Probability and Statistics. IDEA-UAB.
Final Exam 2017-18. Prof. J. Caballé.

The exam lasts for 3 hours

1. **(2 points: 1 point for each sub-question)** Consider a random variable $\widetilde{x}$ having the Poisson distribution with parameter λ.

   (a) Find the moment generating function $M_{\widetilde{x}}(t)$ of the random variable $\widetilde{x}$.

   (b) Use the moment generating function found in part (a) to compute the mean μ and the variance σ^2 of $\widetilde{x}$.

   Hints: (i) The probability function of the Poisson distribution is

   $$p(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \text{ with } \lambda > 0, \quad \text{for } \; x = 0, 1, 2, \ldots$$

   (ii) For part (a) you should use the Taylor expansion of e^y around $y = 0$.

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2. **(4 points: 1 point for each sub-question)** Assume that you have obtained x "successes" in n identical and independent trials.

   (a) Find the value $\hat{\theta}_{ML}$ of the maximum likelihood estimator of the probability θ of success in each trial.

   (b) Prove that the maximum likelihood estimator $\hat{\boldsymbol{\theta}}_{\mathbf{ML}}$ for θ is the most efficient (or the best) estimator in the class of unbiased estimators for θ.

   (c) Does the estimator $\hat{\boldsymbol{\theta}}_{\mathbf{ML}}$ converge in mean square to θ, $\hat{\boldsymbol{\theta}}_{\mathbf{ML}} \xrightarrow{m} \theta$ as $n \to \infty$? Does $\hat{\boldsymbol{\theta}}_{\mathbf{ML}} \xrightarrow{p} \theta$? Does $\hat{\boldsymbol{\theta}}_{\mathbf{ML}} \xrightarrow{d} \theta$? Does $\hat{\boldsymbol{\theta}}_{\mathbf{ML}} \xrightarrow{a.s.} \theta$?

   (d) Using the number of successes in 3 identical and independent trials devise a likelihood ratio test with a level of significance $\alpha = 1/4$ to test the null hypothesis that $\theta = 1/2$ against the alternative hypothesis that $\theta \neq 1/2$. You should provide the test statistic and the critical region for this test.

   Hint: For part (d) note that $\lim\limits_{x \to 0} x^x = 1$ or, equivalently, $\lim\limits_{x \to 0} (x \cdot \ln x) = 0$, which follows from L'Hôpital's rule.

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3. **(4 points: 1 point for each sub-question)** Assume that the density function of the random variable $\widetilde{x}$ is

   $$f_{\widetilde{x}}(x) = \begin{cases} \dfrac{2-x}{4} & \text{for } x \in (0,2) \\[2ex] \dfrac{x+2}{4} & \text{for } x \in (-2,0] \\[2ex] 0 & \text{otherwise.} \end{cases}$$

   (a) Find the distribution function $F_{\widetilde{x}}(x)$ of the random variable $\widetilde{x}$. Draw both $f_{\widetilde{x}}(x)$ and $F_{\widetilde{x}}(x)$.

   (b) Find and draw the distribution function $F_{\widetilde{y}}(y)$ and the density function $f_{\widetilde{y}}(y)$ of the random variable $\widetilde{y} = \widetilde{x}^2$.

   (c) Consider the random vector $(\widetilde{x}, \widetilde{y})$, where $\widetilde{x}$ has the density defined above and $\widetilde{y}$ is defined in part (b). Is the random vector $(\widetilde{x}, \widetilde{y})$ absolutely continuous, i.e., does its distribution have a density function?

   (d) Compute the covariance between $\widetilde{x}$ and $\widetilde{y}$, $\operatorname{Cov}(\widetilde{x}, \widetilde{y})$. Are $\widetilde{x}$ and $\widetilde{y}$ independent? Answer the latter question using the definition of independence of two random variables.

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Answers Final Exam 2017-18. Probability and Statistics.

