$\boxed{\text{EXAM ON NEXT PAGE.}}$

Do not turn the page before solving the exam and practice in an exam-like environment. You should check the time when doing the exercise.

DISCLAIMER

- The topics may be different in the exam.
- The length and number of the exercises may be different.
- The main purpose is to provide extra practice.

PART 1 (50 points). To be handed in after 70 minutes.

1. Consider a binary relation R on a set X. Let $x \in X$ be a least upper bound of a set $A \subset X$.
   (a) (9 points) Provide an example in which x is a maximal element of A.
   (b) (9 points) Provide an example in which x is not a best element of A.
2. (14 points) Show that g increasing and f quasiconcave implies $g \circ f$ quasiconcave.
3. Consider a linear space X, with a basis $B = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.
   (a) (9 points) Given this basis, show that the set $Y = \{\mathbf{y} \in X : \alpha\mathbf{a} + \beta\mathbf{b}$, where $\alpha, \beta \in \mathbb{R}\}$ is a linear subspace of X and determine its dimension.
   (b) (9 points) Find an example in which $\mathbf{a}$, $\mathbf{b}$ belong to the set of functionals on the interval $[0, 1]$, $\mathbf{a}, \mathbf{b} \in \{f | f : [0, 1] \to \mathbb{R}\}$ such that Y, as we defined part (a), is a linear space.

PART 2 (50 points). To be handed in after 140 minutes. Summary sheet allowed after 70 minutes.

4. Consider a sequence (x^n) on the natural numbers $\mathbb{N}$ with the following property: For every number $m \in \{1, 2, 3, 4, 5\}$, there exists an infinite subsequence (x^{n_i}) which converges to m, i.e., $(x^{n_i}) \to m$.
   (a) (4 points) Is this sequence convergent?
   (b) (4 points) Find an example for such a sequence?
5. (a) (6 points) Provide an example of a continuous functional $f : [0, 1] \to \mathbb{R}$ which is a contraction.
   (b) (6 points) Show that any increasing and continuous functional $f : [0, 1] \to \mathbb{R}$ which satisfies $f'(x) \leq k < 1$ for all $x \in [0, 1]$ is a contraction.
6. Consider the correspondence $\phi : \mathbb{R} \rightrightarrows \mathbb{R}$ with

$$\phi(x) = \begin{cases} \{\ln x\} & \text{if } x > 0, \\ [0, 1] & \text{otherwise.} \end{cases}$$

   (a) (7 points) Show that ϕ has a closed graph.
   (b) (7 points) Show that ϕ is not upper hemicontinuous.
7. (16 points) Use the KKT method to find the (global) maximum of $f(x, y) = x + y$ subject to $x^2 + y^2 \leq 1$ and $(x + 1)^2 + y \leq 1$. Justify why the point you obtain is the global maximum. Does this function have a global minimum over the same set?