STATS 206
Applied Multivariate Analysis
Lecture 8: Canonical Correlation Analysis

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Agenda

- Population canonical variates and canonical correlations
- Sample canonical variates and canonical correlations

Introduction

- Objective:
  - to identify and quantify the association between two sets of variables
- Focus:
  - Correlation between a linear combination of variables in set 1 and that in set 2.
- Procedure:
  - first, to find the pair of linear combinations yielding the largest correlation
  - then, to find the pair of linear combinations yielding the largest correlation but uncorrelated with the first pair
  - ...
- Thus:
  - Canonical variables: the pairs of linear combinations described above
  - Canonical correlations: correlations between these pairs

Population Canonical Variates and Canonical Correlations

- Assumptions:
  - Two groups of variables: represented by two random vectors

$$\mathbf{X}^{(1)}_{p\times 1} \text{ and } \mathbf{X}^{(2)}_{q\times 1}$$

  - Without loss of generality: let $p \leq q$
  - Let

$$\mathsf{E}(\mathbf{X}^{(1)}) = \boldsymbol{\mu}^{(1)}, \quad \mathsf{Cov}(\mathbf{X}^{(1)}) = \mathbf{\Sigma}_{11}$$

$$\mathsf{E}(\mathbf{X}^{(2)}) = \boldsymbol{\mu}^{(2)}, \quad \mathsf{Cov}(\mathbf{X}^{(2)}) = \mathbf{\Sigma}_{22}$$

$$\mathsf{Cov}(\mathbf{X}^{(1)}, \mathbf{X}^{(2)}) = \mathbf{\Sigma}_{12} = \mathbf{\Sigma}'_{21}$$

  - Consider $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}$ jointly

$$\mathbf{X}_{(p+q)\times 1} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \mathbf{X}^{(2)} \end{bmatrix} \quad \mathsf{E}(\mathbf{X}) = \begin{bmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{bmatrix} \quad \mathsf{Cov}(\mathbf{X}) = \mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix}$$

Population Canonical Variates and Canonical Correlations

- Given coefficient vectors $\mathbf{a}, \mathbf{b}$, set

$$U = \mathbf{a}'\mathbf{X}^{(1)}, V = \mathbf{b}'\mathbf{X}^{(2)}$$

to obtain

$$\text{Var}(U) = \mathbf{a}'\text{Cov}(\mathbf{X}^{(1)})\mathbf{a} = \mathbf{a}'\mathbf{\Sigma}_{11}\mathbf{a}$$
$$\text{Var}(V) = \mathbf{b}'\text{Cov}(\mathbf{X}^{(2)})\mathbf{b} = \mathbf{b}'\mathbf{\Sigma}_{22}\mathbf{b}$$
$$\text{Cov}(U, V) = \mathbf{a}'\text{Cov}(\mathbf{X}^{(1)}, \mathbf{X}^{(2)})\mathbf{b} = \mathbf{a}'\mathbf{\Sigma}_{12}\mathbf{b}$$

and the correlation between U and V

$$\text{Corr}(U, V) = \frac{\mathbf{a}'\mathbf{\Sigma}_{12}\mathbf{b}}{\sqrt{\mathbf{a}'\mathbf{\Sigma}_{11}\mathbf{a}}\sqrt{\mathbf{b}'\mathbf{\Sigma}_{22}\mathbf{b}}}$$

Population Canonical Variates and Canonical Correlations

- Definitions:

1. First canonical variate pair: $U_1 = \mathbf{a}_1'\mathbf{X}^{(1)}, V_1 = \mathbf{b}_1'\mathbf{X}^{(2)}$; $\mathbf{a}_1, \mathbf{b}_1$ solves:

$$\max_{\mathbf{a}_1,\mathbf{b}_1} \quad \text{Corr}(U_1, V_1) \quad \left[\text{or} \quad \max_{\mathbf{a}_1,\mathbf{b}_1} \frac{\mathbf{a}_1'\boldsymbol{\Sigma}_{12}\mathbf{b}_1}{\sqrt{\mathbf{a}_1'\boldsymbol{\Sigma}_{11}\mathbf{a}_1}\sqrt{\mathbf{b}_1'\boldsymbol{\Sigma}_{22}\mathbf{b}_1}}\right]$$

$$\text{subject to} \quad \mathbf{a}_1'\boldsymbol{\Sigma}_{11}\mathbf{a}_1 = 1,\ \mathbf{b}_1'\boldsymbol{\Sigma}_{22}\mathbf{b}_1 = 1$$