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1. (a) Using the Taylor expansion at $y = 0$, we have

$$e^y = \sum_{x=0}^{\infty} \frac{y^x}{x!}. \tag{1}$$

$$M_{\tilde{x}}(t) = \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} e^{tx} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} \qquad \text{(from (1))}$$

$$= e^{\lambda\left(e^t - 1\right)}.$$

(b)

$$M'_{\tilde{x}}(t) = M_{\tilde{x}}(t) \cdot \lambda e^t \Longrightarrow \mu = M'_{\tilde{x}}(0) = \underbrace{M_{\tilde{x}}(0)}_{1} \cdot \lambda e^0 = \lambda$$

$$M''_{\tilde{x}}(t) = M'_{\tilde{x}}(t) \cdot \lambda e^t + M_{\tilde{x}}(t) \cdot \lambda e^t = M_{\tilde{x}}(t) \cdot \lambda^2 e^{2t} + M_{\tilde{x}}(t) \cdot \lambda e^t$$

$$= M_{\tilde{x}}(t) \lambda e^t \left(\lambda e^t + 1\right)$$

$$\Longrightarrow \mathrm{E}\left(\tilde{x}^2\right) = M''_{\tilde{x}}(0) = \underbrace{M_{\tilde{x}}(0)}_{1} \cdot \lambda e^0 \left(\lambda e^0 + 1\right) = \lambda\left(\lambda + 1\right) = \lambda^2 + \lambda$$

$$\Longrightarrow \sigma^2 = M''_{\tilde{x}}(0) - \left[M_{\tilde{x}}(0)\right]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

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2. (a) We want to maximize

$$L(\theta; x) = b(x; n, \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

$$\hat{\theta}_{ML} = \underset{\theta \in [0,1]}{\arg\max} \ln L(\theta; x) = \underset{\theta \in [0,1]}{\arg\max} \left\{ \ln \binom{n}{x} + x \ln \theta + (n-x) \ln(1-\theta) \right\}$$

$$\frac{d[\ln L(\theta; x)]}{d\theta} = \frac{x}{\theta} - \frac{n-x}{1-\theta} = 0 \Rightarrow \hat{\theta}_{ML} = \frac{x}{n}$$

$$\Rightarrow \hat{\boldsymbol{\theta}}_{ML} = \frac{\tilde{x}}{n}$$

Note that

$$\frac{d^2[\ln L(\theta; x)]}{d\theta^2} = -\frac{x}{\theta^2} - \frac{n-x}{(1-\theta)^2} \leq 0$$

so that we are indeed finding a maximum.

(b)

$$\mathrm{E}(\hat{\boldsymbol{\theta}}_{ML}) = \mathrm{E}\left(\frac{\tilde{x}}{n}\right) = \frac{n\theta}{n} = \theta.$$

Thus, $\hat{\boldsymbol{\theta}}_{ML}$ is an unbiased estimator for θ.

$$\mathrm{Var}(\hat{\boldsymbol{\theta}}_{ML}) = \mathrm{Var}\left(\frac{\tilde{x}}{n}\right) = \frac{n\theta(1-\theta)}{n^2} = \frac{\theta(1-\theta)}{n}.$$

$$\text{Cramer-Rao lower bound} \equiv CR = \frac{1}{-1\mathrm{E}\left[\dfrac{\partial^2 \ln b(\tilde{x}; n, \theta)}{\partial\theta^2}\right]}$$

$$\frac{\partial^2 \ln b(\tilde{x}; n, \theta)}{\partial\theta^2} = -\frac{\tilde{x}}{\theta^2} - \frac{n-\tilde{x}}{(1-\theta)^2}$$

$$\mathrm{E}\left[\frac{\partial^2 \ln b(\tilde{x}; n, \theta)}{\partial\theta^2}\right] = -\frac{n\theta}{\theta^2} - \frac{n-n\theta}{(1-\theta)^2} = -\frac{n}{\theta(1-\theta)}$$

$$CR = \frac{1}{-\left(-\dfrac{n}{\theta(1-\theta)}\right)} = \frac{\theta(1-\theta)}{n} = \mathrm{Var}(\hat{\boldsymbol{\theta}}_{ML}).$$

Therefore, $\hat{\boldsymbol{\theta}}_{ML}$ is the best estimator in the class of unbiased estimators for θ.

