Formula Derivation and Code Analysis of S-MSCKF

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S-MSCKF 论文公式推导与代码解析

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目录

1 概述

1.1 MSCKF

MSCKF 全称 Multi-State Constraint Kalman Filter（多状态约束下的 Kalman 滤波器），是一种基于滤波的 VIO 算法，2007 年由明尼苏达州大学 Mourikis 在 [3] 中首次提出。MSCKF 在 EKF 框架下融合 IMU 和视觉信息，相较于单纯的 VO 算法，MSCKF 能够适应更剧烈的运动、一定时间的纹理缺失等，具有更高的鲁棒性；相较于基于优化的 VIO 算法（VINS，OKVIS），MSCKF 精度相当，速度更快，适合在计算资源有限的嵌入式平台运行。在机器人、无人机、AR/VR 领域，MSCKF 都有较为广泛的运用，如 Google Project Tango 就用了 MSCKF 进行位姿估计。

1.2 MSCKF vs EKF-SLAM

在传统的 EKF-SLAM 框架中，特征点的信息会加入到特征向量和协方差矩阵里, 这种方法的缺点是特征点的信息会给一个初始深度和初始协方差，如果不正确的话，极容易导致后面不收敛，出现 inconsistent 的情况。MSCKF 维护一个 pose 的 FIFO，按照时间顺序排列，可以称为滑动窗口，一个特征点在滑动窗口的几个位姿都被观察到的话，就会在这几个位姿间建立约束，从而进行 KF 的更新。 [4]

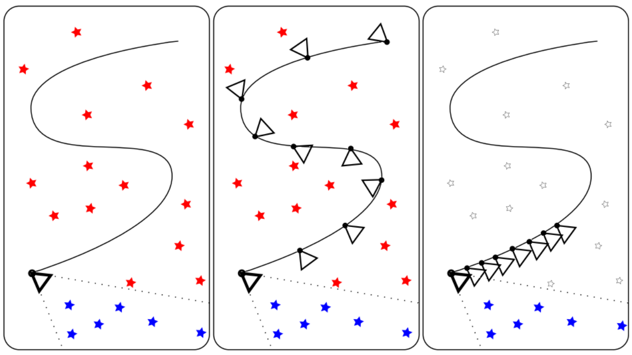
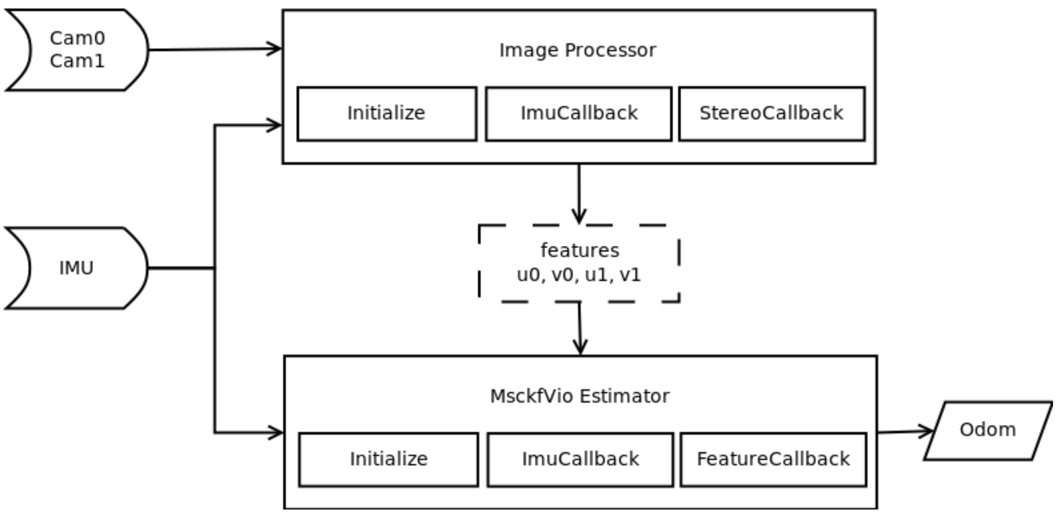


