterion for nonlinear multi-agent systems is derived.

II. Preliminaries and problem description

A. Notations

$\otimes$ denotes the Kronecker product. $\mathbf{1}_M$ is a column vector of size M with 1 as its elements. $\mathscr{V}=\{M+1,M+2,\cdots,M+N\}$ represents the set of follower where M,N respectively represents the total number of leaders and followers in the MAS. $\mathscr{B}\subset\mathscr{V}\times\mathscr{V}$ is the set of undirected edges among the followers in MASs.

B. Lemmas

Lemma 2.1 (see [19]). If a non-negative continuous function $w(t)$ satisfies $\dot{w}(t)\leqslant -bw^{\alpha}(t),\ t\geqslant 0, w(0)\geqslant 0$, where $b>0$ and $0<\alpha<1$ are constants, then it holds that $w^{1-\alpha}(t)\leqslant w^{1-\alpha}(0)-b(1-\alpha)t,\ 0\leqslant t\leqslant\beta$, and $w(t)=0,\ t\geqslant\beta$, in which $\beta=\frac{w^{1-\alpha}(0)}{b(1-\alpha)}$.

Lemma 2.2 (see [20]). Suppose that $W=(w_{ik})_{N\times N}\in\mathbb{R}^{N\times N}$ is irreducible and satisfies $\sum_{k=1}^{N}w_{ik}=0$ with $w_{ik}\geqslant 0 (i\neq k)$. Then, we can find a positive vector $\omega=(\omega_1,\omega_2,\cdots,\omega_N)^T\in\mathbb{R}^N$ satisfying $(i)W^T\omega=0,(ii)\hat{W}=(\hat{w}_{ik})_{N\times N}=\hat{\omega}W+W^T\hat{\omega}$ is symmetric and $\sum_{k=1}^{N}\hat{w}_{ik}=\sum_{k=1}^{N}\hat{w}_{ki}=0$ for any $i=1,2,\cdots,N$, where $\hat{\omega}=\mathrm{diag}(\omega_1,\omega_2,\cdots,\omega_N)$.

Lemma 2.3 (see [21]). Assume that $x_i\in\mathbb{R}, i=1,\cdots,n, 0<\varsigma\leqslant 1$. Then, the following inequality holds: $(\sum_{i=1}^{n}|x_i|)^{\varsigma}\leqslant\sum_{i=1}^{n}|x_i|^{\varsigma}$.

C. Problem description

A kind of nonlinear MASs consisting of M leaders and N followers is considered in this paper. Define $\mathbb{B}_L=\{1,2,\cdots,M\}$ and $\mathbb{B}_F=\{M+1,M+2,\cdots,M+N\}$ represent the leader set and follower set, respectively.

The dynamics of the jth ($j\in\mathbb{B}_L$) leader can be described as follows:

$$\begin{aligned}\dot{x}_j(t) &= f(x_j(t)),\\ y_j(t) &= Ax_j(t),\end{aligned}\tag{1}$$

where $x_j(t)=(x_{j1}(t),x_{j2}(t),\cdots,x_{jn}(t))^T\in\mathbb{R}^n$ denotes the state vector of the jth leader; $y_j(t)=(y_{j1}(t),y_{j2}(t),\cdots,y_{jp}(t))^T\in\mathbb{R}^p (1\leqslant p<n)$ is the output vector of the jth leader; $f(x_j(t))=(f(x_{j1}(t)),f(x_{j2}(t)),\cdots,f(x_{jn}(t)))^T\in\mathbb{R}^n$, and it is assumed that the $\mathrm{rank}(A)=p$. For the ith ($i\in\mathbb{B}_F$) follower, its dynamical model is described as follows:

$$\begin{aligned}\dot{z}_i(t) &= f(z_i(t))+u_i(t),\\ s_i(t) &= Cz_i(t),\end{aligned}\tag{2}$$

where $z_i(t)=(z_{i1}(t),z_{i2}(t),\cdots,z_{in}(t))^T\in\mathbb{R}^n$ denotes the state vector of the ith follower; $s_i(t)=(s_{i1}(t),s_{i2}(t),\cdots,s_{ip}(t))^T\in\mathbb{R}^p$ is the output vector of the ith follower; $u_i(t)=(u_{i1}(t),u_{i2}(t),\cdots,u_{in}(t))^T\in\mathbb{R}^n$ represents the control input of the follower i; and $\mathrm{rank}(C)=p$.