Alternatively,

$$CR = \frac{1}{1 \cdot \mathrm{E}\left[\left(\dfrac{\partial \ln b(\tilde{x}; n, \theta)}{\partial\theta}\right)^2\right]}$$

$$\frac{\partial \ln b(\tilde{x}; n, \theta)}{\partial\theta} = \frac{x}{\theta} - \frac{n-x}{1-\theta}$$

$$\left(\frac{\partial \ln b(\tilde{x}; n, \theta)}{\partial\theta}\right)^2 = \left(\frac{\tilde{x}}{\theta} - \frac{n-\tilde{x}}{1-\theta}\right)^2 = \frac{(\tilde{x}-n\theta)^2}{\theta^2(1-\theta)^2}$$

$$\mathrm{E}\left[\left(\frac{\partial \ln b(\tilde{x}; n, \theta)}{\partial\theta}\right)^2\right] = \mathrm{E}\left[\frac{(\tilde{x}-n\theta)^2}{\theta^2(1-\theta)^2}\right] = \frac{\mathrm{E}\left[(\tilde{x}-n\theta)^2\right]}{\theta^2(1-\theta)^2}$$

$$\frac{\mathrm{E}\left[(\tilde{x}-\mathrm{E}(\tilde{x}))^2\right]}{\theta^2(1-\theta)^2} = \frac{\mathrm{Var}(\tilde{x})}{\theta^2(1-\theta)^2} = \frac{n\theta(1-\theta)}{\theta^2(1-\theta)^2} = \frac{n}{\theta(1-\theta)}.$$

Therefore,

$$CR = \frac{1}{1 \cdot \mathrm{E}\left[\left(\frac{\partial \ln b(\widetilde{x}; n, \theta)}{\partial \theta}\right)^2\right]} = \frac{\theta(1-\theta)}{n} = \mathrm{Var}(\hat{\boldsymbol{\theta}}_{ML}).$$

(c)

$$\mathrm{E}\left[\left(\hat{\boldsymbol{\theta}}_{ML} - \theta\right)^2\right] = \mathrm{E}\left[\left(\hat{\boldsymbol{\theta}}_{ML} - \mathrm{E}\left(\hat{\boldsymbol{\theta}}_{ML}\right)\right)^2\right] = \mathrm{Var}\left(\hat{\boldsymbol{\theta}}_{ML}\right) = \mathrm{Var}\left(\frac{\widetilde{x}}{n}\right) = \frac{\theta(1-\theta)}{n}$$

$$\Rightarrow \lim_{n\to\infty} \mathrm{E}\left[\left(\hat{\boldsymbol{\theta}}_{ML} - \theta\right)^2\right] = \lim_{n\to\infty} \frac{\theta(1-\theta)}{n} = 0 \Longrightarrow \hat{\boldsymbol{\theta}}_{ML} \xrightarrow{m} \theta.$$

$$\Longrightarrow \hat{\boldsymbol{\theta}}_{ML} \xrightarrow{p} \theta \Longrightarrow \hat{\boldsymbol{\theta}}_{ML} \xrightarrow{d} \theta.$$

Moreover, since $\hat{\boldsymbol{\theta}}_{ML}$ is equal to the "average" $\bar{\mathbf{y}}_{\mathbf{n}}$ of a random sample $(\widetilde{y}_1, \widetilde{y}_2, ..., \widetilde{y}_n)$ coming from a Bernoulli population,

$$\hat{\boldsymbol{\theta}}_{ML} = \frac{\widetilde{x}_i}{n} = \frac{\sum_{i=0}^{n} \widetilde{y}_i}{n} = \overline{\boldsymbol{y}}_n,$$

as $\widetilde{x}_i = \sum_{i=0}^{n} \widetilde{y}_i \sim \mathrm{B}\left(n, \theta\right)$, we can apply any of the two Kolmogorov theorems to conclude that $\hat{\boldsymbol{\theta}}_{ML} = \overline{\boldsymbol{y}}_n \xrightarrow{a.s.} \theta$.