图 1 EKF-SLAM, keyframe-based SLAM, MSCKF

1.3 S-MSCKF

S-MSCKF [5] 是宾夕法尼亚大学 Vijay Kumar 实验室开源的双目版本 MSCKF 算法。



2 Image Processor

2.1 Initialize

- load parameters
- create `FastFeatureDetector`

2.2 ImuCallback

第一帧图像后，不断添加 IMU message 到 `imu_msg_buffer`。

2.3 StereoCallback

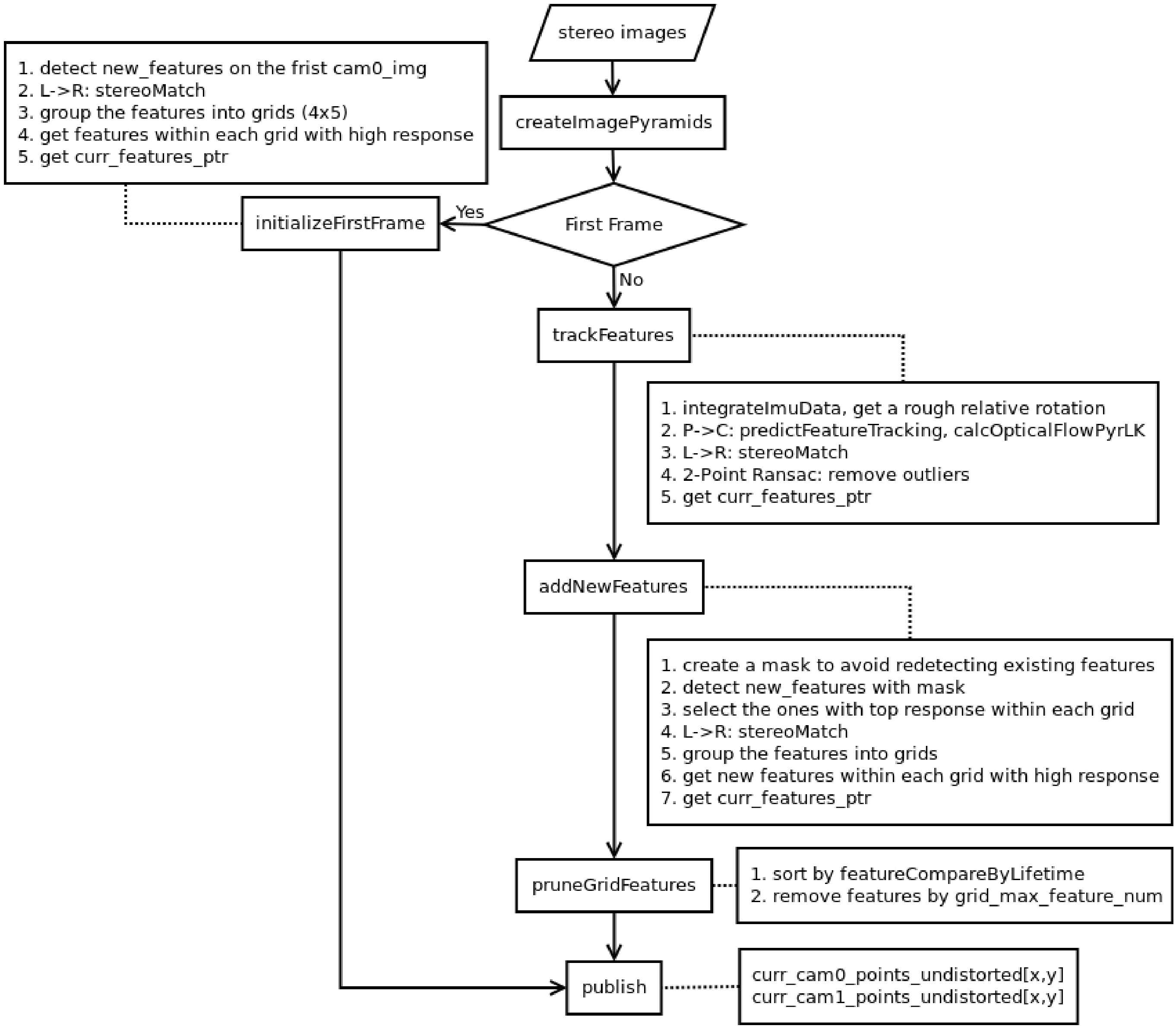


图 2 StereoCallback 流程图

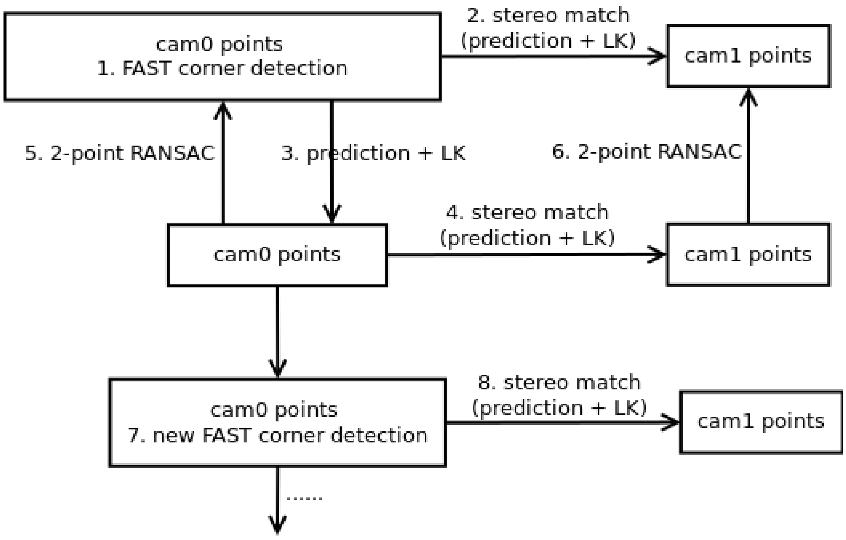


图 3 StereoCallback Circle Match

2.3.1 Feature Tracking (时间)

1. Integrate IMU(Gyroscope) Data
   (a) 利用 IMU 陀螺仪的数据，计算两帧图像之间的平均角速度 $\bar{\boldsymbol{\omega}}_m^I$
   (b) 将其转换到相机坐标系下 $\bar{\boldsymbol{\omega}}_m^C = \mathbf{R}_{CI}\bar{\boldsymbol{\omega}}_m^I$
   (c) 积分得到旋转向量 $\Delta\boldsymbol{\phi} = \bar{\boldsymbol{\omega}}_m^C \cdot \Delta t$
   (d) 利用罗德里格斯公式，得到旋转矩阵 $\Delta\mathbf{R} = \text{Rodrigues}(\Delta\boldsymbol{\phi})$
2. Predict via IMU
$$\mathbf{p}' = \mathbf{K} \cdot \Delta\mathbf{R} \cdot \mathbf{K}^{-1} \cdot \mathbf{p}$$
3. Feature Tracking by LK Optical Flow

2.3.2 Stereo Match (空间)

- curr cam0 points $\longleftrightarrow$ curr cam1 points

2.3.3 2-Point RANSAC

- prev cam0 points $\longleftrightarrow$ curr cam0 points
- prev cam1 points $\longleftrightarrow$ curr cam1 points

3 MsckfVio Estimator/Filter

3.1 Overview

3.1.1 Kalman Filter

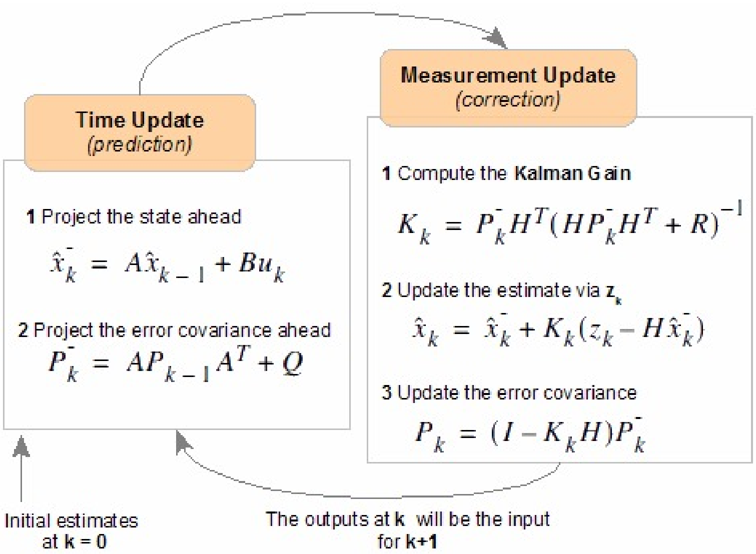


图 4 Kalman Filter 流程图

- 高频预测，低频修正

3.1.2 Multi-State Constraint Filter

Algorithm 1 Multi-State Constraint Filter

Propagation: For each IMU measurement received, propagate the filter state and covariance (cf. Section III-B).

Image registration: Every time a new image is recorded,
- augment the state and covariance matrix with a copy of the current camera pose estimate (cf. Section III-C).
- image processing module begins operation.

Update: When the feature measurements of a given image become available, perform an EKF update (cf. Sections III-D and III-E).