Assumption 2.1. Throughout this paper, we always suppose the function $f(\cdot)$ satisfies the following condition:

$$\|f(\hat{\delta})-\sum_{j=1}^{M}\gamma_jPf(\delta_j)\|\leqslant\rho\|\hat{\delta}-\sum_{j=1}^{M}\gamma_jP\delta_j\|$$

for any $\hat{\delta}\in\mathbb{R}^n,\delta_j\in\mathbb{R}^n,P\in\mathbb{R}^{n\times n},0\leqslant\gamma_j\in\mathbb{R}$ and satisfies $\sum_{j=1}^{M}\gamma_j=1$.

In what follows, the definition of the expected finite-time output containment of the nonlinear MASs (1) and (2) is given.

Definition 2.1. The MASs (1) and (2) realizes the expected finite-time output containment if there exist $0<t_1\in\mathbb{R}$ and $0\leqslant\rho_{ij}\in\mathbb{R}(i\in\mathbb{B}_F,j\in\mathbb{B}_L)$ satisfying $\sum_{j=1}^{M}\rho_{ij}=1$ such that

$$\begin{aligned}\lim_{t\to t_1^-}\left\|s_i(t)-\sum_{j=1}^{M}\rho_{ij}y_j(t)\right\| &= 0,\\ \left\|s_i(t)-\sum_{j=1}^{M}\rho_{ij}y_j(t)\right\| &\neq 0,\quad 0\leqslant t<t_1.\end{aligned}$$

III. Main results

In this section, a finite-time output containment control framework for nonlinear MASs is proposed. Firstly, we design a kind of adaptive distributed observer to estimate the whole states of all the leaders. Then based on the output regulation theory and using designed containment control protocol, it is proved that the expected finite-time output containment can be achieved by the nonlinear MASs (1) and (2).

A. Distributed observer with undirected topology

In order not to lose generality, suppose that the information of each leader can be obtained by at least one follower. If there exists a single leader in MASs, we change the number of leaders to $M-1$ and all the results still hold. In this subsection, the topological structure among the followers is connected and undirected, while the communication topology between the leaders and followers is always directed.

To guarantee the MASs (1) and (2) can realize the finite-time output containment, the adaptive observer of ith follower is designed as follows:

$$\begin{aligned}\dot{\xi}_i(t) &= \bar{f}(\xi_{ij}(t)) - b_1 \operatorname{sign}(\xi_i(t)-\bar{x}(t))|\xi_i(t)-\bar{x}(t)|^\beta \\ &\quad -b_2(t)(\xi_i(t)-\bar{x}(t)) + \sum_{k=M+1}^{M+N} w_{ik}(t)\big(\xi_i(t) \\ &\quad -\xi_k(t)\big) - \begin{bmatrix} w_{i1}\xi_{i1}(t)-x_1(t) \\ w_{i2}\xi_{i2}(t)-x_2(t) \\ \vdots \\ w_{iM}\xi_{iM}(t)-x_M(t)\end{bmatrix}, \quad (3)\end{aligned}$$

$$\dot{b}_2(t) = a\sum_{i=M+1}^{M+N}(\xi_i(t)-\bar{x}(t))^T(\xi_i(t)-\bar{x}(t)) + a, \quad (4)$$

