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基于滑动窗口算法的VIO系统：可观性和一致性

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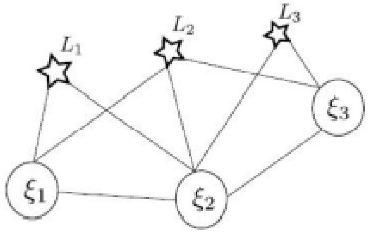
作业内容

① 设某时刻，SLAM系统中相机和路标点的观测关系如下图所示，其中 表示 相机姿态，L表示观测到的路标点。当路标点L表示在世界坐标系下时，第k 个路标被第i时刻的相机观测到，重投影误差为 。另外，相邻相机之 间存在运动约束，如IMU或者轮速计等约束。

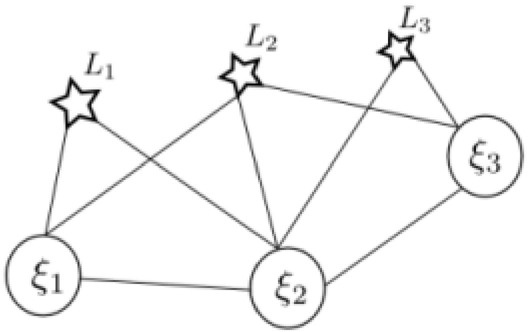
1. 绘制上述系统的信息矩阵
2. 绘制相机 被marg以后的信息矩阵



② 阅读《Relationship between the Hessian and Covariance Matrix for Gaussian RandomVariables》。证明信息矩阵和协方差的逆之间的关系。

③ 补充作业代码中单目Bundle Adjustment信息矩阵的计算，并输出正确的结果。 正确的结果为：奇异值最后7维接近于0，表明零空间的维度为7

题目一



由图中我们可以看到，有6个待优化的变量，其中3个pose、3个landmark。还有9个误差项。 所以雅可比矩阵形式为：

题目一

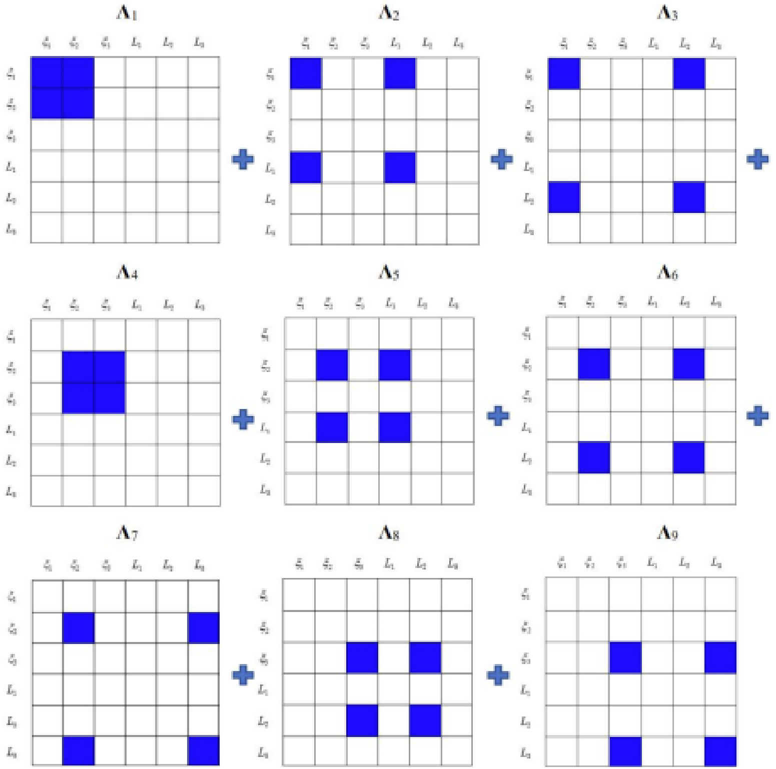
$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \xi} = \begin{bmatrix} \frac{\partial r(\xi 1, \xi 2)}{\partial \xi} \\ \frac{\partial r(\xi 1, L_1)}{\partial \xi} \\ \frac{\partial r(\xi 1, L_2)}{\partial \xi} \\ \frac{\partial r(\xi 2, \xi 3)}{\partial \xi} \\ \frac{\partial r(\xi 2, L_1)}{\partial \xi} \\ \frac{\partial r(\xi 2, L_2)}{\partial \xi} \\ \frac{\partial r(\xi 2, L_3)}{\partial \xi} \\ \frac{\partial r(\xi 3, L_2)}{\partial \xi} \\ \frac{\partial r(\xi 3, L_3)}{\partial \xi} \end{bmatrix}$$