2. Second canonical variate pair: $U_2 = \mathbf{a}_2'\mathbf{X}^{(1)}, V_2 = \mathbf{b}_2'\mathbf{X}^{(2)}$;
   $(\mathbf{a}_2, \mathbf{b}_2)$ is the solution to: (given $\mathbf{a}_1, \mathbf{b}_1$)

$$\max_{\mathbf{a}_2,\mathbf{b}_2} \quad \frac{\mathbf{a}_2'\boldsymbol{\Sigma}_{12}\mathbf{b}_2}{\sqrt{\mathbf{a}_2'\boldsymbol{\Sigma}_{11}\mathbf{a}_2}\sqrt{\mathbf{b}_2'\boldsymbol{\Sigma}_{22}\mathbf{b}_2}}$$

$$\text{subject to} \quad \mathbf{a}_2'\boldsymbol{\Sigma}_{11}\mathbf{a}_2 = 1,\ \mathbf{b}_2'\boldsymbol{\Sigma}_{22}\mathbf{b}_2 = 1$$

$$\mathbf{a}_2'\boldsymbol{\Sigma}_{11}\mathbf{a}_1 = 0,\ \mathbf{b}_2'\boldsymbol{\Sigma}_{22}\mathbf{b}_1 = 0,\ \mathbf{a}_1'\boldsymbol{\Sigma}_{12}\mathbf{b}_2 = 0,\ \mathbf{a}_2'\boldsymbol{\Sigma}_{12}\mathbf{b}_1 = 0$$

Population Canonical Variates and Canonical Correlations

- Definitions:

  3. k-th canonical variate pair: $U_k = \mathbf{a}_k'\mathbf{X}^{(1)}, V_k = \mathbf{b}_k'\mathbf{X}^{(2)}$, where U_k, V_k:

     - having unit variances
     - having the maximum correlation Corr(U_k, V_k)
     - uncorrelated with the previous $(k-1)$ canonical variable pairs

- Results: see next page

Population Canonical Variates and Canonical Correlations

- Results: (to be cont'd on next page)

Let $p \leq q$ and $\mathbf{X} = \begin{bmatrix} \mathbf{X}^{(1)}_{p\times 1} \\ \mathbf{X}^{(2)}_{q\times 1} \end{bmatrix}$ with Cov$(\mathbf{X}) = \mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix}$ (here $\mathbf{\Sigma}$ has full-rank). For coefficient vectors $\mathbf{a}, \mathbf{b}$, form linear combinations $U = \mathbf{a}'\mathbf{X}^{(1)}, V = \mathbf{b}'\mathbf{X}^{(2)}$. Then

$$\max_{\mathbf{a},\mathbf{b}} \ \text{Corr}(U, V) \ = \rho_1^*$$

is attained by the linear combinations (first canonical variate pair)

$$U_1 = \underbrace{\mathbf{e}_1'\mathbf{\Sigma}_{11}^{-1/2}}_{\mathbf{a}_1'}\mathbf{X}^{(1)}, \quad V_1 = \underbrace{\mathbf{f}_1'\mathbf{\Sigma}_{22}^{-1/2}}_{\mathbf{b}_1'}\mathbf{X}^{(2)}.$$

Population Canonical Variates and Canonical Correlations

- Results: (Cont'd and to be cont'd on next page)

The k-th canonical variate pairs, $k = 2, 3, \ldots, p$,

$$U_k = \mathbf{e}_k' \mathbf{\Sigma}_{11}^{-1/2} \mathbf{X}^{(1)}, \quad V_k = \mathbf{f}_k' \mathbf{\Sigma}_{22}^{-1/2} \mathbf{X}^{(2)}$$

maximizes

$$\text{Corr}(U_k, V_k) = \rho_k^*$$

among all linear combinations uncorrelated with $\{(U_j, V_j)\}_{j=1}^{k-1}$.