$$\dot{w}_{ik}(t) = \begin{cases} \varphi_{ik}(\xi_i(t)-\xi_k(t))^T(\xi_i(t)-\xi_k(t)) + \varphi_{ik}, \\ \qquad\qquad\qquad \text{if } \ k \neq i, \\ -\sum_{\substack{\alpha=M+1\\ \alpha\neq i}}^{M+N} \dot{w}_{i\alpha}(t), \quad \text{if } \ k=i, \\ 0, \qquad\qquad \text{otherwise,}\end{cases} \quad (5)$$

where $k \in \mathbb{B}_F, \xi_i(t) = (\xi_{i1}^T(t), \xi_{i2}^T(t), \cdots, \xi_{iM}^T(t))^T$ with $\xi_{ij} \in \mathbb{R}^n$ represents the estimated state of the ith follower for $x_j(t)(i \in \mathbb{B}_F, j \in \mathbb{B}_L)$ at time t; $\bar{x}(t) = \left(x_1^T(t), x_2^T(t), \cdots, x_M^T(t)\right)^T$; $\bar{f}(\xi_{ij}(t)) = (f^T(\xi_{i1}(t)), f^T(\xi_{i2}(t)), \cdots, f^T(\xi_{iM}(t)))^T$; $0 < b_1 \in \mathbb{R}$; $\operatorname{sign}(\cdot)$ represents the sign function; $|\xi_i(t)-\bar{x}(t)|^\beta \in \mathbb{R}^{Mn}$; $0<\beta<1$; $0<\varphi_{ik}=\varphi_{ki}\in\mathbb{R}$; $0<a\in\mathbb{R}$. If there exists a communication path from jth leader to ith follower, then $1 = w_{ij} \in \mathbb{R}(j=1,2,\cdots,M)$; otherwise, $w_{ij}=0(j=1,2,\cdots,M)$. Define $W(t) = (w_{ik}(t))_{N\times N} \in \mathbb{R}^{N\times N}(k\in\mathbb{B}_F)$ is the time-varying matrix, in which $w_{ik}(t)\in\mathbb{R}$ satisfies the following condition: if there is an edge between follower i and follower $k(i\neq k)$, then $0<w_{ik}(t)=w_{ki}(t)\in\mathbb{R}$; otherwise, $0=w_{ik}(t)=w_{ki}(t)\in\mathbb{R}(i\neq k)$, and $w_{ii}(t) = -\sum_{\substack{k=M+1\\ k\neq i}}^{M+N} w_{ik}(t)$.

Define $e_i(t) = (e_{i1}^T(t), e_{i2}^T(t), \cdots, e_{iM}^T(t))^T = \xi_i(t)-\bar{x}(t) \in \mathbb{R}^{Mn}$, in which $e_{ij}(t) = \xi_{ij}(t)-x_j(t)(i\in\mathbb{B}_F, j\in\mathbb{B}_L)$. According to (1) and (3), we can obtain

$$\begin{aligned}\dot{e}_i(t) &= \bar{f}(\xi_{ij}(t)) - \bar{f}(x_j(t)) - b_1\operatorname{sign}(e_i(t))|e_i(t)|^\beta \\ &\quad + \sum_{k=M+1}^{M+N} w_{ik}(t)(e_i(t)-e_k(t)) - W_i e_i(t) \\ &\quad -b_2(t)e_i(t), \quad (6)\end{aligned}$$

where $W_i = \operatorname{diag}(w_{i1}I_n, w_{i2}I_n, \cdots, w_{iM}I_n), i\in\mathbb{B}_F, j\in\mathbb{B}_L$.

The following theorem displays the $\xi_{ij}(t)(i\in\mathbb{B}_F, j\in\mathbb{B}_L)$ converges to the same convex combination of the leader's state.