$$\mathbf{J}_2 = \frac{\partial r(\xi 1, \mathbf{L1})}{\partial \xi} = \begin{bmatrix} \frac{\partial r(\xi 1, L_1)}{\partial \xi 1} & \frac{\partial r(\xi 1, L_1)}{\partial \xi 2} & \frac{\partial r(\xi 1, L_1)}{\partial \xi^3} & \frac{\partial r(\xi 1, L_1)}{\partial L_1} & \frac{\partial r(\xi 1, L_1)}{\partial L_2} & \frac{\partial r(\xi 1, L_1)}{\partial L_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial r(\xi 1, L_1)}{\partial \xi 1} & 0 & 0 & \frac{\partial r(\xi 1, L_1)}{\partial L_1} & 0 & 0 \end{bmatrix}$$

$$\Lambda = \boxed{J^T \Sigma^{-1} J}$$

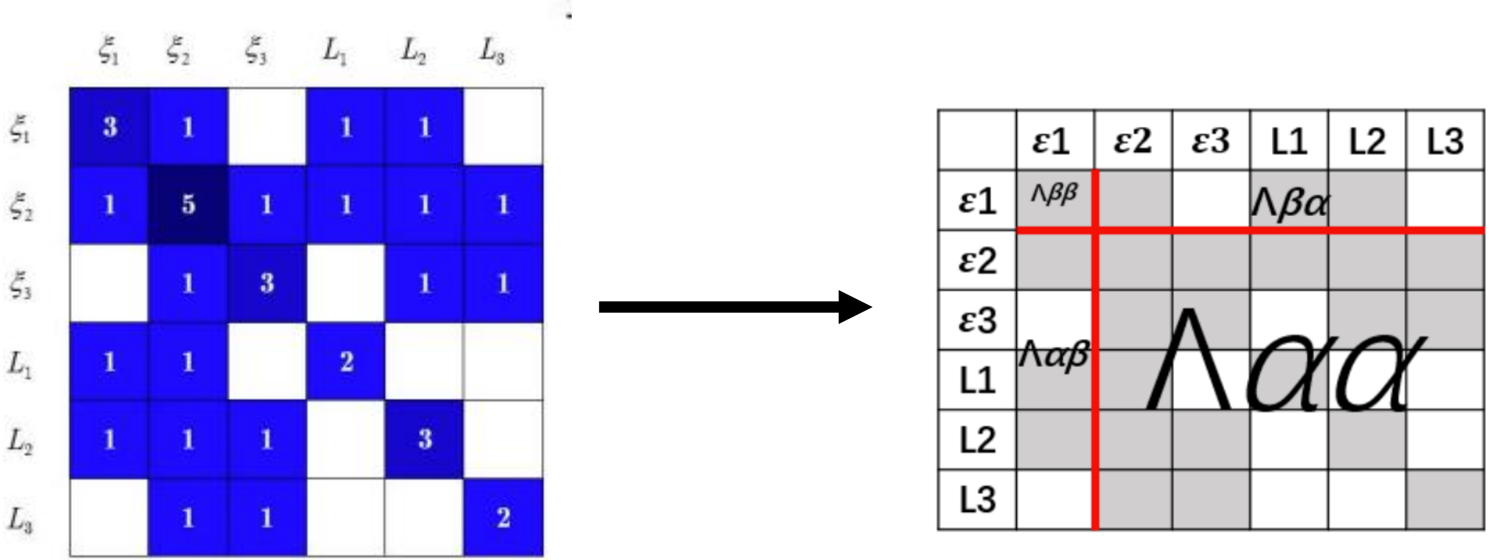
$$\boxed{\sum_{i=1}^{5} \mathbf{J}_i^\top \boldsymbol{\Sigma}_i^{-1} \mathbf{J}_i} \delta \boldsymbol{\xi} = -\sum_{i=1}^{5} \mathbf{J}_i^\top \boldsymbol{\Sigma}_i^{-1} \mathbf{r}_i$$