Here $\rho_1^{*2} \geq \ldots \geq \rho_p^{*2}$ are eigenvalues of $\mathbf{\Sigma}_{11}^{-1/2} \mathbf{\Sigma}_{12} \mathbf{\Sigma}_{22}^{-1} \mathbf{\Sigma}_{21} \mathbf{\Sigma}_{11}^{-1/2}$ and $\mathbf{e}_1, \ldots, \mathbf{e}_p$ are the corresponding eigenvectors.

$\rho_1^{*2} \geq \ldots \geq \rho_p^{*2}$ are also eigenvalues of $\mathbf{\Sigma}_{22}^{-1/2} \mathbf{\Sigma}_{21} \mathbf{\Sigma}_{11}^{-1} \mathbf{\Sigma}_{12} \mathbf{\Sigma}_{22}^{-1/2}$ and $\mathbf{f}_1, \ldots, \mathbf{f}_p$ are the associated eigenvectors. Each $\mathbf{f}_i$ is proportional to $\mathbf{\Sigma}_{22}^{-1/2} \mathbf{\Sigma}_{21} \mathbf{\Sigma}_{11}^{-1/2} \mathbf{e}_i$ (i.e., $\mathbf{f}_i \propto \mathbf{\Sigma}_{22}^{-1/2} \mathbf{\Sigma}_{21} \mathbf{\Sigma}_{11}^{-1/2} \mathbf{e}_i$).

Population Canonical Variates and Canonical Correlations

- Results: (Cont'd)

ρ_k^*: the canonical correlation of $(U_k, V_k), k = 1, \ldots, p$

The canonical variates have the following properties:

$$\text{Var}(U_k) = \text{Var}(V_k) = 1$$

$$\text{Cov}(U_k, U_l) = \text{Corr}(U_k, U_l) = 0, k \neq l$$

$$\text{Cov}(V_k, V_l) = \text{Corr}(V_k, V_l) = 0, k \neq l$$

$$\text{Cov}(U_k, V_l) = \text{Corr}(U_k, V_l) = 0, k \neq l$$

for $k, l = 1, \ldots, p$.

Example 1 – (1)

Calculating canonical variates/correlations for standardized variables

- Given: $\mathbf{Z}^{(1)} = \begin{bmatrix} Z_1^{(1)} \\ Z_2^{(1)} \end{bmatrix}$, $\mathbf{Z}^{(2)} = \begin{bmatrix} Z_1^{(2)} \\ Z_2^{(2)} \end{bmatrix}$: standardized; $\mathbf{Z} = \begin{bmatrix} \mathbf{Z}^{(1)} \\ \mathbf{Z}^{(2)} \end{bmatrix}$

$$\mathrm{Cov}(\mathbf{Z}) = \left[\begin{array}{c|c} \boldsymbol{\rho}_{11} & \boldsymbol{\rho}_{12} \\ \hline \boldsymbol{\rho}_{21} & \boldsymbol{\rho}_{22} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0.4 & 0.5 & 0.6 \\ 0.4 & 1 & 0.3 & 0.4 \\ \hline 0.5 & 0.3 & 1 & 0.2 \\ 0.6 & 0.4 & 0.2 & 1 \end{array}\right]$$

- Then

$$\boldsymbol{\rho}_{11}^{-1/2} = \begin{bmatrix} 1.0681 & -0.2229 \\ -0.2229 & 1.0681 \end{bmatrix}, \quad \boldsymbol{\rho}_{22}^{-1} = \begin{bmatrix} 1.0417 & -0.2083 \\ -0.2083 & 1.0417 \end{bmatrix}$$

$$\boldsymbol{\rho}_{11}^{-1/2}\boldsymbol{\rho}_{12}\boldsymbol{\rho}_{22}^{-1}\boldsymbol{\rho}_{21}\boldsymbol{\rho}_{11}^{-1/2} = \begin{bmatrix} 0.4370 & 0.2178 \\ 0.2178 & 0.1097 \end{bmatrix}$$

From the above, we can obtain $\rho_1^{*2} = 0.5457, \rho_2^{*2} = 0.0009$

Example 1 – (2)