Theorem 3.1. Suppose that the topological structure among the followers is connected and undirected, for the ith follower $(i\in\mathbb{B}_F)$, the $\xi_{ij}(t)(j\in\mathbb{B}_L)$ converges to the same convex combination of the leader's state in a finite time t_0 under the observer (3), that means $\lim\limits_{t\to t_0^-}\|e_i(t)\|=0$, $\|e_i(t)\|\neq 0$, $0\leqslant t<t_0$ e.g.,

$$\begin{aligned}\lim_{t\to t_0^-}\|\xi_i(t)-\bar{x}(t)\| &= 0, \\ \|\xi_i(t)-\bar{x}(t)\| &\neq 0, \quad 0\leqslant t<t_0.\end{aligned}$$

Proof. For the system (6), we construct the following common Lyapunov functional:

$$\begin{aligned} V_3(t) &= V_1(t)+V_2(t), \\ V_1(t) &= \sum_{i=M+1}^{M+N} e_i^T(t)e_i(t), \\ V_2(t) &= \sum_{i=M+1}^{M+N}\sum_{k=M+1}^{M+N}\frac{(w_{ik}(t)-h_{ik})^2}{2\varphi_{ik}}, \quad (7)\end{aligned}$$

where $0\leqslant h_{ik}=h_{ki}\in\mathbb{R}(i\neq k)$, and $h_{ik}=0$ is equivalent to $w_{ik}(t)=0$.

Taking the time derivative of $V_3(t)$ along (6), we can obtain

$$\begin{aligned}\dot{V}_3(t) &= 2\sum_{i=M+1}^{M+N} e_i^T(t)\Big(\bar{f}(\xi_{ij}(t))-\bar{f}(x_j(t)) \\ &\quad -b_1\operatorname{sign}(e_i(t))|e_i(t)|^\beta - W_ie_i(t) \\ &\quad -b_2(t)e_i(t) + \sum_{k=M+1}^{M+N} w_{ik}(t)(e_i(t)-e_k(t))\Big) \\ &\quad + \sum_{i=M+1}^{M+N}\sum_{k=M+1}^{M+N}(w_{ik}(t)-h_{ik})\big(e_i(t) \\ &\quad -e_k(t)\big)^T\big(e_i(t)-e_k(t)\big) \\ &\quad + \sum_{i=M+1}^{M+N}\sum_{k=M+1}^{M+N}(w_{ik}(t)-h_{ik}). \quad (8)\end{aligned}$$

According to the property of $\bar{f}(\cdot)$, it holds that

$$\begin{aligned}& 2e_{ij}^T(t)\left(\bar{f}(\xi_{ij}(t))-\bar{f}(x_j(t))\right) \\ \leqslant\ & \left(1+\rho^2\right)e_i^T(t)e_i(t). \quad (9)\end{aligned}$$

Note that

$$\begin{aligned}& \sum_{i=M+1}^{M+N}\sum_{k=M+1}^{M+N}(w_{ik}(t)-h_{ik})\big(e_i(t) \\ & -e_k(t)\big)^T\big(e_i(t)-e_k(t)\big) \\ =\ & -2\sum_{i=M+1}^{M+N}\sum_{k=M+1}^{M+N}(w_{ik}(t)-h_{ik})e_i^T(t) \\ & \times(e_i(t)-e_k(t)). \quad (10)\end{aligned}$$

In light of Lemma 2.3, it holds that:

$$\begin{aligned}\sum_{i=M+1}^{M+N}|e_i(t)|^{\beta+1} &= \sum_{i=M+1}^{M+N}(e_i^2(t))^{\frac{\beta+1}{2}} \\ &\geqslant V_1^{\frac{\beta+1}{2}}(t). \quad (11)\end{aligned}$$