题目一



	ξ_1	ξ_2	ξ_3	L_1	L_2	L_3
ξ_1	3	1		1	1	
ξ_2	1	5	1	1	1	1
ξ_3		1	3		1	1
L_1	1	1		2		
L_2	1	1	1		3	
L_3		1	1			2

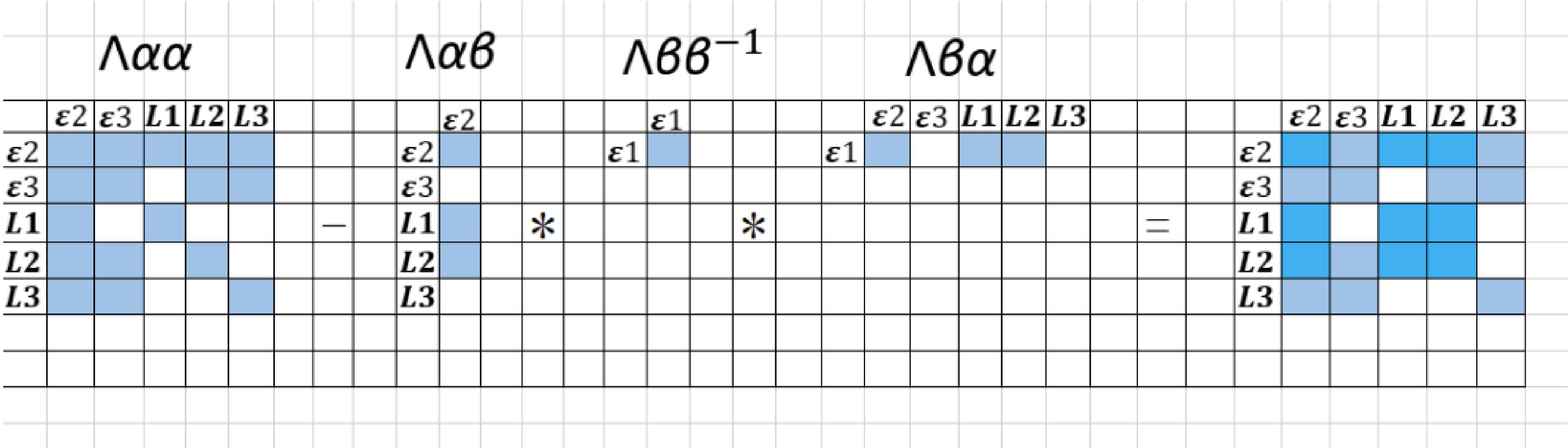
题目一



ξ1 ξ2 ξ3 L1 L2 L3
3 1 1 1
1 5 1 1 1 1
1 3 1 1
1 1 2
1 1 1 3
1 1 2
ε1 ε2 ε3 L1 L2 L3
Λββ
Λβα
Λαβ
Λαα

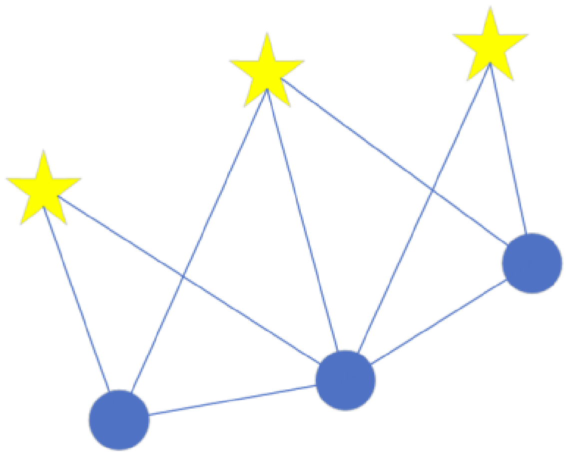
题目一

$$\Lambda = \Lambda_{\alpha\alpha} - \Lambda_{\alpha\beta}\Lambda_{\beta\beta}^{-1}\Lambda_{\beta\alpha}$$