(d) Likekhood function under the null hypothesis:

$$L\left(\frac{1}{2}; x\right) = \binom{3}{x}\left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{3-x} = \binom{3}{x}\left(\frac{1}{2}\right)^3.$$

$$\arg\sup_{\theta\in[0,1]} L\left(\theta; x\right) = \arg\max_{\theta\in[0,1]} \binom{3}{x} \theta^x \left(1-\theta\right)^{3-x} = \hat{\theta}_{ML} = \frac{x}{3}$$

$$\sup_{\theta\in[0,1]} L\left(\theta; x\right) = \binom{3}{x}\left(\frac{x}{3}\right)^x \left(\frac{3-x}{3}\right)^{3-x}$$

Thus, the test statistic is

$$\widetilde{\lambda} = \frac{L\left(\frac{1}{2}; \widetilde{x}\right)}{\sup_{\theta\in[0,1]} L\left(\theta; \widetilde{x}\right)} = \frac{\binom{3}{\widetilde{x}}\left(\frac{1}{2}\right)^3}{\binom{3}{\widetilde{x}}\left(\frac{\widetilde{x}}{3}\right)^{\widetilde{x}}\left(\frac{3-\widetilde{x}}{3}\right)^{3-\widetilde{x}}} = \frac{\left(\frac{1}{2}\right)^3}{\left(\frac{\widetilde{x}}{3}\right)^{\widetilde{x}}\left(\frac{3-\widetilde{x}}{3}\right)^{3-\widetilde{x}}}$$

$$= \frac{\frac{1}{8}}{\frac{\widetilde{x}^{\widetilde{x}}\left(3-\widetilde{x}\right)^{3-\widetilde{x}}}{3^3}} = \frac{\frac{27}{8}}{\widetilde{x}^{\widetilde{x}}\left(3-\widetilde{x}\right)^{3-\widetilde{x}}}.$$

The critical region is defined by a constant $k > 0$ such that we reject the null hypothesis if

$$\widetilde{\lambda} = \frac{\frac{27}{8}}{\widetilde{x}^{\widetilde{x}}\left(3-\widetilde{x}\right)^{3-\widetilde{x}}} \leq k$$

If $\widetilde{x} = 0$, then $\widetilde{\lambda} = \lim_{x\to 0} \frac{27/8}{\widetilde{x}^{\widetilde{x}}\left(3-\widetilde{x}\right)^{3-\widetilde{x}}} = \frac{27/8}{1 \cdot 3^3} = 1/8$. Under the null hypothesis this occurs with probability $\binom{3}{0}\left(\frac{1}{2}\right)^3 = 1/8$.

If $\widetilde{x} = 1$, then $\widetilde{\lambda} = \frac{27/8}{\widetilde{x}^{\widetilde{x}}\left(3-\widetilde{x}\right)^{3-\widetilde{x}}} = 27/32$. Under the null hypothesis this occurs with probability $\binom{3}{1}\left(\frac{1}{2}\right)^3 = 3/8$.

If $\widetilde{x} = 2$, then $\widetilde{\lambda} = \frac{27/8}{\widetilde{x}^{\widetilde{x}}\left(3-\widetilde{x}\right)^{3-\widetilde{x}}} = 27/32$. Under the null hypothesis this occurs with probability $\binom{3}{1}\left(\frac{1}{2}\right)^3 = 3/8$.

If $\widetilde{x} = 3$, then $\widetilde{\lambda} = \lim_{x\to 3} \frac{27/8}{\widetilde{x}^{\widetilde{x}}\left(3-\widetilde{x}\right)^{3-\widetilde{x}}} = \frac{27/8}{3^3 \cdot 1} = 1/8$. Under the null hypothesis this occurs with probability $\binom{3}{0}\left(\frac{1}{2}\right)^3 = 1/8$.

Note that

$$\lim_{x\to 0}\left(x \cdot \ln x\right) = \lim_{x\to 0}\left(\frac{\ln x}{\frac{1}{x}}\right) = \frac{\lim_{x\to 0}\left(\frac{1}{x}\right)}{\lim_{x\to 0}\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0}\left(-\frac{x^2}{x}\right) = \lim_{x\to 0}\left(-x\right) = 0.$$

Thus,

$$\lim_{x\to 0} x^x = \lim_{x\to 0} \exp\left(x \cdot \ln x\right) = \exp\left[\lim_{x\to 0}\left(x \cdot \ln x\right)\right] = e^0 = 1.$$

Therefore, the probability function of $\widetilde{\lambda}$ under the null hypothesis $\theta = \frac{1}{2}$ is $f_{\widetilde{\lambda}}\left(\frac{27}{32}\right) = \frac{3}{8} + \frac{3}{8} = \frac{3}{4}$ and $f_{\widetilde{\lambda}}\left(\frac{1}{8}\right) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.

