Orientation and Rotation

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A rotation is a displacement in which at least one point in the rigid body remains in its initial position and not all lines in the body remain parallel to their initial orientations

1 Rotation Matrices

The orientation of coordinate frame i relative to coordinate frame j can be denoted by expressing the basis vector $(\hat{x}_i \hat{y}_i \hat{z}_i)$ in terms of the basis vectors $(\hat{x}_j \hat{y}_j \hat{z}_j)$.

This yields $(\hat{^jx_i} \hat{^jy_i} \hat{^jz_i})$, which when written together as a 3x3 matrix is known as the rotation matrix. The components of $\hat{^jR_i}$ are the dot products of the basis vectors of the two coordinate frames.

$$[\hat{^jR_i}] = \begin{bmatrix} \hat{x}_i.\hat{x}_j & \hat{y}_i.\hat{x}_j & \hat{z}_i.\hat{x}_j \\ \hat{x}_i.\hat{y}_j & \hat{y}_i.\hat{y}_j & \hat{z}_i.\hat{y}_j \\ \hat{x}_i.\hat{z}_j & \hat{y}_i.\hat{z}_j & \hat{z}_i.\hat{z}_j \end{bmatrix}$$

Because the basis vectors are unit vectors and the dot product of any two unit vectors is the cosine of the angle between them, the components are commonly referred to as direction cosines.

An elementary rotation of frame i about the $\hat{Z}_j$ axis through an angle θ is

$$R_Z(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

while the same rotation about the $\hat{Y}_j$ axis is

$$R_Y(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$$

and about $\hat{X}_j$ axis is

$$R_Y(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}$$

The basis vectors of coordinate frame i are mutually orthonormal, as are the basis vectors of coordinate frame j, the columns of ${}^j\hat{R}_i$ formed from the dot products of these vectors are also mutually orthonormal. A matrix composed of mutually orthonormal vectors is known as an orthogonal matrix and has the property that its inverse is simply its transpose.

1.1 Euler Angles(moving angles)

For a minimal representation, the orientation of coordinate frame i relative to coordinate frame j can be denoted as a vector of three angles $(\alpha, \beta, \gamma)^T$. These angles are known as Euler angles when each represents a rotation about an axis of a moving coordinate. In this way, the location of the axis of each successive rotation depends upon the preceding rotations, so the order of the rotations must accompany the three angles to define the orientation.

Regardless of the order of rotations, an Euler angle representation of orientation always exhibits a singularity when the first and last rotations both occur about the same axis.

1.2 Fixed Angles

A vector of three angles can also denote the orientation of coordinate frame i relative to coordinate frame j when each angle represents a rotation about an axis of a FIXED frame.

1.3 Angle-Axis, Quaternions

A single angle θ in combination with a unit vector $\hat{\omega}$ can also denote the orientation of coordinate frame i relative to coordinate j. In this case, frame i is rotated through the angle θ about an axis defined by the vector $\hat{\omega} = (\omega_x \omega_y \omega_z)^T$ relative to frame j. The vector $\hat{\omega}$ is sometimes referred to as the equivalent axis of a finite rotation. The angle-axis representation, typically written as either $\theta\hat{\omega}$ or $(\theta\omega_x \theta\omega_y \theta\omega_z)^T$, is superabundant by one because it contains four parameters. The auxiliary relationship that resolves this is the unit magnitude of vector $\hat{\omega}$. Even with this auxiliary relationship, the angle-axis representation is not unique because rotation through an angle of $-\theta$ about $-\hat{\omega}$ is equivalent to a rotation through θ about $\hat{\omega}$.

Quaternions do not suffer from singularities as Eular and fixed angles do. A quaternion ϵ is defined to have the from

$$\epsilon = \epsilon_0 + \epsilon_1 i + \epsilon_2 j + \epsilon_3 k$$

Conversions from angle-axis to unit quaternion:

$$\epsilon_0 = cos(\theta/2)$$

$$\epsilon_1 = \omega_x sin(\theta/2)$$

$$\epsilon_2 = \omega_y sin(\theta/2)$$

$$\epsilon_3 = \omega_z sin(\theta/2)$$

unit quaternion to angle-axis:

$$\theta = 2cos^{-1}(\epsilon_0)$$

$$\omega_x = \epsilon_1/sin(\theta/2)$$

$$\omega_y = \epsilon_2/sin(\theta/2)$$

$$\omega_z = \epsilon_3/sin(\theta/2)$$