SOUTH UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA

Methods of Applied Math: Assignment 6

Due Date: December 11, 2019

Read 11.3 in the textbook "Advanced Mathematical Methods for Scientists and Engineers"; read 3.2, 3.3, 3.6, 3.7, 6.2, and 6.3.

✏ **(Exercise 1)** Use the method of strained coordinates to find the leading order approximation to the solution of the following problem so that there is no secular term at $O(\epsilon)$:

$$\begin{cases} \dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 9y = \epsilon y \left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^2, & 0 \leqslant t \leqslant 1, \\ y(0) = 1, y'(0) = 0. \end{cases}$$

✏ **(Exercise 2)** Use multiple scale analysis to find the leading order approximation to the solution of the following problem that is valid for large t:

$$\begin{cases} \dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \epsilon \left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^3 + y = 0, & 0 \leqslant t \leqslant 1, \\ y(0) = 0, y'(0) = 1. \end{cases}$$

✏ **(Exercise 3)** Consider the following problem with slowly-varying coefficient:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(D(\epsilon t) \dfrac{\mathrm{d}y}{\mathrm{d}t}\right) + y = 0, & t > 0, \\ y(0) = \alpha, y'(0) = \beta, \end{cases}$$

where $D = D(\tau)$ is a smooth positive function with $D' > 0$.

1. Find a first-term (leading order) approximation of the solution valid for large t by using multiple scale analysis.

2. Can this problem be solved by using WKB theory? If yes, please do it and compare the result with what you obtained in part 1.

✏ **(Exercise 4)** (Child's Swing) In a swing, a child can control his/her body to change the center of gravity. The problem can be modeled by a pendulum with a time-dependent length $l = l(t)$. The model is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{2l'(t)}{l(t)}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \sin\theta = 0, \quad t \geqslant 0.$$

where θ is the angle between the swing and the vertical line. Suppose that $l = l_0(1 + \epsilon \sin\omega t)$, where ϵ and ω (both nonnegative) are the amplitude and the frequency of movement controlled by the child, respectively.

1. Consider the steady state $\theta_s = 0$. Use linear stability analysis to show that the first order perturbation of $\theta = \theta_s + \delta v + O(\delta^2)$ satisfies the equation:

$$\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + \frac{2l'(t)}{l(t)}\frac{\mathrm{d}v}{\mathrm{d}t} + v = 0, \quad t \geqslant 0.$$

2. Show that if the child does nothing (so that $\omega = 0$), then the steady state $\theta_s = 0$ is stable.

3. What the child must do is to find a frequency ω that makes the steady state $\theta_s = 0$ unstable (so the amplitude grows and the swing goes higher). Find such a frequency ω when $\epsilon \ll 1$. Do this by using multiple scale analysis to find a leading order expansion of v that is valid for large t.
(Hint: You may need to use the following identity:

$$\sin 2\phi = \frac{2\tan\phi}{1+\tan^2\phi}, \qquad \cos 2\phi = \frac{1-\tan^2\phi}{1+\tan^2\phi}$$

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