If we want that the level of significance be 1/4, then

$$P\left(\widetilde{\lambda} \leq k; \frac{1}{2}\right) = \frac{1}{4}.$$

Therefore, we must choose for the threshold value k any real number lying in the left semiclosed interval $\left[\frac{1}{8}, \frac{27}{32}\right)$. Thus, we reject the null hypothesis when

$\widetilde{\lambda} = 1/8$, that is, when we get 0 or 3 successes in 3 trials.

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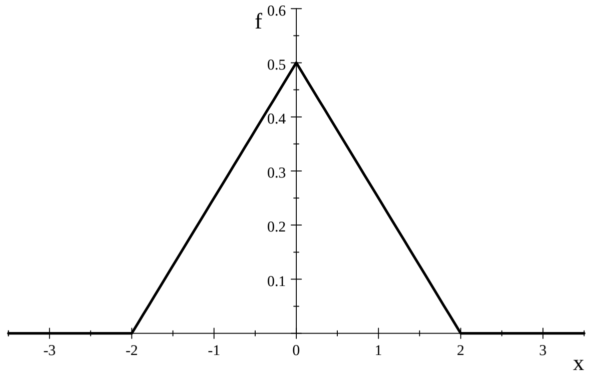
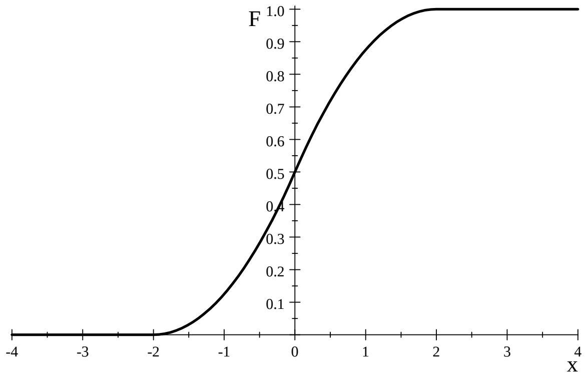
3. (a) $f_{\widetilde{x}}(x)$



$$F_{\widetilde{x}}(x) = \begin{cases} 0 & \text{for } x \leq -2 \\ \int_{-2}^{x} \frac{x+2}{4} dx = \frac{x^2}{8} + \frac{x}{2} + \frac{1}{2} & \text{for } x \in (-2, 0] \\ \int_{-2}^{0} \frac{x+2}{4} dx + \int_{0}^{x} \frac{2-x}{4} dx = \frac{1}{2} + \int_{0}^{x} \frac{2-x}{4} dx = -\frac{x^2}{8} + \frac{x}{2} + \frac{1}{2} & \text{for } x \in (0, 2) \\ 1 & \text{for } x \geq 2. \end{cases}$$



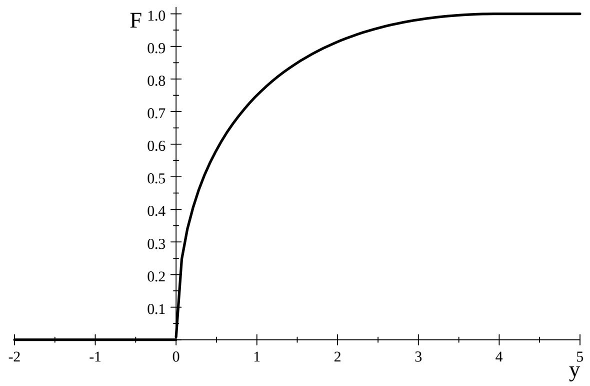
(b) Since

$$F_{\widetilde{y}}(y) = P\left(\widetilde{y} \leq y\right) = P\left(\widetilde{x}^2 \leq y\right) = P\left(-y^{1/2} \leq \widetilde{x} \leq y^{1/2}\right)$$

$$= \int_{-y^{1/2}}^{0} \frac{x+2}{4} dx + \int_{0}^{y^{1/2}} \frac{2-x}{4} dx = y^{1/2} - \frac{y}{4} \quad \text{for } x \in (-2, 2) \Longleftrightarrow y \in (0, 4)\,.$$

