SOUTH UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA

Methods of Applied Math: Assignment 4

Due Date: November 13, 2019

Read 7.1 and 7.2, 9.1 to 9.5 in the textbook "Advanced Mathematical Methods for Scientists and Engineers"; read 2.2 to 2.4, and problem 2.12 about van Dyke's rule in section 2.2 in Holmes' book.

✏ **(Exercise 1)**

1. Compute the first four coefficients in the perturbation series (i.e., $O(1), O(\epsilon), O(\epsilon^2)$, and $O(\epsilon^3)$ terms) to the initial-value problem

$$y' = \frac{3}{2}y + 3\epsilon x y, \quad y(0) = 1.$$

2. Find the exact solution.
3. Use some software, e.g., MATLAB, to plot and compare the exact solution and the n-term perturbation expansion for the solution, $n = 1, 2, 3, 4$ (i.e., $y_0, y_0 + \epsilon y_1, y_0 + \epsilon y_1 + \epsilon^2 y_2$ and $y_0 + \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3$) on $x \in [0, 3]$ when $\epsilon = 0.1$.

✏ **(Exercise 2)** Consider the equation:

$$\begin{cases} \epsilon y'' + (\frac{x}{9} - \frac{2}{3})y' + \frac{1}{9}y = 0, & 0 \leqslant x \leqslant 3, \\ y(0) = 3, y(3) = 2. \end{cases}$$

Assume it is a boundary layer problem. The boundary layer is at $x = 3$, and the boundary layer thickness is ϵ.

1. Find the outer limit, inner limit and the intermediate limit of the solution.
2. Write down a uniform leading order approximation of the solution.

✏ **(Exercise 3)** Consider the equation:

$$\begin{cases} \epsilon y'' + (1+x)^2 y' + y = 0, \quad 0 \leqslant x \leqslant 1, \\ y(0) = 1, y(1) = 1. \end{cases}$$

1. Determine the thickness and location of the boundary layer.
2. Obtain a uniform approximation accurate to order ϵ as $\epsilon \to 0$. Please use three methods to do matching:
   a) The textbook method suggests keeping $O(1)$, $O(\epsilon)$ and $O(x)$ terms;
   b) The van Dyke's matching rule;
   c) The method of intermediate variable $x_\eta = \frac{x}{\eta(\epsilon)}$.

✏ **(Exercise 4)** Find the leading order approximation to the solution of the problem:

$$\begin{cases} \epsilon y'' + x^{\frac{1}{3}} y' - y = 0, \quad 0 \leqslant x \leqslant 1, \\ y(0) = 0, y(1) = e^{\frac{3}{2}}. \end{cases}$$

✏ **(Exercise 5)** Find the leading term of the asymptotic solution of the interior Dirichlet problem

$$\epsilon(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2}) + \frac{\partial u}{\partial y} = 0$$

with $u = y$ on the boundary $C : (x-1)^2 + y^2 = 1$.