SOUTH UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA

Methods of Applied Math: Assignment 2

Due Date: October 14, 2019

Read 3.1 to 3.5 in the textbook "Advanced Mathematical Methods for Scientists and Engineers".

✏ **(Exercise 1)** Find the series solutions about $x = 0$:

$$9y''(x) + \frac{18}{x}y'(x) + y(x) = 0.$$

✏ **(Exercise 2)** Read page 72 to page 76 in 3.3 carefully, show that all solutions of the modified Bessel equation

$$y''(x) + \frac{1}{x}y'(x) - \left(1 + \frac{\nu^2}{x^2}\right)y(x) = 0.$$

with $\nu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ can be expanded in Frobenius series (without logarithmic terms).

✏ **(Exercise 3)** Consider the equation

$$y''(x) + \frac{2}{x}y'(x) - \frac{1}{x^6}y(x) = 0.$$

Find the asymptotic behavior of the solutions as $x \to 0^+$ (Find the first 4 leading terms in $S(x)$).

✏ **(Exercise 4)** Continue the example in the class

$$x^3 y''(x) = y(x)$$

near the irregular singular point $x = 0$. We have calculated that as $x \to 0^+$, $y_1(x) \sim C_1 x^{\frac{3}{4}} e^{2x^{-\frac{1}{2}} - \frac{3}{16}x^{\frac{1}{2}}}$. Show that:

1. If we write $y_1(x) = C_1 x^{\frac{3}{4}} e^{2x^{-\frac{1}{2}}} \left(1 - \frac{3}{16} x^{\frac{1}{2}} + w(x)\right)$, with $w(x) \ll \frac{3}{16} x^{\frac{1}{2}}$ as $x \to 0^+$, then the asymptotic series of $y_1(x)$ is of the form

$$y_1(x) \sim -C_1 x^{\frac{3}{4}} e^{2x^{-\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma(n-\frac{1}{2})\Gamma(n+\frac{3}{2})}{\pi \cdot 4^n \cdot n!} x^{\frac{n}{2}}$$

2. The asymptotic series of the second solution $y_2(x)$ is of the form

$$y_2(x) \sim -C_2 x^{\frac{3}{4}} e^{-2x^{-\frac{1}{2}}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n-\frac{1}{2})\Gamma(n+\frac{3}{2})}{\pi \cdot 4^n \cdot n!} x^{\frac{n}{2}}$$

3. Find the radius of convergence R of the these two series.

✏ **(Exercise 5)** (Behavior of Airy functions for large x) Consider the Airy equation

$$y''(x) = x y(x).$$

1. Show that it has an irregular singular point at ∞.
2. Show that the leading behaviors of the solutions for large x are determined by

$$y_1(x) \sim C_1 x^{-\frac{1}{4}} e^{-\frac{2}{3} x^{\frac{3}{2}}}, \quad x \to +\infty,$$

and

$$y_2(x) \sim C_2 x^{-\frac{1}{4}} e^{\frac{2}{3} x^{\frac{3}{2}}}, \quad x \to +\infty.$$

3. Show that the asymptotic expansion of $y_1(x)$ is

$$y_1(x) \sim C_1 x^{-\frac{1}{4}} e^{-\frac{2}{3} x^{\frac{3}{2}}} \sum_{n=0}^{\infty} \frac{1}{2\pi} \left(-\frac{3}{4}\right)^n \left(\frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!}\right) x^{-\frac{3n}{2}}$$

4. Give the radius of convergence for this series.

✏ **(Exercise 6)** Show that if $f(x) \sim a(x - x_0)^{-b}$ as $x \to x_0^+$, then

1. $\int f \mathrm{d}x \sim \frac{a}{1-b}(x - x_0)^{1-b}$, $x \to x_0^+$ if $b > 1$ and the path of integration does not pass through x_0;
2. $\int f \mathrm{d}x \sim c$, $x \to x_0^+$ where c is a constant if $b < 1$; if $c = 0$, then $\int f \mathrm{d}x \sim \frac{a}{1-b}(x - x_0)^{1-b}$, $x \to x_0^+$;
3. $\int_{x_0}^{x} f \mathrm{d}x \sim \frac{a}{1-b}(x - x_0)^{1-b}$, $x \to x_0^+$ if $b < 1$;
4. $\int f \mathrm{d}x \sim a \ln(x - x_0)$, $x \to x_0^+$ if $b = 1$.

✏ **(Exercise 7)**

1. Give an example of an asymptotic relation $f(x) \sim g(x) (x \to \infty)$ that cannot be exponentiated; that is, $e^{f(x)} \sim e^{g(x)}$ is false.
2. Show that if $f(x) - g(x) \ll 1 (x \to \infty)$, then $e^{f(x)} \sim e^{g(x)} (x \to \infty)$.