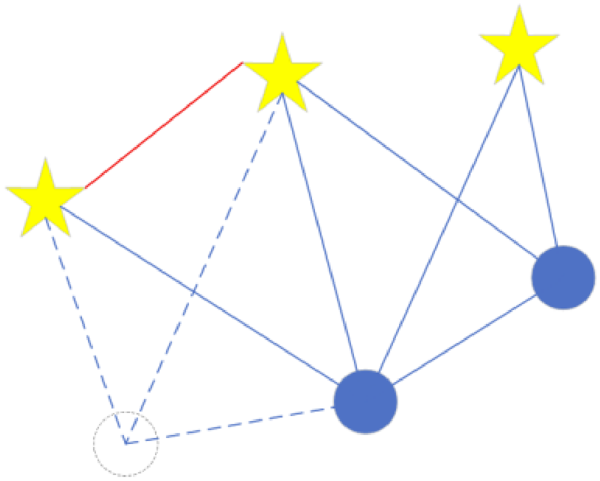


题目一

Marg前：



Marg后：



题目二

根据论文《Relationship between the Hessian and Covariance Matrix for Gaussian Random Variables》可知多元高斯分布负对数似然估计的Hession矩阵等于协方差逆，

证明如下：

对于多元高斯分布，假设随机变量为θ，均值为θ*，协方差矩阵为Σ_θ，其联合概率密度函数为：

$$p(\boldsymbol{\theta}) = (2\pi)^{-\frac{N_\theta}{2}} |\boldsymbol{\Sigma}_\theta|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\theta}^\star)^T \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1} (\boldsymbol{\theta}-\boldsymbol{\theta}^\star)\right]$$

根据极大似然估计，目标函数可以定义为联合概率密度的负对数：

$$J(\boldsymbol{\theta}) \equiv -\ln p(\boldsymbol{\theta}) = \frac{N_\theta}{2}\ln 2\pi + \frac{1}{2}\ln|\boldsymbol{\Sigma}_\theta| + \frac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\theta}^\star)^T \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1} (\boldsymbol{\theta}-\boldsymbol{\theta}^\star)$$

题目二

从上式可以看出，这是关于θ的二次函数，通过对$θ_l$求部分偏导，可以得到在（l, l'）上的 Hessian矩阵。

$$H^{(l,l')}(\theta^*) = \frac{\partial^2 J(\theta)}{\partial\theta_l \partial\theta_{l'}}|_{\theta=\theta^*} = [\frac{\partial}{\partial\theta_l}\left(\frac{\partial J(\theta)}{\partial\theta_l}\right)]|_{\theta=\theta^*} = [\frac{\partial J'(\theta)}{\partial\theta_l}]_{\theta=\theta^*}$$

对联合概率密度函数的负对数J(θ)求一次偏导，得：

$$dJ = (\frac{1}{2}d\theta)^T\left(\Sigma_\theta^{-1}\theta - \Sigma_\theta^{-1}\theta^*\right) + \frac{1}{2}(\theta - \theta^*)^T(\Sigma_\theta^{-1}d\theta)$$

题目二

θ为 n 维行向量，则$\Sigma_\theta^{-1}\theta$仍为 n 维行向量，$d\theta$也为 n 维行向量，则根据两个向量的内积公式$u^T v = v^T u$，可将上式变换为：

$$dJ = \left(\Sigma_\theta^{-1}\theta - \Sigma_\theta^{-1}\theta^*\right)^T (\frac{1}{2}d\theta) + \frac{1}{2}(\theta - \theta^*)^T(\Sigma_\theta^{-1}d\theta)$$

$$= \left(\Sigma_\theta^{-1}(\theta - \theta^*)\right)^T (\frac{1}{2}d\theta) + \frac{1}{2}(\theta - \theta^*)^T(\Sigma_\theta^{-1}d\theta)$$

$$= (\theta - \theta^*)^T(\Sigma_\theta^{-1})^T (\frac{1}{2}d\theta) + \frac{1}{2}(\theta - \theta^*)^T(\Sigma_\theta^{-1}d\theta)$$

$$= \frac{1}{2}(\theta - \theta^*)^T(\Sigma_\theta^{-1})^T(d\theta) + \frac{1}{2}(\theta - \theta^*)^T\Sigma_\theta^{-1}(d\theta)$$