Define $\hat{W} = \mathrm{diag}(W_{M+1}, W_{M+2}, \cdots, W_{M+N}), e = (e_{M+1}^T(t), e_{M+2}^T(t), \cdots, e_{M+N}^T(t))^T \in \mathbb{R}^{NMn}, H = (h_{ik})_{N\times N} \in \mathbb{R}^{N\times N}$ with $h_{ii} = -\sum_{k=M+1}^{M+N} h_{ik}$. Then, it holds that

$$\begin{aligned}\dot{V}_3(t) \leqslant\ & e^T(t)\Big[\big(1+\rho^2-b_2(t)\big)I_{MNn} + 2H\otimes I_{Mn} \\ & -2\hat{W}\Big]e(t) + \sum_{i=M+1}^{M+N}\sum_{k=M+1}^{M+N}(w_{ik}(t)-h_{ik}) \\ & -2b_1 V_1^{\frac{\beta+1}{2}}(t). \end{aligned} \tag{12}$$

On the basis of the definition of H, (4) and (5), there obviously exists $0 < t^* \in \mathbb{R}$ satisfying

$$\big(1+\rho^2-b_2(t)\big)I_{MNn} + 2H\otimes I_{Mn} - 2\hat{W} \leqslant 0, \quad w_{ik}(t) \geqslant h_{ik}$$

for all $t \geqslant t^*$,

Then, it holds that:

$$\dot{V}_3(t) \leqslant -2b_1 V_1^{\frac{\beta+1}{2}}(t), \qquad t \geqslant t^*.$$

Therefore, when $t \geqslant t^*$, one gets

$$\begin{aligned}\dot{V}_1(t) =\ & 2\sum_{i=M+1}^{M+N} e_i^T(t)\dot{e}_i(t) \\ \leqslant\ & e^T(t)\Big[\big(1+\rho^2-b_2(t)\big)I_{MNn} + 2H\otimes I_{Mn} \\ & -2\hat{W}\Big]e(t) - 2b_1 V_1^{\frac{\beta+1}{2}}(t) \\ \leqslant\ & -2b_1 V_1^{\frac{\beta+1}{2}}(t).\end{aligned}$$

In light of Lemma 2.1, it holds that:

$$V_1(t) = 0 \quad t \geqslant t^* + T_1, \tag{13}$$

where $T_1 = \frac{V_1^{\frac{1-\beta}{2}}(t^*)}{b_1(1-\beta)}$.

Apparently, there is a $0 < t_0 \leqslant t^* + T_1$ satisfying

$$\begin{aligned}\lim_{t\to t_0^-}\|\xi_i(t) - \bar{x}(t)\| &= 0, \\ \|\xi_i(t) - \bar{x}(t)\| &\neq 0, \quad 0 \leqslant t < t_0.\end{aligned}$$

Therefore, for the ith follower $(i \in \mathbb{B}_F)$, the $\xi_{ij}(t)(j \in \mathbb{B}_L)$ converges to the same convex combination of the leader's state in a finite time t_0 under the observer (3). This completes the proof of Theorem 3.1.

Remark 2: In this subsection, a kind of adaptive distributed observer is designed to estimate the whole states of all the leaders. From the proof of Theorems 3.1, we can find when $t \geqslant t^*$ the $\xi_{ij}(t)(i \in \mathbb{B}_F, j \in \mathbb{B}_L)$ can converge to the same convex combination of the leader's state in a finite time t_0. Hence, it is unnecessary to update the coupling weights $w_{ik}(t)$ and adaptive parameters $b_2(t)$ when $t \geqslant t^*$, which means $\dot{w}_{ik}(t) = 0, \dot{b}_2(t) = 0$ for $t \geqslant t^*$.

