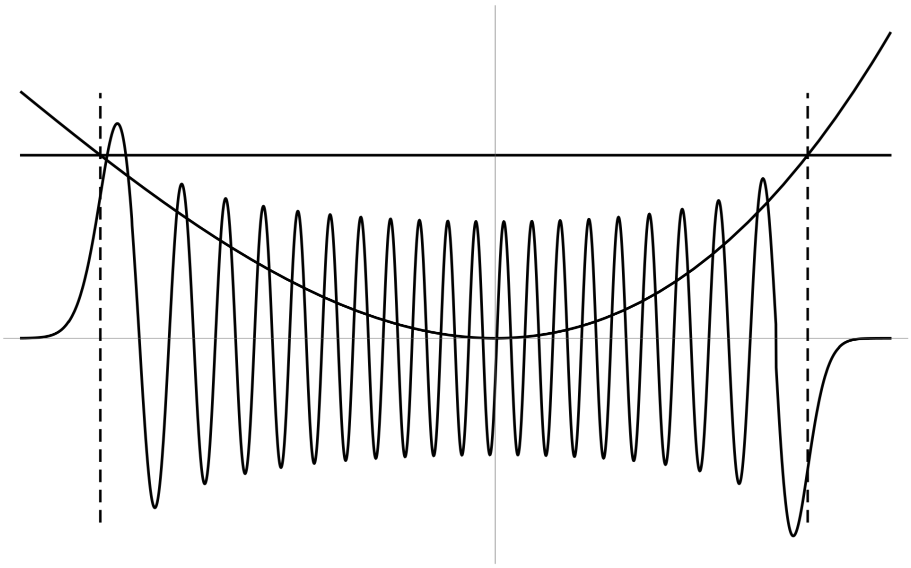
Methods of Applied Mathematics



Zhen Zhang

June 19, 2024

Southern University of Science and Technology

Preface

In recent years, the development and application of mathematical methods have significantly impacted various scientific and engineering disciplines. The course "Methods of Applied Mathematics" is designed to provide a thorough grounding in these essential techniques, focusing on their practical uses and theoretical underpinnings. This course aims to equip students with the skills necessary to the modeling and solutions of complex physical and engineering problems using advanced mathematical tools, preparing them for further study or professional practice in their respective fields.

This book is summarized based on the lecture notes used in my teaching experiences from 2017 until now. It is structured to systematically introduce and explore elementary methods in applied mathematics. Each chapter is crafted to build on the previous one, ensuring a coherent progression of topics. Here's an overview of the chapters covered in this course:

- Perturbation Methods for Algebraic Equations: This chapter begins with the fundamental concepts of perturbation theory, which are crucial for approximating solutions to algebraic equations that cannot be solved exactly. Topics include regular and singular perturbations, and methods for finding approximate roots of algebraic and transcendental equations.

- Asymptotic Expansions: Asymptotic methods are essential for understanding the behavior of functions in limiting cases. This chapter covers the derivation and application of asymptotic expansions in approximating integrals, including techniques like Laplace's method, the method of stationary phase, and the method of steepest descent.

- Introduction to Global Analysis and Perturbation Methods: This section introduces global analysis techniques and their integration with perturbation methods. It addresses the application of these combined methods to solve differential equations and explores their practical significance in various scientific fields.

- WKB Theory: The Wentzel–Kramers–Brillouin (WKB) approximation is a powerful method for analyzing linear differential equations with variable coefficients. This chapter delves into the WKB theory, its derivation, and its applications in differential equations.

- Multiple Scale Analysis: Multiple-scale analysis is a technique used to deal with problems exhibiting behavior on different scales. This chapter explains how to construct multiple-scale expansions and apply them to various physical problems, ensuring a comprehensive understanding of the method.

- Homogenization Method: Homogenization techniques are crucial for understanding the macroscopic properties of heterogeneous materials. This chapter covers the theory and applications of homogenization methods, emphasizing their use in engineering and materials science.

- Bifurcation and Stability: Understanding bifurcation and stability is vital for analyzing dynamic systems. This chapter introduces the concepts of bifurcation theory and stability analysis, providing insights into the behavior of nonlinear systems and their critical points.

- Basic Calculus of Variations: Calculus of variations has found its wide applications in mathematical modeling and analysis in physical and engineering problems. This chapter provides an introduction to the theory of calculus of variations, which may be a little bit formal and lack mathematical rigor in some sense. However, this technique is investigated in detail in modeling physical problems, with the least action principle and Onsager's principle as two representative examples.

The structure of this book ensures that students not only learn the theoretical aspects of applied mathematics but also gain practical experience through examples and exercises. It is expected that a comprehensive study of this book will enrich students with a solid foundation in these advanced mathematical methods and enable them to apply these techniques to real-world problems effectively.

I hope this book serves as a valuable resource in your academic journey and inspires you to delve deeper into the fascinating world of applied mathematics.

Finally, I would like to express my deepest gratitude to Hao Liu, Hongwei Xia, and Zeyu Zhou for their invaluable contributions and support in the development of this book.

Sincerely
Zhen Zhang

Contents

Chapter 1

Perturbation methods for algebraic equations

Many techniques in perturbation analysis can be introduced in the straightforward context of algebraic equations. By starting with particularly simple algebraic equations, such as three quadratic equations, we can take advantage of the exact solutions available, using them to gain useful insights to overcome challenges.

1.1 Iteration method and expansion method

We start with a simple example with a small parameter ε to illustrate the iteration method and the expansion method.

Example 1.1.1 $x^2 + \varepsilon x - 1 = 0.$

Solution: By taking the root formula that the solution to this equation is

$$x = -\frac{1}{2}\varepsilon \pm \sqrt{1 + \frac{1}{4}\varepsilon^2},$$

which can be expended for small ε as

$$x = \begin{cases} 1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 - \frac{1}{128}\varepsilon^4 + O(\varepsilon^6), \\ -1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \frac{1}{128}\varepsilon^4 + O(\varepsilon^6). \end{cases}$$

These expansions are converge if $|\varepsilon| < 2$.

More important than converging, the truncated series gives a good approximation if ε is small. If we choose $\varepsilon = 0.1$, the first terms give

$$x \sim \begin{cases} 1.0, & 1 \text{ term;} \\ 0.95, & 2 \text{ term;} \\ 0.95125, & 3 \text{ term;} \\ 0.95124921, & 4 \text{ term.} \end{cases}$$

and the exact solution is 0.9512492. That implies that the numerical sum of finite-term expansion is more computationally efficient than the evaluation of the exact answer.

We started by finding the exact solution to the quadratic equation and then expanded this exact solution. However, in most problems, finding the exact solution is not feasible. Therefore, we must develop techniques that first make approximations and then involve calculations afterward. There are two distinct methods for first approximating and then calculating: the iterative method and the expansion method. Each method has its own advantages and disadvantages.

① Iterative method

We start with the iterative method because it is a method which is often overlooked, although it has much to offer.

The first step of the iterative method is to find a rearrangement of the original equation, which will become the basis of an iterative process. This first step involves a certain amount of inspiration which must therefore count as a major drawback of the method. A suitable rearrangement of our present quadratic is

$$x = \pm\sqrt{1-\varepsilon x}.$$

Treat it as a fixed-point iteration:

$x = \pm\sqrt{1-\varepsilon x} = g(x) \Rightarrow x_{n+1} = \sqrt{1-\varepsilon x_n}$ near $x = 1$, when $\varepsilon = 0, x = 1$.

Let $x_0 = 1, x_1 = \sqrt{1-\varepsilon} = 1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 - \frac{1}{16}\varepsilon^3 + \cdots$.

We keep the first two terms, i.e.,

$x_1 = 1-\frac{1}{2}\varepsilon, x_2 = \sqrt{1-\varepsilon(1-\frac{1}{2}\varepsilon)} = 1-\frac{1}{2}\varepsilon(1-\frac{1}{2}\varepsilon)-\frac{1}{8}\varepsilon^2(1-\varepsilon+\frac{1}{4}\varepsilon^2)+\cdots = 1-\frac{1}{2}\varepsilon+\frac{1}{8}\varepsilon^2.$

If we keep the first 3 terms, i.e.,

$x_3 = \sqrt{1-\varepsilon(1-\frac{1}{2}\varepsilon+\frac{1}{8}\varepsilon^2)} = 1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + O(\varepsilon^3).$

In order to find more terms, we have to do a lot of work.

It is clear that progressively more work is required to obtain the higher order terms by the iterative method. The method also has the undesirable feature that in the early iterations, it gives erroneous values to the higher terms. One can only check that a term is correct by making one more iteration, which, of course, is usually convincing but no rigorous proof (we leave its proof in future study).

② Expansion method

The first step of the expansion method is to set $\varepsilon = 0$ and find the unperturbed roots $x = \pm 1$. We assume that

$$x(\varepsilon) = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots .$$

Substitute this into the equation

$$1 + \varepsilon(2x_1) + \varepsilon^2(x_1^2 + 2x_2) + \varepsilon^3(2x_1x_2 + 2x_3) + \cdots + \varepsilon + \varepsilon^2 x_1 + \varepsilon^3 x_2 + \cdots = 0.$$

We have

$O(\varepsilon^0) : 1 - 1 = 0;$

$O(\varepsilon^1) : 2x_1 + 1 = 0 \Rightarrow x_1 = -\frac{1}{2};$

$O(\varepsilon^2) : x_1^2 + 2x - 2 + x_1 = 0 \Rightarrow x_2 = \frac{1}{8}$;

$O(\varepsilon^3) : 2x_1x_2 + 2x_3 + x_2 = 0 \Rightarrow x_3 = 0$.

They are the same coefficients as in the exact solution.

1.2 Singular perturbations and rescaling

It is useful to make an imprecise distinction between regular perturbation problems and singular perturbation problems. A regular perturbation problem is one for which the perturbed problem for small, nonzero values of ε is qualitatively the same as the unperturbed problem for $\varepsilon = 0$. One typically obtains a convergent expansion of the solution with respect to ε, consisting of the unperturbed solution and higher-order corrections. In the last section, we give a typical example of a regular perturbation problem.

A singular perturbation problem is one for which the perturbed problem is qualitatively different from the unperturbed problem. One typically obtains an asymptotic, but possibly divergent, expansion of the solution, which depends singularly on the parameter ε. Although singular perturbation problems may appear atypical, they are the most interesting problems to study because they allow one to understand qualitatively new phenomena.

In this section, we give an example of a singular perturbation problem.

Example 1.2.1 $\varepsilon x^2 + x - 1 = 0$.

Solution: When $\varepsilon = 0$, there is one solution $x = 1$. If $\varepsilon \neq 0$, there are two roots:

$$x_{1,2} = -\frac{1}{2\varepsilon} \pm \frac{\sqrt{1+4\varepsilon}}{2\varepsilon} = \begin{cases} 1 - \varepsilon + 2\varepsilon^2 - 5\varepsilon^3 + \cdots, \\ -\frac{1}{\varepsilon} - 1 + \varepsilon - 2\varepsilon^2 + 5\varepsilon^3 + \cdots. \end{cases}$$

As $\varepsilon \to 0$, we have $x_2 \to \infty$. Next, we will use the iterative method and the expansion method to give its asymptotic solution.

① **Iterative method**:

When neglecting -1 term, we can get $\varepsilon x^2 + x \approx 0$ which indicates that $x \approx \frac{1}{\varepsilon}$. Hence, we can assume that

$$x = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon x}.$$

Let $g(x) = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon x}$, $g'(x) = -\frac{1}{\varepsilon x^2}$. When $|g'(x)| < 1$, we can get $|x| > \frac{1}{\sqrt{\varepsilon}}$.

Define $x_{n+1} = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon x_n}$, $x_0 = -\frac{1}{\varepsilon}$.

From computation, we have $x_1 = -\varepsilon^{-1} - 1$, $x_2 = -\varepsilon^{-1} - \frac{1}{1+\varepsilon} = -\varepsilon^{-1} - 1 + \varepsilon + \cdots$

② **Expansion method**:

Assume that

$$x(\varepsilon) = \varepsilon^{-1}x_{-1} + x_0 + \varepsilon x_1 + \cdots .$$

We submit it into the equation:

$$\varepsilon^{-1}x_{-1}^2 + 2x_{-1}x_0 + \varepsilon(2x_{-1}x_1 + x_0^2) + \cdots + \varepsilon^{-1}x_{-1} + x_0 + \varepsilon x_1 + \cdots - 1 = 0.$$

$O(\varepsilon^{-1}) : x_{-1}^2 + x_{-1} = 0 \Rightarrow x_{-1} = -1$ or 0 (regular).

We are looking for a singular root. So $x_{-1} = -1$.

$O(\varepsilon^0) : 2x_{-1}x_0 + x_0 - 1 = 0 \Rightarrow x_0 = -1.$

$O(\varepsilon^1) : 2x_{-1}x_1 + x_0^2 + x_1 = 0 \Rightarrow x_1 = 1.$

How do we know the leading order of x? Previously, let $x = \frac{X}{\varepsilon}$, then we can get $X^2 + X - \varepsilon = 0$; this is a regular form, and it can be solved as before. We assume that

$$X = X_{-1} + \varepsilon X_0 + \varepsilon^2 X_1 + \cdots .$$

In general: $x = \delta X$, $\delta = \delta(\varepsilon)$ may be ε^d. We assume X is strictly of order O(1), i.e. X cannot vanish as $\varepsilon \to 0$ ($X = ord(1)$)

$$\varepsilon\delta^2 X^2 + \delta X - 1 = 0.$$

Case **i)** $\delta \ll 1$. $\varepsilon\delta^2 X^2$ is indefinitely small, δX is also indefinitely small, then we have $small + small - 1$, which is not balance to zero.

ii) $\delta = 1$, $\varepsilon\delta^2 X^2$ is indefinitely small, then we have $small + X - 1 = 0$. Balance both sides, then $X = 1 + small$; this is the regular root.

iii) $1 \ll \delta \ll \varepsilon^{-1}$. $(\varepsilon\delta^2 X^2 + \delta X - 1)/\delta = small + X + small$, balance both side, then $X = 0 + small$, which is contradicts to $X = ord(1)$.

iv) $\delta = \varepsilon^{-1}$. $(\varepsilon\delta^2 X^2 + \delta X - 1)/\varepsilon\delta^2 = X^2 + X + small$, balance both side, then $X = -1 = small$, which is a singular root.

v) $\varepsilon^{-1} \ll \delta$. $(\varepsilon\delta^2 X^2 + \delta X - 1)/\varepsilon\delta^2 = X^2 + small + small$, balance both side, then $X = 0 + small$, which violates $X = ord(1)$.
In summary, $\delta = \varepsilon^{-1}$ for singular root.

In the next example, no three-term dominant balance is possible, but this can occur in other problems

Example 1.2.2 (Wilkinson'spolynomial(1964)) *Find the solution of*

$$(x-1)(x-2)\cdots(x-20) = \varepsilon x^{19}, \quad x_0 = 1, 2, \cdots, 20.$$

Solution: Near $x_0 = 1$, we set that $x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$. Then

$(\varepsilon x_1 + \varepsilon^2 x_2 + \cdots)(-1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)\cdots(-19 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) = \varepsilon(1 + \varepsilon x_1 + \cdots)^{19} = \varepsilon(1 + 19\varepsilon x_1 + O(\varepsilon^2))$.

We have:

$$O(\varepsilon) : (-1)^{19} \cdot 19! x_1 = 1 \Rightarrow x_1 = -\frac{1}{19!};$$
$$O(\varepsilon^2) : x_2 = \frac{1}{19!}[\frac{1}{19!}(1 + \cdots + \frac{1}{19}) - \frac{1}{(18!)^2 \cdot 19}].$$

1.3 Non-integer powers

At this point, we should ask ourselves whether we can always use integer powers. We will soon explore an example involving non-integer powers.

Example 1.3.1 $(1 - \varepsilon)x^2 - 2x + 1 = 0$.

Solution: If $\varepsilon = 0$, we konw that $x = 1$. We use integer powers to expand this equation and get that
$$x(\varepsilon) = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots .$$
Substitute it into the equation:

$$1 + \varepsilon(2x_1) + \varepsilon^2(2x_2 + x_1^2) + \cdots - \varepsilon - \varepsilon^2(x_1) + \cdots - 2 - \varepsilon(2x_1) - \varepsilon^2(2x_2) + \cdots + 1 = 0.$$

Balance both sides:

$O(\varepsilon^0) : 1 - 2 + 1 = 0.$

$O(\varepsilon^1) : 2x_1 - 1 - 2x_1 = 0.$

This is a contradiction, and it is correct only if $x_1 = \infty$. So why is there a conflict? In fact, $x = \frac{1 \pm \varepsilon^{\frac{1}{2}}}{1-\varepsilon}$ is the exact sol. Let

$$x = 1 + x_{\frac{1}{2}}\varepsilon^{\frac{1}{2}} + x_1\varepsilon + x_{\frac{3}{2}}\varepsilon^{\frac{3}{2}} + \cdots .$$

Plug it into the equation:

$O(\varepsilon^0) : 1 - 2 + 1 = 0.$

$O(\varepsilon^{\frac{1}{2}}) : 2x_{\frac{1}{2}} - 2x_{\frac{1}{2}} = 0.$ How can we determine $x^{\frac{1}{2}}$?

$O(\varepsilon^1) : 2x_1 + x_{\frac{1}{2}}^2 - 1 - 2x_1 = 0 \Longrightarrow x_{\frac{1}{2}} = \pm 1$?

x_1 is to be determined in the next level.

$O(\varepsilon^{\frac{3}{2}}) : 2x_{\frac{3}{2}} + 2x_{\frac{1}{2}}x_1 - 2x_{\frac{1}{2}} - 2x_{\frac{3}{2}} = 0 \Rightarrow x_1 = 1$ for both values of $x_{\frac{1}{2}}$.

In general, $x_{\frac{n}{2}}$ is determined at $O(\varepsilon^{\frac{n+1}{2}})$.

In the case where no knowledge of the exact answer is available, we can assume that

$x(\varepsilon) = 1 + \delta_1(\varepsilon)x_1 + \delta_2(\varepsilon)x_2 + \cdots 1 \gg \delta_1(\varepsilon) \gg \delta_2(\varepsilon) \gg \cdots x_1, x_2, \cdots = ord(1)$.

Plug it into the equation:

$$
\begin{aligned}
&1 + 2\delta_1 x_1 + \delta_1^2 x_1^2 + 2\delta_2 x_2 + 2\delta_1\delta_2 x_1 x_2 + \delta_2^2 x_2^2 + \cdots \\
&- \varepsilon - 2\varepsilon\delta_1 x_1 - \varepsilon\delta_1^2 x_1^2 - 2\varepsilon\delta_2 x_2 + \cdots \\
&- 2 - 2\delta_1 x_1 - 2\delta_2 x_2 + \cdots + 1 = 0.
\end{aligned}
$$

Look at the first line: $2\delta_1 x_1 \gg \delta_1^2 x_1^2$; $2\delta_1 x_1 \gg 2\delta_2 x_2$, since $1 \gg \delta_2$, $\delta_1 \gg \delta_2$.

And how to compare $\delta_1^2 x_1^2$ and $2\delta_2 x_2$?

Simplify $\delta_1^2 x_1^2 + 2\delta_1\delta_2 x_1 x_2 + \delta_2^2 x_2^2 + \cdots - \varepsilon - 2\varepsilon\delta_1 x_1 - \varepsilon\delta_1^2 x_1^2 - 2\varepsilon\delta_2 x_2 + \cdots = 0$.

There is no need to compare $\delta_1^2 x_1^2$ and $2\delta_2 x_2$, but we need to compare the first line with the second line! So there are three possibilities compared $\delta_1^2 x_1^2$ with ε.

① $\delta_1^2 x_1^2 = 0$ if $\delta_1^2 \gg \varepsilon$;

② $\delta_1^2 - \varepsilon = 0$ if $\delta_1^2 = \varepsilon$;

③ $-\varepsilon = 0$ if $\delta_1^2 \ll \varepsilon$.

Therefore $\delta_1 = \varepsilon^{\frac{1}{2}}$ and $x_1 = \pm 1$ (corresponding to $x_{\frac{1}{2}} = \pm 1$).

Next we compare $2\delta_1\delta_2 x_1 x_2$ with $-2\varepsilon\delta_1 x_1$.

① if $\delta_2 \gg \varepsilon$, $2\varepsilon^{\frac{1}{2}}\delta_2 x_1 x_2 = 0$;

② if $\delta_2 = \varepsilon$, $2\varepsilon^{\frac{1}{2}}\delta_2 x_1 x_2 - 2\varepsilon^{\frac{3}{2}} x_1$;

③ if $\delta_2 \ll \varepsilon$, $-2\varepsilon^{\frac{3}{2}} x_1 = 0$, $\delta_2 = \varepsilon$, and $x_2 = 1$ (both $x_1 = \pm 1$).

Iterative method:

$(x-1)^2 = \varepsilon x^2 \iff x = 1 \pm \varepsilon^{\frac{1}{2}} x$.

$x_{n+1} = 1 \pm \varepsilon^{\frac{1}{2}} x_n$, $x_0 = 1$, $g(x) = 1 \pm \varepsilon^{\frac{1}{2}} x$.

$x_1 = 1 + \varepsilon^{\frac{1}{2}}$, $x_2 = 1 + \varepsilon^{\frac{1}{2}} + \varepsilon \cdots$.

$|g'(x)| = |\pm \varepsilon^{\frac{1}{2}}| = \varepsilon^{\frac{1}{2}} < 1$.

1.4 Logarithms

Unlike algebraic equations, it is harder to determine the number of solutions of transcendental equations in most cases and we must resort to graphical method. Consider the equation

$$xe^{-x} = \varepsilon.$$

From Figure 1.1, we see that there are two real solutions

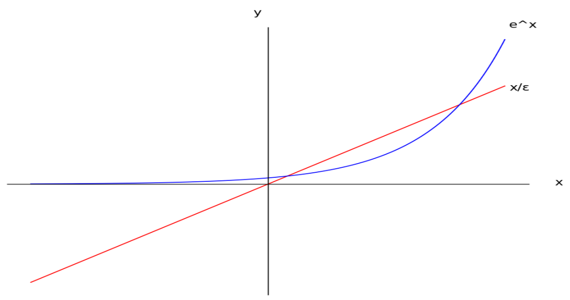


Fig. 1.1: $\frac{1}{\varepsilon}x = e^x$.

Let $f(x) = xe^{-x} - \varepsilon$, $f(0) = -\varepsilon < 0$, $f(1) = e^{-1} - \varepsilon > 0$, $f(\infty) = -\varepsilon < 0$. So one root is near $x_1 = \varepsilon$; the other root $x_2 \to \infty$ as $\varepsilon \to 0$. To find the first root, we assume that $x = \varepsilon e^x = g_1(x)$, $g_1'(x) = \varepsilon e^x < 1$ when $x < \ln \frac{1}{\varepsilon}$.

How can we find another large root?

As $\varepsilon < \frac{1}{4}$, $\ln \frac{1}{\varepsilon} < x_2 < 2\frac{1}{\varepsilon}$ and $f(\ln \frac{1}{\varepsilon}) = -\varepsilon \ln \varepsilon - \varepsilon > 0$, $f(2\ln \frac{1}{\varepsilon}) = -2\varepsilon^2 \ln \varepsilon - \varepsilon < 0$. So the second root$\in (\ln \frac{1}{\varepsilon}, 2\ln \frac{1}{\varepsilon})$.

$$e^{-x} = \frac{\varepsilon}{x} \Rightarrow x = \ln \frac{1}{\varepsilon} + \ln x.$$

Assume that $g_2(x) = \ln \frac{1}{\varepsilon} + \ln x$. We have $g_2'(x) = \frac{1}{x} < 1$ if $x > 1$. Hence,

$$x_{n+1} = \ln \frac{1}{\varepsilon} + \ln x_n \sim \ln \frac{1}{\varepsilon},$$

which indicates that

$$x_0 = \ln \frac{1}{\varepsilon} \Rightarrow x_1 = \ln \frac{1}{\varepsilon} + \ln \ln(\frac{1}{\varepsilon}) = L_1 + L_2.$$

$$x_2 = L_1 + \ln \left(L_1(1 + \frac{L_2}{L_1}) \right) = L_1 + \ln L_1 + \ln(1 + \frac{L_2}{L_1}) = L_1 + L_2 + \frac{L_2}{L_1} - \frac{L_2^2}{2L_1^2} + \cdots .$$

$$\begin{aligned} x_3 &= L_1 + \ln \left(L_1(1 + \frac{L_2}{L_1} + \frac{L_2}{L_1^2} - \frac{L_2^2}{2L_1^3}) \right) \\ &= L1 + L_2 + (\frac{L_2}{L_1} + \frac{L_2}{L_1^2} - \frac{L_2^2}{2L_1^3}) - \frac{1}{2}(\frac{L_2}{L_1} + \frac{L_2}{L_1^2 + \cdots})^2 + \frac{1}{3}(\frac{L_2}{L_1} + \cdots)^3 \\ &= L_1 + L_2 + \frac{L_2}{L_1} + \frac{1}{L_1^2}(-\frac{1}{2}L_1^2 + L_2) + \frac{1}{L_1^3}(\frac{1}{3}L_2^3 - \frac{3}{2}L_2^2 + \cdots). \end{aligned}$$

It is difficult to guess the order of expansion. $L_1 + L_2$ is dominant term as $\varepsilon \to 0$, the other terms such as $\frac{L_2}{L_1}$, $\frac{1}{L_1^2}(-\frac{1}{2}L_2^2 + L_2), \cdots$ shall be small terms as $\varepsilon \to 0$, since $L_1 \gg L_2 \gg 1$. But how small shall they be?

Compare L_2 with 1, $\ln\ln\frac{1}{\varepsilon} > 3 \Rightarrow \varepsilon < 10^{-9}$. Here is a table about the error showing the error.

Table 1.1: error percentage %

ε	L_1	L_2	$\frac{L_2}{L_1}$	$-\frac{1}{2}\frac{L_2^2}{L_1^2}$	$\frac{L_2}{L_1^2}$
10^{-1}	36	12	2	4	0.03
10^{-3}	24	3	0.02	0.04	0.04
10^{-5}	19	1	0.04	0.1	0.001

Numerically, we hope the expansion converges fast as $\varepsilon < 0.1$. There are two options:

① very small ε, $\varepsilon < 10^{-5}$ or ② a great many terms if $\varepsilon \sim 0.1$.

Another feature: $\frac{1}{L^2}(-\frac{1}{2}L_2^2 + L_2) = \frac{1}{L_1^2}(-\frac{1}{2}L_2^2)$(large error)$+\frac{L_2}{L_1^2}$.

Remark: To find a large root, you need to do expansion at least up to the terms such that the remainder is small.

1.5 Convergence

Limitations of the expansion method:

- In some difficult problems, the expansion sequence is not clear, no general form!
- No strong bounds on the expansion terms thus no convergence.

However, the iterative method provides a simple proof of convergence.

Let $x = x^*$ be the root of $x = g(x)$, we have $x_{n+1} = f(x^* + \delta) = x^* + \delta$. This is a fixed-point iteration method. Suppose at step n, $x_n = x^* + \delta$, then $x_{n+1} = f(x^* + \delta) = x^* + \delta g'(x^*) + o(\delta)$.

The error is decreased if $|g'(x^*)| < 1$. Contraction mapping theorem $\Rightarrow x_n \to x^*$.

(1) $g(x) = \sqrt{1 - \varepsilon x}$, $x^* \sim 1$, $g'(x^*) \sim -\frac{1}{2}\varepsilon$.

(2) $g(x) = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon x}$, $x^* \sim -\frac{1}{\varepsilon}$, $g'(x^*) \sim \varepsilon$.

(3) $g(x) = 1 + \varepsilon^{\frac{1}{2}} x$, $x^* \sim 1$, $g'(x^*) \sim \varepsilon^{\frac{1}{2}}$.

(4) $g(x) = \ln \frac{1}{\varepsilon} + \ln x$, $x^* \sim \ln \frac{1}{\varepsilon}$, $g'(x^*) \sim -\frac{1}{\ln \varepsilon}$.

A negative sign of $g'(x^*)$ means an error change sign. $|g'(x^*)|$ can roughly tell how many terms will be correct after a given number of iterations.

1.6 Eigenvalue problems

In this section, we consider a class of eigenvalue problems.

Example 1.6.1 *$Ax + \varepsilon B(x) = \lambda x$, where x is eigenvector and λ is the eigenvalue. Ax is linear whereas $B(x)$ may not be. Assume A is finite-dimensional (also true for compact A).*

Solution: If $\varepsilon = 0$, $Ae = ae$, a is eigenvalue, $A^T\tilde{e} = a\tilde{e}$,

In fact, if e is the first column vector of P.

$$P^{-1}AP = J = \begin{pmatrix} a & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix},$$

then $\tilde{e}$ is the first row vector of P^{-1}. We have $\tilde{e}^T A = a\tilde{e}^T$, $e^+ = \tilde{e}^T$. If a is a single root, $\tilde{e}^T$ is perpendicular to other eigenvectors. Assume that

$$x(\varepsilon) = e + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots,$$
$$\lambda(\varepsilon) = a + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots.$$

We have

$O(\varepsilon^0)$: $Ae = ae$.

$O(\varepsilon)$: $Ax_1 + B(e) = ax_1 + \lambda_1 e$, $(A - aI)x_1 = \lambda_1 e - B(e)$.

Since $\tilde{e} \in ker(A - aI)^T$, $0 = \tilde{e}^T(A - aI)x_1 = \tilde{e}^T\Big(\lambda_1 e - B(e)\Big) \Rightarrow \lambda_1 = \frac{\tilde{e}^T B(e)}{\tilde{e}^T e}$.

$AP = PJ$, $P = (e_1, \cdots, e_n) \Rightarrow A^T P^{-T} = P^{-T} J^T$, $P^{-T} = (\tilde{e}_1, \cdots, \tilde{e}_n)$.

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix}, \ J_1 = \lambda_1.$$

$A^T\tilde{e}_1 = \lambda_1\tilde{e}_1$, in general, $A^T\tilde{e}_i = \lambda_i\tilde{e}_i + \tilde{e}_{i+1}$; $A^T e_i = \lambda_i e_i + e_{i+1}$.

$$P^{-1} = \begin{pmatrix} \tilde{e}_1^T \\ \vdots \\ \tilde{e}_n^T \end{pmatrix}, \ I = P^{-1}P = (\tilde{e}_i^T e_j), \text{ where } \tilde{e}_i^T e_j = \delta_{ij}.$$

$(A-aI)x_1 = -B(e) + \frac{\tilde{e}^T B(e)}{\tilde{e}^T e} e = -B(e)_\perp$, $B(e)_\perp \perp \tilde{e}^T$ by construction.

Hence, $B(e)_\perp$ can be expanded in terms of $e_2, \cdots, e_n$. $(A-aI)x_1 = -B(e)_\perp$ has unique solution up to an additive vector $k_1 e$, i.e. $x_1 = \tilde{x}_1 + k_1 e$, where $\tilde{x}_1$ is unique solution on $\{\tilde{e}^T\}^\perp = \{e_2, \cdots, e_n\} \ni B(e)_\perp$. We have

$$\tilde{x}_1 = \sum_{j\geq 2} v_j e_j, \quad (A-aI)\tilde{x}_1 = \sum_{j\geq 2} v_j (a_j - a) e_j = -B(e)_\perp.$$

That means $v_j(a_j - a) = -\frac{e_j^+ B(e)_\perp}{e_j^+ e_j}$,

Dual basis$\begin{cases} e_j^+ e_i = 0, \text{if } i \neq j \\ e_j^+ e_j = 1 \end{cases}$, $v_j = \frac{e_j^+ B(e)_\perp}{(a-a_j) e_j^+ e_j}$. Requiring $\tilde{e}^T x_1 = 0$, we can remove k_1.

2nd order expansion

$$Ax_2 + B_1 = ax_2 + \lambda_1 x_1 + \lambda_2 e.$$

εB_1 is the $O_s(\varepsilon)$ term in $B(e+\varepsilon x_1)$, i.e. the first order derivative part. If B is linear, $B_1 = Bx$; If B is nonlinear, $B_1 = B'(e)x_1$, $B'(e)$ is the Jacobi matrix. We have

$$(A-aI)x_2 = \lambda_2 e + \lambda_1 x_1 - B_1, \quad \tilde{e}^T (A-aI)x_2 = \tilde{e}^T(\lambda_2 e + \lambda_1 x_1 - B_1),$$

That indicates that $\lambda_2 = \frac{\tilde{e}^T(B_1 - \lambda_1 x_1)}{\tilde{e}^T e} = \frac{\tilde{e}^T B_1}{\tilde{e}^T e}$. Then $x_2 = -(A-aI)^{-1}(B_1 - \lambda_1 x_1)_\perp + k_2 e$, remove k_2 by requiring $\tilde{e}^T x_2 = 0$.

Specifically, if B is linear, $B_1 = Bx_1$. $\lambda_2 = \sum_{j\geq 2} \frac{(e^+ B e_j)(e_j^+ B e)}{(a-a_j)(e_j^+ e_j)(e^+ e)}$.

Multiple roots

$$Ae_i = ae_i, i = 1, \cdots, k.$$

The Jordan block associate to a is diagonal. Assume that

$$x = \sum_{i=1}^k \alpha_i e_i + \varepsilon x_1 + \cdots, \qquad \lambda = a + \varepsilon \lambda_1 + \cdots.$$

Submit it to the equation: $(A-aI)x_1 = \lambda_1 \sum_{i=1}^k \alpha_i e_i - B(\sum_{i=1}^k \alpha_i e_i)$. Still have $\tilde{e}_i^T e_j = \delta_{ij}$. Multiplying by $\tilde{e}_j^T$ on the left, $0 = \lambda_1 \sum_{i=1}^k \alpha_i (\tilde{e}_j^T e_i) - \tilde{e}_j^T B(\sum_i \alpha_i e_i)$

$$\begin{aligned} \lambda_1 \alpha_1 &= \tilde{e}_1^T B(\sum_i \alpha_i e_i)/(\tilde{e}_1^T e_1) \\ &\vdots \qquad\qquad \vdots \\ \lambda_1 \alpha_k &= \tilde{e}_k^T B(\sum_i \alpha_i e_i)/(\tilde{e}_k^T e_k) \end{aligned}$$

where $\alpha = (\alpha_1, \ldots, \alpha_k)^T$, eigenvalue problem, $F(\alpha) = \lambda_1 \alpha$.

If B is linear, $\tilde{e}_j^T B(\sum_i \alpha_i e_i) = \sum_i \alpha_i \tilde{e}_j^T B e_i$, and

$$RHS = (\frac{\tilde{e}_j^T B e_i}{\tilde{e}_j^T e_j}) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix} = (P^{-1})_k B P_k \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix}.$$

There are k eigenvalues, k (indenpendent) eigenvectors.

If B is nonlinear, maybe there are no eigensolutions and no perturbed expansion near the unperturbed problem.

Degenerate roots

$$\begin{aligned} Ae_1 &= ae_1, \\ Ae_2 &= ae_2 + e_1, \\ &\vdots \\ Ae_k &= ae_k + e_{k-1}. \end{aligned}$$

If $\varepsilon B(e_1)$ has component in e_k, say εB_k, then

$$\begin{aligned} &x(\varepsilon) = e_1 + \varepsilon^{\frac{1}{k}} x_2 e_2 + \varepsilon^{\frac{2}{k}} x_3 e_3 + \cdots + \varepsilon^{\frac{k-1}{k}} x_k e_k + \cdots, \\ &\lambda(\varepsilon) = a + \varepsilon^{\frac{1}{k}} \lambda_1 + \cdots, \\ &x_2 = \lambda_1,\ x_3 = \lambda_1^2, \cdots, x_k = \lambda_1^{k-1},\ \lambda_1 = B_k^{\frac{1}{k}} (from O(\varepsilon)). \end{aligned}$$

$O(\varepsilon^{\frac{i}{k}}) : x_i A e_i = x_i(ae_i + e_{i-1}) = a x_i e_i + \lambda_i x_{i-1} e_{i-1} \Rightarrow x_i = \lambda_1 x_{i-1}.$

If $\varepsilon B(e_1)$ has no component in $e_{i+1}, \cdots, e_k$ then expansion in powers of $\varepsilon^{\frac{1}{l}}$. We can try

$$\begin{aligned} x(\varepsilon) &= e_1 + \sum_{i\geq 2} \alpha_i e_i + \delta(\sum_{i \geq 2} e_i) + \cdots, \\ \lambda(\varepsilon) &= a + \delta \lambda_1 + \cdots. \end{aligned}$$

Hence, we have $\alpha_i = 0, i \geq 2;\ \beta_i = 0, i \geq 3$.

Remark:

1.(Fredholm alternatives) H: Hilbert, $K : H \to H$ compact. Then:

(i) $\begin{cases} (I-K)u = f \\ (I-K^*)U = F \end{cases}$ have a unique solution for every $f, F \in H$.

or (ii) $(I-K)V_0 = 0$ and $(I-K^*)v_0 = 0$ have the same finite number of nontrivial solutions:

$$1 \leq dim(N(I-k)) = dim(N(I-K^*)) < \infty$$

In this case,the equations in (i) have a solution if and only if $(f, v_0) = 0$, $\forall v_o \in N(I-K^*)$ and $(F, v_0) = 0$, $\forall v_o \in N(I-K)$ established respectively.

2. Regular v.s. Singular

Regular perturbation: qualitatively similar to the unperturbed solution. Singular perturbation: qualitatively different to the unperturbated solution, usually divergent expansion.

3. General perturbation problem

We have $p^\varepsilon(x) = 0$, N-term asymptotic solution: x^ε_N.

$$p^\varepsilon(x^\varepsilon_N) = O(\varepsilon^{N+1}) \overset{?}{\Rightarrow} x^\varepsilon = x^\varepsilon_N + O(\varepsilon^{N+1}).$$

Small error in equation $\overset{?}{\Rightarrow}$ small error in solution. It can be proved if we have stability estimate $\|x-y\| \le C\|P^\varepsilon(x) - P^\varepsilon(y)\|$. It is hard to establish.

4. Taylor expansion approach

If $x^2 + \varepsilon x - 1 = 0$, then $x(\varepsilon) = x(0) + x(0)\varepsilon + \frac{1}{2}x(0)\varepsilon^2 + \cdots$.

Differentiating w, r, t, ε, we have

$$2xx' + x = 0.$$

Set $\varepsilon = 0$, and $2x(0)x'(0) + x(0) = 0 \Rightarrow x'(0) = -\frac{1}{2}$. Then we can continue to do this$\cdots$

5. Eigenvalue problem: in infinite dimensional cpace

(a) $A^\varepsilon : D(A^\varepsilon) \subset H \to H$ represents self-adjoint and we have $(x, A^\varepsilon y) = (A^\varepsilon x, y), \forall x, y \in D(A^\varepsilon)$.

(b) A^ε has a smooth branch of simple eigenvalue. $\lambda^\varepsilon \in \mathbb{R}$ with $x^\varepsilon \in H$. $A^\varepsilon x^\varepsilon = \lambda^\varepsilon x^\varepsilon$,

$$\begin{aligned} A^\varepsilon &= A_0 + \varepsilon A_1 + \cdots + \varepsilon^n A_n + \cdots, \\ x^\varepsilon &= x_0 + \varepsilon x_1 + \cdots + \varepsilon^n x_n + \cdots, \\ \lambda^\varepsilon &= x_0 + \varepsilon \lambda_1 + \cdots + \varepsilon^n \lambda_n + \cdots. \end{aligned}$$

$O(1) : (A - \lambda_0 I)x_0 = 0 \Rightarrow x \neq 0$ is eigenvector of λ_0.

$O(\varepsilon) : (A_0 - \lambda_0 I)x_0 = -A_1 x_0 + \lambda_1 x_0 \Rightarrow$ singular equation.

$\vdots$

$O(\varepsilon^n) : (A_0 - \lambda_0 I)x_n = \sum_{v=1}^n (-A_i x_{n-i} + \lambda_i x_{n-i})$.

Property: the necessary condition for the existence of $y \in H$ of the equation $(A - \lambda I)y = z$ is that $(x, z) = 0$ for $\forall x \in N(A - \lambda I)$. Then, $\lambda_1 = \frac{(x_0, A_1 x_0)}{(x_0, x_0)}$ is the solvability condition. $x_1 = \bar{x_1} + c_1 x_0$, $\bar{x_1}$ is unique. $c^\varepsilon = 1 + \varepsilon c_1 + O(\varepsilon^2)$, $c^\varepsilon x^\varepsilon = x_0 + \varepsilon(x_1 + c_1 x_0) + O(\varepsilon^2)$ is also a solution.

Example 1.6.2 *Schrodinger Equation of quantum mechanics*

$$\begin{cases} ih\Psi_t = H\Psi, \\ \Psi(t) = e^{\frac{-iEt}{h}}\psi. \end{cases}$$

H is Hamiltonian, $\psi \in H, E \in \mathbb{R}$. Then, $H\psi = E\psi$, where E is the energy eigenstste. Solving this problem leads to the discrete energy levels of atoms.

Solution: Let $H^\varepsilon = H_0 + \varepsilon H_1 + O(\varepsilon^2)$, then $H_0\psi_0 = E_0\psi_0$, $E^\varepsilon = E_0 + \varepsilon \frac{(\psi_0, H_1\psi_0)}{(\psi_0,\psi_0)} + O(\varepsilon^2)$, e.g. $H = -\frac{h^2}{2m}\Delta + V$.
Inner product is complex Hilbert space:$\Psi \in L^2(\mathbb{R}^d)$

$$\begin{aligned} (u, Hv) &= \int_{\mathbf{R^d}} \bar{u}(-\frac{h^2}{2m}\Delta v + Vv)dx = \int_{\mathbf{R^d}} \frac{h^2}{2m}\nabla\cdot(v\nabla\bar{u} - \bar{u}\nabla v) - \frac{h^2}{2m}(\Delta\bar{u}v + V\bar{u}v)dx \\ &= \overline{\int_{\mathbf{R^d}} (-\frac{h^2}{2m}\Delta u + Vu)} v dx = (Hu, v), \end{aligned}$$

where H is self-adjoint. Assume V depends on ε: $V^\varepsilon(x) = V_0(x) + \varepsilon V_1(x) + O(\varepsilon^2)$. That indicates

$$1 = \frac{\int_{\mathbf{R^d}} V_1(x)|\psi_0(x)|^2 dx}{\int_{\mathbf{R^d}} |\psi_0(x)|^2 dx},$$

where ψ_0 satisfies

$$-\frac{h^2}{2m}\Delta\psi_0 + V_0\psi_0 = E_0\psi_0.$$

Example 1.6.3 *Harmonic oscillator in one-dimension*

$$\begin{cases} V_0(x) = \frac{1}{2}kx^2, \\ -\frac{h^2}{2m}\psi'' + \frac{1}{2}kx^2\psi = E\psi. \end{cases}$$

Solution: We can obtain that

$$\begin{cases} E_n = h\omega(n + \frac{1}{2}), n = 0, 1, 2\cdots \\ \omega = \sqrt{\frac{k}{m}} \\ \psi_n(x) = H_n(\alpha x)e^{-\alpha^2x^2/2} \end{cases}$$

where $H_n(\zeta) = (-1)^n e^{\zeta^2}\frac{d^n}{d\zeta^n}e^{-\zeta^2}$ is n-th Hermite polynomial, $\alpha^2 = \frac{\sqrt{mk}}{n}$. Let $V^\varepsilon(x) = \frac{1}{2}kx^2 + \varepsilon\frac{k}{\alpha^2}W(\alpha x) + O(\varepsilon^2), \varepsilon \to 0^+$ and $E_n^\varepsilon = h\omega[n + \frac{1}{2} + \varepsilon \triangle_n + O(\varepsilon^2)], \varepsilon \to 0^+$ $\Delta_n = \frac{\int W(\zeta)H_n^2(\zeta)e^{-\zeta^2}d\zeta}{\int H^2(\zeta)e^{-\zeta^2}d\zeta}$

1.7 Dimension analysis and nondimensionalization

Dimension analysis is important, and we start with a simple example to illustrate it

Example 1.7.1 *Let $f \in C^1[0,1]$ with $f(0) = 0$, show that*

$$\int_0^1 |f(x)|^2 \mathrm{d}x \leqslant \frac{1}{2} \int_0^1 (1 - x^2)|f'(x)|^2 \mathrm{d}x,$$

where "=" iff $f(x) = cx$ for some constant c.

Remark: There is a smart way to prove this by an application of Cauthy-Schwarz inquality. First observe that

$$\Big(\int_0^x f'(t)\mathrm{d}t \Big)^2 \leqslant x \int_0^x |f'(t)|^2 \mathrm{d}t,$$

where "=" if and only if $f'(x) = c$. After an integration, this implies

$$\int_0^1 \Big(\int_0^x f'(t)\mathrm{d}t \Big)^2 \mathrm{d}x \leqslant \int_0^1 x \int_0^x |f'(t)|^2 \mathrm{d}t\mathrm{d}x = \frac{1}{2}x^2 \int_0^x (f'(t))^2 \mathrm{d}t \Big|_{x=0}^1 - \int_0^1 \frac{1}{2}x^2 (f'(x))^2 \mathrm{d}x.$$

Another viewpoint: Assume the quantities are physical with dimensions. Treat the integral as the total number and f^2 as number density (per unit length). From this viewpoint, x should have the dimension of length scale. But the term $1 - x^2$ implies that x has dimension 1. We can then interpret x as the dimensionless variable derived from a dimensional variable.

2nd proof: Let $y = Lx$ be the corresponding dimensional variable where L is the dimension of length scale. Under this change of variable, the objective inequality is equivalent to

$$\frac{1}{L} \int_0^L |g(y)|^2 \mathrm{d}y \leqslant \frac{1}{2} \int_0^L (L^2 - y^2)|g'(y)|^2 \mathrm{d}y$$

where $g(y) = f(\frac{y}{L})$ and $L > 0$ is arbitrary. Note that when $L = 1$, it reduces to the original inequality. It looks like there is a family of inequalities holding for any scaling parameter L, but they are actually equivalent. This is called self-similar property.

By introducing one more variable L, we can prove the result by basic calculus. Let

$$F(L) = \frac{1}{2} \int_0^L (L^2 - y^2)(g'(y))^2 \mathrm{d}y - \frac{1}{L} \int_0^L (g(y))^2 \mathrm{d}y.$$

Since $F(0) = 0$, it is sufficient to show that $F'(L) = L \int_0^L (g'(y))^2 \mathrm{d}y - g(L)^2 \geqslant 0$ for any $L > 0$. This is equivalent to the Cauthy-Schwarz inequality

$$\Big(\int_0^1 f'(x)\mathrm{d}x \Big)^2 = f(1)^2 \leqslant \Big(\int_0^1 f'(x)\mathrm{d}x \Big)^2. \quad \text{(Exactly the key in the first proof)}$$

Again, you can go ahead without using Cauthy-Schwarz inequality. Instead, continue the basic calculus by letting $G(L) = \frac{F'(L)}{L} = \int_0^L (g'(y))^2 \mathrm{d}y - \frac{g(L)^2}{L}$. Since $G(0) = 0$, it is sufficient to show that $G'(L) \geqslant 0$. This is nothing but a complete square.

3rd "proof": Calculus of variation.

Consider the minimization problem of the functional

$$E(f) = \int_0^1 \Big(\frac{1}{2}(1-x^2)(f'(x))^2 - f(x)^2\Big)\mathrm{d}x.$$

in $W = \{f \in C^2[0,1] : f(0) = 0\}$. Assume there is a minimizer at f, then the functional evaluated at any perturbation $f + \epsilon h$ should have a larger value than $E(f)$:

$$0 \leqslant E(f+\epsilon h)-E(f) = \epsilon\Big((1-x^2)f'(x)h(x)\Big|_{x=0}^{x=1} - \int_0^1 (-2xf'(x)+(1-x^2)f''(x)+2f(x))h(x)\Big)+o(\epsilon),$$

for any $h \in W$. The boundary term vanishes since $h(0) = 0$. Then we must have

$$(1-x^2)f''(x) - 2xf'(x) + 2f(x) = 0.$$

Obviously, the linear function $f(x) = cx$ is a solution. Then we can use the Wrongskian to find another independent solution. It can be verified that $f(x) = cx$ minimizes $E(f)$ and the minimum is 0.

To illustrate the nondimensionalization, we introduce the projectile problem.

Example 1.7.2 *First we consider motion of an object projected upward from the surface of Earth, $x(t)$ reprensents the height measured from the surface.*

Solution: By Newton's second law and gravitational law, we have

$$\begin{cases} \dfrac{Gm}{R^2} = g, \\ x''(t) = a, \end{cases}$$

where R is the radius of the earth. Hence,

$$\begin{cases} x''(t) = -g\dfrac{R^2}{(x+R)^2}, t > 0, \\ x(0) = 0, x'(0) = v_0. \end{cases}$$

We use scaling: $\tau = \frac{t}{t_c}, y(\tau) = \frac{x(t)}{x_c}$, $t_c = \frac{v_0}{g}$ (characteristic time), and $x_c = \frac{v_0^2}{g}$ (characteristic height). Then

$$\begin{cases} \dfrac{d^2y}{d\tau^2} = -\dfrac{1}{(1+\varepsilon y)^2}, \quad \tau > 0, \\ y(0) = 0, \ y'(0) = 1, \end{cases},$$

where $\varepsilon = \frac{v_0^2}{R_g}$ is the ratio of height to R, and it's true that $\varepsilon \ll 1$ when the particle is not far from the surface. When $\varepsilon = 0$, we can see $y_0'' = -1$, $y_0(0) = 0$ and $y_0'(0) = 1$. Hence, we have $y_0 = -\frac{1}{2}\tau^2 + \tau$ that indicates $X_0(t) = -\frac{1}{2}gt^2 + V_0 t$. We assume that

$$y = y_0(\tau) + \varepsilon^\alpha y_1(\tau) + \cdots,$$

thus

$$y_0'' + \varepsilon^\alpha y_1(\tau) + \cdots = -\frac{1}{(1+\varepsilon(y_0+\varepsilon^\alpha y_1+\cdots))^2}$$
$$\sim -1 + 2\varepsilon y_0 + \cdots .$$

Also, we have the initial condition,

$$y_0(0) + \varepsilon^\alpha y_1(0) + \cdots = 0,$$
$$y_0'(0) + \varepsilon^\alpha y_1'(0) + \cdots = 1.$$

Then we can get

$$O(1) : y_0''(\tau) = -1, y_0(0) = 0, y_0'(0) = 1 \Rightarrow y_0 = -\frac{1}{2}\tau^2 + \tau;$$
$$O(\varepsilon^\alpha) : \varepsilon^\alpha \sim \varepsilon \Rightarrow \alpha = 1;$$
$$O(\varepsilon) : y_1'' = 2y_0, y_1(0) = y_1'(0) = 0 \Rightarrow y_1 = \frac{1}{3}\tau^3 - \frac{1}{12}\tau^4.$$

Therefore, we have

$$y(\tau) \sim \tau(1-\frac{1}{2}\tau) + \frac{1}{3}\varepsilon\tau^3(1-\frac{1}{4}\tau).$$

Fundamental system of units:

A physical system (model) is always characterized by some basic quantities. $(a_1, \cdots, a_n)$, including independent and dependent variables as well as parameters. Its dimension is $[a]$ and the independent units $(d_1, \cdots, d_r)$ are fundamental system of units. We write $[a] = d_1^{\alpha_1} \cdots d_r^{\alpha_r}$.

Example 1.7.3 (Mechanical problem) *Set $d_1 = M$, $d_2 = L$, $d_3 = T$ reprensents mass, length and time respectively. Then we have $V = \frac{L}{T}$, momentum $P = \frac{ML}{T}$ are derived units. We could also use P, V and T as fundemental units then $M = \frac{P}{V}$ and $L = VT$ are derived units. This could also use K(Kelvin) and A(Amperes).*

The invariance of a model under the change in units $d_j \to \lambda_j d_j$ implies it is invariant under the scaling transformation: $a_i \to \lambda_1^{\alpha_{1,i}} \cdots \lambda_r^{\alpha_{r,i}} a_i$ where $[a_i] = d_1^{\alpha_{1,i}} \cdots d_r^{\alpha_{r,i}}$. If $a = f(a_1, \cdots, a_n)$ is a relation in the model, then f has the scaling property that $\lambda_1^{\alpha_1} \cdots \lambda_r^{\alpha_r} f(a_1, \cdots, a_n) = f(\lambda_1^{\alpha_{1,1}} \cdots \lambda_r^{\alpha_{r,1}} a_1, \cdots, \lambda_1^{\alpha_{1,n}} \cdots \lambda_r^{\alpha_{r,n}} a_n)$.

Specifically, if $f(a_1) = a_1$ then $\lambda_1^{\alpha_1} \cdots \lambda_r^{\alpha_r} a_1 = \lambda_1^{\alpha_{1,1}} \cdots \lambda_r^{\alpha_{r,1}} a_1$. That indicates:

$$\lambda_1^{\alpha_1} \cdots \lambda_r^{\alpha_r} = \lambda_1^{\alpha_{1,1}} \cdots \lambda_r^{\alpha_{r,1}},$$

i.e.,,$\alpha_1 = \alpha_{1,1}, \cdots, \alpha_r = \alpha_{r,1}$. Any two quantities which are equal must have the same dimensions.

Example 1.7.4 *Newton's second law can be written as force=ratio of change of momentum w.r.t. time.*

$$F = \frac{P}{T} = \frac{MV}{T} = \frac{ML}{T^2}.$$

Example 1.7.5 *Fluid mechanics.* $\mathbb{T}$ *is tensor, represented by* τ. $\frac{1}{\tau} = \frac{1}{2}\mu(\bigtriangledown\overrightarrow{u} + (\bigtriangledown\overrightarrow{u})^T)$, μ *is shear viscosity.* $[\tau] = \frac{F}{L^2} = \frac{\mu}{LT^2}$, $[\bigtriangledown\overrightarrow{u}] = \frac{L}{T} \cdot \frac{1}{L} = \frac{1}{T} \Rightarrow [\mu] = \frac{\mu}{LT}$, $[\mu] = \frac{M}{L^3} \cdot \frac{L^2}{T}$. *let* $\mu = \rho_0\nu$, ν *is kinematic viscosity, then* $[\nu] = \frac{L^2}{T}$ *is diffusivity.*

Dimensionless variables:

Suppose $(a_1, \cdots, a_r)$ form a fundamental system of units, remaining quantities are $(b_1, \cdots, b_m)$ and $r+m = n$. Then with suitable exponents $(\beta_{1,i}, \cdots, \beta_{m,i})$, $\pi_i = \frac{b_i}{a_1^{\beta_{1,i}} \cdots a_r^{\beta_{r,i}}}$ is dimensionless. The ratio of two quantities of the same dimension.

Perturbation methods are applicable when some dimensionless quantities are small or large. A relationship of the form $b = f(a_1, \cdots, a_r, b_1, \cdots b_m) \iff \pi = f(1, \cdots, 1, \pi_1, \cdots, \pi_m)$. A number of variables is reduced to a minimal number of dimensionless parameters. if $m = 1$, we can assume the self-similar solutions as $\pi = f(\pi_1)$.

Example 1.7.6 *Initial value problem(IVP)*

$$\begin{cases} u_t = \nu\Delta u, \\ u(x,0) = E\delta(x). \end{cases} \qquad in\ \mathbf{R^d}.$$

The dimensional parameters are ν *and* E, *variables are* x *and* t.

Solution: We assume the self-similar solution is

$$\begin{cases} a_1 = t, a_2 = \nu, a_3 = E, b_1 = |x|, \\ b = u. \end{cases}$$

where $[u] = \theta$ is the temperature, $\pi = \frac{u}{E(\nu t)^{-\frac{d}{2}}}$, $\pi_1 = \frac{|x|}{\sqrt{\nu t}}$. Since $\int_{\mathbb{R}} w(x,0)dx = E$, we can obtain $\theta L^d = [E]$.

From dimension analysis, we have

$$u(x,t) = \frac{E}{(\nu t)^{\frac{d}{2}}} f(\frac{|x|}{\sqrt{\nu t}}).$$

Hence,

$$f'' + (\frac{\zeta}{2} + \frac{d-1}{\zeta})f' + \frac{d}{2}f = 0, \quad \textbf{for} \quad f = f(\zeta).$$

$$(f' + \frac{\zeta}{2}f)' + \frac{d-1}{\zeta}(f' + \frac{\zeta}{2}f) = 0.$$

$$f' + \frac{\zeta}{2}f = \frac{b}{\zeta^{d-1}} \quad (\zeta = 0 \Rightarrow b = 0).$$

$$f(\zeta) = ae^{-\frac{\zeta^2}{4}} + be^{-\frac{\zeta^2}{4}} \int \frac{e^{\frac{\zeta^2}{4}}}{\zeta^{d-1}} d\zeta.$$

Since f is integrable, $b = 0$ is necessary. Hence,

$$w(x,t) = \frac{aE}{(\nu t)^{\frac{d}{2}}} exp(-\frac{|x|^2}{4\nu t}).$$

Since $\int_{\mathbf{R}^{\mathbf{d}}} u(x,t)dx = E$, we have $a = (4\pi)^{-\frac{d}{2}}$, and

$$u(x,t) = \frac{E}{(4\pi\nu t)^{\frac{d}{2}}} exp(-\frac{|x|^2}{4\nu t}).$$

Example 1.7.7 *The radius of Sphere through a fluid is L, the speed is constant, noted by U and there is a primary quantity: drag force D. If the fluid is incompressible, when $u \ll c$ which is the sound speed , then we have $D = f(U, L, \rho_0, \nu)$, $[D] = \frac{ML}{T^2} \Rightarrow D = \rho_0 U^2 L^2 F(\frac{UL}{\nu})$ or $\frac{D}{\rho_0 U^2 L^2} = F(Re)$, where $Re = \frac{UL}{\nu}$ is Reynold's number. F is complicated. F changes rapidlly near some Re for which flow is turbulent nearby. And experiments agree weil with dimensionless analysisi.*

Next, we introduce the incompressible Navier-Stokes Equation.

$$\begin{cases} \rho(u_t + u \cdot \nabla u) + \nabla p = \nu \Delta u, \\ \nabla \cdot u = 0. \end{cases}$$

Its basic dimensions are(u, L, ρ). We can assume $u^* = \frac{u}{U}$, $p^* = \frac{p}{\rho u^2}$, $x^* = \frac{x}{L}$ and $t^* = \frac{u_t}{L}$. So we can wtire the equations as

$$\begin{cases} u^*_{t^*} + u^* \cdot \nabla^* u^* + \nabla^* p^* = \varepsilon \Delta^* u^*, \\ \nabla^* \cdot u^* = 0, \\ \varepsilon = \dfrac{1}{Re} = \dfrac{\nu}{uL}. \end{cases}$$

and the boundary conditions became $u^* = 1$ on $r^* = 1$. Then D depends only on ε. Dimensional analysisi leads to scaling symmetries, other symmetries can be found by Lie group theory.

1.8 Exercise 1

1. Find the rescaling for the roots of

$$\epsilon x^4 - x^2 - x + 2 = 0,$$

and find two terms in the approximation for each root.

2. Find the first two-term asymptotic expansion for the two real roots of

$$\tanh x = \epsilon(\epsilon x^2 - 2x + 1),$$

where $\tanh x = (e^x - e^{-x})/(e^x + e^{-x})$ is the hyperbolic tangent function.

3. Find the first two-term asymptotic expansion for the two real roots of

$$\tanh x = \epsilon(\epsilon x^2 - 2x + 1),$$

where $\tanh x = (e^x - e^{-x})/(e^x + e^{-x})$ is the hyperbolic tangent function.

4. Find the second order perturbations of the eigenvalues of the matrix

$$\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} + \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

for small ω and for large ω.

5. Find by using dimensional analysis the self-similar solution $u(\mathrm{x}, t)$ to the following initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = v\Delta u, & in \quad R^d \\ u(\mathrm{x}, 0) = E\delta(\mathrm{x}), \end{cases}$$

(Hint: you need to assume the rotation invariance in the solution.)

6. Prove, using dimensional analysis, Pythagoras' theorem

$$c^2 = a^2 + b^2,$$

where c is the hypotenuse of the right-angled triangle, a and b are the other two sides.

1.9 Answer 1

1. Find the rescaling for the roots of

$$\epsilon x^4 - x^2 - x + 2 = 0,$$

and find two terms in the approximation for each root.