题目二

由于Σ_θ为协方差矩阵，则Σ_θ为对称矩阵，则有$(\Sigma_\theta^{-1})^T=\Sigma_\theta^{-1}$，则上式可以进一步简化为：

$$dJ=\frac{1}{2}(\theta-\theta^*)^T(\Sigma_\theta^{-1})^T(d\theta)+\frac{1}{2}(\theta-\theta^*)^T\Sigma_\theta^{-1}(d\theta)$$

$$=\frac{1}{2}(\theta-\theta^*)^T\Sigma_\theta^{-1}(d\theta)+\frac{1}{2}(\theta-\theta^*)^T\Sigma_\theta^{-1}(d\theta)=(\theta-\theta^*)^T\Sigma_\theta^{-1}(d\theta)$$

又根据导数与微分的关系$dY=\frac{\partial Y^T}{\partial X}dx$，则有：

$$\frac{\partial J}{\partial\theta}=(\Sigma_\theta^{-1})^T(\theta-\theta^*)=J'(\theta)$$

则对上式$J'(\theta)$再次进行θ求导，则非常容易看到结果就是Σ_θ^{-1}，即：

$\frac{\partial J'}{\partial\theta}=\Sigma_\theta^{-1}$，则证明完毕。

题目二

$$H^{(l,l')}(\theta^*) = \frac{\partial^2 J(\theta)}{\partial\theta_l \partial\theta_{l'}}|_{\theta=\theta^*} = (\Sigma_\theta^{-1})^{(l,l')}$$

即多元高斯分布负对数似然估计的Hession矩阵（负对数的Hession矩阵）等于协方差逆。

题目二

根据Wiki Fisher Information了解信息矩阵的定义，信息矩阵是对分布f(X;θ)取负对数二阶导的期望：

Matrix form [edit]

When there are N parameters, so that θ is an $N \times 1$ vector $\theta = \begin{bmatrix} \theta_1 & \theta_2 & \dots & \theta_N \end{bmatrix}^{\mathrm{T}}$, then the Fisher information takes the form of an $N \times N$ matrix. This matrix is called the **Fisher information matrix** (FIM) and has typical element

$$[\mathcal{I}(\theta)]_{i,j} = \mathrm{E}\left[\left(\frac{\partial}{\partial\theta_i}\log f(X;\theta)\right)\left(\frac{\partial}{\partial\theta_j}\log f(X;\theta)\right)\middle|\theta\right].$$

The FIM is a $N \times N$ positive semidefinite matrix. If it is positive definite, then it defines a Riemannian metric on the N-dimensional parameter space. The topic information geometry uses this to connect Fisher information to differential geometry, and in that context, this metric is known as the Fisher information metric.

Under certain regularity conditions, the Fisher information matrix may also be written as

$$[\mathcal{I}(\theta)]_{i,j} = -\mathrm{E}\left[\frac{\partial^2}{\partial\theta_i\,\partial\theta_j}\log f(X;\theta)\middle|\theta\right].$$

题目二

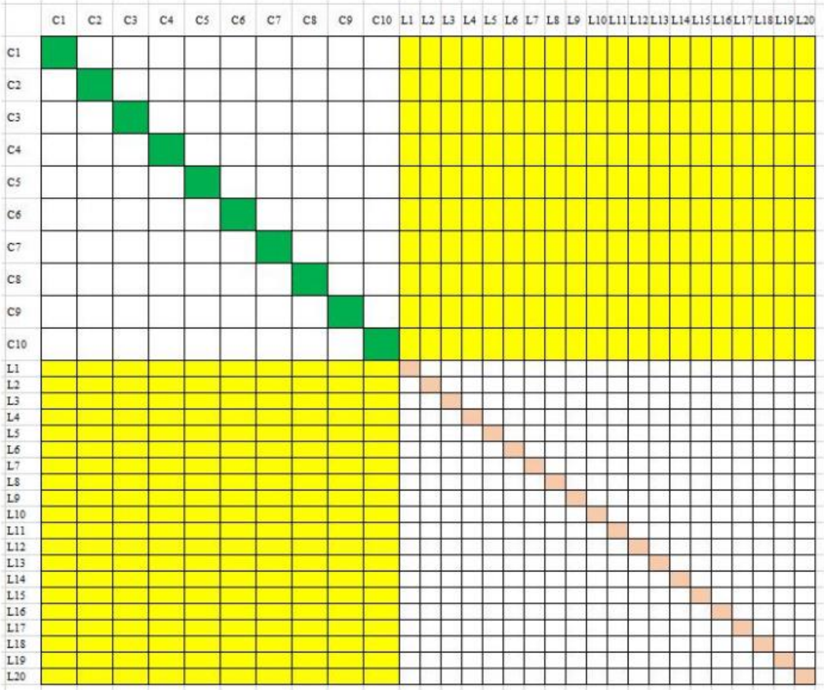
$$[\mathcal{I}(\theta)]_{i,j} = -\operatorname{E}\left[\frac{\partial^2}{\partial\theta_i\,\partial\theta_j}\log f(X;\theta)\middle|\theta\right].$$

