■ **EXAMPLE 5.1**

Suppose we have a noise-free Newtonian system[2] with position r, velocity v, and constant acceleration a. The system can be described as

$$\begin{aligned} \begin{bmatrix} \dot{r} \\ \dot{v} \\ \dot{a} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ v \\ a \end{bmatrix} \\ \dot{x} &= Ax \end{aligned} \tag{5.28}$$

The discretized version of this system (with a sample time of T) can be written as

$$x_{k+1} = Fx_k \tag{5.29}$$

where F is given as

$$\begin{aligned} F &= \exp(AT) \\ &= I + AT + \frac{(AT)^2}{2!} + \cdots \\ &= \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{5.30}$$

The Kalman filter for this system is

$$\begin{aligned} \hat{x}_k^- &= F\hat{x}_{k-1}^+ \\ P_k^- &= FP_{k-1}^+F^T + \underbrace{Q_{k-1}}_{0} \\ &= FP_{k-1}^+F^T \end{aligned} \tag{5.31}$$

We see that the covariance of the estimation error increases between time $(k-1)^+$ [that is, time $(k-1)$ after the measurement at that time is processed], and time k^- (i.e., time k before the measurement at that time is processed). Since we do not obtain any measurements between time $(k-1)^+$ and time k^-, it makes sense that our estimation uncertainty increases. Now suppose that we measure position with a variance of σ^2:

$$\begin{aligned} y_k &= H_k x_k + v_k \\ &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x_k + v_k \\ v_k &\sim (0, R_k) \\ R_k &= \sigma^2 \end{aligned} \tag{5.32}$$

The Kalman gain can be obtained from Equation (5.19) as

$$K_k = P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1} \tag{5.33}$$

If we write out the 3×3 matrix P_k^- in terms of its individual elements, and substitute for H_k and R_k in the above equation, we obtain

[2]The system described in this example is called Newtonian because it has its roots in the mathematical work of Isaac Newton. That is, velocity is the derivative of position, and acceleration is the derivative of velocity.

$$K_k = \begin{bmatrix} P_{k,11}^- \\ P_{k,12}^- \\ P_{k,13}^- \end{bmatrix} \frac{1}{P_{k,11}^- + \sigma^2} \tag{5.34}$$

The *a posteriori* covariance can be obtained from Equation (5.19) as

$$P_k^+ = P_k^- - K_k H_k P_k^- \tag{5.35}$$

If we write out the 3×3 matrix P_k^- in terms of its individual elements, and substitute for H_k and K_k in the above equation, we obtain

$$\begin{aligned} P_k^+ &= P_k^- - \frac{1}{P_{k,11}^- + \sigma^2} \begin{bmatrix} P_{k,11}^- & 0 & 0 \\ P_{k,12}^- & 0 & 0 \\ P_{k,13}^- & 0 & 0 \end{bmatrix} P_k^- \\ &= P_k^- - \frac{1}{P_{k,11}^- + \sigma^2} \begin{bmatrix} (P_{k,11}^-)^2 & P_{k,11}^- P_{k,21}^- & P_{k,11}^- P_{k,31}^- \\ P_{k,12}^- P_{k,11}^- & (P_{k,12}^-)^2 & P_{k,12}^- P_{k,31}^- \\ P_{k,13}^- P_{k,11}^- & P_{k,13}^- P_{k,12}^- & (P_{k,13}^-)^2 \end{bmatrix} \end{aligned} \tag{5.36}$$

We will use this expression to show that from time k^- to time k^+ the trace of the estimation-error covariance decreases. To see this first note that the trace of P_k^- is given as

$$\text{Tr}(P_k^-) = P_{k,11}^- + P_{k,22}^- + P_{k,33}^- \tag{5.37}$$

From Equation (5.36) we see that the trace of P_k^+ is given as

$$\begin{aligned} \text{Tr}(P_k^+) &= P_{k,11}^+ + P_{k,22}^+ + P_{k,33}^+ \\ &= \left(P_{k,11}^- - \frac{(P_{k,11}^-)^2}{P_{k,11}^- + \sigma^2} \right) + \left(P_{k,22}^- - \frac{(P_{k,12}^-)^2}{P_{k,11}^- + \sigma^2} \right) + \\ &\quad \left(P_{k,33}^- - \frac{(P_{k,13}^-)^2}{P_{k,11}^- + \sigma^2} \right) \\ &= \text{Tr}(P_k^-) - \frac{(P_{k,11}^-)^2 + (P_{k,12}^-)^2 + (P_{k,13}^-)^2}{P_{k,11}^- + \sigma^2} \end{aligned} \tag{5.38}$$

When we get a new measurement, we expect our state estimate to improve. That is, we expect the covariance to decrease, and the above equation shows that it does indeed decrease. That is, the trace of P_k^+ is less than the trace of P_k^-.

This system was simulated with five time units between discretization steps ($T = 5$), and a position-measurement standard deviation of 30 units. Figure 5.3 shows the variance of the position estimate ($P_{k,11}^-$ and $P_{k,11}^+$) for the first five time steps of the Kalman filter. It can be seen that the variance (uncertainty) increases from one time step to the next, but then decreases at each time step as the measurement is processed.