图 5 Multi-State Constraint Filter 流程图

3.2 代码流程

3.2.1 Initialize

- Load Parameters
- Initialize state server: `state_server.continuous_noise_cov`

- Initialize the **chi squared test table** with confidence level **P=0.95** for `MsckfVio::gatingTest()`

```
for (int i = 1; i < 100; ++i) {
    boost::math::chi_squared chi_squared_dist(i);
    chi_squared_test_table[i] = boost::math::quantile(chi_squared_dist, 0.05);
}
```

Degrees of freedom (df)	χ^2 value[19]										
1	0.004	0.02	0.06	0.15	0.46	1.07	1.64	2.71	3.84	6.63	10.83
2	0.10	0.21	0.45	0.71	1.39	2.41	3.22	4.61	5.99	9.21	13.82
3	0.35	0.58	1.01	1.42	2.37	3.66	4.64	6.25	7.81	11.34	16.27
4	0.71	1.06	1.65	2.20	3.36	4.88	5.99	7.78	9.49	13.28	18.47
5	1.14	1.61	2.34	3.00	4.35	6.06	7.29	9.24	11.07	15.09	20.52
6	1.63	2.20	3.07	3.83	5.35	7.23	8.56	10.64	12.59	16.81	22.46
7	2.17	2.83	3.82	4.67	6.35	8.38	9.80	12.02	14.07	18.48	24.32
8	2.73	3.49	4.59	5.53	7.34	9.52	11.03	13.36	15.51	20.09	26.12
9	3.32	4.17	5.38	6.39	8.34	10.66	12.24	14.68	16.92	21.67	27.88
10	3.94	4.87	6.18	7.27	9.34	11.78	13.44	15.99	18.31	23.21	29.59
P value (Probability)	0.95	0.90	0.80	0.70	0.50	0.30	0.20	0.10	0.05	0.01	0.001

图 6 chi-square table

3.2.2 ImuCallback

- 添加 IMU message 到 `imu_msg_buffer`
- `initializeGravityAndBias`，要求前 200 帧 IMU 静止不动
  - 将前 200 帧加速度和角速度求平均
  - 平均角速度作为陀螺仪的 bias：`state_server.imu_state.gyro_bias`
  - 平均加速度作为 IMU 系下的重力加速度：`gravity_imu`
  - 平均加速度的模值作为重力加速度模长 g：`IMUState::gravity`
  - ENU 坐标系下，$\mathbf{g} = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & -9.81007 \end{bmatrix}^T$
  - 计算初始时刻 World 系（水平天向）重力向量 **g** 和 IMU 系重力向量`gravity_imu` 之间的姿态 (旋转四元数)：`state_server.imu_state.orientation`

3.2.3 FeatureCallback

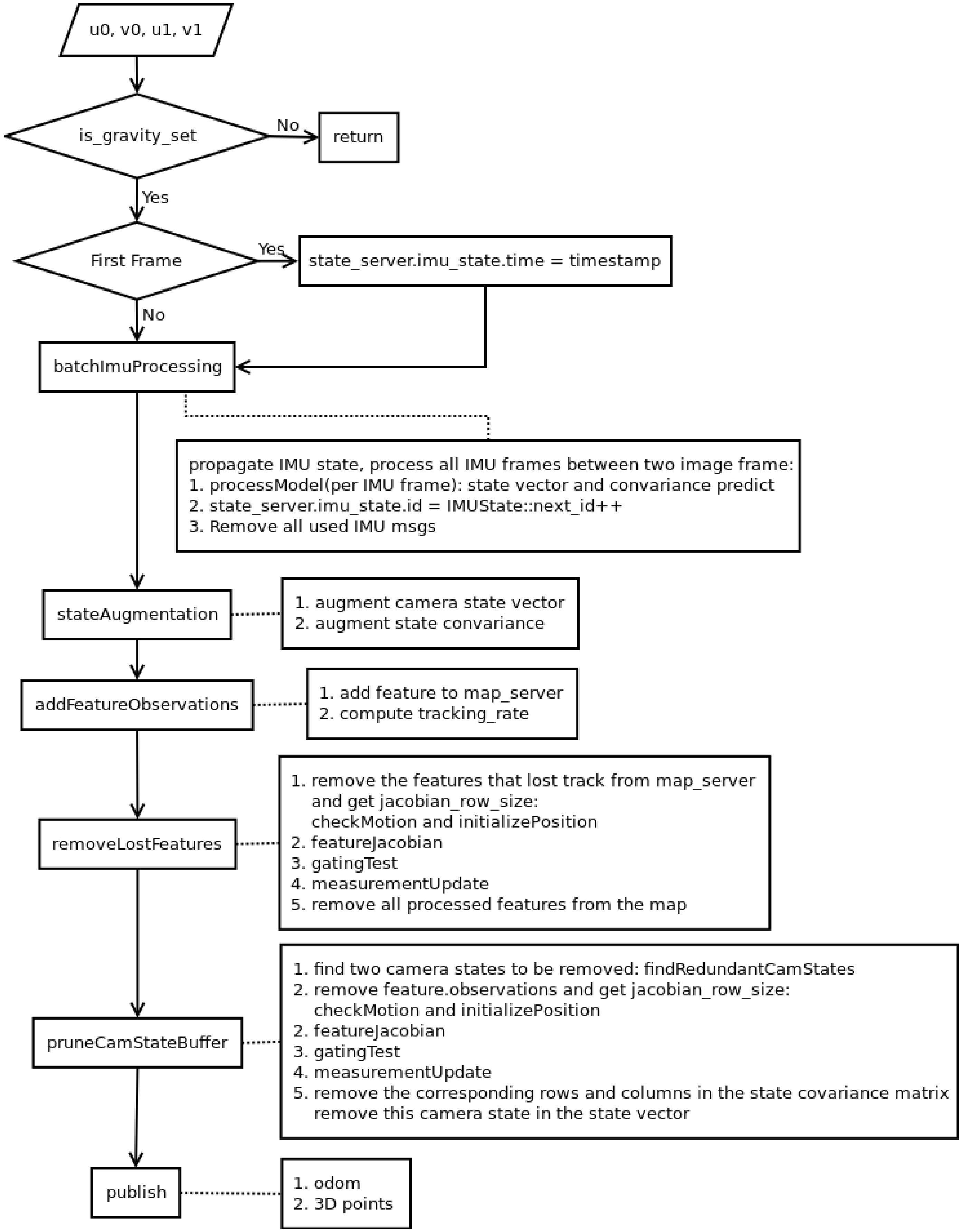


图 7 FeatureCallback 流程图

3.3 状态表示

IMU 状态向量（true-state）

$$\mathbf{x}_I = \left({}_G^I\mathbf{q}^\top \quad \mathbf{b}_g^\top \quad {}^G\mathbf{v}_I^\top \quad \mathbf{b}_a^\top \quad {}^G\mathbf{p}_I^\top \quad {}_C^I\mathbf{q}^\top \quad {}^I\mathbf{p}_C^\top\right)^\top \in \mathbb{R}^{22\times 1}$$

IMU 误差状态向量

$$\tilde{\mathbf{x}}_I = \left({}_G^I\tilde{\boldsymbol{\theta}}^\top \quad \tilde{\mathbf{b}}_g^\top \quad {}^G\tilde{\mathbf{v}}_I^\top \quad \tilde{\mathbf{b}}_a^\top \quad {}^G\tilde{\mathbf{p}}_I^\top \quad {}_C^I\tilde{\boldsymbol{\theta}}^\top \quad {}^I\tilde{\mathbf{p}}_C^\top\right)^\top \in \mathbb{R}^{21\times 1}$$

IMU 噪声向量

$$\mathbf{n}_I^\top = \left(\mathbf{n}_g^\top \ \mathbf{n}_{wg}^\top \ \mathbf{n}_a^\top \ \mathbf{n}_{wa}^\top\right)^\top \in \mathbb{R}^{12\times 1}$$