Solution: Assume $x \sim x_0 + \epsilon x_1 + \cdots$, then we have

$$\begin{aligned} &\epsilon(x_0 + \epsilon x_1 + \cdots)^4 - (x_0 + \epsilon x_1 + \cdots)^2 \\ &- (x_0 + \epsilon x_1 + \cdots) + 2 = 0. \end{aligned}$$

For $O(1)$:

$$-x_0^2 - x_0 + 2 = 0.$$

We get $x_0 = 1 or - 2$ For $O(\epsilon)$

$$x_0^4 - 2x_0 x_1 - x_1 = 0.$$

When $x_0 = 1, x1 = \frac{1}{3}$,When $x_0 = -2, x1 = -\frac{16}{3}$.
Therefore

$$x_1(\epsilon) = 1 + \frac{1}{3}\epsilon + \mathcal{O}(\epsilon^2).$$

$$x_2(\epsilon) = -2 - \frac{16}{3}\epsilon + \mathcal{O}(\epsilon^2).$$

Then assume $x = \delta(\epsilon)X$, where $X = O(1)$. Put it into the equation

$$\epsilon\delta^4 X^4 - \delta^2 X^2 - \delta X + 2 = 0.$$

We look for a solution in which $\delta(\epsilon) \gg 1$, then the dominant terms are $\epsilon\delta^4$ and δ^2. Thus we have

$$\epsilon\delta^4 = \delta^2,$$

which means

$$\delta = \pm\epsilon^{-\frac{1}{2}}$$

Then assume $x \sim 1 + \epsilon^{\frac{1}{2}} x_{\frac{1}{2}} + \cdots$. Thus, we have

$$\begin{aligned} &\epsilon^{-1}(1 + 4\epsilon^{\frac{1}{2}} x_{\frac{1}{2}} + \cdots) \\ &- \epsilon^{-1}(1 + 2\epsilon^{\frac{1}{2}} x_{\frac{1}{2}} + \cdots) \\ &- \epsilon^{-\frac{1}{2}}(1 + \epsilon^{\frac{1}{2}} x_{\frac{1}{2}}) \\ &+ 2 \\ &= 0. \end{aligned}$$

From $O(\epsilon^{-1})$ term, $\epsilon^{-1} - \epsilon^{-1} = 0$, we get nothing, while $O(\epsilon^{-\frac{1}{2}})$ yields that

$$\begin{aligned} &4x_{\frac{1}{2}} - 2x_{\frac{1}{2}} - 1 = 0, \\ &2x_{\frac{1}{2}} = 1, \\ &x_{\frac{1}{2}} = \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned}
&4x_{\frac{1}{2}} - 2x_{\frac{1}{2}} + 1 = 0,\\
&2x_{\frac{1}{2}} = -1,\\
&x_{\frac{1}{2}} = -\frac{1}{2},
\end{aligned}$$

with respect to $\epsilon^{-\frac{1}{2}}$ and $-\epsilon^{-\frac{1}{2}}$. Hence the third and forth roots are

$$x_3(\epsilon) = \epsilon^{-\frac{1}{2}}(1 + \frac{1}{2}\epsilon^{\frac{1}{2}}) = \epsilon^{-\frac{1}{2}} + \frac{1}{2},$$

and

$$x_4(\epsilon) = -\epsilon^{-\frac{1}{2}}(1 - \frac{1}{2}\epsilon^{\frac{1}{2}}) = -\epsilon^{-\frac{1}{2}} + \frac{1}{2}.$$

2. Find the first two terms of $x(\epsilon)$ the solution near 0 of

$$\sqrt{2}\sin(x + \frac{\pi}{4}) - 1 - x + \frac{1}{2}x^2 = -\frac{1}{6}\epsilon.$$

Solution: Since the Taylor expansion of $\sin(x + \frac{\pi}{4})$ near 0 is

$$\sin(x + \frac{\pi}{4}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}x + (-\frac{\sqrt{2}}{4})x^2 + (-\frac{\sqrt{2}}{12})x^3 + \mathcal{O}(x^4),$$

then

$$\begin{aligned}
&\sqrt{2}\sin(x + \frac{\pi}{4}) - 1 - x + \frac{1}{2}x^2\\
&= 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{x^4}{24} - 1 - x + \frac{1}{2}x^2 + \mathcal{O}(x^5)\\
&= -\frac{1}{6}x^3 + \frac{x^4}{24} + \mathcal{O}(x^5).
\end{aligned}$$

Thus we have

$$-\frac{1}{6}x^3 + \frac{x^4}{24} + \mathcal{O}(x^5) = -\frac{1}{6}\epsilon.$$

To find asymptotic expansions of them assume

$$x(\epsilon) \sim \epsilon^{\alpha}x_1 + \epsilon^{2\alpha}x_2 + \dots,$$

substituting this into above equation, we have

$$-\frac{1}{6}(\epsilon^{\alpha}x_1 + \epsilon^{2\alpha}x_2 + \dots)^3 + \frac{1}{24}(\epsilon^{\alpha}x_1 + \epsilon^{2\alpha}x_2 + \dots)^4 + \mathcal{O}(x^5) = -\frac{1}{6}\epsilon.$$

For smallest order:

$$-\frac{1}{6}\epsilon^{3\alpha}x_1^3 = -\frac{1}{6}\epsilon.$$

we get $\alpha = \frac{1}{3}, x_1 = 1$.

For $\mathcal{O}(\epsilon^{4\alpha})$:

$$-\frac{1}{6}\cdot 3\cdot \epsilon^{4\alpha}x_1^2x_2+\frac{1}{24}\epsilon^{4\alpha}x_1^4=0.$$

we get $x_2=\frac{1}{12}$.

Therefore

$$x\sim \epsilon^{\frac{1}{3}}+\frac{1}{12}\epsilon^{\frac{2}{3}}.$$

3. Find the first two-term asymptotic expansion for the two real roots of

$$\tanh x=\epsilon(\epsilon x^2-2x+1),$$

where $\tanh x=(e^x-e^{-x})/(e^x+e^{-x})$ is the hyperbolic tangent function.

Solution: From the figure, we can find this equation has two roots. One is near the 0, the other is at $+\infty$

To find the root near 0, we can assume the asymptotic expansion is

$$x(\epsilon)\sim \epsilon^{\alpha}x_1+\epsilon^{2\alpha}x_2+\ldots,$$

Using Taylor expansion near 0 on $\tanh(x)$, then we have

$$\tanh(x)=x-\frac{1}{3}x^3+\mathcal{O}(x^5).$$

Thus

$$\begin{aligned}&(\epsilon^{\alpha}x_1+\epsilon^{2\alpha}x_2+\ldots)-\frac{1}{3}(\epsilon^{\alpha}x_1+\epsilon^{2\alpha}x_2+\ldots)^3+\mathcal{O}(x^5)\\&=\epsilon(\epsilon(\epsilon^{\alpha}x_1+\epsilon^{2\alpha}x_2+\ldots)^2-2(\epsilon^{\alpha}x_1+\epsilon^{2\alpha}x_2+\ldots)+1).\end{aligned}$$

For smallest order:

$$\epsilon^{\alpha}x_1=\epsilon.$$

We get $\alpha=1, x_1=1$.

For $\mathcal{O}(\epsilon^2)$:

$$\epsilon^{2\alpha}x_2=-2\epsilon^2x_1,$$

we get $x_2=-2$.

So:

$$x\sim \epsilon-2\epsilon^2.$$

Then we try to find the root near the $+\infty$.

$$\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}=1-\frac{2}{e^{2x}+1}.$$

Because $\frac{2}{e^{2x}+1} < 2e^{-2x}$ and $2e^{-2x}$ is exponential to 0.

We can transform the equation to:

$$1 = \epsilon(\epsilon x^2 - 2x + 1).$$

Assume $x \sim \delta_1 x_1 + \delta_2 x_2, \delta_1 \gg \delta_2$

By dominant balance:

$$\epsilon^2 \delta_1^2 x_1^2 - 2\epsilon\delta_1 x_1 = 1.$$

We get $\delta_1 = -1, x_1 = 1 + \sqrt{2}$,(notice $x_1 > 0$).

The other order:

$$2\epsilon\delta_2 x_1 x_2 - 2\epsilon\delta_2 x_2 + \epsilon = 0.$$

We get $\delta_2 = 0, x_2 = -\frac{\sqrt{2}}{4}$.

Therefore

$$x \sim \frac{1+\sqrt{2}}{\epsilon} - \frac{\sqrt{2}}{4}.$$

4. Find the second-order perturbations of the eigenvalues of the matrix

$$\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} + \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

for small ω and for large ω.

Solution: Denote $A = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with the model of $Ax + wBx = \lambda x$.

Firstly, we consider the condition: $E_1 \neq E_2$ [(i)] Assume w is small, then taking basis as $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and consequently dual basis is $\widetilde{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\widetilde{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Assume $\alpha = E_1$ has algebraic multiplicity one, then taking expansion of

$$x(w) \sim e_1 + wx_1 + w^2 x_2 + \cdots,$$
$$\lambda(w) = \alpha + w\lambda_1 + w^2\lambda_2 + \cdots.$$

Form $O(w^0)$ term,we have

$$Ae_1 = \alpha e_1.$$

We get $\alpha = E_1$.

$O(w)$ term,

$$Ax_1 + Be_1 = \alpha x_1 + \lambda_1 e_1,$$
$$\lambda_1 = \frac{\widetilde{e_1}^T Be_1}{\widetilde{e_1}^T e_1} = 0.$$

and $x_1 = \begin{pmatrix} 0 \\ \frac{1}{E_2 - E_1} \end{pmatrix}$.

$O(\epsilon^2)$ term,

$$\begin{aligned}
(A - \alpha I)x_2 &= \lambda_1 x_1 + \lambda_2 e_1 - B x_1, \\
\lambda_2 &= \frac{\tilde{e_1}^T B x_1 - \tilde{e_1}\lambda_1 x_1}{\tilde{e_1}^T e_1} = \frac{1}{E_2 - E_1},
\end{aligned}$$

thus

$$\lambda(w) \sim E_1 + \frac{1}{E_2 - E_1} w^2.$$

$$x(w) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{E_2 - E_1} \end{pmatrix} w + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w^2.$$

Similarly,

$$\lambda(w) \sim E_2 + 0 \cdot w + \frac{1}{E_1 - E_2} w^2.$$

$$x(w) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{E_1 - E_2} \\ 0 \end{pmatrix} w + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w^2.$$

with $\alpha = E_2$ and algebraic multiplicity still one. Assume w is large, then we consider the following problem

$$(\frac{1}{w} A + B)x = \frac{\lambda}{w} x.$$

Since the matrix B have eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$, then taking the eigenvectors as the basis: $e_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$, $e_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$, consequently the dual basis is $\tilde{e_1} = \begin{pmatrix} \frac{1}{2} \\ -\frac{i}{2} \end{pmatrix}$, $\tilde{e_2} = \begin{pmatrix} -\frac{i}{2} \\ \frac{1}{2} \end{pmatrix}$. Assume $\alpha = i$ has algebraic multiplicity one, then taking expansion of

$$\begin{aligned}
x(w) &\sim e_1 + w^{-1} x_1 + w^{-2} x_2 + \cdots, \\
\lambda(w) &= w\alpha + \lambda_1 + w^{-1}\lambda_2 + \cdots.
\end{aligned}$$

Form $O(w^{-1})$ term, we get

$$Ae_1 + Bx_1 = \alpha x_1 + \lambda_1 e_1.$$

$$\lambda_1 = \frac{\tilde{e_1}^T A e_1}{\tilde{e_1}^T e_1} = \frac{E_1 + E_2}{2}.$$

$$x_1 = \frac{E_2 - E_1}{4} \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

From $O(w^{-2})$ term, we get

$$Ax_1 + Bx_2 = \alpha x_2 + \lambda_1 x_1 + \lambda_2 e_1.$$

$$\lambda_2 = \frac{\tilde{e_1}^T A x_1 - \lambda_1 \tilde{e_1}^T x_1}{\tilde{e_1}^T e_1} = -\frac{i}{8}(E_1 - E_2)^2.$$

Thus

$$\lambda(w) \sim iw + \frac{E_1 + E_2}{2} - \frac{i}{8}(E_1 - E_2)^2 w^{-1}.$$

Similarly,

$$\lambda(w) \sim -iw + \frac{E_1 - E_2}{2} + \frac{i}{8}(E_2 - E_1)^2 w^{-1}.$$

with $\alpha = -i$. If $E_1 = E_2$, similarly we can get

$$\lambda = E_1 \pm wi.$$

5. Find by using dimensional analysis the self-similar solution $u(\mathrm{x}, t)$ to the following initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = v\Delta u, & in \quad R^d \\ u(\mathrm{x}, 0) = E\delta(\mathrm{x}), \end{cases}$$

Solution: We have 5 dimensions, they are t, x, u, ν, and E, where $E = \int_{R^d} u dx$, $[E] = [u] \cdot L^d$, $[\nu] = L^2/T$ and $[x] = \sqrt{\nu T}$.

$$u = f(t, \|x\|, \nu, E) = f(t, \nu, E, \|x\|),$$

where $a_1 = t$, $a_2 = \nu$, $a_3 = E$, $b_1 = |x|$ and $b = u$. Then

$$\frac{u}{[u]} = f(1, 1, 1, \frac{\|x\|}{[x]}),$$

$$\pi_1 = \frac{\|x\|}{[x]} = \frac{\|x\|}{\sqrt{\nu t}}.$$

and

$$\pi = \frac{u}{[u]} = \frac{u}{[E]/[x]^d} = \frac{u}{[E]/(\nu t)^{\frac{d}{2}}}.$$

According to $\pi = f(\pi_1)$, we have

$$u(x, t) = \frac{E}{(\nu t)^{\frac{d}{2}}} f(\frac{\|x\|}{\sqrt{\nu T}})$$

is self-similar. Using this expression for $u(x, t)$ in the PDE, we get an ODE for $f(\xi)$

$$f'' + \left(\frac{\xi}{2} + \frac{d-1}{\xi}\right) f' + \frac{d}{2} f = 0,$$

where $\xi = \frac{\|x\|}{\sqrt{\nu t}}$. We can rewrite this equation as a first-order ODE for $f' + \frac{\xi}{2} f$,

$$\left(f' + \frac{\xi}{2} f\right)' + \frac{d-1}{\xi}\left(f' + \frac{\xi}{2} f\right) = 0.$$

Solving this equation, we get

$$f' + \frac{\xi}{2} f = \frac{b}{\xi^{d-1}},$$

where b is a constant of integration. Solving for f, we get

$$f(\xi) = ae^{-\xi^2/4} + be^{-\xi^2/4} \int \frac{e^{\frac{\xi^2}{4}}}{\xi^{d-1}} d\xi.$$

where a is another constant of integration. In order for f to be integrable, we must set $b = 0$. Then

$$u(x,t) = \frac{aE}{(\nu t)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{4\nu t}\right).$$

6. Prove, using dimensional analysis, Pythagoras' theorem

$$c^2 = a^2 + b^2,$$

where c is the hypotenuse of the right-angled triangle, a and b are the other two sides.

Proof: Because the area of a right-angled triangle can be determined by an angle α and the hypotenuse, we can assume $S = f(\alpha, c)$.
By dimensional analysis:
$[S] = L^2, [c] = L$, we can assume:

$$S = c^2\Phi(\alpha).$$

Now, we have

$$A_{ABC} = c^2 \cdot \Phi(\alpha_1),$$
$$A_{BCD} = a^2 \cdot \Phi(\alpha_1),$$
$$A_{ACD} = b^2 \cdot \Phi(\alpha_2).$$

Since CD is the height of the triangle, then $\alpha_1 = \alpha_2 := \alpha$. Then by $A_{ABC} = A_{ACD} + A_{BCD}$, we have

$$c^2 \cdot \Phi(\alpha) = a^2 \cdot \Phi(\alpha) + b^2 \cdot \Phi(\alpha).$$

Divide both sides simultaneously by $\Phi(\alpha)$ (where $\Phi(\alpha) \neq 0 \Longleftrightarrow A \neq 0$) to get

$$c^2 = a^2 + b^2.$$

Chapter **2**

Asymptotic Expansions

2.1 Introduction

We first define what is an asymptotic sequence.

Definition 2.1.1: Look at functions $\varphi_n : \mathbb{R}\backslash\{0\} \to \mathbb{R}$, where $n = 0, 1, 2, \cdots$. If for each n, we have $\varphi_{n+1} = o(\varphi_n)$ as $x \to 0$, then $\{\varphi_n\}$ is called a asymptotic sequence and φ_n is called the gauge function. Considering a function $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$, we can write $f(x) \sim \sum_{n=0}^{\infty} a_n\varphi_n(x)$ as $x \to 0$. If for each $N = 0, 1, 2, \cdots$, we have $f(x) - \sum_{n=0}^{\infty} a_n\varphi_n(x) = o(\varphi_N)$ as $x \to 0$. This is the asymptotic expansion of f w.r.t. $\{\varphi_n\}$ as $x \to 0$.

Example 2.1.1: Power functions: $\varphi_n(x) = x^n$ as $x \to 0$, and $\widetilde{\varphi_n}(x) = x^{-n}$ as $x \to \infty$. Taylor expansion is used to obtain asymptotic expansion, i.e. $\frac{1}{1-x} \sim 1 + x + x^2 + \cdots + x^n + \cdots$ as $x \to 0$.

But how to find the coefficients? If φ_n's do not vanish in a neighborhood of 0, then

$$a_{n+1} = \lim_{x \to 0} \frac{f(x) - \sum_{n=0}^{N} a_n\varphi_n(x)}{\varphi_{N+1}}.$$

Hence the expansion of a function w.r.t. gauge function φ_n is unique. However, different functions may have the same asymptotic expansion. Here is an example to illustrate.

Example 2.1.2: Look at the asymptotic expansions of functions $\frac{1}{1-x} + ce^{-\frac{1}{x}}$ and $\frac{1}{1-x}$, which are same.
$\frac{1}{1-x} + ce^{-\frac{1}{x}} \sim 1 + x + x^2 + \cdots + x^n + \cdots$ as $x \to 0^+$ since $e^{-\frac{1}{x}} = o(x^n)$ as $x \to 0^+$ for $\forall n \in \mathbb{N}$.
$\frac{1}{1-x} \sim 1 + x + x^2 + \cdots + x^n + \cdots$ as $x \to 0$.

Asymptotic expansion can be added and multiplied, so term-by-term integration of asymptotic expansion is valid. But differentiation may not be. (e.g. $x \sin \frac{1}{x}$)

Example 2.1.3: Look at the differentiation of asymptotic expansions of $\frac{1}{1-x}$ and $f(x) = \frac{1}{1-x} + e^{-\frac{1}{x}}$.
$\frac{1}{1-x} \sim 1 + x + x^2 + \cdots + x^n + \cdots$ as $x \to 0$, and $(\frac{1}{1-x})' \sim 1 + 2x + 3x^2 + \cdots$, as $x \to 0$.
$f(x) = \frac{1}{1-x} + e^{-\frac{1}{x}} \sin e^{\frac{1}{x}} \sim 1 + x + x^2 + \cdots$ as $x \to 0$, but $f'(x) \sim -\frac{\cos e^{\frac{1}{x}}}{x^2} + 1 + 2x + 3x^2 + \cdots$ as $x \to 0$.

Remark 2.1.1: This is valid for power series.

Asymptotic power series

Let us consider $\varphi_n(x) = x^n$ and

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^n \ as\ x \to 0.$$

If $f(x) \in C^\infty$, then

$$|f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n| \leq C_{N+1} x^{N+1}.$$

when $|x| \leq r$, and

$$C_{N+1} = \sup_{|x| \leq r} \frac{|f^{(N+1)}(x)|}{(N+1)!}.$$

Hence

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \quad \text{as } x \to 0.$$

Remark 2.1.2: The series converges to f $\iff$ f is analytic at $x = 0$.

If $f \in C^\infty$ but not analytic, the series may converge to a different function(e.g. $\frac{1}{1-x} + Ce^{-\frac{1}{x}}$) or diverge (e.g. $erfx = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$).

Borel-Ritt Theorem: (No restriction on the growth rate of coefficients):

Given any sequence $\{a_n\}$ of real (or complex) coefficients, there exists a C^∞ function $f : \mathbb{R} \to \mathbb{R}$ (or $\mathbb{R} \to \mathbb{C}$) s.t. $f(x) \sim \sum a_n x^n, x \to 0$.

Proof: Let $\eta : \mathbb{R} \to \mathbb{R}$ be a C^∞ function, s.t.

$$\eta(x) = \begin{cases} 1,\ |x| \leq 1, \\ 0,\ |x| \geq 2. \end{cases}$$

We choose a sequence of positive number $\{\delta_n\}$ such that $\delta_n \to 0$ as $n \to \infty$, and $|a_n| \| x^n \eta(\frac{x}{\delta_n}) \|_{C^n} \leq \frac{1}{2^n}$. And $\|f\|_{C^n} = sup_{x \in \mathbb{R}} \sum_{k=0}^{n} |f^{(k)}(0)|$ denotes $C^n - norm$. We define

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \eta(\frac{x}{\delta_n}),$$

which converges pointwise. Since when $x = 0, f(0) = a_0$, and when $x \neq 0$, it consists of finitely many-terms. The sequence converges in C^n for any n, hence f has continuous derivatives for all n.

Note that $\eta(x) = 1 - \int_{-\infty}^{2(x-1)-1} \psi(t) dt$, where

$$\psi(t) = \begin{cases} \qquad e^{-\frac{1}{1-|x|^2}},\ |x| \leq 1 \\ 0,\ otherwise \end{cases}$$

is smooth enough. Hence η is smooth enough, and $\eta(1) = 1,\ \eta(2) = 0$.

Asymptotic v.s. convergent series:

If we have $S_N(x) = \sum_{n=0}^{N} a_n \psi_n(x)$. Then asymptoticity (at $x = 0$) illustrates behavior of $S_N(x)$ as $x \to 0$ with N fixed. While convergence illustrates the behavior of $S_N(x)$ as $N \to \infty$ with x fixed.

We can summarize these properties below.

Convergent series:
① Defines unique limiting sum.
② Does not give the rate of convergence.
③ Does not tell how well the partial sum approximates the limit.

Asymptotic series:
① Does not define a unique sum.
② Does not give an arbitrarily accurate approximation of the value of a function at $x \neq 0$.
③ Gives good approximation by partial sum when $x = 0$.

Example 2.1.4:

$$\begin{aligned}\mathrm{Erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_0^x (1 - t^2 + \frac{1}{2} t^4 + \cdots + \frac{(-1)^n}{n!} 2n + \cdots) dt \\ &= \frac{2}{\sqrt{\pi}} (x - \frac{1}{3} x^3 + \cdots + \frac{(-1)^n}{(2n+1) n!} x^{2n+1} + \cdots).\end{aligned}$$

It converges for $\forall x$, but slow for large x. (Need 31 terms in Taylor series to approximate $\mathrm{Er}f(3)$ to 10^{-5} accuracy). If we wanna know the behavior as $x \to \infty$, we can use integration by parts. If we write

$$\mathrm{Erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

then

$$\begin{aligned}\int_0^x e^{-t^2} dt &= \frac{e^{-x^2}}{2x} - \int_x^\infty \frac{e^{-t^2}}{2t^2} dt \\ &= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{4x^3} + \int_x^\infty \frac{3e^{-t^2}}{4t^4} dt \\ &= e^{-x^2} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(2n-1)!!}{2^{n+1}} \frac{1}{x^{2n+1}}.\end{aligned}$$

Here Erf(3) can be approximated by asymptotic series using two terms within 10^{-5} accuracy. It is divergent, with a convergence radius of 0.

Proof:

$$F_n(x) = \int_{x^2}^\infty s^{-n-\frac{1}{2}} e^{-s} ds (s = t^2).$$

we can get

$$F_n(x) = \frac{e^{-x^2}}{x^{2n+1}} - (n + \frac{1}{2}) F_{n+1}(x). \ (\textit{integration by parts}).$$

Hence

$$\mathrm{Erf}_n(x) = 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \sum_{n=0}^{N} (-1)^n \frac{(2n-1)!!}{2^n x^{2n+1}} + R_{N+1}(x).$$

in which

$$R_{N+1}(x) = (-1)^{N+1}\frac{1}{\sqrt{\pi}}\frac{(2N+1)!!}{2^{N+1}}F_{N+1}(x).$$

To prove it is asymptotic expansion, we have

$$|F_n(x)| = |\int_{x^2}^{\infty} s^{-n-\frac{1}{2}}e^{-s}ds| \leq \frac{1}{x^{2n+1}}\int_{x^2}^{\infty} e^{-s}ds \leq \frac{e^{-x^2}}{x^{2n+1}},$$

and

$$|R_{N+1}| \leq C_{N+1}\frac{e^{-x^2}}{x^{2N+3}},$$

where

$$C_{N+1} = \frac{(2N+1)!!}{2^{N+1}\sqrt{\pi}}$$

We consider the divergence of

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(2n-1)1!}{2^{n+1}}\frac{1}{x^{2n+1}}.$$

$$\frac{(2n-1)1!}{2^{n+1}}\frac{1}{x^{2n+1}} \sim \frac{\sqrt{2}2^n(\frac{n}{e})^n}{2^{n=1}}\cdot\frac{1}{x^{2n+1}} = \frac{1}{\sqrt{2}x}(\frac{n}{x^2e})^n \to \infty.$$

Generalized Expansion:

$f(x) \sim \sum_{n=0}^{\infty} f_n(x)$ and $f(x) - \sum_{n=0}^{\infty} f_n(x) = o(\varphi_{n+1})$.

Example 2.1.5: $f(x) \sim \sum_{n=0}^{\infty} a_n(x)\varphi_n(x)$ in which $a_n(x)$ is bounded. We know that $\frac{1}{1-x} \sim \sum_{n=0}^{\infty} x^n$ when $x \to 0^+$ so $\frac{1}{1-x} \sim \sum_{n=0}^{\infty}(1+x)x^{2n}$.

Nonuniform asymptotic expansions:

We have $u(x,\varepsilon) = \frac{1}{x+\varepsilon} \sim \frac{1}{x}(1 - \frac{\varepsilon}{x} + \frac{\varepsilon^2}{x^2} + \cdots)$ when $\varepsilon \to 0^+$, $x > 0$. When $x = 0$, $u(0,\varepsilon) \sim \frac{1}{\varepsilon}, \varepsilon \to 0^+$. As $y \to \infty$ matches $u(x,\varepsilon)$ when $x > 0$ and $y \to 0$ matches $u(0,\varepsilon)$.

Stokes Phenomenon:

There is a complex function $f : \mathbb{C} \mapsto \mathbb{C}$ has singularity at $z = 0$. It has different expansion in different wedges. Here is an example.

Example 2.1.6: Erf: $\mathbb{C} \mapsto \mathbb{C}$

$$\text{Erf}(z) \sim \begin{cases} 1 - e^{\frac{(-z^2)}{z\sqrt{\pi}}}, & z \to \infty, -\frac{\pi}{4} < arg < \frac{\pi}{4}, \\ -1 - e^{\frac{(-z^2)}{z\sqrt{\pi}}}, & z \to \infty, -\frac{3\pi}{4} < arg < \frac{5\pi}{4}, \\ -e^{\frac{(-z^2)}{z\sqrt{\pi}}}, & z \to \infty, \text{ otherwise.} \end{cases}$$

Another divergent asymptotic expansion

Bessel function:

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k}(k!)^2}.$$

Let $f(\varepsilon) = J_0(\frac{1}{\varepsilon})$

$$f \sim \sqrt{\frac{2\varepsilon}{\pi}}[\alpha \cos(\frac{1}{\varepsilon} - \frac{\pi}{4}) + \beta \sin(\frac{1}{\varepsilon} - \frac{\pi}{4}].$$

2.2 Local analysis

We are looking for approximate analytical solutions using asymptotic methods and perturbation theory (small parameters).

Example 2.2.1: $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ near x_0.

It is local analysis when $x - x_0$ is small, and $y(x) \sim \sum_{n=0}^{N} a_n(x - x_0)^n$(algebraic equation).

Example 2.2.2: $y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x)$, where ε is small.

It is global analysis, and $y(x) \sim \sum_{n=0}^{N} \varepsilon^n y_n(x)$.

Note: singularity is almost invariably a clue.— Shclock Holmes

Example 2.2.3: Airy equation

$$\begin{cases} y'' = xy, \\ y(0) = 1, y'(0) = 2. \end{cases}$$

We will try to find a solution near $x = 0$.

Solution: Assume $y(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n$, where $|x|$ is small. Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n,$$
$$y''(x) = \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n.$$

Hence $xy = \sum_{n=1}^{\infty} a_{n-1} x^n$. Compare the coefficients we get $a - 2 = 0$ and $(n+1)(n+2)a_{n+2} = a_{n-1}$ for $n = 1, 2, 3 \cdots$. That means

$$\begin{cases} a_2 = 0, \\ a_n = \dfrac{1}{n(n-1)} a_{n-3}, \quad n = 3, 4, \cdots. \end{cases}$$

With initial condition, we have

$$y(0) = a_0 = 1, \quad y'(0) = a_1 = 2,$$

which implies

$$a_{3n} = \frac{a_0}{3n(3n-1)(3n-3)(3n-4)\cdots 6\cdot 5\cdot 3\cdot 2}$$
$$3n(3n-3)\cdots 6\cdot 3 = 3^n \cdot n!$$
$$(3n-1)(3n-4)\cdots 5\cdot 2 = 3^n(n-\frac{1}{3})(n-1-\frac{1}{3})\cdots(1-\frac{1}{3}) = 3^n\frac{\Gamma(n+\frac{2}{3})}{\Gamma(\frac{2}{3})}.$$

As we know

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt,$$

and

$$\Gamma(x+1) = x\Gamma(x).$$

So

$$\Gamma(x+n) = (x+n-1)(x+n-2)\cdots x\Gamma(x).$$

In this question, $x = \frac{2}{3}$. Therefore,

$$a_{3n} = \frac{a_0\Gamma(\frac{2}{3})}{9^n n!\Gamma(n+\frac{2}{3})}.$$

Similarly,

$$a_{3n+1} = \frac{a_1}{(3n+1)3n\cdots 7\cdot 6\cdot 4\cdot 3} = \frac{a_1\Gamma(\frac{4}{3})}{9^n n!\Gamma(n+\frac{4}{3})}.$$

As a result,

$$y(x) = y_0\Gamma(\frac{2}{3})\sum_{n=0}^{\infty}\frac{x^{3n}}{9^n n!\Gamma(n+\frac{2}{3})} + y'(0)\Gamma(\frac{4}{3})\sum_{n=0}^{\infty}\frac{x^{3n+1}}{9^n n!\Gamma(\frac{4}{3})}.$$

The two linearly independent solutions are called Airy functions:

$$A_i(x) = 3^{-\frac{2}{3}}\sum_{n=0}^{\infty}\frac{x^{3n}}{9^n n!\Gamma(n+\frac{2}{3})} - 3^{-\frac{4}{3}}\sum_{n=0}^{\infty}\frac{x^{3n+1}}{9^n n!\Gamma(n+\frac{4}{3})},$$
$$B_i(x) = 3^{-\frac{1}{6}}\sum_{n=0}^{\infty}\frac{x^{3n}}{9^n n!\Gamma(n+\frac{2}{3})} - 3^{-\frac{5}{6}}\sum_{n=0}^{\infty}\frac{x^{3n+1}}{9^n n!\Gamma(n+\frac{4}{3})}.$$

We can check for independency: If we have $y(x) = a_0y_0(x) + a_1y_1(x)$ as a solution of the PDE, then y_0 and y_1 are two linearly independent solutions of the ODE and $Wronskian(x)$ =det$\begin{bmatrix} y_0 & y_1 \\ y_0' & y_1' \end{bmatrix} \neq 0$.
$\mathrm{W}\Big(A_i(x) \quad B_i(x)\Big)$ =$\Big[3^{-\frac{2}{3}}y_0(x) - 3^{-\frac{4}{3}}y_1(x), \quad 3^{-\frac{1}{6}}y_0(x) - 3^{-\frac{5}{6}}y_1(x)\Big]$.
We can write $A_i(x)$=$\begin{pmatrix} \frac{3^{-\frac{2}{3}}}{\Gamma(\frac{2}{3})} & -\frac{3^{-\frac{4}{3}}}{\Gamma(\frac{4}{3})} \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$, $B_i(x)$=$\begin{pmatrix} \frac{3^{-\frac{1}{6}}}{\Gamma(\frac{2}{3})} & -\frac{3^{-\frac{5}{6}}}{\Gamma(\frac{4}{3})} \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$.
and $\begin{pmatrix} A_i(x) & B_i(x) \\ A_i'(x) & B_i'(x) \end{pmatrix} = \begin{pmatrix} y_0 & y_1 \\ y_0' & y_1' \end{pmatrix}\begin{pmatrix} \frac{3^{-\frac{2}{3}}}{\Gamma(\frac{2}{3})} & \frac{3^{-\frac{1}{6}}}{\Gamma(\frac{2}{3})} \\ -\frac{3^{-\frac{4}{3}}}{\Gamma(\frac{4}{3})} & -\frac{3^{-\frac{5}{6}}}{\Gamma(\frac{4}{3})} \end{pmatrix} \Rightarrow \mathrm{W}\Big(A_i(x) \quad B_i(x)\Big) \neq 0$ $A_i(x)$: exponentially decays as $x \to 0$ oscillating as $x \to -\infty$.
$B_i(x)$: $\frac{\pi}{2}$ difference in phase with $A_i(x)$ as $x \to \infty$.

2.3 Different types of singularity points of homogenous linear ODEs

Homogenous linear ODEs: $y^{(n)}(x)+P_{n-1}y^{(n-1)}(x)+\cdots+P_1(x)y'(x)+P_0(x)y(x)=0$. There are three types of singularities: ordinary point, regular singular point, irregular regular point.

Definition 2.3.1: A point x_0 is called ordinary point, if all the coefficient functions $P_k(x), k=0,\cdots,n-1$ are analytic in a neighborhood of x_0 in the complex plane.
Note $P_k(z)$ has a derivative complex function, thus $P_k(z)=\sum_{n=0}^{\infty}a_n^{(k)}(z-z_0)^n$ in the neighborhood of z_0.

Example 2.3.1: $x_0=0$ is ordinary point of Airy Equation. Near ordinary point we have solution in the form of $y(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$(Cauchy's Theorem).

Definition 2.3.2: A point is called regular singular point when not all of $P_k(x)$ are analytic but in the form of $(x-x_0)^{n-k}p_k(x), k=0,1,\cdots n-1$, where $p_k(x)$ are analytic in a neighborhood of x_0.

Example 2.3.2: $y''+\frac{1}{x}y'-(1+\frac{\nu^2}{x^2})y=0$, ν is a constant. $x_0=0$ is a regular singular point. Notice that $P_0(x)=-(1+\frac{\nu^2}{x^2})$, $P_1(x)=\frac{1}{x}$ is not analytic at $x=0$, but $x^2P_0(x)=-(x^2+\nu^2)$ and $xP_1(x)=1$ are analytic.

Definition 2.3.3: If a point is neither ordinary point nor regular singular point, it is called irregular singular point.

Example 2.3.3: $x^3y''=y$ near $x=0$, we can write it as $y''-\frac{1}{x^3}y=0$, then $P_0(x)=-\frac{1}{x^3}$, $x^2P_0(x)=-\frac{1}{x}$.
For a point $x_0=\infty$, we let $x=\frac{1}{t}$, then $t=\frac{1}{x}$ and $\frac{d}{dx}=-t^2\frac{d}{dt}$, we consider $t_0=0$ when $x_0=\infty$.

Frobenius Method:

Example 2.3.4: Bessel Equation: $y''+\frac{1}{x}y'-(1+\frac{1}{9x^2})y=0$, near $x=0$.

Solution: We know $x_0=0$ is a regular singular point and $y=0$ is a solution. We want to look for non-trivial solution. Try $y(x)=\sum_{n=0}^{\infty}a_nx^n$, then

$$\sum_{n=2}^{\infty}(n-1)na_nx^{n-2}-\sum_{n=1}^{\infty}na_nx^{n-2}-\sum_{n=2}^{\infty}a_{n-2}x^{n-2}-\sum_{n=2}^{\infty}\frac{1}{9}a_nx^{n-2}=0.$$

we can get

$$\begin{aligned}
&x^{-2}: \quad \frac{a_0}{9}=0\Rightarrow a_0=0,\ n=0.\\
&x^{-1}: \quad 1\cdot a_1=0\rightarrow a_1=0,\ n=1.\\
&(n-1)na_n+na_n-a_{n-2}-\frac{1}{9}a_n=0\Rightarrow(n^2-\frac{1}{9})a_n=a_{n-2}\Rightarrow a_n=0\Rightarrow y(x)\equiv 0,\ n\geq 2.
\end{aligned}$$

Frobenius method: assume $y(x)=(x-x_0)^\alpha\sum_{n=0}^\infty a_n(x-x_0)^n$, in which α is a indicator exponent to be determined. (Require $a_0\neq 0$, otherwise if $a_1\neq 0$, $\sum_{n=0}^\infty a_n(x-x_0)^{n+\alpha}=\sum_{n=0}^\infty a_{n+1}(x-x_0)^{n+\alpha+1}$, we can use $\alpha+1$ instead of α.) Substituting it into $y(x)=\sum_{n=0}^\infty a_n(x)^{n+\alpha}$, $a_0\neq 0$, and we get

$$\begin{aligned}
x^{\alpha-2}: &\quad \alpha(\alpha-1)a_0+\alpha a_0-\frac{a_0}{9}=0\Rightarrow\alpha=\pm\frac{1}{3},\ n=0,\\
x^{\alpha-1}: &\quad (1+\alpha)\alpha a_1+(1+\alpha)a_1-\frac{1}{9}a_1=0\Rightarrow a_1=0,\ n=1,\\
x^{n+\alpha-2}: &\ (n+\alpha)(n+\alpha-1)a_n+(n+\alpha)a_n-a_{n-2}-\frac{1}{9}a_n=0.
\end{aligned}$$

Hence

$$[(n+\alpha)^2-\frac{1}{9}]a_n=a_{n-2},$$

which implies

$$\begin{aligned}
a_{2n}&=\frac{a_0}{\Big((2n+\alpha)^2-\frac{1}{9}\Big)\cdots\Big((2+\alpha^2)-\frac{1}{9}\Big)},\ \text{and}\ a_{2n+1}=0,\\
(i)\alpha&=\frac{1}{3},a_{2n}=\frac{\Gamma(\frac{4}{3})}{2^{2n}n!\Gamma(2n+\frac{4}{3})a_0}\quad (ii)\alpha=-\frac{1}{3},a_{2n}=\frac{\Gamma(\frac{2}{3})}{2^{2n}n!\Gamma(2n+\frac{2}{3})a_0}
\end{aligned}$$

So the linear solutions are $\sum_{n=0}^\infty\frac{(\frac{1}{2}x)^{2n+\frac{1}{3}}}{n!\Gamma(n+\frac{4}{3})}, a_0=\frac{1}{2^{\frac{1}{3}}\Gamma(\frac{4}{3})}$ and $\sum_{n=0}^\infty\frac{(\frac{1}{2}x)^{2n-\frac{1}{3}}}{n!\Gamma(n+\frac{2}{3})}, a_0=\frac{1}{2^{-\frac{1}{3}}\Gamma(\frac{2}{3})}$.

Remark 2.3.1: Modified Bessel Equation of order ν, $y''+\frac{1}{x}y'-(1+\frac{\nu^2}{x^2})y=0$. When $\nu=\frac{1}{3}$, it is the Bessel Equation. If $\alpha_1-\alpha_2=N$, $N=0,1,2,\cdots$, we need more analysis. Look at textbook(Advanced Math Method) Page72–Page76.

Method of dominant balance:

Example 2.3.5: $x^3y''=y$ near $x_0=0$ (irregular singular).

Solution: Using Frobenius method, assume $y(x)=\sum_{n=0}^\infty a_nx^{n+\alpha}, a_0\neq 0$, thus $\sum_{n=1}^\infty(n+\alpha-2)(n+\alpha-1)a_{n-1}x^{n+\alpha}=\sum_{n=0}^\infty a_nx^{n+\alpha}$, hence $a_0=0$, this is a contradiction! In fact, $y(x)=e^{s(x)}$ usually works at irregular singular points. Plugging it into equation, we have $x^3\left(s''e^s+e^s(s')^2\right)=e^s$ and $x^3(s''+(s')^2)=1$.

Dominant balance:

There are three terms: s'', $(s')^2$, and $\frac{1}{x^3}$ when $x\to 0^+$.

(i) Suppose $s''\ll(s')^2\sim\frac{1}{x^3}$, thus $(s')^2\sim\frac{1}{x^3}$ and $s\sim\pm 2x^{-\frac{1}{2}}$(constant can be dropped since $C\ll x^{-\frac{1}{2}}$ as $x\to 0$). We have $s''=\mp\frac{3}{2}x^{-\frac{5}{2}}$, which indeed satisfies $s''\ll\frac{1}{x^3}$ and is consistent with the assumption. So $y(x)\sim e^{\pm 2x^{-\frac{1}{2}}}$ as $x\to 0$.

(ii) Suppose $(s')^2\ll s''\sim\frac{1}{x^3}$, thus $s'\sim\frac{1}{2}x^{-2}$, however $(s')^2\sim\frac{1}{4}x^{-4}\gg x^{-3}$, which is inconsistent with the assumption.

(iii) Suppose $x^{-3} \ll s'' \sim (s')^2$, thus $\frac{ds'}{dx} = -(s')^2$, and $\frac{1}{s'} \sim x + c$, $s' \sim \frac{1}{x+c}$. Since $(s')^2 \gg x^{-3}$ as $x \to 0$, s' must be singular at $x = 0$, we have $C = 0, s' \sim \frac{1}{x}$, hence $s'' \sim -\frac{1}{x^2}$ and $(s')^2 \sim \frac{1}{x^2} \ll \frac{1}{x^3}$, this is inconsistent.

Therefore, $s(x) \sim \pm 2x^{-\frac{1}{2}}$ as $x \to 0^+$, $y(x) \sim e^{2x^{-\frac{1}{2}}}$ or $y(x) \sim e^{-2x-\frac{1}{2}}$ as $x \to 0^+$. This is leading order behavior, controlling fast dynamics.

Now let's go deeper, assume $s(x) = 2x^{-\frac{1}{2}} + c(x)$, where $c(x) \ll x^{-\frac{1}{2}}$ as $x \to 0^+$. Then $s'(x) = -x^{-\frac{3}{2}} + c'(x)$, $s''(x) = \frac{3}{2}x^{-\frac{5}{2}} + c''(x)$, so $c'' + (c')^2 - 2c'x^{-\frac{3}{2}} + \frac{3}{2}x^{-\frac{5}{2}} = 0$. Since $c'(x) \ll x^{-\frac{3}{2}}$, $c''(x) \ll x^{-\frac{5}{2}}$, then $2c'(x)x^{-\frac{3}{2}} \sim \frac{3}{2}x^{-\frac{5}{2}}$, thus $c'(x) \sim \frac{3}{4}x^{-1}$, and $c(x) \sim \frac{3}{4}lnx$ as $x \to 0^+$(integration constant $\ll lnx$). This is consistent with $c(x) \ll x^{-\frac{1}{2}}$. Therefore, $s(x) \sim 2x^{-\frac{1}{2}} + \frac{3}{4}logx$, $x \to 0$, and $y(x) \sim e^{2x^{-\frac{1}{2}}}x^{\frac{3}{4}}$ as $x \to 0^+$.

More accurately, $s(x) \sim 2x^{-\frac{1}{2}} + \frac{3}{4}logx + D(x)$, where $D(x) \ll |logx|\ as\ x \to 0^+$. Hence, $s'(x) \sim -x^{-\frac{3}{2}} + \frac{3}{4}x^{-1} + D'(x)$ and $s'' \sim \frac{3}{2}x^{-\frac{5}{2}} - \frac{3}{4}x^{-2} + D''(x)$. Plugging these two into original equation, we get $D'' + (D')^2 + 2D'(\frac{3}{4}x^{-1} - x^{-\frac{3}{2}}) - \frac{3}{16}x^{-2} = 0$. Since $D' \ll x^{-1}$ and $D'' \ll x^{-2}$, we have $2D'(\frac{3}{4}x^{-1} - x^{-\frac{3}{2}}) \sim \frac{3}{16}x^{-2}$, and $2D'x^{-\frac{3}{2}} \sim -\frac{3}{16}x^{-2}$, hence $D \sim \frac{3}{16}x^{\frac{1}{2}} + d$ as $x \to 0^+$, where d is a constant and can be dropped. This is consistent with $D \ll |logx|$ which means $\lim_{x \to 0^+} \frac{D}{logx} = 0$. Therefore, $s(x) \sim 2x^{-\frac{1}{2}} + \frac{3}{s}logx - \frac{3}{16}x^{\frac{1}{2}} + d \Rightarrow$ $y(x) \sim x^{\frac{3}{4}}e^{2x^{-\frac{1}{2}} - \frac{3}{16}x^{\frac{1}{2}}}$, $x \to 0^+$.

Summary: We could construct a solution,

$$y(x) = e^{f(x)+g(x)+h(x)+\cdots}, \text{ where } f(x) \gg g(x) \gg h(x) \gg \cdots\ as\ x \to 0^+.$$

Since

$$e^{-\frac{3}{16}x^{\frac{1}{2}}} \sim 1 - \frac{3}{16}x^{\frac{1}{2}},$$

then

$$y(x) \sim C_1 x^{\frac{3}{4}} e^{2x^{-\frac{1}{2}}}(1 - \frac{3}{16}x^{\frac{1}{2}}),\ x \to 0^+.$$

We could assume

$$y(x) = C_1 x^{\frac{3}{4}} e^{2x^{-\frac{1}{2}}}(1 - \frac{3}{16}x^{\frac{1}{2}} + w(x)),\ w(x) \ll \frac{3}{16}x^{\frac{1}{2}},\ x \to 0^+.$$

In general, asymptotic series are in the form of $y(x) \sim e^{f(x)}\,(1 + g_1(x) + g_2(x) + \cdots)$. Finally, we can get

$$y(x) \sim -C_1 x^{\frac{3}{4}} e^{2x^{-\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma(n-\frac{1}{2})\Gamma(n+\frac{3}{2})}{\pi \cdot 4^n \cdot n!} x^{\frac{n}{2}}.$$

Similarly,

$$y(z) \sim -C_2 x^{\frac{3}{4}} e^{-2x^{-\frac{1}{2}}} (-1)^n \sum_{n=0}^{\infty} \frac{\Gamma(n-\frac{1}{2})\Gamma(n+\frac{3}{2})}{\pi \cdot 4^n \cdot n!} x^{\frac{n}{2}}.$$

Asymptotic expansion for oscillatory functions:

Example 2.3.6: Airy equation: $y'' = xy$ as $x \to -\infty$.

Solution: let $t = \frac{1}{x}$, then $\frac{d}{dx} = -t^2 \frac{d}{dt}$, we have $y'' + \frac{2}{t} y' - \frac{1}{t^5} y = 0$ and $t \to 0^-$ is a irregular singular point. Let $y(x) = e^{s(x)}$, so $s'' + (s')^2 = x$.

By dominant balance,

$$(s')^2 \sim x \Rightarrow s' \sim \pm i\sqrt{-x} \Rightarrow s \sim \pm \frac{2}{3} i(-x)^{\frac{3}{2}}, x \to -\infty.$$

More terms:

$$s \sim \pm \frac{2}{3} i(-x)^{\frac{3}{2}} - \frac{1}{4} \log(-x) + d,\ x \to -\infty;$$

$$y(x) \sim C(-x)^{-\frac{1}{4}} e^{\pm \frac{2}{3} i(-x)^{\frac{3}{2}}},\ x \to -\infty,$$

or

$$y(x) \sim C_1(-x)^{-\frac{1}{4}} \sin(\frac{2}{3}(-x)^{\frac{3}{2}}) + C_2(-x)^{-\frac{1}{4}} \cos(\frac{2}{3}(-x)^{\frac{3}{2}}).$$

Remark 2.3.2: There is a problem in this expansion, e.g. $|\sin(x + \frac{1}{x}) - \sin x| \to 0$ as $x \to -\infty$, but $\lim\limits_{x \to -\infty} \frac{\sin(x+\frac{1}{x})}{\sin x} \neq 1$. When $\sin x = 0$, $\sin(x + \frac{1}{x}) \neq 0$, we have $\frac{\sin(x+\frac{1}{x})}{\sin x} = \infty$, such x can be arbitrary large. Thus we cannot say rigorously that $\sin(x + \frac{1}{x}) \sim \sin x$ as $x \to -\infty$. While this can be fixed by writing

$$y(x) = w_1(x)(-x)^{-\frac{1}{4}} \sin(\frac{2}{3}(-x)^{\frac{3}{2}}) + w_2(x)(-x)^{-\frac{1}{4}} \cos(\frac{2}{3}(-x)^{\frac{3}{2}}).$$

We then look for asymptotic expansions of $w_1(x)$ and $w_2(x)$ as $x \to -\infty$.

Asymptotic series — Convergence or not?:

Convergence series means when $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$. And $R_N(x) = y(x) - \sum_{n=0}^{N} a_n(x - x_0)^n$ is the remainder. We can use the ratio test.

(1) if $\lim\limits_{n \to \infty} |\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}| < 1 \Rightarrow$ convergent;

(2) if $\lim\limits_{n \to \infty} |\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}| > 1 \Rightarrow$ divergent.

From this test, we know if $|x - x_0| \lim\limits_{n \to \infty} \frac{a_{n+1}}{a_n} < 1$, then the series converges, and the radius of convergence is R, according to $\frac{1}{R} = \lim\limits_{n \to \infty} |\frac{a_{n+1}}{a_n}|$. And the series is convergent when $|x - x_0| < R$.

Example 2.3.7: Bessel Equation $y'' + \frac{1}{x} y' - (1 + \frac{1}{9x^2}) y = 0$, near $x = 0$.

We can get

$$y(x)=\sum_{n=0}^{\infty}\frac{(\frac{1}{2}x)^{2n+\frac{1}{3}}}{n!\Gamma(n+\frac{4}{3})}.$$

Ratio test:

$$\lim_{n\to\infty}\frac{(\frac{1}{2}x)^{2(n+1)+\frac{1}{3}}/(n+1)!\Gamma(n+1+\frac{4}{3})}{(\frac{1}{2}x)^{2n+\frac{1}{3}}/n!\Gamma(n+\frac{4}{3})}=\lim_{n\to\infty}|\frac{\frac{1}{2}x^2}{(n+1)(n+\frac{4}{3})}|=0, \qquad for\ any\ x$$

The series converges for all x.

Example 2.3.8: $x^3y''=y$, $x_0=0$.

We have a solution:

$$y(x)\sim x^{\frac{3}{4}}e^{2x^{-\frac{1}{2}}}\sum_{n=0}^{\infty}\frac{\Gamma(n-\frac{1}{2})\Gamma(n+\frac{3}{2})}{\pi\cdot 4^n\cdot n!}x^{\frac{n}{2}}.$$

Ratio test:

$$\lim_{n\to\infty}|\frac{\Gamma(n+1-\frac{1}{2})\Gamma(n+1+\frac{3}{2})}{4(n+1)\Gamma(n-\frac{1}{2})\Gamma(n+\frac{3}{2})}x^{\frac{1}{2}}|=\lim_{n\to\infty}|\frac{(n-\frac{1}{2})(n+\frac{3}{2})}{4(n+1)}|=\infty, \quad for\ \forall x.$$

The series diverges for all x.

Revisit definition of asymptotic series:

Remainder

$$R_N(x)=y(x)-\sum_{n=0}^{N}a_n(x-x_0)^n\ll(x-x_0)^N,\ as\ x\to x_0.$$

or

$$\lim_{n\to x_0}|\frac{R_N(x)}{(x-x_0)^N}|=0,\ \text{for}\ \forall N.$$

Generally, $y(x)\sim\lim\limits_{n\to\infty}y_n(x)$, $x\to x_0$.
Also

$$R_N(x)=y(x)-y_N(x),\ x\to x_0,$$

or

$$\lim_{n\to x_0}|\frac{R_N(x)}{y_N(x)}|=0.$$

Example 2.3.9:. $x^3y''=y$, $x_0\to 0^+$, $y(x)=e^{s(x)}$.

We also have a solution:

$$s(x)\sim 2x^{-\frac{1}{2}}+\frac{3}{4}\log x+d-\frac{3}{10}x^{\frac{1}{2}}+\cdots,\ x\to 0^+,$$

As $x\to 0^+$,

$$R_0(x)=s(x)-2x^{-\frac{1}{2}}\ll 2x^{-\frac{1}{2}}.$$

$$R_1(x) = s(x) - 2x^{-\frac{1}{2}} + \frac{3}{4}\log x \ll \frac{3}{4}\log x.$$

So $s(x)$ is an asymptotic series.

Example 2.3.10: Rigorous proof of asymptotic series for Taylor series: $e^x = e^{x_0} + e^{x_0}(x-x_0) + \frac{1}{2}e^{x_0}(x-x_0)^2 + \cdots + \frac{1}{n!}e^{x_0}(x-x_0)^n + \cdots$. It is also an asymptotic series as $x \to x_0$.

Proof: $\forall$N, $k_N(z) = \frac{1}{(N+1)!}e^{\zeta}(x-x_0)^{N+1}$, where ζ between x_0 and x, thus e^{ζ} is bounded.

$$\lim_{x\to x_0} \frac{R_N(x)}{(x-x_0)^N} = \lim_{x\to x_0} \frac{e^{\zeta}(x-x_0)}{(N+1)!} = 0.$$

Asymptotic behavior of the remainder :

If we have $R_N(x) = y(x) - \sum_{n=0}^{N} a_n(x-x_0)^n$, then $R_{N+1}(x) = y(x) - \sum_{n=0}^{N+1} a_n(x-x_0)^n = R_N(x) - a_{N+1}(x-x_0)^{N+1}$. Thus $\lim\limits_{x\to x_0} \frac{R_{N+1}(x)}{(x-x_0)^{N+1}} = 0$, and $\lim\limits_{x\to x_0} \frac{R_N(x)}{(x-x_0)^{N+1}} = a_{N+1}$, we have $R_N(x) \sim a_{N+1}(x-x_0)^{N+1}$, $x \to x_0$. So actually if we approximate $y(x)$ by $\sum_{n=0}^{N} a_n(x-x_0)^n$, the error is about $a_{N+1}(x-x_0)^{N+1}$. Recall from the definition we only know that the error is much less than $(x-x_0)^N$!

For a convergent series, the remainder goes to zero as $N \to \infty$, the larger N, the better the approximation. For a divergent asymptotic series, choose the smallest term $y_{N+1}(x)$, then $\sum_{n=0}^{N} y_n(x)$ is the optimal asymptotic approximation (Good in practice).

Other properties:

Power series:

(1) Given $y(x)$, can we find $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ s.t. $y(x) \sim \sum_{n=0}^{\infty} a_n(x-x_0)^n$, $x \to x_0$?

If the answer is yes, then $0 = \lim\limits_{x\to x_0} \frac{R_0(x)}{(x-x_0)^0} = \lim\limits_{x\to x_0} |y(x) - a_0|$, i.e., $a_0 = \lim\limits_{x\to x_0} y(x)$. In general, $a_N = \lim\limits_{x\to x_0} \frac{R_{N-1}(x)}{|x-x_0|^N}$, $N = 1, 2, \cdots$. Conversely, if these limits exist, then $y(x) \sim \sum_{n=0}^{\infty} a_n(x-x_0)^n$, $x \to x_0$ and a_n is uniquely determined by $y(x)$.

(2) Given $\sum_{n=0}^{\infty} a_n x^n$, can we find $y(x)$, s.t. $y(x) \sim \sum_{n=0}^{\infty} a_n(x-x_0)^n$ as $x \to x_0$?

The answer is Yes. For any series, we can find $y(x)$, and such a function is not unique. (See textbook e.g. 2 in section 3.8)

Example 2.3.11: $y(x) + e^{-(x-x_0)^{-2}} \sim \sum_{n=0}^{\infty} a_n(x-x_0)^n$ as $x \to x_0$, as $e^{-(x-x_0)^{-2}} \ll (x-x_0)^N$ for all N.

(3) Operations on asymptotic series similar to convergent series, such as addition, subtraction, multiplication, division, and integration. But differentiation need additional condition to verify.

2.4 Asymptotic expansion of integrals

We have a integration in general form: $I(x) = \int_{a(x)}^{b(x)} f(x,t)dt,\ x \to x_0$.

There are several methods for the asymptotic expansion of integrals:

1. Direct expansion of integrand,
2. Integration by parts,
3. Laplace method,
4. Method of stationary phase,
5. Method of steepest descent.

1. Direct expansion of integrand

If $f(x,t) \sim \sum_{n=0}^{\infty} f_n(x,t),\ x \to x_0$ is uniform for $a(x) \leq t \leq b(x)$ and $\int_{a(x)}^{b(x)} f_n(x,t)$ is finite, $n = 0, 1, 2, \cdots$, then $I(x) = \int_{a(x)}^{b(x)} f(x,t)dt \sim \sum_{n=0}^{\infty} \int_{a(x)}^{b(x)} f_n(x,t)$.

Our purpose is that each $\int_{a(x)}^{b(x)} f_n(x,t)$ can be calculated analytically.

Example 2.4.1: $\int_0^x t^{-\frac{1}{2}} e^{-t} dt,\ x \to 0^+$.

If we have $t^{-\frac{1}{2}} e^{-t} \sim t^{-\frac{1}{2}} \sum_{n=0}^{N} \frac{1}{n!}(-t)^n + t^{-\frac{1}{2}} \frac{e^{\zeta}}{(N+1)!}(-t)^{N+1}$, where ζ between 0 and $-t$. And $\lim\limits_{x\to 0^+} \frac{R_N}{t^{-\frac{1}{2}}(-t)^N/N!} = \lim\limits_{x\to 0^+} \frac{e^{\zeta}}{(N+1)}(-t) = 0$, so it is uniform for $0 \leq t \leq x$, since $|\frac{e^{\zeta}}{(N+1)}(-t)| \leq \frac{x}{(N+1)!} \to 0$ as $x \to 0^+$.

Example 2.4.2: $\int_x^{\infty} e^{-t^4} dt,\ x \to 0$.

We first try $e^{-t^4} \sim \sum_{n=0}^{\infty} \frac{1}{n!}(-t^4)^n$, but this is not correct (finite uniform).

① For $t \in (x, \infty), R_N(t) = \frac{1}{N+1} e^{\zeta}(-t^4)^{N+1}$, $\zeta \in (-t^4, 0)$, and

$$\lim_{x\to 0} \frac{R_N}{(-t^4)^N/N!} = \lim_{x\to 0} \frac{1}{(N+1)} e^{\zeta}(-t^4) \neq 0.$$

This is not an asymptotic expansion.

② $\int_x^{\infty} \frac{1}{n!}(-t^4)^n dt = \frac{(-1)^n}{(4n+1)n!} t^{4n+1}|_x^{\infty}$ is not finite.

So $\int_x^{\infty} e^{(-t)^4} dt = \int_0^{\infty} e^{(-t)^4} dt - \int_0^x e^{(-t)^4} dt = \Gamma(\frac{5}{4}) - \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)n!} x^{4n+1},\ x \to 0$.

In which $\int_x^{\infty} e^{(-t)^4} dt \overset{s=t^4}{=} \int_x^{\infty} e^{-s} \frac{1}{4} s^{\frac{3}{4}} ds = \frac{1}{4}\Gamma(\frac{1}{4}) = \Gamma(\frac{5}{4})$.

2. Integration by parts

Example 2.4.3: $I(x) = \int_x^\infty e^{-t^4}dt$, $x \to \infty$, direct expansion is not working in this problem.

$$I(x) = \int_x^\infty (-\frac{1}{4t^3})de^{-t^4} = -\frac{1}{4t^3}e^{-t^4}|_x^\infty + \int_x^\infty \frac{1}{4}e^{-t^4}d(\frac{1}{t^3}) = \frac{1}{4x^3}e^{-x^4} - \int_x^\infty (\frac{3}{4t^4})e^{-t^4}dt.$$

We know that

$$\int_x^\infty (\frac{1}{t^4})e^{-t^4}dt \le \frac{1}{x^4}\int_x^\infty e^{-t^4}dt = \frac{1}{x^4}I(x) \ll I(x),\ x \to \infty.$$

Therefore $I(x) \sim \frac{1}{4x^3}e^{-x^4}$, $x \to \infty$.

We can repeat this process to get $\frac{3}{4}\int_x^\infty(\frac{1}{t^4})e^{-t^4}dt = \frac{3}{16}\frac{1}{x^7}e^{-x^4} - \frac{21}{16}\int_x^\infty(\frac{1}{t^8})e^{-t^4}dt$, where $\int_x^\infty(\frac{1}{t^8})e^{-t^4}dt \le \frac{1}{x^4}\int_x^\infty(\frac{1}{t^4})e^{-t^4}dt \ll \int_x^\infty(\frac{1}{t^4})e^{-t^4}dt$, $x \to \infty$. Hence $I(x) \sim \frac{1}{4x^3}e^{-x^4} - \frac{3}{16}\frac{1}{x^7}e^{-x^4}$, $x \to \infty$.

Repeating this process, we get $I(x) \sim \frac{1}{4x^3}e^{-x^4}[1 + \sum_{n=1}^\infty (-1)^n \frac{3\cdot 7\cdot 11\cdots(4n-1)}{(4x^4)^n}], x \to \infty$.