The eigenvector $\mathbf{e}_1$ of $\boldsymbol{\rho}_{11}^{-1/2}\boldsymbol{\rho}_{12}\boldsymbol{\rho}_{22}^{-1}\boldsymbol{\rho}_{21}\boldsymbol{\rho}_{11}^{-1/2}$ associated with ρ_1^{*2} is

$$\mathbf{e}_1 = \begin{bmatrix} 0.8947 \\ 0.4468 \end{bmatrix} \Longrightarrow \mathbf{a}_1 = \boldsymbol{\rho}_{11}^{-1/2}\mathbf{e}_1 = \begin{bmatrix} 0.8560 \\ 0.2777 \end{bmatrix}$$

The eigenvector $\mathbf{f}_1$ of $\boldsymbol{\rho}_{22}^{-1/2}\boldsymbol{\rho}_{21}\boldsymbol{\rho}_{11}^{-1}\boldsymbol{\rho}_{12}\boldsymbol{\rho}_{22}^{-1/2}$ associated with ρ_1^{*2} is

$$\mathbf{f}_1 = \begin{bmatrix} 0.6161 \\ 0.7877 \end{bmatrix} \Longrightarrow \mathbf{b}_1 = \boldsymbol{\rho}_{22}^{-1/2}\mathbf{f}_1 = \begin{bmatrix} 0.5448 \\ 0.7366 \end{bmatrix}$$

– $\mathbf{f}_1$ above was obtained using the eigenvalue decomposition of $\boldsymbol{\rho}_{22}^{-1/2}\boldsymbol{\rho}_{21}\boldsymbol{\rho}_{11}^{-1}\boldsymbol{\rho}_{12}\boldsymbol{\rho}_{22}^{-1/2}$.

– Alternatively, use the fact that $\mathbf{f}_1 \propto \boldsymbol{\rho}_{22}^{-1/2}\boldsymbol{\rho}_{21}\boldsymbol{\rho}_{11}^{-1/2}\mathbf{e}_1,\ \mathbf{b}_1 = \boldsymbol{\rho}_{22}^{-1/2}\mathbf{f}_1$

$$\mathbf{b}_1 \propto \boldsymbol{\rho}_{21}\mathbf{a}_1 = \boldsymbol{\rho}_{22}^{-1}\boldsymbol{\rho}_{21}\mathbf{a}_1 = \begin{bmatrix} 0.3959 & 0.2292 \\ 0.5209 & 0.3542 \end{bmatrix} \begin{bmatrix} 0.8560 \\ 0.2777 \end{bmatrix} = \begin{bmatrix} 0.4026 \\ 0.5443 \end{bmatrix}$$

Example 1 – (3)

(Cont'd)

We must scale $\mathbf{b}_1$ such that $\text{Var}(V_1) = \text{Var}(\mathbf{b}_1'\mathbf{Z}^{(2)}) = \mathbf{b}_1'\boldsymbol{\rho}_{22}\mathbf{b}_1 = 1$

The vector $[0.4026 \;\; 0.5443]'$ yields:

$$[0.4026 \quad 0.5443] \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} 0.4026 \\ 0.5443 \end{bmatrix} = 0.5460$$

$$\Longrightarrow \mathbf{b}_1 = \frac{1}{\sqrt{0.5460}} \begin{bmatrix} 0.4026 \\ 0.5443 \end{bmatrix} = \begin{bmatrix} 0.5448 \\ 0.7366 \end{bmatrix}$$

- Therefore, the 1st pair of canonical variates is

  $U_1 = \mathbf{a}_1'\mathbf{Z}^{(1)} = 0.86Z_1^{(1)} + 0.28Z_2^{(1)}, \; V_1 = \mathbf{b}_1'\mathbf{Z}^{(2)} = 0.54Z_1^{(2)} + 0.74Z_2^{(2)}$

  with their canonical correlation being $\rho_1^* = \sqrt{\rho_1^{*2}} = \sqrt{0.5457} = 0.74$
- The 2nd canonical correlation $\rho_2^* = \sqrt{0.0009} = 0.03$ (very small, conveying little info. about the association between sets)

Sample Canonical Variates and Canonical Correlations

- In applications, sample mean and covariances are used instead.
- Random sample of n observations of each of the $(p+q)$ variables $\mathbf{X}^{(1)}_{n\times p}, \mathbf{X}^{(2)}_{n\times q}$

$$\mathbf{X}_{n\times(p+q)} = \left[\mathbf{X}^{(1)} \quad \mathbf{X}^{(2)}\right] = \begin{bmatrix} \mathbf{x}_1^{(1)'} & \mathbf{x}_1^{(2)'} \\ \vdots & \vdots \\ \mathbf{x}_n^{(1)'} & \mathbf{x}_n^{(2)'} \end{bmatrix}$$