其对应的 (连续时间) 噪声协方差矩阵

$$\mathbf{Q}_I = \mathbb{E}\left[\mathbf{n}_I\mathbf{n}_I^\top\right] = \operatorname{diag}\left(\sigma_g^2\mathbf{I}_{3\times 3} \quad \sigma_{bg}^2\mathbf{I}_{3\times 3} \quad \sigma_a^2\mathbf{I}_{3\times 3} \quad \sigma_{ba}^2\mathbf{I}_{3\times 3}\right) \in \mathbb{R}^{12\times 12}$$

Camera 误差状态向量

$$\tilde{\mathbf{x}}_{C_i} = \left({}_G^{C_i}\tilde{\boldsymbol{\theta}}^\top \quad {}^G\tilde{\mathbf{p}}_{C_i}^\top\right)^\top \in \mathbb{R}^{6\times 1}$$

系统 (Imu+Camera) 误差状态向量

$$\tilde{\mathbf{x}} = \left(\tilde{\mathbf{x}}_I^\top \quad \tilde{\mathbf{x}}_{C_1}^\top \quad \cdots \quad \tilde{\mathbf{x}}_{C_N}^\top\right)^\top \in \mathbb{R}^{(21+6N)\times 1}$$

3.4 Propagation/Prediction

3.4.1 IMU true-state kinematics

$$\begin{aligned}
\mathbf{a}_m &= \mathbf{R}_t^\top\left(\mathbf{a}_t - \mathbf{g}_t\right) + \mathbf{a}_{bt} + \mathbf{a}_n \\
\boldsymbol{\omega}_m &= \boldsymbol{\omega}_t + \boldsymbol{\omega}_{bt} + \boldsymbol{\omega}_n
\end{aligned}$$

where in ENU

$$\mathbf{g}_t = \begin{bmatrix}0 & 0 & -9.81007\end{bmatrix}^T$$

3.4.2 IMU nominal-state kinematics

IMU 运动微分方程 (nominal-state 状态方程) [5]

$$\dot{\hat{\mathbf{x}}} = \begin{cases}
{}_G^I\dot{\hat{\mathbf{q}}} = \frac{1}{2}\Omega(\hat{\boldsymbol{\omega}})\,{}_G^I\hat{\mathbf{q}} \\
\dot{\hat{\mathbf{b}}}_g = \mathbf{0}_{3\times 1} \\
{}^G\dot{\hat{\mathbf{v}}} = C\left({}_G^I\hat{\mathbf{q}}\right)^\top\hat{\mathbf{a}} + {}^G\mathbf{g} \\
\dot{\hat{b}}_a = \mathbf{0}_{3\times 1}, \\
{}^G\dot{\hat{\mathbf{p}}}_I = {}^G\hat{\mathbf{v}} \\
{}_C^I\dot{\hat{\mathbf{q}}} = \mathbf{0}_{3\times 1} \\
{}^I\dot{\hat{\mathbf{p}}}_C = \mathbf{0}_{3\times 1}
\end{cases} \tag{1}$$

其中，

$$\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}_m - \hat{\mathbf{b}}_g, \quad \hat{\mathbf{a}} = \mathbf{a}_m - \hat{\mathbf{b}}_a$$

并且（使用 JPL 四元数）

$$\Omega(\hat{\boldsymbol{\omega}}) \triangleq \begin{bmatrix} \hat{\boldsymbol{\omega}} \\ 0 \end{bmatrix}_R = \begin{pmatrix} -[\hat{\boldsymbol{\omega}}_\times] & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^\top & 0 \end{pmatrix}$$

3.4.3 IMU error-state kinematics

根据 [2] ESKF 中 ***5.3.3 The error-state kinematics*** 小节公式，IMU error-state 状态方程

$$\dot{\tilde{\mathbf{x}}}_I = \begin{cases} \delta\dot{\boldsymbol{\theta}} = -[\boldsymbol{\omega}_m - \boldsymbol{\omega}_b]_\times \delta\boldsymbol{\theta} - \delta\boldsymbol{\omega}_b - \boldsymbol{\omega}_n \\ \delta\dot{\boldsymbol{\omega}}_b = \boldsymbol{\omega}_w \\ \delta\dot{\mathbf{v}} = -\mathbf{R}[\mathbf{a}_m - \mathbf{a}_b]_\times \delta\boldsymbol{\theta} - \mathbf{R}\delta\mathbf{a}_b + \delta\mathbf{g} - \mathbf{R}\mathbf{a}_n \\ \delta\dot{\mathbf{a}}_b = \mathbf{a}_w \\ \delta\dot{\mathbf{p}} = \delta\mathbf{v} \\ {}^I_C\delta\dot{\boldsymbol{\theta}} = \mathbf{0}_{3\times1} \\ {}^I\delta\dot{\mathbf{p}}_C = \mathbf{0}_{3\times1} \end{cases} \tag{2}$$

对式 (2)线性化，得到 **IMU(连续时间) 误差状态方程**

$$\dot{\tilde{\mathbf{x}}}_I = \mathbf{F}\tilde{\mathbf{x}}_I + \mathbf{G}\mathbf{n}_I \tag{3}$$

其中，
$\tilde{\mathbf{x}}_I$ 和 $\mathbf{n}_I$ 为

$$\tilde{\mathbf{x}}_I = \begin{pmatrix} {}^I_G\tilde{\boldsymbol{\theta}}^\top & \tilde{\mathbf{b}}_g^\top & {}^G\tilde{\mathbf{v}}_I^\top & \tilde{\mathbf{b}}_a^\top & {}^G\tilde{\mathbf{p}}_I^\top & {}^I_C\tilde{\boldsymbol{\theta}}^\top & {}^I\tilde{\mathbf{p}}_C^\top \end{pmatrix}^\top \in \mathbb{R}^{21\times1}$$

$$\mathbf{n}_I^\top = \begin{pmatrix} \mathbf{n}_g^\top & \mathbf{n}_{wg}^\top & \mathbf{n}_a^\top & \mathbf{n}_{wa}^\top \end{pmatrix}^\top \in \mathbb{R}^{12\times1}$$

F 和 **G** 为（代码在 `MsckfVio::processModel`，文献 [5] 附录中 **G** 计算有误，代码正确）

$$\mathbf{F}_{21\times21} = \begin{pmatrix} -\lfloor\hat{\boldsymbol{\omega}}_\times\rfloor & -\mathbf{I}_3 & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ -C\left({}^I_G\hat{\mathbf{q}}\right)^\top \lfloor\hat{\mathbf{a}}_\times\rfloor & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & -C\left({}^I_G\hat{\mathbf{q}}\right)^\top & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{I}_3 & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{pmatrix}$$

$$\mathbf{G}_{21\times12} = \begin{pmatrix} -\mathbf{I}_3 & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{I}_3 & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & -C\left({}^I_G\hat{\mathbf{q}}\right)^\top & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{I}_3 \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{pmatrix}$$