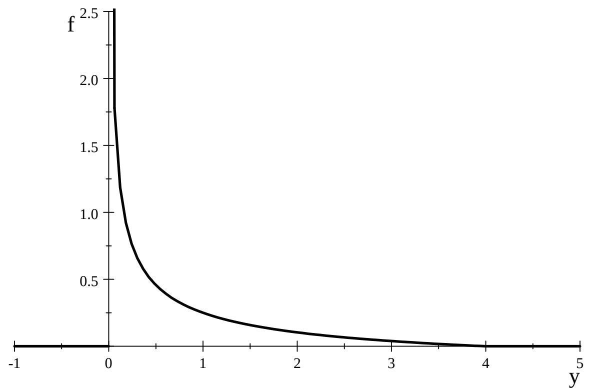
Therefore,

$$F_{\widetilde{y}}(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ y^{1/2} - \frac{y}{4} & \text{for } y \in (0, 4) \\ 1 & \text{for } y \geq 4. \end{cases}$$



We know that $f_{\widetilde{y}}(y) = F'_{\widetilde{y}}(y)$ for all y where the distribution function $F_{\widetilde{y}}$ is differentiable. Thus,

$$f_{\widetilde{y}}(y) = \begin{cases} \dfrac{y^{-1/2}}{2} - \dfrac{1}{4} & \text{for } y \in (0,4) \\ \\ 0 & \text{otherwise} \end{cases}$$



Note that the density $f_{\widetilde{y}}$ is continuous at $y = 4$ and, hence, the distribution

function $F_{\widetilde{y}}$ is differentiable at $y = 4$. However, the density $f_{\widetilde{y}}$ is discontinuous at $y = 0$ and thus the distribution function $F_{\widetilde{y}}$ is non-differentiable at $y = 0$. We see that

$$\lim_{y \to 0^-} f_{\widetilde{y}}(y) = 0 \neq \infty = \lim_{y \to 0^+} f_{\widetilde{y}}(y).$$

Note: This exercise could be also solved using the density function technique instead of the distribution function technique. To this end we first need to make the transformation $\widetilde{z} = |\widetilde{x}|$ and find the distribution function $F_{\widetilde{z}}$ of $\widetilde{z}$. Note that

$$F_{\widetilde{z}}(z) = P\left(\widetilde{z} \leq z\right) = P\left(|\widetilde{x}| \leq z\right) = P\left(-z \leq \widetilde{x} \leq z\right)$$

$$= \int_{-z}^{0} \frac{x+2}{4} dx + \int_{0}^{z} \frac{2-x}{4} dx = z - \frac{z^2}{4} \text{ for } x \in (-2,2) \Longleftrightarrow z \in (0,2)\,.$$

Therefore,

$$F_{\widetilde{z}}(z) = \begin{cases} 0 & \text{for } z \leq 0 \\ \\ z - \dfrac{z^2}{4} & \text{for } z \in (0,2) \\ \\ 1 & \text{for } z \geq 2. \end{cases}$$

Therefore, the density of $\widetilde{z}$ is

$$f_{\widetilde{z}}(z) = \begin{cases} F'_{\widetilde{z}}(z) = 1 - \dfrac{z}{2} & \text{for } z \in (0,2) \\ \\ 0 & \text{otherwise} \end{cases}$$

Then, define $\widetilde{y} = g(\widetilde{z}) = \widetilde{z}^2$. Note that $g : (0,2) \longrightarrow (0,4)$ is a one-to-one correspondence. Thus, $z = g^{-1}(y) = y^{1/2}$ and $\dfrac{dz}{dy} = \dfrac{dg^{-1}(y)}{dy} = \dfrac{y^{-1/2}}{2} > 0$ for

$y \in (0,4)$. Therefore, the density of $\widetilde{y}$ is

$$f_{\widetilde{y}}(y) = \begin{cases} f_{\widetilde{z}}(g^{-1}(y)) \left| \dfrac{g^{-1}(y)}{dy} \right| = \left(1 - \dfrac{y^{1/2}}{2}\right) \dfrac{y^{-1/2}}{2} = \dfrac{y^{-1/2}}{2} - \dfrac{1}{4} & \text{for } y \in (0,4) \\ 0 & \text{otherwise.} \end{cases}$$

and, thus, the distribution function is

$$F_{\widetilde{y}}(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ \displaystyle\int_0^y \left(\frac{y^{-1/2}}{2} - \frac{1}{4} \right) dy = y^{1/2} - \frac{y}{4} & \text{for } y \in (0,4) \\ 1 & \text{for } y \geq 4. \end{cases}$$