Example 2.4.4: $\int_0^x t^{-\frac{1}{2}}e^{-t}dt$, $x \to +\infty$.

$$I(x) = \int_0^x t^{-\frac{1}{2}}e^{-t}dt = \int_0^x t^{-\frac{1}{2}}d(-e^{-t}) = t^{-\frac{1}{2}}(-e^{-t})|_0^x + \int_0^x e^{-t}d(t^{-\frac{1}{2}}).$$

But the first term $t^{-\frac{1}{2}}(-e^{-t})|_0^x \to \infty$, so this method does not work. And we try another way:

$$I(x) = \int_0^x 2e^{-t}dt^{\frac{1}{2}} = 2e^{-t}t^{\frac{1}{2}}|_0^x + 2\int_0^x e^{-t}t^{\frac{1}{2}}dt = 2e^{-x}x^{\frac{1}{2}} + 2\int_0^x e^{-t}t^{\frac{1}{2}}dt.$$

We can split

$$I(x) = \int_0^\infty t^{-\frac{1}{2}}e^{-t}dt - \int_x^\infty t^{-\frac{1}{2}}e^{-t}dt = \Gamma(\frac{1}{2}) - \int_x^\infty t^{-\frac{1}{2}}d(-e^{-t}) = \Gamma(\frac{1}{2}) - x^{-\frac{1}{2}}e^{-x} - \frac{1}{2}\int_x^\infty t^{-\frac{3}{2}}e^{-t}dt.$$

And $\int_x^\infty t^{-\frac{3}{2}}e^{-t}dt \le \frac{1}{x}\int_x^\infty t^{-\frac{1}{2}}e^{-t}dt \ll \int_x^\infty t^{-\frac{3}{2}}e^{-t}dt$, $x \to +\infty$, therefore, $I(x) \sim \Gamma(\frac{1}{2}) - x^{-\frac{1}{2}}e^{-x}$.

2.5 Laplace's Method

Example 2.5.1: $I(x) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-x\sin^2 t}dt$, $x \to +\infty$.

Our main idea is that we write $I(x) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{x\varphi(t)}dt$, $\varphi(t) = -\sin^2 t$. It can be easily verified that $\varphi(t)$ attains its maximum at $t = 0$. Thus $x[\varphi(0) - \varphi(t)]$ is very large as $x \to +\infty$ and $t \in [-\frac{\pi}{4}, \frac{\pi}{4}]$, therefore $e^{-x[\varphi(0)-\varphi(t)]} = e^{x\varphi(t)}/e^{x\varphi(0)}$ is exponentially small for large x. This is the contribution to that $I(x)$ is locally near $t = 0$ as $x \to +\infty$.

Formally, there are three steps in laplace's method:
Step 1: $I(x) = \int_{-\frac{\pi}{4}}^{-\varepsilon} e^{-x\sin^2 t}dt + \int_{-\varepsilon}^{\varepsilon} e^{-x\sin^2 t}dt + \int_{\varepsilon}^{\frac{\pi}{4}} e^{-x\sin^2 t}dt$ for small ε.
Hence $I(x) \sim \int_{-\varepsilon}^{\varepsilon} e^{-x\sin^2 t}dt$, since $\int_{-\frac{\pi}{4}}^{-\varepsilon} e^{-x\sin^2 t}dt \leq e^{-x\sin^2\varepsilon}(\frac{\pi}{4} - \varepsilon) \to 0$ as $x \to +\infty$.
Step 2: When $t \in [-\varepsilon, \varepsilon]$, $\sin^2 t = [t - \frac{t^3}{3!} + O(t^5)]^2 = t^2 + O(t^4)$.
Hence $I(x) \sim \int_{-\varepsilon}^{\varepsilon} e^{-x\sin^2 t}dt \sim \int_{-\varepsilon}^{\varepsilon} e^{-xt^2}dt$, $x \to +\infty$.
Step 3: $\int_{-\varepsilon}^{\varepsilon} e^{-xt^2}dt \sim \int_{-\infty}^{+\infty} e^{-xt^2}dt \overset{s^2=xt^2}{=} \frac{1}{\sqrt{x}}\int_{-\infty}^{+\infty} e^{-s^2}ds = \sqrt{\frac{\pi}{x}}$.
Hence $I(x) \sim \sqrt{\frac{\pi}{x}}$, $x \to +\infty$.

For justification, we need to show that the first and the third terms are much smaller than the second term.
① $\int_{\varepsilon}^{\frac{\pi}{4}} e^{-x\sin^2 t}dt \ll \int_{-\varepsilon}^{\varepsilon} e^{-x\sin^2 t}dt, x \to +\infty$.
Proof: we have $\int_{\varepsilon}^{\frac{\pi}{4}} e^{-x\sin^2 t}dt = \int_{\varepsilon}^{\frac{\pi}{4}} -\frac{d(e^{-x\sin^2 t})}{2x\sin t\cos t} = -\frac{1}{x}e^{-\frac{2}{x}} + \frac{e^{-x\sin^2\varepsilon}}{x\sin 2\varepsilon} - \frac{1}{x}\int_{\varepsilon}^{\frac{\pi}{4}} \frac{2\cos 2t}{\sin^2 t}e^{-x\sin^2 t}dt$,
Since $\frac{1}{x}\int_{\varepsilon}^{\frac{\pi}{4}} \frac{2\cos 2t}{\sin^2 2t}e^{-x\sin^2 t}dt \ll \int_{\varepsilon}^{\frac{\pi}{4}} e^{-x\sin^2 t}dt$, as $x \to +\infty$, $\frac{2\cos 2t}{\sin^2 2t} \leq \frac{2}{\sin^2 2\varepsilon} \leq x$, we have

$$\int_{\varepsilon}^{\frac{\pi}{4}} e^{-x\sin^2 t}dt \sim -\frac{e^{-\frac{x}{2}}}{x} + \frac{e^{-x\sin^2\varepsilon}}{x\sin 2\varepsilon},\ x \to +\infty.$$

when ε is small ($\sin^2\varepsilon < \frac{1}{2}$), we have

$$\int_{\varepsilon}^{\frac{\pi}{4}} e^{-x\sin^2 t}dt \sim \frac{e^{-x\sin^2\varepsilon}}{x\sin 2\varepsilon},\ x \to +\infty.$$

② $\int_{-\varepsilon}^{\varepsilon} e^{-x\sin^2 t}dt \sim \int_{-\varepsilon}^{\varepsilon} e^{-xt^2}dt$ and $\int_{-\varepsilon}^{\varepsilon} e^{-x\sin^2 t}dt = \int_{-\varepsilon}^{-x^{-\frac{3}{8}}} e^{-x\sin^2 t}dt + \int_{-x^{-\frac{3}{8}}}^{x^{-\frac{3}{8}}} e^{-x\sin^2 t}dt + \int_{x^{-\frac{3}{8}}}^{\varepsilon} e^{-x\sin^2 t}dt$
Considering $x\sin^2 t = xt^2 + O(xt^4)$, when $|t| \leq x^{-\frac{3}{8}}$, we have $xt^4 \leq x \cdot x^{-\frac{3}{2}} = x^{-\frac{1}{2}}$.
Thus, when $|t| \leq x^{-\frac{3}{8}}$, we get $e^{-x\sin^2 t} = e^{-xt^2 + O(x^{-\frac{1}{2}})} = e^{-xt^2}\cdot(1 + O(\frac{1}{\sqrt{x}}))$, $x \to +\infty$.
And $\int_{-x^{-\frac{3}{8}}}^{x^{-\frac{3}{8}}} e^{-x\sin^2 t}dt = \int_{-x^{-\frac{3}{8}}}^{x^{-\frac{3}{8}}} e^{-xt^2}(1 + O(\frac{1}{\sqrt{x}}))dt \sim \int_{-x^{-\frac{3}{8}}}^{x^{-\frac{3}{8}}} e^{-xt^2}dt$.
At the same time, $\int_{x^{-\frac{3}{8}}}^{\varepsilon} e^{-x\sin^2 t}dt \sim e^{-x\sin^2(\frac{1}{x})^{\frac{3}{8}}}/x\sin^2(\frac{1}{x})^{\frac{3}{8}} \sim \frac{e^{-x^{\frac{1}{4}}}}{2x^{\frac{1}{4}}} \ll \sqrt{\frac{\pi}{x}}$, thus $\int_{-\varepsilon}^{\varepsilon} e^{-x\sin^2 t}dt \sim \int_{-x^{-\frac{3}{8}}}^{x^{-\frac{3}{8}}} e^{-xt^2}dt \sim \int_{-\varepsilon}^{\varepsilon} e^{-xt^2}dt$.

Generally, in the Laplace method for $\int_a^b f(t)e^{x\varphi(t)}dt$, $x \to +\infty$, $\varphi(t)$ has maximum at $x = C$ (C could be a or b).
Step 1: $\int_a^b f(t)e^{x\varphi(t)}dt \sim \int_{c-\varepsilon}^{c+\varepsilon} f(t)e^{x\varphi(t)}dt\ x \to +\infty$
Step 2:

$$\int_{c-\varepsilon}^{c+\varepsilon} f(t)e^{x\varphi(t)}dt = \int_{c-\varepsilon}^{c+\varepsilon}[f(c) + f'(c)(t-c) + \frac{1}{2}f''(c)(t-c)^2 + \cdots]e^{x[\varphi(c)+\varphi'(c)(t-c)+\frac{1}{2}\varphi''(c)(t-c)^2+\cdots]}dt.$$

We keep leading order terms (more terms in higher order asymptotic expansion), e.g. $C \in (a,b), \varphi(c)$ is maximum thus $\varphi'(c) = 0$ and $\varphi''(c) \leq 0$. When $\varphi(c) \neq 0$ and $\varphi''(c) \neq 0$, we can write $\int_{c-\varepsilon}^{c+\varepsilon} f(t)e^{x\varphi(t)}dt \sim \int_{c-\varepsilon}^{c+\varepsilon} f(c)e^{x[\varphi(c)+\varphi'(c)(t-c)+\frac{1}{2}\varphi''(c)(t-c)^2+\cdots]}dt = f(c)e^{x\varphi(c)}\int_{c-\varepsilon}^{c+\varepsilon} f(c)e^{x\frac{1}{2}\varphi''(c)(t-c)^2}dt$.

Step 3:

$$f(c)e^{x\varphi(c)}\int_{c-\varepsilon}^{c+\varepsilon} e^{x\frac{1}{2}\varphi''(c)(t-c)^2}dt \sim f(c)e^{x\varphi(c)}\int_{-\infty}^{+\infty} e^{x\frac{1}{2}\varphi''(c)(t-c)^2}dt = f(c)e^{x\varphi(t)}\sqrt{\frac{2\pi}{-x\varphi''(c)}}.$$

Watson's Lemma

A simple case with full asymptotic expansion.

Lemma 2.5.1: $I(x) = \int_0^b f(t)e^{-xt}dt,\ x \to +\infty (b > 0)$. If $b = \infty$, we need $f(t) = O(e^{ct}),\ t \to +\infty$ for $c > 0$. If $\varphi(t) = -t$, then it has a unique maximum at $t = 0$ for $t \in [0, b]$. If $f(t) \sim t^\alpha \sum_{n=0}^\infty a_n t^{\beta_n}, t \to 0^+, \alpha > -1, \beta > 0$, then $I(x) \sim \sum_{n=0}^\infty \frac{a_n\Gamma(\alpha+\beta_n+1)}{x^{\alpha+\beta_n+1}}, x \to \infty$

Formally, as $x \to \infty$, $I(x) \sim \int_0^\varepsilon f(t)e^{-xt}dt \sim \int_0^\varepsilon t^\alpha \sum_{n=0}^\infty a_n t^{\beta_n} e^{-xt}dt \sim \int_0^\infty t^\alpha \sum_{n=0}^\infty a_n t^{\beta_n} e^{-xt}dt \sim \sum_{n=0}^\infty a_n \int_0^\infty t^{\alpha+\beta_n} e^{-xt}dt = \sum_{n=0}^\infty \frac{a_n\Gamma(\alpha+\beta_n+1)}{x^{\alpha+\beta_n+1}}$.

Example 2.5.2: $I(x) = \int_0^{\frac{\pi}{2}} e^{-x\sin^2 t}dt$.

Solution: We use Watson's Lemma indirectly. Setting $s = \sin^2 t$, we have $dt = \frac{1}{2}\frac{1}{\sqrt{s(1-s)}}$, so $I(x) = \int_0^1 \frac{1}{2}\frac{1}{\sqrt{s(1-s)}}e^{-xs}ds$.We have $\frac{1}{2}\frac{1}{\sqrt{s(1-s)}} \sim \frac{1}{2}s^{-\frac{1}{2}}(1 + \frac{1}{2}s + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2}s^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1\cdot 2\cdot 3}s^3 + \cdots) = \sum_{n=0}^\infty \frac{\Gamma(n+\frac{1}{2})s^{n-\frac{1}{2}}}{2n!\Gamma(\frac{1}{2})}$, $s \to 0^+$, here $\alpha = -\frac{1}{2}$, $\beta = 1$ in Watson's Lemma. Hence $I(x) \sim \sum_{n=0}^\infty \frac{(\Gamma(n+\frac{1}{2}))^2}{2n!\Gamma(\frac{1}{2})x^{n+\frac{1}{2}}}$, $x \to \infty$.

In general, t is complex in Watson's Lemma. Suppose $f(t) \sim \sum_{n=0}^\infty a_n t^{\alpha_n}$, where $t \to 0$ in S (an internal or sector) and $-1 < \alpha_0 < \alpha_1 < \cdots < \alpha_n$. If $\int_a^b f(t)$ is bounded and analytic (except at $t = 0$) in S, $\frac{1}{x}$ and b are in S and $Re(b) > 0$, then $\int_a^b f(t)e^{-xt}dt \sim \sum_{n=0}^N a_n x^{-\alpha-1}\Gamma(\alpha_n + 1)$ as $x \to \infty$.

Proof: $\forall\varepsilon > 0$, from asymptoticness, $\exists t_0(\varepsilon)$ s.t. $|f(t) - \sum_{n=0}^N a_n t^{\alpha_n}| < \varepsilon|t^{\alpha_n}|$ for all t in S with $|t| < |t_0|$, so $t_0(\varepsilon) = O(\frac{1}{x})$, t_0 is in S, xt_0 is real and $xt_0 > 0$. Because f is analytic, the path of integration can be deformed to go radially from 0 to t_0 and then from t_0 to b, all within S. We can get

$$\int_0^{t_0} t^{\alpha_n}e^{-xt}dt - x^{-\alpha_n-1}\Gamma(\alpha_n + 1) = -\int_{t_0}^\infty t^{\alpha_n}e^{-xt}dt.$$

Since $|e^{-xt}| < |e^{-(x-1)t_0}| \cdot |e^{-t}|$ for t from t_0 to ∞, then

$$\begin{aligned}
&|\int_0^{t_0} e^{-xt}f(t)dt - \sum_{n=0}^N a_n x^{-\alpha_n-1}\Gamma(\alpha_n+1)| \\
&\le |\int_0^{t_0} e^{-xt}f(t)dt - \int_0^{t_0}\sum_{n=0}^N a_n|t^{\alpha_n}e^{-xt}dt + |\int_0^{t_0}\sum_{n=0}^N a_n|t^{\alpha_n}e^{-xt}dt - \sum_{n=0}^N a_n x^{-\alpha_n-1}\Gamma(\alpha_n+1)| \\
&\le \varepsilon|x^{-\alpha_N-1}\Gamma(\alpha_N+1)| + |e^{-(x-1)t_0}|\int_{t_0}^\infty \sum_{n=0}^N a_n x^{-\alpha_n}|e^{-t}||dt| + |e^{-(x-1)t_0}|\int_{t_0}^\infty \varepsilon|t^{\alpha_N}||e^{-t}||dt|.
\end{aligned}$$

On the other hand, $|\int_{t_0}^{b} e^{-xt} f(t)dt| \leq Ce^{-xt_0}$ where $|f(t)| \leq C$ along the contour. The exponentially small terms are $o(x^{-\alpha_n - 1})$, $x \to \infty$, and ε is arbitrary small.

2.6 Method of stationary phase

If we have $I(x) = \int_a^b f(t)e^{ix\phi(t)}dt$, where $x \to \infty$, $\phi(t)$ is real. We want to know that

$$\begin{cases} I_1(x) = \displaystyle\int_a^b f(t)\cos[x\psi(t)]dt \to ? \\ I_2(x) = \displaystyle\int_a^b f(t)\sin[x\psi(t)]dt \to ? \end{cases} \qquad \text{as } x \to \infty.$$

Lemma 2.6.1 (Riemann-Lebesgue lemma):

If $\int_a^b |f(t)|dt$ exists and is finite, then we have that $I_1(x), I_2(x)$ approach to 0 as $x \to \infty$.

Idea of proof: If $f(t) = C \to \int_a^b Ce^{ix\psi(t)}dt = \frac{C}{ix}[e^{ixb} - e^{ixa}]$. From the oscillation property of integration, we can find $\int_a^b Ce^{ix\psi(t)}$ cancels out except near a and b, as $x \to \infty$, the boundary terms vanish. If $f(t)$ continuous, $f(t)$ can be approximated locally by a constant. If $\int_a^b |f(t)|dt$ exists, $f(t)$ can be approximated by continuous function.

Generalization of Riemann-Lebesgue lemma:

We have $I(x) = \int_a^b f(t)e^{ix\psi(t)}dt \to 0, x \to \infty$.

Our idea is that let $u = \psi(t)$, then $t = \psi^{-1}(u)$, $I(x) = \int_{\psi(a)}^{\psi(b)} \frac{f(\psi^{-1}(u))}{\psi'\psi^{-1}(u)}du$. If $\psi'(t) \neq 0$ for $t \in [a, b]$ and meet some other conditions $(\int_{\psi(a)}^{\psi(b)} |\frac{f(\psi^{-1}(u))}{\psi'\psi^{-1}(u)}|du)$.

$I(x) = \int_a^b f(t)e^{ix\psi(t)}dt = \int_a^b \frac{f(t)}{ix\psi'(t)}de^{ix\psi(t)} = \frac{f(t)}{ix\psi'(t)}e^{ix\psi(t)}|_a^b - \int_a^b \frac{1}{ix}\frac{d}{dt}[\frac{f(t)}{\psi'(t)}]e^{ix\psi(t)}dt$. In which the first term $\frac{f(t)}{ix\psi'(t)}e^{ix\psi(t)}|_a^b \sim o(\frac{1}{x})$ as $x \to \infty$ and the second term $\int_a^b \frac{1}{ix}\frac{d}{dt}[\frac{f(t)}{\psi'(t)}]e^{ix\psi(t)}dt \ll \frac{1}{x}$ as $\int_a^b \to 0$ from Generalized Riemann-Lebesgue lemma. So $I(x) = \int_a^b f(t)e^{ix\psi(t)}dt \sim \frac{f(t)}{ix\psi'(t)}e^{ix\psi(t)}|_a^b, x \to \infty$. If $\psi'(t_0) = 0$, t_0 is called a stationary point of ψ or a point of stationary phase ($e^{ix\psi(t)}, psi(t)$ is the phase). In the case, we cannot use Generalized Riemann-Lebesgue lemma and integration by parts.

Example 2.6.1: If $\psi(t) = t^2$, then $e^{ix\psi(t)} = \cos(xt^2) + i\sin(xt^2)$, we have $\psi'(t) = 2t, \psi'(0) = 0$, so $t = 0$ is a stationary point. Near a stationary point, ψ does not oscillate rapidly, hence, Generalized Riemann-Lebesgue lemma and integration fails to apply.

Method of stationary phase

If outside a small region, near the stationary point, we can use integration by parts and

Generalized Riemann-Lebesgue lemma. If inside the small region, we can use Taylor expansion to simplify the integrand. For $I(x) = \int_a^b f(t)e^{ix\psi(t)}dt$ as $x \to +\infty$, we have $\psi'(a) = 0,\ \psi''(a) \neq 0,\ \psi'(t) \neq 0,\ t \in (a, b]$. Formally, $I(x) = \int_a^{a+\varepsilon} f(t)e^{ix\psi(t)}dt + \int_{a+\varepsilon}^b f(t)e^{ix\psi(t)}dt$, we use $G - R - L$ lemma, $\int_{a+\varepsilon}^b f(t)e^{ix\psi(t)}dt \sim \frac{f(t)}{ix\psi'(t)}e^{ix\psi(t)}|_{a+\varepsilon}^b$ and

$$\int_a^{a+\varepsilon} f(t)e^{ix\psi(t)}dt \sim \int_a^{a+\varepsilon} f(a)e^{ix[\psi(a)+\frac{1}{2}\psi''(a)(t-a)^2+\cdots]}dt \sim$$
$$f(a)e^{ix\psi(a)}\int_a^{a+\varepsilon} e^{ix\frac{\psi''(a)}{2}(t-a)^2}dt \sim f(a)e^{ix\psi(a)}\int_a^{\infty} e^{ix\frac{\psi''(a)}{2}(t-a)^2}dt.$$

If $\psi''(a) > 0$, $\int_a^\infty e^{ix\frac{\psi''(a)}{2}(t-a)^2}dt = \frac{\pi}{2x\psi''(a)}e^{ix\cdot\frac{\pi}{4}} = O(\frac{1}{\sqrt{x}})$ is dominant term than $\frac{1}{x}$. Then $I(x) \sim \int_a^{a+\varepsilon} f(t)e^{ix\psi(t)}dt \sim f(a)e^{i[(x\psi(a)+\frac{\pi}{4})]}\sqrt{\frac{\pi}{2x\psi''(a)}}$. Let $z = \sqrt{\frac{\psi''(a)x}{2}}(t-a)$ then $dt = \sqrt{\frac{2}{\psi''(a)x}}dz$. Here we use the fact, $\int_0^\infty e^{iz^2}dz = \frac{\sqrt{\pi}}{2}e^{i\frac{\pi}{4}}$ and $\int_0^\infty e^{-iz^2}dz = \frac{\sqrt{\pi}}{2}e^{-i\frac{\pi}{4}}$ by Cauchy's Theorem. If $\psi''(a) < 0$, we have $I(x) \sim f(a)e^{i[(x\psi(a)-\frac{\pi}{4})]}\sqrt{\frac{\pi}{-2x\psi''(a)}}$.

Theorem 2.6.1: (Cauchy's Theorem) There is a simply connected domain D, r is a closed loop in D and $f(z)$ is analytic in D, then $\int_r f(z)dz = 0$.

For example , we want to calculate $\int_0^\infty e^{iz^2}dz = \frac{\sqrt{\pi}}{2}e^{i\frac{\pi}{4}}$. We use Cauchy's Theorem in this figure

$$\begin{cases} C_1 : z = u, u \in [0, R], \\ C_2 : z = e^{i\frac{\pi}{4}}u, u \in [0, R], \\ C_3 : z = Re^{i\theta}, \theta \in [0, \dfrac{\pi}{4}]. \end{cases}$$

We have $\int_{C_1-C_2-C_3} e^{iz^2}dz = 0$.

Then

$$\int_{C_3} e^{iz^2}dz = -\int_0^{\frac{\pi}{4}} e^{i(Re^{i\theta})^2}iRe^{i\theta}d\theta = -\int_0^{\frac{\pi}{4}} e^{iR^2\cos 2\theta - R^2\sin 2\theta}iR\theta d\theta.$$

which causes

$$|\int_{C_3} e^{iz^2}dz| \leq \int_0^{\frac{\pi}{4}} Re^{-R^2\sin 2\theta}d\theta \leq \int_0^{\frac{\pi}{4}} Re^{-R^2\frac{\pi}{2}2\theta}d\theta = \frac{\pi}{4R}(1-e^{-R^2}) \to 0,\ R \to \infty.$$

Thus,

$$\lim_{R\to\infty}\int_{C_1} = \lim_{R\to\infty}\int_{C_2} = \int_0^\infty e^{iu^2e^{i\frac{\pi}{2}}}d(e^{i\frac{\pi}{4}}u) = \int_0^\infty e^{-u^2}e^{i\frac{\pi}{4}}du = \frac{\sqrt{\pi}}{2}e^{i\frac{\pi}{4}}.$$

2.7 Method of steepest descent

The method of steepest descent means changing the path of the integral in the complex plane.

Example 2.7.1: $I(x) = \int_0^1 \log t e^{ixt} dt, x \to \infty$, while $\psi'(t) = 1 \neq 0$, so there is no stationary point. And $\log t$ is singular at $t = 0$, which means it cannot be integrated by parts. Using Cauchy's Theorem, the singularity at the origin can be removed by

$$\begin{cases} C_1 : t = is, s \in [0, T], \\ C_2 : t = s + iT, s \in [0, 1], \\ C_3 : t = 1 + i(T - s), s \in [0, T]. \end{cases}$$

We have $\int_{C_1 - C_2 - C_3} e^{iz^2} dz = 0$. Then

$$I(x) = \int_{C_1} \log t e^{ixt} dt + \int_{C-2} \log t e^{ixt} dt + \int_{C_3} \log t e^{ixt} dt,$$

and for any fixed x

$$\begin{aligned} \int_{C_2} \log t e^{ixt} dt &= \int_0^1 [\log \sqrt{s^2 + T^2} + i \arg(s + iT)] e^{ix(s+iT)} dS \\ &= e^{-xT} \int_0^1 [\log \sqrt{s^2 + T^2} + i \arg(s + iT)] e^{ixs} ds \to 0,\ T \to \infty. \end{aligned}$$

Hence

$$I(x) = \lim_{T\to\infty} \int_{C_1} + \lim_{T\to\infty} \int_{C_3} = i \int_0^\infty [\log s + i\frac{\pi}{2}] e^{-xs} ds - i \int_0^\infty \log(1 + is) e^{ix - xs} ds,$$

and

$$\int_0^\infty \log s e^{-xs} ds \overset{u=xs}{=} \int_0^\infty (\log u - \log x) e^{-u} \frac{1}{x} du,$$

in which $\int_0^\infty e^{-u} \log u du = \Gamma'(1) = -\zeta$, and ζ is Euler constant which equals to 0.5772. Moreover,

$$\lim_{T\to\infty} \int_{C_1} = -i(\frac{\zeta}{x} + \frac{\log x}{x}) - \frac{\pi}{2}\frac{1}{x} = -i\frac{\log x}{x} - (i\zeta + \frac{\pi}{2})\frac{1}{x}.$$

. Since $\log(1 + iS) \sim -\sum_{n=1}^\infty \frac{(-is)^n}{n}$ as $s \to 0$,

$$\begin{aligned} &\lim_{T\to\infty} \int_{C_3} = -i \int_0^\infty \log(1 + is) e^{-xs} ds \cdot e^{ix} \sim ie^{ix} \sum_{n=1}^\infty (-i)^n \int_0^\infty \frac{s^n}{n} e^{-xs} ds \\ &= ie^{ix} \sum_{n=1}^\infty \frac{(-i)^n}{x^{n+1}} \int_0^\infty \frac{u^n}{n} e^{-u} du = ie^{ix} \sum_{n=1}^\infty \frac{(-i)^n}{x^{n+1}} (n-1)!\ (x \to \infty). \end{aligned}$$

Our idea is from e^{ixs} to e^{-xs} (rapid oscillation); this implies a change of integration path. Rapid decreasing, and the Laplace method can be used. So why is this method called steepest descents? (how we change the integration path?).

If we have $\int e^{x\rho(t)} dt$, $\rho(t) = \varphi(t) + i\psi(t)$, here $\varphi(t)$ and $\psi(t)$ are real. Because $\nabla |e^{x\rho(t)}| = xe^{x\varphi(t)} \nabla \varphi$ with respect to $t = (u, v) \Rightarrow -\nabla\varphi$ is the steepest descent direction. When $\rho(t)$ is analytic on complex plane $(t = u + iv) \Rightarrow \frac{\partial \varphi}{\partial u} = \frac{\partial \psi}{\partial v}, \quad \frac{\partial \varphi}{\partial v} = -\frac{\partial \varphi}{\partial u} \Rightarrow \nabla\varphi \cdot \nabla\psi = 0$.

If $\varphi = constant$, then $\nabla\varphi$ is parallel to the steepest descent direction. $\Rightarrow \psi = constant$ is parallel to the steepest descent direction. This is the desired integration path. In the previous example, $\rho(t) = it, \quad t = u + iv, \begin{cases} \varphi(t) = -v, \\ \psi(t) = u, \end{cases}$ which implies $u = constant$ is parallel to the steepest descent direction, C_1, C_3.

Remark 2.7.1: Euler constant

$$-\gamma = \int_0^{\infty} e^x \ln x dx = \frac{d}{ds}\Gamma(s)|_{s=1} = \Gamma(1)'.$$

We can show that $\zeta = \lim_{n\to\infty}(\sum_{i=1}^{n} \frac{1}{i} - \ln n)$.

Proof:we have

$$\begin{aligned}\gamma &= -\int_0^{\infty} e^x \ln x dx = -\int_0^1 e^x \ln x dx - \int_1^{\infty} e^x \ln x dx \\ &= -\int_0^1 \ln x d(1-e^{-x}) + \int_1^{\infty} \ln x de^{-x} = \int_0^1 \frac{1-e^{-x}}{x} dx - \int_1^{+\infty} \frac{e^{-x}}{x} dx,\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{i} &= \sum_{i=1}^{n} \int_0^1 t^{i-1} dt = \int_0^1 \sum_{i=1}^{n} t^{i-1} dt = \int_0^1 \frac{1-t^n}{1-t} dt \\ &= \int_n^0 \frac{1-(1-\frac{x}{n})^n}{1-(1-\frac{x}{n})} d(1-\frac{x}{n}) = \int_0^n \frac{1-(1-\frac{x}{n})^n}{x} dx.\end{aligned}$$

We know $\ln n = \int_1^n \frac{1}{x} dx$, thus $\lim_{n\to\infty}(\sum_{i=1}^{n} \frac{1}{i} - \ln n) = \lim_{n\to\infty}(\int_n^0 \frac{1-(1-\frac{x}{n})^n}{x} dx - \int_1^n \frac{1}{x} dx) = \lim_{n\to\infty}(\int_n^0 \frac{1-(1-\frac{x}{n})^n}{x} dx - \int_1^n \frac{(1-\frac{x}{n})^n}{x} dx) = \int_0^1 \frac{1-e^{-x}}{x} dx - \int_1^{+\infty} \frac{e^{-x}}{x} dx = \eta$.

Hence we finish it.

2.8 Exercise 2.1

1. Consider the series expansions of the error function $\mathrm{erf}(x)$.

   (a) Show that the following series converges for any $x \in \mathbb{R}$:
   $$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} x^{2n+1}.$$

   (b) Show that the following series diverges for any $x \in \mathbb{R}$:
   $$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)!!}{2^{n+1}x^{2n+1}}.$$

2. Find the series solutions about $x = 0$:
   $$9y''(x) + \frac{18}{x}y'(x) + y(x) = 0.$$

3. Read page 72 to page 76 in 3.3 carefully, show that all solutions of the modified Bessel equation
   $$y''(x) + \frac{1}{x}y'(x) - \left(1 + \frac{v^2}{x^2}\right) y(x) = 0$$
   with $v = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ can be expanded in Frobenius series (without logarithmic terms).

4. Continue the example in the class
   $$x^3 y''(x) = y(x)$$
   near the irregular singular point $x = 0$. We have calculated that as $x \to 0^+, y_1(x) \sim C_1 x^{\frac{3}{4}} e^{2x^{-\frac{1}{2}} - \frac{3}{16}x^{\frac{1}{2}}}$. Show that:

   (a) If we write $y_1(x) = C_1 x^{\frac{3}{4}} e^{2x^{-\frac{1}{2}}} \left(1 - \frac{3}{16}x^{\frac{1}{2}} + w(x)\right)$, with $w(x) \ll \frac{3}{16}x^{\frac{1}{2}}$ as $x \to 0^+$, then the asymptotic series of $y_1(x)$ is of the form
   $$y_1(x) \sim -C_1 x^{\frac{3}{4}} e^{2x^{-\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma\left(n - \frac{1}{2}\right)\Gamma\left(n + \frac{3}{2}\right)}{\pi \cdot 4^n \cdot n!} x^{\frac{n}{2}};$$

   (b) The asymptotic series of the second solution $y_2(x)$ is of the form
   $$y_2(x) \sim -C_2 x^{\frac{3}{4}} e^{-2x^{-\frac{1}{2}}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(n - \frac{1}{2}\right)\Gamma\left(n + \frac{3}{2}\right)}{\pi \cdot 4^n \cdot n!} x^{\frac{n}{2}};$$

   (c) Find the radius of convergence R of the these two series.

5. (Behavior of Airy functions for large x) Consider the Airy equation
   $$y''(x) = xy(x).$$

(a) Show that it has an irregular singular point at ∞.

(b) Show that the leading behaviors of the solutions for large x are determined by

$$y_1(x) \sim C_1 x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}}, \quad x \to +\infty$$

and

$$y_2(x) \sim C_2 x^{-\frac{1}{4}} e^{\frac{2}{3}x^{\frac{3}{2}}}, \quad x \to +\infty.$$

(c) Show that the asymptotic expansion of $y_1(x)$ is

$$y_1(x) \sim C_1 x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}} \sum_{n=0}^{\infty} \frac{1}{2\pi} \left(-\frac{3}{4}\right)^n \left(\frac{\Gamma\left(n+\frac{5}{6}\right)\Gamma\left(n+\frac{1}{6}\right)}{n!}\right) x^{-\frac{3n}{2}}.$$

(d) Give the radius of convergence for this series.

6. Show that if $f(x) \sim a\,(x-x_0)^{-b}$ as $x \to x_0^+$, then

(a) $\int f \mathrm{d}x \sim \frac{a}{1-b}\,(x-x_0)^{1-b}, x \to x_0^+$ if $b > 1$ and the path of integration does not pass through x_0;

(b) $\int f \mathrm{d}x \sim c, x \to x_0^+$ where c is a constant if $b < 1$; if $c = 0$, then $\int f \mathrm{d}x \sim \frac{a}{1-b}(x-x_0)^{1-b}, x \to x_0^+$;

(c) $\int_{x_0}^{x} f \mathrm{d}x \sim \frac{a}{1-b}\,(x-x_0)^{1-b}, x \to x_0^+$ if $b < 1$;

(d) $\int f \mathrm{d}x \sim a \ln(x-x_0), x \to x_0^+$ if $b = 1$.

7. (a) Give an example of an asymptotic relation $f(x) \sim g(x) (x \to \infty)$ that cannot be exponentiated; that is, $e^{f(x)} \sim e^{g(x)}$ is false.

(b) Show that if $f(x) - g(x) \ll 1 (x \to \infty)$, then $e^{f(x)} \sim e^{g(x)} (x \to \infty)$.

2.9 Exercise 2.2

1. Use Taylor expansion of e^{-t^2} to find an asymptotic expansion of the integral:

$$I(x) = \int_0^{\frac{1}{x}} e^{-t^2} dt, \quad x \longrightarrow 0^+.$$

You are required to show that the asymptotic expansion of the integrand holds uniformly for t .

2. Consider the integral:

$$I(x) = \int_0^{\infty} \frac{e^{-t}}{x^2 + t} dt, \quad x \longrightarrow +\infty.$$

(1). Using integration by parts, show that

$$I(x) = \sum_{n=1}^{N} \frac{(-1)^{n-1}(n-1)!}{x^{2n}} + (-1)^N N! \int_0^{\infty} \frac{e^{-t}}{(x^2+t)^{N+1}} dt$$

(2). Using definition, show that

$$I(x) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{x^{2n}}, \quad x \longrightarrow +\infty.$$

3. Consider the asymptotic behavior of the integral:

$$I(x) = \int_1^3 e^{-x(\frac{9}{t}+t)} dt, \quad x \longrightarrow +\infty.$$

(1). Find the leading asymptotic behavior using Laplace's method.
(2). Rewrite the integral using variable $u = \frac{9}{t} + t - 6$ and find the first two leading terms in the asymptotic expansion (usingWatson's lemma).

4. Consider the asymptotic behavior of the integral:

$$I(x) = \int_0^1 \cos(xt^4) \tan t dt, \quad x \longrightarrow +\infty.$$

Find the leading asymptotic behavior using the method of stationary phase (Remember to use the Generalized Riemann-Lebesgue Lemma).

5. Using themethod of steepest descent to find the full asymptotic behaviors of:

$$I(x) = \int_0^1 e^{ixt^3} dt, \quad x \longrightarrow +\infty.$$

6. (1) Show that an integral representation of the Airy function Ai (x) is given by

$$Ai(x) = \frac{1}{2\pi i}\int_C e^{xt-\frac{t^3}{3}}dt$$

where C is a contour which originates at$\infty e^{\frac{-2\pi i}{3}}$ and terminates at$\infty e^{\frac{2\pi i}{3}}$
(2). Use this integral representation to show that the Taylor series expansion of $Ai(x)$ about $x = 0$ is as given in our class notes (or (3.2.1) in the textbook).
(3). Using the method of steepest descents, find the asymptotic behaviour of $Ai(x)$ as $x \longrightarrow +\infty$.

2.10 Answer 2.1

1. Consider the series expansions of the error function $\mathrm{erf}(x)$.

   (a) Show that the following series converges for any $x \in \mathbb{R}$:

   $$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} x^{2n+1}.$$

   (b) Show that the following series diverges for any $x \in \mathbb{R}$:

   $$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)!!}{2^{n+1}x^{2n+1}}.$$

   solution:

   (a) By ratio test, we now have $a_n = \frac{(-1)^n}{(2n+1)n!} x^{2n+1}$, then we need to solve the following inequality

   $$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{2n+3}}{a_n x^{2n+1}} \right| = \lim_{n\to\infty} \left| -\frac{2n+1}{(2n+3)(n+1)} x^2 \right| < 1,$$

   then we have the solution $x \in \mathbb{R}$. Thus we conclude that the series converges for any $x \in \mathbb{R}$.

   (b) Similarly using ratio test, we find that

   $$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{2n+1}}{a_n x^{2n+3}} \right| = \lim_{n\to\infty} \left| -\frac{2n+1}{2x^2} \right| = \infty,$$

   thus the series diverges for any $x \in \mathbb{R}$.

2. Find the series solutions about $x = 0$:

   $$9y''(x) + \frac{18}{x} y'(x) + y(x) = 0.$$

   solution: Since $x = 0$ is an regular singular point of this equation, then we apply the Frobenius series method. Assume that

   $$y(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n,$$

   then

   $$9\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n x^{n+\alpha-2} + 18\sum_{n=0}^{\infty}(n+\alpha)a_n x^{n+\alpha-2} + \sum_{n=2}^{\infty} a_{n-2}x^{n+\alpha-2} = 0.$$

   When $n = 0$, for term $x^{\alpha-2}$, we have

   $$9\alpha(\alpha-1)a_0 + 18\alpha a_0 = 9\alpha a_0(\alpha+1) = 0,$$

   thus $\alpha = 0$ or $\alpha = -1$. When $n = 1$, for term $x^{\alpha-1}$, we have

   $$9\alpha(\alpha+1)a_1 + 18(\alpha+1)a_1 = 9a_1(\alpha+2)(\alpha+1) = 0.$$

When $n \geq 2$, for term $x^{n+\alpha-2}$, we have

$$9(n+\alpha)(n+\alpha-1)a_n + 18(n+\alpha)a_n + a_{n-2} = 0,$$

thus

$$9(n+\alpha)(n+\alpha+1)a_n + a_{n-2} = 0.$$

If $\alpha = 0$, then $a_1 = 0$, and $a_n = -\frac{1}{9n(n+1)}a_{n-2}$, thus $a_{2n+1} = 0$, and $a_{2n} = (-\frac{1}{9})^n \frac{1}{(2n+1)!}a_0$.
If $\alpha = -1$, then for a_1 in-conclusion! Besides, $a_{n-2} = -9n(n-1)a_n$, thus

$$a_{2n} = (-\frac{1}{9})^n \frac{1}{(2n)!}a_0 \quad \text{and} \quad a_{2n+1} = (-\frac{1}{9})^n \frac{1}{(2n+1)!}a_1.$$

Hence we have the conclusion

$$y = \sum_{n=0}^{\infty}(-1)^n \frac{a_0}{9^n(2n)!}x^{2n-1} + \sum_{n=0}^{\infty}(-1)^n \frac{a_1}{9^n(2n+1)!}x^{2n}.$$

3. Read page 72 to page 76 in 3.3 carefully, and show that all solutions of the modified Bessel equation

$$y''(x) + \frac{1}{x}y'(x) - \left(1 + \frac{v^2}{x^2}\right)y(x) = 0$$

with $v = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ can be expanded in Frobenius series (without logarithmic terms).
solution: Assume that

$$y(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n, a_0 \neq 0,$$

then

$$\sum_{n=0}^{\infty} a_n(\alpha+n)(\alpha+n-1)x^{n+\alpha-2} + \sum_{n=0}^{\infty} a_n(\alpha+n)x^{n+\alpha-2} - \sum_{n=0}^{\infty} v^2 a_n x^{n+\alpha-2} - \sum_{n=2}^{\infty} a_{n-2}x^{n+\alpha-2} = 0.$$

There are three cases,

- When $n = 0$, for term $x^{\alpha-2}$, we have

$$a_0\alpha(\alpha-1) + a_0\alpha - v^2 a_0 = a_0(\alpha^2 - v^2) = 0.$$

  Then $\alpha = \pm v$, taking $\alpha_1 = v$ and $\alpha_2 = -v$.
- When $n = 1$, for $x^{\alpha-1}$ term, we have

$$a_1[(\alpha+1)^2 - v^2] = 0.$$

- When $n \geq 2$, for $x^{\alpha+n-2}$ term, we have

$$[(\alpha+n)^2 - v^2]a_n = a_{n-2}.$$

If $\alpha_1 = v$, $a_1 = 0$ and $a_{2n+1} = 0$, then

$$a_{2n} = \frac{a_{2n-2}}{4n(n+v)} = \frac{a_0\Gamma(v+1)}{4^n n!\Gamma(n+v+1)},$$

Thus we have

$$H_v(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{n!\Gamma(n+v+1)} x^v.$$

The convergence radius is

$$R = \frac{1}{\limsup\limits_{n\to\infty} \frac{1}{4(n+1)(n+v+1)}} = \infty,$$

then the series a_n converges for any $x \in \mathbb{R}$.
If $\alpha_2 = -v$, assume $v = \frac{2N+1}{2}$.
for $O(x^{\alpha+2N-1})$: $[(\alpha+2N+1)^2 - v^2]a_{2N+1} = a_{2N-1}$
since $(\alpha+2N+1)^2 - v^2 = 0$, we get $a_{2N-1} = 0$.
easily we know $a_{2n-1} = 0$, if $n <= N$
if $n > N$,

$$\begin{aligned} a_{2n+1} &= \frac{a_{2n-1}}{(2n+1)(2n+1-2v)} \\ &= \frac{a_{2n-3}}{(2n+1)(2n-1)(2n+1-2v)(2n-1-2v)} \\ &= ... \\ &= \frac{a_{2N+1}}{4^{n-N}(n+\frac{1}{2})(n-\frac{1}{2})...(N+\frac{3}{2})(n-N)!} \\ &= \frac{a_{2N+1}}{4^{n-N}} \frac{\Gamma(N+\frac{3}{2})}{\Gamma(n+\frac{3}{2})} \frac{1}{(n-N)!} \end{aligned}$$

Similarly

$$a_{2n} = \frac{a_{2n-2}}{4n(n-+v)} = \frac{a_0\Gamma(-v+1)}{4^n n!\Gamma(n-v+1)},$$

Therefore

$$y = \sum_0^{\infty} \frac{a_0\Gamma(-v+1)}{4^n n!\Gamma(n-v+1)} x^{2n-v} + \sum_{N+1}^{\infty} \frac{a_{2N+1}}{4^{n-N}} \frac{\Gamma(N+\frac{3}{2})}{\Gamma(n+\frac{3}{2})} \frac{1}{(n-N)!} x^{2n+1-v}$$

where $N = 2v - 1$

4. Continue the example in the class

$$x^3 y''(x) = y(x)$$

near the irregular singular point $x = 0$. We have calculated that as $x \to 0^+, y_1(x) \sim C_1 x^{\frac{3}{4}} e^{2x^{-\frac{1}{2}} - \frac{3}{16}x^{\frac{1}{2}}}$. Show that:

(a) If we write $y_1(x) = C_1 x^{\frac{3}{4}} e^{2x^{-\frac{1}{2}}} \left(1 - \frac{3}{16}x^{\frac{1}{2}} + w(x)\right)$, with $w(x) \ll \frac{3}{16}x^{\frac{1}{2}}$ as $x \to 0^+$, then the asymptotic series of $y_1(x)$ is of the form

$$y_1(x) \sim -C_1 x^{\frac{3}{4}} e^{2x^{-\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}{\pi \cdot 4^n \cdot n!} x^{\frac{n}{2}};$$

(b) The asymptotic series of the second solution $y_2(x)$ is of the form

$$y_2(x) \sim -C_2 x^{\frac{3}{4}} e^{-2x^{-\frac{1}{2}}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}{\pi \cdot 4^n \cdot n!} x^{\frac{n}{2}};$$

(c) Find the radius of convergence R of the these two series.

solution:

(a) If we write $y_1(x) = C_1 x^{\frac{3}{4}} e^{2x^{-\frac{1}{2}}} \left(1 - \frac{3}{16} x^{\frac{1}{2}} + w(x)\right)$, with $w(x) \ll \frac{3}{16} x^{\frac{1}{2}}$ as $x \to 0^+$, then substitute it into equation $x^3 y'' = y$, we have $w(x) = -\frac{15}{512} x$ and $w_n = \frac{(2n+1)!(2n+3)!}{16^n n!} = \frac{\Gamma(n-\frac{1}{2})\Gamma(n+\frac{3}{2})}{\pi 4^n n!}$ with $n \geq 2$. Hence, we conclude that

$$y_1(x) \sim -C_1 x^{\frac{3}{4}} e^{2x^{-\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}{\pi \cdot 4^n \cdot n!} x^{\frac{n}{2}};$$

(b) Similarly, we can first calculate that $y_2(x) = C_2 x^{\frac{3}{4}} e^{-2x^{-\frac{1}{2}}} \left(-1 + \frac{3}{16} x^{\frac{1}{2}} + w(x)\right)$, with $w(x) \ll \frac{3}{16} x^{\frac{1}{2}}$ as $x \to 0^+$, substitute it into equation $x^3 y'' = y$, we have $w(x) = -\frac{15}{512} x$ and $w_n = (-1)^n \frac{(2n+1)!(2n+3)!}{16^n n!} = (-1)^n \frac{\Gamma(n-\frac{1}{2})\Gamma(n+\frac{3}{2})}{\pi 4^n n!}$ with $n \geq 2$. Hence, we conclude that

$$y_2(x) \sim -C_2 x^{\frac{3}{4}} e^{-2x^{-\frac{1}{2}}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}{\pi \cdot 4^n \cdot n!} x^{\frac{n}{2}}.$$

(c) Convergence Radius is

$$\begin{aligned} R &= \left[\limsup_{n\to\infty} \frac{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3/2}{)}}{\pi \cdot 4^n \cdot n!}\right]^{-1} \\ &= \left[\limsup_{n\to\infty} \frac{(n-\frac{1}{2})(n+\frac{3}{2})}{4(n+1)}\right]^{-1} = 0. \end{aligned}$$

5. (Behavior of Airy functions for large x) Consider the Airy equation

$$y''(x) = xy(x)$$

(a) Show that it has an irregular singular point at ∞.

(b) Show that the leading behaviors of the solutions for large x are determined by

$$y_1(x) \sim C_1 x^{-\frac{1}{4}} e^{-\frac{2}{3} x^{\frac{3}{2}}}, \quad x \to +\infty$$

and

$$y_2(x) \sim C_2 x^{-\frac{1}{4}} e^{\frac{2}{3} x^{\frac{3}{2}}}, \quad x \to +\infty.$$

(c) Show that the asymptotic expansion of $y_1(x)$ is

$$y_1(x) \sim C_1 x^{-\frac{1}{4}} e^{-\frac{2}{3} x^{\frac{3}{2}}} \sum_{n=0}^{\infty} \frac{1}{2\pi} \left(-\frac{3}{4}\right)^n \left(\frac{\Gamma\left(n+\frac{5}{6}\right)\Gamma\left(n+\frac{1}{6}\right)}{n!}\right) x^{-\frac{3n}{2}}.$$

(d) Give the radius of convergence for this series.

solution:

(a) Possible singular point at $x=\infty$, then taking $x=\frac{1}{t}$. Then

$$\frac{d}{dx}=-t^2\frac{d}{dt}\quad\text{and}\quad\frac{d^2}{dx^2}=t^4\frac{d^2}{dt^2}+2t^3\frac{d}{dt}.$$

Thus

$$t^4y''+2t^3y'=\frac{1}{t}y,$$
$$y''+2\frac{1}{t}y'-\frac{1}{t^5}y=0.$$

Then $t=0$ is an irregular singular point.

(b) Taking $y=e^{s(x)}$, then we have $s''+(s')^2=x$. By dominant method, we have

- If $s''\sim-(s')^2\gg x$, then $s'\sim 1/x$ and $(s')^2\sim 1/x^2\ll x$, when $x\to\infty$. It is inconsistent.
- If $s''\sim x\gg(s')^2$, then $s'\sim\frac{x^2}{2}$ and $(s')^2\sim\frac{x^4}{2}\gg x$, when $x\to\infty$. It is inconsistent.
- If $(s')^2\sim x\gg s''$, then $s'\sim\pm\sqrt{x}$, $s''\sim\pm\frac{1}{2\sqrt{x}}\ll x$, and $s\sim\pm\frac{2}{3}x^{\frac{3}{2}}$, when $x\to\infty$. It is consistent.

Consider that $s(x)\sim-\frac{1}{2\sqrt{x}}$, then we write $s(x)=-\frac{1}{2\sqrt{x}}+c(x)$, where $c(x)\ll x^{\frac{3}{2}}$ as $x\to\infty$. Then we have

$$c''-2\sqrt{x}c'+(c')^2-\frac{1}{2\sqrt{x}}=0.$$

Since $c\ll x^{\frac{3}{2}}$, then $c'\ll\sqrt{x}$, $c''\ll\frac{1}{\sqrt{x}}$, $(c')^2\ll\sqrt{x}c'$ and $c''\ll\sqrt{x}c'$. Thus we have

$$-2\sqrt{x}c'-\frac{1}{2\sqrt{x}}=0,$$

then $c'=-\frac{1}{4x}$, thus $c=-\frac{1}{4}\ln(x)$.
Hence $s(x)\sim-\frac{2}{3}x^{\frac{2}{3}}-\frac{1}{4}\ln(x)$,

$$y_1(x)=e^{s(x)}\sim C_1x^{-\frac{1}{4}}e^{-\frac{2}{3}x^{\frac{3}{2}}},\quad x\to+\infty.$$

Similarly, when $s(x)=\frac{2}{3}x^{\frac{2}{3}}$, we have

$$y_2(x)=e^{s(x)}\sim C_2x^{-\frac{1}{4}}e^{\frac{2}{3}x^{\frac{3}{2}}},\quad x\to+\infty.$$

(c) Suppose that $y_1(x)=C_1x^{-\frac{1}{4}}e^{-\frac{2}{3}x^{\frac{3}{2}}}w(x)$ with $w(x)\sim\sum_{n=0}^{\infty}a_nx^{\alpha n}$ provided $\alpha<0$ and $a_0=1$. Then we have

$$\frac{5}{16}a_0-2\alpha_1x^{\alpha+\frac{3}{2}}+(\alpha^2-\frac{3}{2}\alpha+\frac{5}{16})x^{\alpha}-4\alpha a_2x^{2\alpha+\frac{3}{2}}+\cdots=0$$

when $x\to\infty$. Thus we have $\alpha+\frac{3}{2}=0$, concluded that $\alpha=-\frac{3}{2}$, $a_1=-\frac{5}{48}$ and $a_{n+1}=-\frac{3}{4}\frac{(n+\frac{5}{6})(n+\frac{1}{6})}{n+1}a_n=\frac{1}{2\pi}(\frac{-3}{4})^n\frac{\Gamma(n+5/6)\Gamma(n+1/6)}{n!}$ where $n=0,1,\cdots$. Therefore, we conclude that

$$y_1(x)\sim C_1x^{-\frac{1}{4}}e^{-\frac{2}{3}x^{\frac{3}{2}}}\sum_{n=0}^{\infty}\frac{1}{2\pi}\left(-\frac{3}{4}\right)^n\left(\frac{\Gamma\left(n+\frac{5}{6}\right)\Gamma\left(n+\frac{1}{6}\right)}{n!}\right)x^{-\frac{3n}{2}}.$$

(d) The convergence radius is

$$R = \left[\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\right]^{-1} = \left[\limsup_{n\to\infty}\left|-\frac{3}{4}\frac{n+\frac{5}{6}(n+\frac{1}{6})}{n+1}\right|\right]^{-1} = 0.$$

6. Show that if $f(x) \sim a\,(x-x_0)^{-b}$ as $x \to x_0^+$, then
   (a) $\int f\mathrm{d}x \sim \frac{a}{1-b}\,(x-x_0)^{1-b}, x \to x_0^+$ if $b > 1$ and the path of integration does not pass through x_0;
   (b) $\int f\mathrm{d}x \sim c, x \to x_0^+$ where c is a constant if $b < 1$; if $c = 0$, then $\int f\mathrm{d}x \sim \frac{a}{1-b}(x-x_0)^{1-b}, x \to x_0^+$;
   (c) $\int_{x_0}^{x} f\mathrm{d}x \sim \frac{a}{1-b}\,(x-x_0)^{1-b}, x \to x_0^+$ if $b < 1$;
   (d) $\int f\mathrm{d}x \sim a\ln(x-x_0), x \to x_0^+$ if $b = 1$.

solution: Since $f(x) \sim a\,(x-x_0)^{-b}$ as $x \to x_0^+$, then suppose $f(x) = a\,(x-x_0)^{-b} + g(x)$ where $g(x) \ll a\,(x-x_0)^{-b}$ as $x \to x_0^+$.

(a)

$$\int f dx = \int a\,(x-x_0)^{-b}\,dx + \int g dx = \frac{a}{1-b}\,(x-x_0)^{1-b} + \int g dx + \text{Const}\,.$$

Since $b > 1$, then $\frac{a}{1-b}\,(x-x_0)^{1-b} \to \infty$ and $\int g dx \ll \frac{a}{1-b}\,(x-x_0)^{1-b}$.Thus we can neglect constant and $\int g dx$. Therefore, $\int f\mathrm{d}x \sim \frac{a}{1-b}\,(x-x_0)^{1-b}, x \to x_0^+$.

(b) When $b < 1$, $\int g dx \ll \frac{a}{1-b}\,(x-x_0)^{1-b} \to 0 \ll \text{Const}$. Thus we drop $\int g dx$ and $\frac{a}{1-b}\,(x-x_0)^{1-b}$. Hence $\int f\mathrm{d}x \sim c, x \to x_0^+$.
If $c = 0$, $\int f = \frac{a}{1-b}\,(x-x_0)^{1-b} + \int g$ and $\int g \ll \frac{a}{1-b}\,(x-x_0)^{1-b}$. Thus $\int f\mathrm{d}x \sim \frac{a}{1-b}(x-x_0)^{1-b}$.

(c) When $b < 1$, $\int_{x_0}^{x} f dx \sim \int_{x_0}^{x} a(x-x_0)^{-b}dx + \int_{x_0}^{x} g dx = \frac{a}{1-b}(x-x_0)^{1-b} + \int_{x_0}^{x} g dx$. Since $\int_{x_0}^{x} g dx \ll \frac{a}{1-b}(x-x_0)^{1-b}$, then $\int_{x_0}^{x} f\mathrm{d}x \sim \frac{a}{1-b}\,(x-x_0)^{1-b}, x \to x_0^+$ if $b < 1$.

(d) When $b = 1$, $\int f dx = \int a(x-x_0)^{-1}dx + \int g(x) = a\ln(x-x_0) + \text{Const.} + \int g$. Since $\int g \ll a\ln(x-x_0)$, then $\int f\mathrm{d}x \sim a\ln(x-x_0), x \to x_0^+$ if $b = 1$.

7. (a) Give an example of an asymptotic relation $f(x) \sim g(x)(x \to \infty)$ that cannot be exponentiated; that is, $e^{f(x)} \sim e^{g(x)}$ is false.
   (b) Show that if $f(x) - g(x) \ll 1(x \to \infty)$, then $e^{f(x)} \sim e^{g(x)}(x \to \infty)$.

solution:

(a) Counter example: Let $f(x) = x$ and $g(x) = x+1$, then $f(x) \sim g(x)$ as $x \to \infty$. However, $e^x \sim e^{x+1}$ not true, since $\frac{e^x}{e^{x+1}} = e^{-1}$ not tends to 1.