3.4.4 IMU 状态向量预测

代码在 `MsckfVio::predictNewState`。

- 姿态预测（[6] 中 122 式 0 阶四元数积分，代码据 $|\omega|$ 是否接近 0 分两种情况，接近 0 ??）

$$ {}_G^I\mathbf{q}(t_{k+1}) = \left(\cos\left(\frac{|\omega|}{2}\Delta t\right)\cdot \mathbf{I}_{4\times 4} + \frac{1}{|\omega|}\sin\left(\frac{|\omega|}{2}\Delta t\right)\cdot \boldsymbol{\Omega}(\omega)\right){}_G^I\mathbf{q}(t_k) \tag{4} $$

- 位置和速度预测（4 阶 Runge-Kutta 积分）

$$ y_{n+1} = y_n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{5} $$

3.4.5 系统状态协方差预测

代码在 `MsckfVio::processModel`，处理两帧图像间的每一帧 IMU 数据。
(离散时间) 状态转移矩阵

$$ \begin{aligned} \boldsymbol{\Phi}_k &= \boldsymbol{\Phi}(t_{k+1}, t_k) \\ &= \exp\left(\int_{t_k}^{t_{k+1}} \mathbf{F}(\tau)d\tau\right) \\ &= \exp(\mathbf{F}\Delta t) \\ &\approx \mathbf{I} + \mathbf{F}\Delta t + \frac{1}{2}(\mathbf{F}\Delta t)^2 + \frac{1}{6}(\mathbf{F}\Delta t)^3 \quad (\Delta t\text{比较小时，一般} <0.01\text{s}) \end{aligned} \tag{6} $$

why: Modify the transition matrix to make the observability matrix have proper null space
(离散时间) 噪声协方差矩阵

$$ \begin{aligned} \mathbf{Q}_k &= \int_{t_k}^{t_{k+1}} \boldsymbol{\Phi}(t_{k+1}, \tau)\mathbf{G}\mathbf{Q}_I\mathbf{G}\boldsymbol{\Phi}(t_{k+1}, \tau)^\top d\tau \\ &\approx \boldsymbol{\Phi}_k\mathbf{G}\mathbf{Q}_I\mathbf{G}^T{\boldsymbol{\Phi}_k}^T\Delta t \end{aligned} \tag{7} $$

IMU 状态传播协方差矩阵为

$$ \mathbf{P}_{II_{k+1|k}} = \boldsymbol{\Phi}_k\mathbf{P}_{II_{k|k}}\boldsymbol{\Phi}_k^\top + \mathbf{Q}_k \in \mathbb{R}^{21\times 21} \tag{8} $$

系统状态传播协方差矩阵拆解表示为

$$ \mathbf{P}_{k|k} = \begin{pmatrix} \mathbf{P}_{II_{k|k}} & \mathbf{P}_{IC_{k|k}} \\ \mathbf{P}_{IC_{k|k}}^\top & \mathbf{P}_{CC_{k|k}} \end{pmatrix} \tag{9} $$

其传播形式表示为

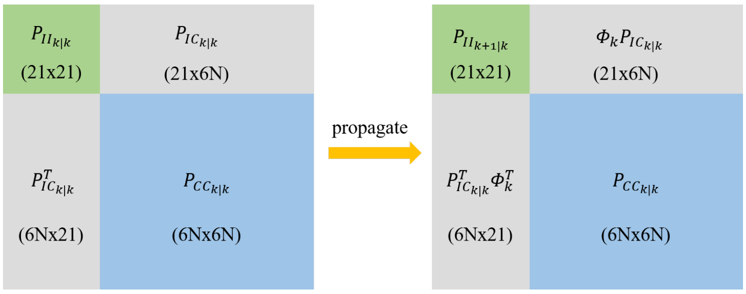
$$ \mathbf{P}_{k+1|k} = \begin{pmatrix} \mathbf{P}_{II_{k+1|k}} & \boldsymbol{\Phi}_k\mathbf{P}_{IC_{k|k}} \\ \mathbf{P}_{IC_{k|k}}^\top\boldsymbol{\Phi}_k^\top & \mathbf{P}_{CC_{k|k}} \end{pmatrix} \in \mathbb{R}^{(21+6N)\times(21+6N)} \tag{10} $$

Fix the covariance to be symmetric

$$ \mathbf{P}_{k+1|k} = \frac{\mathbf{P}_{k+1|k} + \mathbf{P}_{k+1|k}^\top}{2} \tag{11} $$

其中 Camera 的 covariance 暂时还没有变化是因为这个时间段图像还没有到来，只有 IMU 的影响，但是会影响到 IMU 与 Camera 协方差。

整个状态 (IMU+Camera) 的 covariance 传播过程如图所示 [7]：



3.5 State Augmentation

代码在 `MsckfVio::stateAugmentation`。

3.5.1 相机状态向量扩增 (随机克隆)

根据 IMU 预测位姿求取 Camera 位姿

$$
{}^C_G\hat{\mathbf{q}} = {}^C_I\hat{\mathbf{q}} \otimes {}^I_G\hat{\mathbf{q}}, \quad {}^G\hat{\mathbf{p}}_C = {}^G\hat{\mathbf{p}}_I + C\left({}^I_G\hat{\mathbf{q}}\right)^\top {}^I\hat{\mathbf{p}}_C \tag{12}
$$

3.5.2 状态协方差扩增

根据 B.3

$$
y = Ax, \quad \Sigma_y = E(yy^T) = E(Axx^T A^T) = AE(xx^T)A^T = A\Sigma_x A^T
$$

增广的状态协方差矩阵

$$
\mathbf{P}'_{k|k} = \begin{pmatrix} \mathbf{I}_{21+6N} \\ \mathbf{J} \end{pmatrix} \mathbf{P}_{k|k} \begin{pmatrix} \mathbf{I}_{21+6N} \\ \mathbf{J} \end{pmatrix}^\top = \begin{bmatrix} \mathbf{P}_{k|k} & (\mathbf{J}\mathbf{P}_{k|k})^T \\ \mathbf{J}\mathbf{P}_{k|k} & \mathbf{J}\mathbf{P}_{k|k}\mathbf{J}^T \end{bmatrix} \in \mathbb{R}^{(21+6N+6)\times(21+6N+6)} \tag{13}
$$

其中（下式参考 [5]，而代码参考 [3]，哪一个正确），

$$
\mathbf{J} = \frac{\partial \tilde{\mathbf{x}}_{C_i}}{\partial \tilde{\mathbf{x}}} = \frac{\partial \left({}^{C_i}_G\tilde{\boldsymbol{\theta}}^\top \quad {}^G\tilde{\mathbf{p}}_{C_i}^\top\right)^\top}{\partial \left(\tilde{\mathbf{x}}_I^\top \quad \tilde{\mathbf{x}}_{C_1}^\top \quad \cdots \quad \tilde{\mathbf{x}}_{C_N}^\top\right)^\top} = \begin{pmatrix} \mathbf{J}_I & \mathbf{0}_{6\times 6N} \end{pmatrix} \in \mathbb{R}^{6\times(21+6N)}
$$