$$H^{(l,l')}\ (\theta^*) = \frac{\partial^2 J(\theta)}{\partial\theta_l\partial\theta_{l'}}|_{\theta=\theta^*} = (\Sigma_\theta^{-1})^{(l,l')}$$

由此可知，信息矩阵 = Hession矩阵 = 协方差矩阵的逆。

题目三

补充作业代码中单目Bundle Adjustment信息矩阵的计算，验证信息矩阵零空间维度为7.



程序中仿真的单目模型：

10相机Pose,

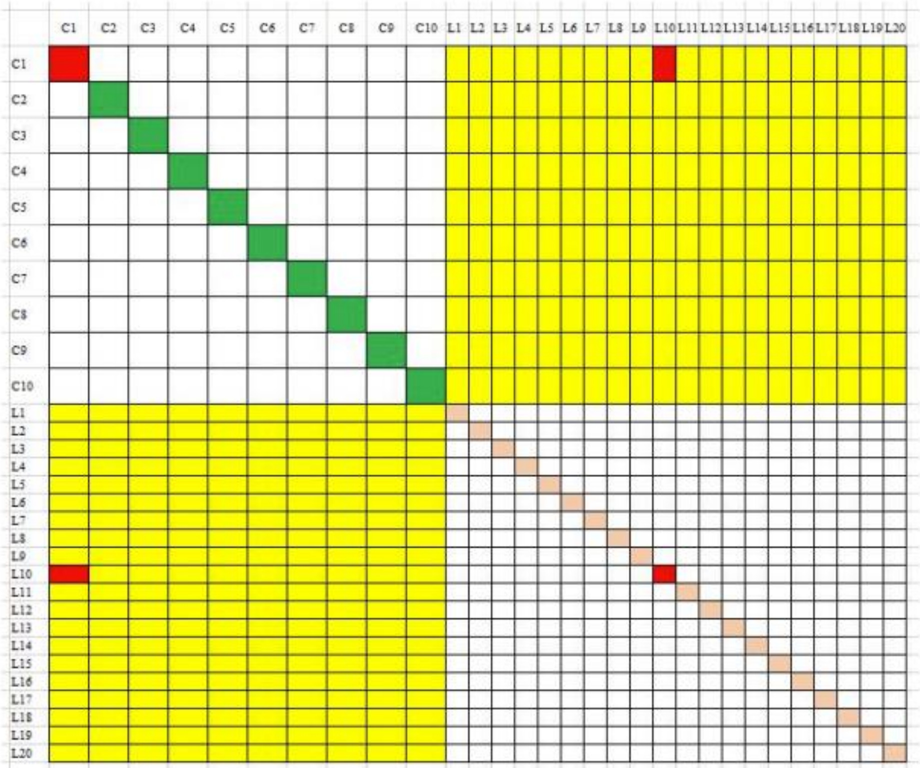
20个Feature Point,

并且每个相机都能观测到所以所有路标点。

相机Pose根据曲线模拟生成，

3D landmark在Pose周围随机生成。

题目三



对于其中一项残差，$r(\xi_i, p_j)$其中$i \in (1 \sim 10), j \in (1 \sim 20)$
对于H的贡献可分成四个小块：

$$J_{T_i}{}^T J_{T_i} (6*6阶)$$

$$J_{T_i}{}^T J_{P_j} (6*3阶)$$

$$J_{P_j}{}^T J_{T_i} (3*6阶)$$

$$J_{P_j}{}^T J_{P_j} (3*3阶)$$

题目三

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对于残差r(ξ_i, p_j)的Jacibian $\quad J_i = (\frac{\partial r(\xi_i, p_j)}{\partial \xi_i}, 0, 0 \ldots\ldots \frac{\partial r(\xi_i, p_j)}{\partial p_j}, 0,0,0,0)$