B. Predefined finite-time containment control protocol

Assumption 3.1. There always exists a matrix K satisfying the following regulation equation:

$$0 = CK - A. \tag{14}$$

On the basis of the distributed observer (3), we design the following finite-time output containment control protocol for the ith follower $(i \in \mathbb{B}_F)$:

$$\begin{aligned}u_i(t) =\ & b_4\sum_{j=1}^{M}\rho_{ij}K\xi_{ij}(t) - b_3\mathrm{sign}(z_i(t) - \sum_{j=1}^{M}\rho_{ij}Kx_j(t)) \\ & \times|z_i(t) - \sum_{j=1}^{M}\rho_{ij}Kx_j(t)|^\beta - b_4 z_i(t),\end{aligned} \tag{15}$$

where $0 < b_3 \in \mathbb{R}, 1+\rho^2 \leqslant b_4 \in \mathbb{R}, K \in \mathbb{R}^{n\times n}$.

Define $\delta_i(t) = (\delta_{i1}(t), \delta_{i2}(t), \cdots, \delta_{in}(t))^T = z_i(t) - \sum_{j=1}^{M}\rho_{ij}Kx_j(t) \in \mathbb{R}^n$. According to (1), (2) and (15), we can obtain

$$\begin{aligned}\dot{\delta}_i(t) =\ & f(z_i(t)) - b_4 z_i(t) - b_3\mathrm{sign}(\delta_i(t))|\delta_i(t)|^\beta \\ & + b_4\sum_{j=1}^{M}\rho_{ij}K\xi_{ij}(t) - \sum_{j=1}^{M}\rho_{ij}Kf(x_j(t)),\end{aligned} \tag{16}$$

where $i \in \mathbb{B}_F$.

Theorem 3.2. Suppose that the Assumptions 2.1 and 3.1 hold, the nonlinear MASs (1) and (2) can realize the desired finite-time output containment under the designed control protocol (15) and distributed observer (3).

Proof. For the system (16), we construct the following common Lyapunov functional:

$$\hat{V}_i(t) = \frac{1}{2}\delta_i^T(t)\delta_i(t). \tag{17}$$

Taking the time derivative of $\hat{V}_i(t)$ along (16), one derives

$$\begin{aligned}\dot{\hat{V}}_i(t) =\ & \delta_i^T(t)\dot{\delta}_i(t) \\ =\ & \delta_i^T(t)\big(f(z_i(t)) - \sum_{j=1}^{M}\rho_{ij}Kf(x_j(t)) - b_4 z_i(t) \\ & + b_4\sum_{j=1}^{M}\rho_{ij}K\xi_{ij}(t) - b_3\mathrm{sign}(\delta_i(t))|\delta_i(t)|^\beta\big),\end{aligned} \tag{18}$$

where $i \in \mathbb{B}_F$.

According to the property of $f(\cdot)$, we can obtain

$$\begin{aligned}& \delta_i^T(t)\left(f(z_i(t)) - \sum_{j=1}^{M}\rho_{ij}Kf(x_j(t))\right) \\ \leqslant\ & \frac{(1+\rho^2)}{2}\delta_i^T(t)\delta_i(t).\end{aligned} \tag{19}$$

Based on Lemma 2.3, it holds that:

$$\delta_i^T(t)\big(\mathrm{sign}(\delta_i(t))|\delta_i(t)|^\beta\big) \geqslant 2\hat{V}_i^{\frac{\beta+1}{2}}(t). \tag{20}$$

Substituting $\xi_{ij}(t) = e_{ij}(t) + x_j(t)$, (20) and (19) into (18) gives

$$\begin{aligned}\dot{\hat{V}}_i(t) \leqslant\ & \frac{(1+\rho^2+b_4)}{2}\delta_i^T(t)\delta_i(t) - b_3 V_i^{\frac{\beta+1}{2}}(t) \\ & + \frac{b_4}{2}\Pi_i^T(t)\Pi_i(t) - b_4\delta_i^T(t)\delta_i(t).\end{aligned} \tag{21}$$

where $i \in \mathbb{B}_F, \Pi_i(t) = \sum_{j=1}^{M} \rho_{ij} K e_{ij}(t)$.