(b) Since $f(x) - g(x) \ll 1(x \to \infty)$, then

$$\lim_{x\to\infty} f(x) - g(x) = 0.$$

Thus

$$\lim_{x\to\infty} e^{f(x)-g(x)} = e^0,$$

$$\lim_{x\to\infty} \frac{e^{f(x)}}{e^{g(x)}} = 1.$$

Thus by definition,

$$e^{f(x)} \sim e^{g(x)}(x \to \infty).$$

2.11 Answer 2.2

1. Use Taylor expansion of e^{-t^2} to find an asymptotic expansion of the integral:

$$I(x) = \int_0^{\frac{1}{x}} e^{-t^2} dt, \quad x \longrightarrow 0^+.$$

You are required to show that the asymptotic expansion of the integrand holds uniformly for t .
Solution:

$$\begin{aligned} I(x) &= \int_0^{\frac{1}{x}} e^{-t^2} dt \\ &= \int_0^{\infty} e^{-t^2} dt - \int_{\frac{1}{x}}^{\infty} e^{-t^2} dt \\ &= \frac{\sqrt{\pi}}{2} - \int_{\frac{1}{x}}^{\infty} e^{-t^2} dt. \end{aligned}$$

Using integration by parts, we get :

$$I(x) = \frac{\sqrt{\pi}}{2} - \frac{1}{2} x e^{-\frac{1}{x^2}} (1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{2^n} x^{2n}$$

2. Consider the integral:

$$I(x) = \int_0^{\infty} \frac{e^{-t}}{x^2 + t} dt, \quad x \longrightarrow +\infty.$$

(1). Using integration by parts, show that

$$I(x) = \sum_{n=1}^{N} \frac{(-1)^{n-1}(n-1)!}{x^{2n}} + (-1)^N N! \int_0^{\infty} \frac{e^{-t}}{(x^2+t)^{N+1}} dt$$

(2). Using definition, show that

$$I(x) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{x^{2n}}, \quad x \longrightarrow +\infty.$$

Solution: (1).

$$I(x) = \int_0^{\infty} \frac{e^{-t}}{x^2 + t} dt$$

Using integration by parts:

$$\begin{aligned} I(x) &= \int_0^\infty \frac{e^{-t}}{x^2+t}dt \\ &= \left.\frac{-e^{-t}}{x^2+t}\right|_0^\infty - \int_0^\infty \frac{-e^{-t}}{(x^2+t)^2}dt \\ &= \frac{1}{x^2} - \frac{1}{x^4} + ... + (-1)^{n-1}\int_0^\infty \frac{-(n-1)!e^{-t}}{(x^2+t)^n}dt \\ &= \sum_{n=1}^{N} \frac{(-1)^{n-1}(n-1)!}{x^{2n}} + (-1)^N \int_0^\infty \frac{N!e^{-t}}{(x^2+t)^{N+1}}dt \end{aligned}$$

(2).We know that:

$$|R_n(x)| = I(x) - \sum_{n=1}^{N} a_n\varphi_n = \left|(-1)^N \int_0^\infty \frac{N!e^{-t}}{(x^2+t)^{N+1}}dt\right| \le \left|\frac{N!}{x^{2N+2}}\int_0^\infty e^{-t}dt\right| = \frac{N!}{x^{2N+2}}$$

$$\sum_{n=1}^{N} a_n\varphi_n = \sum_{n=1}^{N} \frac{(-1)^{n-1}(n-1)!}{x^{2n}} \quad \Longrightarrow \quad \varphi_n(x) = \frac{1}{x^{2n}}$$

Obviously $R_n(x) = O(\varphi_n(x))$ as $x \to +\infty$. So:

$$I(x) \sim \sum_{n=1}^{N} \frac{(-1)^{n-1}(n-1)!}{x^{2n}}, \quad x \to +\infty$$

3. Consider the asymptotic behavior of the integral:

$$I(x) = \int_1^3 e^{-x(\frac{9}{t}+t)}dt, \quad x \longrightarrow +\infty.$$

(1). Find the leading asymptotic behavior using Laplace's method.
(2). Rewrite the integral using variable $u = \frac{9}{t} + t - 6$ and find the first two leading terms in the asymptotic expansion (usingWatson's lemma).
Solution:

$$I(x) = \int_1^3 e^{-x(\frac{9}{t}+t)}dt$$

(1).In this question, we have $f(t) = 1$ and $\phi(t) = -\frac{9}{t} - t$. Here $\phi(t)$ has a maximun the integration region $1 \le t \le 3$ at $t = 3$, we may replace the interal by

$$I(x;\varepsilon) = \int_{3-\varepsilon}^3 e^{-x(\frac{9}{t}+t)}dt$$

for any $\varepsilon > 0$ at the cost of introducing errors which are exponentialy small as $x \to +\infty$. Next using Tayor series about $t = 3$ to replace $\phi(t)$. This replacement makes the integral easy to evaluate. Thus,

$$\int_{3-\varepsilon}^3 e^{-x(\frac{9}{t}+t)}dt \sim \int_{3-\varepsilon}^3 e^{-x(6+\frac{1}{3}(t-3)^2)}dt = e^{-6x}\int_{3-\varepsilon}^3 e^{\frac{-x(t-3)^2}{3}}dt = e^{-6x}\int_0^\varepsilon e^{\frac{-xt^2}{3}}dt.$$

Since we know that

$$\int_0^{\varepsilon} e^{\frac{-xt^2}{3}}dt = \frac{1}{2}\int_{-\varepsilon}^{\varepsilon} e^{\frac{-xt^2}{3}}dt \sim \frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{-xt^2}{3}}dt = \frac{1}{2}\cdot\sqrt{\frac{3}{x}}\int_{-\infty}^{\infty} e^{-(\sqrt{\frac{3}{x}}t)^2}d(\sqrt{\frac{3}{x}}t) = \frac{1}{2}\cdot\sqrt{\frac{3}{x}}\cdot\sqrt{\pi}$$

we obtain the leading behavior

$$I(x) \sim e^{-6x}\sqrt{\frac{3\pi}{4x}}, \quad x \to +\infty.$$

(2).

$$u = \frac{9}{t} + t - 6 \quad \Longrightarrow \quad t^2 - (6+u)t + 9 = 0 \quad \Longrightarrow \quad t = \frac{(6+u) \pm \sqrt{u^2+12u}}{2}.$$

Notice that $t = 1 \Rightarrow u = 4$ and $t = 3 \Rightarrow u = 0$, so the sign is $'-'$, $t = \frac{(6+u)-\sqrt{u^2+12u}}{2}$.

$$\begin{aligned}
I(x) &= \int_1^3 e^{-x(\frac{9}{t}+t)}dt \\
&= \int_4^0 e^{-ux-6x}(\frac{1}{2} - \frac{u+6}{2\sqrt{u^2+12u}})du \\
&= \frac{e^{-6x}}{2}[\int_4^0 e^{-ux}du + \int_0^4 \frac{u}{\sqrt{u^2+12u}}e^{-ux}du + \int_0^4 \frac{6}{\sqrt{u^2+12u}}e^{-ux}du] \\
&= \frac{e^{-6x}}{2}[I_1(x) + I_2(x) + I_3(x)].
\end{aligned}$$

Obviously, $I_1 \sim -\frac{1}{x}$. Using Taylor expansion:

$$\frac{1}{\sqrt{u+12}} = 12^{-\frac{1}{2}} + \frac{1}{2!}(-\frac{1}{2})12^{-\frac{3}{2}}u + \frac{1}{3!}(-\frac{1}{2})(-\frac{3}{2})12^{-\frac{5}{2}}u^2 + ... = \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+\frac{1}{2})12^{-\frac{1}{2}-n}}{(n+1)!\sqrt{\pi}}u^n = \sum_{n=0}^{\infty} a_n u^n$$

So for $I_2(x)$,

$$I_2(x) = \int_0^4 \frac{u}{\sqrt{u^2+12u}}e^{-ux}du = \int_0^4 u^{\frac{1}{2}}\sum_{n=0}^{\infty} a_n u^n e^{-ux}du.$$

We get $\alpha = \frac{1}{2}$ and $\beta = 1$, so according to Watson's Lemma, $(x \to +\infty)$

$$I_2 \sim \sum_{n=0}^{\infty}\frac{a_n\Gamma(\frac{3}{2}+n)}{x^{\frac{3}{2}+n}}.$$

Similarly, using Watson's Lemma in $I_3(x)$, we get $\alpha = -\frac{1}{2}$ and $\beta = 1$,

$$I_3 \sim \sum_{n=0}^{\infty}\frac{6a_n\Gamma(n+\frac{1}{2})}{x^{n+\frac{1}{2}}}.$$

Combining this result with others gives $(x \to +\infty)$.

$$I(x) \sim -\frac{e^{-6x}}{2x} + e^{-6x}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+\frac{1}{2})12^{-\frac{1}{2}-n}}{2(n+1)!\sqrt{\pi}}[\frac{\Gamma(\frac{3}{2}+n)}{x^{\frac{3}{2}+n}} + \frac{6\Gamma(n+\frac{1}{2})}{x^{n+\frac{1}{2}}}].$$

Only consider the first leading terms we have:

$$I(x) = \frac{\sqrt{3\pi}e^{-6x}}{2x^{\frac{1}{2}}} - \frac{e^{-6x}}{2x} \qquad x \to +\infty.$$

4. Consider the asymptotic behavior of the integral:

$$I(x) = \int_0^1 \cos(xt^4) \tan t dt, \quad x \longrightarrow +\infty.$$

Find the leading asymptotic behavior using the method of stationary phase (Remember to use the Generalized Riemann-Lebesgue Lemma).
Solution:

$$I(x) = Re \int_0^1 \tan t e^{ixt^4} dt.$$

Let $T(x) = \int_0^1 \tan t e^{ixt^4} dt$, and we know $T(x)$ has stationary phase at $t = 0$.
So

$$T(x) = \int_0^{\epsilon} tante^{ixt^4} dt + \int_{\epsilon}^1 tante^{ixt^4} dt.$$

By Riemann Lebesgue lemma

$$\int_{\epsilon}^1 \tan t e^{ixt^4} dt = \frac{\tan t}{4ixt^3} e^{ixt^4}\Big|_{t=1}^{t=\epsilon} - \int_1^{\epsilon} \frac{e^{ixt^4}}{4ixt^3} \frac{1}{\cos^2 t} dt \sim O(\frac{1}{x}).$$

And

$$\int_0^{\epsilon} tante^{ixt^4} dt \sim \int_0^{\epsilon} te^{ixt^4} dt \sim \int_0^{\infty} te^{ixt^4} dt.$$

Let $z = \sqrt{x}t^2$:

$$\int_0^{\infty} te^{ixt^4} dt = \frac{1}{2\sqrt{x}} \int_0^{\infty} e^{iz^2} dz = \frac{1}{4}\sqrt{\frac{\pi}{x}} e^{i\frac{\pi}{4}}.$$

Therefore,

$$I(x) \sim Re(\frac{1}{4}\sqrt{\frac{\pi}{x}} e^{i\frac{\pi}{4}}) = \frac{1}{8}\sqrt{\frac{2\pi}{x}}.$$

5. Using the method of steepest descent to find the full asymptotic behaviors of:

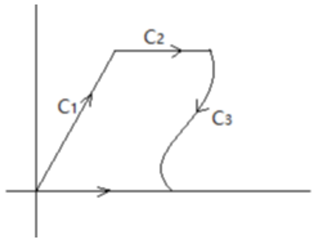
$$I(x) = \int_0^1 e^{ixt^3} dt, \quad x \longrightarrow +\infty.$$

Solution: Let $t = u + iv$, $p(t) = it^3$, then $t = 0 \Rightarrow Im(p(t)) = 0 \Rightarrow u(u^2 - 3v^2) = 0$.

Set $u = \sqrt{3}v$, $|e^{xp(t)}| = |e^{-8xv^2}| \to 0$ as $v \to \infty$. (steepest-descent), So we have $C_1 : t = (\sqrt{3} + i)v$.

Let $t = 1 \Rightarrow Im(p(t)) = 1 \Rightarrow u(u^2 - 3v^2) = 1$, we set the solution of this fuction as C_3. Then there exist a straight line connects between C_1 and C_3 and parallels to the real axis, called C_2. So we have:

$$I(x) = \int_0^1 e^{ixt^3} dt = \int_{C_1} e^{ixt^3} dt + \int_{C_2} e^{ixt^3} dt + \int_{C_3} e^{ixt^3} dt.$$



For C_1, $t = (\sqrt{3}+i)v$, we can get:

$$\int_{C_1} e^{ixt^3}dt = (\sqrt{3}+i)\int_0^\infty e^{-8xv^3}dv = (\sqrt{3}+i)\frac{\Gamma(\frac{1}{3})}{6x^{\frac{1}{3}}} = \frac{e^{\frac{\pi}{6}i}\Gamma(\frac{1}{3})}{3x^{\frac{1}{3}}}.$$

For C_2, $\int_{C_2} e^{ixt^3}dt \to 0$ as $v \to \infty$.

For C_3, $u(u^2-3v^2) = 1 \Rightarrow p(t) = i-(3u^2v-v^3)$. Let $s = 3uv^2-v^3$, $0 \le s$, so $t = (1+is)^{\frac{1}{3}} \Rightarrow \frac{dt}{ds} = \frac{1}{3}i(1+is)^{-\frac{2}{3}}$.

$$\int_{C_3} e^{ixt^3}dt = -\int_0^\infty \frac{ie^{(i-s)x}}{3(1+is)^{\frac{2}{3}}}ds = -\frac{1}{3}ie^{ix}\int_0^\infty (1+is)^{-\frac{2}{3}}e^{-sx}ds.$$

To apply Watson's lemma, we use the Taylor expansion

$$(1+is)^{-\frac{2}{3}} = \sum_{n=0}^\infty (-is)^n \frac{\Gamma(n+\frac{2}{3})}{n!\Gamma(\frac{2}{3})}.$$

We obtain

$$\int_{C_3} e^{ixt^3}dt \sim -\frac{1}{3}ie^{ix}\cdot\sum_{n=0}^\infty (-i)^n\frac{\Gamma(n+\frac{2}{3})\Gamma(n+1)}{n!x^{n+1}\Gamma(\frac{2}{3})} = -\frac{1}{3}ie^{ix}\cdot\sum_{n=0}^\infty(-i)^n\frac{\Gamma(n+\frac{2}{3})}{x^{n+1}\Gamma(\frac{2}{3})}.$$

Combining this result with others gives the full asymptotic expansion of $I(x)$ as $x \to +\infty$:

$$I(x) \sim \frac{e^{\frac{\pi}{6}i}\Gamma(\frac{1}{3})}{3x^{\frac{1}{3}}} - \frac{1}{3}ie^{ix}\cdot\sum_{n=0}^\infty(-i)^n\frac{\Gamma(n+\frac{2}{3})}{\Gamma(\frac{2}{3})x^{n+1}}.$$

6. (1) Show that an integral representation of the Airy function Ai (x) is given by

$$Ai(x) = \frac{1}{2\pi i}\int_C e^{xt-\frac{t^3}{3}}dt$$

where C is a contour which originates at$\infty e^{\frac{-2\pi i}{3}}$ and terminates at$\infty e^{\frac{2\pi i}{3}}$
(2). Use this integral representation to show that the Taylor series expansion of $Ai(x)$ about $x=0$ is as given in our class notes (or (3.2.1) in the textbook).

(3). Using the method of steepest descents, find the asymptotic behaviour of $Ai(x)$ as $x \longrightarrow +\infty$.

Solution: (1). We want to check $Ai(x)$ satisfy: $y'' = xy$

Form $Ai(x) = \frac{1}{2\pi i}\int_C e^{xt-\frac{t^3}{3}}dt$ (C:from $\infty e^{-\frac{2}{3}\pi i}$ to $\infty e^{\frac{2}{3}\pi i}$),we know:

$$Ai'(x) = \frac{1}{2\pi i}\int_C te^{xt-\frac{t^3}{3}}dt,$$
$$Ai''(x) = \frac{1}{2\pi i}\int_C t^2e^{xt-\frac{t^3}{3}}dt.$$

$$\begin{aligned}
Ai''(x) - xAi(x) &= -\frac{1}{2\pi i}\int_C d(e^{xt-\frac{t^3}{3}}) \\
&= -\frac{1}{2\pi i}(e^{xt-\frac{t^3}{3}})\Big|_{\infty e^{-\frac{2}{3}\pi i}}^{\infty e^{\frac{2}{3}\pi i}} \\
&= \lim_{R\to\infty}\frac{1}{2\pi i}\frac{e^{xRe^{-\frac{2}{3}\pi i}} - e^{xRe^{\frac{2}{3}\pi i}}}{e^{\frac{1}{3}R^3}} = 0.
\end{aligned}$$

(2).Let $C_1 : \{t\Big|t = Re^{\frac{2}{3}\pi i}\}$,$C_2 : \{t\Big|t = Re^{-\frac{2}{3}\pi i}\}$

Then:

$$Ai(x) = \frac{1}{2\pi i}(\int_{C1} e^{xt-\frac{t^3}{3}}dt - \int_{C2} e^{xt-\frac{t^3}{3}}dt).$$

$$\begin{aligned}
\int_{C1} e^{xt-\frac{t^3}{3}}dt &= \int_0^\infty e^{xR\frac{2}{3}\pi i}e^{-\frac{1}{3}R^3}e^{\frac{2}{3}\pi i}dR \\
&\overset{R^3=u}{=} \frac{1}{3}e^{\frac{2}{3}\pi i}\int_0^\infty e^{-\frac{1}{3}u}u^{-\frac{2}{3}}e^{xu^{\frac{1}{3}}e^{\frac{2}{3}\pi i}}du \\
&= \frac{1}{3}e^{\frac{2}{3}\pi i}\int_0^\infty e^{-\frac{1}{3}u}u^{-\frac{2}{3}}\sum_{n=0}^\infty \frac{(xu^{\frac{1}{3}}e^{\frac{2}{3}\pi i})^n}{n!}du \\
&\overset{u=3y}{=} \frac{1}{3}e^{\frac{2}{3}\pi i}\int_0^\infty e^{-y}\sum_{n=0}^\infty \frac{x^ne^{\frac{2n}{3}\pi i}3^{\frac{n+1}{3}}y^{\frac{n-2}{3}}}{n!}dy \\
&= \frac{1}{3}e^{\frac{2}{3}\pi i}\sum_{n=0}^\infty \frac{x^n}{n!}e^{\frac{2n}{3}\pi i}3^{\frac{n+1}{3}}\Gamma(\frac{n+1}{3}).
\end{aligned}$$

Similarly:

$$\int_{C2} e^{xt-\frac{t^3}{3}}dt = \frac{1}{3}e^{\frac{4}{3}\pi i}\sum_{n=0}^\infty \frac{x^n}{n!}e^{\frac{4n}{3}\pi i}3^{\frac{n+1}{3}}\Gamma(\frac{n+1}{3}).$$

So

$$\begin{aligned}
Ai(x) &= \frac{1}{2\pi i}(\frac{1}{3}e^{\frac{2}{3}\pi i}\sum_{n=0}^\infty \frac{x^n}{n!}e^{\frac{2n}{3}\pi i}3^{\frac{n+1}{3}}\Gamma(\frac{n+1}{3}) - \frac{1}{3}e^{\frac{4}{3}\pi i}\sum_{n=0}^\infty \frac{x^n}{n!}e^{\frac{4n}{3}\pi i}3^{\frac{n+1}{3}}\Gamma(\frac{n+1}{3})) \\
&= \frac{1}{2\pi i}\sum_{n=0}^\infty \frac{x^n}{n!}\Gamma(\frac{n+1}{3})3^{\frac{n-2}{3}}(e^{\frac{2}{3}\pi i} - e^{\frac{4}{3}\pi i}) \\
&= \frac{sin(\frac{2\pi}{3})}{\pi}\sum_{n=0}^\infty \frac{x^n}{n!}\Gamma(\frac{n+1}{3})3^{\frac{n-2}{3}}.
\end{aligned}$$

because $\Gamma(x)\Gamma(1-x) = \frac{\pi}{sin(\pi x)}$

$$Ai(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\Gamma(\frac{n+1}{3})3^{\frac{n-2}{3}}\frac{1}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})},$$

for $n = 3k$

$$\begin{aligned}
\frac{x^n}{n!}\Gamma(\frac{n+1}{3})3^{\frac{n-2}{3}}\frac{1}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} &= \frac{x^{3k}}{(3k)!}\Gamma(\frac{3k+1}{3})3^{\frac{3k-2}{3}}\frac{1}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} \\
&= 3^{\frac{3k-2}{3}}\frac{x^{3k}}{(3k)!}\frac{(k-\frac{2}{3})(k-\frac{5}{3})...\frac{1}{3}\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} \\
&= 3^{\frac{3k-2}{3}}\frac{x^{3k}}{(3k)(3k-3)...3 \bullet (3k-1)(3k-4)...2 \bullet (3k-2)(3k-5)...1} \\
&\bullet \frac{(k-\frac{2}{3})(k-\frac{5}{3})...\frac{1}{3}\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} \\
&= 3^{-\frac{2}{3}}\frac{x^{3n}}{9^k k!\Gamma(k+\frac{2}{3})}.
\end{aligned}$$

Similarly, for $n = 3k+1$

$$\frac{x^n}{n!}\Gamma(\frac{n+1}{3})3^{\frac{n-2}{3}}\frac{1}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} = -3^{-\frac{4}{3}}\frac{x^{3k+1}}{9^k k!\Gamma(k+\frac{4}{3})}.$$

For $n = 3k+2$

$$\frac{x^n}{n!}\Gamma(\frac{n+1}{3})3^{\frac{n-2}{3}}\frac{1}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} = 0.$$

Therefore, the Taylor series expansion of Ai(x) is (3.2.1) in the textbook.

(3).Let $t = \sqrt{x}s, X = x^{\frac{3}{2}}$, then

$$Ai(x) = \frac{\sqrt{x}}{2\pi i}\int_C e^{X(s-\frac{1}{3}s^3)}ds.$$

Let $p(s) = s - \frac{1}{3}s^3$, then$p'(s) = 1 - s^2, p''(s) = -2s$ We know s=-1 is a saddle point of Ai(x). Let $s = u + iv$, then:

$$s - \frac{1}{3}s^3 = (u - \frac{1}{3}u^3 + uv^2) + i(v - u^2 v + \frac{1}{3}v^3)$$

When s=-1, we get $v - u^2 v + \frac{1}{3}v^3 = 0$. We distort C into:

$$C1 : \{(u,v)|v^2 = 3u^2 - 3\}.$$

Then

$$Ai(x) = \frac{\sqrt{x}}{2\pi i}\int_C e^{X(s-\frac{1}{3}s^3)}ds = \frac{\sqrt{x}}{2\pi i}\int_{C1} e^{X(s-\frac{1}{3}s^3)}ds.$$

let $s = -(\cosh u + i\sqrt(3)\sinh u)$,

$$\begin{aligned} Ai(x) &\sim \frac{\sqrt{x}}{2\pi i}\int_{-\epsilon}^{\epsilon} e^{-X\cosh u(\frac{8}{3}\cosh^2 u-2)}(\sinh u + i\sqrt{3}\cosh u)du \\ &\sim \frac{\sqrt{x}}{2\pi i}\int_{-\epsilon}^{\epsilon} \sqrt{3}ie^{-X(\frac{2}{3}+3u^2)}du \\ &\sim \frac{\sqrt{x}}{2\pi i}\int_{-\infty}^{\infty} \sqrt{3}ie^{-X(\frac{2}{3}+3u^2)}du \\ &= \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{2\sqrt{\pi}x^{\frac{1}{4}}}. \end{aligned}$$

Chapter **3**

Introduction to Global Analysis and Perturbation Methods

3.1 Perturbation method

Perturbation problems can be regarded as problems containing small parameters. Perturbation methods are to find asymptotic approximations of solutions to perturbation problems (including direct expansion, boundary layer theory, WKB theory, multiply-scale analysis, and homogenization).

Asymptotic expansion in ε:

$y(x,\varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n y_n(x)$, $\varepsilon \to 0$, $R_n(x,\varepsilon) = y(x,\varepsilon) - \sum_{n=0}^{N} \varepsilon^n y_n(x)$ and $\lim\limits_{\varepsilon\to 0} \frac{R_n(x,\varepsilon)}{\varepsilon^N} = 0$ for all fied x, N. So $y(x,\varepsilon) = \sum_{n=0}^{N} \varepsilon^n y_n(x) + o(\varepsilon^N)$, $\varepsilon \to 0$. This implies that

$$\begin{cases} \lim\limits_{\varepsilon\to 0} y(x,\varepsilon) = y_0(x), \\ \lim\limits_{\varepsilon\to 0} \dfrac{R_N(x,\varepsilon)}{\varepsilon^{N+1}} = y_{N+1}(x),\ N = 0,1,2,\cdots. \end{cases}$$

Hence, we have $y(x,\varepsilon) = \sum_{n=0}^{N} \varepsilon^n y_n(x) + O(\varepsilon^{N+1})$.

If $y(x,\varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n y_n(x)$ as $\varepsilon \to 0$ exists, then $y_n(x)(n = 0,1,2,\cdots)$ are uniquely determined by $y(x,\varepsilon)$. Generally, $y(x,\varepsilon) \sim \sum_{n=0}^{\infty} \varphi_n(\varepsilon) y_n(x)$, $\varepsilon \to 0$, where $\varphi_n(\varepsilon)$ is a family of function s.t. $\varphi_{n+1}(\varepsilon) \ll \varphi_n(\varepsilon)$ as $\varepsilon \to 0$ and $\lim\limits_{\varepsilon\to 0} \frac{R_n(x,\varepsilon)}{\varphi_N(\varepsilon)} = 0$. Hence, $y(x,\varepsilon) = \sum_{n=0}^{N} \varphi_n(\varepsilon) y_n(x) + o(\varphi_N(\varepsilon))$ $\varepsilon \to 0$ for fixed x, N, which implies that

$$\begin{cases} \lim\limits_{\varepsilon\to 0} \dfrac{y(x,\varepsilon)}{\varphi_0(\varepsilon)} = y_0(x), \\ \lim\limits_{\varepsilon\to 0} \dfrac{R_N(x,\varepsilon)}{\varphi_{N+1}(\varepsilon)} = y_{N+1}(x), \end{cases} \Rightarrow y(x,\varepsilon) = \sum_{n=0}^{N} \varphi_n(\varepsilon) y_n(x) + O(\varphi_{N+1}(\varepsilon)),$$

as $\varepsilon \to 0$. This is similar to expansion $x \to x_0$.

Example using direct expansion:

$$\begin{cases} \dfrac{dv}{dt} = -1 - \varepsilon v, \\ v(0) = 1. \end{cases}$$

We seek solution of form $v(t) \sim\sim \sum_{n=0}^{\infty} \varepsilon^n v_n(t)$. First we substitute this into the differential equation ger $\frac{dv_0}{dt} + \varepsilon \frac{dv_1}{dt} + \varepsilon^2 \frac{dv_2}{dt} + \cdots \sim -1 - \varepsilon v_0 - \varepsilon^2 v_1 - \cdots$ From the initial condition we have $v_0(0) + \varepsilon v_1(0) + \varepsilon^2 v_2(0) + \cdots \sim 1 \Rightarrow v_0(0) = 1,\ v_1(0) = 0,\ v_2(0) = 0 \cdots$.

$$\begin{aligned}
O(\varepsilon^0) &: \frac{dv_0}{dt} = -1 \Rightarrow v_0(t) = 1 - t,\\
O(\varepsilon) &: \frac{dv_1}{dt} = -v_0 = t - 1 \Rightarrow v_1(t) = -t + \frac{1}{2}t^2,\\
O(\varepsilon^2) &: \frac{dv_2}{dt} = -v_1 = t - \frac{1}{2}t^2 \Rightarrow v_2(t) = \frac{1}{2}t^2 - \frac{1}{6}t^3.
\end{aligned}$$

Therefore, $v(t) \sim 1 - t + \varepsilon(-t + \frac{1}{2}t^2) + \varepsilon^2(\frac{1}{2}t^2 - \frac{1}{6}t^3) = \cdots$, $v^m(t) = \sum_{n=0}^{m} \varepsilon^n V_N(t)$ are approximations to the solution.

Comparison with the exact solution:

$$\begin{cases} \dfrac{dv}{dt} = -1 - \varepsilon v \qquad v(0) = 1. \end{cases}$$

The exact solution of this differential equation is $v = \frac{1+\varepsilon}{\varepsilon}e^{-\varepsilon t} - \frac{1}{\varepsilon}$.

Regular and Singular Perturbation Problems:

For a perturbation problem, we are interested in the solution when ε is small but nonzero.

(1) Regular Perturbation Problem:

$v(t,\varepsilon) = v_0(t) + \varepsilon v_1(t) + \varepsilon^2 v_2(t) + \cdots$ has a nonvanishing radius (for ε) of uniform convergence for all t in the given domain ($f(\varepsilon) = \sum_{n=0} \varepsilon^n a_n$ convergence radius $R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{a_n}}$). In this case, we can assume that $v(t,\varepsilon) \sim v_0(t) + \varepsilon v_1(t) + \varepsilon^2 v_2(t) + \cdots$ as $\varepsilon \to 0$, so it is uniformly for all t in the given domain. In particular, $\lim\limits_{\varepsilon\to 0} v(t,\varepsilon) = v_0(t)$, uniformly for t, i.e. the solution of small nonzero ε, converges uniformly to the solution of unperturbed problem ($\varepsilon = 0$) as $\varepsilon \to 0$. In other words, the solution of unperturbed problem is an approximation to the solution of perturbed problem.

(1) Singular Perturbation Problem:

$v(t,\varepsilon) = v_0(t) + \varepsilon v_1(t) + \varepsilon^2 v_2(t) + \cdots$ has a vanishing radius of uniform convergence.

Here are some cases:
① The unperturbed problem has no solution.
② The perturbation series does not take the form of a power series of ε.
③ The solution of small nonzero ε problem does not converge uniformly to the solution of the unperturbed problem.

Example 3.1.1: (Regular perturbation)

$$\begin{cases} \dfrac{df}{dt} + f = \varepsilon f^2, \quad t > 0,\\ f(0) = 1. \end{cases}$$

Assume $f(t) \sim f_0(t) + \varepsilon f_1(t) + \varepsilon^2 f_2(t) + \cdots$. We can find that $f_n(t) = e^{-t}(1 - e^{-t})^n$. Since $|f_n(t)| \leq 1$ for $t \geq 1$, the series converges uniformly for all $t \geq 0$ when $|\varepsilon| < 1$. Thus, this is a regular perturbation problem. In particular, the solution of unperturbed problem $f_0(t) = e^{-t}$ is a uniform approximation to $f(t)$ as $\varepsilon \to 0$ uniformly for $t \geq 0$.

Example 3.1.2: (Singular perturbation: boundary layer)

$$\left\{ \varepsilon y'' - y' = 0, \quad 0 \leq t \leq 1 \quad y(0) = 0, \quad y(1) = 1. \right.$$

When $\varepsilon \to 0$, the unperturbed problem is $\begin{cases} -y' = 0, & 0 \leq t \leq 1 \\ y(0) = 0, & y(1) = 1 \end{cases}$. So $y' = 0$ implies $y = constant$. That means it is impossible for the boundary condition, which implies the unperturbed problem has no solution, and the problem is a singular perturbation problem. Actually, we have $y(x) = \frac{e^{\frac{x}{\varepsilon}} - 1}{e^{\frac{1}{\varepsilon}} - 1}$, and we find that $y(x)$ is almost constant 0 except in a small region (boundary layer) near $x = 1$ and

$$\lim_{\varepsilon \to 0} y(x, \varepsilon) = \begin{cases} 0, & x \in [0, 1], \\ 1, & x = 1. \end{cases}$$

Example 3.1.3: (Singular perturbation: Rapid oscillation on a global scale)

$$\begin{cases} \varepsilon y'' + y' = 0, \\ y(0) = 0, \quad y(1) = 1. \end{cases}$$

The exact solution is $y(x) = \frac{\sin \frac{x}{\sqrt{\varepsilon}}}{\sin \frac{1}{\sqrt{\varepsilon}}}$, when $\varepsilon \neq \frac{1}{n^2\pi^2}, n = 0, 1, 2, \cdots$. When $\varepsilon \neq \frac{1}{n^2\pi^2}$, there is no solution. As ε turns smaller, $y(x)$ oscillates more. As $\varepsilon \to 0$, $y(x)$ becomes discontinuous over the entire interval. (correlated to WKB theory).

3.2 Boundary layer problem

Example 3.2.1: We consider the following equation:

$$\begin{cases} \varepsilon y' + y = 0, \\ y(0) = 0, \quad y(1) = 1. \end{cases}$$

Using the former methods that we introduced, we have $y = Ce^{-x}$. However,

$$\begin{cases} y(0) = 0, \Rightarrow C = 0. \\ y(1) = 1, \Rightarrow C = 1. \end{cases}$$

This leads to a contradiction. The exact solution is $y = \frac{e^{-x} - e^{-\frac{1}{\varepsilon}}}{e^{-1} - e^{-\frac{1}{\varepsilon}}}$. So

$$\lim_{\varepsilon \to 0} y(x, \varepsilon) = \begin{cases} e^{1-x}, & 0 < x \leq 1, \\ 0, & x = 0, \end{cases}$$

is not uniform for $0 \leq x \leq 1$.

(1) When $\varepsilon \ll x \leq 1$, for $\alpha \in (0,1)$. Assume $\varepsilon^{\alpha} \leq x \leq 1$, so $\lim\limits_{\varepsilon\to 0^+} \frac{x}{\varepsilon} = +\infty$, is uniformly for $\varepsilon \ll x \leq 1$, then $\lim\limits_{\varepsilon\to 0^+} e^{-\frac{x}{\varepsilon}} = 0$ and $\lim\limits_{\varepsilon\to 0^+} y(x) = e^{1-x}$ is uniformly, i.e. $y_0(x) = e^{1-x}$ is a uniform approximation to the solution in the region $\varepsilon \ll x \leq 1$. In this region, the solution varies slowly— outer region $y_0(x)$ is the outer limit.

(2)$0 \leq x \leq O(\varepsilon)$, then $\lim\limits_{\varepsilon\to 0^+} x = 0$ is uniformly. So $\lim\limits_{\varepsilon\to 0^+} e^{-x} = e^0 = 1$, it is uniformly $\Rightarrow \lim\limits_{\varepsilon\to 0^+} y(x) = \frac{1-e^{-\frac{x}{\varepsilon}}}{e^{-1}} = e - e^{-\frac{x}{\varepsilon}}$, i.e. $e - e^{-\frac{x}{\varepsilon}}$ is a uniform approximation to the solution in the region $0 \leq x \leq O(\varepsilon)$. In this region, the solution varies rapidlly—inner region or boundary layer region.
Lets's define inner variable: $X = \frac{x}{\varepsilon}$. Then the inner region is $0 \leq X \leq O(1)$, $y(x) = y(\varepsilon X) = Y_{in}(X)$. $\therefore$ $y(x) = Y_{in}(X) = \frac{e^{-\varepsilon X} - e^{-X}}{e^{-1} - e^{-\frac{1}{\varepsilon}}} \to \frac{1-e^{-X}}{e^{-1}} = e - e^{1-X} \equiv Y_0(X)$ as $\varepsilon \to 0^+$, is uniformly for $0 \leq x \leq O(\varepsilon)$. It is a uniform approximation in the inner region. The inner region can be extended to $0 \leq x \leq \eta(\varepsilon)$ with $\varepsilon \ll \eta(\varepsilon) \ll 1$ as $\varepsilon \to 0^+ (\eta(\varepsilon) = \varepsilon^{\alpha}, 0 < \alpha < 1) \Rightarrow 0 \leq X \leq +\infty$, but the inner limit $Y_0(X)$ is still a uniform approximation. (But in this extended inner region, $\lim\limits_{\varepsilon\to 0^+} \varepsilon X = \lim\limits_{\varepsilon\to 0^+} X = 0$, is uniformly for $0 \leq x \leq \eta(\varepsilon)$).

(3) In the overlapping region, e.g. outer region $\varepsilon^{\frac{2}{3}} \leq x \leq 1$ and inner region $0 \leq x \leq \varepsilon^{\frac{1}{3}}$, the overlapping region is $\varepsilon^{\frac{2}{3}} \leq x \leq \varepsilon^{\frac{1}{3}}$. When $\varepsilon \to 0^+$, in this region, $x \to 0$, $X = \frac{x}{\varepsilon} \to +\infty$. $\therefore$ $\lim\limits_{\varepsilon\to 0^+} y(x) = \lim\limits_{\varepsilon\to 0^+} \frac{e^{-x} - e^{-\frac{x}{\varepsilon}}}{e^{-1} - e^{-\frac{1}{\varepsilon}}} = \frac{e^0 - 0}{e^{-1} - 0} = e$, is uniformly in this region.
This limit value e can also be obtained by $\lim\limits_{x\to 0} y_0(x)$($x \to 0$ in the outer region) or $\lim\limits_{X\to +\infty} Y_0(X)$($X \to +\infty$ in the inner region). $\therefore$ $\lim\limits_{\varepsilon\to 0^+} y(x) = \lim\limits_{X\to 0} y_0(x) = \lim\limits_{X\to +\infty} Y_0(X)$ defines intermidiate region.

3.3 Boundary layer theory

Example 3.3.1:

$$\begin{cases} \varepsilon y'' + (1+\varepsilon)y' + y = 0 & x \in [0,1] \\ y(0) = 0, \quad y(1) = 1 \end{cases}$$

Usually, we have the following question:

1. Does the boundary layer exist?
2. Location of the boundary layer?
3. Thickness of the boundary layer?

Tn the outer region: $\underset{thickness}{\varepsilon \ll x \ll 1}$ $\qquad \underset{for\ \varepsilon\to 0^+ uniformly}{y_{out} \approx \sum_{n=0}^{N} y_n(x)\varepsilon^n}$

Substitute this equation into the e.g.(i)$y_0' + y_0 = 0$, and (ii)$y_n' + y_n = -y_{n-1}'' - y_{n-1}'$ $n \geq 1$

B.C. at $x = 1 : y(1) = 1 \Rightarrow \begin{cases} y_0(1) = 1 \\ y_n(1) = 0, \ n \geq 1 \end{cases}$

$$\therefore \begin{cases} y_0' + y_0 = 0 \\ y_0(1) = 1 \end{cases} \quad \textit{and} \quad \begin{cases} y_n' + y_n = -y_{n-1}'' - y_{n-1}' \\ y_n(1) = 1, \ n \geq 1 \end{cases}$$

$\therefore y_0(x) = e^{1-x}$–outer limit, uniform approximation in the outer region.

In the inner region: $0 \leq x \leq O(\varepsilon)$ $[\lim_{\varepsilon \to 0^+} x = 0$ uniformly$]$

Inner variable $X = \frac{x}{\varepsilon}$ $(\varepsilon \leftarrow$ thickness).

$y(x) = y(\varepsilon x) = y_{in}(x) \sim \sum_{n=0}^{N} Y_n(X)\varepsilon^n$ as $\varepsilon \to 0^+$ uniformly

From $Y_{in}(0) = y(0) = 0 \Rightarrow y_n(0) = 0$ for any n.

The equation because

$$\frac{1}{\varepsilon}\frac{d^2Y_{in}}{dX^2} + (1+\varepsilon)\frac{1}{\varepsilon}\frac{dY_{in}}{dX} + Y_{in} = 0 \Rightarrow \begin{cases} Y_0'' + Y_0' = 0 \\ Y_0(0) = 0 \end{cases} \quad \textit{and} \quad \begin{cases} Y_n'' + Y_n' = -Y_{n-1}' - Y_{n-1} \\ Y_n(0) = 0 \end{cases}$$

$\therefore$ Solution $Y_0(X) = A_0(1 - e^{-X})$ –inner limit. uniform approximation in the inner region.

In the overlapping region: $x \to 0, X \to +\infty$ as $\varepsilon \to 0^+$ intermediate limit: $\lim_{\varepsilon \to 0^+} y_0(x) = \lim_{\substack{\varepsilon \to 0^+ \\ X \to +\infty}} Y_0(x)$

$$\Rightarrow \lim_{x \to 0} e^{1-x} = \lim_{X \to +\infty} A_0(1 - e^{-x}) \Rightarrow A_0 = e$$

Uniform approximation

$$y_{unif}(x) = y_{in}^a(x) + y_{out}^a(x) - y_{match}(x)$$

$y_{match}(x)$ leading term matching should calculate the approximate value of $x \to 0^+ or\ X \to +\infty$

$y_{in/out}^a(x)$: uniform approximation in the innerouter region $y_{match}(x)$: intermediate limit.

In this example, the leading order uniform approximation in the whole region is

$$\begin{aligned} y_{unif}(x) &= y_0(x) + Y_0(\frac{x}{\varepsilon}) - y_{match}(x) \\ &= e^{1-x} + (e - e^{1-\frac{x}{\varepsilon}}) - e \\ &= e^{1-x} - e^{1-\frac{x}{\varepsilon}} \end{aligned}$$

Formally, in the outer region $X = \frac{x}{\varepsilon} \to \infty$ uniformly as $\varepsilon \to 0^+$ $Y_0(\frac{x}{\varepsilon}) \to y_{match}(x)$ then $y_{unif}(x) \to y_0(x)$ uniformly as $\varepsilon \to 0^+$ in the inner region, $x \to 0$ uniformly as $\varepsilon \to 0^+$. $y_0(x) \to y_{match}(x)$ then $y_{unif}(x) \to Y_0(\frac{x}{\varepsilon})$ uniformly as $\varepsilon \to 0^+$.

3.4 Location of the boundary layer

$$\begin{cases} \varepsilon y'' + (1+\varepsilon)y' + y = 0 & x \in (0,1) \\ y(0) = 0, \quad y(1) = 1 \end{cases}$$

Assume that the boundary layer is at $x = x_0$

Near x_0, let $x = \frac{x-x_0}{\varepsilon}$($\varepsilon \to$ boundary layer thickness)

$$Y_{in}(X) = y_{in}(x) = y(\varepsilon x + x_0')$$

$\therefore$ Equation: $\frac{1}{\varepsilon}\frac{d^2Y_{in}}{dX^2} + \frac{1+\varepsilon}{\varepsilon}\frac{dY_{in}}{dX} + Y_{in} = 0.$

$$y_{in}(X) \sim \sum_{n=0}^{\infty} Y_n(X)\varepsilon^n \quad \varepsilon \to 0.$$

$$O(\frac{1}{\varepsilon}) : Y_0'' + Y_0' \Rightarrow Y_0 = C_1 + C_2 e^{-x} = C_1 + C_2 e^{-\frac{x-x_0}{\varepsilon}}$$

Intermediate limit If $0 < x_0 \leq 1$, X may go to $-\infty$, $Y_0(X) = C_1 + C_2 e^{-x} \to \infty$.

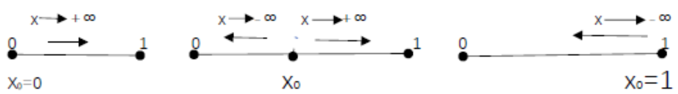


figure 3.4.1

But we know the intermediate limit must be finite to match outer solution. Thus $C_2 = 0$ and $Y_0(X) = C_1$. This solution is not a rapidly varying solution. No boundary layer at x_0 when $0 < x_0 \leq 1$. The only possible boundary layer is at $x_0 = 0$.

3.5 Higher-order boundary layer theory

Example 3.5.1: $\begin{cases} \varepsilon y'' + (1+\varepsilon)y' + y = 0 & 0 \leq x \leq 1 \\ y(0) = 0, \quad y(1) = 1 \end{cases}$

To find an approximation of the solution that is correct up to $O(\varepsilon^2)$. The only possible boundary layer is $x_0 = 0$.
The boundary thickness is ε (need to be determinate).

In the outer region:$\varepsilon \ll x \ll 1$

$$y_{out}(x) \sim y_0(x) + \varepsilon y_1(x) + \varepsilon y_2(x) + \cdots, \ \varepsilon \to 0$$

Outer solution satisfies

$$\begin{cases} (1+x)y_0' + y_0 = 0,\ y_0(1) = 1 \\ (1+x)y_n' + y_n = -y_{n-1}'', y_n(1) = 0,\ n = 1, 2, 3, \cdots \end{cases}$$

$$\frac{d[(1+x)y_0]}{dx} = 0 \Rightarrow (1+x)y_0 = C$$

$$\begin{cases} (1+x)y_0' + y_0 = 0 \\ y_n(1) = 0 \end{cases} \Rightarrow y_0(x) = \frac{2}{1+x}$$

$$\begin{cases} (1+x)y_1' + y_1 = -y_0'' = -\frac{4}{(1+x)^3} \\ y_1(1) = 0 \end{cases} \Rightarrow y_1(x) = \frac{2}{(1+x)^3} - \frac{1}{2(1+x)}$$

$$\begin{cases} (1+x)y_2' + y_2 = -y_1'' \\ y_2(1) = 0 \end{cases} \Rightarrow \frac{d[(1+x)y_2]}{dx} = -y_1'' \Rightarrow (1+x)y_2 = -y_1'$$

$$\Rightarrow y_2 = \frac{6}{(1+x)^5} - \frac{2}{(1+x)^3} - \frac{4}{(1+x)^3}$$

Inner region: $0 \le x \le O(\varepsilon)$ $X = \frac{x}{\varepsilon}$.

$$Y_{in}(X) = y(x) = y(\varepsilon x),\ \frac{d^2Y_{in}}{dX^2} + (1+\varepsilon x)\frac{dY_{in}}{dX} + \varepsilon Y_{in} = 0$$

Assume

$$Y_{in}(X) \sim Y_0(X) + \varepsilon Y_1(X) + \varepsilon^2 Y_2(X) + \cdots, \varepsilon \to 0$$

$$Y_{in}(0) = 1 \sim Y_0(0) + \varepsilon Y_1(0) + \varepsilon^2 Y_2(0) + \cdots, \varepsilon \to 0$$

$Y_0(0) = 1, Y_n(0) = 0, n = 1, 2, 3, \cdots$

$$\begin{cases} Y_0'' + Y_0' = 0 \\ Y_0(0) = 1 \end{cases} \Rightarrow Y_0(X) = 1 + A_0(e^x - 1)$$

$$\begin{cases} Y_n'' + Y_n' = -XY_{n-1}' - Y_{n-1} \\ Y_n(0) = 0,\ n = 1, 2, 3, \cdots \end{cases}$$

$$Y_1(X) = -X + A_0(-\frac{1}{2}X^2e^{-X} + X) + A_1(e^{-x} - 1)$$

$$Y_2(X) = X^2 - 2X + A_0(\frac{1}{8}X^4e^{-X} - X^2 + 2X) + A_1(-\frac{1}{2}X^2e^{-X} + X) + A_1(e^{-x} - 1)$$

Leading order $O(1)$ matching:

$$\lim_{x\to 0} y_0(x) = \lim_{X\to+\infty} Y_0(X)$$

where, $\lim_{x\to 0} y_0(x) = \lim_{x\to 0} \frac{2}{1+x} = 2, \lim_{X\to+\infty} Y_0(X) = 1 - A_0$.

$\Rightarrow A_0 = -1 \quad Y_0(X) = 1 - (e^{-x} - x) = 2 - e^{-x}$

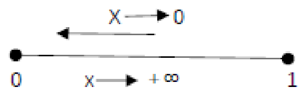


figure 3.5.1

$O(\varepsilon)$ matching:

$$y_{out}(x) \sim y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2)$$
$$= \frac{2}{1+x} + \varepsilon[\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}] + O(\varepsilon^2)$$

$\varepsilon \to 0$ uniformly in the outer region.

$$Y_{in}(X) \sim Y_0(X) + \varepsilon Y_1(X) + O(\varepsilon^2)$$
$$= 2 - e^{-X} + \varepsilon[-2X + \frac{1}{2}X^2 e^{-X} + A_1(e^{-X} - 1)] + O(\varepsilon^2)$$

$\varepsilon \to 0$ uniformly in the inner region.

Then we should have

$$y_0(x) + \varepsilon y_1(x) - Y_0(X) - \varepsilon Y_1(X) = O(\varepsilon)$$

uniformly in the overlapping region.

We cannot use $\lim\limits_{x\to 0}(y_0(x) + \varepsilon y_1(x)) = \lim\limits_{X\to\infty}(Y_0(X) + \varepsilon Y_1(X))$. ($\lim\limits_{x\to 0}(y_0(x) + \varepsilon y_1(x)) = \lim\limits_{X\to\infty}(Y_0(X) + \varepsilon Y_1(X))$ is not enough, we have to see the behavior of this limiting process), because we want to keep $O(\varepsilon)$ terms.

Note that the overlapping region is contained in $\varepsilon \ll x \ll 1$. When $x \to 0$, we have $\varepsilon \to 0$. The $O(x) = O(\varepsilon X)$ terms in the inner region should be kept in $O(\varepsilon)$ matching. $O(\varepsilon)$ term in the outer region should be kept.(We want to see the behavior of $y_{out}(x)$ and $Y_{in}(X)$ in the overlapping.)

Matching method: keep $O(1)$, $O(\varepsilon)$, $O(x)$ terms int the overlapping region.

$$y_{out} \sim y_0(x) + \varepsilon y_1(x) = \frac{2}{1+x} + \varepsilon[\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}]$$
$$= 2(1-x) + \varepsilon(2 - \frac{1}{2}) + O(\varepsilon^2 + \varepsilon x + x^2)$$
$$= 2 - 2x + \frac{3}{2}\varepsilon + O(\varepsilon^2 + \varepsilon x + x^2)$$

$$Y_{in}(X) \sim Y_0(X) + \varepsilon Y_1(X)$$
$$= 2 - e^{-X} + \varepsilon[-2X + \frac{1}{2}X^2 e^{-X} + A_1(e^{-X} - 1)]$$
$$\overset{x=\varepsilon X}{=} 2 - e^{-\frac{x}{\varepsilon}} + \varepsilon[-2\frac{x}{\varepsilon} + \frac{1}{2}(\frac{x}{\varepsilon})^2 e^{-\frac{x}{\varepsilon}} + A_1(e^{-\frac{x}{\varepsilon}} - 1)]$$
$$= 2 - 2x - A_1\varepsilon + O(\varepsilon^2 + \varepsilon x + x^2)$$

where, $e^{-\frac{x}{\varepsilon}} = O(\varepsilon^2 + \varepsilon x + x^2) \Rightarrow A_1 = -\frac{3}{2} \Rightarrow Y_1(X) = -2X + \frac{1}{2}X^2 e^{-X} - \frac{3}{2}(e^{-X} - 1)$

Uniformly approximation up to $O(\varepsilon)$.

$$\begin{aligned} y_{unif}(x) &= [y_0(x) + \varepsilon y_1(x)] + [Y_0(X) + \varepsilon Y_1(X)] - y_{match}(x) \\ &= \frac{2}{1+x} - e^{-\frac{x}{\varepsilon}} + \varepsilon[\frac{2}{(1+x)^3} - \frac{1}{2(1+x)} - \frac{3}{2}e^{-\frac{x}{\varepsilon}} + \frac{1}{2}(\frac{x}{\varepsilon})^2 e^{-\frac{x}{\varepsilon}}] \end{aligned}$$

where, $y_{match}(x) = 2 + \frac{3}{2}\varepsilon - 2x$.

General Framework:

$$\begin{cases} \varepsilon y'' + (1+x)y' + y = 0 & \text{boundary layer at } x = 0 \\ y(0) = 0, \quad y(1) = 1 & \text{thickness } \varepsilon \end{cases}$$

In outer region, $y(x) = y_{out}(x) \sim y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots,\ \varepsilon \to 0$

inner region, $y(x) = Y_{in}(x) \sim Y_0(x) + \varepsilon Y_1(x) + \varepsilon^2 Y_2(x) + \cdots,\ \varepsilon \to 0,\ \ X = \frac{x}{\varepsilon}$.

Assumption matching in the overlapping region:

$O(1) : \lim\limits_{x\to 0} y_0(x) = \lim\limits_{x\to 0} Y_0(x)$

$O(\varepsilon)$:
(1) keep $O(1)$, $O(\varepsilon)$, $O(x)$ terms in $y_0(x) + \varepsilon y_1(x)$
(2) keep $O(1)$, $O(\varepsilon)$, $O(x)$ terms in $Y_0(X) + \varepsilon Y_1(X) = Y_0(\frac{x}{\varepsilon}) + \varepsilon Y_1(\frac{x}{\varepsilon})$
(3) compare coefficients in (1) and (2).

$O(\varepsilon^2)$:
(1) keep $O(1)$, $O(\varepsilon)$, $O(x)$, $O(\varepsilon^2)$, $O(\varepsilon x)$, $O(x^2)$ terms in $y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x)$
(2) keep $O(1)$, $O(\varepsilon)$, $O(x)$, $O(\varepsilon^2)$, $O(\varepsilon x)$, $O(x^2)$ terms in $Y_0(X) + \varepsilon Y_1(X) + \varepsilon^2 Y_2(X) = Y_0(\frac{x}{\varepsilon}) + \varepsilon Y_1(\frac{x}{\varepsilon}) + \varepsilon^2 Y_2(\frac{x}{\varepsilon})$
(3) Compare coefficients in (1) and (2).

$$\begin{aligned} y_{out}(x) &\sim \frac{2}{1+x} + \varepsilon[\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}] + \varepsilon^2[\frac{6}{(1+x)^5} - \frac{1}{2(1+x)^3} - \frac{1}{4(1+x)}] + O(\varepsilon^3) \\ &= 2 - 2x + 2x^2 + \varepsilon(\frac{3}{2} - \frac{11}{2}x) + \frac{21}{4}\varepsilon^2 + O(\varepsilon^3 + \varepsilon^2 x + \varepsilon x^2 + x^3) \end{aligned}$$

$$Y_{in}(X) \sim 2 - e^{-X} + \varepsilon(\frac{3}{2} - 2X - \frac{3}{2}e^{-X} + \frac{1}{2}X^2 e^{-X}) + \varepsilon^2(\frac{21}{4} - \frac{11}{2}X + 2X^2 - \frac{21}{4}e^{-X} + \frac{3}{4}X^2 e^{-X} - \frac{1}{8}X^4 e^{-X})$$

When $x = \varepsilon X$, $Y_{in}(X) = 2 - 2x + \frac{3}{2}\varepsilon + 2x^2 - 4\varepsilon x - \frac{3}{2}\varepsilon x - A_2\varepsilon^2$ ($A_2 = -\frac{21}{4}$).

Remark on overlapping region: When extending the inner region $0 \le X \le O(1)$, a requirement is $Y_{in}(X) \sim Y_0(X) + \varepsilon Y_1(X) + \varepsilon^2 Y_2(X) + \cdots, \varepsilon \to 0$ for $0 \le X \le O(1)$.

$Y_0(X) \gg \varepsilon Y_1(X) \gg \varepsilon^2 Y_2(X)$ should still hold when $X \to +\infty$ in the extending inner region.

Self consistent match principles:
(1) Textbook method: $1 \gg x, \varepsilon \gg x^2, \varepsilon x, \varepsilon^2$ but $\varepsilon \gg x^2$ is an assumption ($x \gg \varepsilon$).

(2) Van Dyke match rules: Derturbation method in fluid mechanics. (See Exercise 2.12 after section 2.2)
(3) To use an intermediate variable $x_\eta = \frac{x}{\eta(\varepsilon)}$ in the overlapping region (Kaplen, see Holmes'book).

Van Dyke's matching rule in $O(\varepsilon)$

$$
\begin{aligned}
y_{out}(x) &\sim \frac{2}{1+x} + \varepsilon[\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}] \\
&\overset{x=\varepsilon X}{=} \frac{2}{1+\varepsilon X} + \varepsilon[\frac{2}{(1+\varepsilon X)^3} - \frac{1}{2(1+\varepsilon X)}](Reurite\,using\,inner\,variable) \\
&= 2 - 2\varepsilon X + \varepsilon[2 - 6\varepsilon X - \frac{1}{2}(1-\varepsilon X) + O(\varepsilon^2)](fix\,X,\,expand\,to\,two\,terms,\ O(\varepsilon)) \\
&= 2 - 2\varepsilon X + \frac{3}{2}\varepsilon + O(\varepsilon^2)
\end{aligned}
$$

$$
\begin{aligned}
Y_{in}(X) &\sim 2 - e^{-X} + \varepsilon[-2X + \frac{1}{2}X^2 e^{-X} + A_1(e^{-x} - 1)] \\
&\overset{X=\frac{x}{\varepsilon}}{=} 2 - e^{-\frac{x}{\varepsilon}} + \varepsilon[-2\frac{x}{\varepsilon} + \frac{1}{2}(\frac{x}{\varepsilon})^2 e^{-\frac{x}{\varepsilon}} + A_1(e^{-\frac{x}{\varepsilon}} - 1)](Reurite\,using\,inner\,variable) \\
&= 2 - 2x - A_1\varepsilon + O(\varepsilon)(fix\,x,\,expand\,to\,two\,terms,\ O(\varepsilon))
\end{aligned}
$$

Matching gives $A_1 = -\frac{3}{2}$

Using intermediate variable $x_\eta = \frac{x}{\eta(\varepsilon)}$, $\eta(\varepsilon) = \varepsilon^\alpha$, $0 < \alpha < 1$, $x_\eta = O_s(1)$.

$$
\begin{aligned}
y_{out}(x) &\sim \frac{2}{1+x_\eta\varepsilon^\alpha} + \varepsilon[\frac{2}{(1+x_\eta\varepsilon^\alpha)^3} - \frac{2}{\varepsilon(1+x_\eta\varepsilon^\alpha)}] \\
&= 2 - 2\varepsilon^\alpha x_\eta + 2\varepsilon^{2\alpha}x_\eta^2 + \cdots + \varepsilon[2 - 6x_\eta\varepsilon^\alpha + \cdots - \frac{1}{2}(1 - \varepsilon^\alpha x_\eta + \cdots)] \\
&= 2 - 2\varepsilon^\alpha x_\eta + 2\varepsilon^{2\alpha}x_\eta^2 + \frac{3}{2}\varepsilon - \frac{11}{2}\varepsilon^{1+\alpha}x_\eta + \cdots
\end{aligned}
$$

$$
\begin{aligned}
Y_{in}(X) &\sim 2 - e^{-\frac{x_\eta}{\varepsilon^{1-\alpha}}} + \varepsilon[-2\frac{x_\eta}{\varepsilon^{1-\alpha}} + \frac{1}{2}\frac{x_\eta^2}{\varepsilon^{2-2\alpha}}e^{-\frac{x_\eta}{\varepsilon^{1-\alpha}}} + A_1(e^{-\frac{x_\eta}{\varepsilon^{1-\alpha}}} - 1)] \\
&= 2 - 2x_\eta\varepsilon^\alpha - A_1\varepsilon + O(\varepsilon)
\end{aligned}
$$

$$\Rightarrow A_1 = -\frac{3}{2}$$

Note that $O(\varepsilon^{2\alpha})$ terms come from the leading order terms in the outer solution. Similar terms should occur from $O(\varepsilon^2)$ sol in the inner solution. (See Holmes'book P_{68} 2.2.6)

3.6 Boundary layer thickness and distinguished limit

$$
\begin{cases}
\varepsilon y'' + (1+x)y' + y = 0 \\
y(0) = 1,\ y(1) = 1
\end{cases}
$$

Assume the thickness of BL is $\delta(\varepsilon) \ll 1$ and boundary location is 0. Let $X = \frac{x}{\delta}$, $Y_{in}(X) = y(\delta x)$.

$$\therefore \frac{\varepsilon}{\delta^2}\frac{d^2Y_{in}}{dX^2} + (\frac{1}{\delta} + x)\frac{dY_{in}}{dX} + Y_{in} = 0$$

We have $O(\frac{\varepsilon}{\delta^2})$, $O(\frac{1}{\delta})$, $O(1)$ terms.

By domineat balance, $\frac{\varepsilon}{\delta^2}\frac{d^2Y_{in}}{dX^2}+\frac{1}{\delta}\frac{dY_{in}}{dX}\sim 0$. (If $\frac{\varepsilon}{\delta^2}\frac{d^2Y_{in}}{dX^2}+X\frac{dY_{in}}{dX}+Y_{in}\sim 0 \Rightarrow \delta=\varepsilon^{\frac{1}{2}}$. then leading order becomes $O(\frac{1}{\delta})=O(\varepsilon^{-\frac{1}{2}})$: $\frac{dY_{in}}{dX}=0\Rightarrow Y_{in}=const\Rightarrow$ no BL!) $\Rightarrow \frac{\varepsilon}{\delta^2}=\frac{1}{\delta}\Rightarrow\delta=\varepsilon$

Then the inner limit $Y_0(X)=\lim\limits_{x\to 0}Y_{in}(X)$ satisfies $\frac{d^2Y_0}{dX^2}+X\frac{dY_0}{dX}=0$ – distinguished limit.

For BL at x_0 with $0<x_0\leq 1$, we can find BL thickness is also ε and distinguished limit is $\frac{d^2Y_{in}}{dX^2}+X\frac{dY_{in}}{dX}(1+x_0)=0$. $\lambda^2+(1+x_0)\lambda=0\,(\lambda=0\,or\,-(1+x_0))$

$Y_0(X)=A+Be^{-(1+x_0)X}\to\infty(X\to+\infty)$ if $B\neq 0$

However, the inner sol is not finite as $X\to-\infty$, thus no BL at x_0 for $0<x_0\leq 1$.

An example for BL thickness$\neq\varepsilon$

$$\begin{cases}\varepsilon y''+x^2y'-y=0 & 0\leq x\leq 1\\ y(0)=1,\ y(1)=1\end{cases}$$

$$y_{out}(x)\sim y_0(x)+\varepsilon y_1(x)+\varepsilon^2y_2(x)+\cdots,\varepsilon\to 0\quad C=e$$

$O(1)$: $x^2y_0'-y_0=0\Rightarrow y_0=Ce^{-\frac{1}{x}}$ (Cannot satisfy both BC). Look for BL at $x=0$: $X=\frac{x}{\eta}$, $\eta\ll 1$. $Y_{in}(X)=y(x)$.

Eq. $\frac{\varepsilon}{\eta^2}\frac{d^2Y_{in}}{dX^2}+\eta X^2\frac{dY_{in}}{dX}-Y_{in}=0$

Look for distinguished limit $O(\frac{\varepsilon}{\eta^2})$, $O(\eta)$, $O(1)$ term. By dominant balance: $\frac{\varepsilon}{\eta^2}\frac{d^2Y_{in}}{dX^2}-Y_{in}\sim 0$

$\Rightarrow\eta=\varepsilon^{\frac{1}{2}}$

The BL thickness is $O(\varepsilon^{\frac{1}{2}})$.

Inner variable $X=\frac{x}{\varepsilon^{\frac{1}{2}}}$

$\Rightarrow\frac{d^2Y_{in}}{dX^2}+\varepsilon^{\frac{1}{2}}X^2\frac{dY_{in}}{dX}-Y_{in}=0$

$$Y_{in}(X)\sim Y_0(X)+\varepsilon^{\frac{1}{2}}Y_1(X)+\varepsilon Y_2(X)+\varepsilon^{\frac{3}{2}}Y_3(X)+\cdots$$

$\varepsilon^{\frac{1}{2}}$ is the small parameter in the inner region from the equation. Not necessary the BL thickness.

(In the outer region, small parameter is still ε)

$Y_0(X)$ satisfies $\frac{d^2Y_0}{dX^2}-Y_0=0$, $Y_o(0)=1$

$\Rightarrow Y_0(X) = Ae^x + (1-A)e^{-x}$

Leading order term matching:

$$\lim_{x\to 0^+} y_0(x) = \lim_{x\to +\infty} Y_0(X) \Rightarrow Y_0(X) = e^{-x}\ (A=0)$$

Uniform approximation: $y_{unif}(x) = e^{1-\frac{1}{x}} + e^{-\frac{x}{\varepsilon^{\frac{1}{2}}}} - O$

If BL is at x_0, $0 < x_0 \leq 1$. $x = \frac{x-x_0}{\eta}$, $\eta \ll 1$.

Inner solution satisfies

$$\frac{\varepsilon}{\eta^2}\frac{d^2Y_{in}}{dX^2} + \frac{(\eta X + x_0)^2}{\eta}\frac{dY_{in}}{dX} - Y_{in} = 0$$

where, $\frac{(\eta X+x_0)^2}{\eta} = \eta X^2 + 2Xx_0 + \frac{x_0^2}{\eta}$.

$O(\frac{\varepsilon}{\eta^2}), O(\eta), O(1), O(\frac{1}{\eta})$ terms.

Since $\eta, 1 \ll \frac{1}{\eta}$, as $\varepsilon \to 0$.

Dominant balance gives $\frac{\varepsilon}{\eta^2}\frac{d^2Y_{in}}{dX^2} + \frac{x_0^2}{\eta}\frac{dY_{in}}{dX} \sim 0$.

$\Rightarrow \frac{\varepsilon}{\eta^2}/\frac{1}{\eta} = O(1)$, $\eta = O_s(\varepsilon)$ choose $\eta = \varepsilon$

$$\frac{d^2Y_{in}}{dX^2} + x_0^2\frac{dY_{in}}{dX} \sim 0 \Rightarrow Y_0 = C_1 + C_2 e^{-x_0^2 X}$$

$$\lim_{x\to -\infty} e^{-x_0 X} = +\infty \Rightarrow C_2 = 0,\ Y_0(X) = C_1$$

Not a rapidly varying function of $x \Rightarrow$ No BL at x_0.

Higher order matching:

$O(\varepsilon)$ matching: To match $y_0(x) + \varepsilon y_1(x)$ and $Y_0(X) + \varepsilon^{\frac{1}{2}}Y_1(X) + \varepsilon Y_2(X)$.

In the overlapping region, keep $O(1), O(\varepsilon^{\frac{1}{2}}), O(x), O(\varepsilon), O(\varepsilon^{\frac{1}{2}}x), O(x^2)$ terms.

