Notes on rigid body transformations and changes of coordinates

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Friday 22nd September, 2017

Imagine you have two orthonormal reference systems in $\mathbb{R}^d$, e.g, one fixed on the floor and one attached to a robot. These two are usually referred to as the *world frame* and the *body frame*, respectively, which will denote as $\mathcal{A}_0$ and $\mathcal{A}_1$. In some situations, we might have additional frames, e.g., corresponding to multiple links in a robotic arm, which we will denote as $\mathcal{A}_2, \mathcal{A}_3, \ldots$ Any physical point x in space can be expressed in coordinates (i.e., vectors of scalars) in any one of these frames. We will denote as $x_i \in \mathbb{R}^d$ the vector of the coordinates of x expressed in the frame $\mathcal{A}_i$. The goal of this note is to explain how to obtain the coordinates in one frame x_i given the coordinates in another frame x_j.

1 Rigid body transformations

The coordinates in a reference frame $\mathcal{A}_i$ can be transformed to the coordinates in a reference frame $\mathcal{A}_j$ using a *rigid body transformation* $g = (R_i^j, T_i^j)$, which is composed of a rotation $R_i^j \in SO(d)$, and a translation $T_i^j \in \mathbb{R}^3$. In practice, the columns of R and T represent, respectively, the axes of the origin reference frame $\mathcal{A}_i$ and it center expressed using coordinates in the target reference frame $\mathcal{A}_j$. The superscript and the subscript keep track of the target and origin reference frames these quantities are referring to.

2 Change of coordinates

The rigid body pose (R_i^j, T_i^j) contains all the information that is needed to transform the coordinates x_i in frame $\mathcal{A}_i$ of a point to the corresponding coordinates in frame $\mathcal{A}_j$. Algebraically, we have the following:

$$x_j = R_i^j x_i + T_i^j \tag{1}$$

For instance, imagine you have a one-link robotic arm, you know the position of a point on that arm in its "rest" position (e.g., $x_i = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$), you know the rotation of the arm R_1^0 with respect to a world reference frame, and you want to know the position of that point in the world reference frame. You can attach the body frame $\mathcal{A}_1$ to the link to its pin (so that it rotates with it), and put the world frame $\mathcal{A}_0$ at the same point (but this reference does not rotate), see Figure 1. Note that since the centers of the two frames coincide, we have $T_1^0 = 0$. The solution is then given by

$$x_1 = R_1^0 x_0. \tag{2}$$

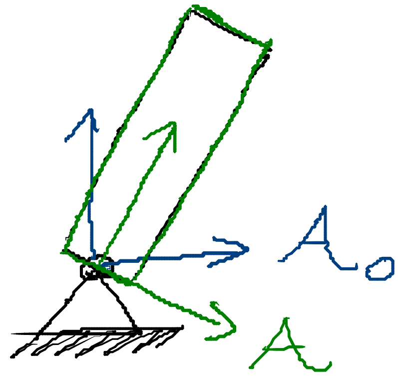
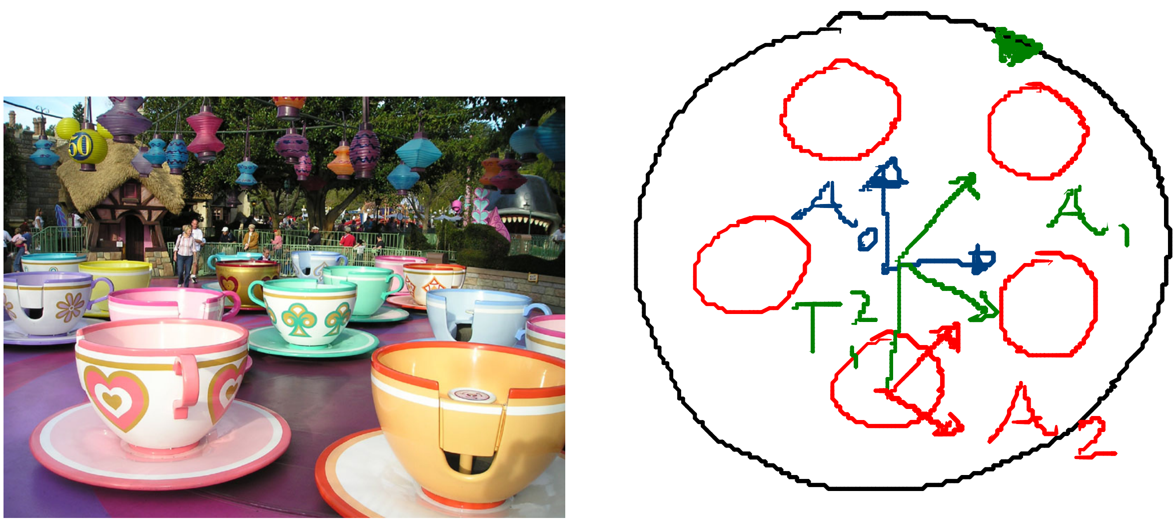


Figure 1: One-link robotic arm example

3 Change of coordinates across multiple reference frames

Relation (1) can be “chained” if you have multiple “nested” reference frames. For instance consider the “Tea Cup Ride” model in Figure 2. The plate turns with respect to the world,



(a) A real “Tea Cup Ride”

(b) A model of the ride with the reference frames attached

Figure 2: “Tea Cup Ride” model example

and the cup turns around a fixed position on the plate. You can affix one reference frame $\mathcal{A}_0$ to the world, one $\mathcal{A}_1$ to the plate, and one $\mathcal{A}_2$ to the cup. Assume you know the rotation of the plate R_1^0 (as in the previous example, $T_1^0 = 0$), the offset T_2^1 and rotation R_2^1 of the cup expressed in the frame of the plate, and the position of a point on the cup in the cup's coordinates x_2. The coordinates of that point in the world reference frame can be obtained by applying (1) twice, passing through the intermediate coordinates x_1 of the point in the plate's reference frame:

$$x_0 = R_0^1 x_1 = R_0^1 (R_1^2 x_2 + T_1^2). \tag{3}$$

Notice that R_1^2 and T_1^2 represents the rotation and translation of the cup $\mathcal{A}_2$ *as if the observer was moving with the plate* $\mathcal{A}_1$, not from an observer positioned in the world $\mathcal{A}_0$.