(c) The distribution of the random vector $(\widetilde{x}, \widetilde{y})$ does NOT have a density $f_{(\widetilde{x},\widetilde{y})}(x,y)$. To see this, let us define the following subset C of $\mathbb{R}^2$:

$$C = \left\{ (x,y) \in \mathbb{R}^2 \middle| \, y = x^2 \right\}$$

The set C is the graph of a parabola on the plane. Then, on the one hand,

$$P\left((x,y) \in C\right) = 1.$$

but, on the other hand, if the density $f_{(\widetilde{x},\widetilde{y})}(x,y)$ exists, we should have

$$\int_C f_{(\widetilde{x},\widetilde{y})}(x,y) d(x,y) = 1,$$

which is a contradiction since the set C has zero Lebesgue measure on $\mathbb{R}^2$.

(d)

$$\mathrm{E}(\widetilde{x}) = \int_{-2}^0 x \cdot \frac{x+2}{4} dx + \int_0^2 x \cdot \frac{2-x}{4} dx = 0,$$

$$\mathrm{E}(\widetilde{y}) = \mathrm{E}\left(\widetilde{x}^2\right) = \int_{-2}^0 x^2 \cdot \frac{x+2}{4} dx + \int_0^2 x^2 \cdot \frac{2-x}{4} dx = \frac{2}{3}$$

or

$$\mathrm{E}(\widetilde{y}) = \int_0^4 y \left(\frac{y^{-1/2}}{2} - \frac{1}{4} \right) dy = \frac{2}{3},$$

$$\mathrm{E}(\widetilde{x} \cdot \widetilde{y}) = \mathrm{E}\left(\widetilde{x}^3\right) = \int_{-2}^0 x^3 \cdot \frac{x+2}{4} dx + \int_0^2 x^3 \cdot \frac{2-x}{4} dx = 0$$

$$\mathrm{Cov}(\widetilde{x}, \widetilde{y}) = \mathrm{E}(\widetilde{x} \cdot \widetilde{y}) - \mathrm{E}(\widetilde{x}) \cdot \mathrm{E}(\widetilde{y}) = 0 - 0 \cdot \frac{2}{3} = 0.$$

However, even if $\widetilde{x}$ and $\widetilde{y}$ are uncorrelated, they are not independent. It is obvious that the values taken by $\widetilde{y}$ depend on the values taken by $\widetilde{x}$. According to the definition of independence between two random variables, if $\widetilde{x}$ and $\widetilde{y}$ are independent we should have that

$$P(\widetilde{x} \in B_1, \widetilde{y} \in B_2) = P(\widetilde{x} \in B_1) \cdot P(\widetilde{y} \in B_2)$$

for all pairs B_1 and B_2 of Borel sets. Then, we can check that the random variables $\widetilde{x}$ and $\widetilde{y}$ are not independent. For instance,

$$P(\widetilde{x} \in (-1,0), \widetilde{y} \in (2,4)) = 0$$

as $\{\widetilde{x} \in (-1,0)\} \cap \{\widetilde{y} \in (2,4)\} = \emptyset$ because $\widetilde{x} \in (-1,0) \Rightarrow \widetilde{y} \in (0,1)$.

Moreover,

$$P(\widetilde{x} \in (-1,0)) = \int_{-1}^0 \frac{x+2}{4} dx = \frac{3}{8}$$

$$P(\widetilde{y} \in (2,4)) = \int_2^4 \left(\frac{y^{-1/2}}{2} - \frac{1}{4} \right) dy = \frac{3}{2} - \sqrt{2},$$

Therefore,

$$P(\widetilde{x} \in (-1,0), \widetilde{y} \in (2,4)) \neq P(\widetilde{x} \in (-1,0)) \cdot P(\widetilde{y} \in (2,4)).$$

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