3.7 BL problem involving $\log\varepsilon$

$$\begin{cases} \varepsilon y'' + xy' - xy = 0\ \ BL\, may\, apper\, at\, x = 0 \\ y(0) = 0,\ y(1) = e\ \ thickness\, \varepsilon^{\frac{1}{2}}\, (\frac{\varepsilon}{\eta^2} \sim X \Rightarrow \eta = \varepsilon^{\frac{1}{2}}) \end{cases}$$

As $\varepsilon \to 0$,

$$y_{out}(X) \sim y_0(x) + \varepsilon y_1(x) + \varepsilon y_2(x) + \cdots$$

$$Y_{in}(X) \sim Y_0(X) + \varepsilon^{\frac{1}{2}}Y_1(X) + \varepsilon Y_2(X) + \cdots$$

$(xy_0' - xy_0 \Rightarrow)(Y_0''(x) + xY_0'(x) \Rightarrow)$

$y_0(x) = e^x$, $Y_0(X) = A_0 \int_0^X e^{-\frac{t^2}{2}} dt \Rightarrow O(l)$ matching

$\Rightarrow \lim\limits_{x\to 0} y_0(x) = 1 = \lim\limits_{X\to+\infty} Y_0(X) \Rightarrow A_0 = \sqrt{\frac{2}{\pi}}$

$O(\varepsilon)$ matching keep $O(1), O(\varepsilon^{\frac{1}{2}}), O(x), O(\varepsilon), O(\varepsilon^{\frac{1}{2}}x), O(x^2)$ terms. $\varepsilon^{\frac{1}{2}} \ll x \ll 1$

$$y_0(x) + \varepsilon y_1(x) = e^x - \varepsilon e^x \log x = 1 + x + \frac{x^2}{2} - \varepsilon \log x + O(\varepsilon),\ \varepsilon \to 0$$

$$Y_0(X) = \sqrt{\frac{2}{\pi}} \int_0^X e^{-\frac{t^2}{2}} dt = \sqrt{\frac{2}{\pi}} (\sqrt{\frac{\pi}{2}} - \int_0^{+\infty} e^{-\frac{t^2}{2}} dt)$$
$$= 1 + O(\frac{1}{X} e^{-\frac{X^2}{2}})\ X \to +\infty$$

$$(\int_0^{+\infty} e^{-\frac{t^2}{2}} dt = \int_X^{+\infty} (-\frac{1}{t}) de^{-\frac{t^2}{2}}) = \frac{1}{X} e^{-\frac{X^2}{2}} + \int_X^{+\infty} (-\frac{1}{t^2}) e^{-\frac{t^2}{2}} dt = \frac{1}{X} e^{-\frac{X^2}{2}} + O(\frac{1}{X^2} \int_X^{+\infty} e^{-\frac{t^2}{2}} dt)$$

$$Y_1(X) = X + C_1 + O(e^{-\frac{x^2}{2}}).\ \ X \to +\infty$$

$$Y_2(X) = \frac{1}{2} X^2 - \log X + C_1 X + C_2 + O(\frac{1}{X^2}).\ \ X \to +\infty$$

$$\therefore Y_0(X) + \varepsilon^{\frac{1}{2}} Y_1(X) + \varepsilon Y_2(X) = 1 + x + \frac{1}{2} x^2 + C_1 + C_2 \varepsilon^{\frac{1}{2}} x - \varepsilon \log x + \frac{\varepsilon}{2} \log \varepsilon + C_1 \varepsilon^{\frac{1}{2}} + C_2 \varepsilon + O(\varepsilon)\ \varepsilon \to 0$$

$(xy_1' - xy_1 = -y_0'' = -e^x \Rightarrow y_1 = -e^x \log x + Ce^x.\ \ C = 0,\ y_1(1) = 0,\ Y_{in}''(X) + XY_{in}''(X) - \varepsilon^{\frac{1}{2}} XY_{in}(X) = 0,\ Y_1''(X) + XY_1'(X) = XY_0(X) = X + O(e^{-\frac{X^2}{2}}))$

$\Rightarrow e^{\frac{X^2}{2}} Y_1'(X) = \int XY_0(X) e^{\frac{X^2}{2}} dX + C = e^{\frac{X^2}{2}} + O(X)$

$$Y_1(X) = X + C_1 + O(e^{-\frac{X^2}{2}})$$

$$Y_2''(X) + XY_2'(X) = XY_1(X) = X^2 + C_1 X + O(Xe^{-\frac{X^2}{2}})$$
$$\Rightarrow e^{\frac{X^2}{2}} Y_2'(X) = \int X^2 e^{\frac{X^2}{2}} dX + \int C_1 X e^{\frac{X^2}{2}} dX + \int O(X) dX$$
$$= \int_1^X t^2 e^{\frac{1}{2}t^2} dt + C_1 e^{\frac{1}{2}t^2} + O(X^2)$$

$$\begin{aligned}
Y_2'(X) &= \int_1^X t^2 e^{-\frac{1}{2}X^2 + \frac{1}{2}t^2} dt + C_1 + O(X^2 e^{-\frac{1}{2}X^2}) \\
&= e^{-\frac{1}{2}X^2} (X e^{\frac{1}{2}X^2} - \int_1^X e^{\frac{1}{2}t^2} dt) + C_1 + O(X^2 e^{-\frac{1}{2}X^2}) \\
&= X - e^{-\frac{1}{2}X^2} \int_1^X \frac{1}{t} d(e^{\frac{1}{2}t^2}) + C_1 + O(X^2 e^{-\frac{1}{2}X^2}) \\
&= X - \frac{1}{X} - e^{-\frac{1}{2}X^2} \int_1^X \frac{1}{t^2} e^{\frac{1}{2}t^2} dt + C_1 + O(X^2 e^{-\frac{1}{2}X^2}) \\
&= X - \frac{1}{X} - e^{-\frac{1}{2}X^2} (\int_1^X \frac{1}{t^3} d(e^{\frac{1}{2}t^2})) + C_1 + O(X^2 e^{-\frac{1}{2}X^2}) \\
&= X - \frac{1}{X} - \frac{1}{X^3} - e^{-\frac{1}{2}X^2} \int_1^X \frac{3}{t^4} e^{\frac{1}{2}t^2} dt + C_1 + O(X^2 e^{-\frac{1}{2}X^2}) \\
&= X - \frac{1}{X} + C_1 + O(\frac{1}{X^3})
\end{aligned}$$

$$Y_2(X) = \frac{1}{2}X^2 - \log X + C_1 X + C_2 + O(\frac{1}{X^2})$$

(Note that in $\varepsilon Y_2(X)$, $\varepsilon O(\frac{1}{X^2}) \ll \varepsilon$, as $X \to +\infty$)

Compare it with the outer expansion, we have $C_1 = 0$, $C_2 = 0$, but there is no corresponding $\frac{\varepsilon}{2}\log\varepsilon$ term in the outer solution. This means the expansions are not correct.

Modify the inner expansion as

$$Y_{in}(X) \sim Y_0(X) + \varepsilon^{\frac{1}{2}}Y_1(X) + \varepsilon\log\varepsilon\bar{Y}_2(X) + \varepsilon Y_2(X) + \varepsilon^{\frac{3}{2}}Y_3(X) + \varepsilon\log\varepsilon\bar{Y}_3(X) + \cdots,\ \varepsilon \to 0$$

where, $\varepsilon\log\varepsilon$ to cancel out $\frac{\varepsilon}{2}\log\varepsilon$ term.

The extra are: $\varepsilon\log\varepsilon(\bar{Y}_2(X)+\varepsilon^{\frac{1}{2}}\bar{Y}_3(X)+\varepsilon\bar{Y}_4(X)+\cdots)$ where, $\varepsilon\log\varepsilon$ to match the outer solution, $\bar{Y}_2(X)+\varepsilon^{\frac{1}{2}}\bar{Y}_3(X)+\varepsilon\bar{Y}_4(X)+\cdots$ because $Y_{in}(X)$ satisfies $\frac{d^2Y_{in}}{dX^2}+X\frac{dY_{in}}{dX}-\varepsilon^{\frac{1}{2}}XY_{in} = 0$.

For $O(\varepsilon)$ matching, we need to match $y_0(x)+\varepsilon y_1(x)$ with $Y_o(X)+\varepsilon^{\frac{1}{2}}Y_1(X)+\varepsilon Y_2(X)+\varepsilon\log\varepsilon\bar{Y}_2(X)$.

$$\bar{Y}_2(X) = \bar{A}_2\int_0^X e^{-\frac{1}{2}t^2}dt = \bar{A}_2\sqrt{\frac{\pi}{2}} + O(\frac{1}{X}e^{-\frac{1}{2}x^2}),\ X \to +\infty$$

$$\begin{aligned}
&Y_o(X) + \varepsilon^{\frac{1}{2}}Y_1(X) + \varepsilon Y_2(X) + \varepsilon\log\varepsilon\bar{Y}_2(X) \\
&= 1 + x + \frac{1}{2}x^2 + C_1\varepsilon^{\frac{1}{2}}x - \varepsilon\log x + \frac{1}{2}\varepsilon\log\varepsilon + C_1\varepsilon^{\frac{1}{2}} + C_2\varepsilon + \varepsilon\log\varepsilon\bar{A}_3\sqrt{\frac{\pi}{2}} + O(\varepsilon),\ \varepsilon \to 0
\end{aligned}$$

Thus $C_1 = C_2 = 0$, $\bar{A}_2 = -\frac{1}{\sqrt{2\pi}}$.

Why not: $y_{out}(x) \sim y_o(x) + \varepsilon y_1(x) + \varepsilon\log\varepsilon\bar{y}_1(x) + \varepsilon^2 y_2(x) + \cdots, \varepsilon \to 0$. because at $O(\varepsilon\log\varepsilon)$: $\begin{cases} x\bar{y}_1'' - x\bar{y}_1 = 0 \\ \bar{y}_1(1) = 0 \end{cases} \Rightarrow \bar{y}_1(x) \equiv 0$.

3.8 Multiple boundary layers

$$\begin{cases} \varepsilon y'' - x^2 y' - y = 0 \\ y(0) = y(1) = 1 \end{cases}$$

$$y_{out}(x) \sim y_o(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots, \varepsilon \to 0$$

$y_0(x) = C_0 e^{\frac{1}{x}}$, $y(0) = y(1) = 0 \Rightarrow l_0 = 0$

$\therefore y_0(x) \equiv 0$ in the outer region $\leftarrow$ we need $\lim\limits_{x\to 0^+} y_0(x)$ to be finite.

Both BLs cannot ne satisfied by this solution.

This suggests that there might exist a BL at each end of the internal.

BL at $x = 1$, thickness is ε. $\therefore X = \frac{1-x}{\varepsilon}$, $Y_{in,right}(X) \sim Y_{0,r}(X) + \varepsilon Y_{1,r}(X) + \cdots, \varepsilon \to 0$, $Y_{0,r} = e^{-\frac{1-x}{\varepsilon}}$ after $O(1)$ matching BL at $x = 0$, thickness is $\varepsilon^{\frac{1}{2}}$.

$\therefore X = \frac{x}{\varepsilon^{\frac{1}{2}}}$, $Y_{in,left}(X) \sim Y_{0,l}(X) + \varepsilon^{\frac{1}{2}} Y_{1,l}(X) + \cdots, \varepsilon \to 0$

$$\therefore Y_{0,l} = e^{-z} = e^{-\frac{x}{\varepsilon^{\frac{1}{2}}}}$$

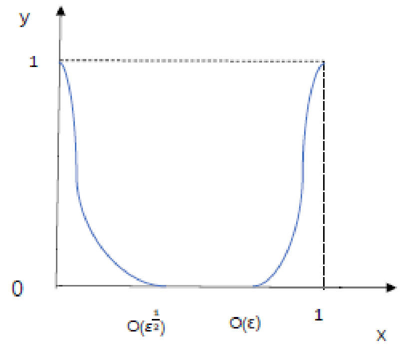


figure 3.8.1

3.9 Internal boundary layer

$$\begin{cases} \varepsilon y'' - xy' + y = 0 & -1 \leq x \leq 1 \\ y(-1) = e, \; y(1) = \frac{2}{e} \end{cases}$$

Outer expansion: $y_{out}(x) \sim y_o(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots, \varepsilon \to 0$

$\therefore y_0' + y_0 = 0 \Rightarrow y_0 = Ce^{-x} \Rightarrow$ cannot satisfies both boundary conditions simultaneously.

Look for a boundary layer solution. BL at $x_0 = 0$ thickness $\varepsilon^{\frac{1}{2}}$

$\therefore$ left:$\begin{cases} y_{0,l}' + y_{o,l} = 0 \\ y_{o,l}(-1) = e \end{cases}$, right$\begin{cases} y_{0,r}' + y_{o,r} = 0 \\ y_{o,r}(1) = \frac{2}{e} \end{cases}$

$y_{o,l} = e^{-x}$, $y_{o,r} = 2e^{-x}$

Inner region: $|x| \leq O(\varepsilon^{\frac{1}{2}})$, $X = \frac{x}{\varepsilon^{\frac{1}{2}}}$

$$\therefore Y_0(X) = A_0 \int_0^X e^{-\frac{t^2}{2}} dt + B_0$$

(Two constants to be determined through matching)

$O(1)$matching: $A_0 = \frac{1}{\sqrt{2\pi}}$, $B_0 = \frac{3}{2}$

3.10 Boundary layer in PDE problem

Steady state $(u, v) = (u_0, 0)$ constant $\neq 0$. solid boundary with no-slip boundary condition $(u, v) = (0, 0)$.

The velocity of steady flow does not satisfy the boundary condition, so there exists a BL in which the small viscosity plays important rule –Prandtl 1905, pioneer work of BL theory.

2D Navier-stokes equation for a steady flow

$$\begin{cases} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \gamma(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \gamma(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) \\ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \end{cases}$$

(u, v): velocity, p: pressure, ρ: density constant, γ: kinematic, $\gamma \ll 1$, $\gamma = \frac{u}{\rho}(m^2/s)$.

The leading order solution satisfies $\begin{cases} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \\ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \end{cases} \Rightarrow \begin{cases} u = u_0 \\ v = 0 \\ p = p_0 \end{cases}$ is a solution.

Assume the boundary is $y = 0$, the flow is in $y > 0$. Look for BL solution (thickness δ)

$\therefore X = x, Y = \frac{y}{\delta}, U = u, V = \frac{v}{\delta}, P = p$ ($V = \frac{v}{\delta}$ comes from 3rd eq.). $\frac{\partial v}{\partial y} = \frac{1}{\delta}\frac{\partial V}{\partial Y} = -\frac{\partial U}{\partial x} = O(1)$.

The system becomes:

$$\begin{cases} U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial X} + \gamma(\frac{\partial^2 U}{\partial X^2} + \frac{1}{\delta^2}\frac{\partial^2 U}{\partial Y^2}) \\ \delta U\frac{\partial V}{\partial X} + \delta V\frac{\partial V}{\partial Y} = -\frac{1}{\delta\rho}\frac{\partial P}{\partial Y} + \gamma(\delta^2\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}) \\ \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \end{cases}$$

In the first equation, $O(1), O(\gamma), O(\frac{\gamma}{\delta^2})$ terms dominaut balance $\Rightarrow \frac{\gamma}{\delta^2} = O(1) \Rightarrow \delta = \gamma^{\frac{1}{2}}$. (with dimensions, it is $\frac{\gamma}{\delta^2} = O(\frac{u_0^2}{L})$. L: typical length.) $\frac{\delta}{L} = O(\frac{\gamma}{u_0^2 L}) = O(\frac{1}{\mathrm{Re}^{\frac{1}{2}}})$. ($\frac{u_0^2}{L}$ comes from $U\frac{\partial U}{\partial X}$, $U = \frac{u}{U_0}$). $\mathrm{Re} = \frac{u_0 L}{\gamma}$ is Reynolds number, large. $\mathrm{Re} \gg 1$. Treat Re as large parameter.)

The leading order equation in the inner region

$$\begin{cases} U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial X} + \frac{\partial^2 U}{\partial Y^2} \\ \frac{\partial P}{\partial Y} = 0 \\ \frac{\partial U}{\partial X} + \frac{\partial U}{\partial Y} = 0 \end{cases}$$

Assume no BL in pressure $\lim\limits_{Y\to+\infty}\frac{\partial P}{\partial X} = \lim\limits_{y\to 0}\frac{\partial p}{\partial x}(Asymptotic\, matching) = 0 \Rightarrow P =$ constant in the BL.

From the 2nd equation $\Rightarrow \begin{cases} U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \frac{\partial^2 U}{\partial Y^2} \\ \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \end{cases} \Rightarrow \begin{cases} U = \frac{\partial \psi}{\partial Y} \\ V = \frac{\partial \psi}{\partial X} \end{cases}$ from some $\psi(X, Y)$ (stream function)

The first equation $\Rightarrow \frac{\partial \psi}{\partial Y}\frac{\partial^2 \psi}{\partial X \partial Y} - \frac{\partial \psi}{\partial X}\frac{\partial^2 \psi}{\partial Y^2} = \frac{\partial^3 \psi}{\partial Y^3}$

Use similarity method to find similarity solution.

$$\bar{X} = \varepsilon^a X, \;\; \bar{Y} = \varepsilon^b Y, \;\; \bar{\psi} = \varepsilon^c \psi$$

When $a = b + c$, self-similar solution exists

$$\varepsilon^{2c-2b-a}\frac{\partial\bar{\psi}}{\partial\bar{Y}}\frac{\partial^2\bar{\psi}}{\partial\bar{X}\partial\bar{Y}} - \varepsilon^{2c-2b-a}\frac{\partial\bar{\psi}}{\partial\bar{X}}\frac{\partial^2\bar{\psi}}{\partial\bar{Y}^2} = \varepsilon^{c-3b}\frac{\partial^3\bar{\psi}}{\partial\bar{Y}^3}$$

self-similar sol means $\bar{\psi} = \bar{\psi}(\bar{X}, \bar{Y})$ satisfies the same equation as $\psi = \psi(X, Y)$ does!

$\Rightarrow 2c - 2b - a = c - 3b \Rightarrow a = b + c$

This is called sealing invatiance property.

$\psi = f(X, Y)$ and $\bar{\psi} = \psi(\bar{X}, \bar{Y})$

$\bar{\psi} = \frac{4}{X^{c/a}},\ X \sim \varepsilon^{-a},\ \bar{Y} = \frac{Y}{X^{b/a}}$

$$\therefore \psi = X^{\frac{a-b}{a}} f(1, \frac{Y}{X^{b/a}}) = X^{\frac{a-b}{a}} f(z),\ z = \frac{Y}{X^{b/a}},\ U = \frac{\partial\psi}{\partial Y} = X^{1-\frac{2b}{a}} f(z)$$

Choose U to be a function of Z only.

$$\therefore 1 - \frac{2b}{a} = 0 \Rightarrow b = \frac{1}{2}a$$

$$\therefore \psi = X^{\frac{1}{2}} f(z), \quad z = \frac{Y}{2X^{\frac{1}{2}}}$$

(just for convenience)

The equation is simplied as: $f''' + ff'' = 0$ with $f(0) = 0,\ f'(0) = 0,\ f'(+\infty) = u_0$

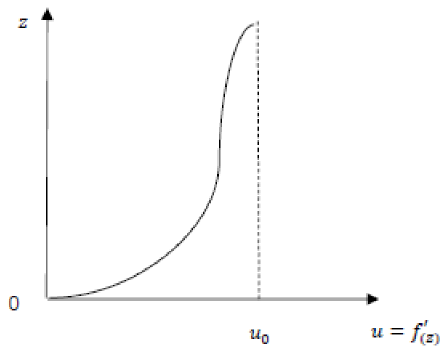


figure 3.10.1

Remark 3.10.1: For dimensionless argument and self-similar solution, see 1.4 of the notes "asy. pdf".

3.11 Exercise 3

1. (1). Compute the first four coefficients in the perturbation series to the initial value problem

$$y' = \frac{3}{2}y + 3\varepsilon xy, \quad y(0) = 1.$$

(2). Find the exact solution.
(3). Use some software, e.g., MATLAB, to plot and compare the exact solution and the n-term perturbation expansion for the solution, $n = 1, 2, 3, 4 (i.e., y_0, y_0 + \varepsilon y_1, y_0 + \varepsilon y_1 + \varepsilon^2 y_2 and y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon^3 y_3)$ on $x \in [0, 3]$ when $\varepsilon = 0.1$.

2. Consider the equation:

$$\begin{cases} \varepsilon y'' + (\frac{x}{9} - \frac{2}{3})y' + \frac{1}{9} = 0, \quad 0 \le x \le 3, \\ y(0) = 3, y(3) = 2. \end{cases}$$

Assume it is a boundary layer problem. The boundary layer is at $x = 3$, and the boundary layer thickness is ε.
(1). Find the outer limit, inner limit and the intermediate limit of the solution.
(2). Write down a uniformleading order approximation of the solution.

3. Consider the equation:

$$\begin{cases} \varepsilon y'' + (1 + x^2)y' + y = 0, \quad 0 \le x \le 1, \\ y(0) = 1, y(1) = 1. \end{cases}$$

(1). Determine the thickness and location of the boundary layer.
(2). Obtain a uniform approximation accurate to order ε as $\varepsilon \to 0$. Please use three methods to do matching:
a) The textbookmethod suggests keeping $O(1), O(\varepsilon)$ and $O(x)$ terms;
b) The van Dyke's matching rule;
c) The method of intermediate variable $x_\eta = \frac{x}{\eta(\varepsilon)}$.

4. Find the leading order approximation to the solution of the problem:

$$\begin{cases} \varepsilon y'' + x^{\frac{1}{3}} y' - y = 0, \quad 0 \le x \le 1, \\ y(0) = 0, y(1) = e^{\frac{3}{2}}. \end{cases}$$

5. Find the leading term of the asymptotic solution of the interior Dirichlet problem

$$\varepsilon(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2}) + \frac{\partial u}{\partial y} = 0$$

with $u = y$ on the boundary $C : (x-1)^2 + y^2 = 1$.

6. A temperature distribution satisfies:

$$\varepsilon(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + a(x)\frac{\partial u}{\partial x} = 0 \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1.$$

with $a(x) > 0$. The boundary conditions are

$$u = \begin{cases} \sin \pi y, & on \quad x = 1, \quad 0 \leq y \leq 1, \\ 0, & on \quad x = 0, \quad 0 \leq y \leq 1 \\ 0, & on \quad y = 1, y = 1 \quad 0 \leq x \leq 1 \end{cases}$$

By using matched asymptotic expansions to obtain a uniformly valid leading order approximation (i.e., composite expansion) to $u(x, y; \varepsilon)$ as $\varepsilon \to 0$.

3.12 Answer 3

1.(1). Compute the first four coefficients in the perturbation series to the initial value problem

$$y' = \frac{3}{2}y + 3\varepsilon xy, \quad y(0) = 1.$$

(2). Find the exact solution.
(3). Use some software, e.g., MATLAB, to plot and compare the exact solution and the n-term perturbation expansion for the solution, $n = 1, 2, 3, 4(i.e., y_0, y_0 + \varepsilon y_1, y_0 + \varepsilon y_1 + \varepsilon^2 y_2 and y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon^3 y_3)$ on $x \in [0, 3]$ when $\varepsilon = 0.1$.

Solution: (1).Set $y = \sum_{n=0}^{\infty} \varepsilon^n y_n$. Substituting this into the equation. In $O(1)$, we get:

$$y_0' = \frac{3}{2}y \quad \Rightarrow \quad y_0 = Ce^{\frac{3}{2}x}.$$

Since $y_0 = 1$, so $y_0 = e^{\frac{3}{2}x}$.
In $O(\varepsilon^k)$, $k = 1, 2, 3, ...$

$$y_k' = \frac{3}{2}y_k + 3xy_{k-1} \quad \Rightarrow \quad (y_k e^{-\frac{3}{2}x})' = 3xy_{k-1}e^{-\frac{3}{2}x} \quad \Rightarrow \quad y_k = e^{\frac{3}{2}x}\int_0^x 3ty_{k-1}(t)e^{-\frac{3}{2}t}dt.$$

Then we can calculate the first four terms:

$$y_0 = e^{\frac{3}{2}x} \quad \Rightarrow \quad y_1 = \frac{3}{2}x^2 e^{\frac{3}{2}x} \quad \Rightarrow \quad y_2 = \frac{9}{8}x^4 e^{\frac{3}{2}x} \quad \Rightarrow \quad y_3 = \frac{9}{16}x^6 e^{\frac{3}{2}x}.$$

The first four coefficients:

$$y \sim e^{\frac{3}{2}x} + \frac{3}{2}\varepsilon x^2 e^{\frac{3}{2}x} + \frac{9}{8}\varepsilon^2 x^4 e^{\frac{3}{2}x} + \frac{9}{16}\varepsilon^3 x^6 e^{\frac{3}{2}x}$$

(2).

$$y' = \frac{3}{2}y + 3\varepsilon xy \quad \Rightarrow \quad y' = (\frac{3}{2} + 3\varepsilon x)y \quad \Rightarrow \quad y = Ce^{\frac{3}{2}x + \frac{3}{2}\varepsilon x^2}.$$

Since $y(0) = 1$, $C = 1$, the exact solution is :

$$y = e^{\frac{3}{2}x + \frac{3}{2}\varepsilon x^2}$$

(3). Set $\varepsilon = 0.1$

$$h_{exact}(x) = e^{\frac{3}{2}x + \frac{3}{2}\varepsilon x^2}$$
$$h_1(x) = y_0 = e^{\frac{3}{2}x}$$
$$h_2(x) = y_0 + \varepsilon y_1 = e^{\frac{3}{2}x} + \frac{3}{2}\varepsilon x^2 e^{\frac{3}{2}x}$$
$$h_3(x) = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 = e^{\frac{3}{2}x} + \frac{3}{2}\varepsilon x^2 e^{\frac{3}{2}x} + \frac{9}{8}\varepsilon^2 x^4 e^{\frac{3}{2}x}$$
$$h_4(x) = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon^3 y_3 = e^{\frac{3}{2}x} + \frac{3}{2}\varepsilon x^2 e^{\frac{3}{2}x} + \frac{9}{8}\varepsilon^2 x^4 e^{\frac{3}{2}x} + \frac{9}{16}\varepsilon^3 x^6 e^{\frac{3}{2}x}$$

2. Consider the equation:

$$\begin{cases} \varepsilon y'' + (\frac{x}{9} - \frac{2}{3})y' + \frac{1}{9} = 0, \quad 0 \le x \le 3, \\ y(0) = 3, y(3) = 2. \end{cases}$$

Assume it is a boundary layer problem. The boundary layer is at $x = 3$, and the boundary layer thickness is ε.
(1). Find the outer limit, inner limit and the intermediate limit of the solution.
(2). Write down a uniformleading order approximation of the solution.

Solution: (1). Set $y = \sum_{n=0}^{\infty} \varepsilon^n y_n$. Subsituting this into the equation, then in $O(1)$:

$$(\frac{x}{9} - \frac{2}{3})y_0' + \frac{1}{9}y_0 = 0 \quad \Rightarrow \quad y_0' = \frac{y_0}{6-x} \quad \Rightarrow \quad y_0 = \frac{C}{6-x}$$

Since $y_0(0) = 3$, $C = 18$. We have outer limit :

$$y_0 = \frac{18}{6-x}$$

Since we have boundary layer at $x = 3$, $3 - \varepsilon \le x \le 3$, set $X = \frac{3-x}{\varepsilon}$. Then we have $y(x) = y(3 - \varepsilon X) = Y_{in}(X) \sim \sum_{n=0}^{\infty} \varepsilon^n Y_n$, also $\frac{dy}{dx} = -\frac{1}{\varepsilon} \cdot \frac{dY_{in}}{dX}$. Then we get the equation:

$$\frac{1}{\varepsilon} \cdot \frac{d^2 Y_{in}}{dX^2} + \frac{1}{\varepsilon} \cdot \frac{\varepsilon X + 3}{9} \cdot \frac{dY_{in}}{dX} + \frac{1}{9} Y_{in} = 0.$$

In $O(\frac{1}{\varepsilon})$, we have :

$$\frac{1}{\varepsilon} Y_0'' + \frac{1}{3\varepsilon} Y_0' = 0 \quad \Rightarrow \quad Y_0 = C_1 + C_2 e^{-\frac{1}{3}X}$$

Since $y(3) = Y_0(0) = 2$, $C_1 + C_2 = 2$. Then according to $lim_{x\to 3} y_0 = lim_{X\to+\infty} Y_0 = 6$, the intermediate limite is 6 and $6 = C_1$, we get the inner limite:

$$Y_0 = 6 - 4e^{-\frac{1}{3}X}.$$

The intermediate limite is 6.

(2). The uniform leading order equal to $y_0 + Y_0 -$intermediate limite, so

$$y = \frac{18}{6-x} + 6 - 4e^{-\frac{1}{3}X} - 6 = \frac{18}{6-x} - 4e^{\frac{x-3}{3\varepsilon}}$$

3. Consider the equation:

$$\begin{cases} \varepsilon y'' + (1 + x^2)y' + y = 0, \quad 0 \le x \le 1, \\ y(0) = 1, y(1) = 1. \end{cases}$$

(1). Determine the thickness and location of the boundary layer.
(2). Obtain a uniform approximation accurate to order ε as $\varepsilon \to 0$. Please use three

methods to do matching:
a) The textbookmethod suggests keeping $O(1), O(\varepsilon)$ and $O(x)$ terms;
b) The van Dyke's matching rule;
c) The method of intermediate variable $x_\eta = \frac{x}{\eta(\varepsilon)}$.

Solution: (1). Assume that there exists a boundary layer in $x = x_0$. Then we have $X = \frac{x-x_0}{\delta(\varepsilon)}$, $Y_{in}(X) = y(\delta X + x_0)$. Then we get the equation:

$$\frac{\varepsilon}{\delta^2}\frac{d^2Y_{in}(X)}{dX^2} + \frac{(\delta X + x_0 + 1)^2}{\delta}\frac{dY_{in}(X)}{dX} + Y_{in} = 0$$

By dominant balance $\frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta} \Rightarrow \varepsilon \sim \delta$. Then we have the equation in $O(\varepsilon^{-1})$:

$$Y_0'' + (1+x_0)^2 Y_0' = 0 \quad \Rightarrow \quad Y_0 = C_1 + C_2 e^{-(1+x_0)^2 X}$$

If $x_0 \neq 0$, Y_0 should be bounded as $X \to -\infty$, so $C_2 = 0$. $Y_0 = C_1$ means there is no boundary layer which is not correct, so $x_0 = 0$.

The thickness is ε and the boundary is at $x = 0$.

(2).(a). First consider the outer limit, set $y = \sum_{n=0}^{\infty} \varepsilon^n y_n$. Subsituting this into the equation.

$$O(1): \quad \begin{cases} (1+x)y_0' - y_0 = 0, \\ y_0(1) = 1, \end{cases} \quad \Rightarrow \quad y_0(x) = e^{\frac{1}{x+1} - \frac{1}{2}},$$

$$O(\varepsilon): \quad \begin{cases} (1+x)^2 y_1' - y_1 = -y_0^2, \\ y_1(1) = 0, \end{cases} \quad \Rightarrow \quad y_1(x) = (\frac{5x+7}{10(x+1)^5} - \frac{3}{80})e^{\frac{1}{x+1} - \frac{1}{2}}.$$

Then consider the inner limit:

$$O(\frac{1}{\varepsilon}): \begin{cases} Y_0'' + Y_0' = 0, \\ Y_0(1) = 1, \\ \lim_{X\to+\infty} Y_0 = \lim_{x\to 0} = e^{\frac{1}{2}}, \end{cases} \quad \Rightarrow Y_0(x) = e^{\frac{1}{2}} + (1 - e^{\frac{1}{2}})e^{-X},$$

$$O(1): \begin{cases} Y_1'' + Y_1' + 2XY_0' + Y_0 = 0, \\ Y_1(0) = 0, \end{cases} \quad \Rightarrow Y_1(x) = (e^{\frac{1}{2}} - 1)(X^2 + X)e^{-X} - e^{\frac{1}{2}}X - Ce^{-X} + C.$$

Matching $y_0 + \varepsilon y_1$ and $Y_0 + \varepsilon Y_1$, replace X as $\frac{x}{\varepsilon}$, and $e^{-\frac{x}{\varepsilon}} \to 0$. In keeping the $O(\varepsilon)$ term we get $C = \frac{53}{80}e^{\frac{1}{2}}$. The matching term is $Y_{match} = e^{\frac{1}{2}}(1 - x + \frac{53}{80}\varepsilon)$

So the uniform approximation is :

$$\begin{aligned} y(x) =& y_{out} + Y_{in} - Y_{match} \\ =& e^{\frac{1}{x+1} - \frac{1}{2}} + \varepsilon(\frac{5x+7}{10(x+1)^2} - \frac{3}{80})e^{\frac{1}{x+1} - \frac{1}{2}} + e^{\frac{1}{2}} + (1 - e^{\frac{1}{2}})e^{-\frac{x}{\varepsilon}} \\ &+ \varepsilon[(e^{\frac{1}{2}} - 1)((\frac{x}{\varepsilon})^2 + \frac{x}{\varepsilon})e^{-\frac{x}{\varepsilon}} - e^{\frac{1}{2}}\frac{x}{\varepsilon} - \frac{53}{80}e^{\frac{1}{2}}e^{-X} + \frac{53}{80}e^{\frac{1}{2}}] - e^{\frac{1}{2}}(1 - x + \frac{53}{80}\varepsilon) \end{aligned}$$

(b). Using van Dyke's matching rule:

For $y_0 + \varepsilon y_1$, set $x = \varepsilon X$ and fixed X, let $\varepsilon \to 0$:

$$\begin{aligned} y_0 + \varepsilon y_1 =& e^{\frac{1}{\varepsilon X+1}-\frac{1}{2}} + \varepsilon(\frac{5\varepsilon X+7}{10(\varepsilon X+1)^2} - \frac{3}{80})e^{\frac{1}{\varepsilon X+1}-\frac{1}{2}} \\ =& e^{\frac{1}{2}} - \varepsilon X e^{\frac{1}{2}} + \frac{53}{80}\varepsilon e^{\frac{1}{2}} - \frac{53}{80}\varepsilon^2 e^{\frac{1}{2}} + ... \end{aligned}$$

Matching this with $Y_0 + \varepsilon Y_1$, also get $C = \frac{53}{80}e^{\frac{1}{2}}$. The uniform approximation is as same as the answer in (a).

(c). Using $x_\eta = \frac{x}{\eta(\varepsilon)}$, $x = x_\eta \varepsilon^\alpha$, so we have:

$$\begin{aligned} y_0 + \varepsilon y_1 =& e^{\frac{1}{x_\eta \varepsilon^\alpha+1}-\frac{1}{2}} + \varepsilon(\frac{5x_\eta\varepsilon^\alpha+7}{10(x_\eta\varepsilon^\alpha+1)^2} - \frac{3}{80})e^{\frac{1}{x_\eta\varepsilon^\alpha+1}-\frac{1}{2}} \\ =& e^{\frac{1}{2}}(1 - x_\eta\varepsilon^\alpha + \frac{3}{2}x_\eta^2\varepsilon^{2\alpha}) + \frac{53}{80}\varepsilon e^{\frac{1}{2}}(1 - x_\eta\varepsilon^\alpha + \frac{3}{2}x_\eta^2\varepsilon^{2\alpha}) + O(\varepsilon) \end{aligned}$$

Also do the replacement $X = x_\eta \varepsilon^{\alpha-1}$ and expand the $Y_0 + \varepsilon Y_1$. Matching these two terms, we also get $C = \frac{53}{80}e^{\frac{1}{2}}$. The uniform approximation is as same as the answer in (a).

4. Find the leading order approximation to the solution of the problem:

$$\begin{cases} \varepsilon y'' + x^{\frac{1}{3}}y' - y = 0, \quad 0 \le x \le 1, \\ y(0) = 0, y(1) = e^{\frac{3}{2}}. \end{cases}$$

Solution: Assume that there exists a boundary layer in $x = x_0$. Then we have $X = \frac{x-x_0}{\delta(\varepsilon)}$, $Y_{in}(X) = y(\delta X + x_0)$. Then we get the equation:

$$\frac{\varepsilon}{\delta^2}\frac{d^2Y_{in}(X)}{dX^2} + \frac{(\delta X + x_0)^{\frac{1}{3}}}{\delta}\frac{dY_{in}(X)}{dX} - Y_{in} = 0$$

If $x_0 \neq 0$, $\frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta}$, then we have $Y_0' + x_0 Y_0 = 0 \Rightarrow Y_0 = C_1 e^{x_0 X} + C_2$. Since Y_{in} should be bounded with $X \to -\infty$, $C_1 = 0$ and $Y_{in} = C_2$. This means there is no boundary layer, so it's not correct, $x_0 = 0$.

The boundary layer is at $x = 0$. $\frac{\varepsilon}{\delta^2} \sim \delta^{-\frac{2}{3}} \Rightarrow \varepsilon = \delta^{\frac{3}{4}}$. Subsituting this into the equation.

$$\varepsilon^{-\frac{1}{2}}\frac{d^2Y_{in}(X)}{dX^2} + \varepsilon^{-\frac{1}{2}}X^{-\frac{1}{3}}\frac{dY_{in}(X)}{dX} - Y_{in} = 0$$

Then in $O(\varepsilon^{-\frac{1}{2}})$, we have the inner limit:

$$Y'' + X^{\frac{1}{3}}Y' = 0 \quad \Rightarrow \quad Y_0 = C\int_0^X e^{-\frac{3}{4}t^{\frac{4}{3}}}dt$$

For outer limit, set $y = \sum_{n=0}^{\infty}\varepsilon^n y_n$, we have $O(1)$:

$$\begin{cases} x^{\frac{1}{3}}y_0' - y_0 = 0 \\ y_0(1) = e^{\frac{3}{2}} \end{cases} \quad \Rightarrow \quad y_0(x) = e^{\frac{3}{2}x^{\frac{2}{3}}}$$

Since $lim_{x\to 0}y_0 = lim_{X\to+\infty}Y_0 = 1$, so we get:

$$C\int_0^{\infty} e^{-\frac{3}{4}t^{\frac{4}{3}}}dt = C(\frac{3}{4})^{\frac{1}{4}}\Gamma(\frac{3}{4}) = 1 \quad \Rightarrow \quad C = \frac{1}{(\frac{3}{4})^{\frac{1}{4}}\Gamma(\frac{3}{4})}$$

The uniform leading order approximation is

$$\begin{aligned} y &= \frac{1}{(\frac{3}{4})^{\frac{1}{4}}\Gamma(\frac{3}{4})}\int_0^{X} e^{-\frac{3}{4}t^{\frac{4}{3}}}dt + e^{\frac{3}{2}x^{\frac{2}{3}}} - 1 \\ &= \frac{1}{(\frac{3}{4})^{\frac{1}{4}}\Gamma(\frac{3}{4})}\int_0^{\varepsilon^{-\frac{4}{3}}x} e^{-\frac{3}{4}t^{\frac{4}{3}}}dt + e^{\frac{3}{2}x^{\frac{2}{3}}} - 1 \end{aligned}$$

5. Find the leading term of the asymptotic solution of the interior Dirichlet problem

$$\varepsilon(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2}) + \frac{\partial u}{\partial y} = 0 \tag{1}$$

with $u = y$ on the boundary $C : (x-1)^2 + y^2 = 1$.

Solution 1: For outer,assume $u \sim u_0 + \epsilon u_1 + \cdots$,then:

$$\frac{\partial u_0}{\partial y} = 0$$

so $u_0 = u_0(x)$
for inner,let

$$\begin{aligned} x =& 1 + r\cos\theta \\ y =& r\sin\theta \end{aligned}$$

(1) is transform to :

$$\epsilon(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}) + \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} \tag{2}$$

Assuming $R = \frac{r-1}{\delta}$,by dominant balacnce we get $\delta = \epsilon$.Then,(2) is transformed to:

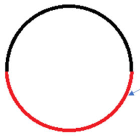
$$\frac{1}{\epsilon}\frac{\partial^2 U}{\partial R^2} + \frac{1}{1+\epsilon R}\frac{\partial U}{\partial R} + \frac{\epsilon}{(1+\epsilon R)^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\sin\theta}{\epsilon}\frac{\partial U}{\partial R} + \frac{\cos\theta}{1+\epsilon R}\frac{\partial U}{\partial \theta}$$

$O(\frac{1}{\epsilon})$:

$$\frac{\partial^2 U}{\partial R^2} + \sin\theta\frac{\partial U}{\partial R} = 0$$

Assume $U \sim U_0 + \epsilon U_1 + \cdots$.

Dedermine where the boundart layer is:
if $\sin\theta > 0$, we have $U_0(R,\theta) = A(\theta) + B(\theta)e^{-R\sin\theta}$. If $R \to -\infty$, then $e^{-R\sin\theta} \to \infty$. We get $B(\theta) = 0$, this means there is no boundary layer at $sin\theta > 0$.
if $\sin\theta = 0$,we have $U_0(R,\theta) = A(\theta) + B(\theta)R$.Using the boundary condition:$U_0(0,\theta) = B(\theta) = y = 0$. So $U_0(R,\theta) = A(\theta)$ doesn't vary fast, and it is not boundary layer.



if $\sin\theta < 0$, we have $U_0(R,\theta) = A(\theta) + B(\theta)e^{-R\sin\theta}$. If $R \to -\infty$, $\lim_{R\to-\infty} U_0(R,\theta) = A(\theta)$
Matching:
1.$\sin\theta > 0$,no boundary layer,$u_0(x) = y = \sqrt{1-(x-1)^2}$
2.$\sin\theta = 0$,as $r \to 1$ and $x \to 0, 2 \Rightarrow u_0(x) = 0$
3.$\sin\theta < 0$,we have $\lim_{r\to1} u_0(r\cos\theta) = \sqrt{1-(x-1)^2} = -\sin\theta$,then:

$$-\sin\theta = \lim_{R\to-\infty} U_0(R,\theta) = A(\theta)$$
$$B(\theta) = \sin\theta - A(\theta) = 2\sin\theta$$
$$u_{inter} = -\sin\theta$$

notice $sin\theta = -\sqrt{1-(x-1)^2}$,we get:

$$u_{inter} = \sqrt{1-(x-1)^2}$$
$$U_0(R,\theta) = \sqrt{1-(x-1)^2} - 2\sqrt{1-(x-1)^2}e^{-\frac{y+\sqrt{1-(x-1)^2}}{\epsilon}}$$

therefore:

$$u \sim u_0 + U_0 - u_{inter} = \sqrt{1-(x-1)^2} - 2\sqrt{1-(x-1)^2}e^{-\frac{y+\sqrt{1-(x-1)^2}}{\epsilon}}$$

Solution 2: Assume $f(x) = \sqrt{1-(x-1)^2}$
when $\epsilon = 0$,(1) is transformed to $\frac{\partial u_0}{\partial y} = 0$. Its characteristic line is $x = constant$, we know its boundary layer is near $f(x)$ or $-f(x)$.
1.if its boundary layer is near f(x):
Outer: $\frac{\partial u}{\partial y} = 0$, we get $u(x,y) = C(x) = -f(x)$
let $Y = \frac{y-f(x)}{\epsilon}$,we have:

$$\epsilon\frac{\partial^2 U}{\partial x^2} + \frac{1}{\epsilon}\frac{\partial^2 U}{\partial Y^2} + \frac{1}{\epsilon}\frac{\partial U}{\partial Y} = 0$$

for $O(\frac{1}{\epsilon})$:

$$\frac{1}{\epsilon}\frac{\partial^2 U}{\partial Y^2} + \frac{1}{\epsilon}\frac{\partial U}{\partial Y} = 0$$

we get $U_{in} = C_1(x)e^{-Y} + C_2(x)$
when $Y \to -\infty$,$e^{-Y} \to \infty$.So $C_1 = 0$, there is no boundary layer.
2.if its boundary layer is near -f(x):
Outer: $\frac{\partial u_0}{\partial y} = 0$, we get $u_0(x,y) = C(x) = f(x)$
let $Y = \frac{y+f(x)}{\epsilon}$,we have:

$$\epsilon\frac{\partial^2 U}{\partial x^2} + \frac{1}{\epsilon}\frac{\partial^2 U}{\partial Y^2} + \frac{1}{\epsilon}\frac{\partial U}{\partial Y} = 0$$

for $O(\frac{1}{\epsilon})$:

$$U''_{YY} + U'_Y = 0$$

we get $U_{in} = C_1(x)e^{-Y} + C_2(x)$
By boundary conditon:$U_{in}(x,0) = C_1(x) + C_2(x) = -f(x)$ Matching:

$$\lim_{Y\to+\infty} U_{in} = C_2(x)$$
$$\lim_{y\to-f(x)} u(x,y) = f(x)$$

We have:

$$C_2(x) = f(x)$$
$$C_1(x) = -f(x) - C_2(x) = -2f(x)$$
$$u_{inter} = C_2(x) = f(x)$$

Therefore

$$u \sim U_{in} + u_0 - u_{inter} = \sqrt{1-(x-1)^2} - 2\sqrt{1-(x-1)^2}e^{-\frac{y+\sqrt{1-(x-1)^2}}{\epsilon}}$$

6. A temperature distribution satisfies:

$$\varepsilon(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + a(x)\frac{\partial u}{\partial x} = 0 \quad 0 \le x \le 1, \quad 0 \le y \le 1.$$

with $a(x) > 0$. The boundary conditions are

$$u = \begin{cases} \sin \pi y, & on \quad x = 1, \quad 0 \le y \le 1, \\ 0, & on \quad x = 0, \quad 0 \le y \le 1 \\ 0, & on \quad y = 1, y = 1 \quad 0 \le x \le 1 \end{cases}$$

By usingmatched asymptotic expansions to obtain a uniformly valid leading order approximation (i.e., composite expansion) to $u(x,y;\varepsilon)$ as $\varepsilon \to 0$.

Solution: O(1): $\frac{\partial^2 u_0}{\partial y^2} + a(x)\frac{\partial^2 u_0}{\partial x^2} = 0$
using the method of Separation of Variables:
let $u_0(x,y) = X(x)Y(y)$,we get:$XY'' + a(x)X'Y = 0$,then

$$\frac{Y''}{Y} = -\frac{a(x)X'}{X} = -\lambda$$

We compute Y first:

$$\lambda_n = n^2\pi^2$$
$$Y_n(y) = \sin(n\pi y)$$

Then, we can get $X_n(x) = exp(n^2\pi^2 \int_0^x \frac{1}{a(s)}ds)$
So,$u_0(x,y) = \sum_0^\infty a_n exp(n^2\pi^2 \int_0^x \frac{1}{a(s)}ds) \sin \pi y$
By boundary condition,$a_n = 0, if n \ge 2, a_1 = exp(\pi^2 \int_0^1 \frac{1}{a(s)}ds)^{-1}$
so $u_0 = exp(-\pi^2 \int_x^1 \frac{1}{a(s)}ds) \sin(\pi y)$
It is not satisfied boundary condition at x=0, so there is a boundary layer at x=0;

let $X=\frac{x}{\epsilon}$,we get $\frac{1}{\epsilon}\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial y^2}+\frac{a(\epsilon X)}{\epsilon}\frac{\partial U}{\partial X}=0$
for $O(\frac{1}{\epsilon}):\frac{1}{\epsilon}\frac{\partial^2 U}{\partial X^2}+\frac{a(\epsilon X)}{\epsilon}\frac{\partial U}{\partial X}=0$, then

$$U_0(X,y)=\int_0^X C_1 e^{-\int_0^t a(\epsilon s)ds}dt+C_2$$

by $U_0(0,y)=0$,we get $C_2=0$
to match inner and outer solution.from $\lim\limits_{X\to\infty}U_0=\lim\limits_{x\to 0^+}u_0$,we get:

$$C_1=\frac{e^{-\pi^2\int_0^1\frac{1}{a(s)}ds}sin(\pi y)}{\int_0^\infty e^{-\int_0^t a(\epsilon s)ds}dt}$$

therefore:

$$u\sim(e^{-\pi^2\int_x^1\frac{1}{a(s)}ds}+\frac{e^{-\pi^2\int_0^1\frac{1}{a(s)}ds}}{\int_0^\infty e^{-\int_0^t a(\epsilon s)ds}dt}\int_0^{\frac{x}{\epsilon}}e^{-\int_0^t a(\epsilon s)ds}dt-e^{-\pi^2\int_0^1\frac{1}{a(s)}ds})\sin(\pi y)$$

Chapter 4

WKB Theory

4.1 Introductory example

In the method of matched asymptotic expansions studied in the last chapter, the dependence of the solution on the boundary-layer coordinate was determined by solving the boundary-layer problem. In a similar way, when using multiple scales, the dependence on the fast time scale was found by solving a differential equation. This does not happen with the WKB method because one begins with the assumption that the dependence is exponential. This is a reasonable expectation since many of the problems we studied in the last chapter ended up having an exponential dependence on the boundary-layer coordinate. Also, with this assumption, the work necessary to find an asymptotic approximation of the solution can be reduced significantly.

In the same manner as was done for boundary layers and multiple scales, the ideas underlying the WKB method will be developed by using it to solve an example problem. The one we begin with is the equation

$$\begin{cases} \varepsilon y'' + y = 0, \\ y(0) = 0, y(1) = 1. \end{cases}$$

whose exact solution is

$$y = \frac{\sin\frac{x}{\sqrt{\varepsilon}}}{\sin\frac{1}{\sqrt{\varepsilon}}}.$$

We can observe that its solution varies rapidly throughout the interval and there is no boundary layer. Hence the method that we take in the last chapter is not proper.

We can assume that an approximation method for such problems

$$y(x) = e^{S(x)}, \quad \begin{cases} S(x) \sim \frac{1}{\delta}S_0(x) + S_1(x) + \delta S_2(x) + \delta^2 S_3(x) + \cdots, \\ \delta = \delta(\varepsilon) \to 0. \end{cases}$$

Consider the problem, $\varepsilon y'' = Q(x)y$, $\varepsilon > 0$, $Q(x) \neq 0$ in the domain. $S(x)$ satisfies $\varepsilon[S'' + (S')^2]e^S = Q(x)e^S$, i.e.,

$$\varepsilon[S'' + (S')^2] = Q(x).$$

Hence, $\varepsilon[\frac{1}{\delta^2}(S_0')^2 + \frac{1}{\delta}S_0'' + \frac{1}{\delta}S_0'S_1' + S_1'' + (S_1')^2 + 2S_0'S_2' + \cdots] \sim Q(x)$, which implies that $\frac{\varepsilon}{\delta^2} \sim O_S(1) \Rightarrow \delta = \sqrt{\varepsilon}$. Then

$$(S_0')^2 + \varepsilon^{\frac{1}{2}}(S_0'' + 2S_0'S_1') + \varepsilon(S_1'' + (S_1')^2 + 2S_0'S_2') + \cdots \sim Q(x).$$

Its leading order equation $(S_0')^2 = Q(x)$ is Eikonal Equation.

$$S_0' = \pm\sqrt{Q(x)} \Rightarrow S_0(x) = \pm\int_0^X \sqrt{Q(x)}dt + A_0.$$

$O(\varepsilon^{\frac{1}{2}})$: $2S_0'S_1'+S_0''=0 \Rightarrow S_1'=-\frac{S_0''}{2S_0'} \Rightarrow S_1(x)=-\frac{1}{2}\log|S_0'(x)|+A_1=-\frac{1}{4}\log Q(x)+A_1$.
Hence

$$S(x)\sim\frac{1}{\sqrt{\varepsilon}}(\pm\int_0^x\sqrt{Q(x)}dt+A_0)-\frac{1}{4}\log Q(x)+A_1,$$

that means

$$y_0(x)=e^{S(x)}\sim C_1[Q(x)]^{-\frac{1}{4}}e^{\pm\int_0^x\sqrt{Q(x)}dt/\sqrt{\varepsilon}}.$$

Therefore,

$$y(x)=C_1[Q(x)]^{-\frac{1}{4}}e^{\frac{1}{\sqrt{\varepsilon}}\int_0^x\sqrt{Q(x)}dt}+C_2[Q(x)]^{-\frac{1}{4}}e^{-\frac{1}{\sqrt{2}}\int_0^x\sqrt{Q(x)}dt}.$$

(Leading order WKB approximation (up to $O(1)$))

For the problem in the introduction: $\varepsilon y''+y=0$, $Q(x)=-1$, $\int_0^x\sqrt{Q(x)}dt=ix$.

Leading order WKB approximation is $y(x)\sim C_1e^{\frac{ix}{\sqrt{\varepsilon}}}+C_2e^{-\frac{ix}{\sqrt{\varepsilon}}}$.

Remark 4.2.1: When $Q(x)=0$ at some x, the above method is no longer working and needs special treatment.

High order approximation: $Q(\varepsilon)$ eq. for $S(x)$.

$$2S_0'S_2'+S_1''+(S_1')^2=0.$$

$$Q(\varepsilon^{\frac{n}{2}}):2S_0S_n'+S_{n-1}''+2\sum_{j=1}^{n-1}S_j'S_{n-j}'=0.$$

$$S_2(x)=\pm\int^x(\frac{Q''}{8Q^{2/3}}-\frac{5(Q')^2}{32Q^{5/2}})dt.$$

$$S_3(x)=-\frac{Q''}{16Q^2}+\frac{5Q'^2}{64Q^3}.$$

Remark 4.2.2:

(1)WKB theory works only for linear equations. Otherwise, e^S cannot be determined on both sides.

Example 4.2.1:$\varepsilon y''=Q(x)y,\ y=e^{S(x)}$, $\varepsilon[S''+(S')^2]=Q(x)$ only forlinear equation. Nonlinear equations do not have such simple equations for $s(x)$.

(2)Some liner boundary layer problems can also be solved by using WKB theory.

4.2 More terms for the asymptotic expansions

$$S(x) \sim \frac{1}{\delta} S_0(x) + S_1(x) + \delta S_2(x) + \delta^2 S_3(x) + \cdots, \varepsilon \to 0$$

$$\delta = \delta(\varepsilon) \to 0 \text{ as } \varepsilon \to 0$$

$$\left.\begin{array}{l} S(x) - \frac{1}{\delta} S_0(x) \sim S_1(x), \varepsilon \to 0 \\ S(x) - [\frac{1}{\delta} S_0(x) + S_1(x)] \sim \delta S_2(x), \varepsilon \to 0 \\ S(x) - [\frac{1}{\delta} S_0(x) + S_1(x) + \delta S_2(x)] \sim \delta^2 S_3(x), \varepsilon \to 0 \end{array}\right\} \tag{4.1}$$

We want to find an approximation to $y(x) = e^{S(x)}$ s.t. the relative error is small $e^{\frac{S_0(x)}{\delta}}$ reflects the main features of $y(x)$ $\lim\limits_{\delta \to 0} \frac{y(x) - e^{\frac{S_0(x)}{\delta}}}{y(x)} = \lim\limits_{\delta \to 0} \frac{e^{S(x) - \frac{S_0(x)}{\delta}} - 1}{e^{S(x) - \frac{S_0(x)}{\delta}}} = \frac{e^{S_1(x)} - 1}{e^{S_1(x)}}$ is not small.

$\Rightarrow y(x) - e^{\frac{S_0(x)}{\delta}}$ is not small compared to $y(x)$.

$$\lim_{\delta \to 0} \frac{y(x) - e^{\frac{S_0(x)}{\delta} + S_1(x)}}{y(x)} = \lim_{\delta \to 0} \frac{e^{S(x) - \frac{S_0(x)}{\delta} - S_1(x)} - 1}{e^{S(x) - \frac{S_0(x)}{\delta} - S_1(x)}} = \lim_{\delta \to 0} \frac{e^{\delta S_2(x)} - 1}{e^{\delta S_2(x)}} = 0$$

($e^{\frac{S_0(x)}{\delta} + S_1(x)}$ is leading order approximation of $y(x)$)

$\Rightarrow$ uniform approximation if (4.1) holds uniformly.

More precisely $\frac{y(x) - e^{\frac{S_0(x)}{\delta} + S_1(x)}}{y(x)} \sim \delta S_2(x), \delta \to 0$. Thus $e^{\frac{S_0(x)}{\delta} + S_1(x)}$ is the leading order WKB approximation.

Genrally, when $N \geq 1$,

$$\frac{y(x) - e^{\frac{1}{\delta} \sum\limits_{n=0}^{N} \delta^n S_n(x)}}{y(x)} \sim \delta^N S_{N+1}(x), \delta \to 0$$

4.3 Problems with turning points

When $Q(x_0) = 0$, x_0 is called a turning point $\to$ WKB approximation doesn't apply. Without loss of generality, assume $Q(0) = 0$ and $Q(x) > 0$ for $x > 0$, $Q(x) < 0$ for $x < 0$.

Further assume that $Q'(0) \neq 0$. Then $Q'(0) > 0$.

(1) $x > 0$: $y_I(x) \sim C_1 [Q(x)]^{-\frac{1}{4}} e^{\frac{1}{\varepsilon} \int_0^x \sqrt{Q(t)} dt} + C_2 [Q(x)]^{-\frac{1}{4}} e^{-\frac{1}{\varepsilon} \int_0^x \sqrt{Q(t)} dt}, \varepsilon \to 0$

(2) $x < 0$: $y_{II}(x) \sim D_1 [-Q(x)]^{-\frac{1}{4}} \cos \frac{\int_0^x \sqrt{-Q(t)} dt}{\varepsilon} + D_2 [-Q(x)]^{-\frac{1}{4}} \sin \frac{\int_0^x \sqrt{-Q(t)} dt}{\varepsilon}$

Near the turning point $x = 0$, the expansion (1) and (2) are not valid.

There is a transition layer.
In the transition layer, let $X = \frac{x}{\eta}$, $\eta = \eta(\varepsilon) \to 0 (\varepsilon \to 0)$

$$\therefore Y(X) \sim Y_0(X) + \eta Y_1(X) + \cdots$$

Equation $\Rightarrow \frac{\varepsilon^2}{\eta^2} Y'' = Q(\eta X)Y,\ Q(\eta X) \approx Q(0) + Q'(0)\eta X + \frac{1}{2}Q''(0)(\eta X)^2 + O(\eta^3)$

Dominaut balance: $\frac{\varepsilon^2}{\eta^2}Y'' = Q'(0)\eta X Y_0 \Rightarrow \frac{\varepsilon^2}{\eta^2} \sim \eta,\ \eta = \varepsilon^{\frac{3}{2}}\ Y_0'' = Q'(0)XY_0$. Let$Z = (Q'(0))^{\frac{1}{3}}X$, then we can get $\frac{d^2Y_0}{dZ^2} = ZY_0$ (Airy equation)

General solution: $Y_0(Z) = E_1 A_i(Z) + E_2 B_i(Z)$

Airy functions $A_i(X)$ and $B_i(X)$ are two particular solutions.

$$x \to +\infty : \begin{cases} A_i(X) \sim \frac{1}{\sqrt{2\pi}} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}} \\ B_i(X) \sim \frac{1}{\sqrt{\pi}} x^{-\frac{1}{4}} e^{\frac{2}{3}x^{\frac{3}{2}}} \end{cases}$$

$$x \to -\infty : \begin{cases} A_i(X) \sim \frac{1}{\sqrt{\pi}}(-x)^{-\frac{1}{4}} \sin[\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}] \\ B_i(X) \sim \frac{1}{\sqrt{\pi}}(-x)^{-\frac{1}{4}} \cos[\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}] \end{cases}$$

A. Matching $y_I(x)$ and $y_{II}(x)$

as $x \to 0^+$, $Q(x) \sim Q'(0)x$, $[Q(x)]^{-\frac{1}{4}} \sim [Q'(0)]^{-\frac{1}{4}}x^{-\frac{1}{4}}$, $\int_0^x \sqrt{Q(x)}dt \sim \frac{2}{3}[Q'(0)]^{\frac{1}{2}}x^{\frac{3}{2}}$.

Thus, $y_I(x) \sim C_1[Q'(0)]^{-\frac{1}{4}}x^{-\frac{1}{4}}e^{\frac{2[Q'(0)]^{\frac{1}{2}}}{3\varepsilon}x^{\frac{3}{2}}} + C_2[Q'(0)]^{-\frac{1}{4}}x^{-\frac{1}{4}}e^{-\frac{2[Q'(0)]^{\frac{1}{2}}}{3\varepsilon}x^{\frac{3}{2}}}$.