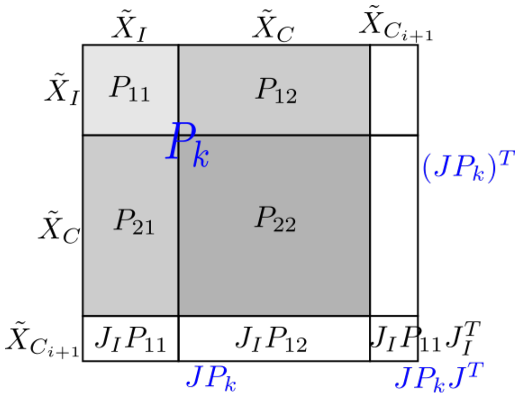
$$
\mathbf{J}_I = \frac{\partial \tilde{\mathbf{x}}_{C_i}}{\partial \tilde{\mathbf{x}}_I} = \begin{pmatrix} C\left({}^I_G\hat{\mathbf{q}}\right) & \mathbf{0}_{3\times 9} & \mathbf{0}_{3\times 3} & \mathbf{I}_3 & \mathbf{0}_{3\times 3} \\ -C\left({}^I_G\hat{\mathbf{q}}\right)^\top \lfloor {}^I\hat{\mathbf{p}}_{C\times} \rfloor & \mathbf{0}_{3\times 9} & \mathbf{I}_3 & \mathbf{0}_{3\times 3} & \mathbf{I}_3 \end{pmatrix} \in \mathbb{R}^{6\times 21}
$$

为表示方便，拆解 $\mathbf{P}_{k|k}$ 为

$$
\mathbf{P}_{k|k} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \tag{14}
$$

则

$$
\mathbf{P}'_{k|k} = \begin{bmatrix} \mathbf{P}_{k|k} & \mathbf{J}_P^T \\ \mathbf{J}_P & \mathbf{J}_I\mathbf{P}_{11}\mathbf{J}_I^T \end{bmatrix} \quad \text{where} \quad \mathbf{J}_P = \begin{pmatrix} \mathbf{J}_I\mathbf{P}_{11} & \mathbf{J}_I\mathbf{P}_{12} \end{pmatrix} \tag{15}
$$



Fix the covariance to be symmetric

$$\mathbf{P}'_{k|k} = \frac{\mathbf{P}'_{k|k} + \mathbf{P}'_{k|k}{}^{\top}}{2} \tag{16}$$

3.6 Measurement Update

3.6.1 Measurement Model

1. 计算获取世界坐标系下的 3D 特征点，代码在 `Feature::initializePosition`
   - 据对极几何原理，三角化计算初始深度，得到初始三维点坐标
   - 通过 L-M 算法迭代优化得到更加精确的世界系三维点
2. 投影世界系三维点到相机系

$$\begin{aligned}
{}^{C_{i,1}}\mathbf{p}_j &= \begin{pmatrix} {}^{C_{i,1}}X_j \\ {}^{C_{i,1}}Y_j \\ {}^{C_{i,1}}Z_j \end{pmatrix} = C\left({}^{C_{i,1}}_{G}\mathbf{q}\right)\left({}^{G}\mathbf{p}_j - {}^{G}\mathbf{p}_{C_{i,1}}\right) \\
{}^{C_{i,2}}\mathbf{p}_j &= \begin{pmatrix} {}^{C_{i,2}}X_j \\ {}^{C_{i,2}}Y_j \\ {}^{C_{i,2}}Z_j \end{pmatrix} = C\left({}^{C_{i,2}}_{G}\mathbf{q}\right)\left({}^{G}\mathbf{p}_j - {}^{G}\mathbf{p}_{C_{i,2}}\right) \\
&= C\left({}^{C_{i,2}}_{C_{i,1}}\mathbf{q}\right)\left({}^{C_{i,1}}\mathbf{p}_j - {}^{C_{i,1}}\mathbf{p}_{C_{i,2}}\right)
\end{aligned}$$

3. 线性化测量模型，视觉测量残差可近似表示为

$$\mathbf{r}_i^j = \mathbf{z}_i^j - \hat{\mathbf{z}}_i^j = \mathbf{H}_{C_i}^j \tilde{\mathbf{x}}_{C_i} + \mathbf{H}_{f_i}^j \, {}^{G}\tilde{\mathbf{p}}_j + \mathbf{n}_i^j \in \mathbb{R}^{4\times 1} \tag{17}$$

   其中，测量雅克比矩阵（代码在 `MsckfVio::measurementJacobian`）

$$\begin{aligned}
\mathbf{H}_{C_i}^j &= \frac{\partial \mathbf{z}_i^j}{\partial {}^{C_{i,1}}\mathbf{p}_j} \cdot \frac{\partial {}^{C_{i,1}}\mathbf{p}_j}{\partial \mathbf{x}_{C_{i,1}}} + \frac{\partial \mathbf{z}_i^j}{\partial {}^{C_{i,2}}\mathbf{p}_j} \cdot \frac{\partial {}^{C_{i,2}}\mathbf{p}_j}{\partial \mathbf{x}_{C_{i,1}}} \in \mathbb{R}^{4\times 6} \\
\mathbf{H}_{f_i}^j &= \frac{\partial \mathbf{z}_i^j}{\partial {}^{C_{i,1}}\mathbf{p}_j} \cdot \frac{\partial {}^{C_{i,1}}\mathbf{p}_j}{\partial {}^{G}\mathbf{p}_j} + \frac{\partial \mathbf{z}_i^j}{\partial {}^{C_{i,2}}\mathbf{p}_j} \cdot \frac{\partial {}^{C_{i,2}}\mathbf{p}_j}{\partial {}^{G}\mathbf{p}_j} \in \mathbb{R}^{4\times 3}
\end{aligned} \tag{18}$$

   why: Modifty the measurement Jacobian to ensure observability constrain, Ref: OC-VINS

4. 叠加对同一特征点的多个观测（代码在 `MsckfVio::featureJacobian`）

$$\mathbf{r}^j = \mathbf{H}_{\mathbf{x}}^j \tilde{\mathbf{x}} + \mathbf{H}_f^j {}^G\tilde{\mathbf{p}}_j + \mathbf{n}^j \in \mathbb{R}^{4M} \quad \text{with} \quad \mathbf{H}_{\mathbf{x}}^j \in \mathbb{R}^{4M\times 6}, \mathbf{H}_f^j \in \mathbb{R}^{4M\times 3}$$

为避免 ${}^G\mathbf{p}_j$ 的不确定性对测量残差的影响，将残差投影到 $\mathbf{H}_{f_i}^j$ 的左零空间 $\mathbf{V} \in \mathbb{R}^{4M\times(4M-3)}$

$$\mathbf{r}_o^j = \mathbf{V}^\top \mathbf{r}^j = \mathbf{V}^\top \mathbf{H}_{\mathbf{x}}^j \tilde{\mathbf{x}} + \mathbf{V}^\top \mathbf{n}^j = \mathbf{H}_{\mathbf{x},o}^j \tilde{\mathbf{x}} + \mathbf{n}_o^j \in \mathbb{R}^{(4M-3)\times 1} \quad \text{with} \quad \mathbf{H}_{\mathbf{x},o}^j \in \mathbb{R}^{(4M-3)\times(21+6n)} \tag{19}$$