Sample Canonical Variates and Canonical Correlations

- Given $\mathbf{X}_{n\times(p+q)}$:

$$\underbrace{\overline{\mathbf{x}}}_{\text{sample mean}} = \begin{bmatrix} \overline{\mathbf{x}}^{(1)} \\ \overline{\mathbf{x}}^{(2)} \end{bmatrix} = \begin{bmatrix} \frac{1}{n}\sum_{j=1}^{n} \mathbf{x}_j^{(1)} \\ \frac{1}{n}\sum_{j=1}^{n} \mathbf{x}_j^{(2)} \end{bmatrix}, \qquad \underbrace{\mathbf{S}}_{\substack{(p+q)\times(p+q) \\ \text{sample} \\ \text{covariance} \\ \text{matrix}}} = \begin{bmatrix} \underbrace{\mathbf{S}_{11}}_{p\times p} & \underbrace{\mathbf{S}_{12}}_{p\times q} \\ \underbrace{\mathbf{S}_{21}}_{q\times p} & \underbrace{\mathbf{S}_{22}}_{q\times q} \end{bmatrix}$$

$$\mathbf{S}_{kl} = \frac{1}{n-1}\sum_{j=1}^{n}(\mathbf{x}_j^{(k)} - \overline{\mathbf{x}}^{(k)})(\mathbf{x}_j^{(l)} - \overline{\mathbf{x}}^{(l)})', k, l = 1, 2$$

- Linear combinations $\widehat{U} = \widehat{\mathbf{a}}'\mathbf{x}^{(1)}$, $\widehat{V} = \widehat{\mathbf{b}}'\mathbf{x}^{(2)}$ have sample correlation

$$r_{\widehat{U},\widehat{V}} = \frac{\widehat{\mathbf{a}}'\mathbf{S}_{12}\widehat{\mathbf{b}}}{\sqrt{\widehat{\mathbf{a}}'\mathbf{S}_{11}\widehat{\mathbf{a}}}\sqrt{\widehat{\mathbf{b}}'\mathbf{S}_{22}\widehat{\mathbf{b}}}}$$

Sample Canonical Variates and Canonical Correlations

- The first pair of sample canonical variates is the pair of linear combinations $\widehat{U}_1, \widehat{V}_1$
  - having unit sample variances
  - and having maximum $r_{\widehat{U},\widehat{V}}$
- In general, the k-th pair of sample canonical variates is the pair of linear combinations $\widehat{U}_k, \widehat{V}_k$
  - having unit sample variances
  - having maximum $r_{\widehat{U},\widehat{V}}$
  - and uncorrelated with the previous $(k-1)$ sample canonical variates
- Results: see next page

Sample Canonical Variates and Canonical Correlations

$\widehat{\rho}_1^{*2} \geq \ldots \geq \widehat{\rho}_p^{*2}$: the p ordered eigenvalues of $\mathbf{S}_{11}^{-1/2}\mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21}\mathbf{S}_{11}^{-1/2}$
$\widehat{\mathbf{e}}_1, \ldots, \widehat{\mathbf{e}}_p$: the corresponding eigenvectors;
$\widehat{\mathbf{f}}_1, \ldots, \widehat{\mathbf{f}}_p$: the eigenvectors of $\mathbf{S}_{22}^{-1/2}\mathbf{S}_{21}\mathbf{S}_{11}^{-1}\mathbf{S}_{12}\mathbf{S}_{22}^{-1/2}$
Then the k-th sample canonical variate pair is

$$\widehat{U}_k = \underbrace{\widehat{\mathbf{e}}_k'\mathbf{S}_{11}^{-1/2}}_{\widehat{\mathbf{a}}_k'}\mathbf{x}^{(1)}, \quad \widehat{V}_k = \underbrace{\widehat{\mathbf{f}}_k'\mathbf{S}_{22}^{-1/2}}_{\widehat{\mathbf{b}}_k'}\mathbf{x}^{(2)}$$