Note that $1+\rho^2 \leqslant b_4$ and when $t \geqslant t_0, \left\|e_{ij}(t)\right\| = 0,\ (i \in \mathbb{B}_F, j \in \mathbb{B}_L)$, we get $\|\Pi_i(t)\| = 0,\ t \geqslant t_0$.

Consequently, when $t \geqslant t_0$, one gets

$$\dot{\hat{V}}_i(t) \quad \leqslant \quad -2b_3 \hat{V}_i^{\frac{\beta+1}{2}}(t). \tag{22}$$

In light of Lemma 2.1, it holds that:

$$\hat{V}_i(t) = 0 \quad t \geqslant t_0 + T_2, \tag{23}$$

where $T_2 = \frac{\hat{V}_i^{\frac{1-\beta}{2}}(t_0)}{b_3(1-\beta)}$.

Apparently, there is a $0 < t_1 \leqslant t_0 + T_2$ satisfying

$$\begin{aligned} \lim_{t \to t_1^-} \|\delta_i(t)\| &= 0, \\ \|\delta_i(t)\| &\neq 0, \quad 0 \leqslant t < t_1. \end{aligned}$$

Define the finite-time output containment error $\tilde{e}_i(t) = s_i(t) - \sum_{j=1}^{M} \rho_{ij} y_j(t)\,(i \in \mathbb{B}_F)$. Since $CK - A = 0$, one derives

$$\tilde{e}_i(t) = C\delta_i(t). \tag{24}$$

Obviously, it holds that

$$\begin{aligned} \lim_{t \to t_1^-} \|\tilde{e}_i(t)\| &= 0, \\ \|\tilde{e}_i(t)\| &\neq 0, \quad 0 \leqslant t < t_1. \end{aligned}$$

Therefore, we can conclude that the desired finite-time output containment is reached by the nonlinear MASs (1) and (2) under designed control protocol (15) and distributed observer (3).

IV. NUMERICAL EXAMPLES

A nonlinear MASs consisting of 3 leaders and 3 followers is considered in the simulation, where $j \in \mathbb{B}_L = \{1,2,3\}, i \in \mathbb{B}_F = \{4,5,6\}$. In MASs (1) and (2), assume that $x_1(0) = [11,10,11,12]^T, x_2(0) = [12,10,14,12]^T, x_3(0) = [13,10,19,12]^T, z_1(0) = [1,1,1,1]^T, z_2(0) = [4,2,2,\sqrt{2}]^T, z_3(0) = [9,3,3,\sqrt{3}]^T$;

$$A = C \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Apparently, we can find the following matrix I_4 satisfying the regulation equation (14).

And $f(x_j(t)) = (3\sin(x_{j1}(t)), 3\sin(x_{j2}(t)), 3\sin(x_{j3}(t)), 3\sin(x_{j4}(t))), f(z_j(t)) = ((3\sin(z_{j1}(t)), 3\sin(z_{j2}(t)), 3\sin(z_{j3}(t)), 3\sin(z_{j4}(t)))\,(j \in \mathbb{B}_L, i \in \mathbb{B}_F)$. Obviously, $f(\cdot)$ fulfills Assumption 2.1 condition with $\rho = 3$. The topological structure of the nonlinear MASs (1) and (2) is described by Fig.1, and the coupling weight among the followers is selected as follows:

$$W(0) \;=\; \begin{pmatrix} -0.5 & 0.2 & 0.3 \\ 0.2 & -0.2 & 0 \\ 0.3 & 0 & -0.3 \end{pmatrix}.$$

Select $b_1 = 2.8, b_5 = 11, b_6 = 13, \beta = 0.4, a = 0.2$, and the initial values of $\xi_{ij}(0)\,(j \in \mathbb{B}_L, i \in \mathbb{B}_F)$ are generated by random numbers in $[0.4, 0.5]$,

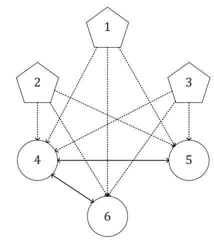

Fig. 1. Topological structure

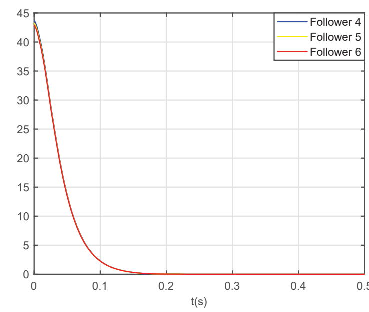

Fig. 2. Estimation errors $\|e_i(t)\|$, $i = 4,5,6$.