零空间 $\mathbf{V}$ 通过 SVD 分解 (或 QR 分解) 得到（代码在 `MsckfVio::featureJacobian`）

$$\mathbf{H}_f^j = UDV^T$$

```
// Project the residual and Jacobians onto the nullspace of H_fj.
JacobiSVD<MatrixXd> svd_helper(H_fj, ComputeFullU | ComputeThinV);
MatrixXd A = svd_helper.matrixU().rightCols(jacobian_row_size - 3);
H_x = A.transpose() * H_xj;
r   = A.transpose() * r_j;
```

5. Mahalanobis gating test (Probability = 0.95)

$$\gamma_j = \mathbf{r}_o^{j\,T} (\mathbf{H}_{\mathbf{x},o}^j \mathbf{P} \mathbf{H}_{\mathbf{x},o}^{j\;T} + \sigma^2 \mathbf{I})^{-1} \mathbf{r}_o^j$$

```
if (gatingTest(H_xj, r_j, cam_state_ids.size()-1)) {
  H_x.block(stack_cntr, 0, H_xj.rows(), H_xj.cols()) = H_xj;
  r.segment(stack_cntr, r_j.rows()) = r_j;
  stack_cntr += H_xj.rows();
}
```

6. 叠加所有特征点的多个观测，得到

$$\mathbf{r}_o = \mathbf{H}_{\mathbf{X}} \widetilde{\mathbf{X}} + \mathbf{n}_o \in \mathbb{R}^{N(4M-3)\times 1} \quad \text{with} \quad \mathbf{H}_{\mathbf{X}} \in \mathbb{R}^{N(4M-3)\times(21+6n)} \tag{20}$$

3.6.2 能观性约束 (OC)

OC-EKF

3.6.3 ESKF Update

代码在 `MsckfVio::measurementUpdate`。
根据 [3]，为降低 EKF 更新的计算复杂度，对 $\mathbf{H}_{\mathbf{X}}$ 进行 QR 分解

$$\mathbf{H}_{\mathbf{X}} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} \mathbf{T}_H \\ \mathbf{0} \end{bmatrix} \tag{21}$$

带入式 (20)，得

$$\mathbf{r}_o = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} \mathbf{T}_H \\ \mathbf{0} \end{bmatrix} \tilde{\mathbf{X}} + \mathbf{n}_o \tag{22}$$

$$\begin{bmatrix} \mathbf{Q}_1^T \mathbf{r}_o \\ \mathbf{Q}_2^T \mathbf{r}_o \end{bmatrix} = \begin{bmatrix} \mathbf{T}_H \\ \mathbf{0} \end{bmatrix} \tilde{\mathbf{X}} + \begin{bmatrix} \mathbf{Q}_1^T \mathbf{n}_o \\ \mathbf{Q}_2^T \mathbf{n}_o \end{bmatrix} \tag{23}$$

上式 $\mathbf{Q}_2^T\mathbf{r}_o$ 仅含有噪声，对其忽略，取 $\mathbf{Q}_1^T\mathbf{r}_o$ 用于 EKF 更新

$$\mathbf{r}_n = \mathbf{Q}_1^T\mathbf{r}_o = \mathbf{T}_H\widetilde{\mathbf{X}} + \mathbf{n}_n \in \mathbb{R}^{(21+6n)\times 1}, \quad \mathbf{n}_n \sim \mathcal{N}(0, \mathbf{R}_n) \tag{24}$$

with

$$\mathbf{T}_H = \mathbf{Q}_1^T \cdot \mathbf{H}_{\mathbf{X}} \in \mathbb{R}^{(21+6n)\times(21+6n)} \tag{25}$$

$$\mathbf{R}_n = \mathbf{Q}_1^T\mathbf{R}_o\mathbf{Q}_1 = \sigma_{\text{im}}^2\mathbf{I}_r \quad \text{with} \quad \sigma_{\text{im}} = 0.01 \tag{26}$$

测量协方差矩阵

$$\mathbf{S} = \mathbf{T}_H\mathbf{P}\mathbf{T}_H^T + \mathbf{R}_n \tag{27}$$

Kalman 增益

$$\mathbf{K} = \mathbf{P}\mathbf{T}_H^T \cdot \mathbf{S}^{-1} \tag{28}$$

更新系统误差状态

$$\Delta\mathbf{X} = \mathbf{K}\mathbf{r}_n \tag{29}$$

更新系统状态

$$\mathbf{X}_{k+1} = \mathbf{X}_k \oplus \Delta\mathbf{X} \tag{30}$$

更新系统状态协方差矩阵 (参见附录A)

$$\mathbf{P}_{k+1|k+1} = (\mathbf{I} - \mathbf{K}\mathbf{T}_H)\,\mathbf{P}_{k+1|k} \tag{31}$$