($\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$: values of $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}$ for a particular experiment unit)

The first sample canonical variate pair has the maximum sample correlation $r_{\widehat{U}_1,\widehat{V}_1} = \widehat{\rho}_1^*$; and for the k-th pair, $r_{\widehat{U}_k,\widehat{V}_k} = \widehat{\rho}_k^*$ is the largest possible correlation among linear combinations uncorrelated with the preceding $(k-1)$ sample canonical variates.
$\widehat{\rho}_1^*, \ldots, \widehat{\rho}_p^*$: sample canonical correlations

Example 2 – (1)
Canonical Correlation Analysis of Chicken-bone Data

- The chicken-bone measurements used here:

$$\underbrace{\mathbf{X}^{(1)}}_{\text{Head}} : \begin{cases} X_1^{(1)} = \text{skull length} \\ X_2^{(1)} = \text{skull breadth} \end{cases} \qquad \underbrace{\mathbf{X}^{(2)}}_{\text{Leg}} : \begin{cases} X_1^{(2)} = \text{femur length} \\ X_2^{(2)} = \text{tibia length} \end{cases}$$

Standardizing the data to obtain $\mathbf{Z}^{(1)}$, $\mathbf{Z}^{(2)}$, and $\mathbf{Z} = [\mathbf{Z}^{(1)} | \mathbf{Z}^{(2)}]$

The sample correlation matrix $\mathbf{R}$ (in place of $\mathbf{S}$) is given by

$$\mathbf{R} = \left[\begin{array}{c|c} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \hline \mathbf{R}_{21} & \mathbf{R}_{22} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0.505 & 0.569 & 0.602 \\ 0.505 & 1 & 0.422 & 0.467 \\ \hline 0.569 & 0.422 & 1 & 0.926 \\ 0.602 & 0.467 & 0.926 & 1 \end{array}\right]$$

See the textbook, p. 552, Example 10.4.

Example 2 – (2)

- Thus, $\mathbf{R}_{11}^{-1/2}\mathbf{R}_{12}\mathbf{R}_{22}^{-1}\mathbf{R}_{21}\mathbf{R}_{11}^{-1/2} = \begin{bmatrix} 0.2844 & 0.1789 \\ 0.1789 & 0.1170 \end{bmatrix}$

  and $\widehat{\rho}_1^{*2} = 0.3893, \widehat{\rho}_2^{*2} = 0.0032$. The eigenvector $\widehat{\mathbf{e}}_1$ (associated with $\widehat{\rho}_1^{*2}$) of $\mathbf{R}_{11}^{-1/2}\mathbf{R}_{12}\mathbf{R}_{22}^{-1}\mathbf{R}_{21}\mathbf{R}_{11}^{-1/2}$ is

$$\widehat{\mathbf{e}}_1 = \begin{bmatrix} 0.8437 \\ 0.5468 \end{bmatrix} \Longrightarrow \widehat{\mathbf{a}}_1 = \mathbf{R}_{11}^{-1/2}\widehat{\mathbf{e}}_1 = \begin{bmatrix} 0.7808 \\ 0.3445 \end{bmatrix}$$

  The eigenvector $\widehat{\mathbf{f}}_1$ (associated with $\widehat{\rho}_1^{*2}$) of $\mathbf{R}_{22}^{-1/2}\mathbf{R}_{21}\mathbf{R}_{11}^{-1}\mathbf{R}_{12}\mathbf{R}_{22}^{-1/2}$ is

$$\widehat{\mathbf{f}}_1 = \begin{bmatrix} 0.5766 \\ 0.8170 \end{bmatrix} \Longrightarrow \widehat{\mathbf{b}}_1 = \mathbf{R}_{22}^{-1/2}\widehat{\mathbf{f}}_1 = \begin{bmatrix} 0.0603 \\ 0.9439 \end{bmatrix}$$

- The 1st sample canonical variate pair and $\widehat{\rho}_1^*$ are obtained:

$$\widehat{U}_1 = 0.781 z_1^{(1)} + 0.345 z_2^{(1)}, \ \widehat{V}_1 = 0.060 z_1^{(2)} + 0.944 z_2^{(2)}, \widehat{\rho}_1^* = 0.631$$

  Similarly, we can work out the 2nd pair and their sample canonical correlation.