The desired finite-time output containment for the nonlinear MASs (1) and (2) is specified by $\rho_{41} = \frac{1}{4}, \rho_{42} = \frac{1}{4}, \rho_{43} = \frac{1}{2}, \rho_{51} = \frac{1}{3}, \rho_{52} = \frac{1}{2}, \rho_{53} = \frac{1}{6}, \rho_{61} = \frac{1}{2}, \rho_{62} = \frac{1}{6}, \rho_{63} = \frac{1}{3}$.

According to the Theorems 3.1, 3.2, the nonlinear MASs (1) and (2) can realize the desired finite-time output containment under the designed distributed observer (3) and control protocol (15). Figs.2-6 show the results of simulation.

V. CONCLUSION

The finite-time output containment problem for nonlinear MASs with undirected topology has been studied in this paper. In order to estimate the whole states of all the leaders, we have designed a kind of adaptive distributed observer. With the help of inequality techniques and common Lyapunov theory, the finite-time convergence of proposed

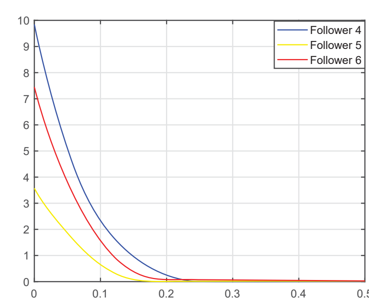

Fig. 3. Output containment errors $\|\tilde{e}_i(t)\|$, $i = 4,5,6$.

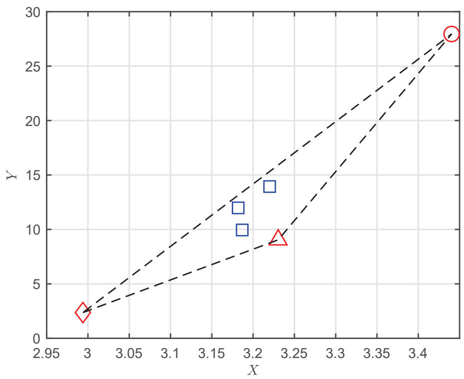

Fig. 4. Output snapshots of the MASs

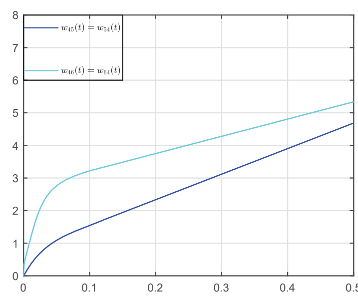

Fig. 5. $w_{ik}(t),\ (i,k) \in \mathscr{B}$.

distributed observer has been proved. In addition, we have also designed an output containment control protocol for nonlinear MASs such that the outputs of followers can converge to the expected convex hull formed by the multiple leaders in a finite time.

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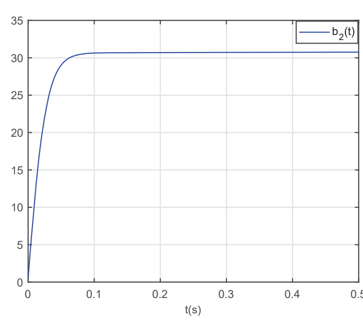

Fig. 6. $b_2(t),\ t \in [0, 0.5s]\ (b_2(0) = 0.5)$.

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