SOUTH UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA

Methods of Applied Math: Assignment 5

Due Date: December 27, 2019

Read 10.1 and 10.2, 11.1 and 11.2 in the textbook "Advanced Mathematical Methods for Scientists and Engineers"; read 3.2 and 3.3, 4.2 and 4.3, in Holmes' book.

✏ **(Exercise 1)** Use the boundary layer theory to find the uniform approximation of the solution to the following ODE up to $O(\epsilon)$:

$$\begin{cases} \epsilon y''(x) + xy'(x) - xy(x) = 0, \\ y(0) = 0, \quad y(1) = e. \end{cases}$$

You need to determine the location and the thickness of the boundary layer, find the outer and inner solutions up to $O(\epsilon)$, and do matching.

✏ **(Exercise 2)** Find the WKB approximation correct up to $O(\epsilon^{\frac{1}{2}})$ to the the general solution of the problem

$$\epsilon y'' + \frac{1}{8}(2x^2 + \epsilon x)y = 0, \quad x > 0,$$

where $\epsilon > 0$.

✏ **(Exercise 3)** Use the WKB method to find an approximate solution to the problem:

$$\begin{cases} \epsilon y'' + (x + \frac{1}{2})y' + y = 0, \quad 0 \leqslant x \leqslant 1, \\ y(0) = 2, y(1) = 3. \end{cases}$$

Compare your answer with the composite expansion obtained using matched asymptotic expansions.

✏ **(Exercise 4)** Use the WKB method to find the leading order asymptotic behavior of the large eigenvalue E^2 and the corresponding eigenfunction of the boundary value problem:

$$\begin{cases} y'' + E^2 e^{2x} y = 0, \quad 1 \leqslant x \leqslant 2, \\ y(1) = 0, y(2) = 0. \end{cases}$$

✏ **(Exercise 5)** (Slender body approximation) Consider the vertical displacement of an elastic membrane:

$$\epsilon^2 u_{xx} + u_{yy} = \mu^2(x) u_{tt}, \qquad \text{for} \quad 0 < x < +\infty, |y| < G(x), t > 0,$$

where $u(x, y, t)$ at $y = \pm G(x)$, $u(0, y, t) = f(y)\cos(\omega t)$. Use the WKB method to construct a traveling wave solution for small $\epsilon \ll 1$ (Note that ω is not a small parameter). Find the leading order approximation to the solution so that it is a convergent approximation. Note that the leading order equation is an eigenvalue problem. You need to find the general solution in terms of the combination of all the eigenvalues and eigenfunctions.