Inside the transition layer, as $\frac{x}{\varepsilon^{\frac{2}{3}}} \to +\infty$, $X \to +\infty$, $Z = [Q^{'}(0)]^{\frac{1}{3}}X \to +\infty$

$$\therefore y_{II}(x) \sim \frac{E_1}{2\sqrt{\pi}}[Q'(0)]^{-\frac{1}{12}}x^{-\frac{1}{4}}\varepsilon^{\frac{1}{6}}e^{-\frac{2[Q'(0)]^{\frac{1}{2}}}{3\varepsilon}x^{\frac{3}{2}}} + \frac{E_2}{\sqrt{\pi}}[Q'(0)]^{-\frac{1}{12}}x^{-\frac{1}{4}}\varepsilon^{\frac{1}{6}}e^{\frac{2[Q'(0)]^{\frac{1}{2}}}{3\varepsilon}x^{\frac{3}{2}}}$$

Compare the coefficients

$$\begin{cases} C_1 = \frac{1}{\sqrt{\pi}}[Q'(0)]^{\frac{1}{6}}\varepsilon^{\frac{1}{6}}E_2 \\ C_2 = \frac{1}{2\sqrt{\pi}}[Q'(0)]^{\frac{1}{6}}\varepsilon^{\frac{1}{6}}E_1 \end{cases} \quad (E_1\ and\ E_2\ depend\ on\ \varepsilon) \tag{4.2}$$

B. Matching $y_{III}(x)$ and $y_{II}(x)$

As $x \to 0^-$, $Q(x) \sim Q'(0)x$,

$$y_{III}(x) \sim D_1[Q'(0)]^{-\frac{1}{4}}(-x)^{-\frac{1}{4}}\cos\frac{2[Q'(0)]^{\frac{1}{2}}}{3\varepsilon}(-x)^{\frac{3}{2}} + D_2[Q'(0)]^{-\frac{1}{4}}(-x)^{-\frac{1}{4}}\sin\frac{2[Q'(0)]^{\frac{1}{2}}}{3\varepsilon}(-x)^{\frac{3}{2}}$$

Inside the transition layer, as $X = \frac{x}{\varepsilon^{2/3}} \to -\infty$, $Z = [Q'(0)]^{\frac{1}{3}}X \to -\infty$,

$$\begin{aligned} y_{II}(x) \sim &(E_1 + E_2)\frac{1}{\sqrt{2\pi}}[Q'(0)]^{-\frac{1}{12}}\varepsilon^{\frac{1}{6}}(-x)^{-\frac{1}{4}}\cos\frac{2[Q'(0)]^{\frac{1}{2}}(-x)^{\frac{3}{2}}}{3\varepsilon} \\ &+ (E_1 - E_2)\frac{1}{\sqrt{2\pi}}[Q'(0)]^{-\frac{1}{12}}\varepsilon^{\frac{1}{6}}(-x)^{-\frac{1}{4}}\sin\frac{2[Q'(0)]^{\frac{1}{2}}(-x)^{\frac{3}{2}}}{3\varepsilon} \end{aligned}$$

Compare coefficients

$$\left\{\begin{array}{l} D_1 = (E_1 + E_2)\frac{1}{\sqrt{2\pi}}[Q'(0)]^{\frac{1}{6}}\varepsilon^{\frac{1}{6}} \\ D_2 = (E_1 - E_2)\frac{1}{\sqrt{2\pi}}[Q'(0)]^{\frac{1}{6}}\varepsilon^{\frac{1}{6}} \end{array}\right. \tag{4.3}$$

We have four conditions for 6 constants $C_1, C_2, D_1, D_2, E_1, E2$ (Normalization determines the other 2 conditions). 2 Boundary conditions determine thst relationship between C_1, C_2 and D_1, D_2.

$$\left\{\begin{array}{l} D_1 = \sqrt{2}C_2 + \frac{1}{\sqrt{2}}C_1 \\ D_2 = \sqrt{2}C_2 - \frac{1}{\sqrt{2}}C_1 \end{array}\right.$$

4.4 Eigenvalue problems (Sturm-Liville problem)

$$\left\{\begin{array}{ll} y'' + EQ(x)y(x) = 0. & 0 \le x \le \pi,\ E\,is\,a\,constant \\ y(0) = y(\pi) = 0. & where\ \ Q(x) > 0 \end{array}\right.$$

Obviously, it has the trivial solution, $y \equiv 0$. There exists an infinite number of values of $E_1, E_2, \cdots$ s.t. the problem has non-trivial solution.

E_n: $n-$th eigenvalue $0 < E_1 < E_2 < \cdots,$

$y_n(x)$: non-trivial solution corresponding to E_n.

Look for large E_n and corresponding $y_n(x)$. Let $\varepsilon = \frac{1}{E} \ll 1$, then $\left\{\begin{array}{l} \varepsilon y'' + Q(x)y = 0 \\ y(0) = y(\pi) = 0 \end{array}\right.$

Leading order WKB approximation for $\varepsilon y'' + Q(x)y = 0$ is

$$\begin{aligned} y(x) \sim & C_1[Q(x)]^{-\frac{1}{4}}e^{\sqrt{E}\int_0^x i\sqrt{Q(t)}dt} + C_2[Q(x)]^{-\frac{1}{4}}e^{-\sqrt{E}\int_0^x i\sqrt{Q(t)}dt} \\ = & C_1[Q(x)]^{-\frac{1}{4}}\cos[\sqrt{E}\int_0^x \sqrt{Q(t)}dt] + C_2[Q(x)]^{-\frac{1}{4}}\sin[\sqrt{E}\int_0^x \sqrt{Q(t)}dt] \end{aligned}$$

$$y(0) = 0 \Rightarrow C_1 = 0$$

$$y(\pi) = 0 \Rightarrow C_2[Q(\pi)]^{-\frac{1}{4}}\sin[\sqrt{E}\int_0^\pi \sqrt{Q(t)}dt] \sim 0, E \to +\infty$$

$$\therefore \sqrt{E_n}\int_0^\pi \sqrt{Q(t)}dt \sim n\pi,\ E_n \to +\infty \Rightarrow E_n \sim [\frac{n\pi}{\int_0^\pi \sqrt{Q(t)}dt}]^2$$

$$\therefore y(x) \sim C[Q(\pi)]^{-\frac{1}{4}}\sin[n\pi\frac{\int_0^x \sqrt{Q(t)}dt}{\int_0^\pi \sqrt{Q(t)}dt}], n \to +\infty$$

Constant C can be determined by other conditions, e.g. normalization.

Application of WKB Method to Wave Equation.

$$\left\{\begin{array}{l} u_{xx} = \mu^2(x)u_{tt} + \alpha(x)u_t + \beta(x)u, \quad \mu(x) > 0 \\ u(0,t) = \cos(wt),\ \ w \gg 1\ at\ x = 0 \end{array}\right.$$

$\mu(x),\ \alpha(x),\ \beta(x)$ are smooth, $\mu(x) > 0$

If $\alpha = \beta = 0$, $\mu =$ constant, then $u(x,t) = e^{i(wt-kx)}$, where $k = \pm w\mu$. This is a plane wave solution.

For high frequency waves, we have a shorter wavelength, $\left|\frac{2\pi}{k}\right| \ll 1 \Rightarrow$ small parameter $\varepsilon = \frac{1}{w} \ll 1$ (optics, electromagretics acoustics: ultrasand $10^6 H_z \Leftrightarrow 3\times 10^{-2} m \approx \frac{3\times 10^4}{10^6}$, visalde light $10^{15} H_z \Leftrightarrow 5\times 10^{-7} m \approx \frac{3\times 10^8}{10^{15}}$)

Constant an asymptotic approximation of traveling wave solution. in the case of high frequency.

Assume the solution has the expansion

$$u(x,t) \sim e^{iwt}\cdot e^{-iw\gamma Q(x)}(u_0(x) + \frac{1}{w^\gamma}u_1(x) + \cdots)$$

Subsituting into equation $\Rightarrow$

$$\begin{aligned}&-w^{2\gamma}Q_x^2(u_0 + w^{-\gamma}u_1(x) + \cdots) + iw^\gamma Q_x(u_0' + \cdots) + \tfrac{d}{dx}(iw^\gamma Q_x u_0 + \cdots)\\ &= \mu^2 w^2(u_0 + w^{-\gamma}u_1 + \cdots) - iw\alpha(u_0 + \cdots) + \beta(u_0 + \cdots)\end{aligned}$$

By dominaut balance $\Rightarrow \gamma = 1$, $w^{2\gamma} \sim w^2$

$$\begin{aligned}O(w^2) : Q_x^2 = \mu^2(x) &\Rightarrow Q = \pm\int_0^x \mu(t)dt\\ O(w) : Q_{xx}u_0 + 2Q_x\partial x u_0 &= -\alpha u_0\\ \Rightarrow 2\mu\frac{\partial u_0}{\partial x} &= -(u_x + \alpha)u_0\\ \log u_0 &= -\frac{1}{2}\log\mu - \frac{1}{2}\int_0^x \frac{\alpha(t)}{\mu(t)}dt + C_0\\ \Rightarrow u_0(x) &= \frac{A_0}{\sqrt{\mu}}\exp(-\frac{1}{2}\int_0^x \frac{\alpha(t)}{\mu(t)}dt)\end{aligned}$$

We want the right trveling wave, then $Q > 0$.

By BC, $u(0,t) = \cos wt \Rightarrow e^{iwt}\frac{A_0}{\sqrt{\mu(0)}}\cos wt$

$$A_0 = \sqrt{\mu(0)},\ \mu(0) = \sqrt{\frac{\mu(0)}{\mu(x)}}\exp(-\frac{1}{2}\int_0^x \frac{\alpha(s)}{\mu(s)}ds)$$

$$u(x,t) \sim \sqrt{\frac{\mu(0)}{\mu(x)}}\exp(-\frac{1}{2}\int_0^x \frac{\alpha(s)}{\mu(s)}ds)\cos(wt - w\int_0^x \mu(s)ds)$$

4.5 Application to slender body approximation

$u_{xx} + u_{yy} = \mu^2(x)u_{tt}$ vertical displacement of an elastic membrane. $0 < x < \infty,\ |y| < \varepsilon C_1(x), t > 0$

$\varepsilon \ll 1 \Rightarrow$ membrane is much longer than it is wide $\bar{y} = \frac{y}{\varepsilon}$, $\bar{\mu} = \varepsilon\mu$.

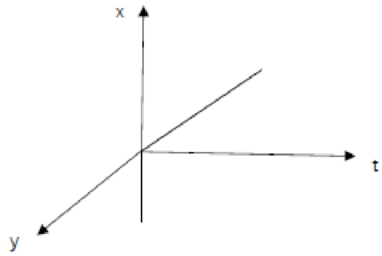


figure 4.6.1

Equation $\Rightarrow u_{xx} + \frac{1}{\varepsilon^2} u_{yy} = \frac{1}{\varepsilon^2} \bar{\mu}^2 u_{tt}$ (dragging bras)

$\Rightarrow \varepsilon^2 u_{xx} + u_{yy} = \mu^2 u_{tt},\ 0 < x < \infty,\ |y| < \varepsilon C_1(x)$

$$\begin{cases} u(x, y, t) = 0 \ for \ y = \pm C_1(x) \\ u(0, y, t) = f(y)\cos(wt) \end{cases}$$

Use WKB theory to constant the traveling wave solutions for small ε.

$$u(x, y, t) \sim e^{iwt} \cdot e^{-iQ(x)/\varepsilon}(u_0(x, y) + \varepsilon u_1(x, y) + \cdots)$$

(You need to determine $\gamma = 1$, when assuming $e^{-iQ(x)/\varepsilon^\gamma}$)

Subsituting into equation $\Rightarrow$

$$(-Q_x^2 u_0 - \varepsilon(Q_{xx}u_0 + 2iQ_x \partial x u_0 + Q_x^2 u_1) + \cdots) + \partial_y^2 u_0 + \varepsilon \partial_y^2 u_1 + \cdots = -w^2\mu^2(u_0 + \varepsilon u_1 + \cdots)$$

This is similar to ODE case.

$$O(1) : \ \partial_y^2 u_0 + (w^2\mu^2 - Q_x^2)u_0 = 0$$

$$u_0 = 0 \ for \ y = \pm a(x)$$

This is an eigenvalue problem. Let $\lambda = \sqrt{w^2\mu^2 - Q_x^2}$, then $u_0(x, y) = A(x)\sin(\lambda(y + a))$. $A(x)$ is to be determined by BC.

For $u_0|_{y=a(x)} = 0$ to be satisfied, we need $2\lambda a = n\pi$.

$\Rightarrow \lambda_n(x) = \frac{n\bar{a}}{2a(x)}$

$Q_x = \pm\sqrt{w^2\mu^2 - \lambda_n^2},\ \ n = 1, 2, \cdots$ (Eikonal Problem)

$$O(\varepsilon) : \ \partial_y^2 u_1 + \lambda_n^2 u_1 = i(Q_{xx}u_0 + 2Q_x \partial x u_0)$$

$$u_1 = 0 \ for \ y = \pm a(x)$$

For the equation to have a solution, we need a solubility condition on RHS.

Multiply equation by u_0 and then integrate w.r.t. y

$$\begin{aligned} \int_{-a}^{a} u_0(\partial_y^2 u_1 + \lambda_n^2 u_1)dy &= i\int_{-a}^{a} u_0(Q_{xx}u_0 + 2Q_x \partial x u_0)dy \\ &= -\int_{-a}^{a} \partial y u_0 \partial y u_1 dy + \int_{-a}^{a} \lambda_n^2 u_0 u_1 dy \\ &= \int_{-a}^{a} u_1 \partial_y^2 u_0 dy + \int_{-a}^{a} \lambda_n^2 u_0 u_1 dy \\ &= \int_{-a}^{a} u_1 dy \end{aligned}$$

$\Rightarrow \int_{-a}^{a} \partial x(Q_x u_0^2)dy = 0(solvability\ condition) = \frac{d}{dx}\int_{-a}^{a} Q_x u_0^2 dy - a^1$

$\Rightarrow \frac{d}{dx}\int_{-a}^{a} Q_x u_0^2 dy = a^2 = Q_x \int_{-a}^{a} A^2(x)\sin^2(\lambda(y+a))dy$

$\Rightarrow A(x) = a(Q_x a(x))^{-\frac{1}{2}}$

$\therefore u_n(x,y,t) \sim \frac{a}{\sqrt{Q_x a(x)}} e^{i(wt \pm \frac{Q(x)}{\varepsilon})} \sin(\lambda_n(y+a))$

where, $Q(x) = \int^x \sqrt{w^2\mu^2 - \lambda_n^2}dx,\ \lambda_n = \frac{n\pi}{2a(x)},\ n = 1, 2\cdots.$
By superposition: $u(x,y,t) = \sum_n a_n u_n(x,y,t)$
where a_n's are determined by initial condition.

Remark 4.6.1: The turning points in the WKB approximation occur when $Q_x = 0$, i.e. $\mu(xt) = \frac{n\pi}{2wa(xt)}$.

Single turning point $0 < xt < \infty$. (Assume $Q_x^2 = w^2\mu^2 - \lambda_n^2$ decreases with x)

Introduce $X = \frac{x - x_0}{\varepsilon^\beta}$,

$$u(X,y,t) \sim \varepsilon^\gamma e^{iwt}\{u_0(x)\sin[\lambda_n(y + a(x)) + \cdots\}$$

4.6 Inhomgeneous linear equations

We consider the Schrodinger Equation:

$$\varepsilon^2 y'' - Q(x)y(x) + R(x) = 0$$

where, $y(\pm\infty) = 0$, assume $Q(x) > 0$ for all x not turning points.

$$\varepsilon \to 0 \Rightarrow y(x) \sim \frac{R(x)}{Q(x)} \quad \varepsilon \to 0 \tag{4.4}$$

If R and Q are smooth, and turning pts, then (4.4) is valid everywhere.
In order to find the solution of satisfy BC, we require $R(x) \ll Q(x)$ as $x \to \pm\infty$.

This approximation breaks down at the pts of discontinuity R and at the turning pts of $Q(x)$. The WKB analysis given next yields a uniformly valid approximation even in the neighborhood of discontinuity of $R(x)$

(1) Use WKB to solve Green's function equation.

$$\begin{cases} \varepsilon^2 \frac{\partial^2 a}{\partial x^2}(x,x') - Q(x)a(x,x') = -\delta(x - x') \\ a(\pm\infty, x') = 0 \end{cases} \tag{4.5}$$

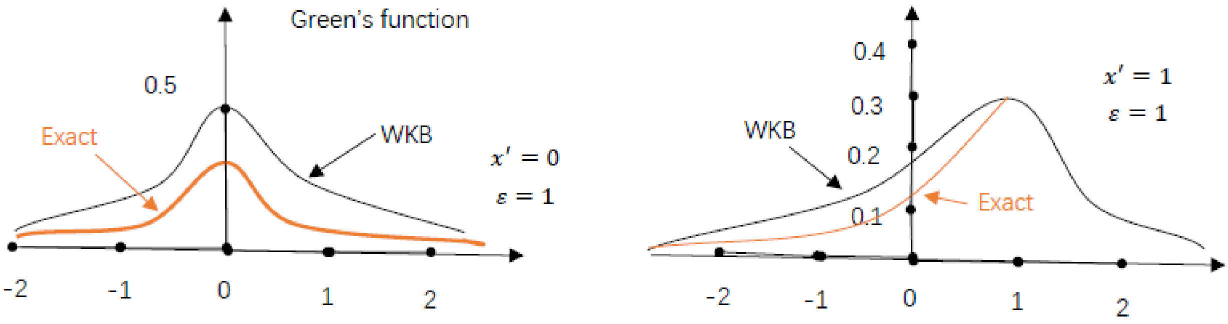


figure 4.7.1

(2) Use Green's function to constant the solution.

$$y(x) = \int_{-\infty}^{+\infty} a(x,x')R(x')dx'$$

To solve (1), consider two regions: Region I: $x > x'$, Region II: $x < x'$. WKB can be used in the two regions.

$$a_I(x,x') \sim C_1 Q(x)^{-\frac{1}{4}} e^{-\frac{1}{\varepsilon}\int_x^{x'} \sqrt{Q(t)}dt} \ (a \to 0,\ as\, x \to +\infty)$$

$$a_{II}(x,x') \sim C_2 Q(x)^{-\frac{1}{4}} e^{-\frac{1}{\varepsilon}\int_x^{x'} \sqrt{Q(t)}dt} \ (a \to 0,\ as\, x \to -\infty)$$

C_1 and C_2 are determined by patching a_I and a_{II} at $x = x'$

i) Continuity of $a(x,x')$:

$$\lim_{\eta \to 0^+} (a_I(x'+\eta, x') - a_{II}(x'-\eta, x')) = 0 \Rightarrow C_1 = C_2$$

ii) Jump of derivative of $a(x,x')$:

$$\lim_{\eta \to 0^+} (\frac{\partial}{\partial x} a_I(x,x')|_{x=x'+\eta} - \frac{\partial}{\partial x} a_{II}(x,x')|_{x=x'-\eta}) = -\frac{1}{\varepsilon^2} \Rightarrow C_1 = C_2 = \frac{1}{2\varepsilon} Q(x')^{-\frac{1}{4}}$$

Uniformly approximation:

$$a(x,x') \sim \frac{1}{2\varepsilon}(Q(x)Q(x'))^{-\frac{1}{4}} e^{-\frac{1}{\varepsilon}\left|\int_{x'}^{x} \sqrt{Q(t)}dt\right|}$$

4.7 Exercise 4

1. Find the WKB approximation correct up to $O(\varepsilon^{\frac{1}{2}})$ to the the general solution of the problem

$$\varepsilon y'' + \frac{1}{8}(2x^2 + \varepsilon x)y = 0, \quad x > 0.$$

where $\varepsilon > 0$

2. Use the WKBmethod to find an approximate solution to the problem:

$$\begin{cases} \varepsilon y'' + (x + \frac{1}{2})y' + y = 0, & 0 \le x \le 1, \\ y(0) = 2, y(1) = 3. \end{cases}$$

Compare your answer with the composite expansion obtained using matched asymptotic expansions.

3. Use the WKB method to find the leading order asymptotic behavior of the large eigenvalue E^2 and the corresponding eigenfunction of the boundary value problem:

$$\begin{cases} y'' + E^2 e^{2x} y = 0, & 1 \le x \le 2, \\ y(1) = 0, y(2) = 0. \end{cases}$$

4. The classical Boussinesq equations for water waves have the form

$$\begin{aligned} u_t + \epsilon u u_x + \eta_x &= \frac{1}{2}\epsilon u_{xxt} \\ \eta_t + [(1 + \epsilon\eta)u]_x &= \frac{1}{6}\epsilon\lambda u_{xxx} \end{aligned} \qquad (*)$$

where $0 < \epsilon \ll 1$ and λ is a constant.
(1). Consider waves in the near field (x=O(1) and t=O(1)), and show that the asymptotic expansion is not valid for large t.
(2). Study right-going waves in the far field and deduce the equation for the leading term of U.

4.8 Answer 4

1. Find the WKB approximation correct up to $O(\varepsilon^{\frac{1}{2}})$ to the the general solution of the problem

$$\varepsilon y'' + \frac{1}{8}(2x^2 + \varepsilon x)y = 0, \quad x > 0.$$

where $\varepsilon > 0$

Solution: Using WKB method to solve the problem.

$$y \sim e^{\frac{1}{\delta}S_0(x)+S_1(x)+\delta S_2(x)+\dots}$$

Subsituting this into the equation:

$$\varepsilon(\frac{1}{\delta}S_0'' + S_1'' + \delta S_2'') + \varepsilon(\frac{1}{\delta}S_0' + S_1' + \delta S_2')^2 = -\frac{x^2}{4} - \frac{\varepsilon x}{8}$$

Then dominant balance shows that $\frac{\varepsilon}{\delta^2} \sim 1 \quad \Rightarrow \quad \delta = \varepsilon^{\frac{1}{2}}$.

$$\begin{aligned}
O(1): &\quad (S_0')^2 = -\frac{x^2}{4} \quad \Rightarrow \quad S_0 = \pm i\frac{x^2}{4} + C \\
O(\varepsilon^{\frac{1}{2}}): &\quad S_0'' + 2S_0S_1' = 0 \quad \Rightarrow \quad S_1' = \frac{-S_0''}{2S_0'} \quad \Rightarrow \quad S_1 = \mp\frac{1}{2}lnx + C
\end{aligned}$$

Since we only need the approximation correct up to $O(\varepsilon^{\frac{1}{2}})$,

$$\begin{aligned}
y \sim& e^{\frac{1}{\delta}S_0(x)+S_1(x)} \\
=& C_1e^{\frac{ix^2}{4\sqrt{\varepsilon}}-\frac{1}{2}lnx} + C_2e^{-\frac{ix^2}{4\sqrt{\varepsilon}}+\frac{1}{2}lnx} \\
=& C_1x^{-\frac{1}{2}}e^{\frac{ix^2}{4\sqrt{\varepsilon}}} + C_2x^{\frac{1}{2}}e^{-\frac{ix^2}{4\sqrt{\varepsilon}}}
\end{aligned}$$

2. Use the WKBmethod to find an approximate solution to the problem:

$$\begin{cases} \varepsilon y'' + (x + \frac{1}{2})y' + y = 0, \quad 0 \le x \le 1, \\ y(0) = 2, y(1) = 3. \end{cases}$$

Compare your answer with the composite expansion obtained using matched asymptotic expansions.

Solution: (1). Using WKB method to solve the problem.

$$y \sim e^{\frac{1}{\delta}S_0(x)+S_1(x)+\delta S_2(x)+\dots}$$

Subsituting this into the equation:

$$\varepsilon(\frac{1}{\delta}S_0'' + S_1'' + \delta S_2'') + \varepsilon(\frac{1}{\delta}S_0' + S_1' + \delta S_2')^2 + (x + \frac{1}{2})(\frac{1}{\delta}S_0' + S_1' + \delta S_2') + 1 = 0$$

Then dominant balance shows that $\frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta} \quad \Rightarrow \quad \delta = \varepsilon.$

$$O(\varepsilon^{-1}): \quad (S_0')^2 + (x+\frac{1}{2})S_0' = 0 \quad \Rightarrow \quad S_0 = C \quad or \quad S_0 = -\frac{x^2}{2} - \frac{x}{2} + C$$
$$O(1): \quad S_0'' + 2S_0'S_1' + (x+\frac{1}{2})S_1' + 1 = 0 \quad \Rightarrow \quad S_1 = -ln(x+\frac{1}{2}) + C \quad or \quad S_1 = C$$

Then we get the approximation by WKB

$$y \sim C_1 e^{-ln(x+\frac{1}{2})} + C_2 e^{-\frac{x^2}{2\varepsilon} - \frac{x}{2\varepsilon}}$$

Using the boundary condition:

$$\begin{cases} y(0) = 2 \\ y(1) = 3 \end{cases} \Rightarrow \begin{cases} 2C_1 + C_2 = 2 \\ \frac{2}{3}C_1 = 3 \end{cases} \Rightarrow \begin{cases} C_1 = \frac{9}{2} \\ C_2 = -7 \end{cases} \Rightarrow \quad y \sim \frac{9}{2x+1} - 7e^{-\frac{x^2+x}{2\varepsilon}}$$

(2) Then we use matched asymptotic expansions. Set $y = \sum_{n=0}^{\infty} \varepsilon^n y_n$. Subsituting this into the equation, in $O(1)$:

$$(x+\frac{1}{2})y_0' + y_0 = 0 \quad \Rightarrow \quad y_0 = \frac{C}{-x-\frac{1}{2}}$$

Since $y_0(1) = 3$, $C = -\frac{9}{2}$. We have outer limit :$y_0 = \frac{9}{2x+1}$.
The boundary layer is at $x = 0$. Then we set $X = \frac{x}{\delta(\varepsilon)}$, $Y_{in}(X) = y(\delta X)$. The equation becomes:

$$\frac{\varepsilon}{\delta^2}\frac{d^2Y_{in}(X)}{dX^2} + \frac{1}{\delta}(\delta X + \frac{1}{2})\frac{dY_{in}(X)}{dX} + Y_{in} = 0$$

By dominant balance $\frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta} \Rightarrow \varepsilon \sim \delta$.
In $O(\frac{1}{\varepsilon})$, we have :

$$\frac{1}{\varepsilon}Y_0'' + \frac{1}{2\varepsilon}Y_0' = 0 \quad \Rightarrow \quad Y_0 = C_1 + C_2 e^{-\frac{1}{2}X}$$

Since $y(0) = Y_0(0) = 2$, $C_1 + C_2 = 2$. Then according to $lim_{x\to 0} y_0 = lim_{X\to+\infty} Y_0 = 9$, the intermediate limite is 9 and $C_1 = 9$, we get the inner limite $Y_0 = 9 - 7e^{-\frac{1}{2}X}$. The uniform leading order equal to $y_0 + Y_0-$intermediate limite, so

$$y = \frac{9}{2x+1} + 9 - 7e^{-\frac{1}{2}X} - 9 = \frac{9}{2x+1} - 7e^{-\frac{1x}{2\varepsilon}}$$

3. Use the WKB method to find the leading order asymptotic behavior of the large eigenvalue E^2 and the corresponding eigenfunction of the boundary value problem:

$$\begin{cases} y'' + E^2 e^{2x} y = 0, \quad 1 \le x \le 2, \\ y(1) = 0, y(2) = 0. \end{cases}$$

Solution: Since the eigenvalue E^2 is large, set $\varepsilon = \frac{1}{E^2}$, $\varepsilon \ll 1$. Using WKB method to solve the result, let $y \sim e^{\frac{1}{\delta}S_0(x) + S_1(x) + \delta S_2(x) + \cdots}$. Subsituting this into the equation:

$$\varepsilon(\frac{1}{\delta}S_0'' + S_1'' + \delta S_2'') + \varepsilon(\frac{1}{\delta}S_0' + S_1' + \delta S_2')^2 + e^{2x}(\frac{1}{\delta}S_0 + S_1 + \delta S_2) = 0$$

Then dominant balance shows that $\frac{\varepsilon}{\delta^2} \sim 1 \quad \Rightarrow \quad \delta = \varepsilon^{\frac{1}{2}}$.

$$
\begin{aligned}
O(1):&\quad (S_0')^2 = -e^{2x} \quad \Rightarrow \quad S_0 = \pm \int_1^x ie^t dt = \pm i(e^x - e) + C \\
O(\varepsilon^{\frac{1}{2}}):&\quad S_0'' + 2S_0 S_1' = 0 \quad \Rightarrow \quad S_1' = \frac{-S_0''}{2S_0'} \quad \Rightarrow \quad S_1 = -\frac{x}{2} + C
\end{aligned}
$$

$$
\begin{aligned}
y \sim & C_1 e^{\frac{1}{\delta} i(e^x - e) - \frac{x}{2}} + C_2 e^{-\frac{1}{\delta} i(e^x - e) - \frac{x}{2}} \\
= & e^{-\frac{x}{2}} (C_1 \cos[E(e^x - e)] + C_2 \sin[E(e^x - e)])
\end{aligned}
$$

Since $y(1) = 0$, then $e^{-\frac{1}{2}}(C_1 \cos[E(e-e)] + C_2 \sin[E(e-e)]) = e^{-\frac{1}{2}} C_1 = 0 \Rightarrow C_1 = 0$.
Then $y(2) = 0$ shows: $C_2 e^{-\frac{1}{2}} \sin[(e^2 - e)E] = 0 \Rightarrow E = \frac{k\pi}{e^2 - e}$.

$$
\begin{aligned}
\int_1^2 e^{2x} y^2 dx &= C_2^2 \int_1^2 e^{2x} e^{-x} \sin^2[(e^2 - e)E] dx \\
&= C_2^2 \int_1^2 e^x \frac{1 - \cos[2(e^2 - e)E]}{2} dx \\
&= C_2^2 \int_1^{2(e^2 - e)E} \frac{1 - \cos(t)}{4E} dt \qquad\qquad , t = 2(e^2 - e)E \\
&= C_2^2 \frac{e^2 - e}{2} = 1
\end{aligned}
$$

So $C_2 = \sqrt{\frac{2}{e^2 - e}}$, the leading order asymptotic behavior is:

$$
y \sim \sqrt{\frac{2}{e^2 - e}} \cdot e^{-\frac{x}{2}} \cdot \sin\left(\frac{(e^x - e)k\pi}{e^2 - e}\right) \qquad k = 1, 2, 3...
$$

4. The classical Boussinesq equations for water waves have the form

$$
\begin{aligned}
u_t + \epsilon u u_x + \eta_x &= \frac{1}{2} \epsilon u_{xxt} \\
\eta_t + [(1 + \epsilon\eta)u]_x &= \frac{1}{6} \epsilon \lambda u_{xxx}
\end{aligned}
\qquad (*)
$$

where $0 < \epsilon \ll 1$ and λ is a constant.
(1). Consider waves in the near field (x=O(1) and t=O(1)), and show that the asymptotic expansion is not valid for large t.
(2). Study right-going waves in the far field and deduce the equation for the leading term of U.

Solution: (1). let $\epsilon = 0$, we get:

$$
\begin{aligned}
u_t + \eta_x &= 0 \\
\eta_t + u_x &= 0
\end{aligned}
$$

From this equation system, we konw the directions of the wave propagation are $x + t$ and $x - t$.

Therefore, we assume $\theta_1 = x + t, \theta_2 = x - t$.Then (*) transform to:

$$\frac{\partial u}{\partial \theta_1} - \frac{\partial u}{\partial \theta_2} + \epsilon u \frac{\partial u}{\partial \theta_1} + \epsilon u \frac{\partial u}{\partial \theta_2} + \frac{\partial \eta}{\partial \theta_1} + \frac{\partial \eta}{\partial \theta_2} = \frac{1}{2}\epsilon\lambda(\frac{\partial}{\partial \theta_1} - \frac{\partial}{\partial \theta_2})(\frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2})^2 u$$
$$\frac{\partial \eta}{\partial \theta_1} - \frac{\partial \eta}{\partial \theta_2} + \frac{\partial u}{\partial \theta_1} + \frac{\partial u}{\partial \theta_2} + \epsilon \frac{\partial (\eta u)}{\partial \theta_1} + \epsilon \frac{\partial (\eta u)}{\partial \theta_2} = \frac{1}{6}\epsilon\lambda(\frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2})^3 u$$

assume $u = u_0 + \epsilon u_1 + \ldots, \eta = \eta_0 + \epsilon \eta_1 + \ldots$
for O(1):

$$\frac{\partial u_0}{\partial \theta_1} - \frac{\partial u_0}{\partial \theta_2} + \frac{\partial \eta_0}{\partial \theta_1} + \frac{\partial \eta_0}{\partial \theta_2} = 0$$
$$\frac{\partial \eta_0}{\partial \theta_1} - \frac{\partial \eta_0}{\partial \theta_2} + \frac{\partial u_0}{\partial \theta_1} + \frac{\partial u_0}{\partial \theta_2} = 0$$

form this equation system, we get:

$$u_0 = f(\theta_1) + g(\theta_2)$$
$$\eta_0 = g(\theta_2) - f(\theta_1)$$

for O(ϵ):

$$\frac{\partial u_1}{\partial \theta_1} - \frac{\partial u_1}{\partial \theta_2} + \frac{\partial \eta_1}{\partial \theta_1} + \frac{\partial \eta_1}{\partial \theta_2} = -u_0 \frac{\partial u_0}{\partial \theta_1} - u_0 \frac{\partial u_0}{\partial \theta_2} + \frac{1}{2}\lambda(\frac{\partial}{\partial \theta_1} - \frac{\partial}{\partial \theta_2})(\frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2})^2 u_0$$
$$\frac{\partial \eta_1}{\partial \theta_1} - \frac{\partial \eta_1}{\partial \theta_2} + \frac{\partial u_1}{\partial \theta_1} + \frac{\partial u_1}{\partial \theta_2} = -\frac{\partial (\eta_0 u_0)}{\partial \theta_1} - \frac{\partial (\eta_0 u_0)}{\partial \theta_2} + \frac{1}{6}\lambda(\frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2})^3 u_0 \tag{1}$$

let

$$T_1 = -u_0 \frac{\partial u_0}{\partial \theta_1} - u_0 \frac{\partial u_0}{\partial \theta_2} + \frac{1}{2}\lambda(\frac{\partial}{\partial \theta_1} - \frac{\partial}{\partial \theta_2})(\frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2})^2 u_0$$
$$T_2 = -\frac{\partial (\eta_0 u_0)}{\partial \theta_1} - \frac{\partial (\eta_0 u_0)}{\partial \theta_2} + \frac{1}{6}\lambda(\frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2})^3 u_0$$

then

$$T_1 = -(f+g)(\frac{\partial f}{\partial \theta_1} + \frac{\partial g}{\partial \theta_2}) + \frac{1}{2}\lambda(\frac{\partial^3 f}{\partial \theta_1^3} - \frac{\partial^3 g}{\partial \theta_2^3})$$
$$T_2 = 2f\frac{\partial f}{\partial \theta_1} - 2g\frac{\partial g}{\partial \theta_2} + \frac{1}{6}\lambda(\frac{\partial^3 f}{\partial \theta_1^3} + \frac{\partial^3 g}{\partial \theta_2^3}) \tag{2}$$

from (1):

$$u_1 = \int (T_1 + T_2) d\theta_1 + \int (T_2 - T_1) d\theta_2 + \phi_1(\theta_1) + \phi_2(\theta_2)$$

pluging (2) into it:

$$u_1 = (\frac{1}{2}f^2 - fg - \frac{\partial g}{\partial \theta_2}\int f d\theta_1 - 3g\frac{\partial g}{\partial \theta_2}\theta_1 + \frac{2}{3}\lambda\frac{\partial^2 f}{\partial \theta_1^2} - \frac{1}{3}\lambda\frac{\partial^3 g}{\partial \theta_2^3}\theta_1)$$
$$+(3f\frac{\partial f}{\partial \theta_1}\theta_2 + \frac{\partial f}{\partial \theta_1}\int g d\theta_2 - fg + \frac{1}{2}g^2 - \frac{1}{3}\lambda\frac{\partial^3 f}{\partial \theta_1^3}\theta_2 + \frac{2}{3}\lambda\frac{\partial^2 g}{\partial \theta_2^2})$$
$$+\phi_1(\theta_1) + \phi_2(\theta_2)$$

Because $3g\frac{\partial g}{\partial \theta_2}\theta_1, \frac{1}{3}\lambda\frac{\partial^3 g}{\partial \theta_2^3}\theta_1, 3f\frac{\partial f}{\partial \theta_1}\theta_2, \frac{1}{3}\lambda\frac{\partial^3 f}{\partial \theta_1^3}\theta_2$ are secular terms, the asymptotic expansion is not valid for large t.

(2). Because only consider the right-going waves in the far field, let $\theta_2 = x - t, X = \epsilon x, T = \epsilon t$. we get:

$$
\begin{aligned}
&-\frac{\partial u}{\partial \theta_2} + \epsilon\frac{\partial u}{\partial T} + \epsilon u\frac{\partial u}{\partial \theta_2} + \epsilon^2 u\frac{\partial u}{\partial X} + \frac{\partial \eta}{\partial \theta_2} + \epsilon\frac{\partial \eta}{\partial X} = \frac{1}{2}\epsilon\lambda(\frac{\partial}{\partial \theta_2} + \epsilon\frac{\partial}{\partial X})^2(-\frac{\partial}{\partial \theta_2} + \epsilon\frac{\partial}{\partial T})u \\
&-\frac{\partial \eta}{\partial \theta_2} + \epsilon\frac{\partial \eta}{\partial T} + \frac{\partial u}{\partial \theta_2} + \epsilon\frac{\partial u}{\partial X} + \epsilon\frac{\partial(\eta u)}{\partial \theta_2} + \epsilon^2\frac{\partial(\eta u)}{\partial X} = \frac{1}{6}\epsilon\lambda(\frac{\partial}{\partial \theta_2} + \epsilon\frac{\partial}{\partial X})^3 u
\end{aligned}
$$

for O(1):

$$
-\frac{\partial u_0}{\partial \theta_2} + \frac{\partial \eta_0}{\partial \theta_2} = 0
$$

we get:$u_0 - \eta_0 = f(X, T)$.
for O(ϵ):

$$
-\frac{\partial u_1}{\partial \theta_2} + \frac{\partial \eta_1}{\partial \theta_2} = -\frac{\partial u_0}{\partial T} - u_0\frac{\partial u_0}{\partial \theta_2} - \frac{\partial \eta_0}{\partial X} - \frac{1}{2}\lambda\frac{\partial^3 u_0}{\partial \theta_2^2 \partial T} \tag{3}
$$

$$
-\frac{\partial \eta_1}{\partial \theta_2} + \frac{\partial u_1}{\partial \theta_2} = -\frac{\partial \eta_0}{\partial T} - \frac{\partial u_0}{\partial X} - \frac{\partial(\eta_0 u_0)}{\partial \theta_2} + \frac{1}{6}\lambda\frac{\partial^3 u_0}{\partial \theta_2^3} \tag{4}
$$

(3)-(4):

$$
\begin{aligned}
2(-\frac{\partial u_1}{\partial \theta_2} + \frac{\partial \eta_1}{\partial \theta_2}) &= -\frac{\partial(u_0 - \eta_0)}{\partial T} - (u_0 - \eta_0)\frac{\partial u_0}{\partial \theta_2} + \frac{\partial(u_0 - \eta_0)}{\partial X} + u_0\frac{\partial \eta_0}{\partial \theta_2} - \frac{1}{2}\lambda\frac{\partial^3 u_0}{\partial \theta_2^2 \partial T} + \frac{1}{6}\lambda\frac{\partial^3 u_0}{\partial \theta_2^3} \\
&= -\frac{\partial f}{\partial T} - f\frac{\partial u_0}{\partial \theta_2} + \frac{\partial f}{\partial X} + u_0\frac{\partial \eta_0}{\partial \theta_2} - \frac{1}{2}\lambda\frac{\partial^3 u_0}{\partial \theta_2^2 \partial T} + \frac{1}{6}\lambda\frac{\partial^3 u_0}{\partial \theta_2^3}
\end{aligned}
$$

In order to avoid secular term, we must let:

$$
-\frac{\partial f}{\partial T} + \frac{\partial f}{\partial X} = 0
$$

just:

$$
-\frac{\partial(u_0 - \eta_0)}{\partial T} + \frac{\partial(u_0 - \eta_0)}{\partial X} = 0 \tag{5}
$$

(3)+(4):

$$
0 = -\frac{\partial u_0}{\partial T} - u_0\frac{\partial u_0}{\partial \theta_2} - \frac{\partial \eta_0}{\partial X} - \frac{1}{2}\lambda\frac{\partial^3 u_0}{\partial \theta_2^2 \partial T} + -\frac{\partial \eta_0}{\partial T} - \frac{\partial u_0}{\partial X} - \frac{\partial(\eta_0 u_0)}{\partial \theta_2} + \frac{1}{6}\lambda\frac{\partial^3 u_0}{\partial \theta_2^3} \tag{6}
$$

(5) and (6) are the equations for the leading term of U.

Chapter 5

Multiple Scale Analysis

5.1 Introductory example

When one uses matched asymptotic expansions, the solution is constructed in different regions that are then patched together to form a composite expansion. The method of multiple scales differs from this approach in that it essentially starts with a generalized version of a composite expansion. In doing this, one introduces coordinates for each region (or layer), and these new variables are considered to be independent of one another. A consequence of this is that what may start out as an ordinary differential equation is transformed into a partial differential equation. Exactly why this helps to solve the problem, rather than make it harder, will be discussed as the method is developed in this chapter.

As in the last chapter, we will introduce the ideas underlying the method by going through a relatively simple example. The problem to be considered is to find the function $y(t)$ that satisfies

$$\begin{cases} \frac{d^2y}{dt^2} + y + \varepsilon y^3 = 0,\ t \geq 0, \\ y(0) = 1,\ y'(0) = 0. \end{cases}$$

We assume that $y(t) \sim y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots,\ \varepsilon \to 0$. Then

$$\begin{cases} y_0 = \cos t, \\ y_1 = \frac{1}{32}\cos 3t - \frac{1}{32}\cos t - \frac{3}{8} t \sin t. \end{cases}$$

Hence,

$$y(t) \sim \cos t + \varepsilon[\frac{1}{32}\cos 3t - \frac{1}{32}\cos t - \frac{3}{8}t\sin t] + \cdots,\ \varepsilon \to 0.$$

It contains term $\varepsilon t \sin t$, which is small conpared with $O(1)$ term when $t \ll \frac{1}{\varepsilon}$, compared with $O(1)$ term when $t = O(\frac{1}{\varepsilon})$. Therefore, when $t \gg O(\frac{1}{\varepsilon})$, $\cos t$ is not an approximation to $y(t)$.

$\varepsilon t \sin t$ is oscillatory amplitude grows. These kinds of terms are secular terms. After a very long time, the effect of these terms becomes important. $y_0(t) + \varepsilon y_1(t)$ is unbounded due to the secular term. However, the exact solution is bounded. We illustrate it as follows. Multiply the equation by $\frac{dy}{dt}$

$$\frac{d}{dt}[\frac{1}{2}(\frac{dy}{dt})^2] + \frac{d}{dt}[\frac{y^2}{2} + \frac{1}{4}\varepsilon y^4] = 0.$$

Then

$$\frac{1}{2}(\frac{dy}{dt})^2 + \frac{1}{2}y^2 + \frac{1}{4}\varepsilon y^4 = C \Rightarrow \frac{y^2}{2} \leq C.$$

From the initial condition,

$$y(0) = 1\ y'(0) = 0 \Rightarrow C = \frac{1}{2} + \frac{1}{4}\varepsilon \Rightarrow y^2 \leq 1 + \frac{1}{2}\varepsilon,$$

which indicates $y(t)$ is bounded.

One source of the secular term is resonance

$$\frac{d^2y}{dt^2} + y = \cos wt$$

$$\Rightarrow \begin{cases} w \neq \pm 1, \ y = A\cos t + B\sin t + \frac{1}{1-w^2}\cos(wt). & \textit{bounded} \\ w = \pm 1, \ y = A\cos t + B\sin t + \frac{1}{2}t\sin t. & \textit{contain secular term} \end{cases}$$

In the previous example: $y_0(t) = \cos t$, and

$$y_1'' + y_1 = -\cos^3 t = -\frac{1}{4}\cos 3t - \frac{3}{4}\cos t.$$

5.2 Method of strained coordinates

Consider a simpler problem:

$$\begin{cases} \frac{d^2y}{dt^2} + y + \varepsilon y = 0 \\ y(0) = 1 \ y'(0) = 0 \end{cases} \qquad y(t) \sim \cos t - \frac{\varepsilon}{2}t\sin t + \cdots, \varepsilon \to 0$$

Exact solution: $y = \cos(\sqrt{1+\varepsilon t}) = \cos([1 + \frac{\varepsilon}{2} - \frac{\varepsilon^3}{8} + \cdots]t) \approx \cos t - \frac{\varepsilon}{2}t\sin t$.

This is valid when $\varepsilon t \ll 1$.

From the example, we can see that $\tau = \sqrt{1+\varepsilon}t$ is a better independent variable than t, solution $y = \cos\tau$. Without knowing the exact solution, we don't know the expression of τ.

We assume that $\tau = (1 + w_1\varepsilon + w_2\varepsilon^2 + \cdots)t$
τ: strained coordinate.
C_{wal}: to eliminate the secular terms.

In practice, we compute only finite terms in τ. $\tau = (1+w_1\varepsilon+\cdots+w_N\varepsilon^N)t$ to eliminate secular terms up to $O(\varepsilon^N)$ in the expansion of solution. Using $\tau = (1 + w_1\varepsilon + \cdots)t$ to solve the original problem:

$$\frac{d^2y}{dt^2} = (\frac{d\tau}{dt})^2\frac{d^2y}{d\tau^2} = (1 + w_1\varepsilon + \cdots)^2\frac{d^2y}{d\tau^2}$$

Back to the original problem

$$\begin{cases} \frac{d^2y}{dt^2} + y + \varepsilon y^3 = 0 \\ y(0) = 1, \ y'(0) = 0 \end{cases}$$

To eliminate secular term at $O(\varepsilon)$. Let $\tau = (1 + w_1\varepsilon)t$, $(1 + w_1\varepsilon)^2\frac{d^2y}{d\tau^2} + y + \varepsilon y^3 = 0$.

$O(1): \ \frac{d^2y_0}{dt^2} + y_0 = 0, \ y_0(0) = 1, \ y_0'(0) = 0$

$\Rightarrow y_0(\tau) = \cos\tau$

$O(\varepsilon)$: $\frac{d^2y'}{d\tau^2} + 2w_1\frac{d^2y_0}{d\tau^2} + y_1 + y_0^3 = 0$

$$\frac{d^2y_1}{d\tau^2} + y_1 = 2w_1\cos\tau - \cos^3\tau = (2w_1 - \frac{3}{4})\cos\tau - \frac{1}{4}\cos^3\tau$$

$\cos\tau$ terms on RHS will give a secular term $\tau\cos\tau$.

If $2w_1 - \frac{3}{4} = 0 \Rightarrow w_1 = \frac{3}{8}$, then there are no secular terms.

Therefore, the leading order approximation is $y(\tau) \sim \cos\tau$, $\varepsilon \to 0$ with $\tau = (1+\frac{3}{8}\varepsilon)t$ and there is no secular term at $O(\varepsilon)$.

Remarks 5.2.1:
i) For this leading order approximation $y(\tau) \sim \cos\tau$, $\tau = (1+\frac{3}{8}\varepsilon)t$, there is no secular term at $O(\varepsilon)$, but there is no guarantee that the secular terms at $O(\varepsilon^2)$ or higher orders are also canceled. If there is some secular terms at $O(\varepsilon^2)$, e.g. $\varepsilon^2 t\sin t$, then the leading order approximation $y \sim \cos\tau$ is valid when $\varepsilon^2 t \ll 1$.

ii) More general strained coordinates: $t \sim \tau + \varepsilon f_1(\tau) + \varepsilon^2 f_2(\tau) + \cdots, \varepsilon \to 0$

5.3 Multiple scale analysis

$$\begin{cases} \frac{d^2y}{dt^2} + 2\varepsilon\frac{dy}{dt} + y = 0 & \textit{Strained coordinate} \\ y(0) = 0 \;\; y'(0) = 1 & \tau = (1+w_1\varepsilon)t \end{cases}$$

$$y \sim y_0(\tau) + \varepsilon y_1(\tau), \;\; \varepsilon \to 0$$

$$y_0(\tau) = \sin\tau, \;\; y_1'' + y_1 = w_1\sin\tau - 2\cos\tau$$

both $\sin\tau$ and $\cos\tau$ are secular terms. cannot be eliminated both.

Exact solution: $y = \frac{e^{-\varepsilon t}}{\sqrt{1-\varepsilon^2}}\sin(\sqrt{1-\varepsilon^2}t) \Rightarrow$ time scale: $\varepsilon t\sqrt{1-\varepsilon^2}t$.

$\therefore$ We use multiple scale analysis $y = y(t,\tau)$, let $\tau = \varepsilon t$ (from the physics).

$$y(t,\tau) \sim Y_0(t,\tau) + \varepsilon Y_1(t,\tau) + \varepsilon^2 Y_2(t,\tau) + \cdots, \varepsilon \to 0$$

$$\frac{dY_j}{dt} = \frac{\partial Y_j}{\partial t} + \frac{\partial Y_j}{\partial \tau}\frac{d\tau}{dt} = \frac{\partial Y_j}{\partial t} + \varepsilon\frac{\partial Y_j}{\partial \tau}$$

$$\begin{aligned} \therefore \frac{d^2Y_j}{dt^2} =& (\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial \tau})(\frac{\partial Y_j}{\partial t} + \varepsilon\frac{\partial Y_j}{\partial \tau}) \\ =& \frac{\partial^2 Y_j}{\partial t^2} + 2\varepsilon\frac{\partial^2 Y_j}{\partial t\partial \tau} + \varepsilon^2\frac{\partial^2 Y_j}{\partial \tau^2} \end{aligned}$$

Then, we can get:

$$\begin{aligned}\frac{dy}{dt} &\sim \frac{dY_0(t,\varepsilon t)}{dt} + \varepsilon\frac{dY_1(t,\varepsilon t)}{dt} + \varepsilon^2\frac{dY_2(t,\varepsilon t)}{dt} + \cdots \\ &\sim \frac{\partial Y_0}{\partial t} + \varepsilon(\frac{\partial Y_0}{\partial \tau} + \frac{\partial Y_1}{\partial t}) + \varepsilon^2(\frac{\partial Y_1}{\partial \tau} + \frac{\partial Y_2}{\partial t}) + \cdots\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dt^2} &\sim \frac{d^2Y_0(t,\varepsilon t)}{dt^2} + \varepsilon\frac{d^2Y_1(t,\varepsilon t)}{dt^2} + \varepsilon^2\frac{d^2Y_2(t,\varepsilon t)}{dt^2} + \cdots \\ &\sim \frac{\partial^2 Y_0}{\partial t^2} + \varepsilon(2\frac{\partial^2 Y_0}{\partial t\partial \tau} + \frac{\partial^2 Y_1}{\partial t^2}) + \varepsilon^2(\frac{\partial^2 Y_1}{\partial \tau^2} + 2\frac{\partial^2 Y_1}{\partial t\partial \tau} + \frac{\partial^2 Y_2}{\partial t^2}) + \cdots\end{aligned}$$

$$O(1):\ \frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0 \Rightarrow Y_0(t,\tau) = A(\tau)e^{it} + A^*(\tau)e^{-it}$$

where A^* is the complex conjugate of A. (to look for real solution)

$A(\tau)$ is to be determined to eliminate secular terms at $O(\varepsilon)$.

$$\begin{aligned}O(\varepsilon):\ \frac{\partial^2 Y_1}{\partial t^2} + Y_1 &= -2\frac{\partial^2 Y_0}{\partial t\partial \tau} - 2\frac{\partial Y_0}{\partial t} \\ &= -2i[(\frac{dA}{d\tau} + A)e^{it} - (\frac{dA^*}{d\tau} + A^*)e^{-it}]\end{aligned}$$

To eliminate secular terms, we need $\frac{dA}{d\tau} + A = 0$ (Then $\frac{dA^*}{d\tau} + A^* = 0$ automatically).

$$A(\tau) = A(0)e^{-\tau}$$

Therefore,

$$Y_0(t,\tau) = A(0)e^{-\tau+it} + A^*(0)e^{-\tau-it} = e^{-\tau}(C_1\cos t + C_2\sin t)$$

$$y(0) = 0 \Rightarrow Y_0(0,0) = 0 \Rightarrow C_1 = 0$$

$$y'(0) = 0 \Rightarrow \frac{\partial Y_0}{\partial t}(0,0) = 1 \Rightarrow C_2 = 1$$

$$(\frac{\partial Y_0}{\partial t}(0,0) + \varepsilon(\frac{\partial Y_0}{\partial \tau}(0,0) + \frac{\partial Y_1}{\partial t}(0,0)) + O(\varepsilon^2) = 0)$$

$$\therefore y(t) \sim e^{-\tau}\sin t = e^{-\varepsilon t}\sin t,\ \varepsilon \to 0$$

This is leading order approximation. The uniformly valid region is $t \ll \frac{1}{\varepsilon^2}$.

Remark 5.3.1:
i) In principle, higher order multiple scale analysis can be used, e.g.t, εt, εt^2. But in practice, it is complicated, because of two much freedoms.
ii) The two scales $(t, \varepsilon t)$ may be generalized to $(t, f(t,\varepsilon))$ or $(f(t,\varepsilon), \varepsilon t)$, $\cdots$.

5.4 Slowly varying coefficients

$$\begin{cases} y'' + w^2(\varepsilon t)y = 0 \\ y(0) = a,\ y'(0) = b \end{cases} \qquad a, b\, not\, both\, 0,\ \varepsilon t = O(1)\, or\, t = O(\frac{1}{\varepsilon})$$

Profile of exact solution.

Two time sacles: t, εt. Need to resolve both time scales: $t = O(1)$ and $t = O(\frac{1}{\varepsilon})$. Multiple scale analysis: (t, τ), $\tau = \varepsilon t$.

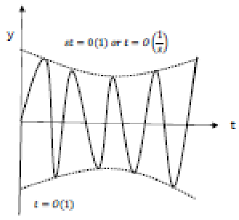


figure 5.4.1

$O(1)$: $\frac{\partial^2 Y_0}{\partial t^2} + w^2(\tau) Y_0 = 0$

Solution $\Rightarrow Y_0 = A(\tau)e^{iw(\tau)t} + A^*(\tau)e^{-iw(\tau)t}$.

$$\begin{aligned} O(\varepsilon): \ & \frac{\partial^2 Y_1}{\partial t^2} + w^2(\tau)Y_1 = -2\frac{\partial^2 Y_1}{\partial t \partial \tau} \\ & = -2i[\frac{d(A(\tau)w(\tau))}{d\tau} + itA(\tau)w(\tau)\frac{dw(\tau)}{d\tau}]e^{iw(\tau)t} \\ & + 2i[\frac{d(A(\tau)w(\tau))}{d\tau} - itA^*(\tau)w(\tau)\frac{dw(\tau)}{d\tau}]e^{-iw(\tau)t} \end{aligned} \tag{5.1}$$

No secular terms in $Y_1 \Rightarrow$

$$\frac{d(Aw)}{d\tau} + itAw\frac{dw}{d\tau} = 0 \Rightarrow Aw = Ce^{-iw(\tau)t} \tag{5.2}$$

Aw doesn't depends on $t \Rightarrow C = 0$.

$\Rightarrow Aw = 0 \Rightarrow A = 0 \Rightarrow Y_0 \equiv 0$ does not satisfy the initial conditions $\to$ method fails.

The "t" in Eq(5.2) comes from $\frac{\partial Y_0}{\partial \tau}$ in Eq(5.1):

$$\frac{\partial}{\partial \tau}e^{iw(\tau)t} = it\frac{dw(\tau)}{d\tau}e^{iw(\tau)t}$$

If we doesn't depend on "τ", there is no such term scales $\to (T, \tau)$, $T = f(t)$ is another time scale. The equation becomes

$$\frac{d^2y}{dT^2} + \frac{f''}{(f')^2}\frac{dy}{dT} + \frac{w^2(\varepsilon t)}{(f')^2}y = 0$$

Let $f'(t) = w(\varepsilon t)$, then $f(t) = \int_0^t w(\varepsilon u)du = \frac{1}{\varepsilon}\int_0^{\varepsilon t} w(s)ds$, $f''(t) = \varepsilon w'(\varepsilon t)$.

$$\frac{dy}{dt} = \frac{dy}{dT}\frac{dT}{dt} = f''(t)\frac{dy}{dT}$$

$$\frac{d^2y}{dt^2} = f'(t)\frac{d}{dT}(f''(t)\frac{dy}{dT}) = f''(t)\frac{dy}{dT} + (f')^2\frac{d^2y}{dT^2}$$

New time scales:

$T = \frac{1}{\varepsilon}\int_0^{\varepsilon t} w(s)ds$, $\tau = \varepsilon t$,

$$y \sim Y_0(T, \tau) + \varepsilon Y_1(T, \tau) + \varepsilon^2 Y_2(T, \tau) + \cdots, \varepsilon \to 0$$

Then $T = \frac{1}{\varepsilon}\int_0^{\tau} w(s)ds$, $\frac{dT}{d\tau} = \frac{1}{\varepsilon}w(\tau)$

$$\frac{dY_i(T, \tau(T))}{dT} = \frac{\partial Y_i}{\partial T} + \frac{\partial Y_i}{\partial \tau}\frac{d\tau}{dT}|_{\tau=\tau(T)} = \frac{\partial Y_i}{\partial T} + \frac{\varepsilon}{w(\tau)}\frac{\partial Y_i}{\partial \tau}|_{\tau=\tau(T)}$$

$$\frac{d^2Y_j(T, \tau(T))}{dT^2} = \frac{\partial^2 Y_j}{\partial T^2} + \frac{2\varepsilon}{w(\tau)}\frac{\partial^2 Y_j}{\partial T \partial \tau} + \frac{\varepsilon^2}{w(\tau)}\frac{\partial}{\partial \tau}(\frac{1}{w(\tau)}\frac{\partial Y_j}{\partial \tau})|_{\tau=\tau(T)}$$

$$\frac{\partial^2 Y}{\partial T^2} + \frac{2\varepsilon}{w}\frac{\partial^2 Y}{\partial T \partial \tau} + \frac{\varepsilon^2}{w}\frac{\partial}{\partial \tau}(\frac{1}{w}\frac{\partial Y}{\partial \tau}) + \frac{\varepsilon w'(\tau)}{w^2}(\frac{\partial Y}{\partial T} + \frac{\partial Y}{\partial \tau}\frac{\varepsilon}{w}) + Y = 0$$

$$O(1): \ \frac{\partial^2 Y_0}{\partial T^2} + Y_0 = 0 \Rightarrow Y_0 = A(\tau)e^{iT} + A^*(\tau)e^{-iT}$$

$$\begin{aligned} O(\varepsilon): \ \frac{\partial^2 Y_1}{\partial T^2} + Y_1 =& \frac{2\varepsilon}{w(\tau)}\frac{\partial^2 Y_0}{\partial T \partial \tau} - \frac{w'(\tau)}{w^2(\tau)}\frac{\partial Y_0}{\partial T} \\ =& - i\frac{2}{w(\tau)}[\frac{dA(\tau)}{d\tau} + \frac{w'(\tau)}{2w(\tau)}A(\tau)]e^{iT} \\ &+ i\frac{2}{w(\tau)}[\frac{dA^*(\tau)}{d\tau} + \frac{w'(\tau)}{2w^2(\tau)}A^*(\tau)]e^{-iT} \end{aligned}$$

No secular term $\Rightarrow \frac{dA}{d\tau} + \frac{w'}{2w}A = 0$

Solution $A(\tau) = \frac{C}{w(\tau)}$

Real solution is $Y_0(T,\tau) = \frac{1}{\sqrt{w(\tau)}}[C_1 \cos T + C_2 \sin T]$,

$$y(t) \sim \frac{1}{\sqrt{w(\varepsilon t)}}[C_1 \cos \frac{1}{\varepsilon}\int_0^{\varepsilon t} w(s)ds + C_2 \sin \frac{1}{\varepsilon}\int_0^{\varepsilon t} w(s)ds] \tag{5.3}$$

This problem can also be should by WKB theory using "τ" as variable:

$$\varepsilon^2 \frac{\partial^2 y}{\partial \tau^2} + w^2(\tau)y = 0$$

leading order WKB approximation is exactly (5.3)

Comparison: i) Multiple scale method may work for nonlinear problems. Appropriate scales need to be identified.
ii) WKB theory works only for linear problems.

Some BL problems can also be solved using multiple scale analysis $\begin{pmatrix} \text{out variable} \\ \text{inner variable} \end{pmatrix}$

$$\begin{cases} \varepsilon y'' + ay' + by = 0 & 0 < x < 1 \\ y(0) = A \ \ y(1) = B & a > 0 \end{cases}$$

BL is at $x = 0$ with thickness ε.

Slowly varying region $\varepsilon \ll x \ll 1$.

Two Scales: short scale (or fast scale) $X = \frac{x}{\varepsilon}$ describe inner solution, long scale (or slow scale) x describe outer solution.

$$y = Y(X, x)$$

where, $X = \frac{x}{\varepsilon}$, ODE becomes PDE

$$(\frac{\partial^2}{\partial X^2} + 2\varepsilon\frac{\partial^2}{\partial X \partial x} + \varepsilon\frac{\partial^2}{\partial x^2})Y + a(\frac{\partial}{\partial X} + \varepsilon\frac{\partial}{\partial x})Y + \varepsilon bY = 0$$

$$Y \sim Y_0(X,x) + \varepsilon Y_1(X,x) + \cdots$$
$$O(1): \frac{\partial^2 Y_0}{\partial X^2} + a\frac{\partial Y_0}{\partial X} = 0 \Rightarrow Y_0(X,x') = A_1(x) + A_2(x)e^{-aX}$$
$$O(\varepsilon): \frac{\partial^2 Y_1}{\partial X^2} + a\frac{\partial Y_1}{\partial X} = -2\frac{\partial^2 Y_0}{\partial X \partial x} - a\frac{\partial Y_0}{\partial x} - bY_0$$
$$= -[aA_1'(x) + bA_1(x)] + [aA_2'(x) + bA_2(x)]e^{-ax}$$

To eliminate secular term that grows like X. (Characteristic roots $\lambda_1 = 0,\ \lambda_2 = -a$) We need $aA_1'(x) + bA_1(x) = 0 \Rightarrow A_1(x) = C_1 e^{-\frac{b}{a}x}$. Note that to eliminate secular term Xe^{-X}, we need $aA_2' - bA_2(x) = 0$.