Figure 5.4 shows the variance of the position estimate ($P_{k,11}^-$ and $P_{k,11}^+$) for the first 60 time steps of the Kalman filter. This shows that the variance increases between time steps, and then decreases at each time step. But it

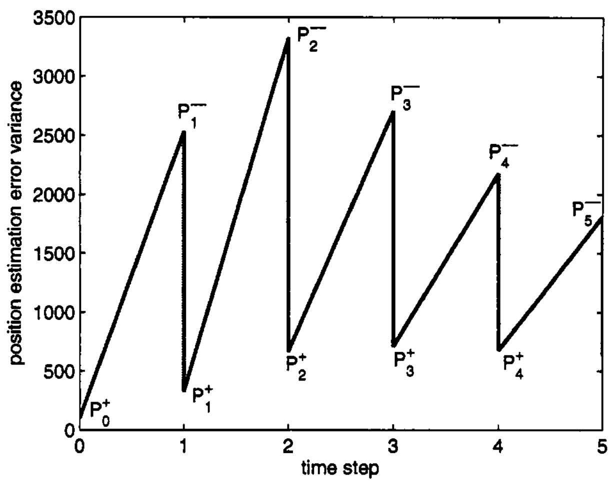

Figure 5.3 The first five time steps of the *a priori* and *a posteriori* position-estimation-error variances for Example 5.1.

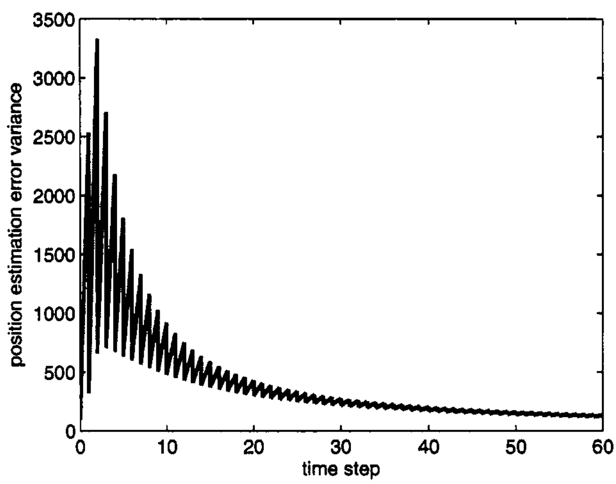

Figure 5.4 The first 60 time steps of the *a priori* and *a posteriori* position-estimation-error variances for Example 5.1.

can also be seen from this figure that the variance converges to a steady-state value.

Figure 5.5 shows the position-measurement error (with a standard deviation of 30) and the error of the *a posteriori* position estimate. The estimation error starts out with a standard deviation close to 30, but by the end of the simulation the standard deviation is about 11.

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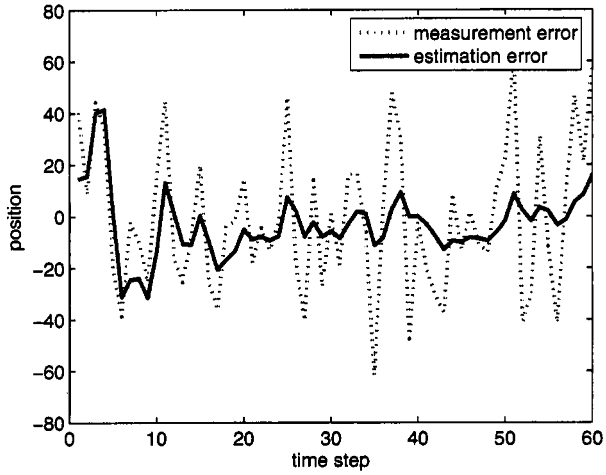

Figure 5.5 The position-measurement error and position estimation error for Example 5.1.

5.4 ALTERNATE PROPAGATION OF COVARIANCE

In this section, we derive an alternate equation for the propagation of the estimation-error covariance P. This alternate equation, based on [Gre01], can be used to find a closed-form equation for a scalar Kalman filter.[3] It can also be used to find a fast solution to the steady-state estimation-error covariance.

5.4.1 Multiple state systems

Recall from Equation (5.19) the update equations for the estimation-error covariance:

$$\begin{aligned} P_k^- &= F_{k-1}P_{k-1}^+F_{k-1}^T + Q_{k-1} \\ P_k^+ &= P_k^- - P_k^-H_k^T(H_kP_k^-H_k^T + R_k)^{-1}H_kP_k^- \end{aligned} \tag{5.39}$$

If the $n \times n$ matrix P_k^- can be factored as

$$P_k^- = A_kB_k^{-1} \tag{5.40}$$

where A_k and B_k are $n \times n$ matrices to be determined, then P_{k+1}^- satisfies

$$P_{k+1}^- = A_{k+1}B_{k+1}^{-1} \tag{5.41}$$

where A and B are propagated as follows:

$$\begin{bmatrix} A_{k+1} \\ B_{k+1} \end{bmatrix} = \begin{bmatrix} (F_k + Q_kF_k^{-T}H_k^TR_k^{-1}H_k) & Q_kF_k^{-T} \\ F_k^{-T}H_k^TR_k^{-1}H_k & F_k^{-T} \end{bmatrix} \begin{bmatrix} A_k \\ B_k \end{bmatrix} \tag{5.42}$$

[3]The equations given in [Gre01] have some typographical errors that have been corrected in this section.