残差对于Pose的导数：$J_{T_i} = \frac{\partial r(\xi_i, p_j)}{\partial \delta \xi_i} = \lim_{\partial \xi \to 0} \frac{r(\xi_i \oplus \partial \xi)}{\partial \delta \xi} = \frac{\partial \mathrm{r}}{\partial P'} \frac{\partial P'}{\partial \delta \xi}$

残差对于landmark的导数：$J_{P_j} = \frac{\partial r(\xi_i, p_j)}{\partial \mathrm{P}} = \frac{\partial \mathrm{r}}{\partial P'} \frac{\partial P'}{\partial P}$

题目三

残差关于Pose的导数：

首先定义一个中间变量—相机坐标系下的空间点坐标P’，并将其前3维取出来：

$$\boldsymbol{P}' = \left(\exp\left(\boldsymbol{\xi}^{\wedge}\right)\boldsymbol{P}\right)_{1:3} = [X', Y', Z']^{\mathrm{T}}.$$

那么，相机投影模型相对于P’为：

$$s\boldsymbol{u} = \boldsymbol{K}\boldsymbol{P}'. \qquad u = f_x \frac{X'}{Z'} + c_x, \quad v = f_y \frac{Y'}{Z'} + c_y.$$

当我们求误差时，可以把这里的 u, v 与实际的测量值比较， 求差。在定义了中间变量后，我们对 ξ ^ 左乘扰动量 δξ，然后考虑 e 的变化关于扰动量的导数。利用链式法则，可以列写如下：

$$\frac{\partial e}{\partial \delta \boldsymbol{\xi}} = \lim_{\delta \boldsymbol{\xi} \to 0} \frac{e\left(\delta \boldsymbol{\xi} \oplus \boldsymbol{\xi}\right)}{\delta \boldsymbol{\xi}} = \frac{\partial e}{\partial \boldsymbol{P}'} \frac{\partial \boldsymbol{P}'}{\partial \delta \boldsymbol{\xi}}.$$

题目三

这里的 ⊕ 指李代数上的左乘扰动。第一项是误差(测量值 - 投影点)关于投影点的导数，易得：

$$\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{P}'} = -\begin{bmatrix} \frac{\partial u}{\partial X'} & \frac{\partial u}{\partial Y'} & \frac{\partial u}{\partial Z'} \\ \frac{\partial v}{\partial X'} & \frac{\partial v}{\partial Y'} & \frac{\partial v}{\partial Z'} \end{bmatrix} = -\begin{bmatrix} \frac{f_x}{Z'} & 0 & -\frac{f_x X'}{Z'^2} \\ 0 & \frac{f_y}{Z'} & -\frac{f_y Y'}{Z'^2} \end{bmatrix}$$

第二项为变换后的点关于李代数的导数：

$$\frac{\partial (\boldsymbol{T}\boldsymbol{P})}{\partial \delta \boldsymbol{\xi}} = (\boldsymbol{T}\boldsymbol{P})^{\odot} = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{P}'^{\wedge} \\ \boldsymbol{0}^{\mathrm{T}} & \boldsymbol{0}^{\mathrm{T}} \end{bmatrix}.$$

题目三

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将这两项相乘，就得到了 2×6 的雅可比矩阵：

$$\frac{\partial \boldsymbol{e}}{\partial \delta \boldsymbol{\xi}} = -\begin{bmatrix} \frac{f_x}{Z'} & 0 & -\frac{f_x X'}{Z'^2} & -\frac{f_x X' Y'}{Z'^2} & f_x + \frac{f_x X^2}{Z'^2} & -\frac{f_x Y'}{Z'} \\ 0 & \frac{f_y}{Z'} & -\frac{f_y Y'}{Z'^2} & -f_y - \frac{f_y Y'^2}{Z'^2} & \frac{f_y X' Y'}{Z'^2} & \frac{f_y X'}{Z'} \end{bmatrix}$$