But it is not necessary to do this since Xe^{-X} decays exponentially with X.

$$Y_0(X,x) = C_1 e^{-\frac{b}{a}x} + A_2(x)e^{-\frac{a}{\varepsilon}x}$$

$$A_2(x)e^{-\frac{a}{\varepsilon}x} = A_2(\varepsilon X)e^{-aX} = A_2(0)e^{-aX} + O(\varepsilon)$$

Set $A_2(0) = C_2 \Rightarrow$

$$y(x) \sim C_1 e^{-\frac{b}{a}x} + C_2(x)e^{-\frac{a}{\varepsilon}x} + O(\varepsilon)$$
$$BC \Rightarrow C_1 = Be^{b/a}, \quad C_2 = A - Be^{b/a}$$

Multiple scale analysis $\begin{cases} t: & \text{factly varying} \\ \varepsilon t: & \text{slouly varying} \end{cases}$

With secular terms at $O(\varepsilon)$: leading order approximation $t \ll \frac{1}{\varepsilon}$

without secular terms at $O(\varepsilon)$: leading order approximation $t \ll \frac{1}{\varepsilon}$

Sometimes: $\frac{t}{\varepsilon}$: fastly varying e.g. microscopic atomistic model. t: slowly varuing e.g. macroscopic, continuous model.

With secular terms at $O(\varepsilon)$: leading order approximation $t \ll 1$

With secular terms at $O(\varepsilon)$: leading order approximation $t \ll \frac{1}{\varepsilon}$

5.5 Method of averaging

Example 5.5.1: $\begin{cases} \varphi_t = \frac{1}{\varepsilon}w(I) + f(\varphi, I) & \varphi: \textit{fastly varying} \\ I_t = g(\varphi, I) & I: \textit{slowly varying} \end{cases}$

We have interested in the leading order behavior of I. Multiple scales: t, $\tau = \frac{t}{\varepsilon}$

f, g: period in variable φ period 2π

I: sub-linear growth, i.e. there exists a constant α, $0 \le \alpha \le 1$, s.t. $|I(t)| \le C|t|^\alpha$.

As $\varepsilon \to 0$,

$$\varphi \sim \varphi_0(t,\tau) + \varepsilon\varphi_1(t,\tau) + \varepsilon^2\varphi_2(t,\tau) + \cdots$$
$$I \sim I_0(t,\tau) + \varepsilon I_1(t,\tau) + \varepsilon^2 I_2(t,\tau) + \cdots$$
$$\varphi_t = \frac{\partial\varphi}{\partial t} + \frac{1}{\varepsilon}\frac{\partial\varphi}{\partial\tau} \qquad I_t = \frac{\partial I}{\partial t} + \frac{1}{\tau}\frac{\partial I}{\partial\tau}$$

and

$$w(I) \sim w(I_0) + \varepsilon w'(I_0)I_1 + \cdots$$
$$f(\varphi_1, I) \sim f(\varphi_0, I) + \varepsilon\left(\frac{\partial f}{\partial\varphi}(\varphi_0, I_0)\varphi_1 + \frac{\partial f}{\partial I}(\varphi_0, I_0)I_1\right) + \cdots$$
$$g(\varphi, I) \sim g(\varphi_0, I_0) + \varepsilon\left(\frac{\partial g}{\partial\varphi}(\varphi_0, I_0)\varphi_1 + \frac{\partial g}{\partial I}(\varphi_0, I_0)I_1\right) + \cdots$$
$$O(\frac{1}{\varepsilon}) : \begin{cases} \frac{\partial\varphi_0}{\partial\tau} = w(I_0) \\ \frac{\partial I_0}{\partial\tau} = 0 \end{cases} \Rightarrow I_0 = I_0(t),\ \varphi_0(t,\tau) = w(I_0(t))\tau + A(t)$$
$$O(1) : \begin{cases} \dfrac{\partial\varphi_0}{\partial t} + \dfrac{\partial\varphi_1}{\partial\tau} = w'(I_0)I_1 + f(\varphi_0, I_0) & (5.4) \\ \dfrac{\partial I_0}{\partial t} + \dfrac{\partial I_1}{\partial\tau} = g(\varphi_0, I_0) & (5.5) \end{cases}$$
$$(5.5) \Rightarrow I_0'(t) + \frac{\partial I_1}{\partial t} = g(w(I_0(t))\tau + A(t), I_0(t))$$

Perform average over fast varying scale τ, for the second equation

$$I_0'(t) = -\frac{1}{T}\int_0^T \frac{\partial I_1}{\partial\tau}d\tau + \frac{1}{T}\int_0^T g(\varphi_0, I_0)d\tau$$
$$\begin{aligned} \frac{1}{T}\int_0^T \frac{\partial I_1}{\partial\tau}d\tau =& \frac{I_1(t,\tau) - I_1(t,0)}{T} \to 0 \ \ as\ T \to +\infty \ \ (\mathrm{sub-linear}) \\ &\le C\frac{T^\alpha}{T} \to 0 \end{aligned}$$
$$\frac{1}{T}\int_0^T g d\tau \to \frac{1}{2\pi}\int_0^{2\pi} g(z, I_0(t))dz \ \ as\ T \to +\infty \ \ \ (\mathrm{periodicity})$$
$$I_0'(t) = \frac{1}{2\pi}\int_0^{2\pi} g(z, I_0(t))dz$$

$$\begin{aligned} (*).\ \frac{1}{T}\int_0^T g(w\tau + A, I_0)d\tau &\overset{Z=w\tau+A}{=} \frac{1}{wT}\int_0^T g(Z, I_0)dZ\ \ (\left[\frac{wT}{2\pi}\right] = n,\ \frac{wT}{2\pi} = n + \beta,\ \alpha \le \beta < 1) \\ &= \frac{1}{2\pi(n+\beta)}(\int_A^{2\pi n + A} g(Z, I_0)dZ + \int_{2\pi n + A}^{2\pi n + A + 2\pi\beta} g(Z, I_0)dZ) \\ &= \frac{n}{2\pi(n+\beta)}\int_0^{2\pi} g(Z, I_0)dZ + \frac{n}{2\pi(n+\beta)}\int_A^{A+2\pi\beta} g dZ \end{aligned}$$
$$\left|\int_A^{A+2\pi\beta} g dZ\right| \le \int_A^{A+2\pi\beta} |g| dZ \le \int_0^{2\pi} |g| dZ \le C$$

5.6 Exercise 5

1. Use the method of strained coordinates to find the leading order approxi- mation to the solution of the following problem so that there is no secular termat $O(\epsilon)$:

$$\begin{cases} \dfrac{d^2y}{dt^2} + 9y = \varepsilon y(\dfrac{dy}{dt})^2, \quad 0 \leq t \leq 1, \\ y(0) = 1, y'(0) = 0. \end{cases}$$

2. Use multiple scale analysis to find the leading order approximation to the solution of the following problem that is valid for large t :

$$\begin{cases} \dfrac{d^2y}{dt^2} + \varepsilon(\dfrac{dy}{dt})^3 + y = 0, \quad 0 \leq t \leq 1, \\ y(0) = 0, y'(0) = 1. \end{cases}$$

3. Consider the following problem with slowly-varying coefficient:

$$\begin{cases} \dfrac{d}{dt}(D(\varepsilon t)\dfrac{dy}{dt}) + y = 0, \quad t > 0, \\ y(0) = \alpha, y'(0) = \beta. \end{cases}$$

where $D = D(\tau)$ is a smooth positive function with $D' > 0$.
(1). Find a first-term(leading order) approximation of the solution valid for large t by using multiple scale analysis.
(2). Can this problem be solved by using WKB theory? If yes, please do it and compare the result with what you obtained in part 1.

4. (1). Show that for a given periodic function $g(y)$ with period 2π, the equation $\frac{dz}{dy} = g(y)$ has a periodic solution $z(y)$ with period 2π if and only if$\int_0^{2\pi} g(y)dy = 0$, assuming that $g(y)$ and $z(y)$ are smooth.
(2). Find the homogenized equation of the problem

$$\begin{cases} \dfrac{d}{dx}(D(x, \dfrac{x}{\varepsilon})\dfrac{du}{dx}) = f(x, \dfrac{x}{\varepsilon}), \quad 0 \leq x \leq 1, \\ u(0) = a, u(1) = b. \end{cases}$$

where $D(x, y)$ and $f(x, y)$ are periodic in y with period 2π, and $0 < D_m(x) \leq D(x, y) \leq D_M(x)$, for $0 \leq x \leq 1$ and $0 \leq y \leq 2\pi$.

5.7 Answer 5

1. Use the method of strained coordinates to find the leading order approxi- mation to the solution of the following problem so that there is no secular termat $O(\epsilon)$:

$$\begin{cases} \dfrac{d^2y}{dt^2} + 9y = \varepsilon y(\dfrac{dy}{dt})^2, \quad 0 \le t \le 1, \\ y(0) = 1, y'(0) = 0. \end{cases}$$

Solution: Use the method of strained coordinates to find the leading order approximation. Set $\tau = (1 + \omega_1\varepsilon)t$, and $y = y(\tau) \sim \sum_{i=0}^{\infty} \varepsilon y_i(\tau)$. Subsituting this into the equation, then we get:

$$\begin{aligned} O(1): &\quad \frac{d^2y_0}{d\tau^2} + 9y_0 = 0 \\ O(\varepsilon): &\quad \frac{d^2y_1}{d\tau^2} + 2\omega_1\frac{d^2y_0}{d\tau^2} + 9y_1 - y_0(\frac{dy_0}{d\tau})^2 = 0. \end{aligned}$$

In $O(1)$, we can get the solution $y_0(\tau) = cos(3\tau)$. Subsituting this into the equation in $O(\varepsilon)$:

$$\begin{aligned} \frac{d^2y_1}{d\tau^2} + 9y_1 =& y_0(\frac{dy_0}{d\tau})^2 - 2\omega_1\frac{d^2y_0}{d\tau^2} \\ =& 9sin^2(3\tau)cos(3\tau) + 18\omega_1 cos(3\tau) \\ =& 9cos(3\tau) + 18\omega_1 cos(3\tau) - 9cos^3(3\tau) \\ =& 9cos(3\tau) + 18\omega_1 cos(3\tau) - 9[\frac{1}{2}(cos(3\tau) + cos(9\tau))] \\ =& (18\omega_1 + \frac{9}{4})cos(3\tau) - \frac{9}{4}cos(9\tau) \end{aligned}$$

Then $18\omega_1 + \frac{9}{4} = 0 \quad \Rightarrow \quad \omega_1 = -\frac{1}{8}$, the leading order approximation of this equation is

$$y_0 = cos[3(1 - \frac{1}{8}\varepsilon)t].$$

2. Use multiple scale analysis to find the leading order approximation to the solution of the following problem that is valid for large t :

$$\begin{cases} \dfrac{d^2y}{dt^2} + \varepsilon(\dfrac{dy}{dt})^3 + y = 0, \quad 0 \le t \le 1, \\ y(0) = 0, y'(0) = 1. \end{cases}$$

Solution: This question is for large t, so using multiple-scale analysis. Firstly we assume a perturbation expansion for $y(t)$ of the form

$$y(t) \sim Y_0(t, \tau) + \varepsilon Y_1(t, \tau) + ..., \qquad \varepsilon \to 0+$$

where $\tau = \varepsilon t$, so $\frac{d\tau}{dt} = \varepsilon$. Then we get y_t and y_{tt}:

$$\begin{aligned}
\frac{dy}{dt} &= (\frac{\partial Y_0}{\partial t} + \frac{\partial Y_0}{\partial \tau}\frac{d\tau}{dt}) + \varepsilon(\frac{\partial Y_1}{\partial t} + \frac{\partial Y_1}{\partial t}\frac{d\tau}{dt}) + ... \quad = \frac{\partial Y_0}{\partial t} + \varepsilon(\frac{\partial Y_0}{\partial \tau} + \frac{\partial Y_1}{\partial t}) + O(\varepsilon^2) \\
\frac{d^2y}{dt^2} &= \frac{\partial^2 Y_0}{\partial t^2} + \varepsilon(2\frac{\partial^2 Y_0}{\partial \tau \partial t} + \frac{\partial^2 Y_1}{\partial t^2}) + O(\varepsilon^2).
\end{aligned}$$

Subsituting this into the equation gives two equations in $O(1)$ and $O(\varepsilon)$:

$$\begin{aligned}
O(1): \quad & \frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0 \\
O(\varepsilon): \quad & \frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -2\frac{\partial^2 Y_0}{\partial t \partial \tau} - (\frac{\partial Y_0}{\partial t})^3.
\end{aligned}$$

In the equation from $O(1)$, we can draw a general real solution

$$Y_0(t, \tau) = A(\tau)e^{it} + A^*(\tau)e^{-it}.$$

Subsituting this into the second equation gives

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -e^{it}[2i\frac{dA}{d\tau} + 3iA^2A^*] - e^{-it}[-2i\frac{dA^*}{d\tau} - 3i(A^*)^2A] + ie^{3it}A^3 - ie^{-3it}(A^*)^3.$$

Since the Y_1 is not secular, the secular term should equal to zero:

$$2i\frac{dA}{d\tau} + 3iA^2A^* = -2i\frac{dA^*}{d\tau} - 3i(A^*)^2A = 0.$$

Set $A(\tau) = R(\tau)e^{i\theta(\tau)}$, where $R(\tau)$ and $\theta(\tau)$ are real. Subsitituting this expression into the equation gives $R(\tau)$ and $\theta(\tau)$:

$$\begin{cases} \frac{dR}{d\tau} = -\frac{3}{2}R^3 \\ \frac{d\theta}{d\tau} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} R(\tau) = \frac{R(0)}{\sqrt{3\tau R(0)+1}} \\ \theta(\tau) = \theta(0) \end{cases}$$

$R(0)$ and $\theta(0)$ are determined by the initial conditions:

$$\begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases} \quad \Rightarrow \quad \begin{cases} Y_0(0,0) = 0 \\ \frac{\partial Y_0}{\partial t}(0,0) = 1 \end{cases} \quad \Rightarrow \quad \begin{cases} R(0)(e^{i\theta(0)} + e^{-i\theta(0)}) = 0 \\ iR(0)(e^{i\theta(0)} - e^{-i\theta(0)}) = 1 \end{cases} \quad \Rightarrow \quad \begin{cases} R(0) = -\frac{1}{2} \\ \theta(0) = \frac{\pi}{2} \end{cases}.$$

Thus, to leading order in ε,

$$\begin{aligned}
y(t) \sim & \frac{-\frac{1}{2}}{\sqrt{\frac{3}{4}\tau + 1}} e^{i\frac{\pi}{2}} e^{it} + \frac{-\frac{1}{2}}{\sqrt{\frac{3}{4}\tau + 1}} e^{-i\frac{\pi}{2}} e^{-it} \\
= & \frac{2sint}{\sqrt{3\varepsilon t + 4}}, \qquad \varepsilon \to 0+, \varepsilon t = O(1).
\end{aligned}$$

3. Consider the following problem with slowly-varying coefficient:

$$\begin{cases} \frac{d}{dt}(D(\varepsilon t)\frac{dy}{dt}) + y = 0, \quad t > 0, \\ y(0) = \alpha, y'(0) = \beta. \end{cases}$$

where $D = D(\tau)$ is a smooth positive function with $D' > 0$.
(1). Find a first-term(leading order) approximation of the solution valid for large t by using multiple scale analysis.
(2). Can this problem be solved by using WKB theory? If yes, please do it and compare the result with what you obtained in part 1.

Solution: (1).Let $T = f(t)$, the original equation transform to:

$$D(f')^2\frac{d^2y}{dT^2} + Df''\frac{dy}{DT} + \epsilon D'f'\frac{dy}{DT} + y = 0 \tag{1}$$

assume $D(f')^2 = 1$, we get $T = f(t) = \frac{1}{\epsilon}\int \frac{1}{\sqrt{D(s)}}ds$.
let $\tau = \epsilon t$,$y = Y(T,\tau) = Y_0(T,\tau)$ (1) transform to:

$$\frac{\partial^2 Y}{\partial T^2} + 2\epsilon\sqrt{D}\frac{\partial^2 Y}{\partial T\partial\tau} + \epsilon^2 D\frac{\partial^2 Y}{\partial\tau^2} + \epsilon\frac{D'}{2\sqrt{D}}\frac{\partial Y}{\partial T} + \epsilon^2\frac{D'}{2}\frac{\partial Y}{\partial\tau} + Y = 0$$

for O(1), we get $\frac{\partial^2 Y_0}{\partial T^2} + Y_0 = 0$. Therefore $Y_0 = Ae^{iT} + A^*e^{-iT}$
for O(ϵ):, we get $\frac{\partial^2 Y_1}{\partial T^2} + Y_1 = -2\sqrt{D}\frac{\partial^2 Y_0}{\partial T\partial\tau} - \frac{D'}{2\sqrt{D}}\frac{\partial Y_0}{\partial T}$
Notice:

$$-2\sqrt{D}\frac{\partial^2 Y_0}{\partial T\partial\tau} - \frac{D'}{2\sqrt{D}}\frac{\partial Y_0}{\partial T} = -(2\sqrt{D}A' + \frac{D'}{2\sqrt{D}}A)ie^{iT} + (2\sqrt{D}(A^*)' + \frac{D'}{2\sqrt{D}}(A^*))ie^{-iT}$$

in order to eliminate the secular term, let $(2\sqrt{D}A' + \frac{D'}{2\sqrt{D}}A) = 0$, we get $A = CD^{-\frac{1}{4}}$.
so $Y_0 = CD^{-\frac{1}{4}}e^{iT} + CD^{-\frac{1}{4}}e^{-iT}$
no matter to assume $Y_0 = (C_1 cosT_0 + C_2 sinT_0)D^{-\frac{1}{4}}$, where $T_0 = \int_0^t \frac{1}{\sqrt{D(\epsilon s)}}ds$
because $Y_0(0) = \alpha, Y_0'(0) = \beta$, we can get:

$$y \sim (\alpha D^{\frac{1}{4}}(0)cosT_0 + \beta D^{\frac{3}{4}}(0)sinT_0)D^{-\frac{1}{4}}(\epsilon t)$$

(2). Let $\tau = \epsilon t$, the original equation transform to:

$$\epsilon^2 D(\tau)\frac{d^2y}{d\tau^2} + \epsilon^2 D'(\tau)\frac{dy}{d\tau} + y = 0 \tag{2}$$

assume $y = e^{S(x)}$, (2) transform to:

$$\epsilon^2 D(S')^2 + \epsilon^2 DS'' + \epsilon^2 D'S' + 1 = 0$$

let $S = \frac{S_0}{\epsilon} + S_1 + \cdots$,
for O(1): $D(S_0')^2 + 1 = 0$, we get $S_0 = \pm i\int_0^t \frac{1}{D(s)}ds + C$
for O(ϵ): $2DS_0'S_1' + DS_0'' + D'S_0' = 0$, we get $S_1 = C_2 lnD^{-\frac{1}{4}}$.
therefore $y \sim e^{S(x)} = A_1 D^{-\frac{1}{4}}cosT_0 + A_D^{-\frac{1}{4}}sinT_0$, T_0 and T_1 are constant.
because $y(0) = \alpha, y'(0) = \beta$, we can get:

$$y \sim (\alpha D^{\frac{1}{4}}(0)cosT_0 + \beta D^{\frac{3}{4}}(0)sinT_0)D^{-\frac{1}{4}}(\epsilon t)$$

it is the same as the part 1.

4. (1). Show that for a given periodic function $g(y)$ with period 2π, the equation $\frac{dz}{dy} = g(y)$ has a periodic solution $z(y)$ with period 2π if and only if $\int_0^{2\pi} g(y)dy = 0$, assuming that $g(y)$ and $z(y)$ are smooth.
(2). Find the homogenized equation of the problem

$$\begin{cases} \frac{d}{dx}(D(x,\frac{x}{\varepsilon})\frac{du}{dx}) = f(x,\frac{x}{\varepsilon}), \quad 0 \le x \le 1, \\ u(0) = a, u(1) = b. \end{cases}$$

where $D(x,y)$ and $f(x,y)$ are periodic in y with period 2π, and $0 < D_m(x) \le D(x,y) \le D_M(x)$, for $0 \le x \le 1$ and $0 \le y \le 2\pi$.

Solution: (1). Firstly, we prove ($\Rightarrow$). Since $z(y)$ is with period 2π,

$$\int_0^{2\pi} g(y)dy = \int_0^{2\pi} \frac{dz}{dy}dy = \int_0^{2\pi} dz = z(2\pi) - z(0) = 0.$$

For ($\Leftarrow$), since $g(y)$ is with period 2π:

$$\int_x^{x+2\pi} g(y)dy = \int_0^{2\pi} g(y)dy = 0 = z(2\pi + x) + z(x) = 0.$$

This implies $z(y)$ is with period 2π.

(2). Set $y = \frac{x}{\varepsilon}$, so $\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}$, then we get:

$$(\frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y})[D(x,y)(\frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y})u] = f(x,y)$$
$$u \sim u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + ...$$

(a). In $O(\frac{1}{\varepsilon^2})$, we get the equation:

$$\frac{\partial}{\partial y}(D(x,y)\frac{\partial u_0}{\partial y}) = 0 \quad \Rightarrow \quad u_0 = C_1(x) + C_0 \int_{y_0}^{y} \frac{ds}{D(x,s)}$$

In fact, as $y \to +\infty$,

$$0 < \frac{1}{D_M(x)} \le \frac{1}{D(x,s)} \le \frac{1}{D_m(x)} < +\infty \quad \Rightarrow \quad \frac{y - y_0}{D_M(x)} \le \int_{y_0}^{y} \frac{ds}{D(x,s)} \le \frac{y - y_0}{D_m(x)}$$

In order to look for bounded problem, $C_0 = 0$, which implies $u_0(x,y) = u_0(x)$.

(b). In $O(\frac{1}{\varepsilon})$, we get the equation:

$$\frac{\partial}{\partial y}(D(x,y)\frac{\partial u_1}{\partial y}) = -\frac{\partial}{\partial y}(D\frac{du_0}{dx}) \quad \Rightarrow \quad D\frac{\partial u_1}{\partial y} = -D\frac{du_0}{dx} + b_0(x)$$

This equation can gives $u_1(x,y)$:

$$u_1(x,y) = b_1(x) + b_0 \int_{y_0}^{y} \frac{ds}{D(x,s)} - (y - y_0)\frac{du_0}{dx}$$

Since the solution should be bounded, this condition should be satisfied:

$$\lim_{y\to+\infty} \frac{b_0(x)\int_{y_0}^{y} \frac{ds}{D(x,s)}}{(y - y_0)\frac{du_0}{dx}} = 1 \quad \Rightarrow \quad \frac{du_0}{dx} = b_0(x) \cdot \lim_{y\to+\infty} (y - y_0)\int_{y_0}^{y} \frac{ds}{D(x,s)}$$

(c). In $O(1)$, we get the equation:

$$\frac{\partial}{\partial y}(D\frac{\partial u_2}{\partial y}) = f(x,y) - \frac{\partial}{\partial x}(D\frac{\partial u_1}{\partial y}) - \frac{\partial}{\partial y}(D\frac{\partial u_1}{\partial x}) - \frac{\partial}{\partial x}(D\frac{\partial u_1}{\partial x}) = f(x,y) - b_0'(x) - \frac{\partial}{\partial x}(D\frac{\partial u_1}{\partial x}).$$

So we can solve $u_2(x,y)$:

$$u_2(x,y) = d_1(x) + d_0(x)\int_{y_0}^{y}\frac{ds}{D(x,s)} + \int_{y_0}^{y}\frac{\partial u_1}{\partial x}ds + \int_{y_0}^{y}(\frac{\int_{s_0}^{s} f(x,t)dt}{D(x,s)} - b_0'(x)\frac{s}{D(x,s)})ds.$$

In oder to prevent the secular term, let

$$\lim_{s\to+\infty}\frac{\frac{\int_{s_0}^{s} f(x,t)dt}{D(x,s)} - b_0'(x)\frac{s}{D(x,s)}}{s} = 0$$

By L' Hospital, we get:

$$b_0'(x) = \lim_{y\to+\infty}\frac{\int_{y_0}^{y} f(x,t)dt}{y} = \frac{1}{2\pi}\int_0^{2\pi} f(x,s)ds := < f >_\infty$$

Form the result in **b** and **c** , we can draw the homogenized equation and homogenized coefficient:

$$\begin{aligned}
&\frac{d}{dx}(\bar{D}(x)\frac{du_0}{dx}) = < f >_\infty \\
&\bar{D}(x) = \frac{1}{\lim_{y\to+\infty}\frac{1}{y-y_0}\int_{y_0}^{y}\frac{ds}{D(x,s)}} = \frac{2\pi}{\int_0^{2\pi}\frac{1}{D(x,s)}ds}.
\end{aligned}$$

This equation can be solved with the bounded condition $u(0) = a$ and $u(1) = b$.

Chapter **6**

Homogenization Method

6.1 Background

It is common in engineering and scientific problems to have to deal with materials that are formed from multiple constituents. Solving a mathematical problem that includes such variations in the structure can be very difficult. It is therefore natural to try to find simpler equations that effectively smooth out whatever substructure variations there may be. We give an example to illustrate it:

$$D : (D(x_1, x_2)\nabla T) = f(x_1, x_2),$$

where T is temperature, $D(x_1, x_2)$ is conductivity, and $f(x_1, x_2)$ is heat source.

For composite material

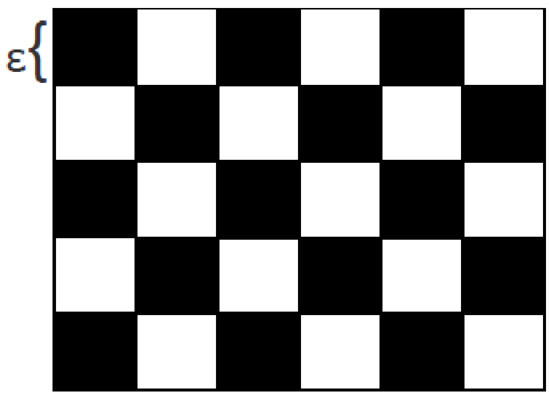


figure 6.1.1

Conductivity: $D = D(x_1, x_2, \frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})$ varies fast near length scale of $O(\varepsilon)$.

Question:
(1) What is the property of T, when $\varepsilon \to 0$?
(2) Can we find an effective conductivity $\bar{D}(x_1, x_2)$ s.t. as $\varepsilon \to 0$, T satisfies $\nabla(\bar{D}(x_1, x_2)\nabla T) = f(x_1, x_2)$ –Homogenation problem.

6.2 1D problem

To illustrate it, we first consider the 1D problem.

$$\begin{cases} \frac{d}{dx}(D(x,\frac{x}{\varepsilon})\frac{du}{dx}) = f(x), \ \ 0 \le y \le +\infty \\ u(0) = a, \ u(1) = b \end{cases}$$

We assume that

$$\begin{cases} y = \frac{x}{\varepsilon} : \ microscale \ \ 0 \le y < +\infty \\ x : \qquad macroscale \ \ 0 \le x \le 1 \end{cases}$$

Assumptions: There exist $D_m(x)$, $D_u(x)$ s.t. $0 < D_m(x) \le D(x,y) \le D_u(x)$.

Multiple scale analysis: $\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}$

$$(\frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y})(D(x,y)(\frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y})u) = f(x)$$

Let $u \sim u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \cdots, \varepsilon \to 0$

$$O(\frac{1}{\varepsilon^2}) : \frac{\partial}{\partial y}(D(x,y)\frac{\partial u_0}{\partial y}) = 0 \Rightarrow D\frac{\partial u_0}{\partial y} = C_0(x)$$

$$\Rightarrow u_0(x,y) = C_1(x) + C_0(x)\int_{y_0}^{y}\frac{ds}{D(x,s)}$$

In fact, $0 < \frac{1}{D_M(x)} \le \frac{1}{D(x,s)} \le \frac{1}{D_m(x)}$

$$\Rightarrow \frac{y-y_0}{D_M(x)} \le \int_{y_0}^{y}\frac{ds}{D(x,s)} \le \frac{y-y_0}{D_m(x)} \Rightarrow \lim_{y\to+\infty}\int_{y_0}^{y}\frac{ds}{D(x,s)} = +\infty$$

We look for bounded solution, thus $C_0(x) \equiv 0$

$\therefore u_0 = u_0(x)$

$$O(\frac{1}{\varepsilon}) : \ \frac{\partial}{\partial y}(D\frac{\partial u_1}{\partial y}) = -\frac{\partial}{\partial y}(D\frac{du}{dx}) \ \ (\mathcal{L}_y u_1 = \mathcal{K} u_0)$$

$$\Rightarrow D\frac{\partial u_1}{\partial y} = -D\frac{du_0}{dx} + b_0(x) \Rightarrow \frac{\partial u_1}{\partial y} = -\frac{du_0}{dx} + \frac{b_0(x)}{D(x,y)}$$

$$\Rightarrow u_1(x,y) = b_1(x) + b_0(x)\int_{y_0}^{y}\frac{ds}{D(x,s)} - (y-y_0)\frac{du_0}{dx}$$

For bounded $u_1(x,y)$, they must cancel out as $y \to +\infty$

$$\therefore \lim_{y\to+\infty}\frac{b_0(x)\int_{y_0}^{y}\frac{ds}{D(x,s)}}{(y-y_0)\frac{du_0}{dx}} = 1 \Rightarrow \frac{du_0}{dx} = \lim_{y\to+\infty}\frac{b_0(x)}{(y-y_0)}\int_{y_0}^{y}\frac{ds}{D(x,s)} \tag{6.1}$$

where, $b_0(x)$ to be determined.

$$O(1) : \ \frac{\partial}{\partial y}(D\frac{\partial u_2}{\partial y}) = f(x) - \frac{\partial}{\partial x}(D\frac{\partial u_1}{\partial y}) - \frac{\partial}{\partial y}(D\frac{\partial u_1}{\partial x}) - \frac{\partial}{\partial x}(Du_0'(x))$$

$$(\mathcal{L}_y u_2 = \mathcal{H} u_1 + \mathcal{J} u_0 + f(x))$$

since $\frac{\partial}{\partial x}(D\frac{\partial u_1}{\partial y}) + \frac{\partial}{\partial x}(Du_0'(x)) = \frac{\partial}{\partial x}(D\frac{\partial u_1}{\partial y} + D\frac{du_0}{dx}) = b_0'(x)$

$\Rightarrow \frac{\partial}{\partial y}(D\frac{\partial u_2}{\partial y}) = f(x) - \frac{\partial}{\partial y}(D\frac{\partial u_1}{\partial x}) - b_0'(x)$

$$\therefore D\frac{\partial u_2}{\partial y} = d_0(x) + [f(x) - b_0'(x)](y - y_0) - D\frac{\partial u_1}{\partial x}$$

$$u_2(x,y) = d_1(x) + d_0(x)\int_{y_0}^{y}\frac{ds}{D(x,s)} + [f(x) - b_0'(x)]\int_{y_0}^{y}\frac{s-y_0}{D(x,s)}ds - \int_{y_0}^{y}\frac{\partial u_1}{\partial x}(x,s)ds$$

$$\int_{y_0}^{y}\frac{s-y_0}{D(x,s)}ds \geq \int_{y_0}^{y}\frac{s-y_0}{D_\mu(x)}ds = \frac{1}{2D_\mu(x)}(y-y_0)^2$$

$$d_0(x)\int_{y_0}^{y}\frac{ds}{D(x,s)} = O(y), \quad \int_{y_0}^{y}\frac{\partial u_1}{\partial x}(x,s)ds = O(y) \quad as\ y \to +\infty$$

$$u_1(x,y) \to b_1(x), \ \frac{\partial u_1}{\partial x} \to b_1'(x), \ as\ y \to +\infty$$

Cannot cancel $O(y^2)$ terms as $y \to +\infty$. So

$$f(x) = b_0'(x) \tag{6.2}$$

to cancel $O(y^2)$ term.

From (6.1) and (6.2),

$$\frac{d}{dx}(\bar{D}(x)\frac{du_0}{dx}) = f(x) \tag{6.3}$$

where,

$$\bar{D}(x) = \frac{1}{\lim\limits_{y\to+\infty}\frac{1}{y-y_0}\int_{y_0}^{y}\frac{ds}{D(x,s)}} \tag{6.4}$$

Eq (6.3) can be solved with BC: $u_0(0) = a, \ u_1(1) = b$. Eq (6.3) is called the homogenited eq. containing no fast-varying scale. $\bar{D}(x)$ is called the effective chomogenited coefficient.

$\bar{D}(x)$ is sometimes called effective conductivity for heat conduction problem.
Properties of $\bar{D}(x)$:
1. If $\lim\limits_{y\to+\infty} D(x,y) = D_\infty(x)$, then

$$\lim_{y\to+\infty}\frac{\int_{y_0}^{y}\frac{ds}{D(x,s)}}{y-y_0} \overset{L'Hospital's\ Rule}{=} \lim_{y\to+\infty}\frac{1}{D(x,y)} = \frac{1}{D_\infty(x)} = \bar{D}(x) = D_\infty(x)$$

2. If $D(x,y)$ is periodic in y and $D(x, y+y_p) = D(x,y)$

$$\lim_{y\to+\infty}\frac{1}{y-y_0}\int_{y_0}^{y}\frac{ds}{D(x,s)} = \frac{1}{y_p}\int_0^{y_p}\frac{ds}{D(x,s)} \Rightarrow \bar{D}(x) = \frac{y_p}{\int_0^{y_p}\frac{ds}{D(x,s)}}$$

Remark 6.2.1:
In the periodic case, by Fredholm Alternative:

i) $\mathcal{L}_y u_1 = \mathcal{K} u_0$ has solution iff $(K u_0, v) = 0$ for $V u \in V(\mathcal{L}_y^*)$. v =constant, since $N(L_y^*)$ is 1-D, then $\frac{1}{y_p}\int_0^{y_p} \mathcal{K} u_0 ds = 0$

$$\frac{1}{y_p}\int_0^{y_p} K u_0 ds = -D(x, y_p)u_0'(x) + D(x,0)u_0'(x) = 0 \;\; by\, periodicity$$

ii) $\mathcal{L}_y u_2 = \mathcal{H} u_1 + \mathcal{J} u_0 + f(x)$ has solution iff $(\mathcal{H} u_1 + \mathcal{J} u_0 + f(x), v)_{\mathcal{L}^2_{per}} = 0$ for $\forall\, v \in N(\mathcal{L}_y^*)$.

$$\Rightarrow f(x) = \frac{1}{y_p}\int_0^{y_p} \frac{\partial}{\partial x}[D(\frac{\partial u_1}{\partial y} + u_0'(x))]dy + \frac{1}{y_p}\int_0^{y_p} \frac{\partial}{\partial y}[D\frac{\partial u_1}{\partial x}]dy = \frac{1}{y_p}\int_0^{y_p} b_0'(x)dy = b_0'(x)$$

6.3 Multidimensional problems

In this section, we consider the multidimensional Problem:

$$\begin{cases} \nabla \cdot (D(\vec{x}, \frac{\vec{x}}{\varepsilon})\nabla u) = f(\vec{x}) & \vec{x} \in \Omega \\ u = g(\vec{x}),\ \vec{x} \in \partial\Omega \end{cases}$$

Assumption: There exist two constant $\lambda, \Lambda > 0$, s.t. $\lambda \le D(\vec{x}, \frac{\vec{x}}{\varepsilon}) \le \Lambda$.

Let $\vec{y} = \frac{\vec{x}}{\varepsilon}$, $D(\vec{x}, \vec{y})$ is periodic in $\vec{y}$. i.e. $\exists \vec{y}_p$ s.t. $D(\vec{x}, \vec{y} + \vec{y}_p) = D(\vec{x}, \vec{y})$, $\vec{y}_p = (p_1, p_2)$.

Goal: To find an effective $\bar{D}(\vec{x})$ s.t. as $\varepsilon \to 0$, the solution of the original problem can be approximated by the sol of the homegenited eg. $\nabla \cdot (\bar{D}(\vec{x})\nabla u) = f(\vec{x})$.

Using variable $\vec{x}$ and $\vec{y}$, $\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y$. Original equation $\Rightarrow (\nabla_x + \frac{1}{\varepsilon}\nabla_y)[D(\nabla_x + \frac{1}{\varepsilon}\nabla_y)u] = f(\vec{x})$

$$u(\vec{x}, \vec{y}) \sim u_0(\vec{x}, \vec{y}) + \varepsilon u_1(\vec{x}, \vec{y}) + \varepsilon^2 u_2(\vec{x}, \vec{y}) + \cdots, \varepsilon \to 0$$

$$O(\frac{1}{\varepsilon^2}) : \nabla_y \cdot (D\nabla_y u_0) = 0$$

Multiply u_0 and integrate is over one cell Ω_0: $\int\limits_{\Omega_0} u_0 \nabla_y \cdot (D\nabla_y u_0) d\vec{y} = 0$, using integration by parts,

$$\int\limits_{\Omega_0} u_0 \nabla_y \cdot (D\nabla_y u_0) d\vec{y} = \int\limits_{\partial\Omega_0} \frac{\partial u_0}{\partial n_y}\nabla u_0 dy - \int\limits_{\Omega_0} D(\nabla_y u_0)^2 d\vec{y}$$

$$\nabla_y \cdot (D\nabla_y u_0) = D\nabla_y u_0 \cdot \nabla_y u_0 + u_0 \nabla_y \cdot (D\nabla_y u_0)$$

$$\int\limits_{\Omega_0} u_0 \nabla_y \cdot (D\nabla_y u_0) d\vec{y} = \int\limits_{\partial\Omega_0} u_0 D\vec{n}_y \cdot \nabla_y u_0 dl_y - \int\limits_{\Omega_0} D\|\nabla_y u_0\|^2 d\vec{y} = -\int\limits_{\Omega_0} D\|\nabla_y u_0\|^2 d\vec{y}$$

where, $\int\limits_{\partial\Omega_0} u_0 D\vec{n}_y \cdot \nabla_y u_0 dl_y = 0$ $(by\, periodicity)$.

$\therefore \int\limits_{\Omega_0} D\|\nabla_y u_0\|^2 d\vec{y} = 0 \Rightarrow \|\nabla_y u_0\| = 0 \Rightarrow \nabla_y u_0 = 0$

$\therefore u_0 = u_0(x)$

$$O(\frac{1}{\varepsilon}) : \nabla_y(D\nabla_y u_1) = -\nabla_y(D\nabla_x u_0) \overset{u_0=u_0(\vec{x})}{=} -\nabla_y D \cdot \nabla_x u_0$$

Method of superposition.

If $-\frac{\partial p}{\partial y_i}$ generates a solution a_i for the equation $\nabla_y(D\nabla_y a_i) = -\frac{\partial D}{\partial y_i}$ in Ω_0 for $i = 1, 2$, then $\nabla_y \cdot (D\nabla_y(u_1 - \sum\limits_{i=1}^{2} a_i \frac{\partial u_0}{\partial x_i})) = 0$.

Similar to above discussion, we have

$$u_1 - \sum a_i \frac{\partial u_0}{\partial x_i} = C(\vec{x}) \Rightarrow u_1 = \vec{a} \cdot \nabla_x u_0 + C(\vec{x})$$

$$O(1) : \nabla_y \cdot (D\nabla_y u_2) = -\nabla_y \cdot (D\nabla_x u_1) - \nabla_x \cdot (D\nabla_y u_1) - \nabla_x \cdot (D\nabla_x u_0) + f(x) \qquad (6.5)$$

$\therefore \frac{1}{|\Omega_0|} \int\limits_{\Omega_0} Eq.(6.5) d\vec{y}$

Donate $< v >_p = \frac{1}{|\Omega_0|} \int\limits_{\Omega_0} v(\vec{x}, \vec{y}) d\vec{y}$. Then $< \nabla_y \cdot (D\nabla_y u_2) >_p = 0$ by Gauss theory and periodic BC.

Similarly, $< \nabla_y \cdot (D\nabla_x u_1) >_p = 0$

$$\begin{aligned} < \nabla_x \cdot (D\nabla_y u_1) > &= \sum_{i=1}^{2} \frac{\partial}{\partial x_i} < \frac{\partial u_1}{\partial y_i} >_p \\ &\overset{u_1=\vec{a}\cdot\nabla_x u_0 + C(\vec{x})}{=} \sum_{i=1}^{2} \frac{\partial}{\partial x_i} (< \frac{\partial u_1}{\partial y_i} >_p \cdot \nabla_x u_0) \\ &= \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial}{\partial x_i} (< \frac{\partial a_i}{\partial y_i} >_p \cdot \frac{\partial u_0}{\partial x_j}) \end{aligned}$$

$$< \nabla_x \cdot (D\nabla_x u_0) >_p = \nabla_x \cdot (< D >_p \nabla_x u_0), \quad < f(\vec{x}) >_p = f(\vec{x})$$

$\therefore$ Eq(6.5) becomes $\sum\limits_{i,j=1}^{2} \frac{\partial}{\partial x_i} (\bar{D}_{ij}(\vec{x}) \frac{\partial u_0}{\partial x_j} = f(\vec{x})$. (homogenized equation)

where, $\bar{D}_{ij}(\vec{x}) =< D >_p \cdot \delta_{ij} + < \frac{\partial a_i}{\partial y_i} >_p$ and $\vec{a} = (a_1, a_2)$ is periodic and satisfies $\nabla_y \cdot (D\nabla_y a_i) = -\frac{\partial D}{\partial y_i}$ in Ω.

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

6.4 Dorous medium flow–Darcy's Law

Consider the flow of an incompressible viscous fluid through a porous medium, e.g. water or oil through sand.

Rs: solid region. Rf: fluid region

Stokes eg. $\begin{cases} \varepsilon^2 \Delta \vec{v} = \nabla p \\ \Delta \vec{v} = 0 \end{cases}$ ε^2 is used to reach the velocity $\vec{v}$ s.t. is has a limit.

L: macroscopic length scale. l: microscopic length scale foe the pore and substruture.

At fluid-solid interface R_{fs}, $v/R_\beta = 0$.

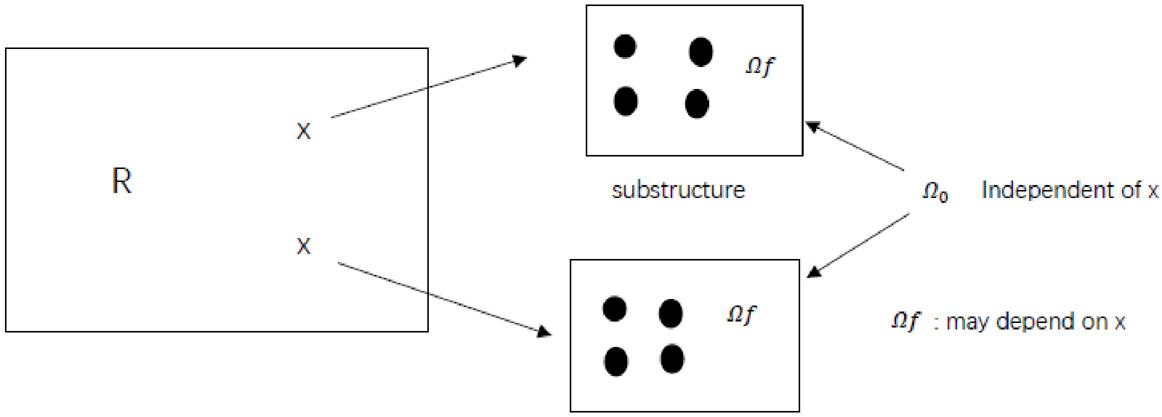


figure 6.4.1

Assume at each x on the macroscopic scale, the substructure is periodic.

Let $\vec{y} = \frac{\vec{x}}{\varepsilon}$, Eqs becomes $(\nabla = (\nabla_x + \frac{1}{\varepsilon}\nabla_y))$

$$\begin{cases} \varepsilon^2 (\nabla_x + \frac{1}{\varepsilon}\nabla_y)^2 \vec{v} = (\nabla_x + \frac{1}{\varepsilon}\nabla_y) p & (6.6) \\ (\nabla_x + \frac{1}{\varepsilon}\nabla_y)\vec{v} = 0 & (6.7) \end{cases}$$

$$\vec{v} \sim \vec{v}_0(\vec{x}, \vec{y}) + \varepsilon \vec{v}_1(\vec{x}, \vec{y}) + \cdots \qquad \varepsilon \to 0$$

$$p \sim p_0(\vec{x}, \vec{y}) + \varepsilon p_1(\vec{x}, \vec{y}) + \cdots \qquad \varepsilon \to 0$$

$\vec{v}_i$, p_{i} periodic in $\vec{y}$.

$O(\frac{1}{\varepsilon})$ in (6.6), $\nabla_y p_0 = 0 \Rightarrow p_0 = p_0(\vec{x})$
in (6.7) $\nabla_y p_0 = 0 \Rightarrow p_0 = p_0(\vec{x})$.

$O(1)$ in (6.6) $\nabla_y p_1 + \nabla_x p_0 = \nabla_y^2 \vec{v}_0$. solved for

$$y \in \Omega_f \tag{6.8}$$

in (6.7),

$$\nabla_y \vec{v}_1 + \nabla_x \vec{v}_0 = 0. \tag{6.9}$$

(6.8) is linear problem, $\vec{v}_0$ depends on y. $\nabla_x p_0$ does not depend on y, so we assume $\vec{v}_0 = \vec{B}\nabla_x p_0$, $p_1 = \vec{g}\nabla_x p_0$ (rescale by $\nabla_x p_0$). $\vec{B} \in R^{3\times 3}$, $\vec{g} \in R^{3\times 3}$, $B = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$, $g = (g_1, g_2, g_3)^T$.

For $i = 1, 2, 3$,

$$\begin{cases} \nabla_y g_i + \vec{e}_i = \nabla_y^2 \vec{b}_i & \textit{periodic BL on } \partial\Omega \\ \nabla_y \vec{b} = 0 & \vec{b}_i = 0 \textit{ on } \partial\Omega_{fs} \end{cases}$$

$\vec{b}_i$ can be written in terms of Green's function.

cell average: $< \zeta > (x) = \frac{1}{|\Omega_0|}\int\limits_{\Omega_f} \zeta(\vec{x}, \vec{y})d\vec{y}$, $< \vec{v}_0 >= -L\nabla_x p_0$, where, $L(x) = -\frac{1}{|\Omega_0|}\int\limits_{\Omega_f} \vec{B}d\vec{y}$.

Honogenuized equation for pressure is obtain by average (6.9).

$$\begin{aligned} O &= \int\limits_{\Omega_f} \nabla_y \vec{v}_1 + \nabla_x \vec{v}_0 d\vec{y} \\ &= \int\limits_{\partial\Omega_f} \vec{\eta}_y \vec{v}_1 d\vec{y} + \nabla_x \int\limits_{\Omega_f} \vec{v}_0 d\vec{y} - \sum_{i=1}^{3} \int\limits_{\Omega_f} v_{0i}(\frac{\partial \vec{r}}{\partial x_i}) \cdot \vec{\eta}_y dS_y \end{aligned} \tag{6.10}$$

$\vec{\eta}_y$ is outward normal to $\partial\Omega_f = \{\vec{r}(\vec{x}, \vec{y})\}$.

Reynelds Transport Theory:

$$f = f(\vec{x}, t) \qquad \Omega = \Omega(t)$$

$$\frac{d}{dt}\int\limits_{\Omega(t)} f(\vec{x}, t)d\vec{x} = \int\limits_{\Omega(t)} \frac{\partial f}{\partial t}d\vec{x} + \int\limits_{\Omega(t)} (\vec{v}^b, \vec{n})f dA$$

$\vec{v}^b$: velocity of surface element.

$\vec{n}$: outward unit normal.

Remark 6.4.1: ID →Leibmz Rule

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x, t)dx = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}dx + b'(t)f(b(t), t) - a'(t)f(a(t), t)$$

$t \to x_i \quad \vec{x} \to \vec{y} \quad \Rightarrow (5)$

no-slip BC$\Rightarrow$ $\vec{v}_1 = 0$, $\vec{v}_0 = 0$ on $\partial\Omega_f$.

(5)$\Rightarrow$ $\nabla_x < v_0 >= 0$

The homogenized problem:

$$\begin{cases} \vec{v}_h = -\vec{L}\nabla_x p_h & (Darcy's\ Law) \\ \nabla_x \cdot \vec{v}_h = 0 \end{cases}$$

$\vec{v}_h$ and p_h: on macroscopic domain depend only on x. $\vec{L}$: permeability tensor contains information of substructure to be obtained by solring cell problem.

6.5 Exercise 6

1. Consider the 1D transport equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - b(\dfrac{x}{\varepsilon})\dfrac{\partial u}{\partial x} = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = g(x). \end{cases}$$

where $b = b(y) > 0$ is smooth and periodic in y with period 1. Use the multiple scale expansion to find the homogenized equation for small $\varepsilon \leq 1$.

2. A classic model in the study of oscillatory systems is the Brusselator. The equations are

$$\begin{cases} \dfrac{dx}{dt} = \mu - (1+\alpha)x + x^2 y, \\ \dfrac{dy}{dt} = x - x^2 y. \end{cases}$$

Here μ and α are positive constants.
(1). Find the steady state and determine its stability in the case where $\alpha \geq 1$.
(2). Suppose $0 \leq \alpha \leq 1$ and μ is the bifurcation parameter. Determine the stability of the steady state and describe what happens near the point where the steady state loses stability.
(3). Assuming $\alpha = \frac{3}{4}$, use multiple scales to find the solution near the bifurcation point.

3. (Child's Swing) In a swing, a child can control his/her body to change the center of gravity. The problem can be modeled by a pendulum with a time-dependent length $l = l(t)$. The model is

$$\frac{d^2\theta}{dt^2} + \frac{2l'(t)}{l(t)}\frac{d\theta}{dt} + \sin\theta = 0, \quad t \geq 0$$

where θ is the angle between the swing and the vertical line. Suppose that $l = l_0(1 + \varepsilon \sin \omega t)$, where ε and ω (both nonnegative) are the amplitude and the frequency of movement controlled by the child, respectively.
(1). Consider the steady state $\theta_s = 0$. Use linear stability analysis to show that the first order perturbation of $\theta = \theta_s + \delta\nu + O(\delta^2)$ satisfies the equation:

$$\frac{d^2\nu}{dt^2} + \frac{2l'(t)}{l(t)}\frac{d\nu}{dt} + \nu = 0, \quad t \geq 0$$

(2). Show that if the child does nothing (so that $\omega = 0$), then the steady state $\theta_s = 0$ is stable.
(3). What the child must do is to find a frequency ω that makes the steady state $\theta_s = 0$ unstable (so the amplitude grows and the swing goes higher). Find such a frequency ω when $\varepsilon \ll 1$. Do this by using multiple scale analysis to find a leading order expansion of ν that is valid for large t .
Hint: You may need to use the following identity:

$$\sin 2\varphi = \frac{2\tan\varphi}{1+\tan^2\varphi}, \qquad \cos 2\varphi = \frac{1-\tan^2\varphi}{1+\tan^2\varphi}$$

6.6 Answer 6

1. Consider the 1D transport equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - b(\dfrac{x}{\varepsilon})\dfrac{\partial u}{\partial x} = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = g(x). \end{cases}$$

where $b = b(y) > 0$ is smooth and periodic in y with period 1. Use the multiple scale expansion to find the homogenized equation for small $\varepsilon \leq 1$.

Solution: let $y = \frac{x}{\epsilon}$, we get:

$$\epsilon\frac{\partial u}{\partial t} - b(y)(\epsilon\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}) = 0$$

for O(1):$-b(y)\frac{\partial u_0}{\partial y} = 0$, we get $u_0(x,y,t) = u_0(x,t)$.
for O(ϵ):$-b(y)\frac{\partial u_1}{\partial y} = -\frac{\partial u_0}{\partial t} + b(y)\frac{\partial u_0}{\partial x}$, we get :

$$u_1 = C_1(x,t) + \frac{\partial u_0}{\partial t}\int_0^y \frac{1}{b(s)}ds - y\frac{\partial u_0}{\partial x}$$

U_1 should be bounded in y, we get:

$$\lim_{y\to+\infty} \frac{\frac{\partial u_0}{\partial t}\int_0^y \frac{1}{b(s)}ds - y\frac{\partial u_0}{\partial x}}{y} = 0$$

so $\frac{\partial u_0}{\partial x} = \frac{\partial u_0}{\partial t} < b^{-1} >_\infty$, where $< b^{-1} >_\infty = \lim_{y\to+\infty} \frac{\int_0^y \frac{1}{b(s)}ds}{y} = \int_0^1 \frac{1}{b(s)}ds$. Therefore, the homogenized equation is:

$$\frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial x} < b^{-1} >_\infty^{-1} = 0$$

2. A classic model in the study of oscillatory systems is the Brusselator. The equations are

$$\begin{cases} \dfrac{dx}{dt} = \mu - (1+\alpha)x + x^2 y, \\ \dfrac{dy}{dt} = x - x^2 y. \end{cases}$$

Here μ and α are positive constants.
(1). Find the steady state and determine its stability in the case where $\alpha \geq 1$.
(2). Suppose $0 \leq \alpha \leq 1$ and μ is the bifurcation parameter. Determine the stability of the steady state and describe what happens near the point where the steady state loses stability.
(3). Assuming $\alpha = \frac{3}{4}$, use multiple scales to find the solution near the bifurcation point.

Solution: (1). steady state:

$$\begin{cases} \mu - (1+\alpha)x + x^2 y = 0 \\ x - x^2 y = 0 \end{cases}$$

we get steady state is$x=\frac{\mu}{\alpha}, y=\frac{\alpha}{\mu}$.

$$A := \nabla f = \begin{bmatrix} -(1+\alpha)+2xy & x^2 \\ 1-2xy & -x^2 \end{bmatrix} = \begin{bmatrix} 1-\alpha & (\frac{\mu}{\alpha})^2 \\ -1 & -(\frac{\mu}{\alpha})^2 \end{bmatrix}$$

because $tr(A) < 0, \det(A) > 0$, it is stable
(2).

$$|\lambda I - \nabla f| = \begin{vmatrix} \lambda+(1+\alpha)-2xy & -x^2 \\ -1+2xy & x^2 \end{vmatrix} = \lambda^2 + (\alpha - 1 + (\frac{\mu}{\alpha})^2)\lambda + \frac{\mu^2}{\alpha}$$

we have $\lambda_1\lambda_2 > 0, \lambda_1+\lambda_2 = -(\alpha-1+(\frac{\mu}{\alpha})^2)$
if $\lambda_1+\lambda_2 < 0$, it is stable.
if $\lambda_1+\lambda_2 > 0$, it is unstable.
Therefore:
if $\mu > \alpha\sqrt{1-\alpha}$, it is stable.
if $0 < \mu < \alpha\sqrt{1-\alpha}$, it is unstable.
near the point $\mu = \alpha\sqrt{1-\alpha}$, there will happen hopf bifurcation.
(3).when $\alpha = \frac{3}{4}$, the bifurcation point $\mu = \frac{3}{8}$
let $\mu = \frac{3}{8} - \epsilon$, then the steady state can transfrom to:

$$x_s = \frac{\mu}{\alpha} = \frac{4}{3}\mu = \frac{4}{3}(\frac{3}{8} - \epsilon) = \frac{1}{2} - \frac{4}{3}\epsilon$$
$$y_s = \frac{\alpha}{\mu} = \frac{3}{\frac{3}{2} - 4\epsilon} = \frac{2}{1 - \frac{8\epsilon}{3}} = 2(1 - \frac{8}{3}\epsilon + \frac{64}{9}\epsilon^2 + \cdots)$$

assume the solution near the bifurcation point is:

$$x \sim \frac{1}{2} + \sqrt{\epsilon}x_1 - \frac{4}{3}\epsilon + \epsilon^{\frac{3}{2}}x_2 + \cdots$$
$$y \sim 2 + \sqrt{\epsilon}y_1 - \frac{16}{3}\epsilon + \epsilon^{\frac{3}{2}}y_2 + \cdots$$

let $t_1 = t, t_2 = \sqrt{\epsilon}t$, we have

$$(\frac{\partial}{\partial t_1} + \sqrt{\epsilon}\frac{\partial}{\partial t_2})(\frac{1}{2} + \sqrt{\epsilon}x_1 - \frac{4}{3}\epsilon + \epsilon^{\frac{3}{2}}x_2 + \cdots) = \frac{3}{8} - \epsilon - \frac{7}{4}(\frac{1}{2} + \sqrt{\epsilon}x_1 - \frac{4}{3}\epsilon + \epsilon^{\frac{3}{2}}x_2 + \cdots)$$
$$+ (\frac{1}{2} + \sqrt{\epsilon}x_1 - \frac{4}{3}\epsilon + \epsilon^{\frac{3}{2}}x_2 + \cdots)^2(2 + \sqrt{\epsilon}y_1 - \frac{16}{3}\epsilon + \epsilon^{\frac{3}{2}}y_2 + \cdots)$$
$$(\frac{\partial}{\partial t_1} + \sqrt{\epsilon}\frac{\partial}{\partial t_2})(2 + \sqrt{\epsilon}y_1 - \frac{16}{3}\epsilon + \epsilon^{\frac{3}{2}}y_2 + \cdots) = (\frac{1}{2} + \sqrt{\epsilon}x_1 - \frac{4}{3}\epsilon + \epsilon^{\frac{3}{2}}x_2 + \cdots)$$
$$- (\frac{1}{2} + \sqrt{\epsilon}x_1 - \frac{4}{3}\epsilon + \epsilon^{\frac{3}{2}}x_2 + \cdots)^2(2 + \sqrt{\epsilon}y_1 - \frac{16}{3}\epsilon + \epsilon^{\frac{3}{2}}y_2 + \cdots)$$

for O($\sqrt{\epsilon}$):

$$\frac{\partial x_1}{\partial t_1} = -\frac{7}{4}x_1 + \frac{1}{4}y_1 + 2x_1 = \frac{1}{4}x_1 + \frac{1}{4}y_1$$
$$\frac{\partial y_1}{\partial t_1} = x_1 - \frac{1}{4}y_1 - 2x_1 = -x_1 - \frac{1}{4}y_1$$

we get:

$$x_1 = A(t_2)\sin(\frac{\sqrt{3}}{4}t_1 + \theta(t_2))$$
$$y_1 = \frac{\sqrt{3}}{4}A(t_2)\cos(\frac{\sqrt{3}}{4}t_1 + \theta(t_2)) - \frac{1}{4}A(t_2)\sin(\frac{\sqrt{3}}{4}t_1 + \theta(t_2))$$

for O(ϵ):

$$\frac{\partial x_2}{\partial t_1} + \frac{\partial x_1}{\partial t_2} = \frac{1}{4}x_2 + \frac{1}{4}y_2 + 2x_1^2 + x_1y_1 - \frac{8}{3}$$
$$\frac{\partial y_2}{\partial t_1} + \frac{\partial y_1}{\partial t_2} = -x_2 - \frac{1}{4}y_2 + -2x_1^2 - x_1y_1 + \frac{8}{3}$$

try to find the secular term of the first equation:

$$2x_1^2 + x_1y_1 - \frac{8}{3} - \frac{\partial x_1}{\partial t_2} = \frac{7}{8}A^2(1-\cos 2(\frac{\sqrt{3}}{4}t_1 + \theta(t_2))) + \frac{\sqrt{3}}{8}A^2 \sin 2(\frac{\sqrt{3}}{4}t_1 + \theta(t_2))$$
$$- \frac{8}{3} - A' \sin(\frac{\sqrt{3}}{4}t_1 + \theta(t_2)) - A\frac{\sqrt{3}}{4}\cos(\frac{\sqrt{3}}{4}t_1 + \theta(t_2))$$

we have:

$$A' \sin(\frac{\sqrt{3}}{4}t_1 + \theta(t_2)) = 0$$
$$A\frac{\sqrt{3}}{4}\cos(\frac{\sqrt{3}}{4}t_1 + \theta(t_2)) = 0$$

we konw A and θ are const

$$\frac{7}{8}A^2 - \frac{8}{3} = 0$$
$$\frac{\sqrt{3}}{8}A^2 \sin 2(\frac{\sqrt{3}}{4}t_1 + \theta(t_2)) - \frac{7}{8}A^2 \cos 2(\frac{\sqrt{3}}{4}t_1 + \theta(t_2)) = 0$$

we get $A = \sqrt{\frac{64}{21}}$,$\theta = \theta_0$
similar we can eliminate the secular term of equation 2, we get the same result.
therefore:

$$x \sim \frac{1}{2} + \sqrt{\frac{64}{21}}\sin(\frac{\sqrt{3}}{4}t_1 + \theta_0)\sqrt{\frac{3}{8} - \mu}$$
$$y \sim 2 + (\frac{\sqrt{3}}{4}\sqrt{\frac{64}{21}}\cos(\frac{\sqrt{3}}{4}t_1 + \theta_0) - \frac{1}{4}\sqrt{\frac{64}{21}}\sin(\frac{\sqrt{3}}{4}t_1 + \theta_0))\sqrt{\frac{3}{8} - \mu}$$

3. (Child's Swing) In a swing, a child can control his/her body to change the center of gravity. The problem can be modeled by a pendulum with a time-dependent length $l = l(t)$. The model is

$$\frac{d^2\theta}{dt^2} + \frac{2l'(t)}{l(t)}\frac{d\theta}{dt} + \sin\theta = 0, \quad t \geq 0$$

where θ is the angle between the swing and the vertical line. Suppose that $l = l_0(1 + \varepsilon \sin \omega t)$, where ε and ω (both nonnegative) are the amplitude and the frequency of movement controlled by the child, respectively.
(1). Consider the steady state $\theta_s = 0$. Use linear stability analysis to show that the first order perturbation of $\theta = \theta_s + \delta\nu + O(\delta^2)$ satisfies the equation:

$$\frac{d^2\nu}{dt^2} + \frac{2l'(t)}{l(t)}\frac{d\nu}{dt} + \nu = 0, \quad t \geq 0$$

(2). Show that if the child does nothing (so that $\omega = 0$), then the steady state $\theta_s = 0$ is stable.
(3). What the child must do is to find a frequency ω that makes the steady state $\theta_s = 0$ unstable (so the amplitude grows and the swing goes higher). Find such a frequency ω when $\varepsilon \ll 1$. Do this by using multiple scale analysis to find a leading order expansion of ν that is valid for large t .
Hint: You may need to use the following identity:

$$\sin 2\varphi = \frac{2\tan\varphi}{1+\tan^2\varphi}, \qquad \cos 2\varphi = \frac{1-\tan^2\varphi}{1+\tan^2\varphi}$$

Solution: (1). because $\theta = \theta_s + \delta v + O(\delta^2) = \delta v + O(\delta^2)$, we have

$$\delta\frac{d^2v}{dt^2} + \delta\frac{2l'(t)}{l(t)}\frac{dv}{dt} + \delta v + O(\delta^2) = 0$$

for O(δ), we get the equation.