3.6.4 Post EKF Update

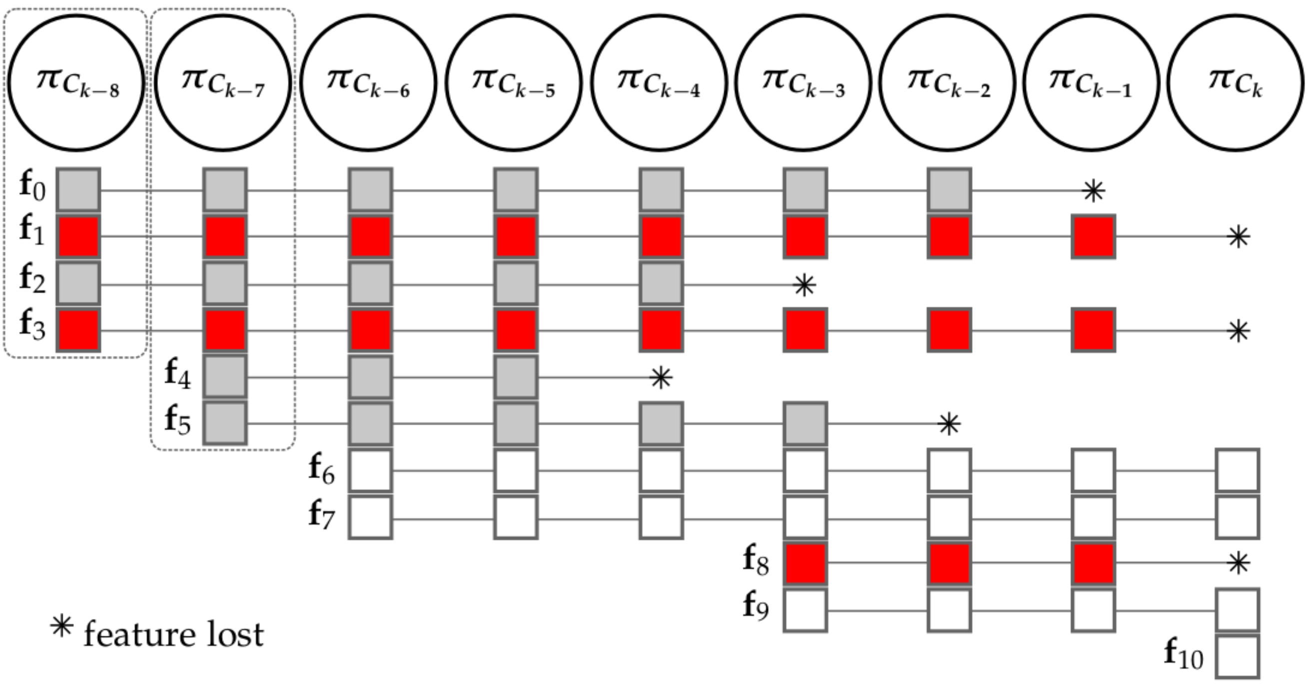


图 8 Post EKF Update

4 Result Analysis

5 Stereo-MSCKF to Mono-MSCKF

6 Mono-MSCKF + Depth

7 Hybrid MSCKF (MSCKF + EKF) [1]

A Three forms of P [2]

- the simplest form which is known to have poor numerical stability, as its outcome is not guaranteed to be symmetric nor positive definite

$$\mathbf{P} \leftarrow (\mathbf{I} - \mathbf{K}\mathbf{H})\mathbf{P}$$

- the symmetric form

$$\mathbf{P} \leftarrow \mathbf{P} - \mathbf{K}\left(\mathbf{H}\mathbf{P}\mathbf{H}^{\top} + \mathbf{V}\right)\mathbf{K}^{\top}$$

- the symmetric and positive Joseph form

$$\mathbf{P} \leftarrow (\mathbf{I} - \mathbf{K}\mathbf{H})\mathbf{P}(\mathbf{I} - \mathbf{K}\mathbf{H})^{\top} + \mathbf{K}\mathbf{V}\mathbf{K}^{\top}$$

B Multidimensional Normal Distribution [4]

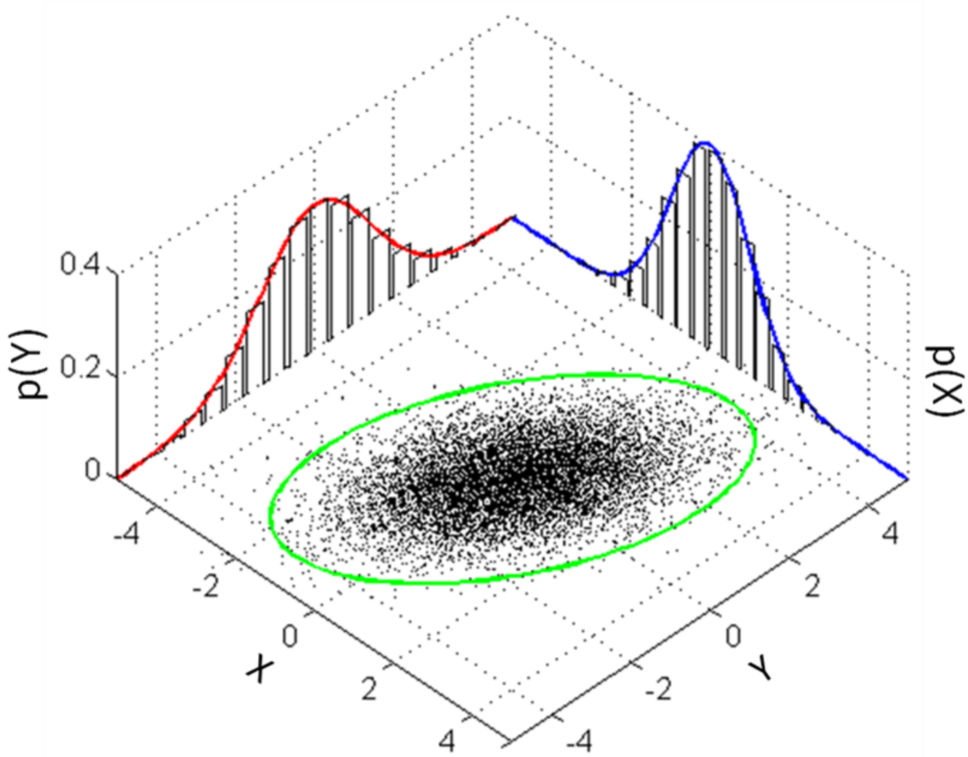


图 9 A two-dimensional normal distribution

$$\left[\begin{array}{c} x \\ y \end{array}\right] \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \mathcal{N}\left(\left[\begin{array}{c} \mu_x \\ \mu_y \end{array}\right], \left[\begin{array}{c} \Sigma_{xx}\Sigma_{xy} \\ \Sigma_{yx}\Sigma_{yy} \end{array}\right]\right)$$

$$p\left(\left[\begin{array}{c} x \\ y \end{array}\right]; \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) = \eta \exp\left(-\frac{1}{2}\left(\left[\begin{array}{c} x \\ y \end{array}\right] - \left[\begin{array}{c} \mu_x \\ \mu_y \end{array}\right]\right)^T \left[\begin{array}{cc} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{array}\right]^{-1} \left(\left[\begin{array}{c} x \\ y \end{array}\right] - \left[\begin{array}{c} \mu_x \\ \mu_y \end{array}\right]\right)\right)$$

$$\text{with } \eta = \frac{1}{2\pi|\boldsymbol{\Sigma}|^{1/2}} \text{ and } \Sigma_{xy} = \Sigma_{yx}^T$$

B.1 Marginalization

If we wish to find the pdf of x alone, we must marginalize out y.

$$\begin{aligned} p(x) &= \int p(x, y) dy \\ &= \int p(x \mid y) p(y) dy \\ &= \mathcal{N}\left(\mu_x, \Sigma_{xx}\right) \end{aligned}$$

B.2 Conditioning

$$x|_{y=y_0} \sim \mathcal{N}(\mu_x + \underbrace{\sum_{xy} \Sigma_{yy}^{-1}\left(y_0 - \mu_y\right)}_{\text{mean offset}}, \Sigma_{xx} - \underbrace{\sum_{xy} \Sigma_{yy}^{-1} \Sigma_{yx}}_{\text{covariance offset}})$$

B.3 Intersection

we find the joint probability $p(x, y)$, also called the intersection. Let $y = Ax + b$ where A is a constant and $b \sim \mathcal{N}(0, Q)$.

$$\begin{aligned} p\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) &= \mathcal{N}\left(\left[\begin{array}{c} \mu_x \\ \mu_y \end{array}\right], \left[\begin{array}{cc} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{array}\right]\right) \\ &= \mathcal{N}\left(\left[\begin{array}{c} \mu_x \\ A\mu_x + b \end{array}\right], \left[\begin{array}{cc} \Sigma_{xx} & \Sigma_{xx} A^T \\ A\Sigma_{xx} & A\Sigma_{xx} A^T + Q \end{array}\right]\right) \end{aligned}$$

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