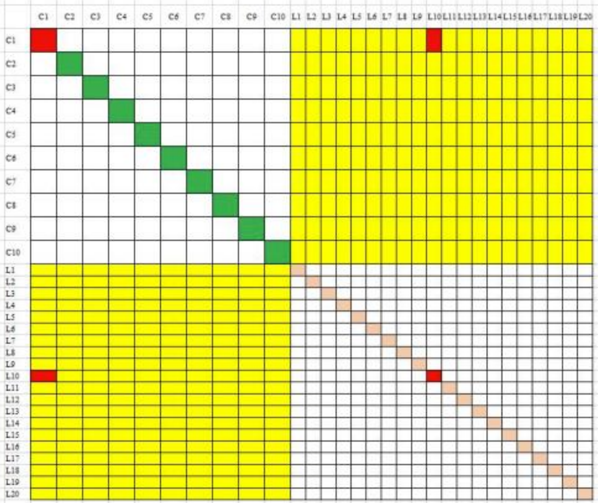
残差关于空间点P的导数：

$$\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{P}} = \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{P}'} \frac{\partial \boldsymbol{P}'}{\partial \boldsymbol{P}}.$$

$$\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{P}} = -\begin{bmatrix} \frac{f_x}{Z'} & 0 & -\frac{f_x X'}{Z'^2} \\ 0 & \frac{f_y}{Z'} & -\frac{f_y Y'}{Z'^2} \end{bmatrix} \boldsymbol{R}$$

$$\boldsymbol{P}' = \exp(\boldsymbol{\xi}^\wedge)\boldsymbol{P} = \boldsymbol{R}\boldsymbol{P} + \boldsymbol{t},$$

题目三



$J_{T_i}^T J_{T_i}$ (6*6阶)　$J_{T_i}^T J_{P_j}$ (6*3阶)

$J_{P_j}^T J_{T_i}$ (3*6阶)　$J_{P_j}^T J_{P_j}$ (3*3阶)

```
for (int i = 0; i < poseNums; ++i) {
    Eigen::Matrix3d Rcw = camera_pose[i].Rwc.transpose();
    Eigen::Vector3d Pc = Rcw * (Pw - camera_pose[i].twc);

    double x = Pc.x();
    double y = Pc.y();
    double z = Pc.z();
    double z_2 = z * z;
    Eigen::Matrix<double,2,3> jacobian_uv_Pc;
    jacobian_uv_Pc<< fx/z, 0 , -x * fx/z_2,
                     0, fy/z, -y * fy/z_2;
    Eigen::Matrix<double,2,3> jacobian_Pj = jacobian_uv_Pc * Rcw;
    Eigen::Matrix<double,2,6> jacobian_Ti;

    jacobian_Ti << -x* y * fx/z_2, (1+ x*x/z_2)*fx, -y/z*fx, fx/z, 0 , -x * fx/z_2,
                   -(1+y*y/z_2)*fy, x*y/z_2 * fy, x/z * fy, 0,fy/z, -y * fy/z_2;

    H.block(i*6,i*6,6,6) += jacobian_Ti.transpose() * jacobian_Ti;
    H.block(j*3 + 6*poseNums,j*3 + 6*poseNums,3,3) += jacobian_Pj.transpose() *jacobian_Pj ;
    H.block(i*6,j*3 + 6*poseNums, 6,3) += jacobian_Ti.transpose() * jacobian_Pj;
    H.block(j*3 + 6*poseNums,i*6 , 3,6) += jacobian_Pj.transpose() * jacobian_Ti;
}
```

对信息矩阵进行 SVD 分解后，

发现特征值的最后 7 维接近于零，

即表示原始的 H 矩阵零空间维度为7维。

3.21708e-17

2.06732e-17

1.43188e-17

7.66992e-18

6.08423e-18

6.05715e-18

3.94363e-18

在线问答

Q&A

感谢各位聆听！

Thanks for Listening