SOUTH UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA

Methods of Applied Math: Assignment 7

Due Date: Dec 25, 2019

Read 5.2 to 5.4 in Holmes' book.

✏ **(Exercise 1)** A classic model in the study of oscillatory systems is the Brusselator. The equations are

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \mu - (1+\alpha)x + x^2 y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = x - x^2 y. \end{cases}$$

Here μ and α are positive constants.

1. Find the steady state and determine its stability in the case where $\alpha \geqslant 1$.
2. Suppose $0 \leqslant \alpha \leqslant 1$ and μ is the bifurcation parameter. Determine the stability of the steady state and describe what happens near the point where the steady state loses stability.
3. Assuming $\alpha = \frac{3}{4}$, use multiple scales to find the solution near the bifurcation point.

✏ **(Exercise 2)**

1. Show that for a given periodic function $g(y)$ with period 2π, the equation $\frac{\mathrm{d}z}{\mathrm{d}y} = g(y)$ has a periodic solution $z(y)$ with period 2π if and only if $\int_0^{2\pi} g(y)\mathrm{d}y = 0$, assuming that $g(y)$ and $z(y)$ are smooth.
2. Find the homogenized equation of the problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\left(D(x, \frac{x}{\epsilon})\frac{\mathrm{d}u}{\mathrm{d}x}\right) = f(x, \frac{x}{\epsilon}), \quad 0 \leqslant x \leqslant 1, \\ u(0) = a, u(1) = b. \end{cases}$$

where $D(x,y)$ and $f(x,y)$ are periodic in y with period 2π, and $0 < D_m(x) \leqslant D(x,y) \leqslant D_M(x)$, for $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 2\pi$.

✏ **(Exercise 3)** Consider the 1D transport equation

$$\begin{cases} \frac{\partial u}{\partial t} - b(\frac{x}{\epsilon})\frac{\partial u}{\partial x} = 0, & (x,t) \in \mathbb{R}\times\mathbb{R}^+, \\ u(x,0) = g(x), \end{cases}$$

where $b = b(y) > 0$ is smooth and periodic in y with period 1. Use the multiple scale expansion to find the homogenized equation for small $\epsilon \ll 1$.

✏ **(Exercise 4)** Consider the functional

$$F(u) = \int_{-1}^{1}\Big(\frac{\epsilon}{2}u_x^2 + \frac{1}{\epsilon}f(u)\Big)\mathrm{d}x,$$

for any function $u(x) \in S = \{w \in H^1(-1,1), w(-1) = -1, w(1) = 1\}$, where $f(u) = \frac{1}{4}(1-u^2)^2$.

1. Assume $u \in S$ is the minimizer of the functional $F(u)$. Show that u must satisfy the equation
$$-\epsilon u_{xx} + \frac{1}{\epsilon}f'(u) = 0. \tag{0.1}$$
(Hint: computing the following quantity:
$$\lim_{\delta\to 0}\frac{F(u+\eta v) - F(u)}{\eta},$$
where for any $v \in H_0^1(-1,1) = \{w \in H^1(-1,1), w(-1) = 0, w(1) = 0\}$.)
2. Assume there is a boundary layer at $x = 0$. Use the matched asymptotic theory (outer and inner expansions) to show that the leading order solution to the differential equation (0.1) has the form
$$u \sim \tanh(\frac{x}{\sqrt{2\epsilon}}), \quad \epsilon \to 0.$$