(2).when w=0, we get:

$$\frac{d^2v}{dt^2} + v = 0$$

$v = C_1e^{iv} + C_2e^{-iv}$, so $Re(\pm i) = 0$. The steady state is stable.

(3).let $\tau = \epsilon t, v = v_1(t,\tau) + \epsilon v_2(t,\tau) + \cdots$, we get

$$\frac{\partial^2 v}{\partial t^2} + 2\epsilon\frac{\partial^2 v}{\partial t\partial\tau} + \epsilon^2\frac{\partial^2 v}{\partial\tau^2} + \frac{2\epsilon w\cos wt}{1+\epsilon\sin wt}\epsilon\frac{\partial v}{\partial\tau} + v = 0$$

for O(1):

$$\frac{\partial^2 v_0}{\partial t^2} + v_0 = 0$$

we get $v_0 = C_1\cos t + C_2\sin t$
for O(ϵ)

$$\frac{\partial^2 v_1}{\partial t^2} + v_1 = -2\frac{\partial^2 v_0}{\partial t\partial\tau} - 2w\cos wt\frac{\partial v_0}{\partial t}$$

we want to find the secular term:

$$\begin{aligned}-2\frac{\partial^2 v_0}{\partial t\partial\tau} - 2w\cos wt\frac{\partial v_0}{\partial t} &= -2(-C_1'\sin t + C_2'\cos t) - 2w\cos wt(-C_1\sin t + C_2\cos t)\\ &= 2C_1'\sin t - 2C_2'\cos t + wC_1(\sin(w+1)t - \sin(w-1)t) - wC_2(\cos(w+1)t + \cos(w-1)t)\end{aligned}$$

notice $w > 0$
if $w \neq 2$, then $w+1 \neq 1$ and $w-1 \neq 1$, in order to eliminate the secular term, we have:

$$C_1' = 0$$
$$C_2' = 0$$

then C_1, C_2 are constant. So $Re(v) = 0, \theta_s = 0$ is stable. contradict!
if $w = 2$,

$$-2\frac{\partial^2 v_0}{\partial t\partial\tau} - 2w\cos wt\frac{\partial v_0}{\partial t} = (2C_1' - 2C_1)\sin t + (-2C_2' - 2C_2)\cos t + 2C_1\sin 3t - 2C_2\cos 3t$$

eliminate the secular term:

$$
\begin{aligned}
2C_1' - 2C_1 &= 0 \\
-2C_2' - 2C_2 &= 0
\end{aligned}
$$

we get: $C_1 = A_1 e^{\tau}, C_2 = A_2 e^{-\tau}, (A_1, A_2$ are constant)
therefore $v = A_1 e^{\tau} \cos t + A_2 e^{-\tau} \sin t$

Chapter 7

Bifurcation and Stability

7.1 Linearized stability of steady states

We illustrate how to determine the stability through Van-Der-Pol's Equation:

$$y'' - \lambda(1-y^2)y' + y = 0, \quad t > 0,$$

where λ is a parameter. We can observe that its steady state is $y_s = 0$. In this section, we will determine the stability of the system near $y_s = 0$. Initial conditions are

$$y(0) = \alpha_0\delta, \;\; y^{'}(0) = \beta_0\delta,$$

where δ is a small perturbation parameter.

Expand $y(t)$ around y_s in terms of δ:

$$y(t) \sim y_s + \delta y_1(t) + \cdots .$$

Substituting into the equation:

$$\delta y_1'' + O(\delta^2) - \lambda(1 - y_s^2 - 2\delta y_s y_1 + O(\delta^2))(\delta y_1' + O(\delta^2)) + y_s + \delta y_1 + O(\delta^2) = 0.$$

$O(1) : y_s = 0.$

$O(\delta) : {y_1}'' - \lambda(1 - y_s^2)y_1' + y_1 = 0. \quad \Rightarrow \quad y_1'' - \lambda y_1' + y_1 = 0.$

$\alpha_0\delta = y(0) \sim 0 + \delta y_1(0) + \cdots \Rightarrow y_1(0) = \alpha_0.$

$\beta_0\delta = y^{'}(0) \sim \delta y_1^{'}(0) + \cdots \Rightarrow y_1(0) = \beta_0.$

$y_1(t) = a_0 e^{r_+ t} + a_1 e^{r_- t}$, where $r_\pm = \frac{1}{2}(\lambda \pm \sqrt{\lambda^2 - 4})$. a_0 and a_1 could be determined using α_0, β_0.

7.2 Limit circle and Hopf bifurcation

What happens to the solution when $\lambda > 0$?

Total energy os the oscillator: $E(t) = \frac{1}{2}y'^2 + \frac{1}{2}y^2$, $\frac{dE}{dt} = y'(y'' + y) = \lambda(1 - y^2)y'^2$.

i) $|y| < 1 \Rightarrow \frac{dE}{dt} > 0 \Rightarrow$ unbounded sol.

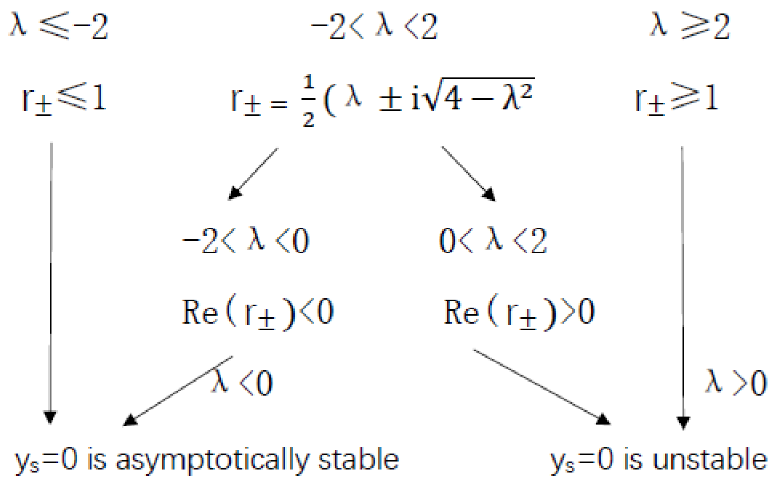


Figure 7.1.1

ii) $|y| > 1 \Rightarrow \frac{dE}{dt} < 0 \Rightarrow$ stabilize the sol.

When i) happens, it is possible that $|y|$ grows to > 1, and falls into ii).

What happens as λ goes from negative to positive?

Let $\varepsilon = \lambda - \lambda_b$ where $\lambda_b = 0$ is a bifarcation point.

$$y'' - \varepsilon(1 - y^2) + y = 0, \ t > 0.$$

This equation has an oscillator with weak damping. To analyze the long time behavior, we need to introduce a different time scale

$$\tau = \varepsilon t, \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \frac{1}{\varepsilon}\frac{\partial}{\partial \tau}.$$

Then, this equation becomes

$$(\frac{\partial^2}{\partial t^2} + 2\varepsilon\frac{\partial^2}{\partial t \partial \tau} + \varepsilon^2\frac{\partial^2}{\partial \tau^2})y - \varepsilon(1 - y^2)(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau})y = 0.$$

Assume that

$$y \sim y_0(t, \tau) + \varepsilon y_1(t, \tau) + \cdots, \qquad \varepsilon \to 0,$$

then

$$O(1): \frac{\partial^2 y_0}{\partial t^2} + y_0 = 0 \Rightarrow y_0 = A(\tau)\cos(t + \phi(\tau)), \ \ (WLOG, \ assume \ A > 0)$$

$$\begin{aligned} O(\varepsilon): \frac{\partial^2 y_1}{\partial t^2} + y_1 &= (1 - y_0^2)\frac{\partial y_0}{\partial t} - 2\frac{\partial^2 y_0}{\partial t \partial \tau} \\ &= [2A'(\tau) - A(1 - \frac{1}{4}A^2)]\sin(t + \phi(\tau)) + 2A\phi'(\tau)\cos(t + \phi(\tau)) + \frac{1}{4}A^3\sin[3(t + \phi(\tau))]. \end{aligned}$$

To eliminate the secular terms in $O(\varepsilon)$

$$\begin{cases} A\phi' = 0 \quad \Rightarrow \phi(\tau) = \phi_0, \\ 2A' - A(1 - \frac{1}{4}A^2) = 0 \Rightarrow A(\tau) = \frac{2}{\sqrt{1+ce^{-\tau}}} > 0. \end{cases}$$

Leading order approximation (for λ close to $\lambda_b = 0)|\lambda - \lambda_b| \ll 1$

$$y(t) \sim \frac{2}{\sqrt{1+ce^{-\lambda t}}} \cos(t + \phi_0), \ \lambda \to 0$$

$$y'(t) \sim \frac{2}{\sqrt{1+ce^{-\lambda t}}} \sin(t + \phi_0), \ \lambda \to 0$$

C and ϕ_0 could be determined by initial conditionals. As $t \to \infty$, $y(t) \to 2\cos(t + \phi_0)$.

On the phase plane, $y^2 + y'^2 \sim \frac{4}{1+ce^{-\lambda t}} \to 4$ as $t \to \infty$.

The limit solution is periodic (limit cycle):

$$(y(t), y'(t)) = (2\cos(t + \phi_0), -2\sin(t + \phi_0))$$

Conclusion:

Steady state $y_s = 0$ loses stability at $\lambda_b = 0$, and a stable unit cycle appears for $0 < \lambda - \lambda_b \ll 1 \leftarrow$(degenerate) Holf bifurcation.

7.3 Systems of ODEs

In this section, we will give an example of a system of ODE:

$$\vec{y}'(t) = \vec{f}(\lambda, \vec{y}), \ \ \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \ \vec{f} = \begin{pmatrix} f_1(\lambda_1 y_1) & \cdots & y_n \\ \vdots & \ddots & \vdots \\ f_n(\lambda_1 y_1) & \cdots & y_n \end{pmatrix}$$

Assume $\vec{y}_s(\lambda)$ is a steady state solution, i.e. $\vec{f}(\lambda, \vec{y}_s(\lambda)) = 0$.

We take small perturbations:

$\vec{y}(0) = \vec{y}_s + \delta\vec{a}$, $\delta \ll 1$, $\vec{y}(t) \sim \vec{y}_s + \delta\vec{v}(t) + O(\delta^2)$, $\delta \to 0$.

Substituting into equations, we have

$$\vec{f}(\lambda, \vec{y}(t)) \sim \vec{f}(\lambda, \vec{y}_s) + \delta\nabla\vec{f} \cdot \vec{v}(t) + \cdots = \delta\nabla\vec{f} \cdot \vec{v}(t) + O(\delta^2)$$

$O(\delta) : \vec{v}'(t) = \nabla\vec{f} \cdot \vec{v}(t) = Av$, $A = \nabla\vec{f}(Jacobian\ \vec{y} = \vec{y}_s)$, $\vec{v}(0) = \vec{a}$.

Assume the solution has the form $\vec{v} = \vec{x}e^{rt} \Rightarrow r\vec{x}e^{rt} = A\vec{x}e^{rt} \Rightarrow A\vec{x} = r\vec{x}$.
i) If A has n distinct eigenvalues $r_1, \cdots, r_n$ with corresponding eigenvectors $\vec{x}_1, \cdots, \vec{x}_n$, then the general solution is

$$\vec{v} = \alpha_1\vec{x}_1e^{r_1t} + \cdots + \alpha_n\vec{x}_ne^{r_nt}. \tag{7.1}$$

ii) A does not have n distinct eigenvalues, solve eigenvalues have metric multiplicity $k \geq 2$ corresponding independent solutions have the form of t^ie^{rt}, $0 \leq i \leq k$.

Remark 7.3.1: If A has n distinct eigenvalue or each eigenvalue has geometric multiplicity =1 for those eigenvalue having algebraic $multiplicity^m > 1$. We can find m independent eigenvectors s.t. they also span the whole eigenspace. (7.1) is still valid.

If eigenvalue r has genmetric multiplicity $k > 1$, k = order of Jordan block, then $\exists\vec{e}, \cdots, \vec{e}$ indeoendent s.t.

$$A\vec{e}_1 = r\vec{e}_1,\ A\vec{e}_2 = r\vec{e}_2 + \vec{e}_1,\ \cdots, A\vec{e}_k = r\vec{e}_k + \vec{e}_{k-1}.$$

Then consider the linear condition of the form $\vec{v}_i = (\alpha_1(t)\vec{e}_1 + \alpha_2(t)\vec{e}_2 + \cdots + \alpha_k(t)\vec{e}_k)e^{rt}$ to be a solution.

$$\begin{aligned}A\vec{v}_i &= [\alpha_1(t)r\vec{e}_1 + \alpha_2(t)(r\vec{e}_2 + \vec{e}_1) + \cdots + \alpha_i(t)(r\vec{e}_i + \vec{e}_{i-1})e^H \\ &= [(\alpha_1(t)r + \alpha_2(t))\vec{e}_1 + (\alpha_2(t)r + \alpha_3(t))\vec{e}_2 + \cdots + (\alpha_{i-1}(t)r + \alpha_i(t))\vec{e}_{i-1} + \alpha_i(t)r\vec{e}_i]e^{rt}\end{aligned}$$

$$\begin{aligned}\vec{v}_i =& [(\alpha_1'(t) + r\alpha_1(t))\vec{e}_1 + (\alpha_2'(t) + r\alpha_2(t))\vec{e}_2 + \cdots \\ &+ (\alpha_{i-1}'(t) + r\alpha_{i-1}(t))\vec{e}_{i-1} + (\alpha_{i-1}'(t) + r\alpha_{i-1}(t))\vec{e}_{i-1}]e^{rt}\end{aligned}$$

Hence, we have

$$\alpha_1'(t) = \alpha_2(t) \quad \alpha_2'(t) = \alpha_3(t), \cdots,\ \alpha_{i-1}'(t) = \alpha_i(t)$$

$$\alpha_i'(t) = 0 \Rightarrow \alpha_i(t) = 1\ \ \alpha_{i-1}(t) = t\ \ \alpha_{i-2}(t) = \frac{1}{2}t^2, \cdots, \alpha_2(t) = \frac{1}{(i-2)}t^{i-2}, \alpha_1(t) = \frac{1}{(i-1)}t^{i-1}$$

$\sum\limits_{i=1}^{k} C_i\vec{V}_i$ is a general solution.

Theorem 7.3.1(Linear stability):

The steady-state y_s is asymptotically stable if all the eigenvalues of A have a negative real part, i.e., $R_e(r) < 0$; it is unstable if even one eigenvalue r has a positive real part. i.e. $R_e(r) > 0$.

Example 7.3.1:

$$\begin{cases} y' = v - y[v^2ty^2 - \lambda(1-\lambda)], \\ v' = -y - v[v^2 + y^2 - \lambda(1-\lambda)]. \end{cases}$$

Steady state: $(y_s, v_s) = (0, 0)$.

$$\nabla\vec{f}\cdot(y_s, v_s) = \begin{pmatrix} \lambda(-\lambda) & 1 \\ -1 & \lambda(-\lambda) \end{pmatrix}$$

$r_{\pm} = \lambda(1-\lambda) \pm i$ steady state is stable if $\lambda < 0$ or $\lambda > 1$.

Analysis near bifurcation point $\lambda_b = 0$, set $\in= \lambda - \lambda_b$, $|\varepsilon| \ll 1$.

Let $\tau = \varepsilon t$, (t, τ) multiple expansion.

$$\vec{y} \sim \vec{y}_s + \varepsilon^{\alpha}\vec{y}_0(t,\tau) + \varepsilon^{\beta}\vec{y}_0(t,\tau) + \cdots, \beta > \alpha,$$

Substituting it into equation

$$(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial \tau})(\varepsilon^{\alpha} y_0 + \varepsilon^{\beta} y_1 + \cdots) = \varepsilon^{\alpha} v_0 + \varepsilon^{\beta} v_1 - \varepsilon^{\alpha} y_0[\varepsilon^{2\alpha}(v_0^2 + y_0^2) - \varepsilon + \varepsilon^2] + \cdots .$$

$$(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial \tau})(\varepsilon^{\alpha} v_0 + \varepsilon^{\beta} v_1 + \cdots) = -\varepsilon^{\alpha} y_0 - \varepsilon^{\beta} y_1 - \varepsilon^{\alpha} v_0[\varepsilon^{2\alpha}(v_0^2 + y_0^2) - \varepsilon + \varepsilon^2] + \cdots .$$

We have

$$O(\varepsilon^{\alpha}) : \begin{cases} \frac{\partial y_0}{\partial t} = v_0 \Rightarrow y_0 = A(\tau)\sin(t + O(\tau)), \\ \frac{\partial v_0}{\partial t} = -y_0 \Rightarrow v_0 = A(\tau)\cos(t + O(\tau)), \end{cases}$$

which is a Hamiltonian system.

By dominant balance

$$\varepsilon^{\alpha+1}\frac{\partial y_0}{\partial \tau} + \varepsilon^{\beta}\frac{\partial y_1}{\partial t} + \cdots = \varepsilon^{\beta} v_1 + \varepsilon^{\alpha+1} y_0 - \varepsilon^{3\alpha} y_0(v_0^2 + y_0^2) + \cdots .$$

Hence

$$\varepsilon^{\alpha+1} = \varepsilon^{\beta} = \varepsilon^{3\alpha} \Rightarrow \beta = \alpha + 1 = 3\alpha \Rightarrow \alpha = \frac{1}{2} \;\; \beta = \frac{3}{2}.$$

At the next order, we have

$$O(\varepsilon^{\frac{3}{2}}) : \begin{cases} \frac{\partial y_1}{\partial t} = v_1 - y_0[v_0^2 + y_0^2 - 1] - \frac{\partial y_0}{\partial \tau}, \\ \frac{\partial v_1}{\partial t} = -y_1 - v_0[v_0^2 + y_0^2 - 1] - \frac{\partial v_0}{\partial \tau}, \end{cases}$$

which implies that

$$\begin{aligned} \frac{\partial^2 y_1}{\partial t^2} + y_1 =& \frac{\partial}{\partial \tau}[y_0(v_0^2 + y_0^2 - 1)] - \frac{\partial^2 y_0}{\partial t \partial \tau} - v_0[v_0^2 + y_0^2 - 1] - \frac{\partial v_0}{\partial \tau} \\ =& -2[A(A^2 - 1) + A']\cos(t + \theta(\tau)) + 2A\theta'(\tau)\sin(t + \theta(\tau)). \end{aligned}$$

To eliminate the secular terms

$$\begin{cases} A(A^2 - 1) + A' = 0, \\ A\theta' = 0, \end{cases}$$

i.e.,

$$\begin{cases} A(\tau) + A' = \frac{1}{\sqrt{1 + ce^{-2\lambda t}}}, \\ \theta(\tau) = \theta_0, \end{cases}$$

where θ_0 are determined from initial conditions.

$$\vec{y} \sim \sqrt{\frac{\lambda}{\sqrt{1 + ce^{-2\lambda t}}}}(\sin(t + \theta_0), \cos(t + \theta_0)), \quad (0 < \lambda \ll 1).$$

As $t \to \infty$, leading order approximation is periodic.

$$\vec{y} \sim \sqrt{\lambda}(\sin(t + \theta_0), \cos(t + \theta_0)),$$

which is a limit cycle.

7.4 Exercise 7

1. Consider the equation for $y(t)$:

$$y'' - \lambda \left(3 - e^{\lambda} (y')^2\right) y' + y = 1, \quad t > 0.$$

1. Find the fixed point (i.e., stationary solution) and determine the range of λ on which the fixed point is stable. 2. Find a leading order approximation for the limit cycle that appears at the Hopf bifurcation point. (Hint: $\sin 3\theta = -4\sin^3\theta + 3\sin\theta, \cos 3\theta = 4\cos^3\theta - 3\cos\theta$.)

7.5 Answer 7

1. Consider the equation for $y(t)$:

$$y'' - \lambda\left(3 - e^{\lambda}\left(y'\right)^{2}\right) y' + y = 1, \quad t > 0.$$

1. Find the fixed point (i.e., stationary solution) and determine the range of λ on which the fixed point is stable. 2. Find a leading order approximation for the limit cycle that appears at the Hopf bifurcation point. (Hint: $\sin 3\theta = -4\sin^3\theta + 3\sin\theta, \cos 3\theta = 4\cos^3\theta - 3\cos\theta$.)

Solution:

(1) $y_s = 1$. Assume $y \sim 1 + \delta y_1 + \delta^2 y_2 + \cdots$, then

$$\left(\delta y_1'' + \delta^2 y_2'' + \cdots\right) - \lambda\left(3 - e^{\lambda}\left(\delta y_1' + \delta^2 y_2' + \cdots\right)^2\right)\left(\delta y_1' + \delta^2 y_2' + \cdots\right) + \left(1 + \delta y_1 + \delta^2 y_2 + \cdots\right) = 1.$$

For O(δ) :

$$y_1'' - 3\lambda y_1' + y_1 = 0$$

Assume the eigenvalues of this equation are: λ_1, λ_2, then:

$$\begin{cases} \lambda_1 + \lambda_2 = 3\lambda, \\ \lambda_1 \lambda_2 = 1. \end{cases}$$

If $\lambda < 0$, the fixed point is stable. If $\lambda = 0$, we solve the ode and get $y_1 = C_1 \sin x + C_2 \cos x$. It is also stable. Hence, when $\lambda \leq 0$, the fixed point is stable.

(2) let $0 < \epsilon = \lambda - 0 \ll 1$, and $\tau = \epsilon t$, then:

$$\frac{\partial^y}{\partial t^2} + 2\epsilon\frac{\partial^2 y}{\partial t \partial \tau} + \epsilon^2\frac{\partial^y}{\partial \tau^2} - \epsilon\left(3 - (1 + \epsilon + \cdots)\left(\frac{\partial y}{\partial t} + \epsilon\frac{\partial y}{\partial \tau}\right)\right) + y = 1.$$

Assume $y \sim y_0 + \epsilon y_1 + \cdots$. For O(1) :

$$\frac{\partial^2 y_0}{\partial t^2} + y_0 = 1$$

We get $y_0 = A(\tau)\cos(t + \theta(\tau)) + 1$. For O($\epsilon$) :

$$\frac{\partial^2 y_1}{\partial t^2} + y_1 = -2\frac{\partial^2 y_0}{\partial t \partial \tau} + 3\frac{\partial y_0}{\partial t} - \left(\frac{\partial y_0}{\partial t}\right)^3.$$

In order to eliminate the secular term, we compute:

$$-2\frac{\partial^2 y_0}{\partial t \partial \tau} + 3\frac{\partial y_0}{\partial t} - \left(\frac{\partial y_0}{\partial t}\right)^3$$

$$= \left(2A'(\tau) - 3A(\tau) + \frac{3}{4}A^3(\tau)\right)\sin(t + \theta(\tau)) + 2A(\tau)\theta'(\tau)\cos(t + \theta(\tau)) - \frac{1}{4}A^3(\tau)\sin 3(t + \theta(\tau)).$$

Hence,

$$\begin{cases} 2A'(\tau) - 3A(\tau) + \frac{3}{4}A^3(\tau) = 0, \\ 2A(\tau)\theta'(\tau) = 0 \end{cases}$$

Compute this equation:

$$\begin{cases} A = \dfrac{2}{\sqrt{1 + Ce^{-3\tau}}}, \theta = \theta_0, \end{cases}$$

where θ_0 and C are constant. Therefore, the leading order approximation is

$$y(t) \sim \frac{2}{\sqrt{1 + Ce^{-3\lambda t}}} \cos(t + \theta_0) + 1.$$

Chapter 8

Basic Calculus of Variations

In this chapter, we establish a necessary condition for a function to produce an extremum for a functional. The core focus is a second-order differential equation known as the Euler-Lagrange equation, which serves a role similar to the gradient of a function. The derivation of the Euler-Lagrange equations is presented in Section 8.1, followed by a discussion of special cases where the differential equation can be simplified in Section 8.2. The subsequent three sections explore more qualitative topics.

8.1 The Euler-Lagrange equation

We first consider a particular class of problem called the fixed endpoint variational problem to introduce the Euler-Lagrange Equation:

$$\min J[y] = \int_{x_0}^{x_1} f(x, y, y')\, dx.$$

subject to $y(x_0) = y_0$, $y(x_1) = y_1$. We have

$$\delta J = \int_{x_0}^{x_1} \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right) dx.$$

Let η be the perturbation around the local extrema $y = \bar{y}$. Then $\eta \in \mathcal{H}$, where

$$\mathcal{H} := \{\eta \in C^2[x_0, x_1] : \ \eta(x_0) = y_0,\, \eta(x_1) = y_1\}.$$

We assume that $y = \bar{y} + \epsilon\eta$, then

$$J[\bar{y} + \epsilon\eta] = \int_{x_0}^{x_1} f(x, \bar{y} + \epsilon\eta, \bar{y}' + \epsilon\eta')\, dx.$$

Since y is a minima, then

$$\frac{d}{d\epsilon} J[\bar{y} + \epsilon\eta] \bigg|_{\epsilon=0} = 0.$$

This implies

$$\delta J[y] = \int_a^b \left(\frac{\partial f}{\partial y} \eta + \frac{\partial f}{\partial y'} \eta' \right) dx = 0 \quad \forall, \eta \in \mathcal{H}.$$

$$\Rightarrow \int_a^b \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \eta\, dx = 0.$$

Hence, we obtain the Euler-Lagrange equation:

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0.$$

Next, we give an example to illustrate the Euler-Lagrange equation.

Example: (Geodesics in the Plane) Let $(x_0, y_0) = (0,0)$ and $(x_1, y_1) = (1,1)$. The arclength of a curve described by $y(x), x \in [0,1]$ is given by

$$J(y) = \int_0^1 \sqrt{1+y'^2}dx.$$

The geodesic problem in the plane entails determining the function y such that the arclength is minimum. We limit our investigation to functions in $C^2[0,1]$ such that

$$y(0) = 0, \quad y(1) = 1.$$

If y is an extremal for J then the Euler-Lagrange equation must be satisfied; hence,

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) - 0 = 0$$

$$\frac{y'}{\sqrt{1+y'^2}} = \text{const.}$$

The last equation is equivalent to the condition that $y' = c_1$, where c_1 is a constant. Consequently, an extremal for J must be of the form

$$y(x) = c_1 x + c_2,$$

where c_2 is another constant of integration. Since $y(0) = 0$, we see that $c_2 = 0$, and since $y(1) = 1$, we see that $c_1 = 1$. Thus, the only extremal y is given by $y(x) = x$, which describes the line segment from $(0,0)$ to $(1,1)$ in the plane (as expected).

8.2 Some special cases

The Euler-Lagrange equation is a second-order nonlinear differential equation, and such equations are usually difficult to simplify. There are, however, certain cases when this differential equation can be simplified. We examine two such cases in this section.

Case I: No Explicit y Dependence Suppose that the functional is of the form

$$J(y) = \int_{x_0}^{x_1} f\left(x, y'\right) dx,$$

where the variable y does not appear explicitly in the integrand. Evidently, the Euler-Lagrange equation reduces to

$$\frac{\partial f}{\partial y'} = c_1, \tag{8.1}$$

where c_1 is a constant of integration. Now $\partial f/\partial y'$ is a known function of x and y', so that equation (8.1) is a first-order differential equation for y. In principle, equation (8.1) is solvable for y', provided $\partial^2 f/\partial y'^2 \neq 0$, so that equation (8.1) could be recast in the form

$$y' = g\left(x, c_1\right),$$

for some function g and then integrated. In practice, however, solving equation (8.1) for y' can prove formidable if not impossible, and there may be several solutions available.

Nonetheless, the absence of y in the integrand simplifies the problem of solving a second-order differential equation to solving an implicit equation and quadrature.

Case II: No Explicit x Dependence

Another simplification is available when the integrand does not contain the independent variable x explicitly. Let J be a functional of the form

$$J(y) = \int_{x_0}^{x_1} f\left(y, y'\right) dx,$$

and define the function H by

$$H\left(y, y'\right) = y' \frac{\partial f}{\partial y'} - f.$$

Then H is constant along any extremal y. We give its proof as follows:

Suppose that y is an extremal for J. Now,

$$\begin{aligned} \frac{d}{dx} H\left(y, y'\right) &= \frac{d}{dx}\left(y' \frac{\partial f}{\partial y'} - f\right) \\ &= y'' \frac{\partial f}{\partial y'} + y' \frac{d}{dx} \frac{\partial f}{\partial y'} - \left(y' \frac{\partial f}{\partial y} + y'' \frac{\partial f}{\partial y'}\right) \\ &= y'\left(\frac{d}{dx} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y}\right), \end{aligned}$$

and since y is an extremal, the Euler-Lagrange equation (2.9) is satisfied; hence,

$$\frac{d}{dx} H\left(y, y'\right) = 0.$$

Consequently, H must be constant along an extremal. Note that the function H depends only on y and y', and thus the equation

$$H\left(y, y'\right) = \text{ const},$$

is a first-order differential equation for the extremal y.

Example: (Catenary)

The catenary problem (Section 1.2) has a functional of the form

$$J(y) = \int_{x_0}^{x_1} y\sqrt{1 + y'^2} dx.$$

The above integrand does not contain x explicitly and therefore

$$\begin{aligned} H\left(y, y'\right) &= y' \frac{\partial f}{\partial y'} - f \\ &= y' \frac{y y'}{\sqrt{1 + y'^2}} - y\sqrt{1 + y'^2} \end{aligned}$$

is constant along an extremal. Any extremal y must consequently satisfy the first-order differential equation

$$\frac{y^2}{1+y'^2} = c_1^2, \tag{8.2}$$

where c_1 is a constant. If $c_1 = 0$, then the only solution to equation (8.2) is $y = 0$. Suppose that $c_1 \neq 0$; then equation (8.2) can be replaced by

$$y' = \sqrt{\frac{y^2}{c_1^2} - 1}. \tag{8.3}$$

We integrate equation (8.3) for x as a function of y, viz.,

$$x = \int \frac{dy}{\sqrt{\frac{y^2}{c_1^2} - 1}} = c_1 \ln\left(\frac{y + \sqrt{y^2 - c_1^2}}{c_1}\right) + c_2,$$

where c_2 is a constant of integration. Now,

$$c_1 e^{(x-c_2)/c_1} = y + \sqrt{y^2 - c_1^2},$$

and

$$c_1 e^{-(x-c_2)/c_1} = \frac{c_1^2}{y + \sqrt{y^2 - c_1^2}}.$$

Therefore,

$$\begin{aligned} c_1\left(e^{(x-c_2)/c_1} + e^{-(x-c_2)/c_1}\right) &= y + \sqrt{y^2 - c_1^2} + \frac{c_1^2}{y + \sqrt{y^2 - c_1^2}} \\ &= 2y. \end{aligned}$$

The extremals are thus given by

$$y(x) = c_1 \cosh\left(\frac{x - c_2}{c_1}\right).$$

8.3 Isoperimetric problems

Variational problems are often accompanied by one or more constraints, so in the section we give a example. Consider a cable $\{y(s) : 0 \leq s \leq L\}$, s: arclegth. We introduce some notations:

1. m: mess per unit length.
2. g: gravitational constant.
3. Potential energy: $W_p(y) = \int_0^L mgy(s)ds$.

We back to Cartesian coordinates (x,y): $W_p(y) = \int_{x_0}^{x_1} mgy(x)\sqrt{1+(y'(x))^2}dx$, cable $\{y(x) : x_0 \le x \le x_1\}$

$$\min_y J(y) = \int_{x_0}^{x_1} \sqrt{1+y'(x)}dx,\ y(x_0) = y_0,\ y(x_1) = y_1,$$

with $x_0 = 0,\ x_1 = 1,\ y_0 > y_1 = h$. More constraint fixed length: $I(y) = \int_0^1 \sqrt{1+y'^2}dx = L$.

Theorem 7.4.1: Let $J : C^2[x_0, x] \to R$, and

$$J(y) = \int_{x_0}^{x_1} f(x,y,y')dx \quad y(x_0) = y_0\ y(x_1) = y_1,$$

subject to $I(y) = \int_{x_0}^{x_1} f(x,y,y')dx = L$ (Isopermetric constraint).

Let $\hat{y} = y + \varepsilon_1\eta_1 + \varepsilon_2\eta_2\ \eta_k \in C^2[x_0, x_1],\ \eta_k(x_0) = \eta_k(x_1) = 0$. Since there are one more constraint. $\varepsilon_2\eta_2$ is a^2 correction term, must be selected s.t. $\hat{y}$ satisfies the isoperimetric condition η_1 is arbitrary.

$$I(\hat{y}) = \int_{x_0}^{x_1} g(x, y+\varepsilon_1\eta_1+\varepsilon_2\eta_2 y' + \varepsilon_1\eta_1' + \varepsilon_2\eta_2')dx = \Theta(\varepsilon_1, \varepsilon_2),$$

$\Theta(0,0) = L$ by implicit function theorem.

when $\|\varepsilon\| = \max(|\varepsilon_1|, |\varepsilon_2|)$ is small, $\exists$ a curve $\varepsilon_2 = \varepsilon_2(\varepsilon_1)\ s.t.\ \Theta(\varepsilon_1, \varepsilon_2(\varepsilon_1)) = L$ provided. $\nabla\Theta(0,0) \neq \vec{0}$ ($\nabla\Theta(0,0) = \vec{0}$. is called rigid external).

Let $J(g) = \Lambda(\varepsilon_1, \varepsilon_2)$, Equivalent problem:

$$\min_{\varepsilon_1,\varepsilon_2} \Lambda(\varepsilon_1, \varepsilon_2)\ s.t.\ \Theta(\varepsilon_1, \varepsilon_2) - L = 0.$$

By method of Lagrange multiplier, $\exists\lambda$ constant s.t. $\nabla(\Lambda(\varepsilon_1,\varepsilon_2)) - \lambda(\Theta(\varepsilon_1,\varepsilon_2) - L)|_{\vec{\varepsilon}=\vec{0}} = \vec{0}$, $\nabla = (\frac{\partial}{\partial\varepsilon_1}, \frac{\partial}{\partial\varepsilon_2}) \Rightarrow 2$ equations.

$$\begin{aligned}\frac{\partial}{\partial\varepsilon_1}\Lambda(\varepsilon_1,\varepsilon_2)|_{\vec{\varepsilon}=\vec{0}} &= \int_{x_0}^{x_1} \eta_1\frac{\partial f}{\partial y} + \eta_1'\frac{\partial f}{\partial y'}dx \\ &= \int_{x_0}^{x_1} \eta_1(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'})dx.\end{aligned}$$

$$\frac{\partial}{\partial\varepsilon_1}\Lambda(\varepsilon_1,\varepsilon_2)|_{\vec{\varepsilon}=\vec{0}} = \int_{x_0}^{x_1} \eta_1(\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'})dx.$$

Hence

$$\int_{x_0}^{x_1} \eta_1\{\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y}\}dx = 0 \quad F = f - \lambda g,$$

which implies $\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$ since η_1 is abitrary.

Similarly, the second optimality condition $\Rightarrow \int_{x_0}^{x_1} \eta_2\{\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y}\}dx = 0$ is automatically satisfied for any η_2.

Remark 7.4.1: If $\nabla\Theta|_{\vec{\varepsilon}=\vec{0}} = \vec{0}$, then similar calculation yields.

$$\int_{x_0}^{x_1} \eta_k\{\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'}\}dx = 0, \quad k = 1, 2.$$

Since η_1 is arbitrary, $\frac{d}{dx}\frac{\partial g}{\partial y'} - \frac{\partial g}{\partial y} = 0$. Therefore, y is external for I.

Theorem 7.4.2: Suppose J has an extremum at $y \in C^2[x_0, x_1]$ subject to BC and permetric constraint $I(y) = L$. Suppose also that y is not an external for I. Thm $\exists\lambda$ s.t. y satisfies

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0, \quad F = f - \lambda g.$$

Theorem 7.4.3: Suppose J has an extremum at $y \in C^2[x_0, x_1]$ subject to BC and isopcrimetric constraint. Then $\exists\lambda_0, \lambda_1$, not both zero, s.t.

$$\frac{d}{dx}\frac{\partial k}{\partial y'} - \frac{\partial k}{\partial y} = 0 \;\; k = \lambda_0 f - \lambda_1 g.$$

For rigid extremal, $\lambda_0 = 0$; otherwise, $\lambda_0 = 1$.

Example 7.4.1: (Catenary) $F = (y - \lambda)\sqrt{1 + y'^2}$ no explicit x dependence.

Let $H = y'\frac{\partial F}{\partial y'} - F$ (Hamiltonian). we hvae

$$\begin{aligned}\frac{d}{dx}H(y, y') =& y''\frac{\partial F}{\partial y'} + y'\frac{d}{dx}\frac{\partial F}{\partial y'} - (y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'})\\ =& y'(\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y}) = 0.\end{aligned}$$

This implies

$$H(y, y') = constant = \frac{(y - \lambda)y'^2}{\sqrt{1 + y'^2}} - (y - \lambda)\sqrt{1 + y'^2}$$

$$u = y - \lambda \Rightarrow \frac{u^2}{1 + u'_2} = c_1^2.$$

$$U(x) = c_1\cosh(\frac{x - c_2}{c_1}) = y(x) - \lambda.$$

$$k_1 = c_1 \;\; k_2 = -\frac{c_2}{c_1} \quad BC \Rightarrow \cosh(k_2) = \cosh(k_2 + \frac{1}{k_1}) \Rightarrow k_2 = -\frac{1}{2k_1}.$$

$$L = \int_0^1 \sqrt{1 + y'^2}dx = k_1\sinh(\frac{x}{k_1} + k_2)|_0^1.$$

Hence,

$$L = 2k_1\sinh(2\frac{1}{k_1}) \overset{\xi=\frac{1}{2k_1}}{\Rightarrow} L_\xi = \sinh(\xi).$$

Two nonzero solutions $\hat{\xi}$ and $-\hat{\xi}$

$$y(x) = h + \frac{1}{2\hat{\xi}}\{\cosh(\hat{\xi}(2x - 1)) - \cosh(\hat{\xi})\}.$$

8.4 Optimal control problem using variational analysis

We consider the following control system in this section

$$\begin{cases} x'(t) = & f(t,x,u), \\ x(t_0) = & x_0, \end{cases}$$

where $u = u(t)$ is control variable, $x \in \mathbb{R}^n$ is state parameter, and $u \in U$ is control set. We assume that the Lipschitz continuity for f to guarantee existence and uniqueness, i.e.,

$$f(t,x_1) - f(t,x_2) \leq L|x_1 - x_2|.$$

The cost functional is defined by

$$J(u) = \int_{t0}^{t_f} L(t,x(t),u(t))dt + K(t_f,x_f),$$

where $x(f) = x(t_f)$. $L(t,x(t),u(t))$ is running cost, and $K(t_f,x_f)$ is terminal cost. It is clear that J depends on $u(t)$ through $x(t)$. We assume L and K are smooth enough. For $k = 0$, this problem is also called the Lagrange problem. For $L = 0$, it is called the Mayer problem. These two problems can be transformed into each other.

We consider that the target set is $S \subset [t_0,+\infty) \times \mathbb{R}^n$, and t_f is the smallest time s.t. $(t_f,x_f) \in S$. $S = \{t_1\} \times \mathbb{R}^n$, fixed time free-endpoint problem:

$$\min J(u) = \int_{t0}^{t_1} L(t,x(t),u(t))dt + K(x(t_1)),$$

under the constraint

$$\begin{cases} x'(t) = & f(t,x(t),u(t)), \\ x(t_0) = & x_0. \end{cases}$$

Let $x = x^* + \varepsilon\eta \qquad u = u^* + \varepsilon\xi$, then

$$\begin{cases} \eta'(t) = & \frac{\partial f}{\partial x}(t,x^*,u^*)\eta + \frac{\partial f}{\partial u}(t,x^*,u^*)\xi, \\ \eta(t_0) = & 0. \end{cases}$$

With Lagrange multiplier $p(t)$:

$$\begin{aligned} J(u) &= \int_{t0}^{t_1} (L + p(x'(t) - f))dt + K(x(t_1)) \\ &= \int_{t0}^{t_1} (p \cdot x'(t) - H(t,x,u,p))dt + K(x(t_1)), \end{aligned}$$

Where $H(t,x,u,p) = p \cdot f - L$ is Hamiltonian, which is slightly different from that in mechanics. From computation:

$$\begin{aligned}
J(u) - J(u^*) &= \delta J\Big|_{u^*}(\xi)\varepsilon + o(\varepsilon),\\
K(x(t_1)) - K(x^*(t_1)) &= \varepsilon\frac{\partial K}{\partial x}(x^*(t_1))\eta(t_1) + o(\varepsilon),\\
H(t,x,u,p) - H(t,x^*,u^*,p) &= \varepsilon\frac{\partial H}{\partial x}(t,x^*,u^*,p)\eta + \varepsilon\frac{\partial H}{\partial x}(t,x^*,u^*,p)\xi + o(\varepsilon),\\
\int_{t_0}^{t_1} p(\dot{x} - \dot{x^*})dt &= \varepsilon p(t_1)\eta(t_1) - \varepsilon\int_{t_0}^{t_1} p'(t)\eta(t)dt,\\
\delta J\Big|_{u^*}(\xi) = -\int_{t_0}^{t_1}[(p'(t) + \frac{\partial H}{\partial x})\eta + \frac{\partial H}{\partial u}\xi]dt &+ (\frac{\partial K}{\partial x} + p(t_1))\eta(t_1) = 0.
\end{aligned}$$

Hence

$$\begin{cases} p'(t) = & \frac{\partial H}{\partial x}(t,x^*,u^*,p).\\ p(t_1) = & \frac{\partial K}{\partial x}(x*(t_1). \end{cases}$$

If $K = 0$, $p(t_1) = 0$, then $p = p^*(t)$ depends on x^*, u^*. And $\frac{\partial H}{\partial u}(t,x^*,u^*,p) = 0$ for any $t \in [t_0,t_1]$. $H(t,x^*,\cdot,p^*(t))$ has a stationary point at $u^*(t)$. Actually, it is a maxima.

In summary,

$$\begin{cases} \dot{x^*} = & Hp\Big|_* = \frac{\partial H}{\partial p}(t,x^*,u^*,p^*),\\ \dot{p^*} = & -Hx\Big|_* = -\frac{\partial H}{\partial x}(t,x^*,u^*,p^*), \end{cases}$$

that indicates

$$\dot{p^*} = -(fx)^T\Big|_* p^* + Lx\Big|_*,$$

where p^* is adjoint vector or costate.

Maximum Principle. (for $S = [t_0,\infty) \times \{x_1\}$)

The optimal solution $(x^*(t), u^*(t), p^*(t))$ and a constant $p_0^* \leq 0$ satisfy $(p_0^*, p_t^*) \neq (0,0)$. For $\forall[t_0,t_f]$, and
i)

$$\begin{cases} \dot{x^*} = & Hp(x^*,u^*,p^*,p_0^*)\\ \dot{p^*} = & -Hx(x^*,u^*,p^*,p_0^*) \end{cases}$$

Where $H = p\cdot f(x,u) + p_0 L(x,u)$
ii)For each fixed t,the function $u \mapsto H(x^*(t),u,p^*(t),p_0^*)$ has a global maximum at $u = u^*(t)$.i.e.$H(x^*,u^*,p^*,p_0^*) \geq H(x^*,u,p^*,p_0^*)$
iii)$H(x^*,u^*,p^*,p_0^*) = 0$ for $t \in [t_0,t_f]$.

Remark 7.6.1: This is necessary condition.
For sufficient condition, consider the value function $V(t,x) = \inf\limits_{u_{[t,t_1]}} J(t,x,u)$ and HJB

(Hamilton-Jiacobi-Bellman) Equation:

$$\begin{cases} -V_t(t,x) = & \inf_{u \in U} \{L(t,x,u) + V_x(t,x) \cdot f(t,x,u)\}, \\ V(t_1,x) = & K(x). \end{cases}$$

$V_t = \sup_{u \in U} H(t,x,u,-V_x(t,x))$, if we can optimize inf or sup.

8.5 Exercise 8

1. In $\mathbb{R}^2$-space with Euclidean distance, the are infinitely many smooth curves go through two points. Prove that the shortest length is the straight line.

2. (Brachystochrones) Consider two points on the plane: $A=(x_0, y_0)$ and $B=(x_1, y_1)$ where $x_0<x_1$ and $y_0>y_1$. A ball starting from A runs towards B along a smooth (monotonically decreasing) curve $(x, u(x))$ with $u \in M=\left\{w \in C^1\left[x_0, x_1\right], w\left(x_i\right)=y_i, i=0,1\right\}$. The initial velocity of the ball is 0 . Assume that the ball runs subject to the gravity with acceleration g with no friction.

(1). Show that the total time that the ball travels from A to B along the curve is

$$T(u)=\int_{x_0}^{x_1} \sqrt{\frac{1+\left(u^{\prime}(x)\right)^2}{2 g\left(y_0-u(x)\right)}} \mathrm{d} x .$$

(2). Find the differential equation satisfied by the minimizer of $T(u)$.

3. For the functional J defined by

$$J(y)=\int_0^1 y^{\prime}(x) \sqrt{1+\left(y^{\prime \prime}(x)\right)^2} \mathrm{~d} x,$$

find an extremal satisfying the conditions $y(0)=0, y^{\prime}(0)=0, y(1)=1, y^{\prime}(1)=2$.

4. Consider minimizing the following functional

$$J(y)=\int_{x_0}^{x_1}\left(p(x)\left(y^{\prime}(x)\right)^2+q(x) y(x)^2\right) \mathrm{d} x,$$

subject to the boundary conditions

$$y\left(x_0\right)=y\left(x_1\right)=0,$$

and isoperimetric constraint

$$I(y)=\int_{x_0}^{x_1} r(x) y(x)^2 \mathrm{~d} x=1,$$

where $p(x)>0, q(x)$ and $r(x)>0$ are given continuous functions, and in addition $p \in C^1\left[x_0, x_1\right]$. (1). Show that the solution to this problem satisfies the eigenvalue problem

$$\mathcal{L} y:=\left(-p(x) y^{\prime}(x)\right)^{\prime}+q(x) y(x)=\lambda r(x) y(x),$$

where λ is the smallest (positive) eigenvalue. (2). In the case that $x_0=0, x_1=\pi, p(x)=r(x)=1$ and $q(x)=0$, solve this eigenvalue problem and thus find the minimum of $J(y)$.

5. Consider the functional

$$F(u)=\int_{\Omega}\left(\frac{\epsilon}{2}|\nabla u|^2+\frac{1}{\epsilon} f(u)\right) \mathrm{d} \mathbf{x},$$

for any function $u \in U(u_0) = \left\{ w \in C^2(\Omega) \cap C(\bar{\Omega}),\, w|_{\partial\Omega} = u_0 \right\}$ defined on a bounded domain $\Omega \in \mathbb{R}^2$. Here $f \in C^1(\mathbb{R})$ is a given function and ϵ is a positive number. Suppose Ω has smooth boundary $\partial\Omega$. (1). Show that any minimizer $u \in U(u_0)$ of F must satisfy the equation

$$-\epsilon \Delta u + \frac{1}{\epsilon} f'(u) = 0.$$

(2). We consider in addition that there is a boundary contribution to the functional F, i.e.,

$$F(u) = \int_\Omega \left(\frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} f(u) \right) \mathrm{d}\mathbf{x} + \int_{\partial\Omega} \gamma(u) \mathrm{d}s,$$

where $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ with no fixed value on $\partial\Omega$ and $\gamma \in C^1(\mathbb{R})$ is a given function. Here $\mathrm{d}s$ is the differential of arclength. Show that any minimizer $u \in U(u_0)$ of F must satisfy the equation (0.1) in Ω with the following boundary condition

$$\epsilon \frac{\partial u}{\partial n} + \gamma'(u) = 0,$$

where $\frac{\partial u}{\partial n} = \mathbf{n} \cdot \nabla u$ is the normal derivative of u with $\mathbf{n}$ being the outward normal to Ω.

8.6 Answer 8

1. Let $P(x_0, y_0)$, $Q(x_1, y_1)$ be two points on $\mathbb{R}^2$. No matter to assume $x_0 \neq x_1$, otherwise we can rotate the axis.

   The length of the path from P to Q is

$$S(y) = \int_{x_0}^{x_1} \sqrt{1 + (y')^2} dx$$

   And we let $L(x, y, y') = \int_{x_0}^{x_1} \sqrt{1 + (y')^2} dx$, then the Euler-Lagrange equation is

$$\frac{\partial}{\partial y} L(x, y, y') = \frac{d}{dx}(\frac{\partial}{\partial y'} L(x, y, y'))$$

   that is

$$0 = \frac{d}{dx} \frac{y'}{\sqrt{1 + (y')^2}}$$

   then

$$\frac{d}{dx} \frac{y'}{\sqrt{1 + (y')^2}} = C \Rightarrow \frac{(y')^2}{1 + (y')^2} = C^2 \Rightarrow y'(x) = \frac{C^2}{1 - C^2} \Rightarrow y(x) = Ax + B$$

   where $A = \dfrac{C^2}{1 - C^2}(-1 < C < 1), B$ are constant.

   And $y(x_0) = y_0$, $y(x_1) = y_1$, so we have $y(x) = \dfrac{y_1 - y_0}{x_1 - x_0} x + \dfrac{x_1 y_0 - x_0 y_1}{x_1 - x_0}$.

   Therefore, the shortest length which goes through two points is the straight line.

2.

   (1). Let $v(x)$ be the velocity of the ball, since the ball runs with no friction, then we have

$$\frac{1}{2} m v^2(x) = mg(y_0 - u(x)) \Rightarrow v(x) = \sqrt{2g(y_0 - u(x))}$$

   Let $s(x)$ be the length of the path from $(x_0, u(x_0))$ to $(x, u(x))$, then

$$s(x) = \int_{x_0}^{x} \sqrt{1 + (u'(t))^2} dt \Rightarrow \frac{ds}{dx} = \sqrt{1 + (u'(x))^2}$$

   So the total time that the ball travels from A to B along the curve is

$$T(u) = \int_{x_0}^{x_1} \frac{ds}{v} = \int_{x_0}^{x_1} \frac{1}{\sqrt{2g(y_0 - u(x))}} \frac{ds}{dx} dx = \int_{x_0}^{x_1} \frac{\sqrt{1 + (u'(x))^2}}{\sqrt{2g(y_0 - u(x))}} dx$$

   (2). Let $L(u, u') = \dfrac{\sqrt{1 + (u'(x))^2}}{\sqrt{2g(y_0 - u(x))}}$, by the Euler-Lagrange equation, we have

$$\frac{\partial}{\partial u} L(u, u') - \frac{d}{dx}(\frac{\partial}{\partial u'} L(u, u')) = 0$$

$$\frac{\sqrt{1 + (u')^2}}{2\sqrt{2g}(y_0 - u)^{3/2}} = \frac{d}{dx}(\frac{u'}{\sqrt{1 + (u')^2}} \cdot \frac{1}{\sqrt{2g(y_0 - u)}})$$

so we get
$$\frac{\sqrt{1+(u')^2}}{2\sqrt{2g}(y_0-u)^{3/2}} = \frac{2u''(y_0-u)+(u')^2(1+(u')^2)}{2\sqrt{2g}(1+(u')^2)^{3/2}(y_0-u)^{3/2}}$$
then
$$\frac{2u''(y_0-u)-(1+(u')^2)}{2\sqrt{2g}(1+(u')^2)^{3/2}(y_0-u)^{3/2}} = 0$$
thus the minimizer of $T(u)$ satisfies the differential equation:
$$u'' = \frac{1+(u')^2}{2(y_0-u)}.$$

Moreover, if we observe $L(u,u')$ does not contain the independent variable x explicitly, we can get
$$(1+(u')^2)(y_0-u) = C,$$
where C is a constant.

3.

Since J does not depend explicitly on y, there exists a constant c_1 such that
$$\frac{d}{dx}\frac{\partial f}{\partial y''} - \frac{\partial f}{\partial y'} = c_1$$

But, it's also true that since J does not depend explicitly on x there exist a constant c_2 such that
$$y''\frac{\partial f}{\partial y''} - y'\left(\frac{d}{dx}\frac{\partial f}{\partial y''} - \frac{\partial f}{\partial y'}\right) - f = c_2$$
That is, we have that
$$y''\frac{\partial f}{\partial y''} - y'c_1 - f = c_2$$
Therefore,
$$-c_1y' - \frac{y'}{\sqrt{1+y'^2}} = c_2$$
The boundary condition $y'(0) = 0$ implies that $c_2 = 0$, and thus, we only need to solve
$$-c_1y' = \frac{y'}{\sqrt{1+y''^2}}$$
This last equation leads to $y''^2 = \frac{1}{c_1^2} - 1$. Integrating and using the others boundary conditions we see that $y(x) = x^2$.

4. (1). If y is an extremal for the isoperimetric problem, then there is a constant λ such that y satisfies the Euler-Lagrange equation:
$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$
where $F = f - \lambda g = p(y')^2 + (q-\lambda r)y^2$, then
$$\frac{d}{dx}(2py') - 2(q-\lambda r)y = 0 \Rightarrow \frac{d}{dx}(py') - qy + \lambda ry = 0 \Rightarrow \frac{d}{dx}(-py') + qy = \lambda ry$$

Therefore, the solution to this problem satisfies the eigenvalue problem.

(2). If $p(x) = r(x) = 1$, $q(x) = 0$, $x_0 = 0$, $x_1 = \pi$, then the Euler-Lagrange equation reduces to: $y''(x) + \lambda y(x) = 0$ with the boundary conditions $y(0) = y(\pi) = 0$.

Case 1: If $\lambda < 0$, then the general solution is

$$y(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$$

And $y(0) = A + B = 0, y(\pi) = Ae^{\sqrt{-\lambda}\pi} + Be^{-\sqrt{-\lambda}\pi} = 0 \Rightarrow A = B = 0$, then $y = 0$ is trivial solution.

Case 2: If $\lambda = 0$, then $y''(x) = 0 \Rightarrow y'(x) = C_1 \Rightarrow y(x) = C_1x + C_2$. And $y(0) = C_2 = 0, y(\pi) = C_1\pi + C_2 = 0 \Rightarrow C_1 = C_2 = 0$, then $y = 0$ is trivial solution.

Case 3: If $\lambda > 0$, then the general solution is

$$y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

And $y(0) = A = 0, y(\pi) = A\cos\sqrt{\lambda}\pi + B\sin\sqrt{\lambda}\pi = 0 \Rightarrow B = \sin\sqrt{\lambda}\pi = 0$.

Then the eigenvalues for this problem are $\lambda_n = n^2, n = 1, 2, \cdots$, so the smallest eigenvalue is $\lambda = 1$, and the minimizer is $y(x) = B\sin x$.

Since

$$I(B\sin x) = \int_0^\pi B^2\sin^2 xdx = \frac{B^2}{2}\int_0^\pi 1 - \cos 2xdx = \frac{\pi B^2}{2} = 1 \Rightarrow B = \sqrt{\frac{2}{\pi}}$$

then $y(x) = \sqrt{\frac{2}{\pi}}\sin x$, and the minimum of $J(y)$ is

$$\begin{aligned}
J(\sqrt{\frac{2}{\pi}}\sin x) &= \int_0^\pi (\sqrt{\frac{2}{\pi}}\cos x)^2dx \\
&= \frac{2}{\pi}\int_0^\pi \cos^2 xdx \\
&= \frac{1}{\pi}\int_0^\pi 1 + \cos 2xdx \\
&= 1
\end{aligned}$$

5.

(1). Let $\hat{u} = u + \varepsilon'\eta$, $\eta \in C^2(\Omega) \cap C(\bar{\Omega})$ and $\eta|_{\partial\Omega} = 0$, then

$$\begin{aligned}
F(\hat{u}) - F(u) &= \int_\Omega \frac{\varepsilon}{2}|\nabla u + \varepsilon'\nabla\eta|^2 + \frac{1}{\varepsilon}f(u + \varepsilon'\eta) - \frac{\varepsilon}{2}|\nabla u|^2 - \frac{1}{\varepsilon}f(u)d\vec{x} \\
&= \varepsilon'\int_\Omega \varepsilon\nabla u\nabla\eta d\vec{x} + \frac{1}{\varepsilon}\int_\Omega f(u + \varepsilon'\eta) - f(u)d\vec{x} + o(\varepsilon') \\
&= \varepsilon'\Big(\int_\Omega \varepsilon\eta(-\Delta u)d\vec{x} + \int_{\partial\Omega}\eta\frac{\partial u}{\partial\vec{n}}ds\Big) + \varepsilon\int_\Omega \frac{\eta}{\varepsilon}f'(u)d\vec{x} + o(\varepsilon') \\
&= \varepsilon'\int_\Omega \eta\Big(-\varepsilon\Delta u + \frac{1}{\varepsilon}f'(u)\Big)d\vec{x} + o(\varepsilon')
\end{aligned}$$

so the first variation is

$$\delta F(u,\eta) = \int_\Omega \eta\Big(-\varepsilon\Delta u + \frac{1}{\varepsilon}f'(u)\Big)d\vec{x}$$

Since the minimizer of F must satisfy the equation $\delta F(u,\eta)=0$, and η is arbitrary, then

$$-\varepsilon\Delta u+\frac{1}{\varepsilon}f'(u)=0$$

(2). Let $\hat{u}=u+\varepsilon'\eta$, $\eta\in C^2(\Omega)\cap C(\bar{\Omega})$ and $\eta|_{\partial\Omega}=0$, then

$$\begin{aligned}F(\hat{u})-F(u)&=\int_\Omega\frac{\varepsilon}{2}\left|\nabla u+\varepsilon'\nabla\eta\right|^2+\frac{1}{\varepsilon}f(u+\varepsilon'\eta)-\frac{\varepsilon}{2}\left|\nabla u\right|^2-\frac{1}{\varepsilon}f(u)d\vec{x}+\int_{\partial\Omega}\gamma(u+\varepsilon'\eta)-\gamma(u)ds\\&=\varepsilon'\int_\Omega\varepsilon\nabla u\nabla\eta d\vec{x}+\frac{1}{\varepsilon}\int_\Omega f(u+\varepsilon'\eta)-f(u)d\vec{x}+\int_{\partial\Omega}\gamma(u+\varepsilon'\eta)-\gamma(u)ds\\&=\varepsilon'\Big[\varepsilon\Big(\int_\Omega-\eta\Delta ud\vec{x}+\int_{\partial\Omega}\eta\frac{\partial u}{\partial\vec{n}}ds\Big)+\int_\Omega\frac{1}{\varepsilon}\eta f'(u)d\vec{x}+\int_{\partial\Omega}\eta\gamma'(u)ds\Big]+o(\varepsilon')\end{aligned}$$

so the first variation is

$$\delta F(u,\eta)=\int_\Omega\eta\Big[-\varepsilon\Delta u+\frac{1}{\varepsilon}f'(u)\Big]d\vec{x}+\int_{\partial\Omega}\eta\Big[\varepsilon\frac{\partial u}{\partial\vec{n}}+\gamma'(u)\Big]ds$$

Since the minimizer of F must satisfy the equation $\delta F(u,\eta)=0$, and η is arbitrary, then we have

$$-\varepsilon\Delta u+\frac{1}{\varepsilon}f'(u)=0\quad\text{and}\quad\varepsilon\frac{\partial u}{\partial\vec{n}}+\gamma'(u)=0$$